Introduction to **Linear Algebra**

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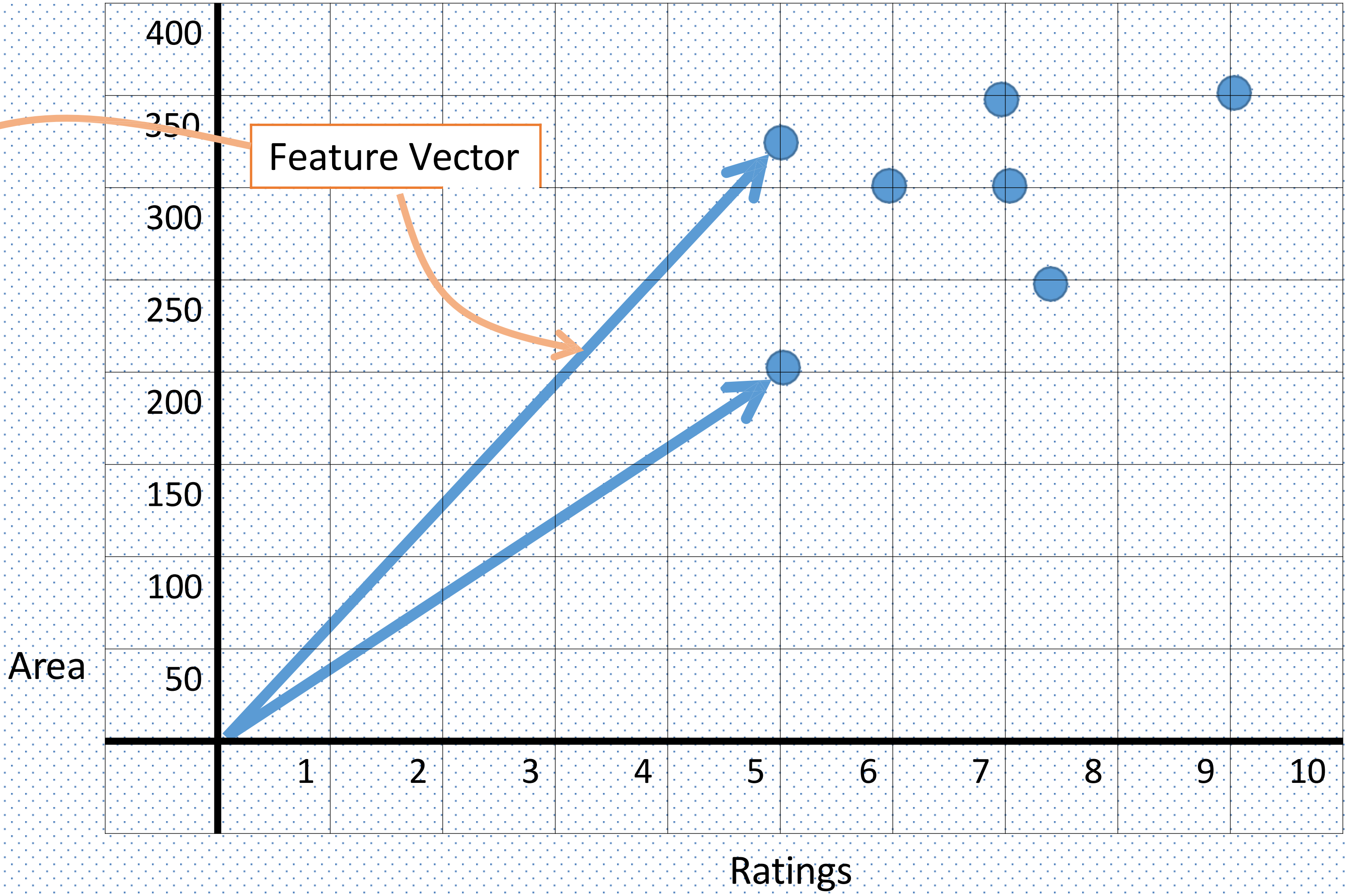
Vectors

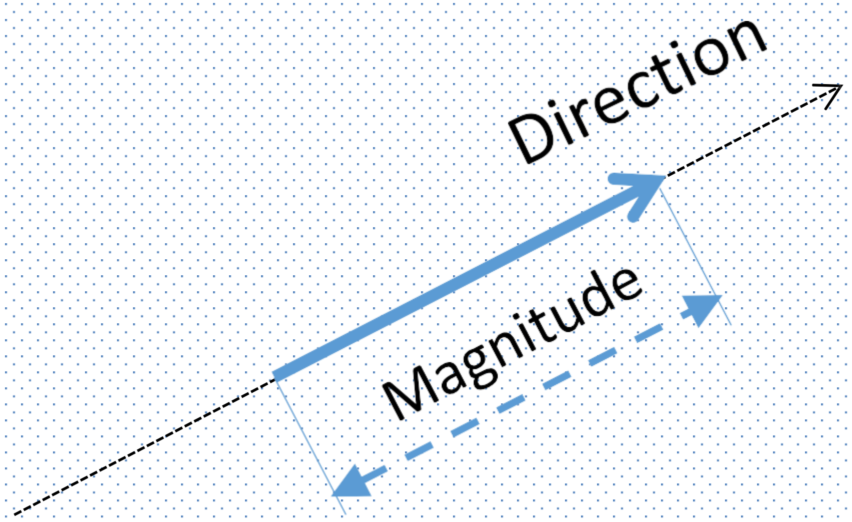
Vectors in Machine Learning

Rating	Area sq. mtr
5	200
7	300
5	325
8	250
6	300
7	350
7.5	250
9	350



What is a vector?

$\vec{V}$



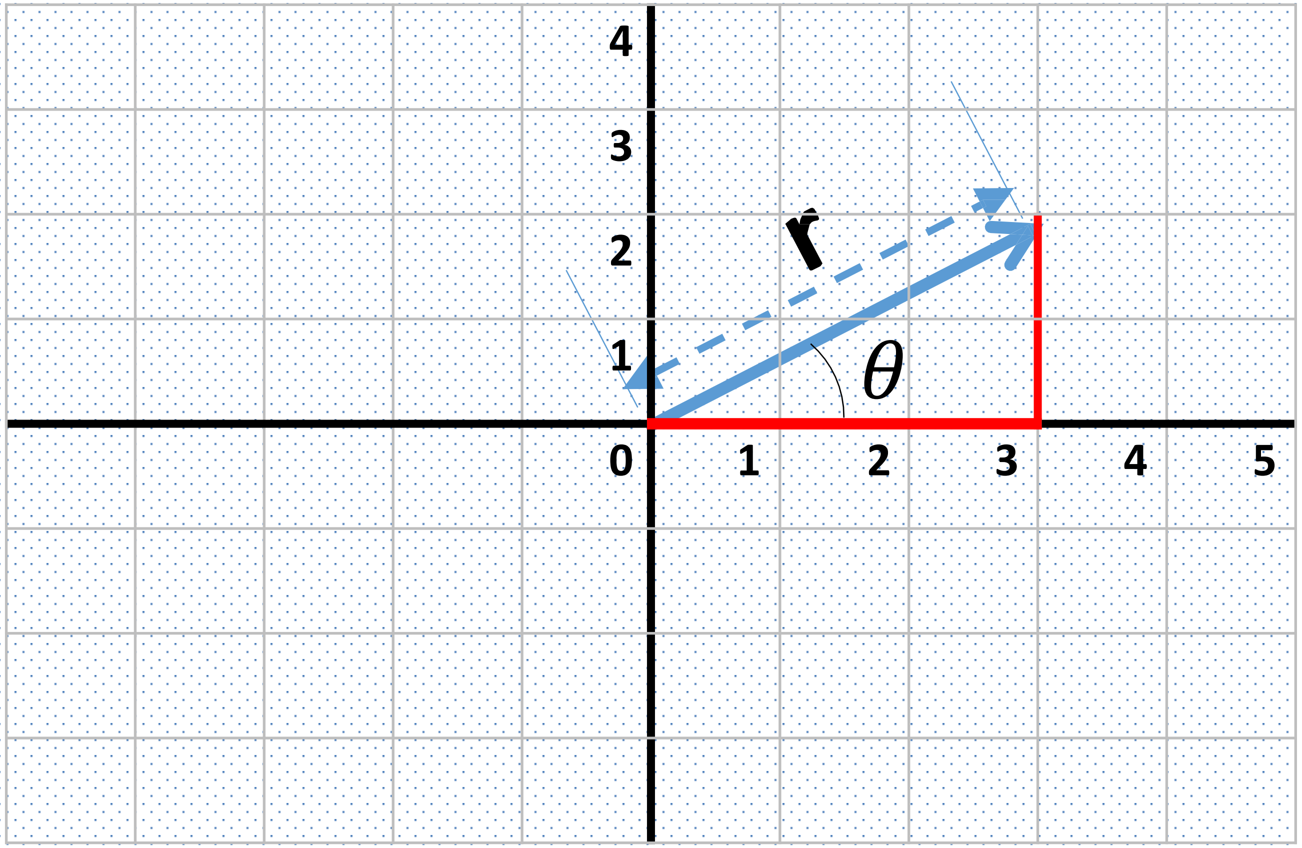
What is a vector?

Cartesian:

$$\vec{V} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

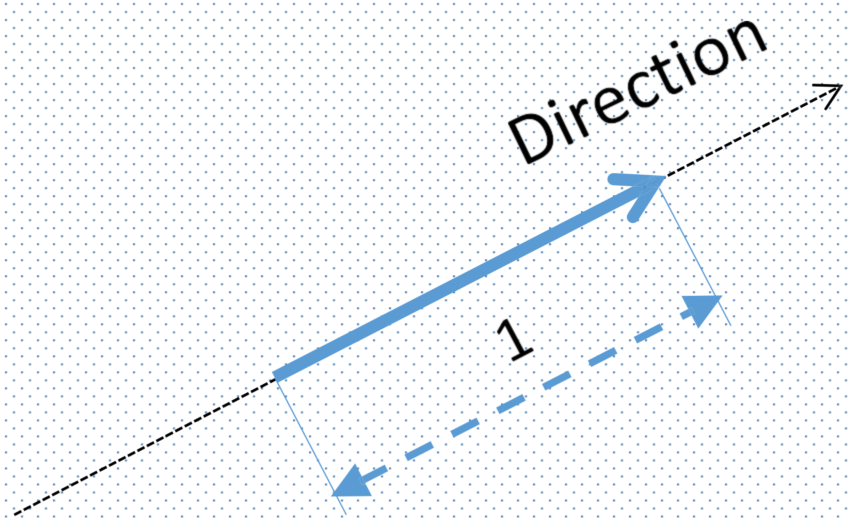
Polar:

$$\vec{V} = (r, \ \theta)$$



Unit Vector

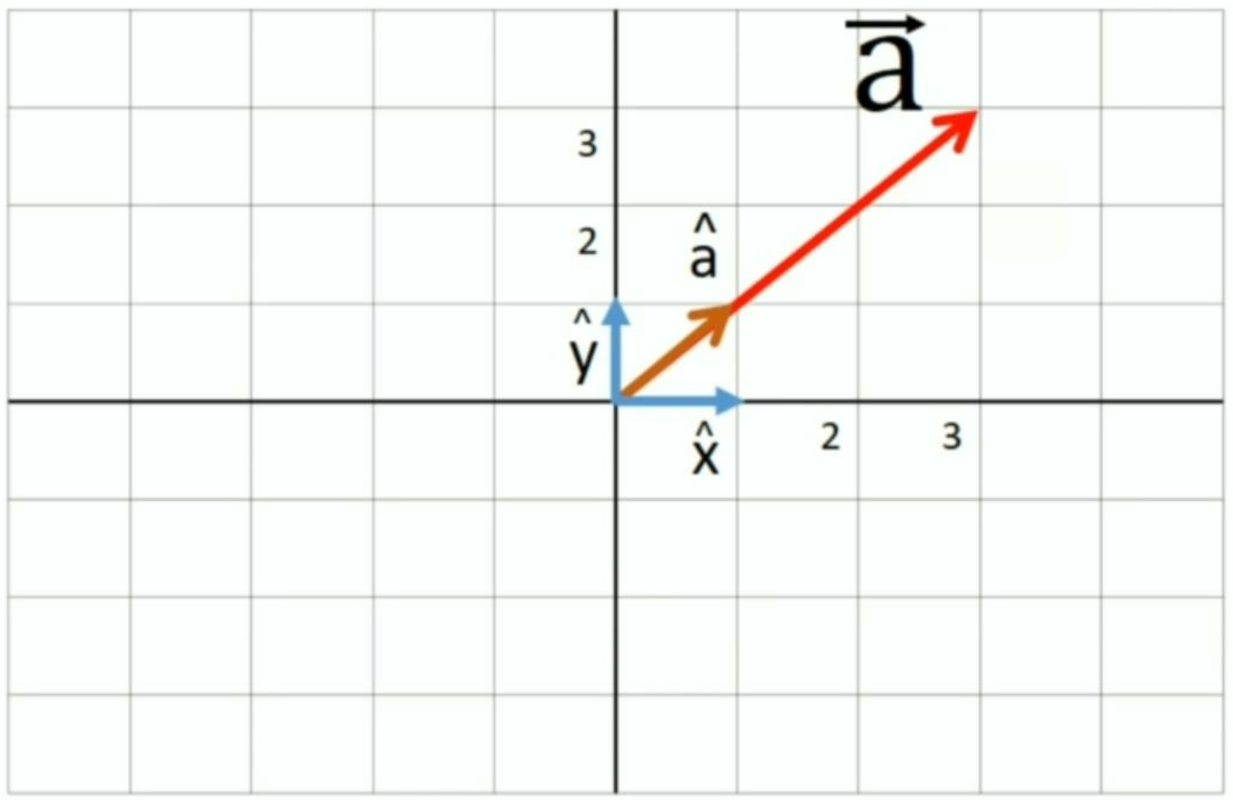
$$\hat{a}$$



Unit Vector

$$\vec{a} = 3 * \hat{a}$$

$$\vec{a} = 3*\hat{x} + 3*\hat{y}$$

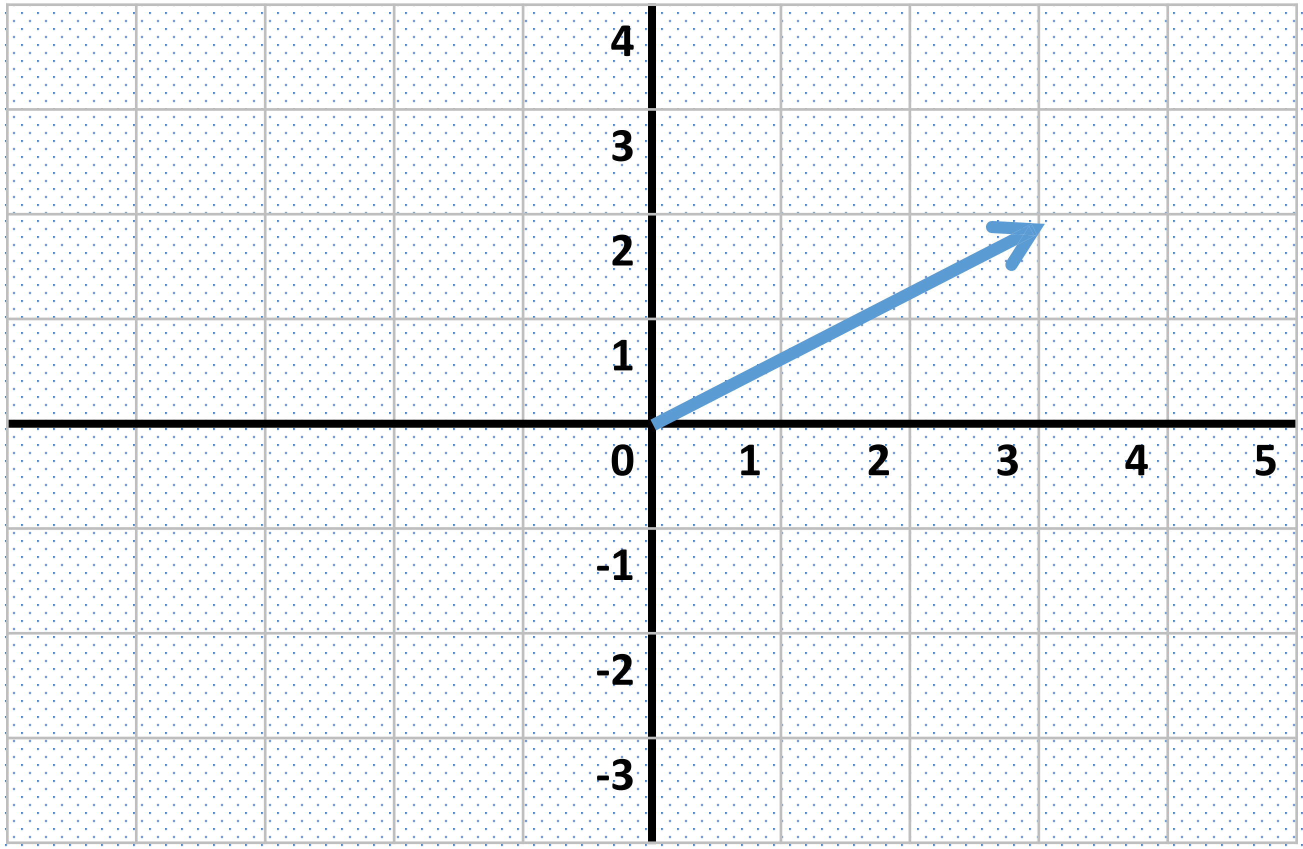


Vector Arithmetic

- Addition
- Subtraction
- Multiplication

Vector Addition

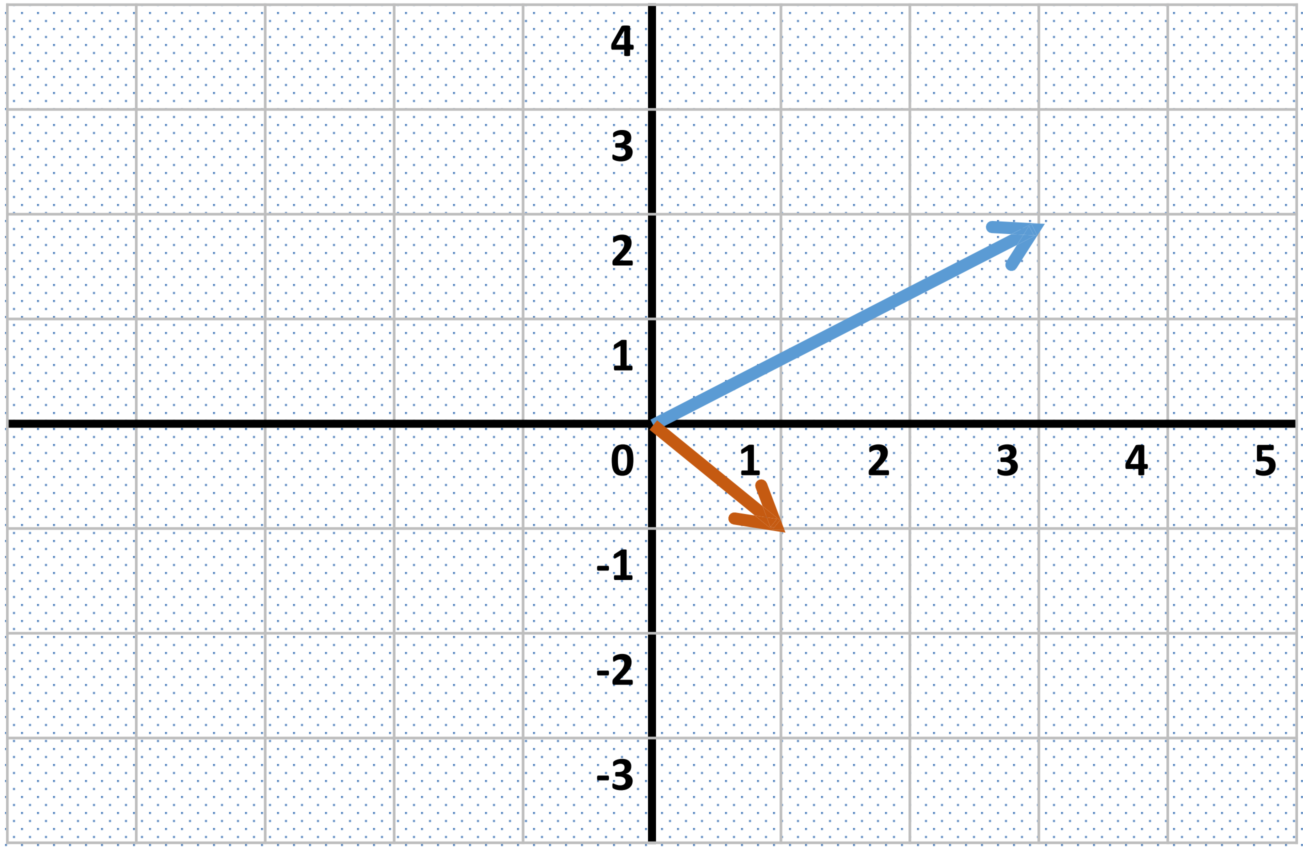
$$\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Vector Addition

