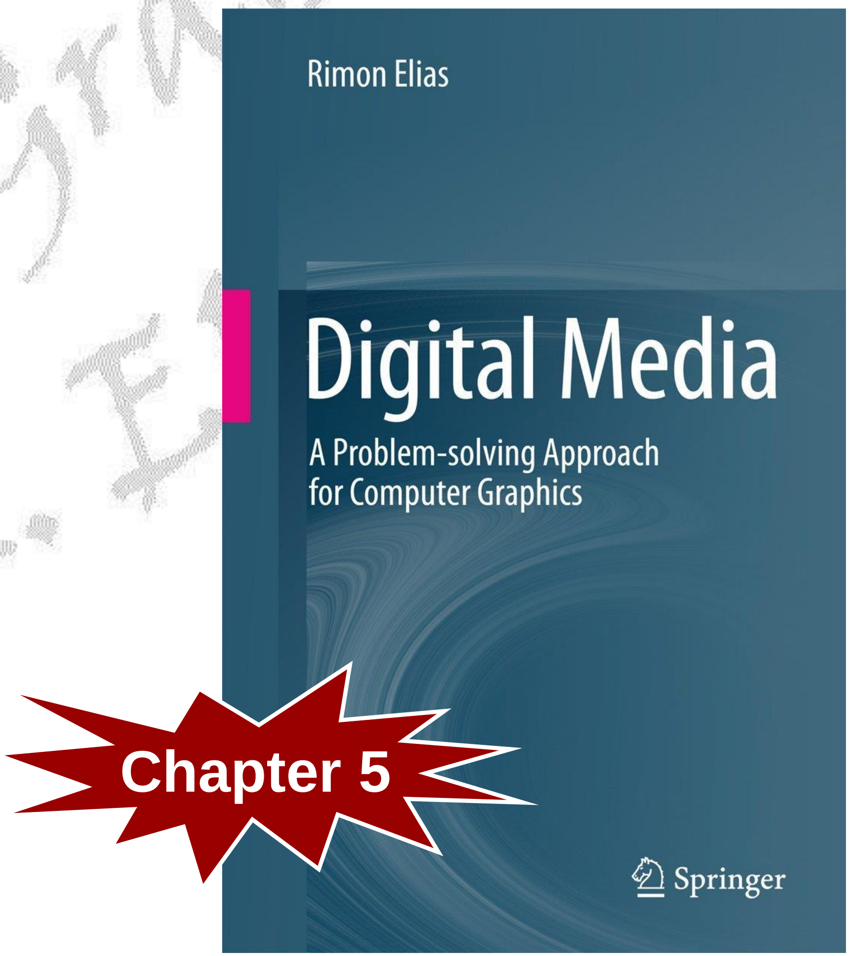
DMET 502/701
Computer Graphics

3D Transformations

Assoc. Prof. Dr. Rimon Elias

Contents

- 3D transformation operations
  - Translation
  - Rotation
  - Scaling
  - Reflection
  - Shearing
  - Composite transformations
  - Axes transformations

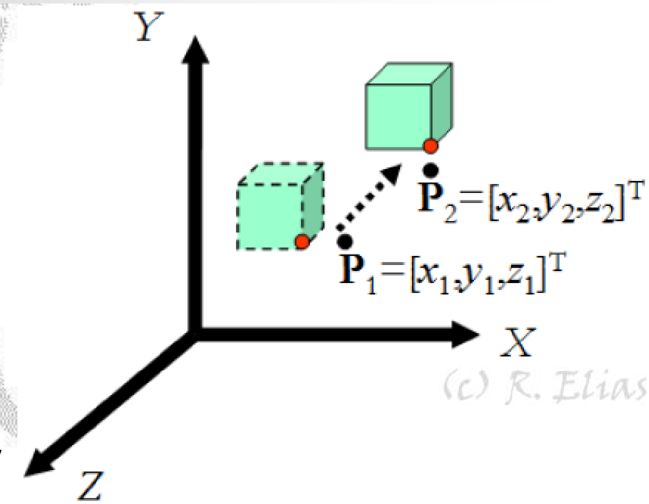


3D Transformations

- 3D transformations can be regarded as extensions to 2D transformations.
- As with 2D transformations, we will be concerned with object vertices.
- The main 3D transformation operations:
  - Translation
  - Rotation
  - Scaling
  - Reflection
  - Shearing

3D Translation

- The translation operation in 3D space is performed when a 3D point is moved or *translated* from a position $[x_1, y_1, z_1]^T$ to another position $[x_2, y_2, z_2]^T$.



- The magnitude and direction of translation are characterized by the *translation vector* $\mathbf{t} = [t_x, t_y, t_z]^T = [x_2 - x_1, y_2 - y_1, z_2 - z_1]^T$.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathbf{P}}_2} = \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathbf{P}}_1} + \underbrace{\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}_{\mathbf{t}}$$

translation vector

3D Translation

- The same translation operation can be performed on homogeneous points.

