DMET 502/701
Computer Graphics

Projections

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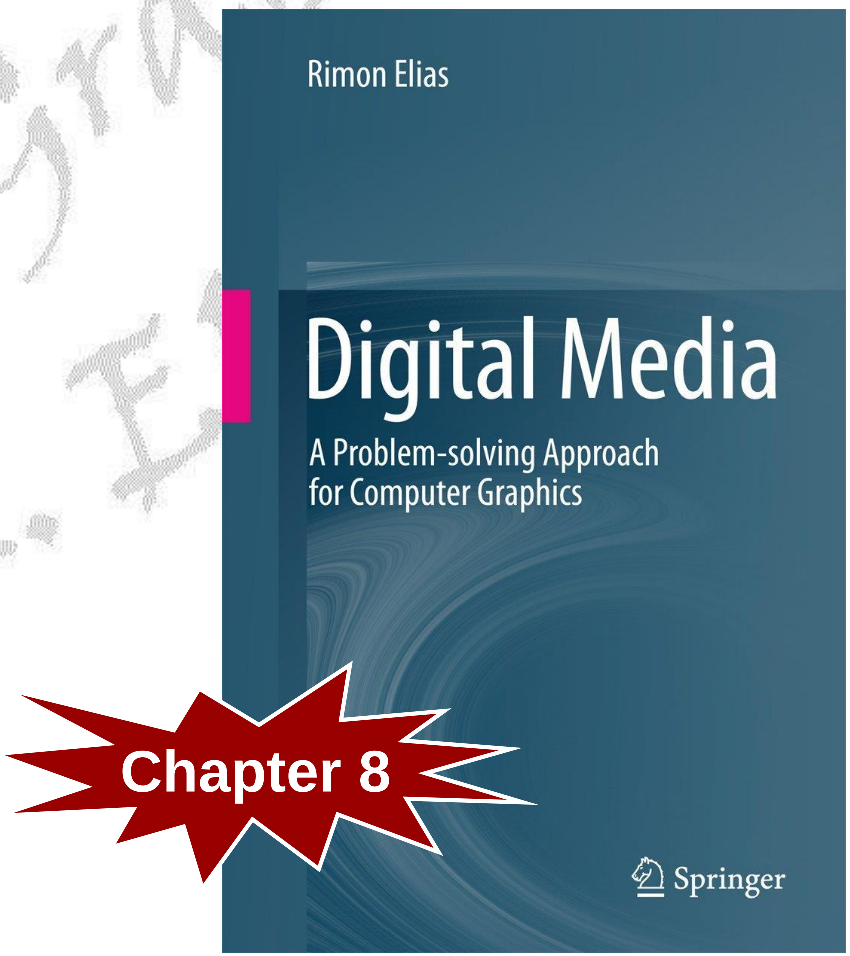
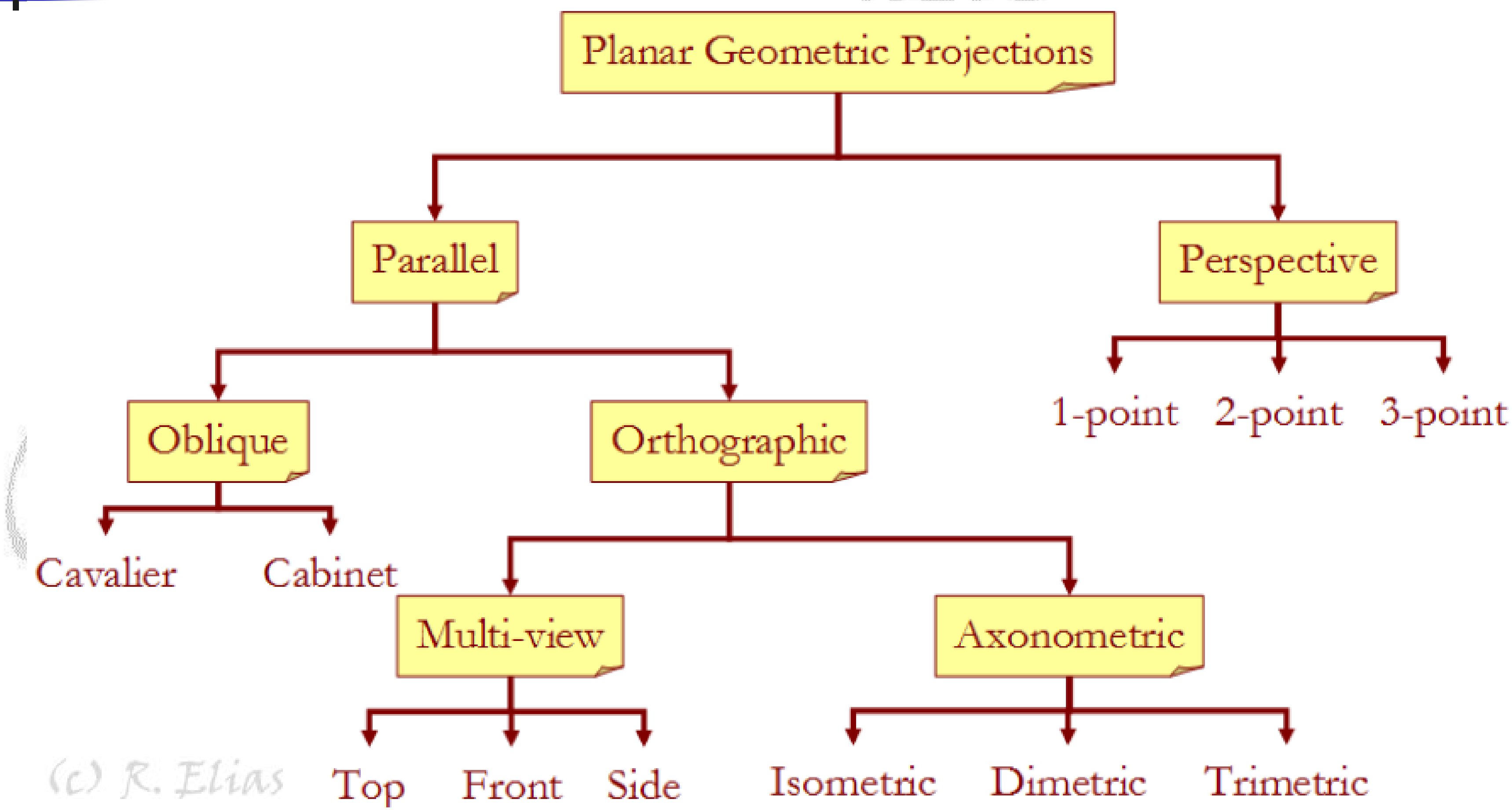


Main Types of Projections

- Types of projections can be split into two main categories:
  - Planar projection
  - Non-planar projection
- The differences between them are in the type of viewing surface and the projectors.

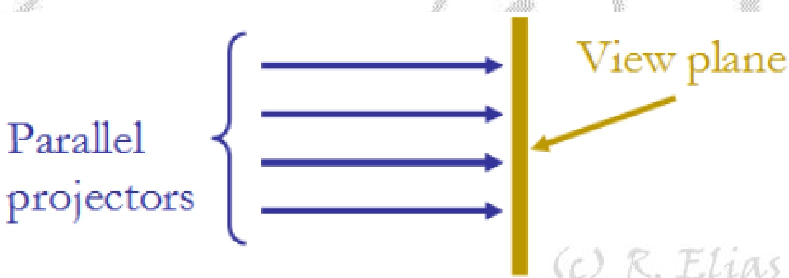
Type	**Viewing surface**	**Projectors**
Planar projection	Planar (view plane)	Straight lines
Non-planar projection	Curved	Curved

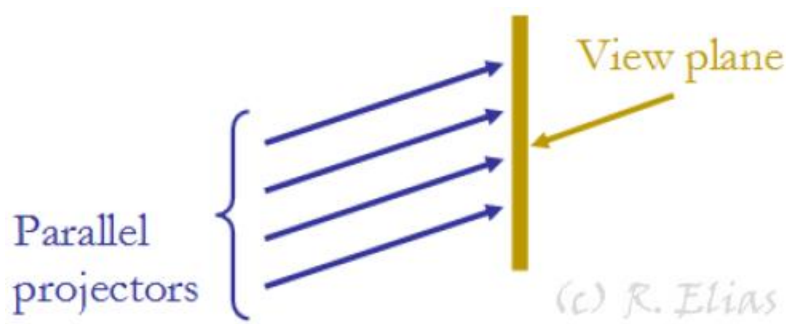
Subclasses of Planar Geometric Projections



Parallel Projections

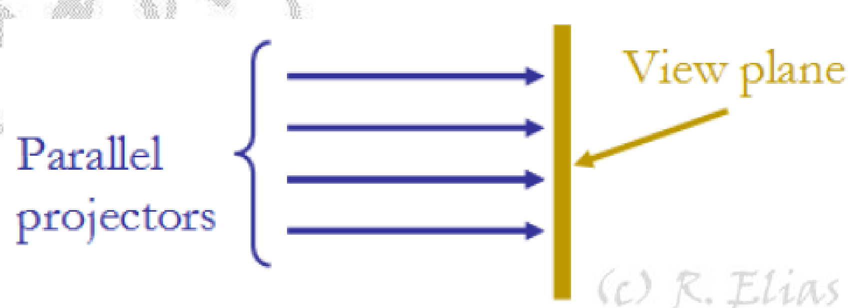
- Two main characteristics appear in **parallel projections**:
  1. The viewpoint or the center of projection is placed at infinity.
  2. Consequently, projectors are parallel to each other; hence comes the term "parallel projection"
- This results in a parallelepiped view volume.
- The proportions of an object (not necessarily the actual measurements) are maintained through parallel projections.