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

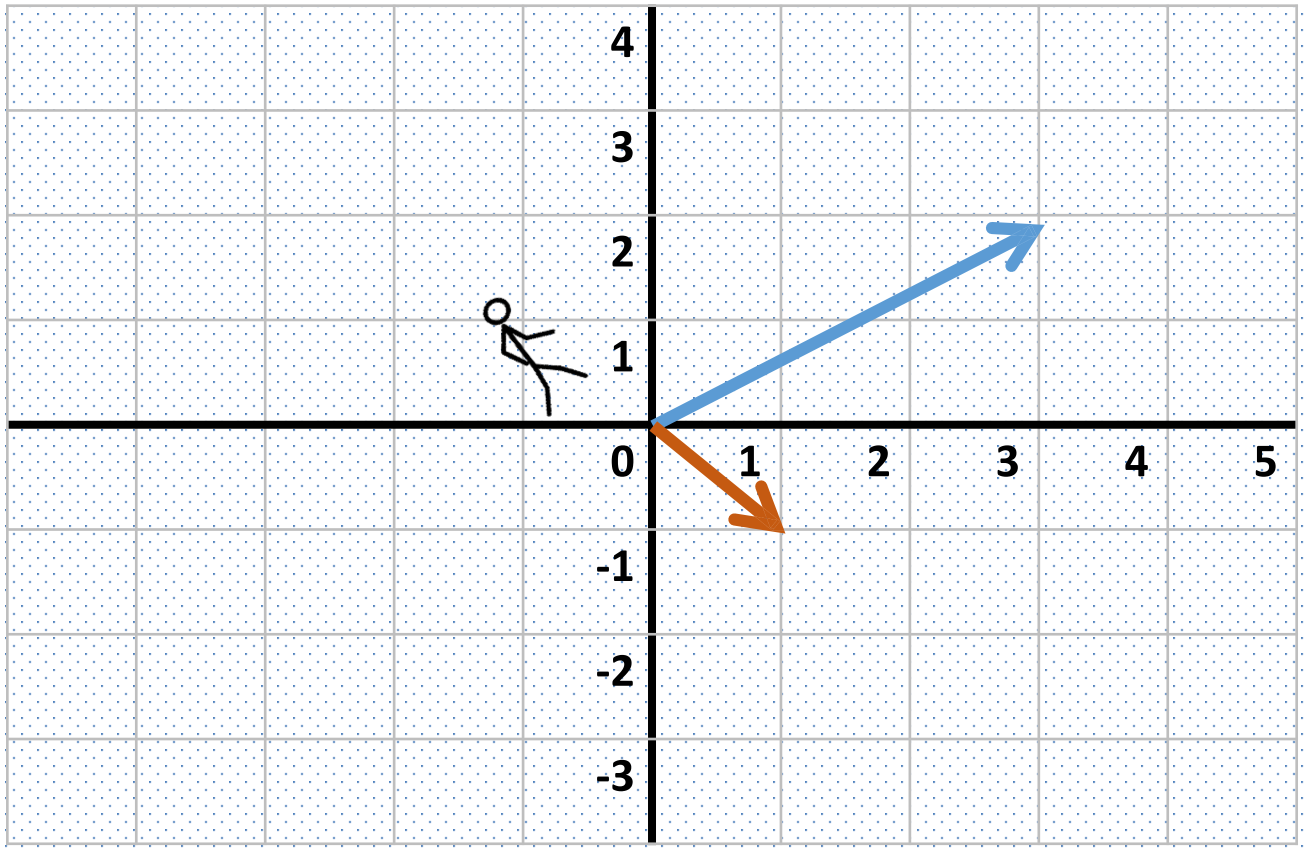


Vector Addition

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{V}_1 + \vec{V}_2$$

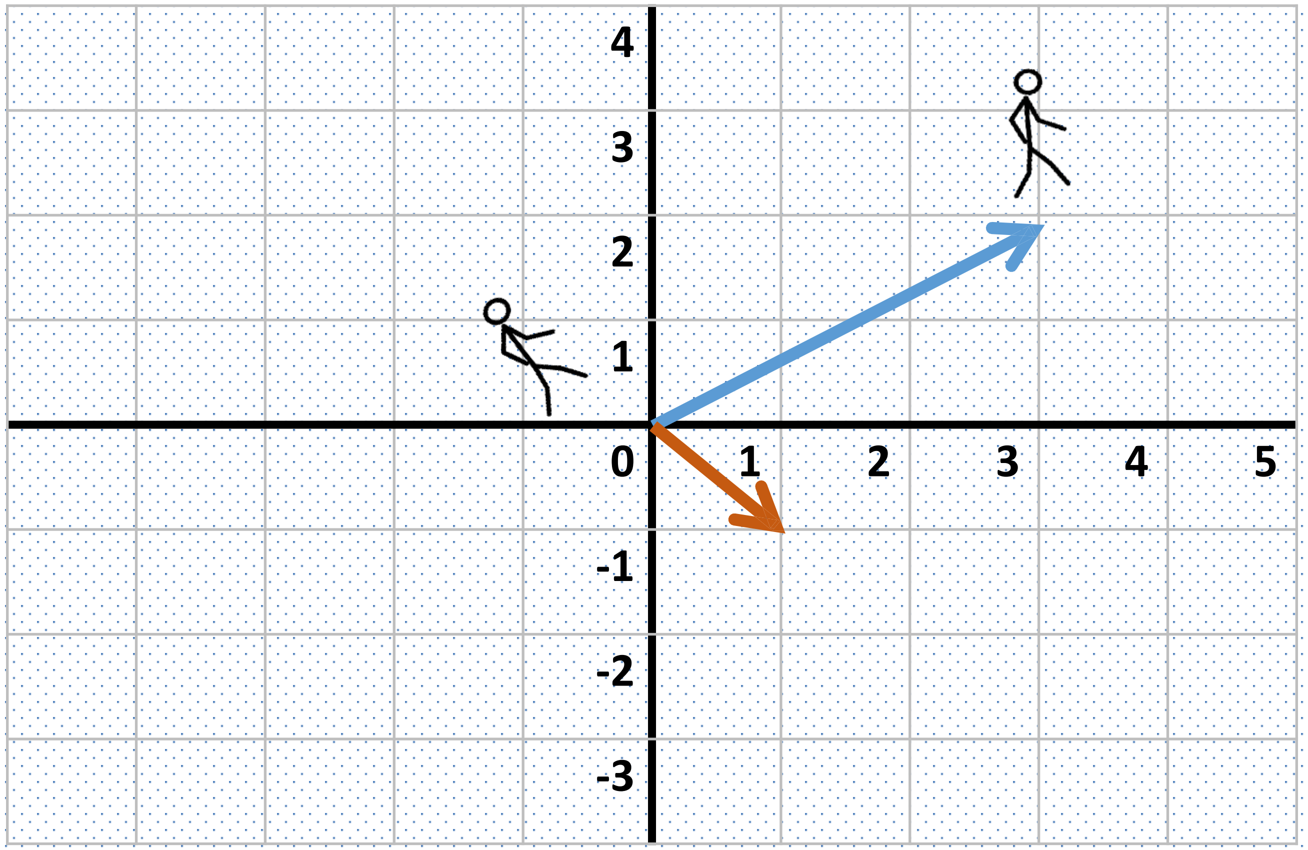


Vector Addition

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

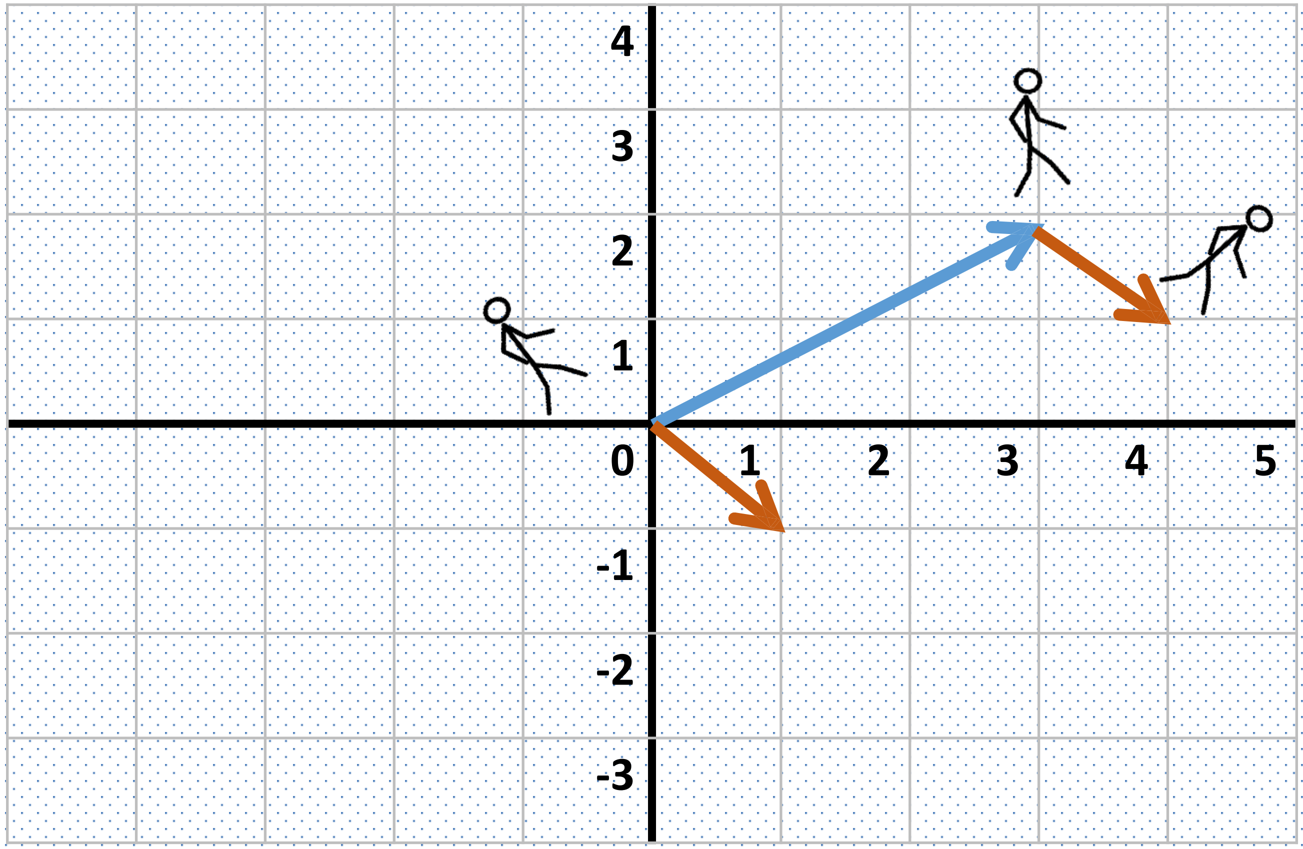
$$\vec{V}_1 + \vec{V}_2$$



Vector Addition

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



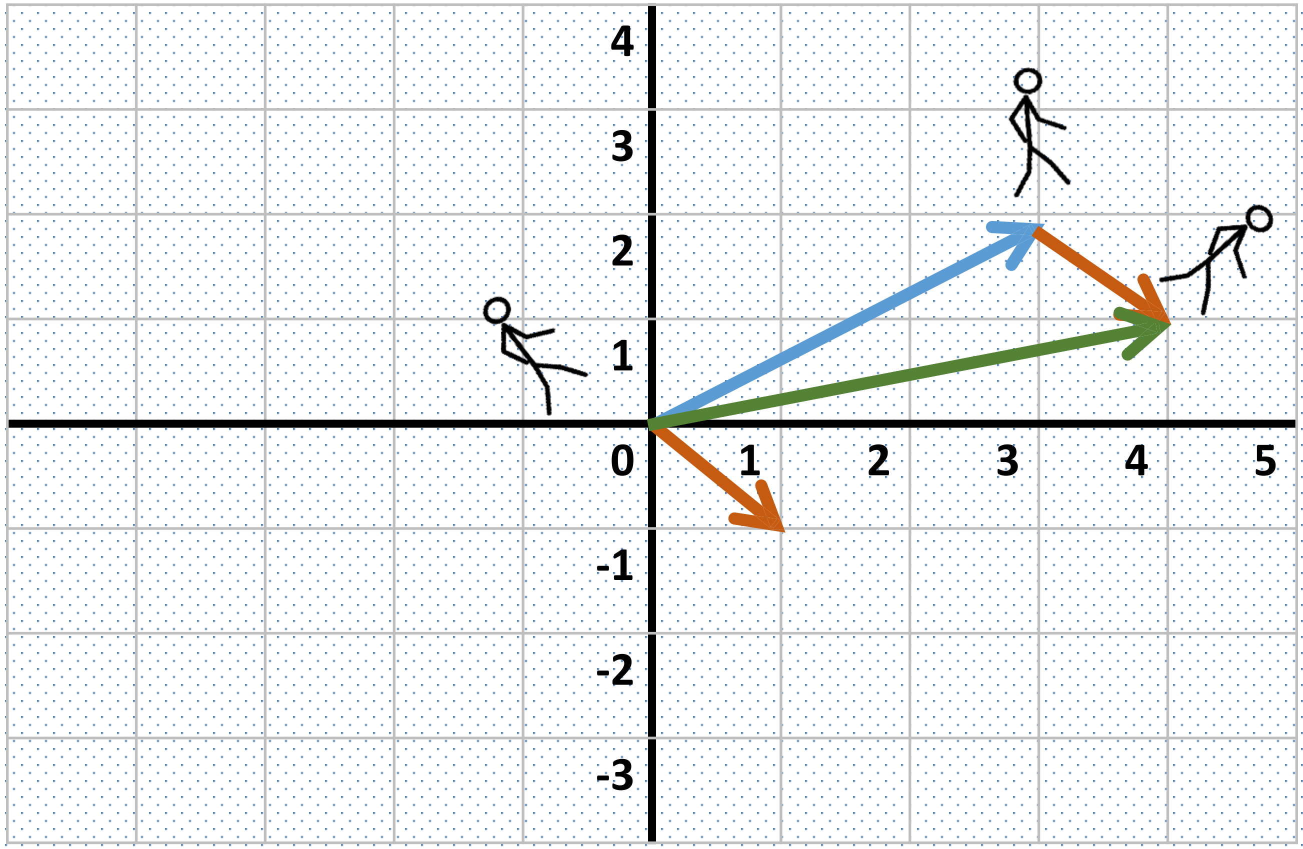
$$\vec{V}_1 + \vec{V}_2$$

Vector Addition

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{V}_1 + \vec{V}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