- A homogeneous point at $\mathbf{P_1}=[x_1, y_1, z_1, 1]^T$ is translated to another position $\mathbf{P_2}= [x_2, y_2, z_2, 1]^T$ using

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}([t_x,t_y,t_z]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

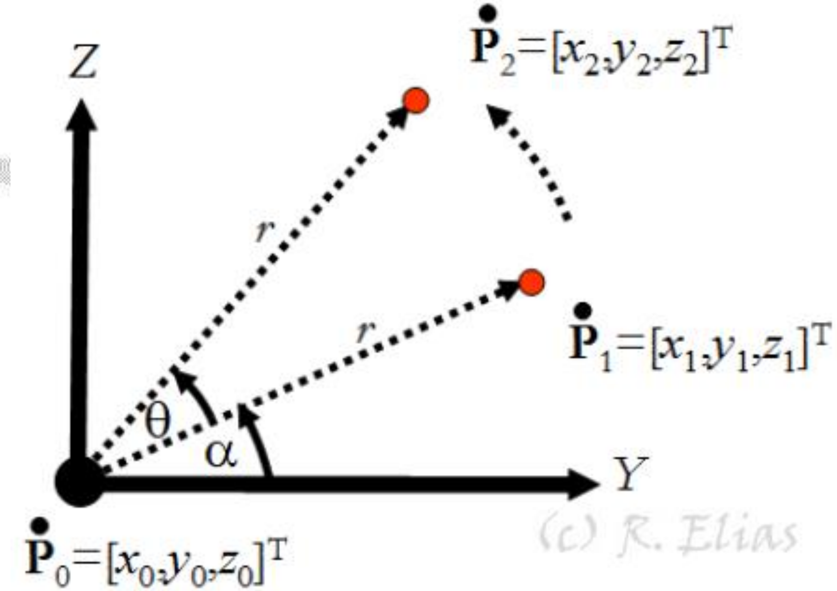
translation matrix

3D Rotation about the *x*-axis

- Suppose that a point $[x_1,y_1,z_1]^T$ is to be rotated through an angle θ about the *x*-axis to another point $[x_2,y_2,z_2]^T$ (where $x_2=x_1$).
- Consider the projection of the rotation operation onto the *yz*-plane as depicted here. (The *x*-axis is perpendicular to both *y*- and *z*-axes and pointing outwards.)
- Notice that

$$\dot{\mathbf{P}}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ r\cos(\alpha) \\ r\sin(\alpha) \end{bmatrix}$$

$$\dot{\mathbf{P}}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}$$



$$\begin{aligned}\sin(\alpha+\theta) &= \sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta)\\ \cos(\alpha+\theta) &= \cos(\alpha)\cos(\theta)-\sin(\alpha)\sin(\theta)\end{aligned}$$

3D Rotation about the *x*-axis

- Thus,

$$\begin{aligned}
\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathbf{P}}_2} &= \begin{bmatrix} x_1 \\ r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ r\left(\cos(\alpha)\cos(\theta)-\sin(\alpha)\sin(\theta)\right) \\ r\left(\sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta)\right) \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ \underbrace{r\cos(\alpha)}_{y_1}\cos(\theta)-\underbrace{r\sin(\alpha)}_{z_1}\sin(\theta) \\ \underbrace{r\sin(\alpha)}_{z_1}\cos(\theta)+\underbrace{r\cos(\alpha)}_{y_1}\sin(\theta) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\dot{\mathrm{R}}_x(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathbf{P}}_1}
\end{aligned}$$

Inhomogeneous coordinates

Rotation matrix (about the *x*-axis)

3D Rotation about the *x*-axis

- The same operation is performed on homogeneous points as

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}_x(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

Rotation matrix (about the *x*-axis)
for homogeneous points

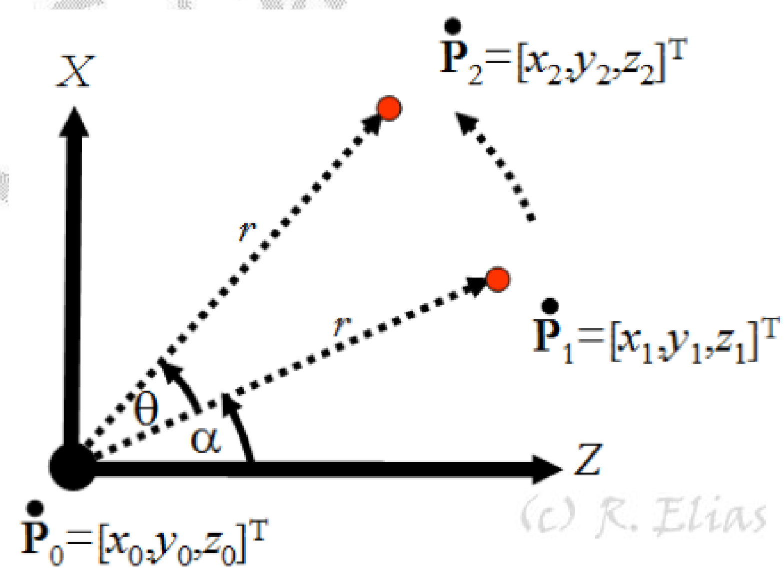
Notice that both rotation matrices assume that the axis of rotation is the *x*-axis.

