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Homework #2

1. (20 points) Consider a room with 50 people. 10 are wearing red hats, 20 are wearing blue pants, 8 are wearing red hats and blue pants, 6 are wearing red hats and green t-shirts, 13 are wearing blue pants and green t-shirts, 3 are wearing red hats, blue pants and green t-shirts. 15 people are wearing non of these pieces of clothing. How many are wearing green t-shirts?

1) let A = red hats
B = Blue Pants
C = Green T-shirt

*Note:
$(A \cup B \cup C) = 50 - 15$
$= 35$

Using the Formula:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$35 = 10 + 20 + x - 8 - 13 - 6 + 3$$

$\Rightarrow$ we get $\boxed{x = 29}$ People are wearing Green T-shirts

2. (20 points) Let A = {a, b, c} and B = {x, y}.

• What is |A × B|?

• How many surjective functions we have from A to B?

• How many injective functions we have from A to B?

2) $A = \{a, b, c\}$, $B = \{x, y\}$

$|A| = 3$ $\quad |B| = 2$

- $|A \times B| = |A||B| = (3)(2) = \boxed{6}$
- Since $|B| \leq |A|$, there will be surjective functions between A to B.

For $|A| = m$ $\quad |B| = n$, the number of onto functions can be founde be $\sum_{r=1}^{n} (-1)^{n-r}\, {}^{n}C_r\, r^m$, $m \geq n$

$\Rightarrow$ Hence, $= \sum_{r=1}^{2} (-1)^{n-r}\, {}^{2}C_r\, r^3$

$= (-1)^1\, {}^{2}C_1\, (1)^3 + (-1)^0\, {}^{2}C_2\, (2)^3$

$= -2 + 8 = \boxed{6}$

- Since $|A| \not\leq |B|$, there will be no injective functions between A to B.

3. (10 points) Use the Euclidean algorithm to compute gcd(90, 64).

3) gcd(90, 64), where $A = 90$, $B = 64$ and $A \neq 0$, $B \neq 0$ in gcd(A, B).

⇒ Using long division:

$$\begin{array}{r} 1 \\ 64 \overline{)\,90} \\ \underline{64} \\ 26 \end{array}$$

26 → remainder.

This can be written as:

$$90 = 64(1) + 26$$

⇒ Finding gcd(64, 26), where $A = 64$, $B = 26$ and $A \neq 0$, $B \neq 0$

Using long division:

$$\begin{array}{r} 2 \\ 26 \overline{)\,64} \\ \underline{52} \\ 12 \end{array}$$

12 → remainder

This can be written as:

$$64 = 26(2) + 12$$

⇒ Finding gcd(26, 12), where $A = 26$, $B = 12$ and $A \neq 0$, $B \neq 0$

using long division:

$$\begin{array}{r} 2 \\ 12 \overline{)\,26} \\ \underline{24} \\ 2 \end{array}$$

2 → remainder

This can be written as:

$$26 = 12(2) + 2$$

⇒ Finding gcd(12, 2), where $A = 12$, $B = 2$ and $A \neq 0$, $B \neq 0$

Using long division:

$$\begin{array}{r} 6 \\ 2 \overline{)\,12} \\ \underline{12} \\ 0 \end{array}$$

0 → remainder

This can be written as:

$$12 = 2(6) + 0$$

⇒ gcd(2, 0), where $A = 2$, $B = 0$, since $B = 0 \rightarrow$ gcd(A, B) = A, where $B = 0$

↳ gcd(2, 0) = 2 ∴ therefore $\boxed{\gcd(90, 64) = 2}$

4. (30 points) Use the mathematical induction to prove the following statements:

- $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$

4) a) $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$

let $n=1$, then LHS is $\sum_{i=1}^{n}(8i-5) \Rightarrow (8-5) = 3$

and RHS $\Rightarrow (4(1)^2-(1)) = 3$

for $n=1$, LHS = RHS

let K be any Positive integer and the result is true for $n=K \Rightarrow \sum_{i=1}^{K}(8i-5) = 4K^2-K$