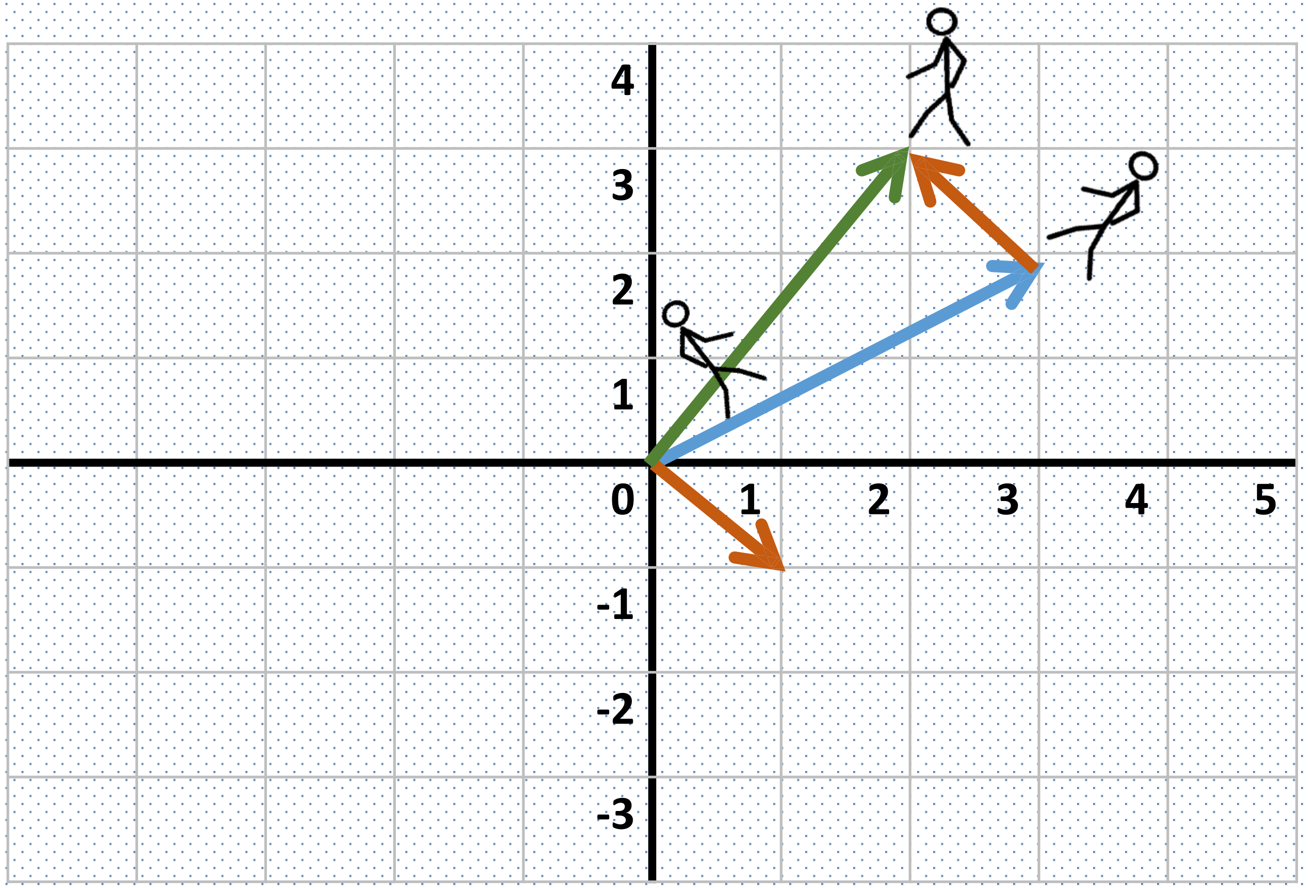


Vector Subtraction

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

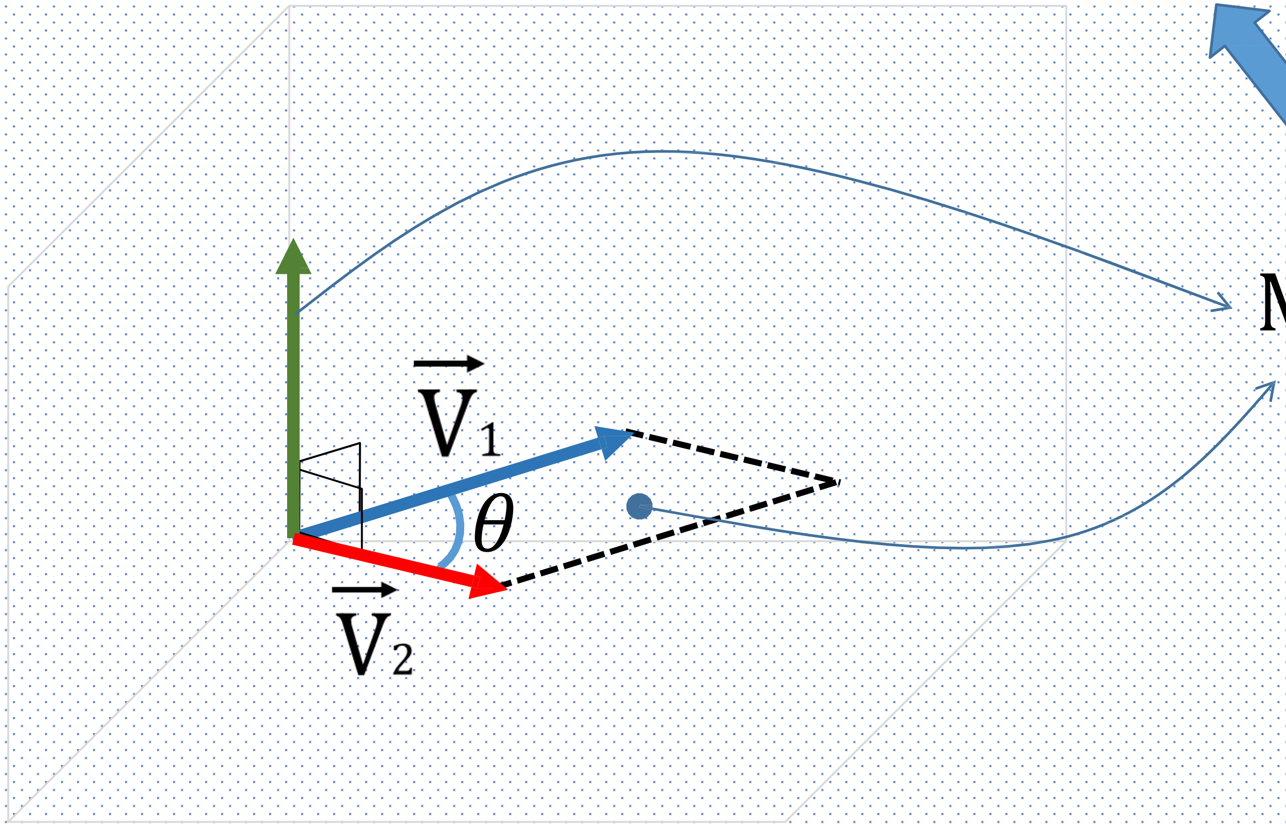
$$\vec{V}_1 - \vec{V}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



Vector Multiplication (Cross Product)

$$\vec{V}_1 * \vec{V}_2 = \boxed{|V_1| * |V_2| \sin(\theta)} * \hat{n}$$

Unit vector



Magnitude

Matrices

What is a Matrix?

$$X = \underbrace{\begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix}}_{\text{Columns}} \left.\vphantom{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}\right\} \text{Rows}$$

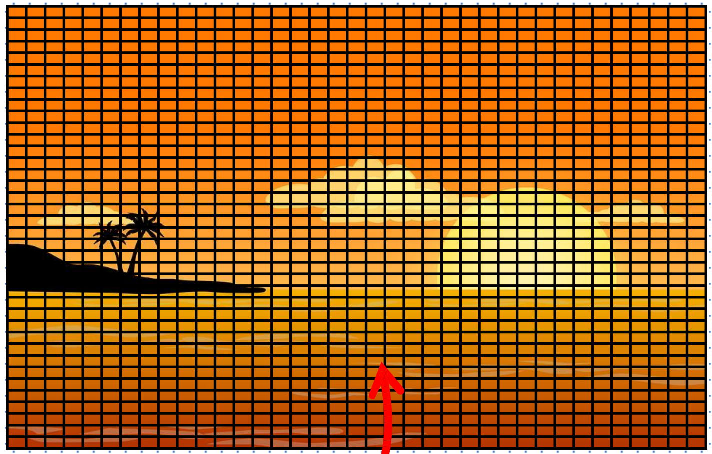
What is a Matrix?

Datasets treated as Matrix that have many rows, each row represents a feature vector.

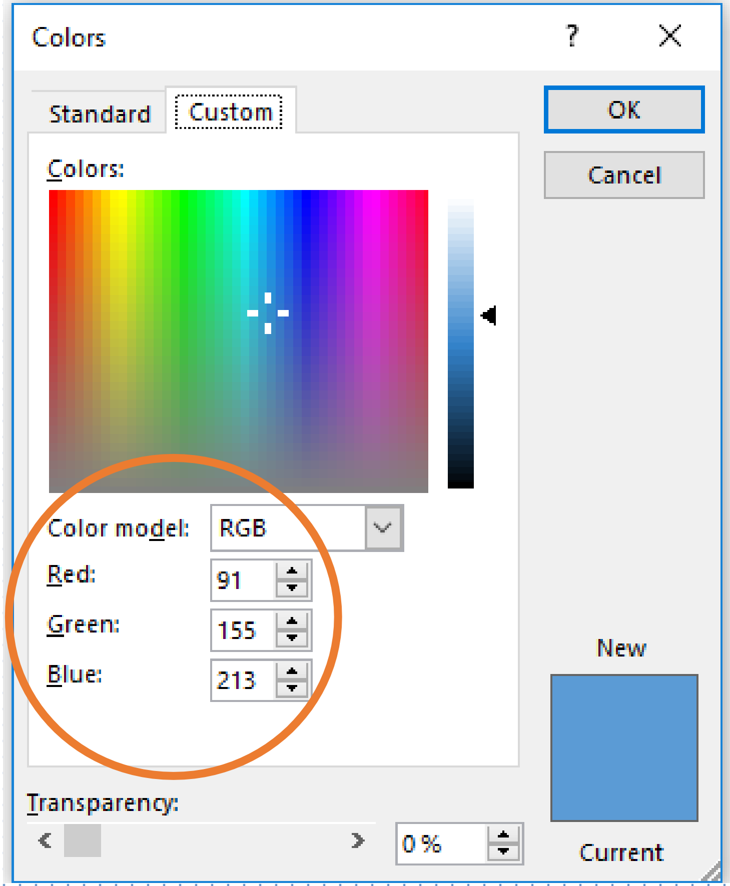
Fixed Acidity	Volatile Acidity	Citric Acid	Residual Sugar	Chlorides	Free Sulfur Dioxide	Total Sulfur Dioxide	Density	pH	Sulphates	Alcohol	Quality
7.4	0.7	0	1.9	0.076	11	34	0.9978	3.51	0.56	9.4	5
7.8	0.88	0	2.6	0.098	25	67	0.9968	3.2	0.68	9.8	5
7.8	0.76	0.04	2.3	0.092	15	54	0.997	3.26	0.65	9.8	5
11.2	0.28	0.56	1.9	0.075	17	60	0.998	3.16	0.58	9.8	6
7.4	0.7	0	1.9	0.076	11	34	0.9978	3.51	0.56	9.4	5
7.4	0.66	0	1.8	0.075	13	40	0.9978	3.51	0.56	9.4	5
7.9	0.6	0.06	1.6	0.069	15	59	0.9964	3.3	0.46	9.4	6
7.3	0.65	0	1.2	0.065	15	21	0.9946	3.39	0.47	10	7
7.8	0.58	0.02	2	0.073	9	18	0.9968	3.36	0.57	9.5	7

Why should we learn Matrices?

Matrix of Pixels



$$\begin{array}{c} R \\ G \\ B \end{array} \begin{bmatrix} 230 \\ 169 \\ 43 \end{bmatrix}$$



Matrix Arithmetic

- Addition
- Subtraction
- Multiplication

Matrix Addition

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 8 & -1 \\ 5 & -2 & -3 \end{bmatrix}$$

$$X + Y = \begin{bmatrix} 2+1 & 3+8 & 4+(-1) \\ 1+5 & 6+(-2) & 7+(-3) \end{bmatrix} = \begin{bmatrix} 3 & 11 & 3 \\ 6 & 4 & 4 \end{bmatrix}$$

Matrix Subtraction

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 8 & -1 \\ 5 & -2 & -3 \end{bmatrix}$$

$$X - Y = \begin{bmatrix} 2 - 1 & 3 - 8 & 4 - (-1) \\ 1 - 5 & 6 - (-2) & 7 - (-3) \end{bmatrix} = \begin{bmatrix} 1 & -5 & 5 \\ -4 & 8 & 10 \end{bmatrix}$$

Matrix Multiplication – Scalar

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix}$$

$$2 * X = 2 * \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 6 & 8 \\ 2 & 12 & 14 \end{bmatrix}$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix} \qquad X\,.\,A = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$$

$$2 \; X \; \boxed{3 \qquad\qquad 3} \; X \; 2 \qquad\qquad 2 \; X \; 2$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$X\,.\,A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$X \,.\, A = \begin{bmatrix} \color{orange}{2} & \color{orange}{3} & \color{orange}{4} \\ 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} \color{orange}{2} & 3 \\ \color{orange}{1} & 6 \\ \color{orange}{2} & 3 \end{bmatrix}$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$X \,.\, A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 15 & \\ & \end{bmatrix}$$

$$(2*2) + (3*1) + (4*2) = 15$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$X\,.\,A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 15 & 36 \\ & \\ & \end{bmatrix}$$

$$(2*3) + (3*6) + (4*3) = 36$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$X \, . \, A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 15 & 36 \\ 22 & \end{bmatrix}$$

$$(1*2) + (6*1) + (7*2) = 22$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$X\,.\,A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 15 & 36 \\ 22 & 60 \end{bmatrix}$$

$$(1*3) + (6*6) + (7*3) = 60$$

Matrix Multiplication – Dot product

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 2 & 3 \end{bmatrix} \qquad X \,.\, A = \begin{bmatrix} 15 & 36 \\ 22 & 60 \end{bmatrix}$$

$$2 \text{ X } \boxed{3 \qquad\qquad 3} \text{ X } 2 \qquad\qquad 2 \text{ X } 2$$

Matrix Multiplication – Example

	Average Price
Sports Shoes	$ 40
Formal	$ 30
Sandals	$ 20

	2016	2017	2018
Sports Shoes	2	3	3
Formal	3	4	3
Sandals	6	8	9

	2016	2017	2018
Sports Shoes	2 * 40	3 * 40	3 * 40
Formal	3 * 30	4 * 30	3 * 30
Sandals	6 * 20	8 * 20	9 * 20

	2016	2017	2018
Sports Shoes	80	120	120
Formal	90	120	90
Sandals	120	160	180
Total	**290**	**400**	**390**

Matrix Division

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 7 & -3 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix}$$

$$\frac{A}{X} = A \, . \, X^{-1}$$

We will see soon how to get the inverse of a Matrix

Important Matrix Terms

Matrix Terms

- Determinant of the Matrix
- Inverse of Matrix
- Identity Matrix
- Transpose of the Matrix

Determinant of a Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{Determinant} = ad - bc$$

Inverse of a Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$1/A = \text{Inverse of } A = A^{-1}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Identity Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 3 & 8 \end{bmatrix}$$

$$A * I = A$$

Transpose of a matrix

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \Rightarrow X^T = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$$

Transpose of a matrix

$$X = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix} \Rightarrow X^T = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$$

