CSI 2101 Lecture Notes

Mohamed Shakir

March 27, 2023

Contents

Definitions, Theorems, Lemmas, and Corollaries

Definition 4.1.1. *Let a and b be two integers such that $a \neq 0$. We say that a divides b if there exists c such that $b = ac$. If a divides be we say a is a factor or divisor of b. We also can say b is a multiple of a.*

Theorem 4.1.1. *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$.*

1. *If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$*

2. *If $a \mid b$, then $a \mid bc$ for every integer*

3. *If $a \mid b$ and $b \mid c$, then $a \mid c$*

Corollary 4.1.1. *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $a \mid b$ and $a \mid c$, $a \mid (mb + nc)$ for all integers m and n*

Theorem 4.1.2 (The Division Algorithm)**.** *Let $a, d \in \mathbb{Z}$ with $d > 0$. There exists a unique q and r such that*

$$0 \leq r \leq d$$

and

$$a = dq + r$$

We write

$$q = a \ div \ d$$

$$r = a \ mod \ a$$

Definition 4.1.2. *Let $a, b, m \in \mathbb{Z}$ with $m \geq 2$. We say a is congruent to b modulo m if $m \mid (a - b)$. We write $a \equiv b \ (mod \ m)$*

Theorem 4.1.3. *Let $a, b, c, d, m \in \mathbb{Z}$ with $m \geq 2$. If $a \equiv b \ (mod \ m)$ and $c \equiv d \ (mod \ m)$, then*

1. *$a + c \equiv b + d \ (mod \ m)$*

2. *$ac \equiv bd \ (mod \ m)$*

Definition 5.1.1. *A positive integer p is prime if it admits exactly two divisors.*

Theorem 5.1.1 (Fundamental Theorem of Arithmetic)**.** *All integers greater than 1 can be written as a product of prime numbers. This representation is unique if we write the prime numbers in non-decreasing order.*

Theorem 5.1.2. *Let $n > 1$ be an integer. If n is not prime, then n has a prime divisor p such that $p \leq \sqrt{n}$.*

Corollary 6.0.1. *Let*
$$a = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$
$$a = p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_k^{b_k}$$

Where p_i is prime, $a_i \geq 0$ and $b_i \geq 0$, $1 \leq i \leq k$. Then

$$gcd(a, b) = p_1^{\min(a_1, b_1)} \cdot p_2^{\min(a_2, b_2)} \cdot \ldots \cdot p_k^{\min(a_k, b_k)}$$

$$lcm(a, b) = p_1^{\max(a_1, b_1)} \cdot p_2^{\max(a_2, b_2)} \cdot \ldots \cdot p_k^{\max(a_k, b_k)}$$

$$gcd(a, b) \cdot lcm(a, b) = ab$$

Lemma 6.0.1. *Let a, b, q, r be integers such that*

$$a = b \cdot q + r$$

Then

$$gcd(a, b) = gcd(b, r)$$

Definition 6.0.1 (Euclidean Algorithm)**.**

$$x = a$$

$$y = b$$

while $y \neq 0$

$$r = x \mod y$$

$$x = y$$

$$y = r$$

return x

Theorem 6.0.1 (Bézout)**.** *Let $a, b \in \mathbb{Z}$ be positive integers. There exists $s, t \in \mathbb{Z}$ such that*

$$s \cdot a + t \cdot b = gcd(a, b)$$

Lemma 6.0.1. *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $gcd(a, b) = 1$ and $a \mid (bc)$, then $a \mid c$.*

Lemma 8.0.1. *Let $a, b, c \in \mathbb{Z}$, with $a \neq 0$. If $gcd(a, b) = 1$, and $a \mid (bc)$, then $a|c$.*

Theorem 8.0.1. *Let $a, b, c, m \in \mathbb{Z}$, with $m \geq 2$. Assume $ac \equiv bc \pmod{m}$ and $gcd(c, m) = 1$. Then $a \equiv b \pmod{m}$.*

Lemma 8.0.2. *Let p be a prime number and $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. If $p \mid (a_1 \cdot a_2 \cdot \cdots \cdot a_n)$, then there exists $1 \leq i \leq n$ such that $p \mid a_i$.*

Theorem 8.0.2. *Let $m \in \mathbb{Z}$ with $m \geq 2$ and let $a \in \mathbb{Z}_m$. The multiplicative inverse of $a \pmod{n}$ exists if and only if $gcd(a, m) = 1$. When it exists, the inverse of $a \pmod{n}$ is unique.*

Lecture 1

Logic and Proof Techniques

TBC.

Lecture 2

Proof Examples

TBC.