Orthographic Projections

- In **orthographic projections**, the projectors are:
  - Parallel to each other and
  - perpendicular to view plane.

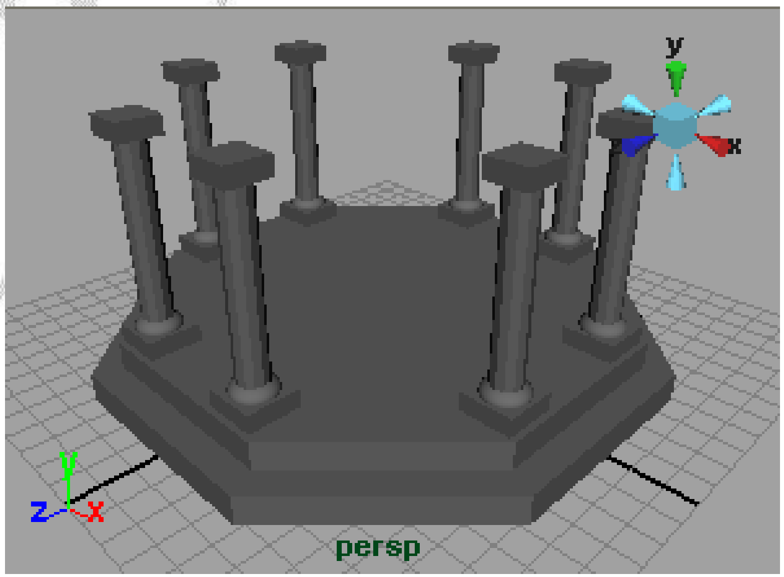


Depending on the normal to the view plane

- Orthographic projections are categorized as:
  - Multi-view or
  - Axonometric

Multi-view Projections

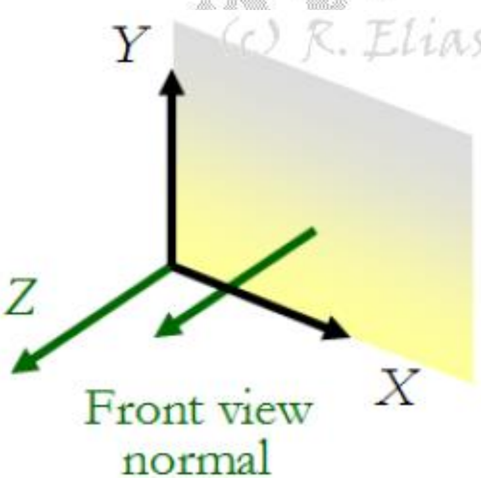
- They do not show the object as a 3D model.
- A multi-view projection may display a single face of a 3D model.
- Examples: top, front and side views.
- The view plane normal is parallel to one of the principal axes.
- They do preserve the dimensions and the angles. Thus, they are used in engineering and architectural drawings.

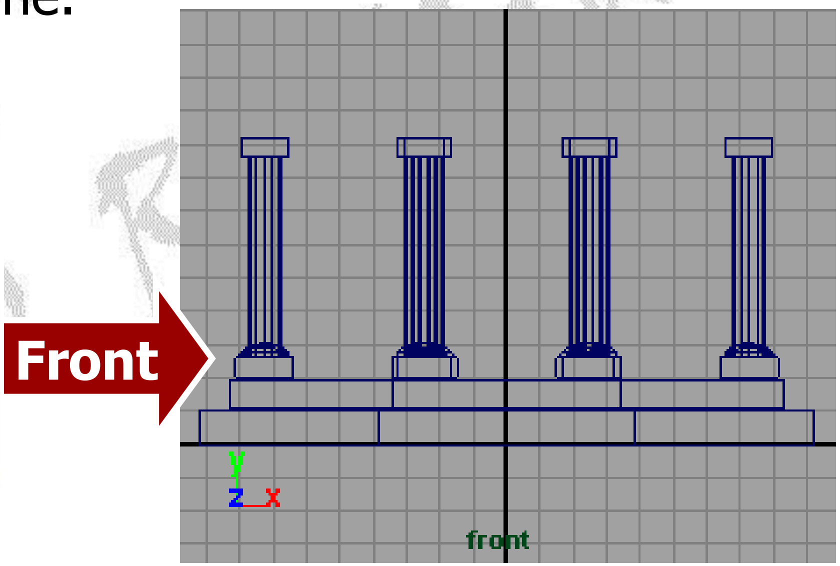


Perspective of a 3D model (**not** a multi-view projection)

Multi-view Projections: Front View

- In a right-handed coordinate system where the *y*-axis points upwards, the front view normal is parallel to the positive *z*-axis. In other words, the view plane is parallel to the *xy*-plane.
- The *z*-coordinates are discarded and the *x*- and *y*-coordinates are mapped to the view plane.





Front view

Multi-view Projections: Front View

- We can estimate the location of a 3D point $[x, y, z]^T$ after front projection onto the *xy*-plane as

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\dot{\mathbf{P}}}$$

point after front projection

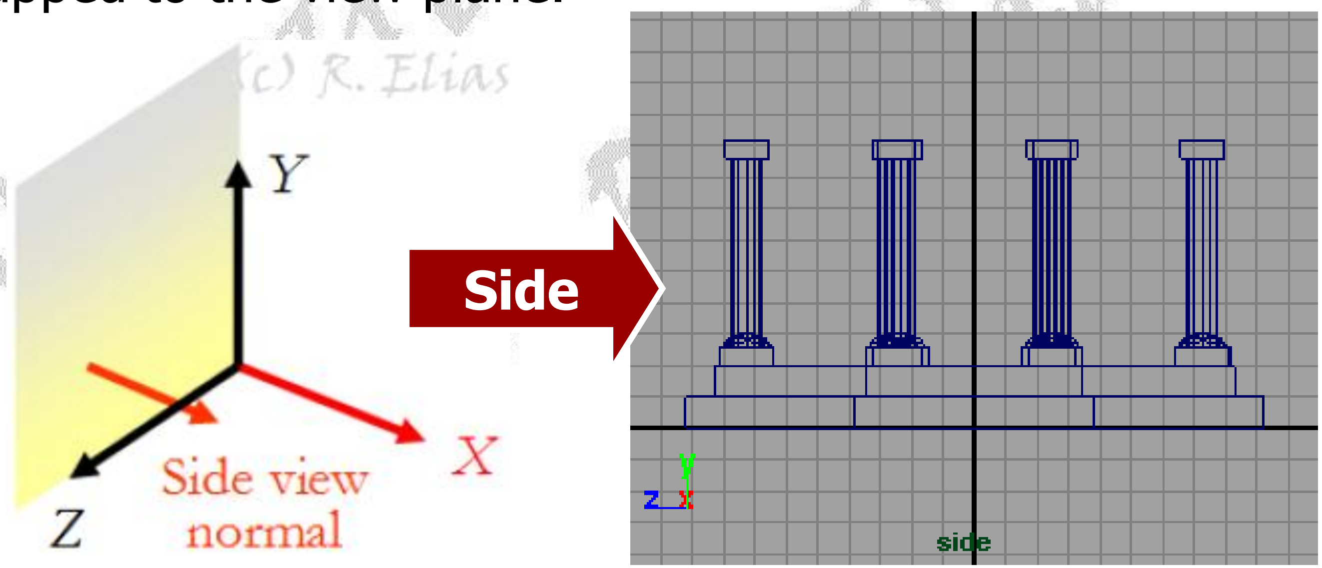
point before front projection

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{P}'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\mathbf{P}}$$

Multi-view Projections: Side View

- In a right-handed coordinate system, the side view normal is parallel to the positive *x*-axis. In other words, the view plane is parallel to the *yz*-plane.
- The *x*-coordinates are discarded and the *y*- and *z*-coordinates are mapped to the view plane.



Side view

Multi-view Projections: Side View

- We can estimate the location of a 3D point $[x, y, z]^T$ after side projection onto the *yz*-plane as

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix}0\\y'\\z'\end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\dot{\mathbf{P}}}$$

point after side projection

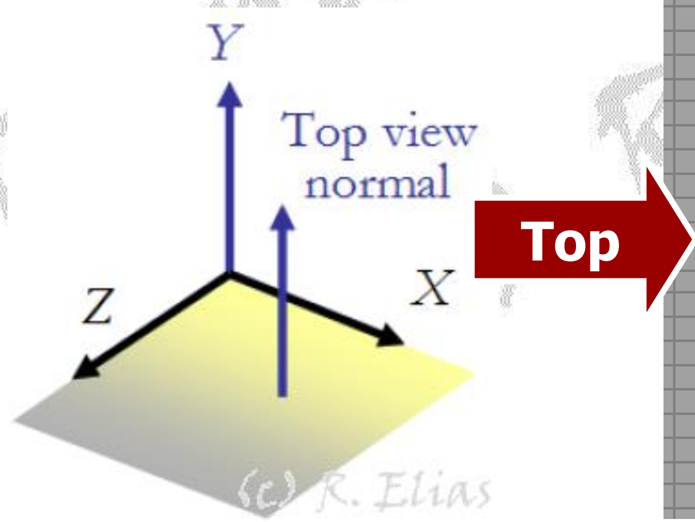
point before side projection

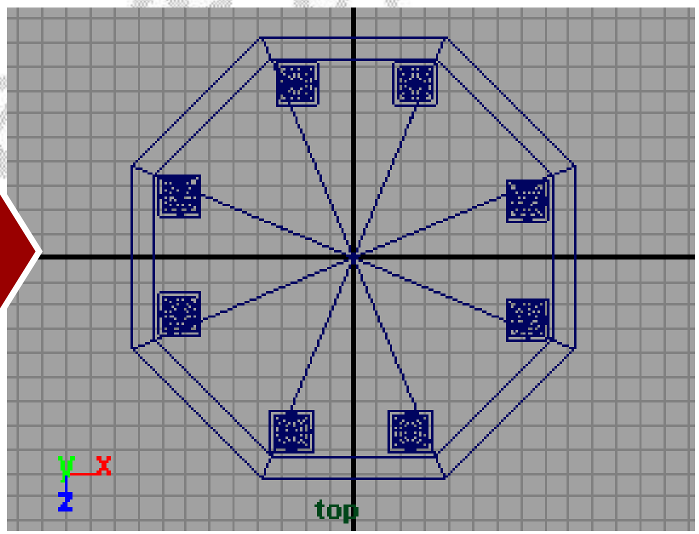
Homogeneous coordinates

$$\underbrace{\begin{bmatrix}0\\y'\\z'\\1\end{bmatrix}}_{\mathbf{P}'} = \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}x\\y\\z\\1\end{bmatrix}}_{\mathbf{P}}$$