Vector Transformation using Matrix

Vector Transformation

$$T = \begin{bmatrix} \boxed{1 \quad -1} \\ \boxed{1 \quad 2} \end{bmatrix}$$

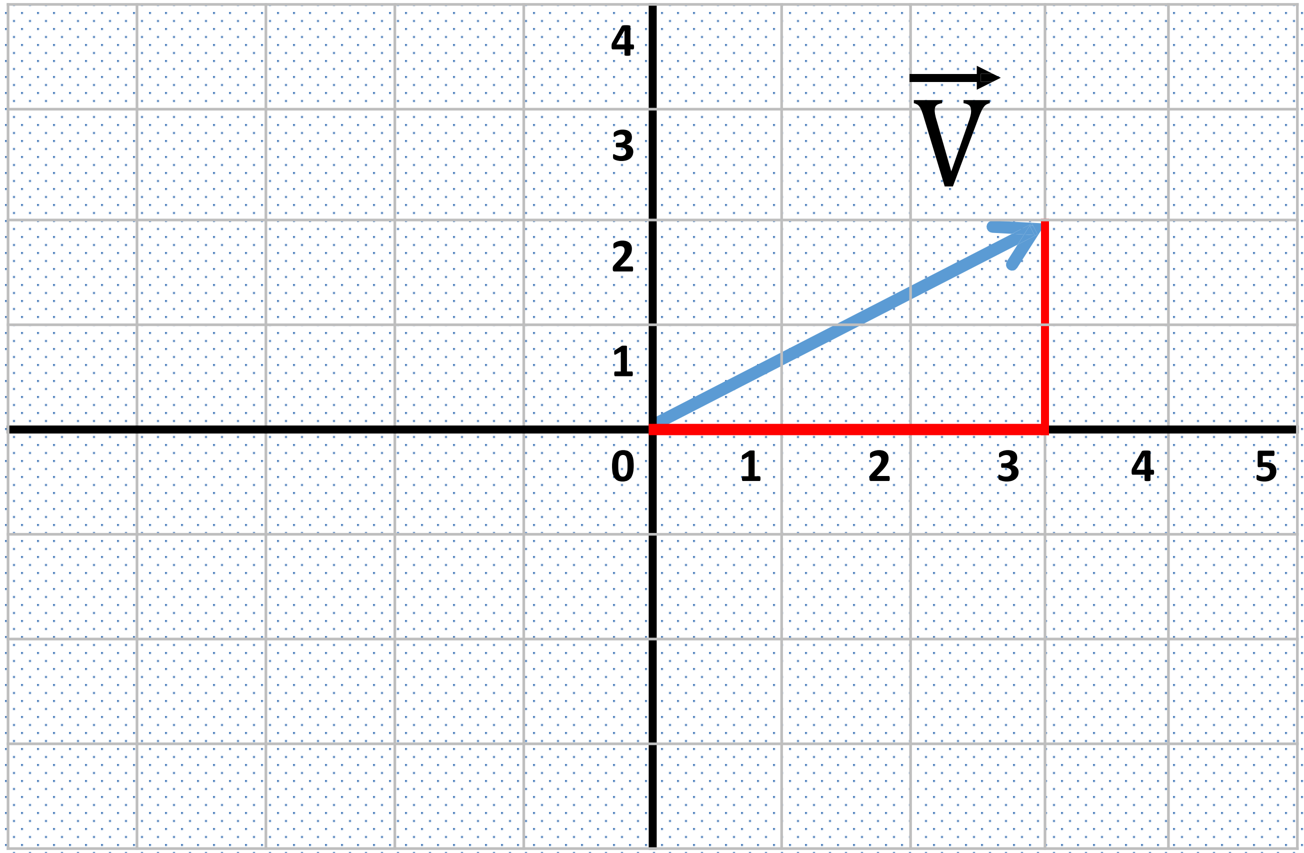
2 x 2

$$\vec{V} = \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 2 \end{matrix}} \end{bmatrix}$$

2 x 1

$$T \, . \, \vec{V} = \begin{bmatrix} (1*3) + (-1*2) \\ (1*3) + (2*2) \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 2 \\ 3 + 4 \end{bmatrix}$$

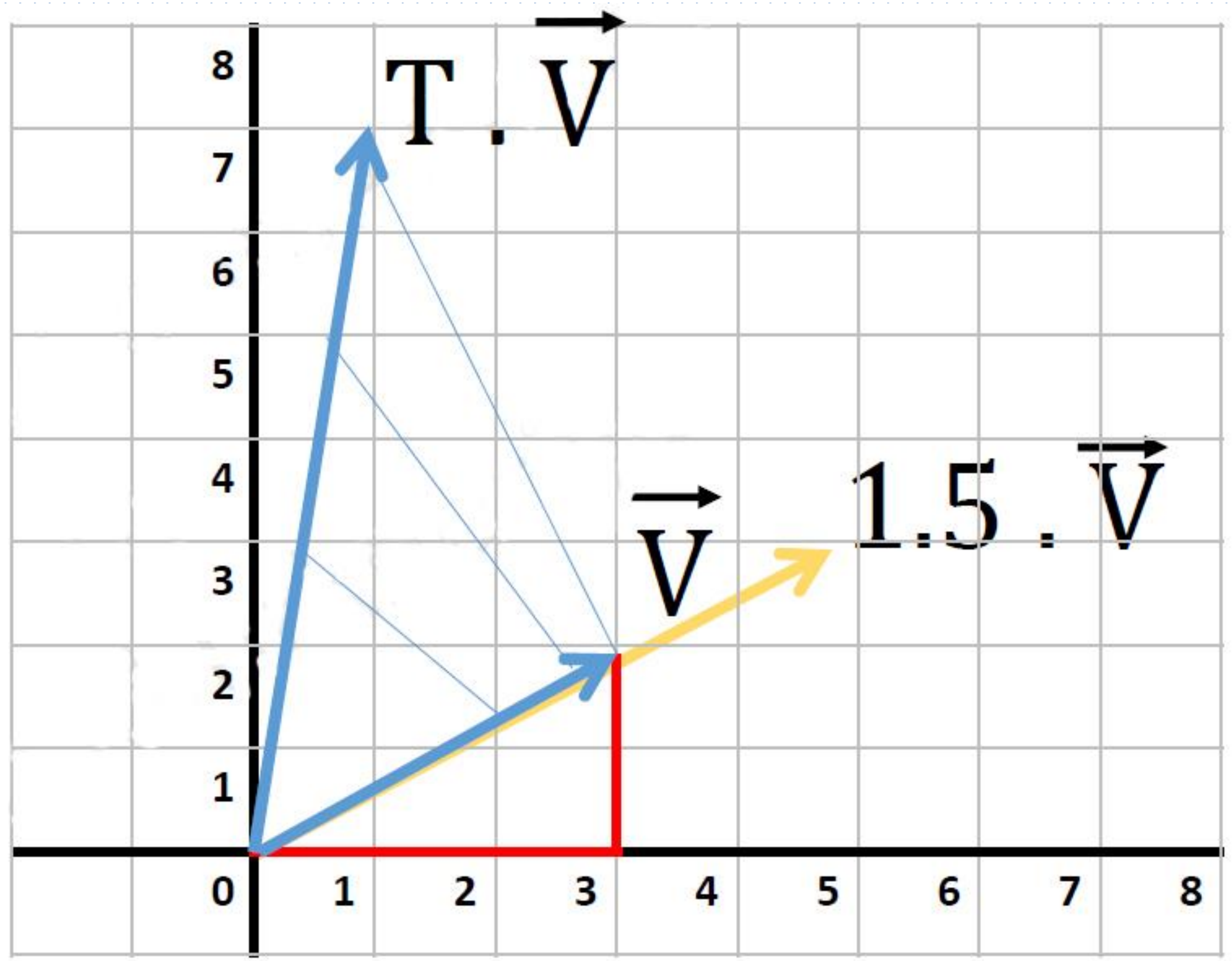


Vector Transformation

$$T = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \qquad \vec{V} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T \, . \, \vec{V} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

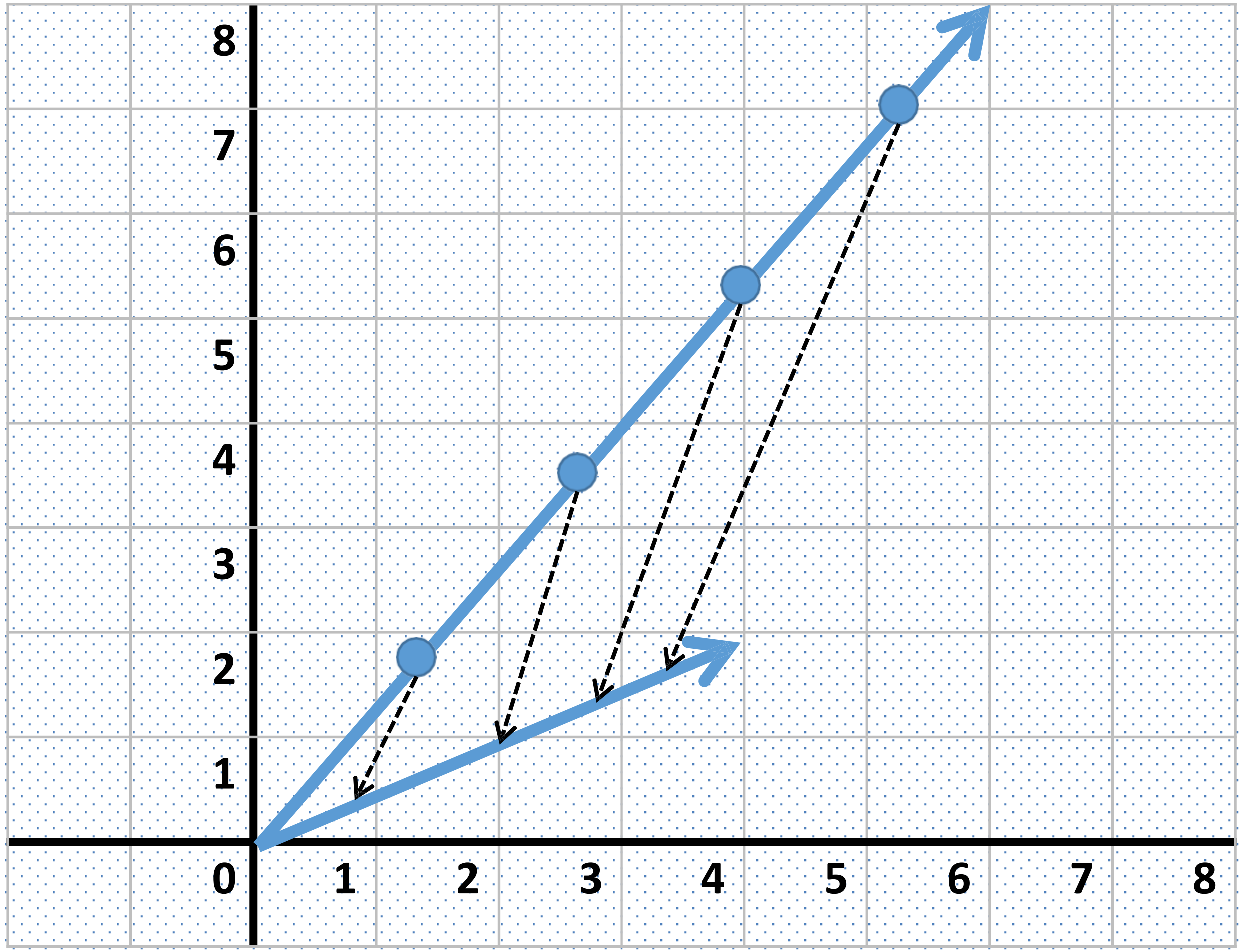
$$1.5 \, . \, \vec{V} = \begin{bmatrix} 4.5 \\ 3 \end{bmatrix}$$



Vector Transformation

$$T = \begin{bmatrix} 2 & -1 \\ 1 & -0.5 \end{bmatrix} \qquad \vec{V} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

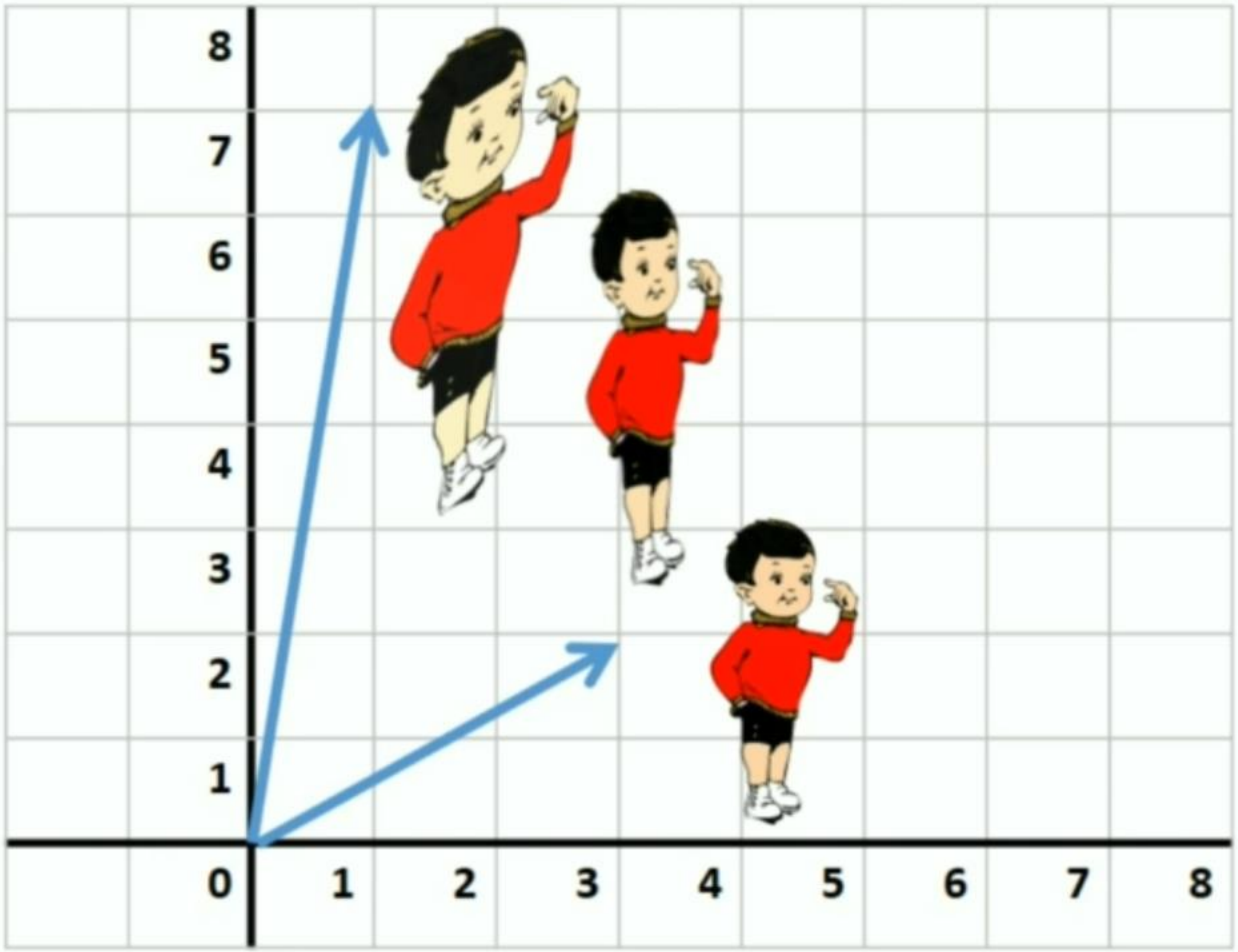
$$T \cdot \vec{V} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



