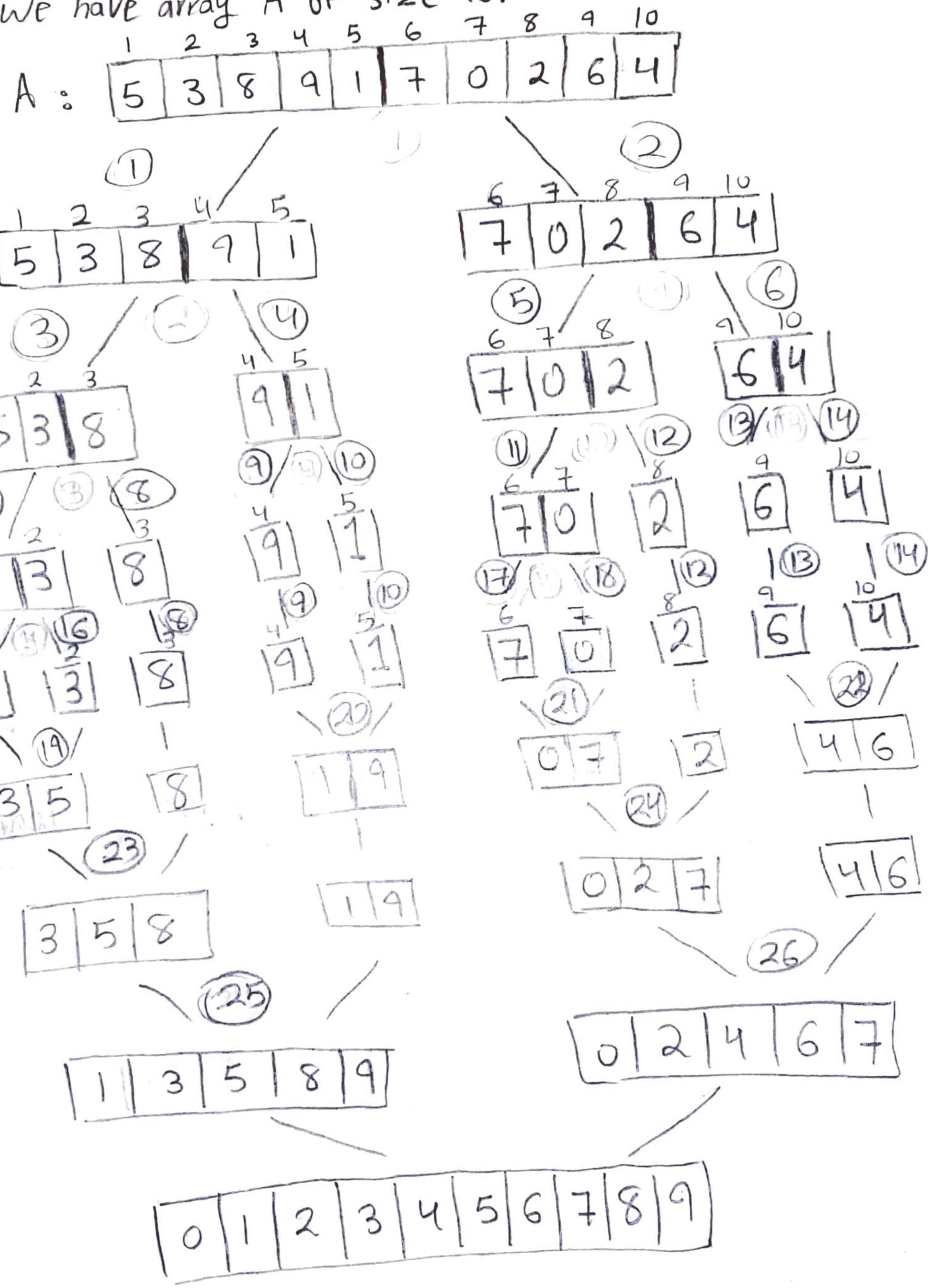
1] Merge sort Order of Recursion:

⇒ We have array A of size 10.