Multi-view Projections: Top View

- In a right-handed coordinate system, the top view normal is parallel to the positive *y*-axis. In other words, the view plane is parallel to the *zx*-plane.
- The *y*-coordinates are discarded and the *z*- and *x*-coordinates are mapped to the view plane.





Top view

Multi-view Projections: Top View

- We can estimate the location of a 3D point $[x, y, z]^T$ after top projection onto the *zx*-plane as

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ 0 \\ z' \end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\dot{\mathbf{P}}},$$

point after top projection

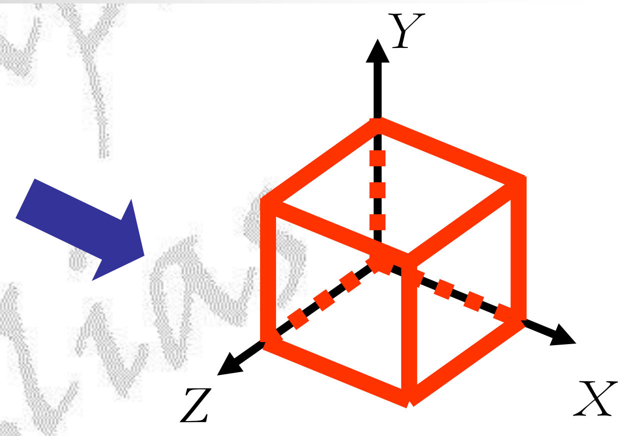
point before top projection

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ 0 \\ z' \\ 1 \end{bmatrix}}_{\mathbf{P}'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\mathbf{P}}$$

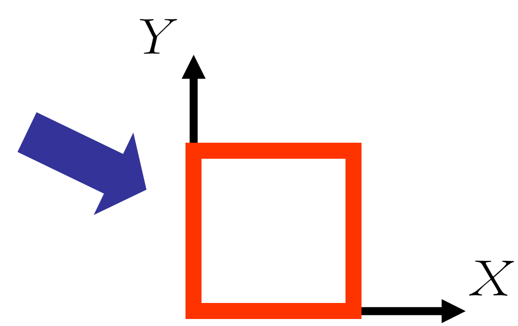
Axonometric Projections

- In **axonometric projection**, the view plane normal is placed in any direction such that the three axes may be visible.



- In this case, the view plane should intersect *at least* ***two*** of the principal axes.

- If the view plane normal is placed parallel to *one* principal axis, the projection turns to a multi-view projection.



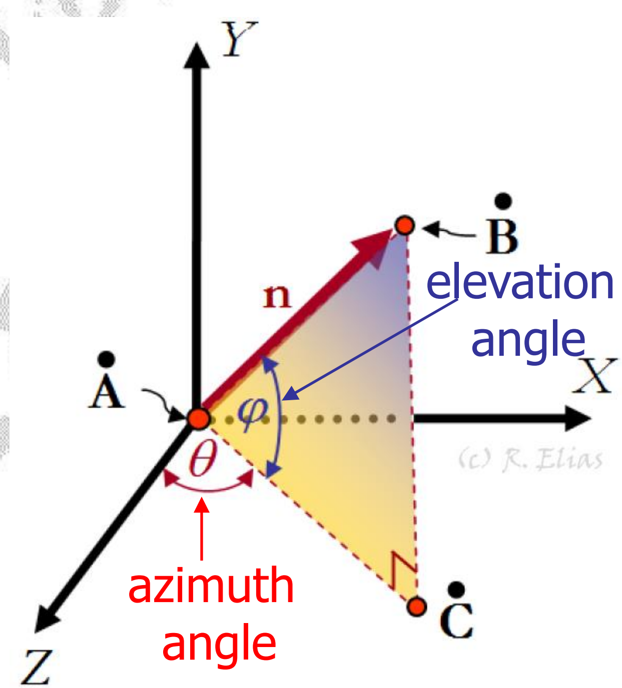
Axonometric Projections

- In an axonometric projection:
  - Parallel lines remain parallel after projection.

  - In addition:
    - If a line is parallel to the view plane:
      - Line length is preserved.
    - If a line is not parallel to the view plane:
      - Line proportions (not lengths) are maintained.
      - Equal lengths of parallel lines will be foreshortened equally.

Axonometric Projections

- The axonometric projection matrix can be obtained by rotating the normal vector **n** to coincide with the *z*-axis; hence, the front projection matrix can be used.
  1. Rotate through an angle $-\theta$ about the *y*-axis.
  2. Rotate through an angle φ about the *x*-axis.
  3. Use the front view projection matrix.



Axonometric Projections

3 Project 2 Rotate 1 Rotate

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi) & -\sin(\varphi)\cos(\theta) \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\dot{\mathbf{P}}}$$

Axonometric projection matrix

Inhomogeneous coordinates

Axonometric Projections

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{P}'} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi) & -\sin(\varphi)\cos(\theta) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\mathbf{P}}$$

Axonometric projection matrix

Homogeneous coordinates

Axonometric Projections: An Example

- **Example:** When omitting the front view projection, estimate the projection matrix in this case for inhomogeneous points.
- **Answer:**

$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi) & -\sin(\varphi)\cos(\theta) \\ \cos(\varphi)\sin(\theta) & \sin(\varphi) & \cos(\varphi)\cos(\theta) \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\dot{\mathbf{P}}}$$

Inhomogeneous coordinates

Isometric Projection

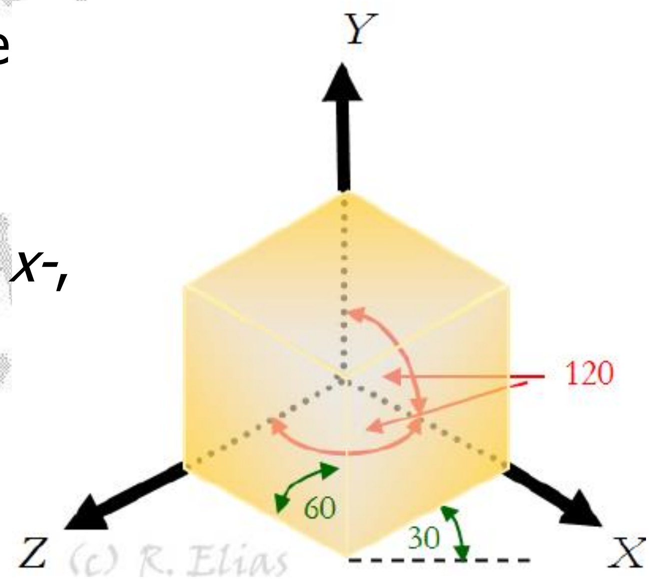
- In **isometric projection**, the view plane normal makes equal angles with the three principal axes.

- The angles between the projection of the *x*-, *y*-, and *z*- axes are all the same, or 120°.

- The normal **n** is expressed as:

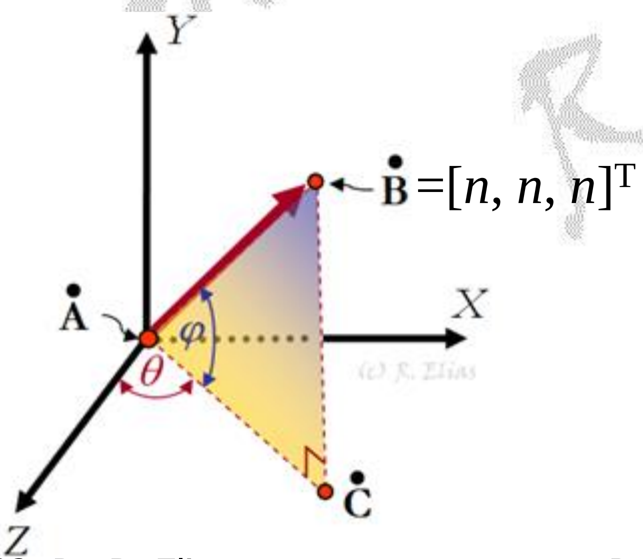
$$\mathbf{n} = [x_n, y_n, z_n]^T \quad \text{such that} \quad |x_n| = |y_n| = |z_n|$$

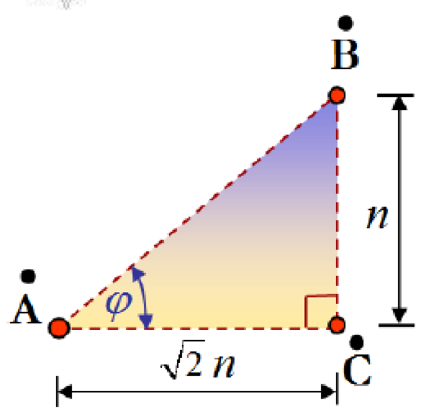
- This restricts the view plane to only 8 directions; one for each octant.