Vector Transformation Applications

Computer Graphics

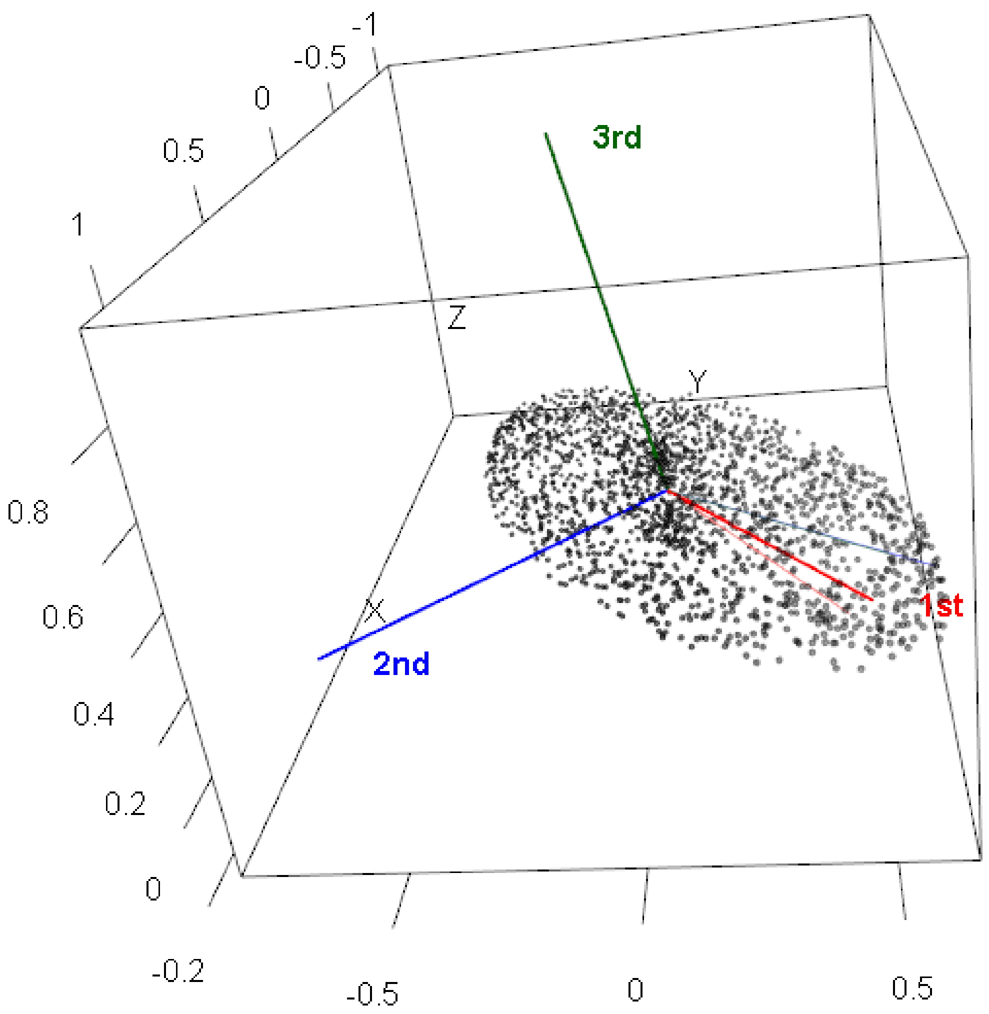
Used a lot in computer graphics and video games to process moving objects in 3D space.



Vector Transformation Applications

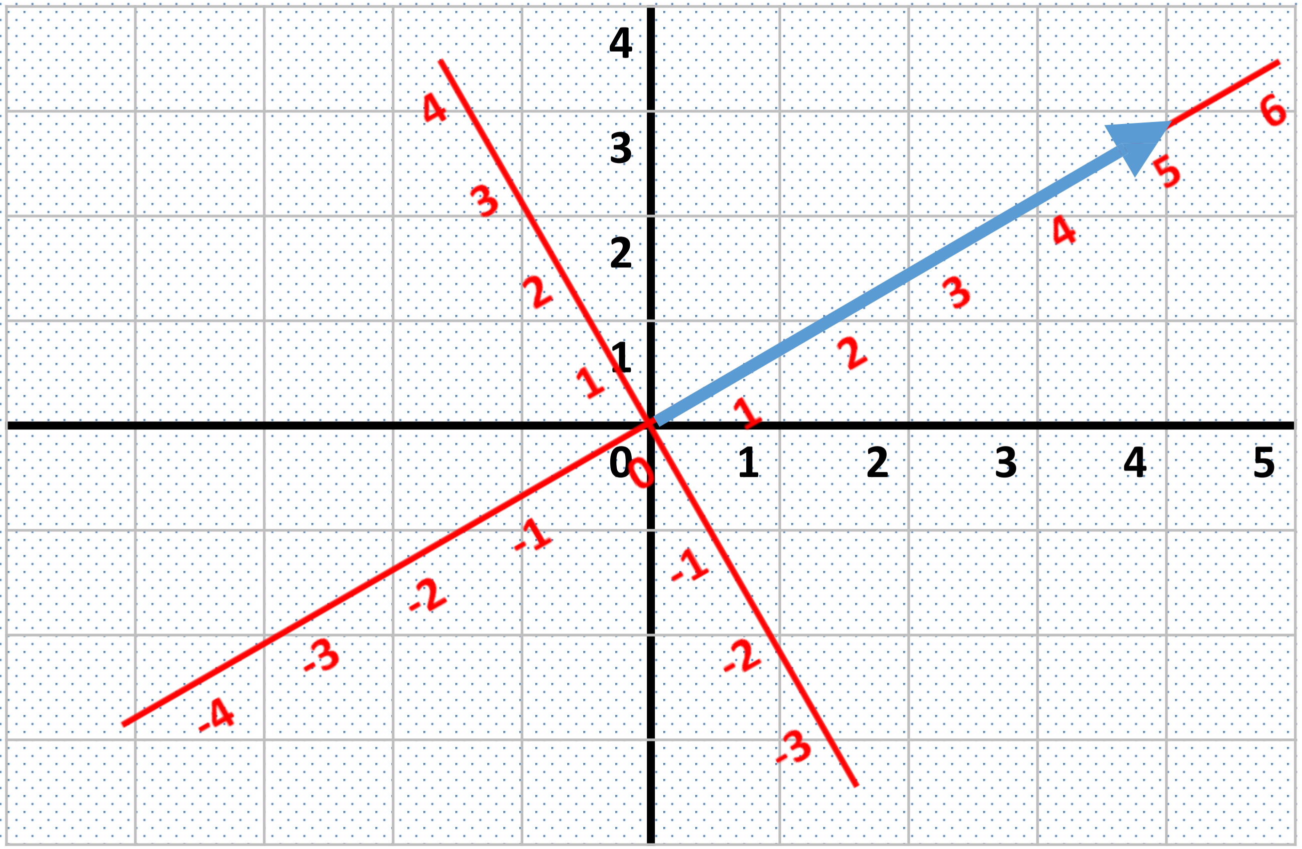
Dimensions Reduction

Used for dimension reduction techniques like PCA.



Change of Basis

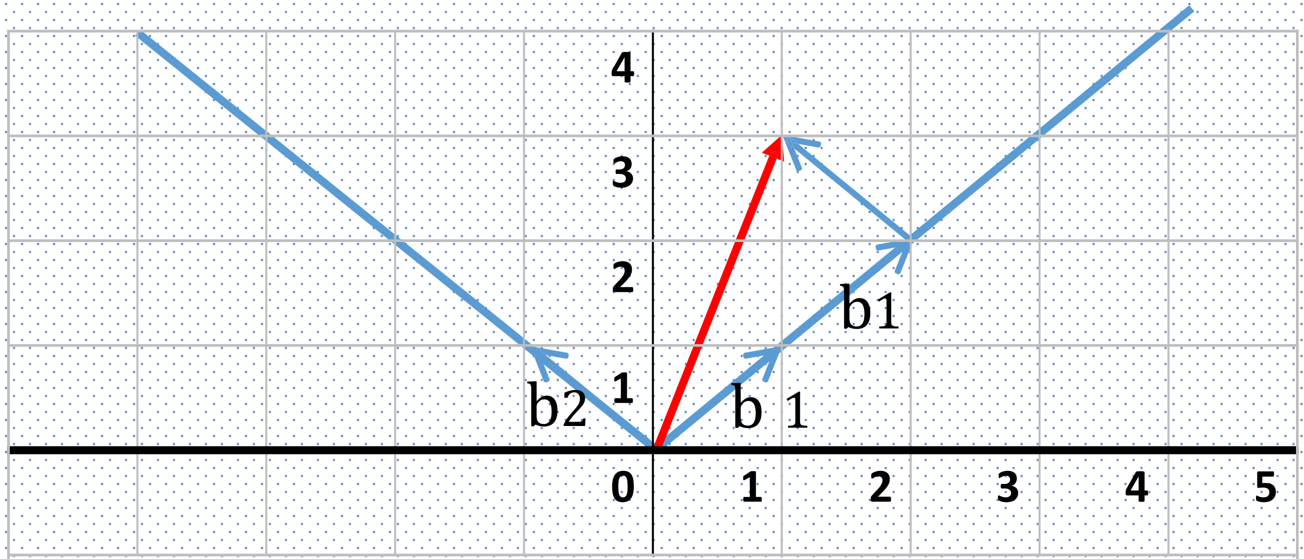
Change of Basis – Alternate Coordinates



$$\vec{V} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\vec{V} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Change of Basis – Alternate Coordinates

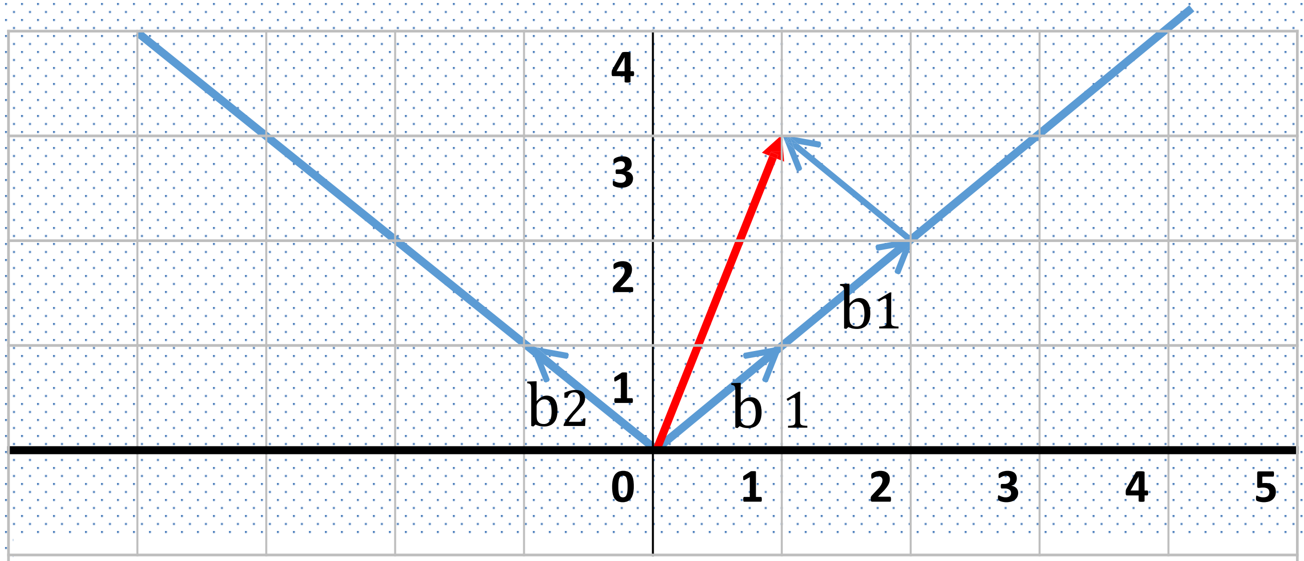


$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad b_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{W} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2b_1 + b_2$$

$$\vec{V} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Change of Basis – Alternate Coordinates