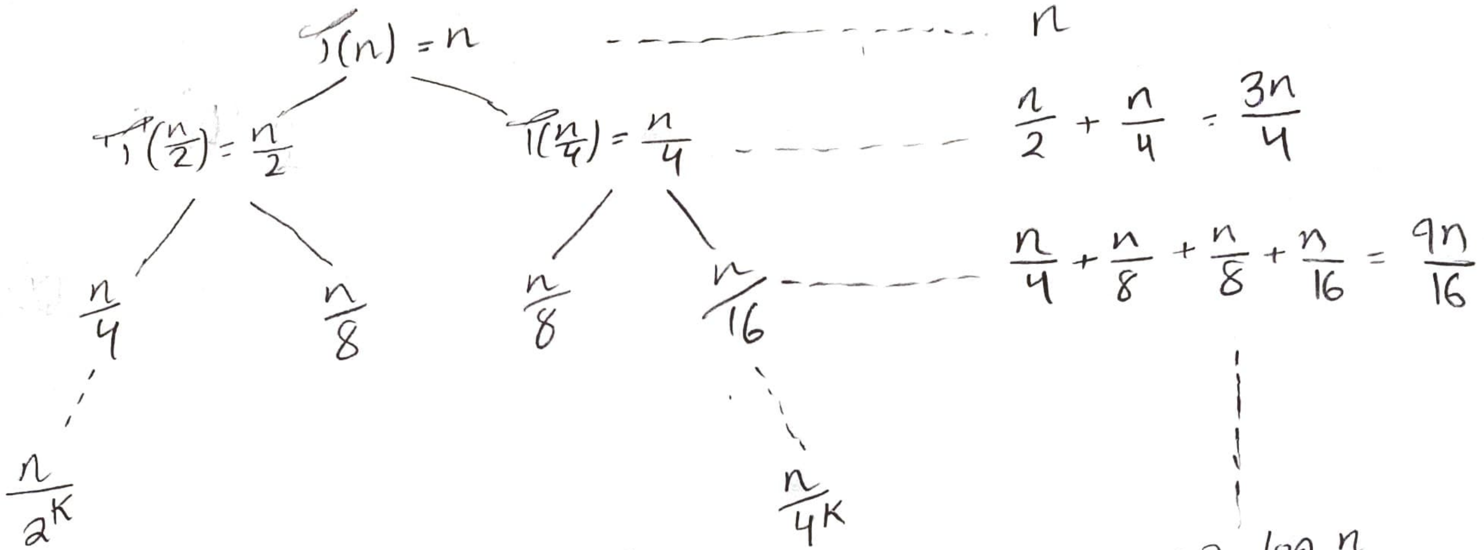


* Final answer:
An array A of size 10 have 18 recursive calls to merge sort, but to sort the array, it have 26 calls to merge sort.

2 Recursion Tree:

a) $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + \Theta(n)$

$= T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + n$



$\Rightarrow$ Total $= n + \frac{3n}{4} + \left(\frac{3}{4}\right)^2 n + \left(\frac{3}{4}\right)^3 n + \cdots + \left(\frac{3}{4}\right)^{\log_2 n} n$

$= n\left(\underbrace{1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots \left(\frac{3}{4}\right)^{\log_2 n}}_{\text{decreasing geometric series } = 1}\right)$

$= n(1) = n$

$\Rightarrow \boxed{T(n) = \Theta(n)}$

b) $T(n) = 3T\left(\frac{n}{2}\right) + 4T\left(\frac{n}{2}\right) + \Theta(n^2)$