Isometric Projection

- Isometric projection properties:
  - Parallel lines remain parallel.
  - Vertical lines remain vertical.
  - Horizontal lines are drawn at 30° to the horizontal.
  - Line lengths are preserved or scaled equally along each axis.

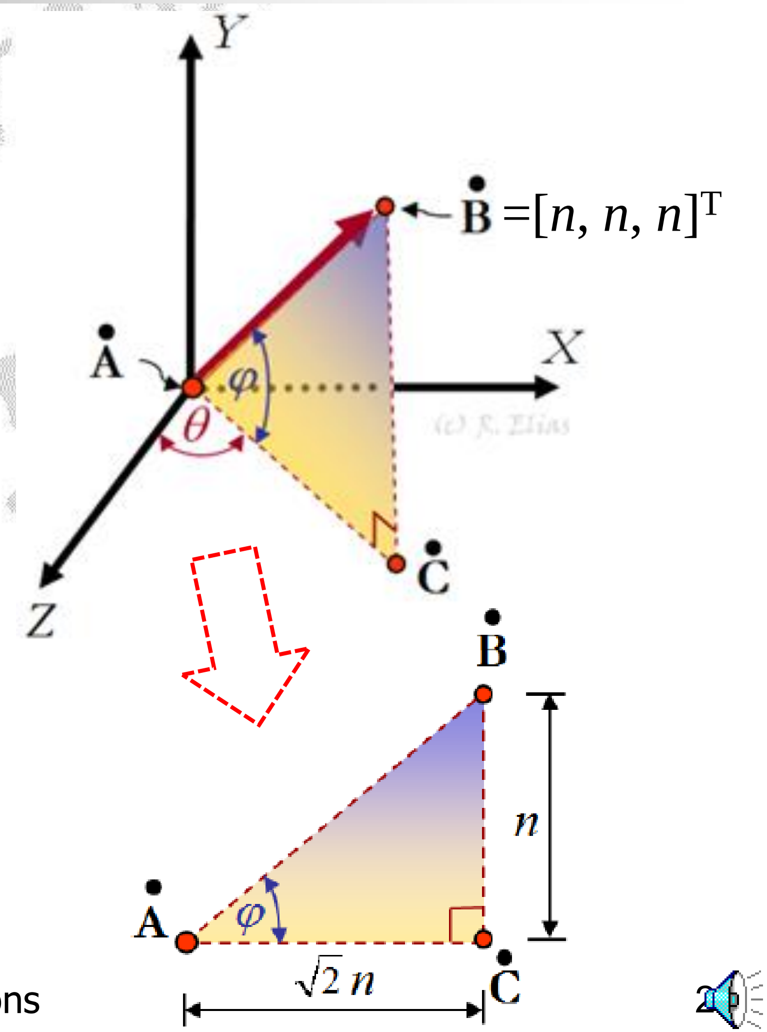




Isometric Projection: What are the Angles θ and φ?

- Assuming that the normal vector to the view plane is $[n, n, n]^T$:
  - the azimuth angle θ must be 45°.
  - the elevation angle φ is estimated as

$$\varphi = \tan^{-1}\left(\frac{n}{\sqrt{2}n}\right) = 35.2644^\circ$$



Isometric Projection: An Example

- **Example:** Based on the previous angle values, estimate the isometric projection matrix for the octant where the *x*-, *y*- and *z*-coordinates are all positive.

- **Answer:**

$$\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi) & -\sin(\varphi)\cos(\theta) \\ \cos(\varphi)\sin(\theta) & \sin(\varphi) & \cos(\varphi)\cos(\theta) \end{bmatrix}$$

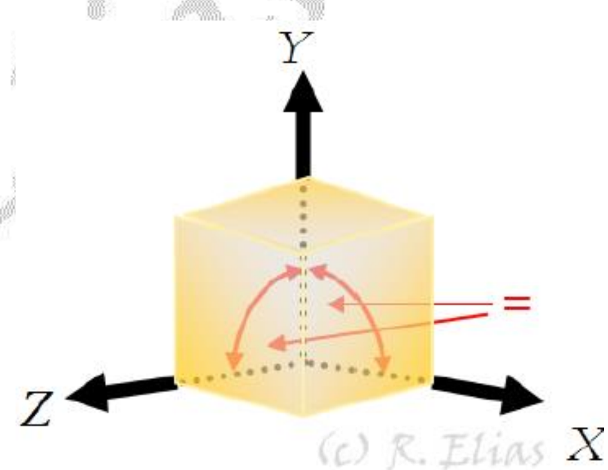
$$= \begin{bmatrix} \cos(45) & 0 & -\sin(45) \\ -\sin(35.2644)\sin(45) & \cos(35.2644) & -\sin(35.2644)\cos(45) \\ \cos(35.2644)\sin(45) & \sin(35.2644) & \cos(35.2644)\cos(45) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Inhomogeneous coordinates

Dimetric Projection

- In **dimetric projection**, the view plane normal makes equal angles with only two of the principal axes.
- Consequently, two of the angles enclosed between the projections of the principal axes are equal.



- If the normal **n** is expressed as:

$$\mathbf{n} = [x_n, y_n, z_n]^T$$

- then

$$x_n = |y_n| \text{ or } x_n = |z_n| \text{ or } y_n = |z_n|$$

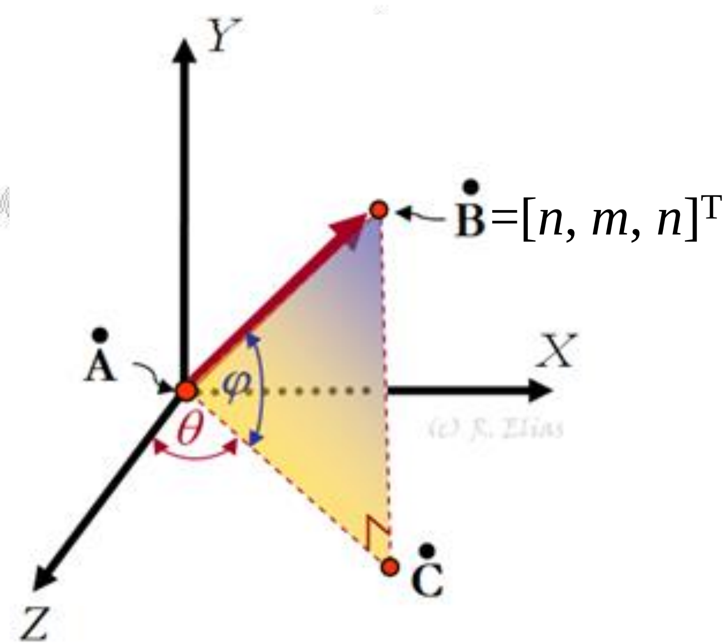
- Parallel lines remain parallel and their lengths are preserved or scaled equally along the two equally foreshortened axes.

Dimetric Projection: An Example

- **Example:** Assuming that the normal vector to the view plane is $[n, m, n]^T$, determine the values of the azimuth and elevation angles (i.e., θ and φ).

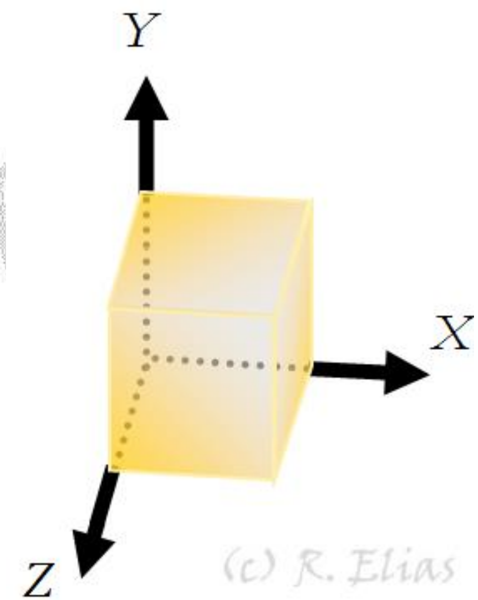
- **Answer:**
  - The azimuth angle θ must be 45°.
  - The elevation angle φ is estimated as

$$\varphi = \tan^{-1}\left(\frac{m}{\sqrt{2}n}\right).$$



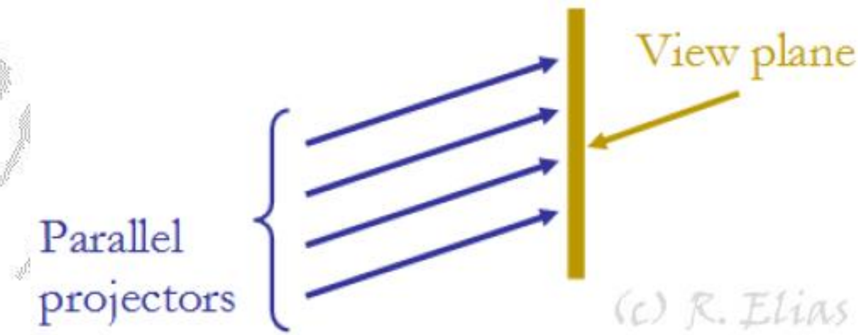
Trimetric Projection

- In **trimetric projection**, the view plane normal makes different angles with the three principal axes.
- No two components of **n** have the same value.
- None of the angles enclosed between the projections of the principal axes are equal.
- Lines are scaled differently along principal axes.