Lecture 3

Proof by Induction and More Examples

Lecture 4

Intro to Number Theory

4.1 Divisibility

Definition 4.1.1. *Let a and b be two integers such that $a \neq 0$. We say that a divides b if there exists c such that $b = ac$. If a divides b we say a is a factor or divisor of b. We also can say b is a multiple of a.*

Theorem 4.1.1. *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$.*

1. *If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$*

2. *If $a \mid b$, then $a \mid bc$ for every integer*

3. *If $a \mid b$ and $b \mid c$, then $a \mid c$*

Proof.   1. We have to prove if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Assume that $a \mid b$ and $a \mid c$, then for some $k, l \in \mathbb{Z}$

$$b = k \cdot a$$

$$c = l \cdot a$$

Thus, we have
$$b + c = k \cdot a + l \cdot a = a(k + l)$$

So $a \mid (b + c)$

2. We have to prove if $a \mid b$, $a \mid bc$ for every c. Let $a, b \in \mathbb{Z}$ with $a \neq 0$. Assume that $a \mid b$. Then for some $k \in \mathbb{Z}$,

$$b = k \cdot a$$

Let $c \in \mathbb{Z}$, so
$$bc = k \cdot a \cdot c = a \cdot (kc)$$

Therefore, $a \mid bc$

9

3. We have to prove if $a \mid b$ and $b \mid c$, then $a \mid c$. Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Assume $a \mid b$ and $b \mid c$. Then we have for some $k, l \in \mathbb{Z}$

$$b = k \cdot a$$

$$c = l \cdot b$$

So,

$$c = l \cdot b = l \cdot (k \cdot a) = (lk)a$$

Therefore $a \mid c$

$\square$

Corollary 4.1.1. *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $a \mid b$ and $a \mid c$, $a \mid (mb + nc)$ for all integers m and n*

Proof. Let $a, bc \in \mathbb{Z}$ with $a \neq 0$. Assume $a \mid b$ $a \mid c$. By the previous theorem (part 2), we have $a \mid mb$ and $a \mid nc$. Therefore, by the previous theorem (part 1), $a \mid (mb + nc)$ $\square$

Theorem 4.1.2 (The Division Algorithm). *Let $a, d \in \mathbb{Z}$ with $d > 0$. There exists a unique q and r such that*

$$0 \leq r \leq d$$

and

$$a = dq + r$$

We write

$$q = a \ div \ d$$

$$r = a \ mod \ a$$

Definition 4.1.2. *Let $a, b, m \in \mathbb{Z}$ with $m \geq 2$. We say a is congruent to b modulo m if $m \mid (a - b)$. We write $a \equiv b \ (mod \ m)$*

   Example: Prove or disprove. We have $a \equiv b \ (mod \ m)$ if and only if $b \equiv a \ (mod \ m)$

$$
\begin{aligned}
& a \equiv b \ (mod \ m) & \\
\Longleftrightarrow \ & m \mid (a - b) & \text{(by definition)} \\
\Longleftrightarrow \ & a - b = km & (k \in \mathbb{Z}) \\
\Longleftrightarrow \ & b - a = -km & \\
\Longleftrightarrow \ & m \mid (b - a) & \\
\Longleftrightarrow \ & b \equiv a \ (mod \ b) & \text{(by definition)}
\end{aligned}
$$

Theorem 4.1.3. *Let $a, b, c, d, m \in \mathbb{Z}$ with $m \geq 2$. If $a \equiv b \ (mod \ m)$ and $c \equiv d \ (mod \ m)$, then*

   1. $a + c \equiv b + d \ (mod \ m)$

2. $ac \equiv bd \ (mod \ m)$

Proof.   1. We have to prove $a + c \equiv b + d \ (mod \ m)$. Since $a \equiv b$ and $c \equiv d$, we have

$$m \mid (a - b)$$

$$m \mid (c - c)$$

By theorem 4.1.1 (part 1),we have

$$m \mid ((a - b) + (c - d)$$

$$m \mid ((a + c) - (b + d))$$

Therefore,

$$a + c \equiv b + d \ (mod \ m)$$

2. We have to prove $ac \equiv cd \ (mod \ m)$

Since $a \equiv b$ and $c \equiv d$, we have $m \mid (a - b)$ and $m \mid (c - d)$. By Corollary 4.1.1, we have

$$m \mid (c(a - b) + b(c - d)$$

$$m \mid (ac - bc + bc - bd)$$

$$m \mid (ac - bd)$$

Therefore $ac \equiv bd$.

