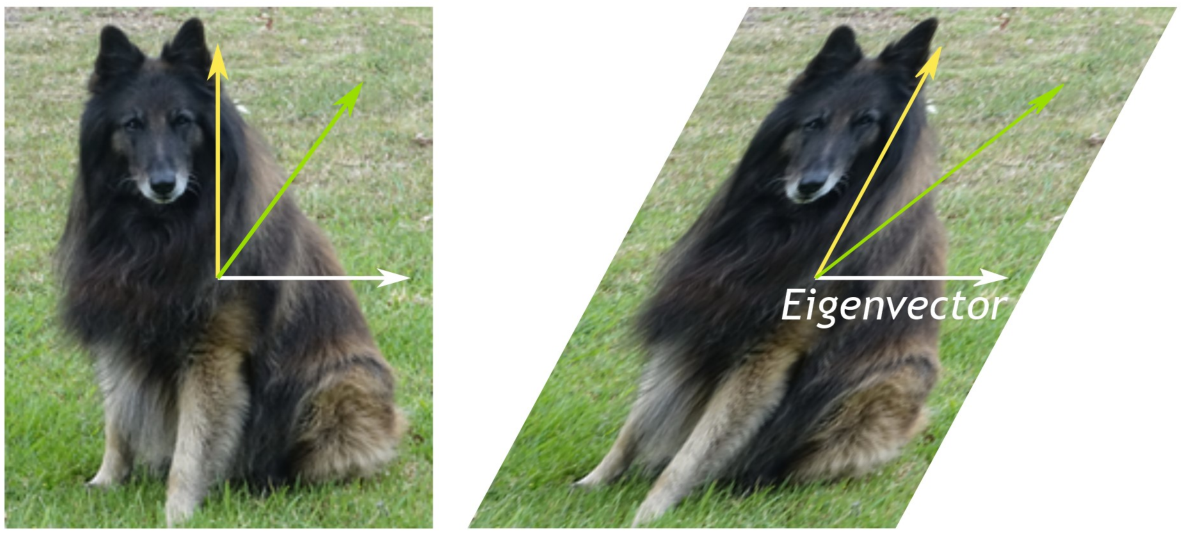
Eigenvector and Eigenvalue

They have many uses!

A simple example is that an eigenvector **does not change direction** in a transformation:



How do we find that vector?

The Mathematics Of It

For a square matrix **A**, an Eigenvector and Eigenvalue make this equation true:

$$\mathbf{A}v = \lambda v$$

Matrix *Eigenvector* *Eigenvalue*

Let us see it in action:

Example: For this matrix

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$$

an eigenvector is

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

with a matching eigenvalue of 6

Let's do some matrix multiplies to see if that is true.

Av gives us:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -6\times1+3\times4 \\ 4\times1+5\times4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

λv gives us :

$$6 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

Yes they are equal!

So we get $Av = \lambda v$ as promised.

Notice how we multiply a **matrix** by a **vector** and get the same result as when we multiply a **scalar** (just a number) by that **vector**.

How do we find these eigen things?

We start by finding the **eigenvalue**. We know this equation must be true:

$$Av = \lambda v$$

Next we put in an identity matrix so we are dealing with matrix-vs-matrix:

$$Av = \lambda Iv$$

Bring all to left hand side:

$$Av - \lambda Iv = 0$$

If **v** is non-zero then we can (hopefully) solve for λ using just the determinant:

$$| A - \lambda I | = 0$$

Let's try that equation on our previous example:

Example: Solve for λ

Start with $| A - \lambda I | = 0$

$$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Which is:

$$\begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

Calculating that determinant gets:

$$(-6-\lambda)(5-\lambda) - 3\times 4 = 0$$

Which simplifies to this Quadratic Equation:

$$\lambda^2 + \lambda - 42 = 0$$

And solving it gets:

$$\lambda = -7 \text{ or } 6$$

And yes, there are **two** possible eigenvalues.

Now we know **eigenvalues**, let us find their matching **eigenvectors**.

Example (continued): Find the Eigenvector for the Eigenvalue **λ = 6**:

Start with:

$$Av = \lambda v$$

Put in the values we know:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

After multiplying we get these two equations:

$$\begin{aligned} -6x + 3y &= 6x \\ 4x + 5y &= 6y \end{aligned}$$

Bringing all to left hand side:

$$\begin{aligned} -12x + 3y &= 0 \\ 4x - 1y &= 0 \end{aligned}$$

Either equation reveals that **y = 4x**, so the **eigenvector** is any non-zero multiple of this:

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

And we get the solution shown at the top of the page:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -6\times1+3\times4 \\ 4\times1+5\times4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

... and also ...

$$6 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

So $Av = \lambda v$, and we have success!

Now it is **your turn** to find the eigenvector for the other eigenvalue of -7

Why?

What is the purpose of these?

One of the cool things is we can use [matrices] to do [transformations] in space, which is used a lot in computer graphics.

In that case the eigenvector is "the direction that doesn't change direction" !

And the eigenvalue is the scale of the stretch:

- **1** means no change,
- **2** means doubling in length,
- **−1** means pointing backwards along the eigenvalue's direction
- etc

There are also many applications in physics, etc.

Why "Eigen"

Eigen is a German word meaning "own" or "typical"

*"das ist ihnen **eigen**"* is German for *"that is **typical** of them"*

Sometimes in English we use the word "characteristic", so an eigenvector can be called a "characteristic vector".

Not Just Two Dimensions

Eigenvectors work perfectly well in 3 and higher dimensions.

Example: find the eigenvalues for this 3x3 matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix}$$

First calculate $A - \lambda I$:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 5 \\ 0 & 4 & 3-\lambda \end{bmatrix}$$

Now the determinant should equal zero:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 5 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0$$

Which is:

$$(2-\lambda)\ [\ (4-\lambda)(3-\lambda) - 5\times4\ ] = 0$$

This ends up being a cubic equation, but just looking at it here we see one of the roots is **2** (because of $2-\lambda$), and the part inside the square brackets is Quadratic, with roots of **−1** and **8**.

So the Eigenvalues are **−1**, **2** and **8**

Example (continued): find the Eigenvector that matches the Eigenvalue **−1**

Put in the values we know:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

After multiplying we get these equations:

$$\begin{aligned} 2x &= -x \\ 4y + 5z &= -y \\ 4y + 3z &= -z \end{aligned}$$

Bringing all to left hand side:

$$\begin{aligned} 3x &= 0 \\ 5y + 5z &= 0 \\ 4y + 4z &= 0 \end{aligned}$$

So **x = 0**, and **y = −z** and so the **eigenvector** is any non-zero multiple of this:

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

TEST Av:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4-5 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

And λv:

$$-1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So $Av = \lambda v$, yay!

(You can try your hand at the eigenvalues of **2** and **8**)