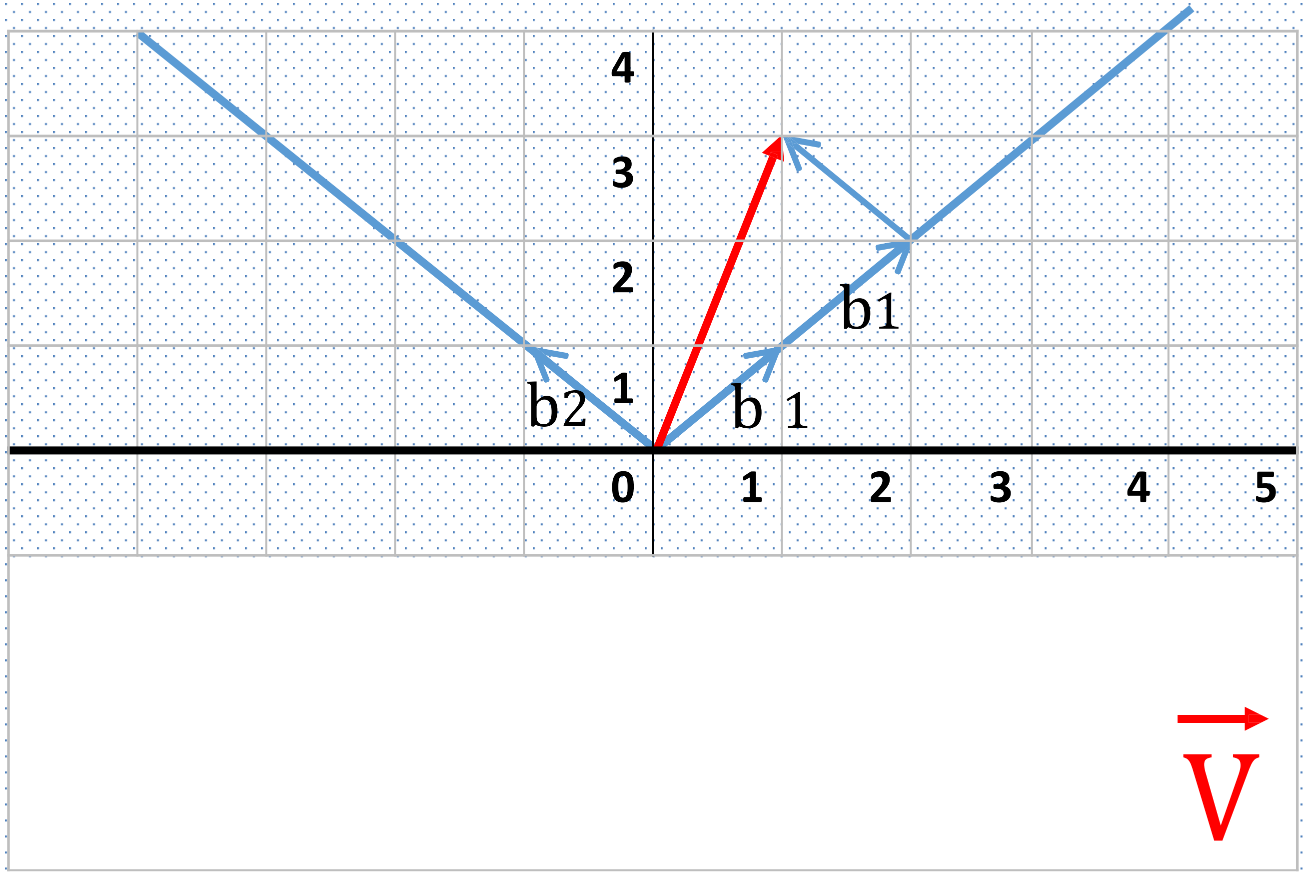


$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad b_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{W} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2b_1 + b_2$$

$$\vec{V} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Change of Basis – Alternate Coordinates

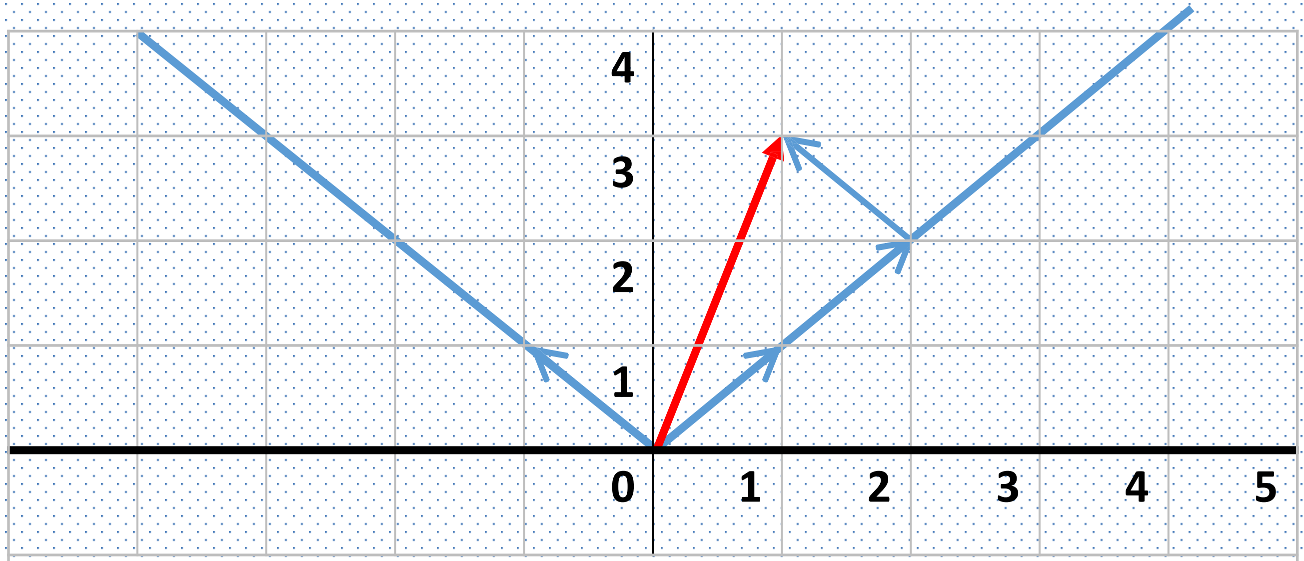


$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad b_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{W} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2b_1 + b_2$$

$$\vec{V} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Change of Basis – Alternate Coordinates



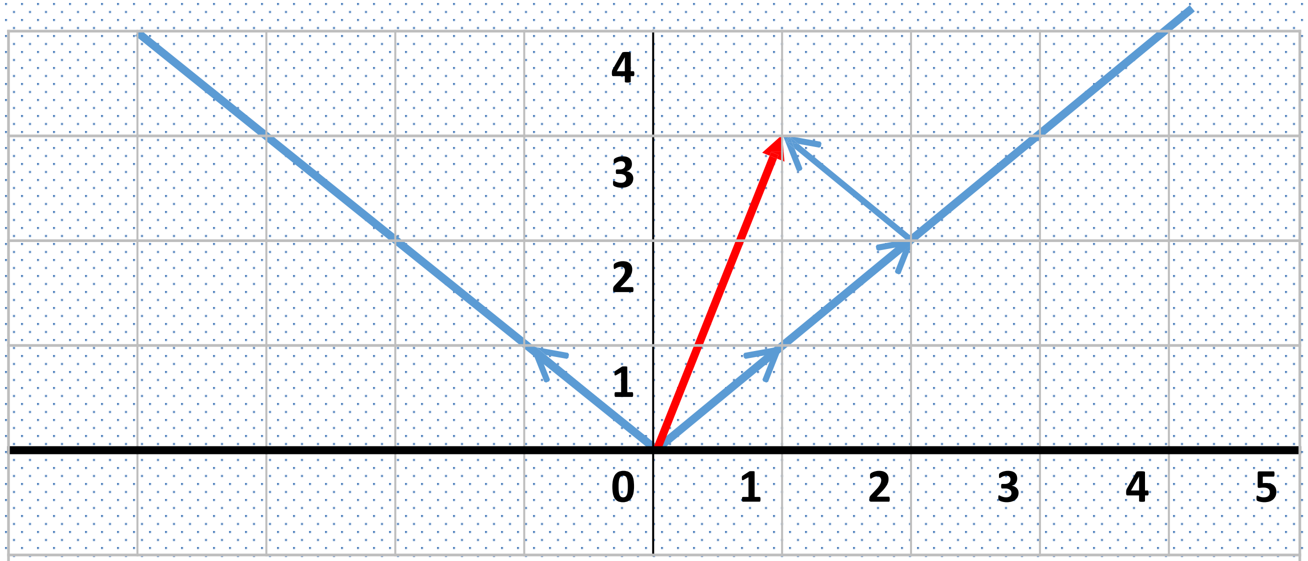
$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad b_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{W} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2b_1 + b_2$$

$$\vec{V} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} . \vec{W}$$

Matrix Transformation of $\vec{W}$

Change of Basis – Alternate Coordinates

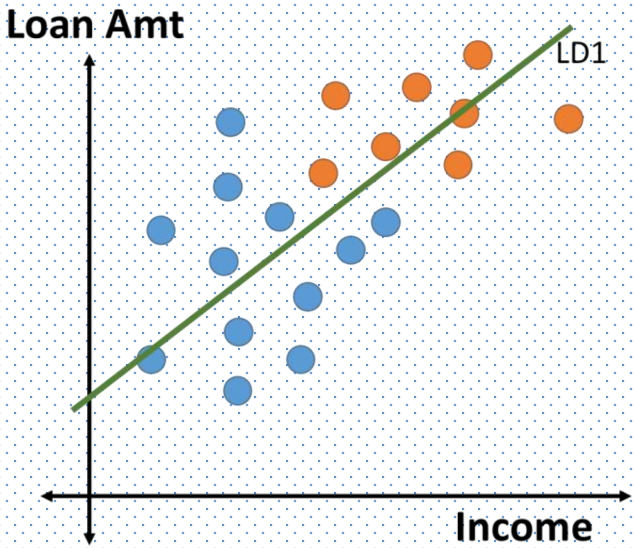


$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad b_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

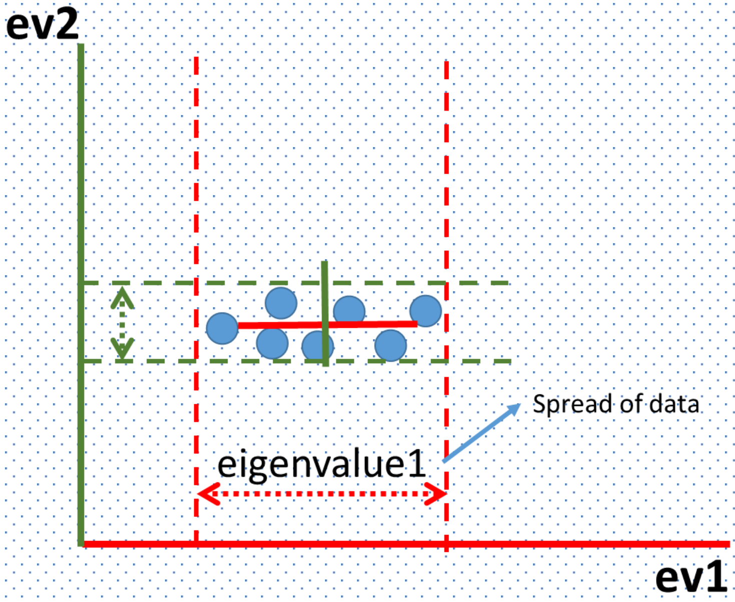
$$\vec{W} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2b_1 + b_2$$

$$\vec{W} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} . \vec{V}$$

Why we are learning this?



Linear Discriminant Analysis



Principal Component Analysis

Eigenvectors and Eigenvalues

Eigenvector and Eigenvalues?

- A non-zero vector that changes by a scalar during linear transformation.

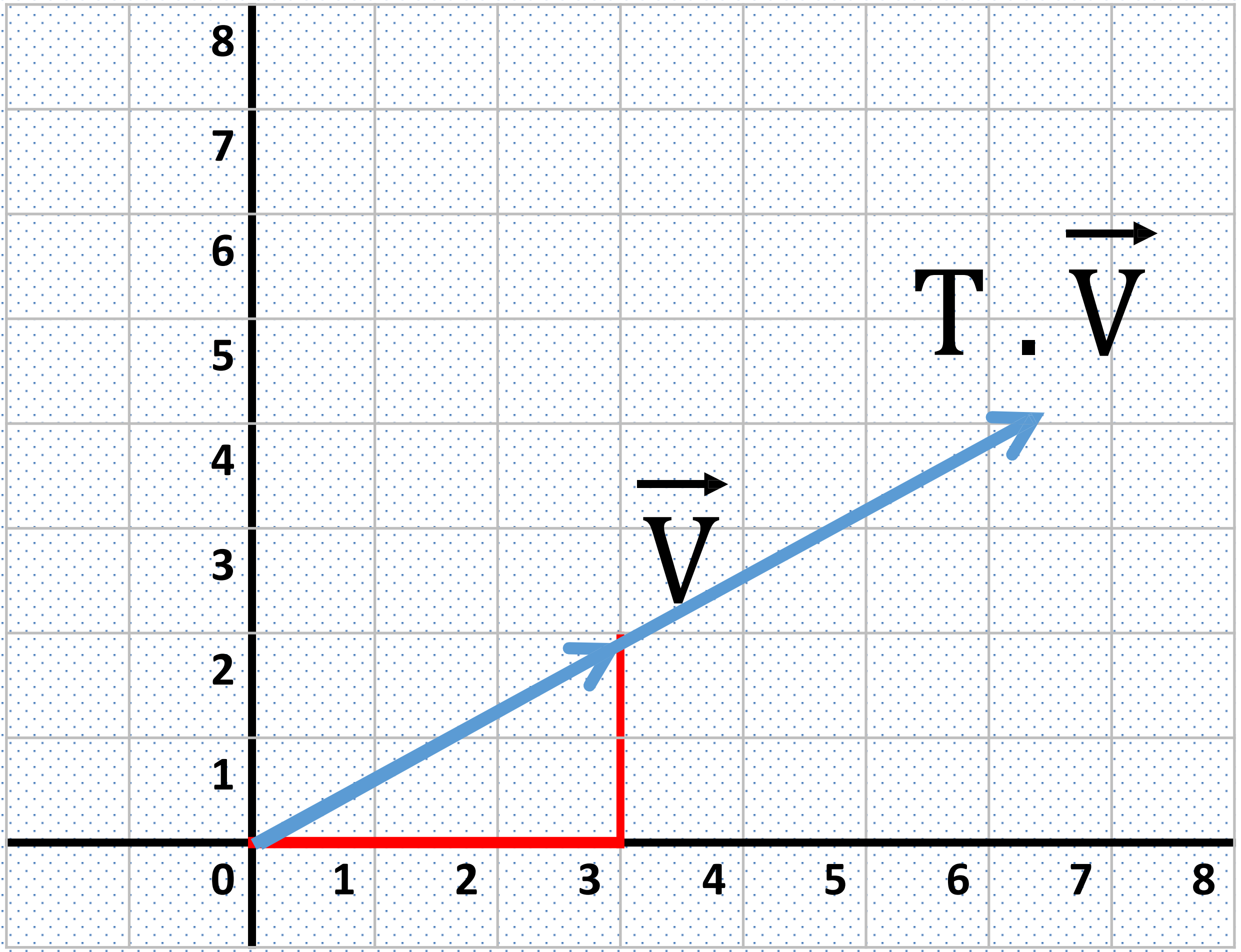
- Scalar value by which it changes its magnitude is eigenvalue