3D Rotation about the *y*-axis

- Suppose that a point $[x_1,y_1,z_1]^T$ is to be rotated through an angle θ about the *y*-axis to another point $[x_2,y_2,z_2]^T$ (where $y_2=y_1$).
- Consider the projection of the rotation operation onto the *zx*-plane as depicted here. (The *y*-axis is perpendicular to both *z*- and *x*-axes and pointing outwards.)

$$\dot{\mathbf{P}}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} r\sin(\alpha) \\ y_1 \\ r\cos(\alpha) \end{bmatrix}$$

$$\dot{\mathbf{P}}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} r\sin(\alpha+\theta) \\ y_2 \\ r\cos(\alpha+\theta) \end{bmatrix}$$



$$\begin{aligned}\sin(\alpha+\theta) &= \sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta)\\ \cos(\alpha+\theta) &= \cos(\alpha)\cos(\theta)-\sin(\alpha)\sin(\theta)\end{aligned}$$

3D Rotation about the *y*-axis

- Thus,

$$\underbrace{\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}}_{\dot{\mathbf{P}}_2} = \begin{bmatrix}r\sin(\alpha+\theta)\\ y_1\\ r\cos(\alpha+\theta)\end{bmatrix}$$

$$= \begin{bmatrix}r\,(\sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta))\\ y_1\\ r\,(\cos(\alpha)\cos(\theta)-\sin(\alpha)\sin(\theta))\end{bmatrix}$$

$$= \begin{bmatrix}\underbrace{r\sin(\alpha)}_{x_1}\cos(\theta)+\underbrace{r\cos(\alpha)}_{z_1}\sin(\theta)\\ y_1\\ \underbrace{r\cos(\alpha)}_{z_1}\cos(\theta)-\underbrace{r\sin(\alpha)}_{x_1}\sin(\theta)\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta) & 0 & \cos(\theta)\end{bmatrix}}_{\dot{\mathrm{R}}_y(\theta)} \underbrace{\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}}_{\dot{\mathbf{P}}_1}$$

Inhomogeneous coordinates

Rotation matrix (about the *y*-axis)

3D Rotation about the *y*-axis

- The same operation is performed on homogeneous points as

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathrm{R}_y(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

Rotation matrix (about the *y*-axis) for homogeneous points

Notice that both rotation matrices assume that the axis of rotation is the *y*-axis.

3D Rotation about the *z*-axis

- Applying the same procedure as done before, the rotation about the *z*-axis can be expressed as:

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathbf{P}}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\dot{\mathrm{R}}_z(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathbf{P}}_1}$$

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathrm{R}_z(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

Properties of the Rotation Matrix

- The rotation matrix is a special orthogonal matrix that has the properties:
  - $\mathbf{R}\mathbf{R}^{-1} = \mathbf{R}\mathbf{R}^{T} = \mathbf{I}$ $\quad$ $\boxed{\mathbf{R}^{-1} = \mathbf{R}^{T}}$
  - $\det(\mathbf{R}) = +1$
- R is normalized: the squares of the elements in any row or column sum to 1.
- R is orthogonal: the dot product of any pair of rows or any pair of columns is 0.
- You can utilize these properties to make sure that you are on the right track when calculating a rotation matrix.

3D Rotation: General Case

- In order to rotate a point about an arbitrary axis that does not coincide with any of the main coordinate axes, do the following:
  1. If the arbitrary axis does not pass through the origin $[0, 0, 0]^T$, translate the point/object and the arbitrary axis by a translation vector that causes this axis to pass through the origin.
  2. If the arbitrary axis does not coincide with a principal axis, rotate the point/object as well as the arbitrary axis so that the arbitrary axis coincide with one of the principal axes.
  3. Perform the specified point rotation about the selected principal axis.
  4. Apply inverse rotation (if Step 2 has been performed).
  5. Apply inverse translation (if Step 1 has been performed).

3D Rotation: An Example

- **Example:** Derive a matrix that rotates a point about a 2D line having the equation $x = y$ through an angle of 45°.

- **Answer:** 3 steps
  1. Rotate through an angle of 45° about the *z*-axis.
  2. Rotate through an angle of 45° about the *y*-axis.
  3. Rotate through an angle of –45° about the *z*-axis.

  or

  1. Rotate through an angle of –45° about the *z*-axis.
  2. Rotate through an angle of 45° about the *x*-axis.
  3. Rotate through an angle of 45° about the *z*-axis.

3D Rotation: An Example

$$
\mathtt{M}_1 = \mathtt{R}_z(45) = \begin{bmatrix} \cos(45) & -\sin(45) & 0 & 0 \\ \sin(45) & \cos(45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathtt{M}_2 = \mathtt{R}_y(45) = \begin{bmatrix} \cos(45) & 0 & \sin(45) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(45) & 0 & \cos(45) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathtt{M}_3 = \mathtt{R}_z(-45) = \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 & 0 \\ \sin(-45) & \cos(-45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The overall transformation

$$
\begin{aligned}
\mathtt{M} &= \mathtt{M}_3\mathtt{M}_2\mathtt{M}_1 \\
&= \mathtt{R}_z(-45)\mathtt{R}_y(45)\mathtt{R}_z(45) \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
= \begin{bmatrix} \frac{\sqrt{2}+1}{2\sqrt{2}} & \frac{\sqrt{2}-1}{2\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{\sqrt{2}-1}{2\sqrt{2}} & \frac{\sqrt{2}+1}{2\sqrt{2}} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