$= 3T\left(\frac{n}{2}\right) + 4T\left(\frac{n}{2}\right) + n^2$

n^2 ------------------ n^2

$\left(\frac{n}{2}\right)^2 \quad \left(\frac{n}{2}\right)^2 \quad \left(\frac{n}{2}\right)^2$ ----- $\left(\frac{n}{2}\right)^2$ ----- sum all $= \frac{7}{4}n^2$

$\left(\frac{n}{4}\right)^2$ ------------------ $\left(\frac{n}{4}\right)^2$ ---- sum all $= \frac{7^2}{16}n^2$

$\left(\frac{n}{2^k}\right)$

$\Rightarrow$ Total $= n^2 + \frac{7}{4}n^2 + \frac{7^2}{4^2}n^2 + \cdots$

$= n^2 \underbrace{\left(1 + \frac{7}{4} + \frac{7^2}{4^2} + \cdots\right)}_{(1)}$

$= n^2$

$\boxed{T(n) = \Theta n^2}$

- Check using Master Theorem:

$T(n) = 3T\left(\frac{n}{2}\right) + 4T\left(\frac{n}{2}\right) + \Theta(n^2)$

$= 7T\left(\frac{n}{2}\right) + \Theta(n^2)$

$a = 7, \; b = 2, \; d = 2$

Since $d < \log_b a$, then $\Theta(n^{\log_b a})$

$\Rightarrow n^{\log_2 7} = n^{2.8} = n^{2+0.8}$

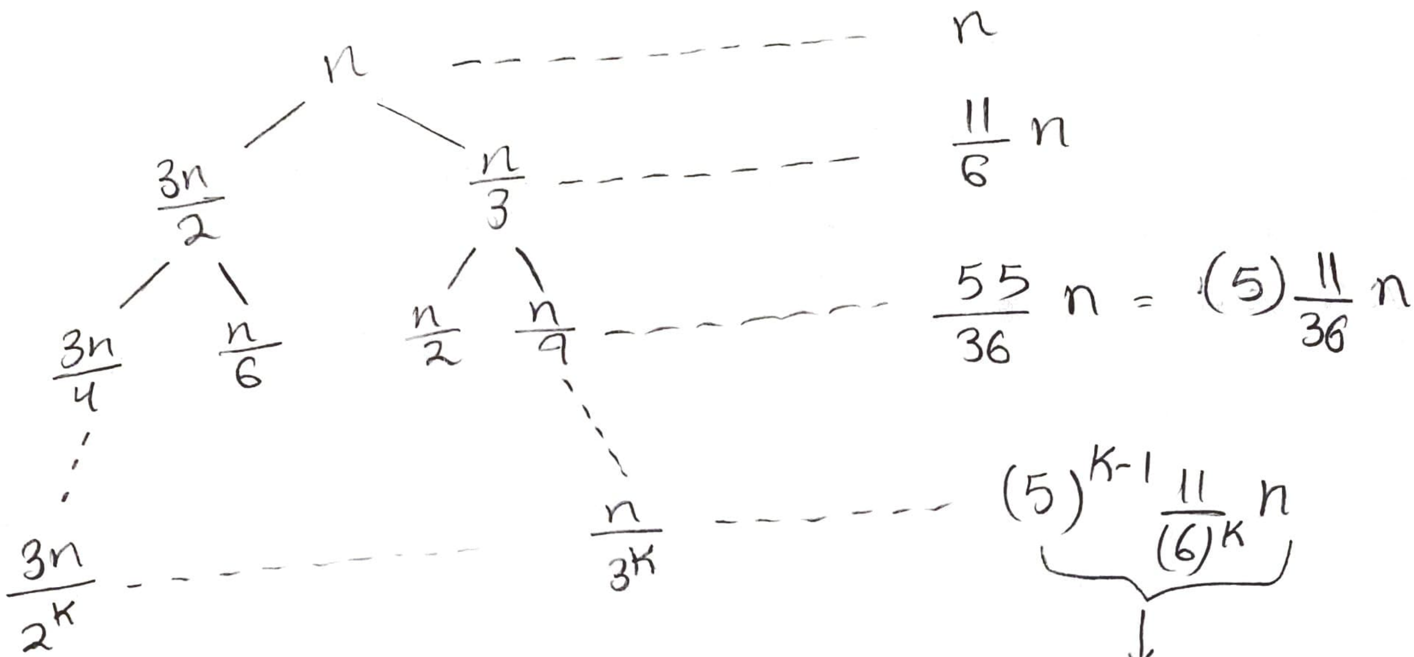
Also, $f(n) = \Theta(n^2)$ and since $\therefore f(n) = \Omega(n^{\log_b a + \epsilon})$ for $\epsilon > 0$