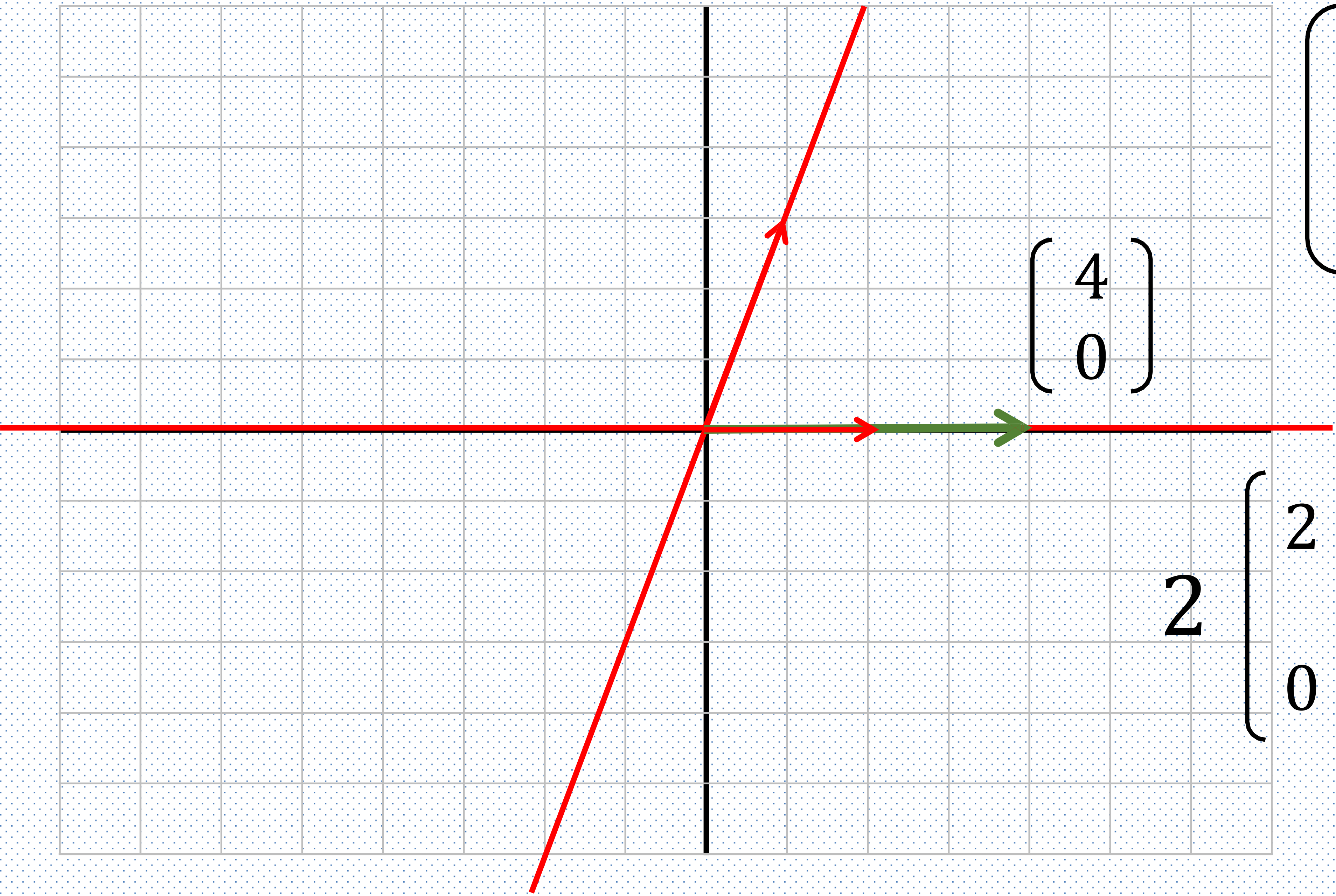
Note: Only thing that's changing is our perception of the coordinates.

What is an Eigenvector and Eigenvalues?

- A non-zero vector that changes by a scalar during linear transformation

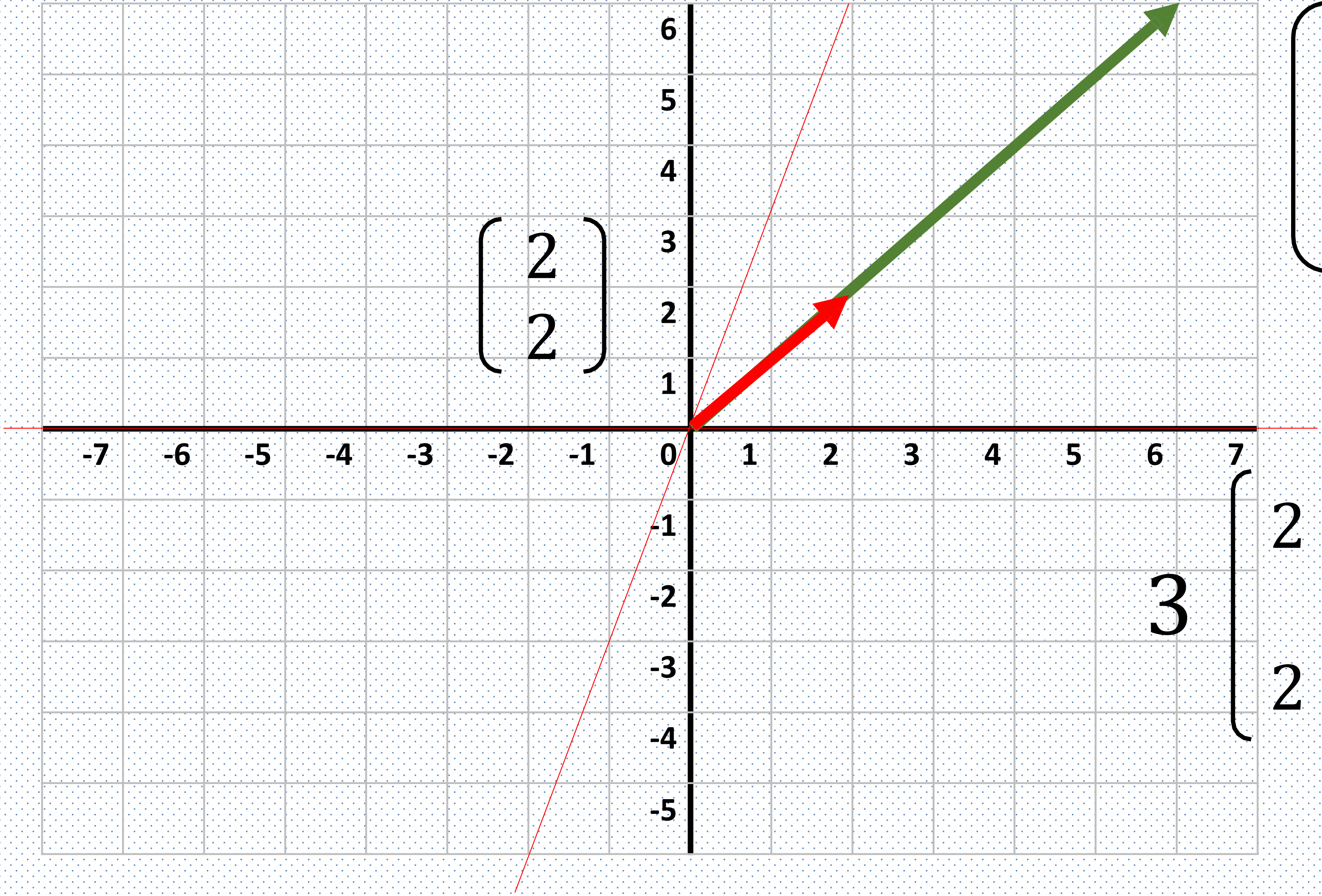
$$T.\vec{V} = \lambda.\vec{V}$$





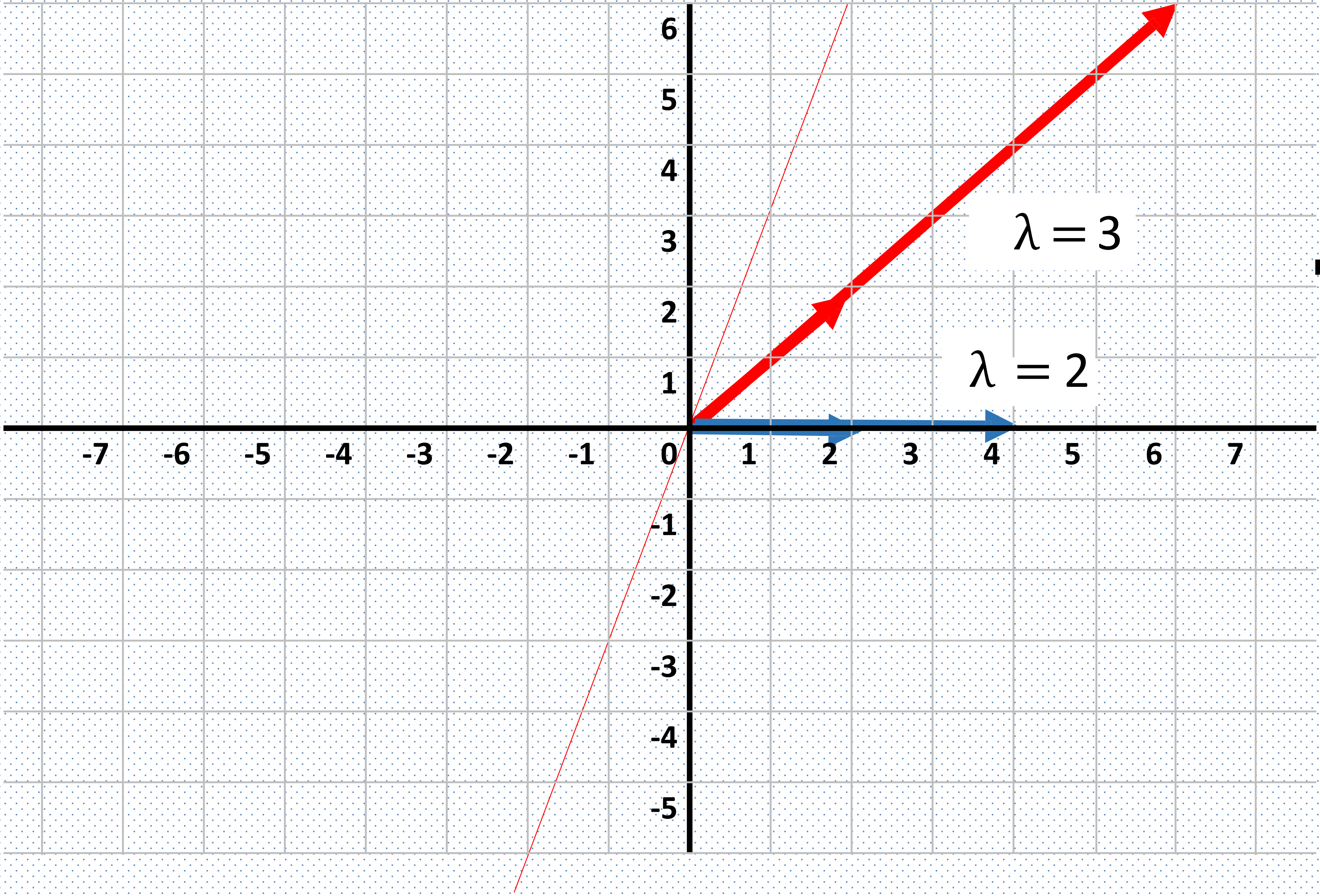
$$\begin{pmatrix}2 & 1\\0 & 3\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}$$

$$2\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}$$



$$\begin{pmatrix}2 & 1\\0 & 3\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}$$

$$3\begin{pmatrix}2\\2\end{pmatrix}=\begin{pmatrix}6\\6\end{pmatrix}$$



$$T.\overrightarrow{V} = \lambda.\overrightarrow{V}$$

Thank You!