□

4.2   Arithmetic Modulo m

Let $m \geq 2$ be an integer and

$$\mathbb{Z}_m = \{0, 1, 2, \ldots, m - 1\}$$

We define

$$a +_m b = (a + b) \ (mod \ m)$$

$$a \cdot_m b = (a \cdot b) \ (mod \ m)$$

in $\mathbb{Z}_m$, this is arithmetic modulo m. TBC

Lecture 5

Prime Numbers and GCD

5.1 Prime Numbers

Definition 5.1.1. *A positive integer p is prime if it admits exactly two divisors.*

Theorem 5.1.1 (Fundamental Theorem of Arithmetic)**.** *All integers greater than 1 can be written as a product of prime numbers. This representation is unique if we write the prime numbers in non-decreasing order.*

Proof. (**Existence**) By induction,

- **Base Case:** Take $n = 2$. We have $2 = 2$, the product of 1 prime number.

- **Induction Hypothesis:** Let $k \geq 2$ be an integer. Suppose that all numbers $2, 3, 4, \ldots, k - 1, k$ can be written as a product of primes.

- **Induction Step:** Consider $k + 1$. If $k + 1$ is prime, then we're done. If not, then $K + 1 = d \cdot e$ for integers $1 < d < k + 1$ and $1 < e < k + 1$ By the induction hypothesis, d and e can be written as products of prime. So $k + 1 = d \cdot e$ can be written as a product of primes.

(**Uniqueness**) to be seen later. $\square$

Theorem 5.1.2. *Let $n > 1$ be an integer. If n is not prime, then n has a prime divisor p such that $p \leq \sqrt{n}$.*

Proof. Let $n > 1$, if n is not prime, then $n = a \cdot b$ for two integers $1 < a < n$ and $1 < b < n$. We will show that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ by contradiction. Assume $a > \sqrt{n}$ and $b > \sqrt{n}$. Then $n = a \cdot b > \sqrt{n} \cdot \sqrt{n} = n$. This is a contradiction so $a \leq \sqrt{n}$.

Assume without loss of generality that $a \leq \sqrt{n}$. If a is prime, we're done. If not, then by the fundamental theorem of arithmetic, a is divisible by a prime number p $\square$

Theorem 5.1.3. *There exists an infinite number of prime numbers.*

Proof. By contradiction, suppose there exists a finite number of prime numbers, say k prime numbers, and we order them

$$p_1 < p_2 < p_3 < \cdots < p_k$$

Consider the number
$$Q = p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1 \in \mathbb{Z}$$

Since $Q > p_k$, then Q is not prime by our assumption. By Theorem 5.1.2, Q is divisible by a prime number. So $p_i \mid Q$ for some $1 \leq i \leq k$. We also have that

$$p_i \mid (p_1 \cdot p_2 \cdot \ldots \cdot p_i \cdot \ldots \cdot p_k)$$

By Corollary 4.1.1, we get

$$p_i \mid (Q - p_1 \cdot p_2 \cdot \ldots \cdot p_k)$$

$p_i \mid 1$ Therefore $p_i = 1$, this is a contradiction since we assumed p_k is the largest prime but $Q > p_k$ is prime. $\qquad \square$

Lecture 6

Euclidean Algoirthm and Bézout's Theorem

Corollary 6.0.1. *Let*

$$a = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$

$$a = p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_k^{b_k}$$

Where p_i is prime, $a_i \geq 0$ and $b_i \geq 0$, $1 \leq i \leq k$. Then

$$gcd(a,b) = p_1^{\min(a_1,b_1)} \cdot p_2^{\min(a_2,b_2)} \cdot \ldots \cdot p_k^{\min(a_k,b_k)}$$

$$lcm(a,b) = p_1^{\max(a_1,b_1)} \cdot p_2^{\max(a_2,b_2)} \cdot \ldots \cdot p_k^{\max(a_k,b_k)}$$

$$gcd(a,b) \cdot lcm(a,b) = ab$$

Example:

$$24 = 2^3 \cdot 3$$

$$36 = 2^2 \cdot 3^2$$

$$gcd(24,36) = 2^2 \cdot 3^1 = 12$$

$$lcm(24,36) = 2^3 \cdot 3^2 = 72$$

$$12 \cdot 72 = 864 = 24 \cdot 36$$

Lemma 6.0.1. *Let a, b, q, r be integers such that*

$$a = b \cdot q + r$$

Then

$$gcd(a,b) = gcd(b,r)$$

Proof. Let a, b, q, r be integers such that

$$a = bq + r$$

Let $d \in \mathbb{Z}$. We will prove that

$$d \mid a \wedge d \mid b \iff d \mid b \wedge d \mid r$$

($\implies$) Let $d \in \mathbb{Z}$. Assume $d \mid a$ and $d \mid b$. Then $d \mid (1 \cdot a + (-q) \cdot b)$, by Corollary 4.1.1. Then $a = bq + r \implies r = a - bq$, so $d \mid (1 \cdot a + (-q) \cdot b) \implies d \mid r$.