Thus, $\therefore T(n) = \Theta(f(n))$

$\boxed{= \Theta(n^2)}$

c) $T(n) = T\left(\frac{3n}{2}\right) + T\left(\frac{n}{3}\right) + \Theta(n)$

$= T\left(\frac{3n}{2}\right) + T\left(\frac{n}{3}\right) + n$



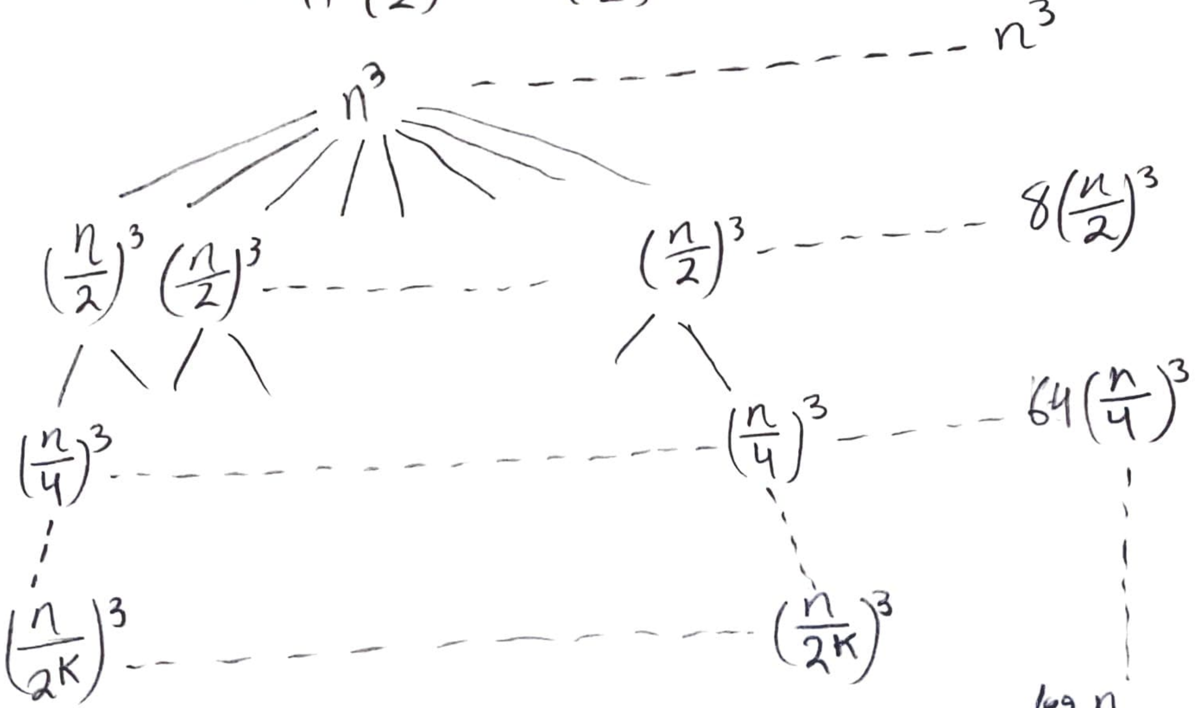
$\Rightarrow \text{total} = n + \frac{11}{6}n + (5)\frac{11}{36}n + \cdots$