3D Rotation: An Example

$$M_1 = R_z(-45) = \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 & 0 \\ \sin(-45) & \cos(-45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_2 = R_x(45) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(45) & -\sin(45) & 0 \\ 0 & \sin(45) & \cos(45) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_3 = R_z(45) = \begin{bmatrix} \cos(45) & -\sin(45) & 0 & 0 \\ \sin(45) & \cos(45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation

$$\begin{aligned} M &= M_3 M_2 M_1 \\ &= R_z(45) R_x(45) R_z(-45) \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}+1}{2\sqrt{2}} & \frac{\sqrt{2}-1}{2\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{\sqrt{2}-1}{2\sqrt{2}} & \frac{\sqrt{2}+1}{2\sqrt{2}} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3D Scaling

- In order to scale an object in 3D space by a factor *s* along all directions, the positions of its vertices $[x_i, y_i, z_i]^T$ are multiplied by this scaling factor to get $[s\,x_i, s\,y_i, s\,z_i]^T$.

- Hence, to scale a point $[x_1, y_1, z_1]^T$ using scaling factors s_x, s_y and s_z to get $[x_2, y_2, z_2]^T$, we may use

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathrm{P}}_2} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}}_{\dot{\mathrm{S}}(s_x, s_y, s_z)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathrm{P}}_1}$$

Scaling matrix

- Scaling is **uniform** when $s_x = s_y = s_z$.
- Otherwise scaling is **non-uniform**.

3D Scaling

- The same operation is performed on homogeneous points as

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}(s_x, s_y, s_z)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

Scaling matrix

Notice that both scaling matrices perform the operation with respect to the origin (i.e., the fixed point).

3D Scaling: General Case

- Scaling operations may be performed with respect to a general fixed point.
- As done with general rotation, the following three steps should be performed in case of general scaling:
  1. Translate so that the fixed point coincides with the origin.
  2. Scale as done before.
  3. Translate back.

Fixed point

Point to scale

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2\\ y_2\\ z_2\end{bmatrix}}_{\dot{P}_2} = \underbrace{\begin{bmatrix} s_x & 0 & 0\\ 0 & s_y & 0\\ 0 & 0 & s_z\end{bmatrix}}_{\dot{S}(s_x,s_y,s_z)} \underbrace{\begin{bmatrix} x_1-x_0\\ y_1-y_0\\ z_1-z_0\end{bmatrix}}_{\dot{P}_1-\dot{P}_0} + \underbrace{\begin{bmatrix} x_0\\ y_0\\ z_0\end{bmatrix}}_{\dot{P}_0}$$

2 Scale 1 Translate 3 Translate back

3D Scaling: General Case

- The same operation is performed on homogeneous points as

3 Translate back   2 Scale   1 Translate

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([x_0,y_0,z_0]^T)} \underbrace{\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathtt{S}(s_x,s_y,s_z)} \underbrace{\begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([-x_0,-y_0,-z_0]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

$$= \underbrace{\begin{bmatrix} s_x & 0 & 0 & x_0 - s_x x_0 \\ 0 & s_y & 0 & y_0 - s_y y_0 \\ 0 & 0 & s_z & z_0 - s_z z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([x_0,y_0,z_0]^T)\mathtt{S}\mathtt{T}([-x_0,-y_0,-z_0]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1},$$

Homogeneous coordinates

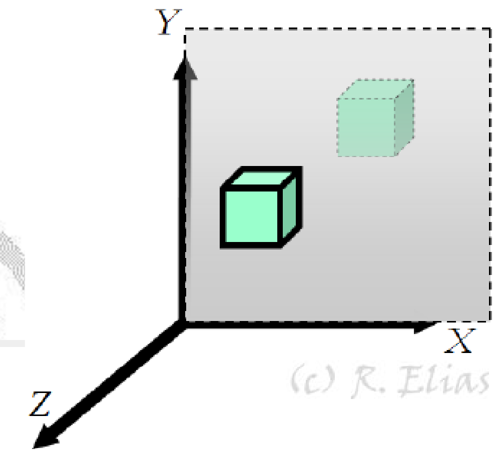
3D Reflection

- In 3D space, an object may be reflected about an axis or a plane.
- 3D reflection about an axis is similar to the 2D case.
- 3D reflection with respect to an axis is equivalent to 180°-rotation about that axis.
- 3D reflection with respect to one of the coordinate planes (i.e. *xy*-, *yz*- or *zx*-plane) is equivalent to a conversion between a right-handed frame and a left-handed frame.

- There are three basic reflection operations:
  1. About the *xy*-plane.
  2. About the *yz*-plane.
  3. About the *zx*-plane.

3D Reflection: About the *xy*-plane

- Reverses the sign of the *z*-coordinates.