($\impliedby$) Let $d \in \mathbb{Z}$. Assume $d \mid b$ and $d \mid r$. Then $d \mid (q \cdot b + 1 \cdot r)$ by Corollary 4.1.1. Then $d \mid a$, therefore $d \mid a$ and $d \mid b$ $\qquad \square$

Example: $gcd(414, 662)$, $662 = 1 \cdot 414 + 248$

$$662 = 1 \cdot 414 + 248$$
$$414 = 1 \cdot 248 + 166$$
$$248 = 1 \cdot 166 + 82$$
$$166 = 2 \cdot 82 + 2$$
$$82 = 41 \cdot 2 + 0$$

The last none-zero remainder of this sequence is the gcd of 414 and 662 by the previous lemma. (can someone find which lemma this is!)

Definition 6.0.1 (Euclidean Algorithm)**.**

$$x = a$$
$$y = b$$

while $y \neq 0$

$$r = x \mod y$$
$$x = y$$
$$y = r$$

return x

This algorithm returns the gcd of a and b.

Example:$gcd(465, 144)$

$$465 = 3 \cdot 144 + 33$$
$$144 = 4 \cdot 33 + 12$$
$$33 = 2 \cdot 12 + 9$$
$$12 = 1 \cdot 9 + 3$$
$$9 = 3 \cdot 3 + 0$$

Therefore $gcd(465, 144) = 3$.

Theorem 6.0.1 (Bézout). *Let $a, b \in \mathbb{Z}$ be positive integers. There exists $s, t \in \mathbb{Z}$ such that*

$$s \cdot a + t \cdot b = gcd(a, b)$$

Proof. Let $a, b \in \mathbb{N} \setminus \{0\}$. Run Euclidian algorithm, and assume without loss of generality $b \leq a$.

$$a = q \cdot b + r$$
$$r_0 = q_1 \cdot r_1 + r_2$$
$$r_1 = q_2 \cdot r^2 + r_3$$
$$r_2 = q_3 \cdot r_3 + r_4$$
$$\vdots$$
$$r_{n-3} = q_{n-2} \cdot r_{n-2} + r_{n-1}$$
$$r_{n-2} = q_{n-1} \cdot r_{n-1} + r_n$$
$$r_{n-1} = q_n \cdot r_n + 0$$

Then, we have

$$gcd(a, b) = r_n$$
$$= r_{n-2} - q_{n-1} \cdot r_{n-1}$$
$$= r_{n-2} - q_{n-1}(r_{n-3} - q_{n-2} r_{n-2})$$
$$= r_{n-2} - q_{n-1}(r_{n-3} - q_{n-2} r_{n-2})$$
$$= -q_{n-1} \cdot + (1 + q_{n-2} q_{n-1}) \cdot r_{n-2}$$
$$\vdots$$
$$= s \cdot r_0 + t \cdot r_1$$
$$= s \cdot a + t \cdot tb$$

So we read the trace of Euclid's algorithm backward while keeping $gcd(a, b)$ on the same side of the equality. $\square$

TBC.

Lemma 6.0.2. *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $gcd(a, b) = 1$ and $a \mid (bc)$, then $a \mid c$.*

Proof. Assume $gcd(a, b) = 1$ and $a \mid (bc)$. By Bézout, there exist $s, t \in \mathbb{Z}$ such that

$$s \cdot a + t \cdot b = gcd(a, b) = 1$$
$$s \cdot a \cdot c + t \cdot b \cdot c = c \qquad \qquad (*)$$

Since $a \mid a$ and $a \mid (bc)$, we have

$$a \mid (s \cdot c \cdot a + t \cdot b \cdot c$$

By Corollary 4.1.1. Then from (*), this means

$$a \mid c$$

$\square$

Lecture 7

Applications of Bézout's Theorem

TBC.

Lecture 8

GCD and Modulo n, Multiplicative Inverses in Modulo n

Lemma 8.0.1. *Let $a, b, c \in \mathbb{Z}$, with $a \neq 0$. If $gcd(a, b) = 1$, and $a \mid (bc)$, then $a \mid c$.*

Proof. Seen last week. □

Theorem 8.0.1. *Let $a, b, c, m \in \mathbb{Z}$, with $m \geq 2$. Assume $ac \equiv bc \ (mod \ m)$ and $gcd(c, m) = 1$. Then $a \equiv b \ (mod \ m)$.*

Proof. Let $a, b, c, m \in \mathbb{Z}$ with $m \geq 2$. Assume $ac \equiv bc \ (mod \ m)$ and $gcd(c, m) = 1$.

$$
\begin{aligned}
m &\mid (ac - bc) && \text{(def of mod)} \\
m &\mid (c(a - b)) && \\
m &\mid (a - b) && \text{(by previous lemma)} \\
a &\equiv b \ (mod \ m) && \text{(def of mod)}
\end{aligned}
$$

□

Lemma 8.0.2. *Let p be a prime number and $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. If $p \mid (a_1 \cdot a_2 \cdot \cdots \cdot a_n)$, then there exists $1 \leq i \leq n$ such that $p \mid a_i$.*

Proof. By induction on n.