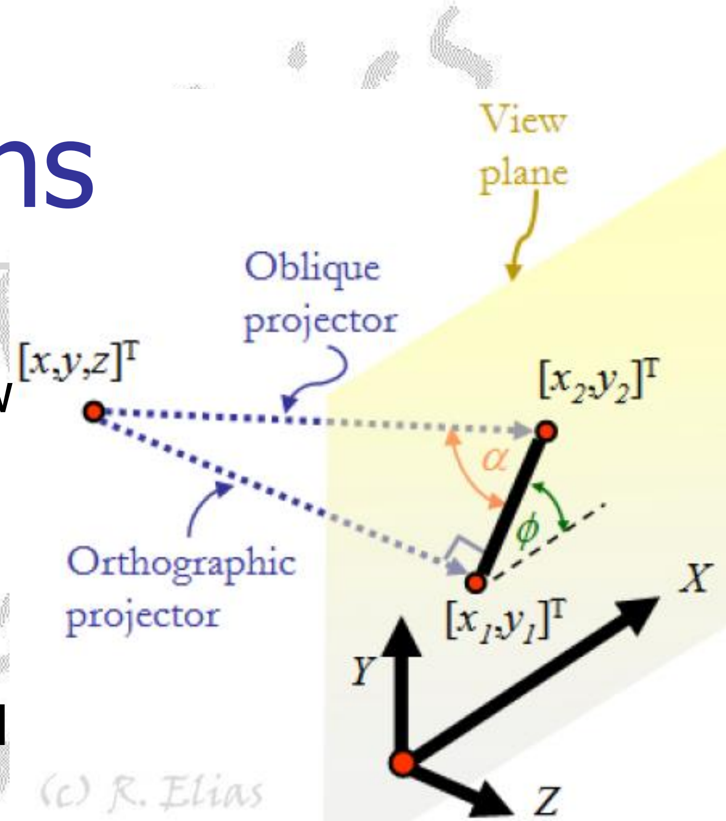
Oblique Projections

- In **oblique projections**, the projectors are:
  - Parallel to each other but
  - **NOT perpendicular** to view plane.



- Similar to axonometric projections, oblique projections present 3D models.
- Similar to multi-view projections, oblique projections displays the exact shapes of faces parallel to the view plane.
- Oblique projections are categorized as:
  - Cavalier or
  - Cabinet

Oblique Projections

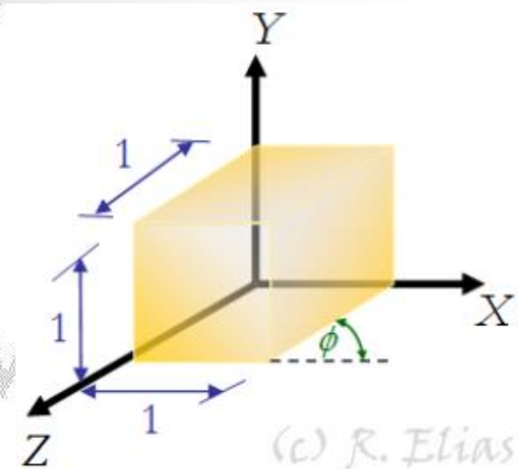


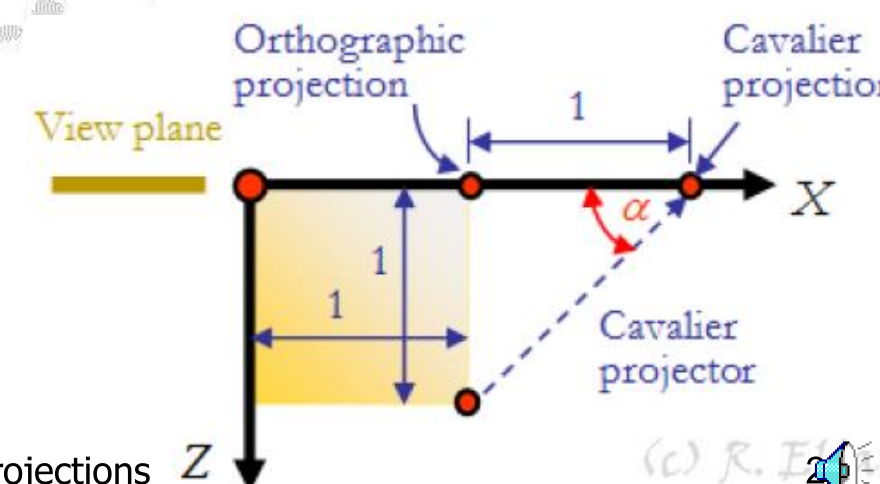
How is oblique projection achieved?

- Consider a 3D point $[x,y,z]^T$ and a view plane.
- If the point is orthographically projected (i.e., the projectors make an angle of 90° with the view plane) onto the plane, the location of the projected point will appear at point $[x_1, y_1]^T$.
- Alternatively, if the projectors form another angle with the plane, the projected point will reside at another location $[x_2,y_2]^T$ on this plane.
- The angle α between the oblique projector and the line connecting $[x_1,y_1]^T$ and $[x_2,y_2]^T$ determines the category of the oblique projection (i.e., cavalier or cabinet).

Cavalier Projection

- The projection is a **cavalier projection** when **$\alpha = 45°$**.
- Consider a cube with faces parallel to the principal planes in the world coordinate system.
- One face (parallel to the view plane) will be orthographically projected as in multi-view projections.
- The lines perpendicular to that face will maintain their actual lengths since $\tan(\alpha) = 1$.
- However, it does not seem to be a real 3D model!





Cavalier Projection

- The location of $[x, y, z]^T$ after cavalier projection onto the *xy*-plane:

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix} 1 & 0 & \cos(\phi) \\ 0 & 1 & \sin(\phi) \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\dot{\mathbf{P}}}$$

point after projection

Cavalier projection matrices

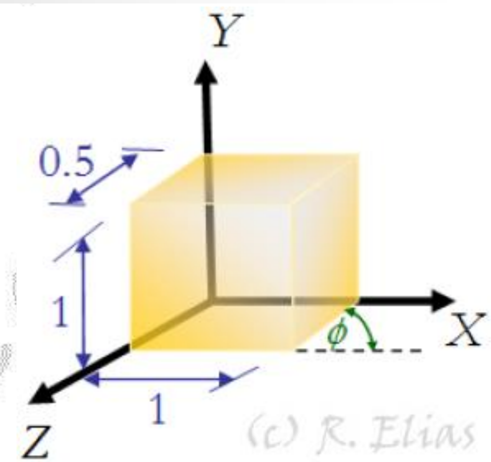
point before projection

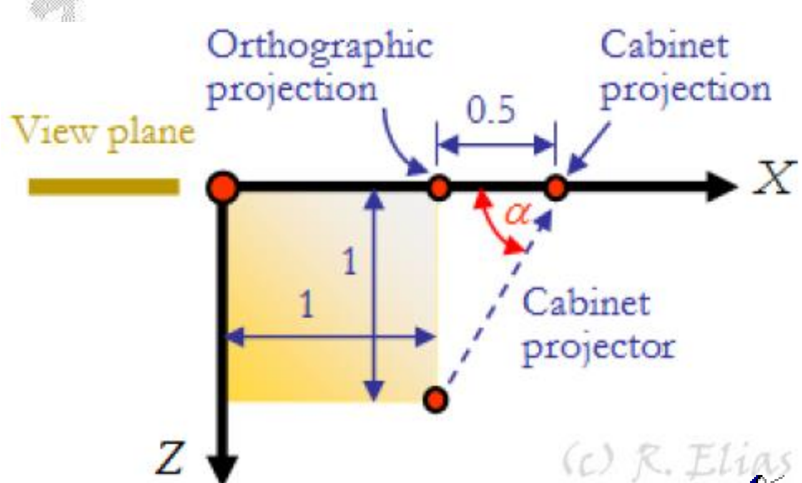
Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{P}'} = \begin{bmatrix} 1 & 0 & \cos(\phi) & 0 \\ 0 & 1 & \sin(\phi) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\mathbf{P}}$$

Cabinet Projection

- The projection is a **cabinet projection** if $\alpha = 63.4^\circ$.
- Unlike the previous case, lines perpendicular to the view plane are displayed at one-half of their actual lengths since $\tan(\alpha) = 2$.
- It does seem to be a real 3D model!