Inhomogeneous coordinates

$$\begin{bmatrix} x_2\\ y_2\\ z_2\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}}_{\mathbf{Ref}_{xy}} \underbrace{\begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix}}_{\mathbf{P}_1}$$

=S (1,1,-1)

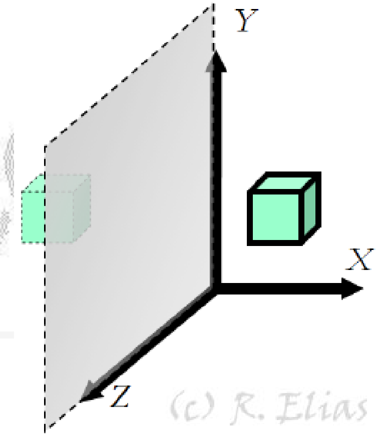
Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2\\ y_2\\ z_2\\ 1\end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}}_{\mathbf{Ref}_{xy}} \underbrace{\begin{bmatrix}x_1\\ y_1\\ z_1\\ 1\end{bmatrix}}_{\mathbf{P}_1}$$

=S (1,1,-1)

Reflection matrices about *xy*-plane

3D Reflection: About the *yz*-plane



- Reverses the sign of the *x*-coordinates.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{Ref}_{yz}} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\mathbf{P}_1}$$

=S (-1,1,1)

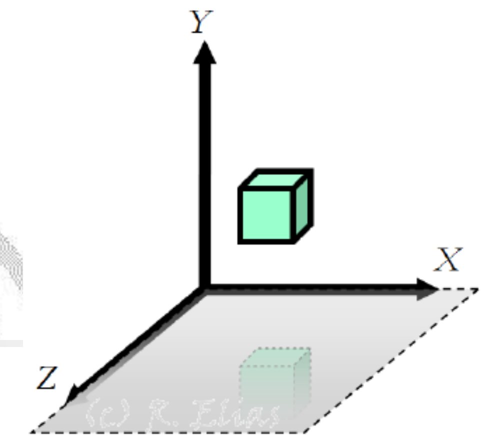
Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathtt{Ref}_{yz}} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

=S (-1,1,1)

Reflection matrices about *yz*-plane

3D Reflection: About the *zx*-plane



- Reverses the sign of the *y*-coordinates.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathbf{P}}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{Ref}_{zx}} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathbf{P}}_1}$$

$=\dot{S}\ (1,-1,1)$

Reflection matrices about *zx*-plane

$=S\ (1,-1,1)$

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathtt{Ref}_{zx}} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

3D Reflection: General Case

- In the general case, the reflection operation can be performed about any arbitrary plane.

- We may perform the following steps in case of general reflection:
  1. Translate so that the reflection plane passes through the origin.
  2. Rotate so that the reflection plane coincides with one of the principal planes.
  3. Reflect about that principal plane as done before.
  4. Rotate back through the same angle of Step 2 in the opposite direction.
  5. Translate back using the same vector of Step 1 in the opposite direction.

3D Reflection: An Example

- **Example:** Derive a matrix that reflects a point $[x, y, z]^T$ about the plane $y = \delta$ where δ is a real number.

- **Answer:**
- Notice that the plane $y = \delta$ is parallel to the *zx*-plane.
- The steps:
  1. Translate the plane $y = \delta$ so that it coincides with the *zx*-plane (i.e., using the vector $[0, -\delta, 0]^T$).
  2. Reflect about the *zx*-plane.
  3. Translate back using the vector $[0, \delta, 0]^T$.

3D Reflection: An Example

1 Translate

$$\mathtt{M}_1 = \mathtt{T}([0,-\delta,0]^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2 Reflect

$$\mathtt{M}_2 = \mathtt{Ref}_{zx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 Translate back

$$\mathtt{M}_3 = \mathtt{T}([0,\delta,0]^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation

$$\begin{aligned} \mathtt{M} &= \mathtt{M}_3\mathtt{M}_2\mathtt{M}_1 \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3D Shearing

- In 3D space, one can push in two coordinate axis directions and keep the third one fixed.

- Thus, with respect to the principal axes, there are three different shearing situations relative to these axes:
  1. Relative to the *x*-axis
  2. Relative to the *y*-axis
  3. Relative to the *z*-axis

3D Shearing: Relative to the *x*-axis

- The *x*-coordinates are kept the same.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathbf{P}}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ sh_{xy} & 1 & 0 \\ sh_{xz} & 0 & 1 \end{bmatrix}}_{\dot{\mathbf{S}}\mathbf{h}_x(sh_{xy}, sh_{xz})} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathbf{P}}_1}$$

Shearing factors

Shearing matrices relative to the *x*-axis

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ sh_{xy} & 1 & 0 & 0 \\ sh_{xz} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}\mathbf{h}_x(sh_{xy}, sh_{xz})} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

3D Shearing: Relative to the *y*-axis

- The *y*-coordinates are kept the same.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathbf{P}}_2} = \underbrace{\begin{bmatrix} 1 & sh_{yx} & 0 \\ 0 & 1 & 0 \\ 0 & sh_{yz} & 1 \end{bmatrix}}_{\dot{\mathbf{S}}\mathbf{h}_y(sh_{yx}, sh_{yz})} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathbf{P}}_1}$$

Shearing factors

Shearing matrices relative to the *y*-axis

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} 1 & sh_{yx} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & sh_{yz} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}\mathbf{h}_y(sh_{yx}, sh_{yz})} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1}$$

3D Shearing: Relative to the z-axis

- The z-coordinates are kept the same.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{\dot{\mathbf{P}}_2} = \underbrace{\begin{bmatrix} 1 & 0 & sh_{zx} \\ 0 & 1 & sh_{zy} \\ 0 & 0 & 1 \end{bmatrix}}_{\dot{\mathbf{S}}\mathbf{h}_z(sh_{zx},sh_{zy})} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\dot{\mathbf{P}}_1}$$

Shearing factors

Shearing matrices relative to the z-axis

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{bmatrix} 1 & 0 & sh_{zx} & 0 \\ 0 & 1 & sh_{zy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{Sh}_z(sh_{zx},sh_{zy})} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{\mathbf{P}_1},$$

3D Shearing: General Case

- In the general case, the shearing operation can be performed relative to any arbitrary axis.