$= n\left(1 + \underbrace{\frac{11}{6} + (5)\frac{11}{36} + \cdots (5)^{K-1}\left(\frac{11}{6^K}\right)}_{1}\right)$

$\boxed{T(n) = \Theta(n)}$

d) $T(n) = 4T\left(\frac{n}{2}\right) + 4T\left(\frac{n}{2}\right) + \Theta(n^3)$

$= 4T\left(\frac{n}{2}\right) + 4T\left(\frac{n}{2}\right) + n^3$



$\Rightarrow$ Total $= n^3 + 8\left(\frac{n}{2}\right)^3 + 64\left(\frac{n}{4}\right)^3 + \cdots \frac{8^{\log_2 n}}{8^{\log_2 n}} n^3$

$= n^3 (\underbrace{1 + 1 + 1 + 1 + \cdots}_{\log n = \text{hight times}})$

$\boxed{T(n) = \Theta(n^3 \log n)}$

- check using Master theorem:

$T(n) = 4T\left(\frac{n}{2}\right) + 4T\left(\frac{n}{2}\right) + \Theta(n^3)$

$= 8T\left(\frac{n}{2}\right) + \Theta(n^3)$

$a = 8, \ b = 2, \ d = 3$

since $d = \log_b a$, then $\Theta(n^d \log n)$

$\boxed{\Rightarrow T(n) = \Theta(n^3 \log n)}$

3] Master theorem:

*Note: Master theorem:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$T(n) = \begin{cases} \Theta(n^d) & a < b^d \\ \Theta(n^d \log n) & a = b^d \\ \Theta(n^{\log_b a}) & a > b^d \end{cases} \quad a \geq 1,\ b \geq 2,\ d \geq 0$$

a) $T(n) = 25T\left(\frac{n}{2}\right) + \Theta\left(\frac{n^3}{\log n}\right)$

$a = 25,\ b = 2,\ d = 3$

$\Rightarrow 25 > 2^3$

$25 > 8 \rightarrow$ case 3 $\Theta(n^{\log_b a})$

$\boxed{T(n) = \Theta(n^{\log_2 25})}$

b) $W(n) = 5W\left(\frac{n}{3}\right) + \Theta(n^3)$

$a = 5,\ b = 3,\ d = 3$

$\Rightarrow 5 < 3^3$

$5 < 27 \rightarrow$ case 1 $\Theta(n^d)$

$\boxed{W(n) = \Theta(n^3)}$

c) $T(n) = T\left(\frac{9n}{13}\right) + \Theta(n)$

$a = 1,\ b = \frac{9}{13},\ d = 1$

$\Rightarrow 1 < \frac{9}{13}^1 \rightarrow$ case 1 $\Theta(n^d)$

$\boxed{T(n) = \Theta(n)}$

d) $T(n) = \sqrt{2}\, T\left(\frac{n}{2}\right) + \Theta(n^2)$

$a = 2^{\frac{1}{2}},\ b = 2,\ d = 2$

$\Rightarrow 2^{\frac{1}{2}} < 2^2$

$2^{\frac{1}{2}} < 4 \rightarrow$ case 1 $\Theta(n^d)$

$\boxed{T(n) = \Theta(n^2)}$

e) $T(n) = 4T\left(\frac{n}{4}\right) + \Theta(\sqrt{n})$

$a = 4,\ b = 4,\ d = n^{\frac{1}{2}}$

$\boxed{\rightarrow f(n) \text{ is not a Polynomial}}$

f) $T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n \log_2 n)$

$a = 3,\ b = 2,\ K = 1$

$\Rightarrow$ Using case 4 of Master theorem

$T(n) = \Theta(n^{\log_b a} \log^{K+1} n)$

we get: $\boxed{T(n) = \Theta(n^{\log_2 3} \log^2 n)}$

g) $T(n) = 16T\left(\frac{n}{4}\right) + \Theta(n^2\sqrt{n})$

$a = 16,\ b = 4,\ d = \frac{5}{2}$

$\boxed{\rightarrow f(n) \text{ is not a Polynomial}}$

Answer Version 2:

- According to an example we did in the lecture, Problem (e) and (g) can be solved using Master theorem.

e) $T(n) = 4T\left(\frac{n}{4}\right) + \Theta(\sqrt{n})$

$a=4, \quad b=4, \quad d=\frac{1}{2}$

$\Rightarrow 4 > 4^{\frac{1}{2}} \rightarrow$ case 3, $\Theta(n^{\log_b a})$

$$\boxed{T(n) = \Theta(n^{\log_4 4}) = \Theta(n)}$$

g) $T(n) = 16T\left(\frac{n}{4}\right) + \Theta(n^2\sqrt{n})$

$a=16, \quad b=4, \quad d=\frac{5}{2}$

$\Rightarrow 16 < 4^{\frac{5}{2}} \rightarrow$ Case 1, $\Theta(n^d)$

$$\boxed{T(n) = \Theta(n^{\frac{5}{2}})}$$

4] Solving Recurrence Relations:

a) $W(n) = 2W(n/2) + \Theta(n)$

$\Rightarrow t_n - 2t_{\frac{n}{2}} = n$

let $n = 2^K$, where $(K = \log n)$, we get:

$\Rightarrow t_K - 2t_{K-1} = 2^K$ — ①

Replace K with K+1 in eq ①, we get:

$\Rightarrow t_{K+1} - 2t_K = 2^{K+1}$ — ②

Multiply eq ① by 2, we get:

$\Rightarrow 2t_K - 4t_{K-1} = 2^{K+1}$ — ③

Subtract eq ③ from eq ②, we get:

$\Rightarrow t_{K+1} - 4t_K + 4t_{K-1} = 0$

Replace the t_i's with x^i, we get:

$\Rightarrow x^{K+1} - 4x^K + 4x^{K-1} = x^{K-1}(x^2 - 4x + 4) = x^{K-1}(x-2)(x-2) = 0$

Thus, the roots are: $r_1 = 2$
$r_2 = 2$

Therefore:

$\Rightarrow t_K = C_1 2^K + C_2 (K)(2^K)$

Since $n = 2^K$, and $K = \log n$, then t_n will be:

$\Rightarrow t_n = C_1 n + C_2 (\log n)(n)$

$\boxed{\Rightarrow \text{order of } \Theta(n \log n)}$

b) $w(n) = 4w\left(\frac{n}{4}\right) + \Theta(n)$

$\Rightarrow t_n - 4t_{\frac{n}{4}} = n$

Let $n = 4^K$, where ($K = \log n$), we get:

$\Rightarrow t_K - 4t_{K-1} = 4^K$ — ①

Replace K with K+1 in eq ①, we get:

$\Rightarrow t_{K+1} - 4t_{K} = 4^{K+1}$ — ②

Multiply eq ① by 4, we get:

$\Rightarrow 4t_K - 16t_{K-1} = 4^{K+1}$ — ③

Subtract eq ③ from eq ②, we get:

$\Rightarrow t_{K+1} - 8t_K + 16t_{K-1} = 0$

Replace the t_i's with x^i, we get:

$\Rightarrow x^{K+1} - 8x^K + 16x^{K-1} = x^{K-1}(x^2 - 8x + 16) = x^{K-1}(x-4)(x-4) = 0$

Thus, the roots are: $r_1 = 4$

$r_2 = 4$

Therefore:

$\Rightarrow t_K = C_1 2^K + C_2 (K)(2^K)$

Since $n = 2^K$ and $K = \log n$, then t_n will be:

$\Rightarrow t_n = C_1 n + C_2 (\log n)(n)$

$\boxed{\Rightarrow \text{Order of } \Theta(n\log n)}$