Cabinet Projection

- The location of $[x, y, z]^T$ after cabinet projection onto the *xy*-plane:

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix} 1 & 0 & \frac{1}{2}\cos(\phi) \\ 0 & 1 & \frac{1}{2}\sin(\phi) \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\dot{\mathbf{P}}}$$

point after projection

Cabinet projection matrices

point before projection

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{P}'} = \begin{bmatrix} 1 & 0 & \frac{1}{2}\cos(\phi) & 0 \\ 0 & 1 & \frac{1}{2}\sin(\phi) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\mathbf{P}}$$

Oblique Projections: An Example

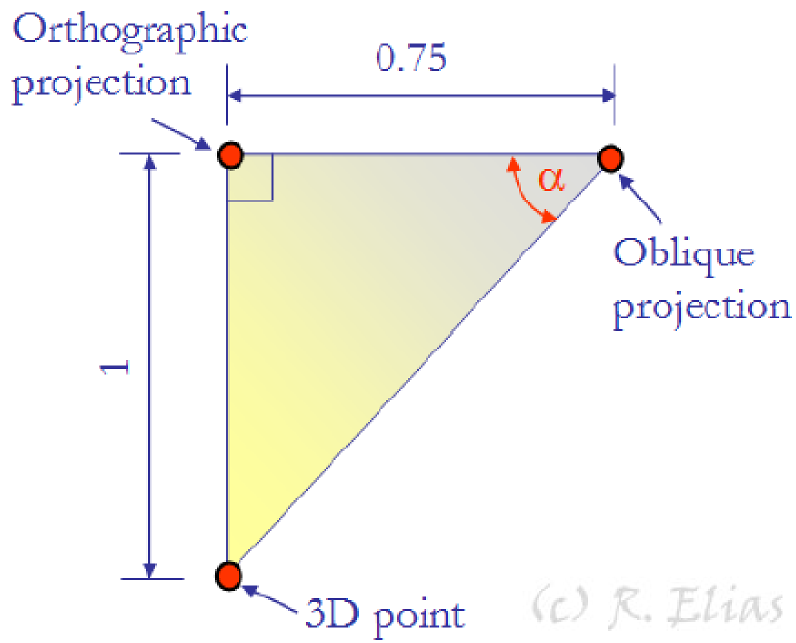
- **Example:** Consider a cube whose faces are parallel to the three principal planes. If an oblique projection is applied to this cube such that the view plane is parallel to the *xy*-plane, one face of the cube that is parallel to the view plane is orthographically projected.

- The lines perpendicular to that face maintain their actual lengths if $\alpha = 45°$ in case of cavalier projection. The same lines may be displayed at one-half of their actual lengths if $\alpha = 63.4°$ in case of cabinet projection.

- What would the value of angle α be if those lines are to be projected at 0.75 of their actual lengths?

Oblique Projections: An Example

- **Answer:**
- Consider the triangle shown whose vertices are
  1. the 3D point
  2. its orthographic projection
  3. its oblique projection.

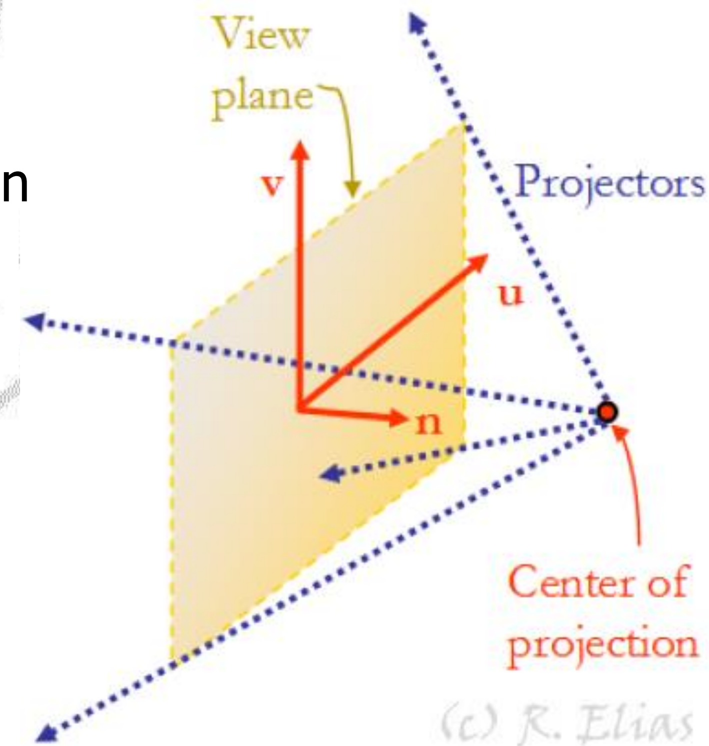


- The angle α is obtained as

$$\alpha = \cot^{-1}(0.75) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1301^\circ$$

Perspective Projections

- **Perspective projections** represent the 2^{nd} subclass of planar geometric projections.
- The two main characteristics appearing in **perspective projections** are:
  - The viewpoint or the center of projection (COP) is placed at a finite distance from the view plane.
  - Consequently, projectors are **NOT** parallel to each other as in case of parallel projections.



- This results in a pyramidal view volume.

Perspective Projections

- In this class of projections, lines not parallel to the view plane converge to a distant point (called a *vanishing point*).
- Objects far from the center of projection appear smaller comparing to identical objects closer to the center of projection.
- This gives a more realistic look!

Source: Google images

Perspective Projections: Categories

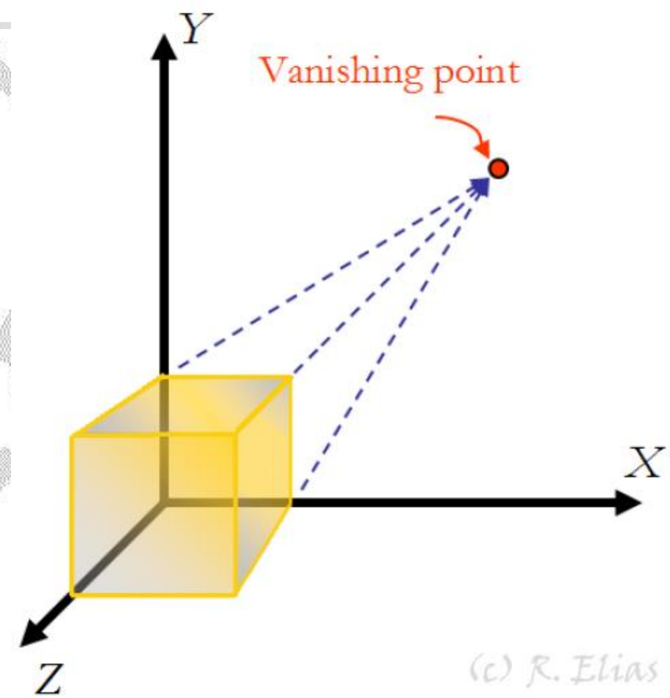
- Perspective projections can be categorized into:
  - One-point projections.
  - Two-point projections.
  - Three-point projections.

- The differences between these categories are in the orientation of the view plane and the number of vanishing points.

The terms one-, two- and three-point perspective projections do not mean that the exact numbers of vanishing points in these projections are one, two and three points respectively.

One-point Perspective Projection

- **One-point perspective** exists when the view plane is parallel to two principal axes (*x*- and *y*-axes in the figure shown).
- In this case:
  - The view plane intersects the *z*-axis.
  - The normal to the view plane is parallel to the *z*-axis.
  - Two components of the view plane normal **n** are zeros.



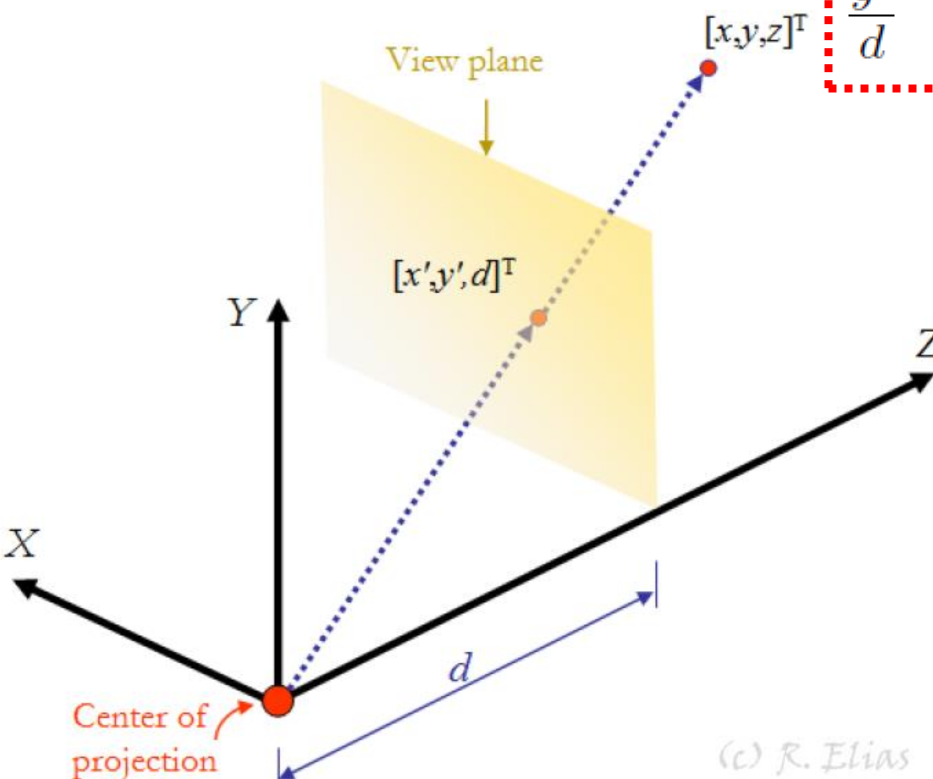
  - Lines parallel to the view plane will be orthographically projected as a multi-view projection.
  - Lines parallel to the 3rd axis (which is parallel to the view plane normal) converge to a single vanishing point.