- We may perform the following steps in case of general shearing:
  1. If the axis does not pass through the origin, translate by a translation vector that causes this axis to pass through the origin.
  2. If the axis does not coincide with one of the principal axes, rotate so that the axis coincides with one of the principal axes. Note that more than a single rotation may be needed to satisfy this step.
  3. Perform the shearing operation relative to the selected principal axis.
  4. Apply inverse rotation (if Step 2 has been performed).
  5. Apply inverse translation (if Step 1 has been performed).

Composite 3D Transformations

- In general, 3D transformations of one type (e.g., rotation) will not be commutative with 3D transformations of another type (e.g., translation).

- Furthermore, a rotation about one axis will not, <u>in general</u>, be commutative with a rotation about a different axis.
  - For example, a rotation of 90° about the *x*-axis followed by a rotation of 90° about the *y*-axis is $\neq$ a rotation of 90° about the *y*-axis followed by a rotation of 90° about the *x*-axis.

Composite 3D Transformations: An Example

- **Example:** Derive the transformation matrix that performs this series of 3D transformations applied to a 3D object.

1. Scale the object using a factor 5 in the *x*-direction.
2. Rotate it through 30° about the *z*-axis.
3. Shear it in the *x*- and *y*-directions with shearing factors 2 and 3, respectively.
4. Translate it using a translation vector $[2,1,2]^T$.

Composite 3D Transformations: An Example

1 Scale

$$\mathtt{M}_1 = \mathtt{S}(5,1,1) = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2 Rotate

$$\mathtt{M}_2 = \mathtt{R}_z(30^\circ) = \begin{bmatrix} \cos(30) & -\sin(30) & 0 & 0 \\ \sin(30) & \cos(30) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 Shear

$$\mathtt{M}_3 = \mathtt{Sh}_z(2,3) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4 translate

$$\mathtt{M}_4 = \mathtt{T}([2,1,2]^T) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation

$$\begin{aligned} \mathtt{M} &= \mathtt{M}_4\mathtt{M}_3\mathtt{M}_2\mathtt{M}_1 \\ &= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(30) & -\sin(30) & 0 & 0 \\ \sin(30) & \cos(30) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{3}}{2} & -\frac{1}{2} & 2 & 2 \\ 2\frac{1}{2} & \frac{\sqrt{3}}{2} & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Axes Transformations

- Similar to 2D space, axes may be transformed in 3D space.
- Although objects in space do not transform; however, their vertex coordinates get affected by axes transformation.

Axes Translation

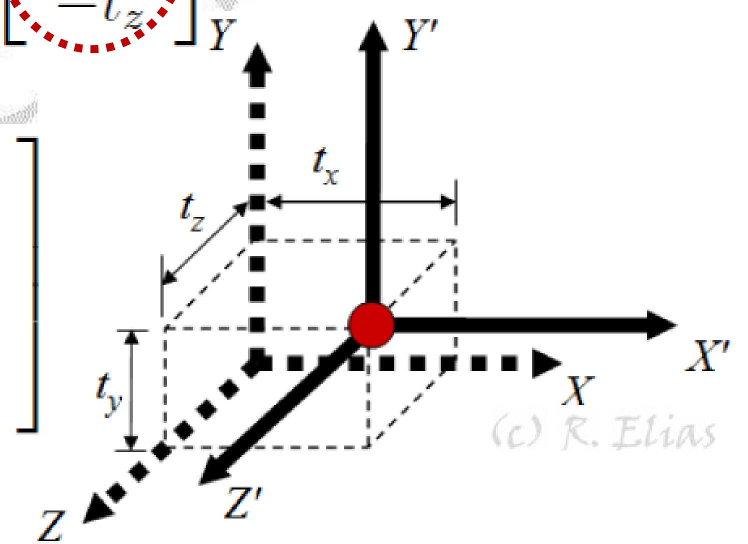
- When the three axes are translated using a vector **$[t_x, t_y, t_z]^T$**, a point $[x, y, z]^T$ will have new coordinates $[x', y', z']^T$ expressed as

Inhomogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -t_x \\ -t_y \\ -t_z \end{bmatrix}$$

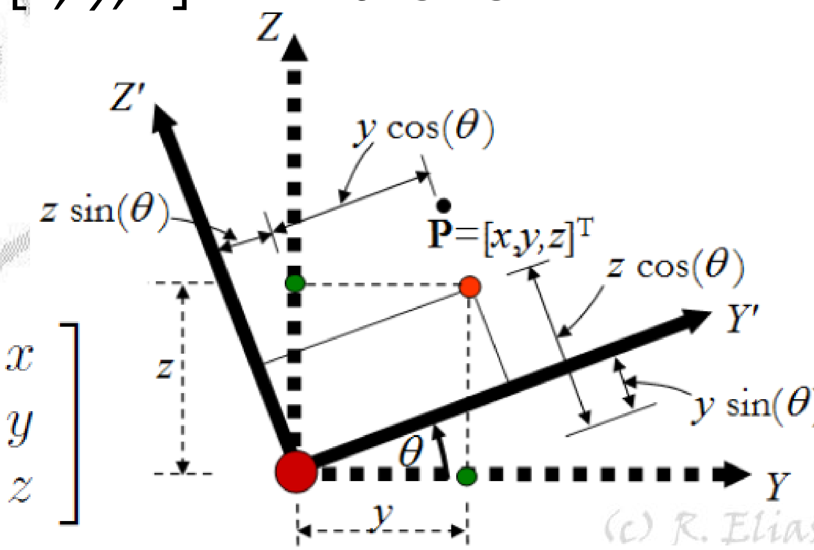
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Homogeneous coordinates