- **Base Case:** $n = 1$. Let p be a prime number, if $p \mid a_1$, then $p \mid a_1$

- **Induction Hypothesis:** Let $k \geq 1$ be an integer. Suppose that for all integers $a_1, a_2, \ldots, a_k$

$$
p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k) \implies \exists 1 \leq i \leq k \ s.t \ p \mid a_i
$$

If $p \mid a_{k+1}$, then we're done. If not, then

$$gcd(p, a_{k+1}) = 1$$

So $p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k)$ by the previous lemma. By the induction hypthesis, ther exists $1 \leq i \leq k$ such that $p \mid a_i$.

Induction Step: Suppose

$$p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k \cdot a_{k+1})$$

$\square$

Theorem 8.0.2. *Let $m \in \mathbb{Z}$ with $m \geq 2$ and let $a \in \mathbb{Z}_m$. The multiplicative inverse of a (mod n) exists if and only if $gcd(a, m) = 1$. When it exists, the inverse of a (mod n) is unique.*

Proof. Let $m \in \mathbb{Z}$ with $m \geq 2$ and $a \in \mathbb{Z}_m$

($\implies$): Assume the multiplicative inverse of a (mod n) exists. Let $\bar{a}$ be this inverse. By definition,

$$a \cdot \bar{a} \equiv 1 \ (mod \ m)$$
$$m \mid (a \cdot \bar{a} - 1) \qquad\qquad \text{(def. of modulo)}$$

Then, $a \cdot \bar{a} - 1 = k \cdot m$ for some $m \in \mathbb{Z}$. Let $d = gcd(a, m)$ Then $d|a$ and $d|m$. By a result seen in class,

$$d \mid (\bar{a} \cdot a + (-k)m)$$
$$d \mid 1$$

So, $d = 1$

($\impliedby$): Assume $gcd(a, m) = 1$. By Bézout, there exists $s, t \in \mathbb{Z}$ such that

$$s \cdot a + t \cdot m = gcd(a, m) = 1$$

$$s \cdot a + t \cdot m \equiv 1 \ (mod \ m)$$
$$s \cdot a + t \cdot 0 \equiv 1 \ (mod \ m)$$
$$s \cdot a \equiv 1 \ (mod \ m)$$

So, we can take $\bar{a} \equiv s \ (mod \ m)$

(Uniqueness): Consider two arbitrary multiplicative inverses of a (mod m). Denote them by, $s, s' \in \mathbb{Z}_m$. So by definition

$$sa \equiv 1 \ (mod \ m) \text{ and } s'a \equiv 1 \ (mod \ m)$$

Then $gcd(a, m) = 1$ by the previous proof, also we have

$$sa \equiv s'a \ (mod \ m)$$
$$m \mid (sa - s'a) \qquad \qquad \text{(def. of modulo)}$$
$$m \mid (a(s - s'))$$
$$m \mid (s - s') \qquad \qquad \text{(since } gcd(a, m) = 1)$$
$$s \equiv s' \ (mod \ m) \qquad \qquad \text{(def. of modulo)}$$

Therefore, s and s' are the same in $\mathbb{Z}_m$. $\qquad\qquad\qquad\qquad$ $\square$

Example: Find the multiplicative inverse of 101 (mod 4620).

Euclid:

$$4620 = 45 \cdot 101 + 75$$
$$101 = 1 \cdot 75 + 26$$
$$75 = 2 \cdot 26 + 23$$
$$26 = 1 \cdot 23 + 3$$
$$23 = 7 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$
$$2 = 2 \cdot 1 + 0$$

Bézout:

$$1 = 3 - 1 \cdot 2$$
$$1 = 3 - 1 \cdot (23 - 7 \cdot 3)$$
$$1 = 3 - 1 \cdot 23 + 7 \cdot 3$$
$$1 = 8 \cdot 3 - 1 \cdot 23$$
$$1 = -1 \cdot 23 + 8 \cdot 3$$
$$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23)$$
$$1 = -1 \cdot 23 + 8 \cdot 26 - 8 \cdot 23$$
$$1 = -9 \cdot 23 + 8 \cdot 26$$
$$1 = 8 \cdot 26 - 9 \cdot 23$$
$$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26)$$
$$1 = 8 \cdot 26 - 9 \cdot 75 + 18 \cdot 26$$
$$1 = -9 \cdot 75 + 26 \cdot 26$$
$$1 = -9 \cdot 75 + 26 \cdot (101 - 1 \cdot 75)$$
$$1 = -9 \cdot 75 + 26 \cdot 101 - 26 \cdot 75$$
$$1 = 26 \cdot 101 - 35 \cdot 75$$
$$1 = 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$$
$$1 = 26 \cdot 101 - 35 \cdot 4620 + 1575 \cdot 21$$
$$1 = -35 \cdot 4620 + 1601 \cdot 101$$