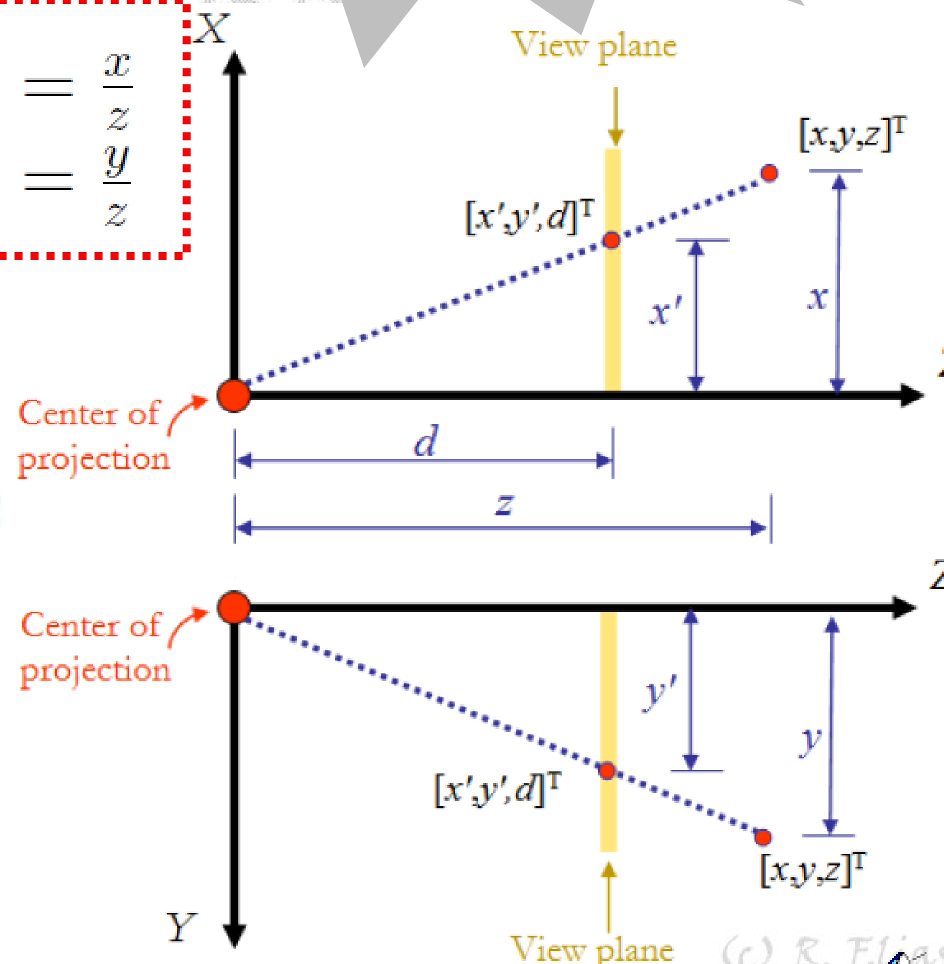
One-point Perspective Projection

COP is at the origin and view plane is at $z=d$

- Considering similar triangles:

$$\frac{x'}{d} = \frac{x}{z}$$
$$\frac{y'}{d} = \frac{y}{z}$$





One-point Perspective Projection

COP is at the origin and view plane is at $z=d$

$$\frac{x'}{d} = \frac{x}{z} \quad \frac{y'}{d} = \frac{y}{z} \quad \equiv \quad x' = \frac{x}{z/d}, \quad y' = \frac{y}{z/d}.$$

- Thus

One-point perspective projection matrix

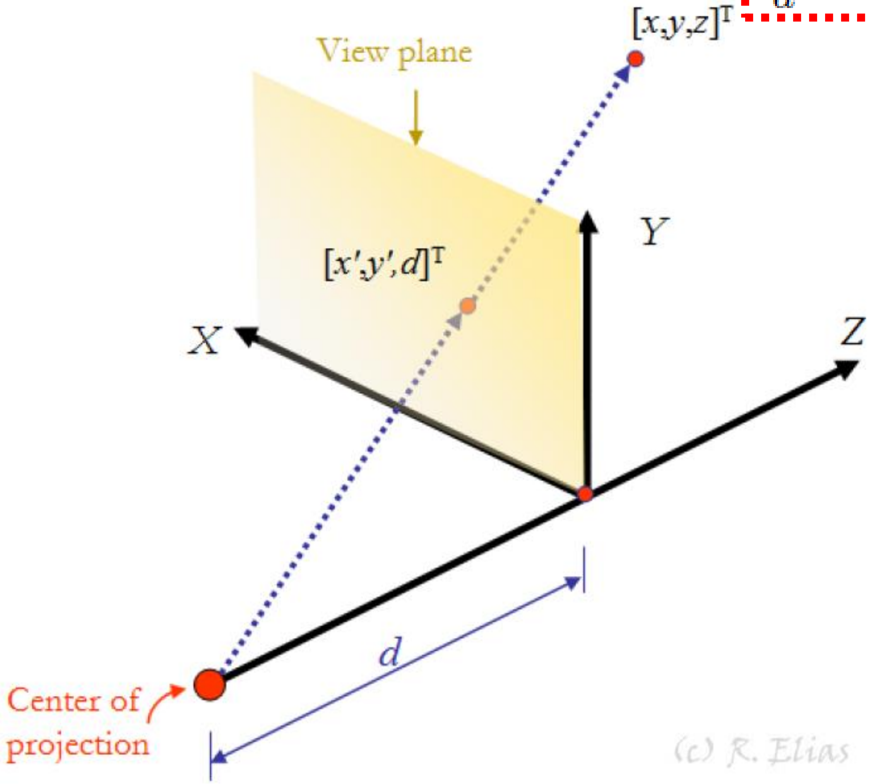
$$\mathrm{P}' = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix}}_{\mathrm{P}_{per}} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\mathrm{P}} = \begin{bmatrix} x \\ y \\ z \\ \frac{z}{d} \end{bmatrix} \quad \text{or} \quad \underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{\dot{\mathrm{P}}'} = \begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ d \end{bmatrix}$$

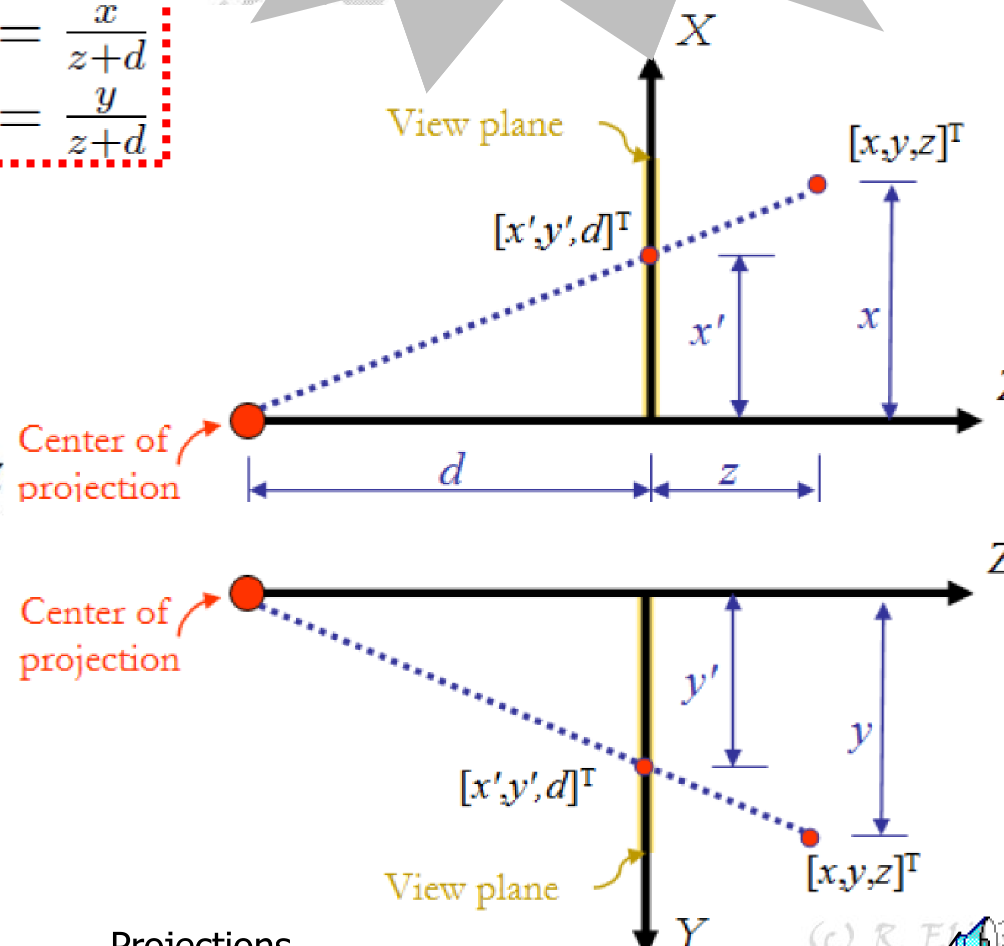
One-point Perspective Projection

COP is at $[0, 0, -d]^T$ and view plane is at $z=0$

- Considering similar triangles:

$$\boxed{\begin{aligned}\frac{x'}{d} &= \frac{x}{z+d}\\ \frac{y'}{d} &= \frac{y}{z+d}\end{aligned}}$$





One-point Perspective Projection

COP is at $[0, 0, -d]^T$ and view plane is at $z=0$

$$\frac{x'}{d} = \frac{x}{z+d} \qquad \frac{y'}{d} = \frac{y}{z+d} \quad \equiv \quad x' = \frac{x}{(z/d)+1} \qquad y' = \frac{y}{(z/d)+1}$$