For $n=K+1$

LHS $\Rightarrow 3+11+9 \ldots + (8(K+1)-5)$

$\Rightarrow (3+11+9+\ldots+8K)+(8(K+1)-5)$

since $n=K$ is true,

hence, $(3+11+9+\ldots+8K) = 4K^2-K$

LHS $\Rightarrow 4K^2-K+(8(K+1)-5)$

$4K^2-K+8K+8-5$

$4K^2+7K+3$

The RHS of $n=K+1$ equall,

$4(K+1)^2-(K+1) = 4K^2+7K+3$

LHS = RHS

- By the methematical induction, since $n=K+1$ is true, this means that $P(n)$ is true for all Positive integers n.

- $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

b) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Let $n=1$, then LHS is $\sum_{i=1}^{n} \frac{1}{i(i+1)} \Rightarrow \frac{1}{1(1+1)} = \frac{1}{2}$

RHS $\Rightarrow \frac{n}{n+1} \Rightarrow \frac{1}{1+1} = \frac{1}{2}$

For $n=1$, LHS = RHS

Next, suppose that $\sum_{i=1}^{K} \frac{1}{i(i+1)} = \frac{K}{K+1}$ — ①

Let $n = K+1$

$\hookrightarrow \sum_{i=1}^{K+1} \frac{1}{i(i+1)} = \frac{(K+1)}{(K+1)+1}$

LHS $\Rightarrow \sum_{i=1}^{K+1} \frac{1}{i(i+1)}$

$\Rightarrow \sum_{i=1}^{K} \frac{1}{i(i+1)} + \frac{1}{(K+1)(K+2)}$

By ①, we have

$\Rightarrow \frac{K}{K+1} + \frac{1}{K+1(K+2)} \rightarrow \frac{K^2+2K+1}{(K+1)(K+2)}$

$\Rightarrow \frac{K(K+2)+1}{(K+1)(K+2)}$

$\Rightarrow \frac{(K+1)^2}{(K+1)(K+2)} = \frac{K+1}{K+2} \Rightarrow$ RHS

Therefore, LHS = RHS.

- By the methematical induction, since $n=K+1$ is true, this means that $P(n)$ is true for all positive integers n.

5. (20 points) Let $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2x - 1 & \text{if } x > 0 \\ -2x & \text{if } x \leq 0 \end{cases}$$

1. Prove that f is one-to-one and onto.
2. Determine f^{-1}.

5) $f : A \to B$, where $y = f(x)$, $x \in A$ and $y \in B$

1) One-to-one functions: Function $f : A \to B$ is one to one image of distinct element of A are distinct under f.

onto functions: function $f : A \to B$ is onto if every element of B is image of some element of A.

$\Rightarrow$ We have Domain $(\mathbb{Z}) = (-\infty, \infty)$

(range) Co-domain$(\mathbb{N}) = (1, \infty)$

$\Rightarrow f(x) = -2x$

$\left.\begin{array}{l} f(-1) = 2 \\ f(-2) = 4 \\ f(-3) = 6 \end{array}\right\} f(-\infty) = \text{even numbers}$

$\Rightarrow f(x) = 2x - 1$

$\left.\begin{array}{l} f(1) = 1 \\ f(2) = 3 \\ f(3) = 5 \end{array}\right\} f(\infty) = \text{odd numbers}$

$\Rightarrow$ since every element in domain have a unique number, it's one-to-one and every element in co-domain have the numbers pre-image, therefore it's onto.

2) $f^{-1}(x)$ for $x > 0$

$y = 2x - 1$

$x = \dfrac{(y+1)}{2}$

$\boxed{f^{-1}(x) = \dfrac{(y+1)}{2}}$

for $x \leq 0$

$y = -2x$

$\boxed{f^{-1}(x) = \dfrac{-y}{2}}$