So,

$$-35 \cdot 4620 + 1601 \cdot 101 \equiv 1 \ (mod \ 4620)$$
$$-35 \cdot 0 + 1601 \cdot 101 \equiv 1 \ (mod \ 4620)$$
$$1601 \cdot 101 \equiv 1 \ (mod \ 4620)$$
$$101 \equiv 1601 \ (mod \ 4620)$$

Therefore, the inverse of 101 in $\mathbb{Z}_{4620}$ is 1601.

Example: Find the multiplicative inverses in $\mathbb{Z}_{10}$.

- $\bar{0}$ does not exist since $gcd(0, 10) = 10 \neq 1$

- $\bar{1} \equiv 1 \ (mod \ 10)$

- $\bar{2}$ does not exist since $gcd(2, 10) = 2 \neq 1$

- $\bar{3} \equiv 7 \ (mod \ 10)$

- $\bar{4}$ does not exist since $gcd(4, 10) = 2 \neq 1$

- $\bar{5}$ does not exist since $gcd(5, 10) = 5 \neq 1$

- $\bar{6}$ does not exist since $gcd(6, 10) = 2 \neq 1$

- $\bar{7} \equiv 3 \ (mod \ 10)$

- $\bar{8}$ does not exist since $gcd(8, 10) = 2 \neq 1$

- $\bar{9} \equiv 9 \ (mod \ 10)$

This concludes the material for midterm 1.

Lecture 9

Solving Congruences

Definition 9.0.1 (Linear Congruence). *$ax \equiv b \pmod{m}$*

Example:
$$3x \equiv 5 \pmod{7}$$
$$x \equiv 0 \pmod{7}$$
$$x - 0 = 7k$$

Question: What is the multiplicative inverse of 3 (mod 7) So we have $3x \equiv 5$ (mod 7).
$$15x \equiv 25 \pmod{7}$$
$$x \equiv 4 \pmod{7}$$
$$3 \cdot 4 = 12 \equiv 5 \pmod{7}$$

9.1   Linear Congruence System

Find x such that
$$x \equiv a_1 \pmod{m_1}$$
$$x \equiv a_2 \pmod{m_n}$$
$$\vdots$$
$$x \equiv a_n \pmod{m_n}$$

Example:
$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 5 \pmod{7}$$

Try $x = 68$
$$68 \equiv 2 \pmod{3}$$

$$68 \equiv 3 \ (mod \ 5)$$

$$68 \equiv 5 \ (mod \ 7)$$

So, $x = 68$ is a solution to the system.

9.1.1   Substitution Method

$$x \equiv 2 \ (mod \ 3)$$

$$x = 3 \cdot t + 2$$

For some $t \in \mathbb{Z}$

$$x \equiv \ (mod \ 5)$$

$$3t + 2 \equiv 3 \ (mod \ 5)$$

$$3t \equiv 1 \ (mod \ 5)$$

Multiply $3x$ by the multiplicative inverse of 3 in $\mathbb{Z}_5$.

$$2 \cdot 3t \equiv 2 \cdot 1 \ (mod \ 5)$$

$$t \equiv 2 \ (mod \ 5)$$

$$t = 5u + 2 \ (mod \ 5)$$

For an $u \in \mathbb{Z}$

$$\left. \begin{array}{l} x = 3t + 2 \\ t = 5u + 2 \end{array} \right\}$$

$$\implies x = ?$$

$$x = 3(5u + 2) + 2 = 15u + 8$$

$$15u + 8 \equiv 5 \ (mod \ 7)$$

$$15u \equiv -3 \ (mod \ 7)$$

$$15u \equiv 4 \ (mod \ 7)$$

$$15u - 14u \equiv 4 \ (mod \ 7)$$

$$u \equiv 4 \ (mod \ 7)$$

So $u = 7v + 4$ for some $v \in \mathbb{Z}$. Thus,

$$x = 15u + 8$$

$$= 15(7v + 4) + 8$$

$$= 105v + 68$$

So,

$$105v + 68 \equiv 2 \ (mod \ 3)$$
$$105v + 68 \equiv 3 \ (mod \ 5)$$
$$105v + 68 \equiv 5 \ (mod \ 7)$$