Axes Rotation about the *x*-axis

- When the *y*- and *z*-axes are rotated through an angle θ to the y'- and z'-axes about the *x*-axis, a point $[x, y, z]^T$ will have new coordinates $[x', y', z']^T$ expressed as



Inhomogeneous coordinates

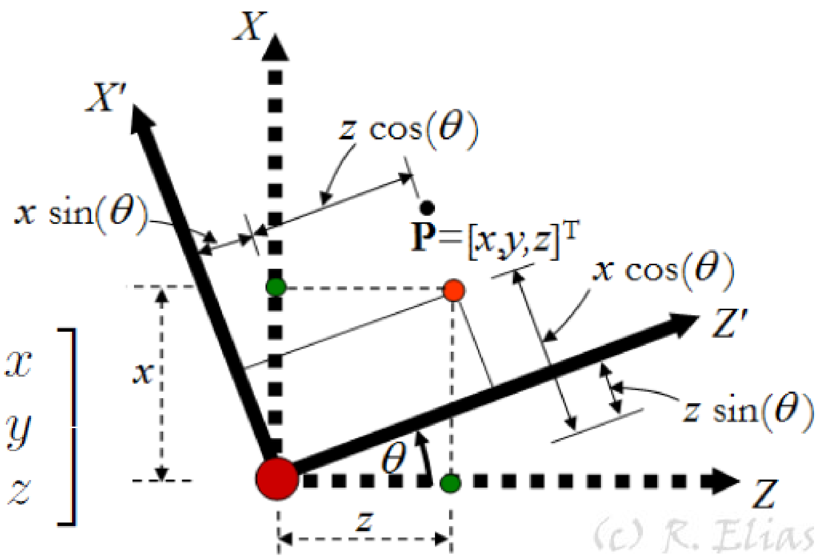
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Axes Rotation about the *y*-axis

- When the *z*- and *x*-axes are rotated through an angle θ to the z'- and x'-axes about the *y*-axis, a point $[x, y, z]^T$ will have new coordinates $[x', y', z']^T$ expressed as



Inhomogeneous coordinates

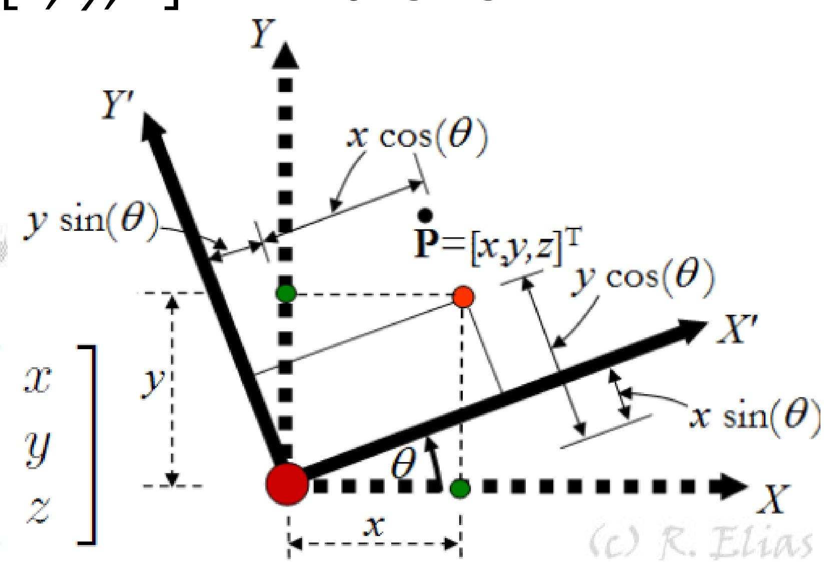
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Axes Rotation about the *z*-axis

- When the *x*- and *y*-axes are rotated through an angle θ to the x'- and y'-axes about the *z*-axis, a point $[x, y, z]^T$ will have new coordinates $[x', y', z']^T$ expressed as



Inhomogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Axes Scaling

- The axes scaling process affects the units of the coordinate system.
- When the *x*-, *y*- and *z*-axes are scaled to the *x'*-, *y'*- and *z'*-axes, a point $[x, y, z]^T$ will have new coordinates $[x', y', z']^T$ expressed as

Inhomogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & \frac{1}{s_z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Axes scaling matrices

Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

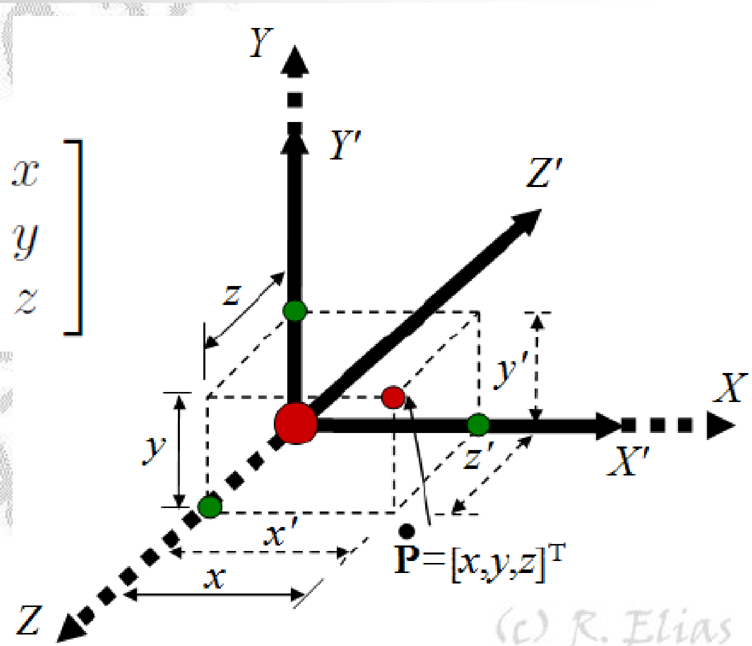
Axes Reflection about the *xy*-plane

- The *z*-coordinates are affected.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Inhomogeneous coordinates

Axes reflection matrices



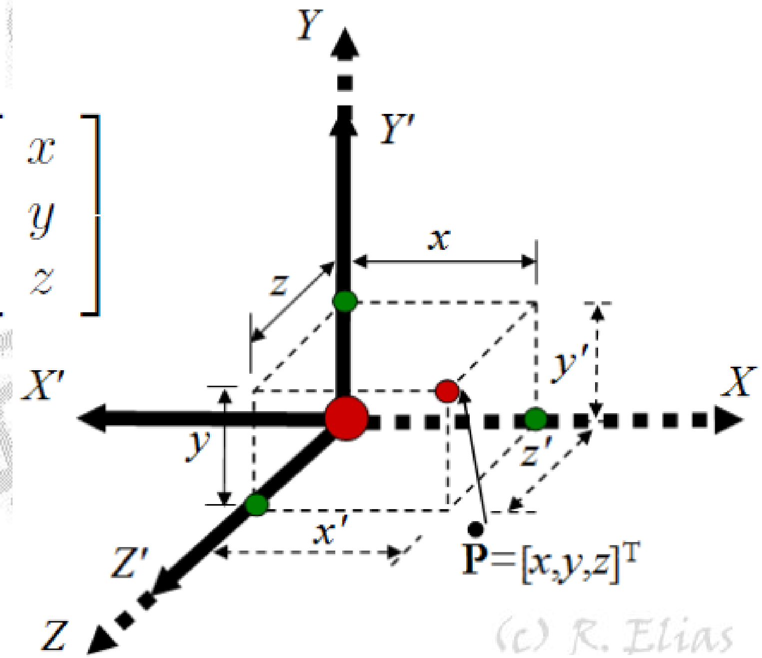
Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Axes Reflection about the *yz*-plane

- The *x*-coordinates are affected.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



Inhomogeneous coordinates

Axes reflection matrices

Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

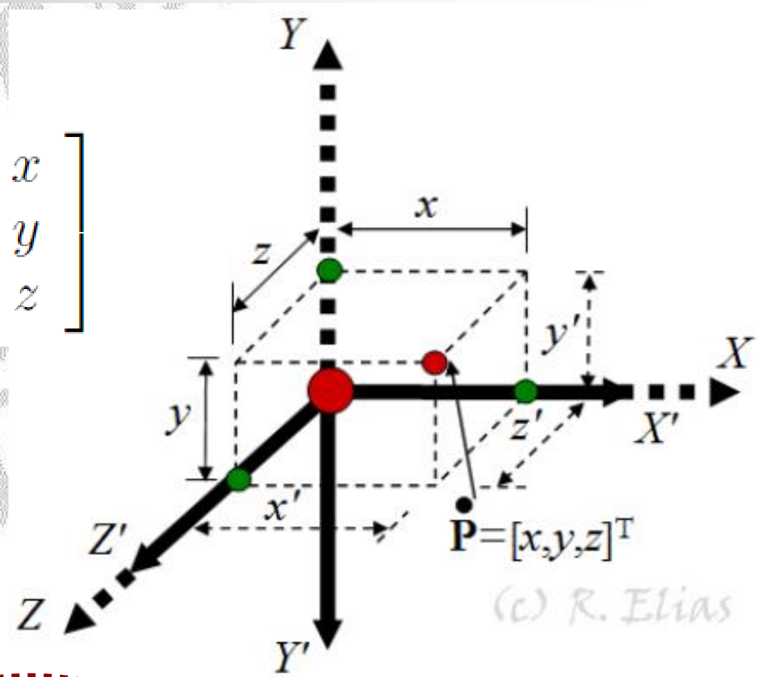
Axes Reflection about the *zx*-plane

- The *y*-coordinates are affected.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Inhomogeneous coordinates

Axes reflection matrices



Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Summary

- 3D transformation operations
  - Translation
  - Rotation
  - Scaling
  - Reflection
  - Shearing
  - Composite transformations
  - Axes transformations