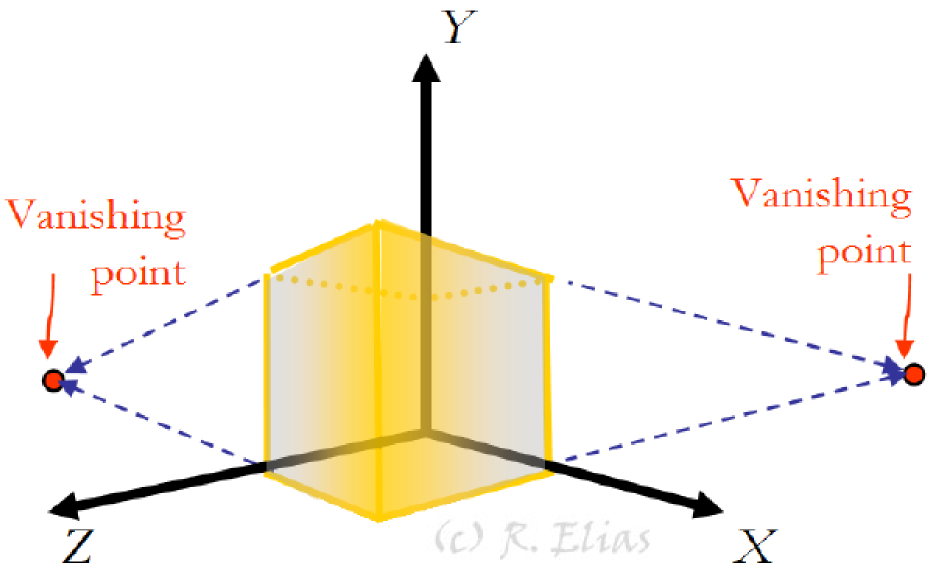
- Thus

one-point perspective projection matrix

$$\mathbf{P}' = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix}}_{\mathbf{P}'_{per}} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\mathbf{P}} = \begin{bmatrix} x \\ y \\ 0 \\ \frac{z+d}{d} \end{bmatrix} \quad \text{or} \quad \underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{\dot{\mathbf{P}}'} = \begin{bmatrix} \frac{x}{(z/d)+1} \\ \frac{y}{(z/d)+1} \\ 0 \end{bmatrix}$$

Two-point Perspective Projection

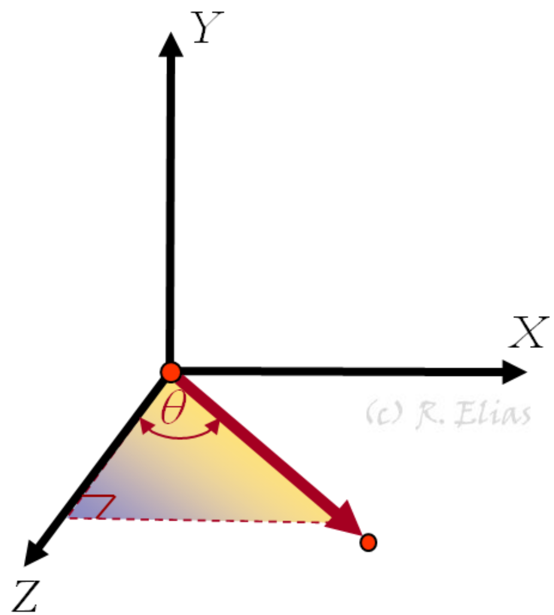
- **Two-point perspective** exists when the view plane is parallel to a principal axis (y-axis in the figure shown).



- In this case:
  - The view plane intersects the x- and z-axes.
  - The normal to the view plane is perpendicular to the y-axis.
  - One component of the view plane normal **n** is zero.
  - Lines parallel to the y-axis remain parallel to this axis.
  - Lines parallel to the other two axes converge to two different vanishing points.

Two-point Perspective Projection

- When
  - the center of projection is at the origin and
  - the normal to the view plane makes an angle θ with the *yz*-plane,



- the overall two-point perspective projection process can be achieved in three steps.
  1. Rotate through an angle $-\theta$ about the *y*-axis so that the normal to the view plane coincides with the *z*-axis.
  2. Perform a one-point perspective projection.
  3. Rotate back through an angle θ about the *y*-axis.

Two-point Perspective Projection

$$\mathrm{M}_1 = \mathrm{R}_y(-\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{M}_2 = \mathrm{P}_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix}$$

$$\mathrm{M}_3 = \mathrm{R}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

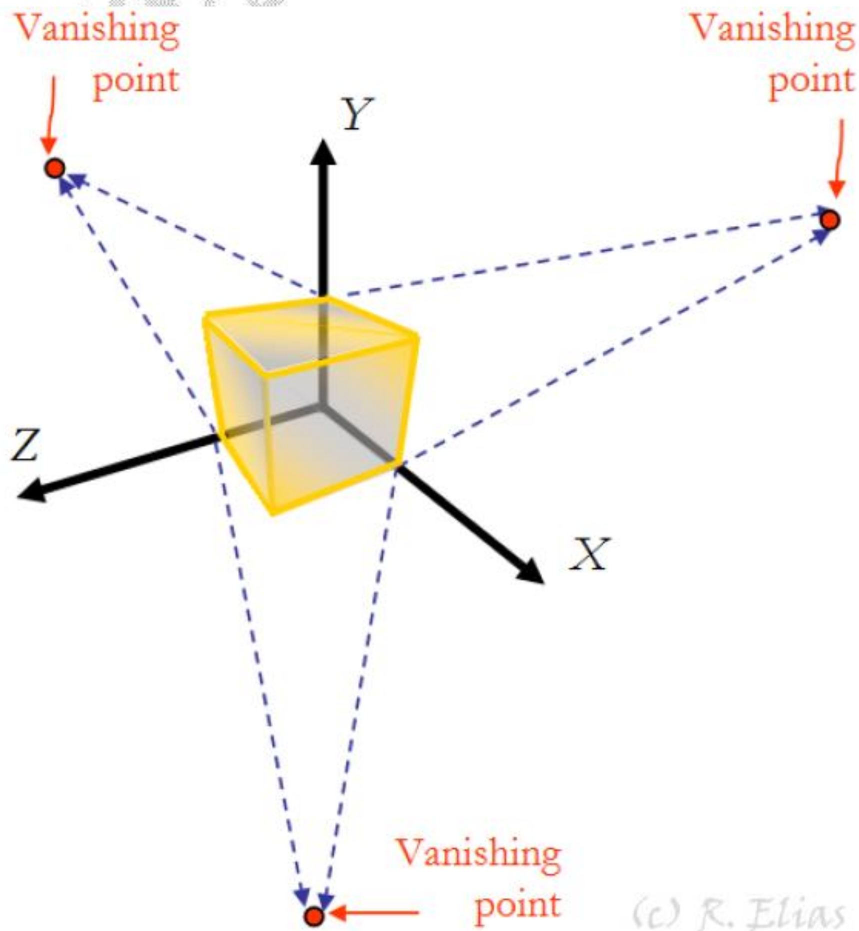
The two-point perspective projection matrix

$$\begin{aligned} \mathrm{P}_{per2} &= \mathrm{M}_3\mathrm{M}_2\mathrm{M}_1 \\ &= \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix} \end{aligned}$$

Three-point Perspective Projection

- **Three-point perspective** exists when the view plane is not parallel to any of the principal axes.

- In this case:
  - The view plane intersects all three axes.
  - No component in the view plane normal **n** is zero.
  - Lines parallel to the three principal axes converge to three different vanishing points.



Three-point Perspective Projection

- When
  - the center of projection is at the origin and
  - the normal to the view plane makes an angle θ with the *yz*-plane and an angle φ with the *zx*-plane,

- the overall three-point perspective projection process can be achieved in three steps.
  1. Rotate through an angle φ about the *x*-axis so that the normal to the view plane coincides with the *zx*-plane.
  2. Perform a two-point perspective projection.
  3. Rotate back through an angle $-\varphi$ about the *x*-axis.

Three-point Perspective Projection

$$M_1 = R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_2 = P_{per2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & 0 & \frac{\cos(\theta)}{d} & 0 \end{bmatrix}$$

$$M_3 = R_x(-\varphi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) & 0 \\ 0 & -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The three-point perspective projection matrix

$$P_{per3} = M_3 M_2 M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin(\theta)}{d} & \frac{\cos(\theta)\sin(\varphi)}{d} & \frac{\cos(\theta)\cos(\varphi)}{d} & 0 \end{bmatrix}$$

An Example

- **Example:** If the normal to the view plane is $[0,0,1]^T$, determine whether or not the following projections can be produced:
  - Multi-view projection as front view
  - Cabinet projection
  - Cavalier projection
  - Isometric projection
  - One-point perspective projection

- If more than one projection type can be produced using that same normal to the view plane, which is $[0,0,1]^T$, what are the other settings that would make such differences among these projections?

An Example

- **Answer:**

- **Front view**: Can be produced using $[0,0,1]^T$ if the projectors are:
  - Parallel and
  - perpendicular to view plane.

- **Cabinet projection**: Can be produced using $[0,0,1]^T$ if projectors:
  - Parallel,
  - not perpendicular to view plane and
  - $\alpha=63.4^\circ$.

An Example

- **Cavalier projection**: Can be produced using $[0,0,1]^T$ if projectors:
  - Parallel,
  - not perpendicular to view plane and
  - $\alpha=45^{o}$.

- **Isometric projection**: Cannot be produced using $[0,0,1]^T$.

- **One-point perspective projection**: Can be produced using $[0,0,1]^T$ if:
  - The viewpoint or the center of projection is placed at a finite distance from the view plane (i.e., if the projectors are not parallel to each other as in case of parallel projections).

Summary

Types of projections:

- Non-planar projection
- Planar projection
  - Parallel projection
    - Orthographic
      - Multi-view
      - Axonometric
    - Oblique
      - Cavalier
      - Cabinet
  - Perspective projection
    - One-point
    - Two-point
    - Three-point