Example:

$$x \equiv 1 \ (mod \ 4)$$
$$x \equiv 3 \ (mod \ 5)$$

Then $x = 4t + 1$ for some $t \in \mathbb{Z}$. Then from the second equation, we get

$$4t + 1 \equiv 3 \ (mod \ 5)$$
$$4t + 1 - 1 \equiv 3 - 1 \ (mod \ 5)$$
$$4t \equiv 2 \ (mod \ 5)$$
$$4 \cdot 4t \equiv 4 \cdot 2 \ (mod \ 5)$$
$$16t \equiv 8 \ (mod \ 5)$$
$$16t \equiv 8 \ (mod \ 5)$$
$$16t - 15t \equiv 8 - 5 \ (mod \ 5)$$
$$t \equiv 3 \ (mod \ 5)$$

Thus, $t = 5u + 3$ for some $u \in \mathbb{Z}$. So $x = 20u + 13$ is a solution to the system.

$$20u + 13 \equiv 1 \ (mod \ 4)$$
$$20u + 13 \equiv 3 \ (mod \ 5)$$

Question: Are there systems that admit no solution? Consider

$$x \equiv 2 \ (mod \ 4)$$

$$x \equiv 3 \ (mod \ 6)$$

So $x = 4t + 2$ for some $t \in \mathbb{Z}$

$$4t + 2 \equiv 3 \ (mod \ 6)$$

$$4t \equiv 1 \ (mod \ 6)$$

But, 4 does not have a multiplicative inverse in $\mathbb{Z}_6$ since $gcd(4, 6) \neq 1$.

Theorem 9.1.1 (Chinese Remainder Theorem)**.** *Let $m_1, m_2, \ldots, m_r \in \mathbb{Z}$ be pairwise co-prime integers such that $m_i \geq 2$ for $1 \leq i \leq r$*

Definition 9.1.1 (Pairwise Co-prime). $gcd(m_i, m_j) = 1$

Let $a_1, a_2, \ldots, a_r \in \mathbb{Z}$, then the system

$$x \equiv a_1 \ (mod \ m_1)$$
$$x \equiv a_2 \ (mod \ m_2)$$
$$\vdots$$
$$x \equiv a_r \ (mod \ m_r)$$

admits a unique solution $(mod \ m_1 \cdot m_2 \cdots m_r)$. In other words, the solution exists and is unique in $\mathbb{Z}_{m_1 \cdot m_2 \cdots m_r}$

Consider the system

$$x \equiv 2 \ (mod \ 3)$$
$$x \equiv 3 \ (mod \ 5)$$
$$x \equiv 5 \ (mod \ 7)$$

So we have $\mathbb{Z}_{3 \cdot 5 \cdot 7} = \mathbb{Z}_{105}$, $68 \in \mathbb{Z}_{105}$ and $x = 105u + 68$.

Lecture 10

Fermat's Theorem

Theorem 10.0.1 (Fermat's Theorem). *Let $p, a \in \mathbb{Z}$ such that p is prime, then*

1.
$$a^p \equiv a \ (mod \ p)$$

2. *If $gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \ (mod \ p)$*

Example:
$$1534^{2016} \pmod{2017}$$

2017 is prime and $1534 < 2017$, so $gcd(1534, 2017) = 1$ and $1534^{2016} \equiv 1 \ (\ mod \ 2017)$

Proof. For (2), we need to usthe following property

$$1 \cdot a, 2 \cdot a, 3 \cdot a, \ldots, (p-1) \cdot a$$

are all different $(mod \ p)$.Consider $s, e \in \{1, 2, \ldots, p-1\}$ such that

$$ra \equiv sa \pmod{p}$$

Since $\gcd(a, p) = 1$, we can divide both sides by a to get

$$r \equiv s \pmod{p}$$

Then since $r, s < p$, then $r = s$''

$\square$

27

Lecture 12

Intro to Cryptography

Lecture 13

Asympotic Notation

13.1 Big-O Notation

The *O-notation* describes an asympotic upper bound.

Definition 13.1.1. *Let*

$$f : \mathbb{N} \to \mathbb{R}^+$$

$$g : \mathbb{N} \to \mathbb{R}^+$$

be two functions. We say that f is $\mathrm{O(g)}$ if there exists a real number $c > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$,

$$f(n) \leq c \cdot g(n)$$

Notation:

$$f(n) \leq c \cdot g(n)$$
$$f = O(g)$$
$$\exists c \exists k \forall n (n \geq k \implies f(n) \leq c \cdot g(n))$$

Domain: $k, n \in \mathbb{N}$, $c \in \mathbb{R}^+ \setminus \{0\}$

Example: $13x^3 + 12x^2 + 5 = O(x^3)$. We have

$$13x^3 + 12x^2 + 5 \leq 13x^3 + 12x^3 + 5x^2 = 30x^3$$

Take $c = 30$ and $k = 1$. So

$$13x^3 + 12x^2 + 5 \leq 30 \cdot x^3$$

for all $x \geq 1$. Therefore $13x^3 + 12x^2 + 5 = O(x^3)$.

Example: $x^2 = O\left(\frac{1}{2}x^2 - 10x\right)$. We have

$$x^2 \leq 2 \left(\frac{1}{2}x^2 - 10x\right)$$

Now we want
$$x^2 \geq 40x$$
so that that $x^2 - 40x$ is positive. So
$$x > 40$$
Then,
$$
\begin{aligned}
x^2 &= 2x^2 - x^2 \\
&\leq 2x^2 - 40x \\
&= 4\left(\frac{1}{2}x^2 - 10x\right)
\end{aligned}
$$

So take $c = 4$ and $k = 40$. Then $x^2 = O\left(\frac{1}{2}x^2 - 10x\right)$ for all $x \geq 40$.

Proposition 13.1.1. *Let $a > 0$ and $b > 0$. be two rael numbers. We have*
$$log^a(x) = O(x^b)$$

Proof. Let $a > 0$ and $b > 0$ be two real numbers. We'll use that fact that $\forall x \geq 0$, we have $x \leq e^x$. From which, we have $log(x) \leq x$. Let x be an integer. We have, by the previous property,
$$
\begin{aligned}
log(x^{\frac{b}{a}}) &\leq x^{\frac{b}{a}} \\
\frac{b}{a}log(x) &\leq x^{\frac{b}{a}} \\
\left(\frac{b}{a}\right)^a log^a(x) &\leq x^b \\
log^a(x) &\leq \left(\frac{a}{b}\right)^a x^b
\end{aligned}
$$

So we take $c = \left(\frac{a}{b}\right)^a$ and $k = 1$. $\qquad\square$

13.2 Big-Omega Notation

The Ω-*notation* describes an asympotic lower bound.

Definition 13.2.1. *Let*
$$f : \mathbb{N} \to \mathbb{R}^+$$
$$g : \mathbb{N} \to \mathbb{R}^+$$
be two functions. We say that f is $\Omega(g)$ if there exists a real number $c > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$,
$$f(n) \geq c \cdot g(n)$$

Notation:

$$f(n) = \Omega(g(n))$$
$$f = \Omega(g)$$
$$\exists c \exists k \forall n (n \geq k \implies f(n) \geq c \cdot g(n))$$

Domain: $k, n \in \mathbb{N}$, $c \in \mathbb{R}^+ \setminus \{0\}$

Example: $13x^3 + 12x^2 + 5 = \Omega(x^3)$.

$$13x^3 + 12x^2 + 5 \geq 13x^3$$

Take $c = 13$ and $k = 0$. So $13x^3 + 12x^2 + 5 = \Omega(x^3)$.

Example: $x^2 = \Omega\left(\frac{1}{2}x^2 - 10x\right)$.

$$x^2 \geq \frac{1}{2}x^2$$
$$\geq \frac{1}{2}x^2 - 10x$$
$$= 1 \cdot \left(\frac{1}{2}x^2 - 10x\right)$$

Take $c = 1$ and $k = 0$. So $x^2 = \Omega\left(\frac{1}{2}x^2 - 10x\right) \forall x \geq k$.

Proposition 13.2.1. *Let $f(n)$ and $g(n)$ be two functions.*

$$f(n) = O(g(n)) \iff g(n) - \Omega(f(n))$$

Proof. ($\implies$) Let $f(n)$ and $g(n)$ be two functions. Assume $f(n) = O(g(n))$. Then there exists $c > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$, we have $f(n) \leq c \cdot g(n)$. So,

$$f(n) \leq c \cdot g(n)$$

given that $n \geq g(n)$, then

$$g(n) \geq \frac{1}{c}f(n)$$

($\impliedby$) The proof follows the same. $\qquad\square$

13.3   Big-Theta Notation

The Θ-*notation* describes an asympotic upper and lower bound.

Definition 13.3.1. *Let*

$$f : \mathbb{N} \to \mathbb{R}^+$$
$$g : \mathbb{N} \to \mathbb{R}^+$$

be two functions. We say that f is $\Theta(g)$ if there exists a real number $c_1 > 0$, $c_2 > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$. In otherwords,

$$f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$

Notation:

$$f(n) = \Theta(g(n))$$

$$f = \Theta(g)$$

Proposition 13.3.1. *Let $f(n)$ and $g(n)$ be two functions. $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $g(n) = \Omega(f(n))$.*

Proof.

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$

By the definition of theta, so

$$g(n) = \Omega(f(n)) \text{ and } g(n) = O(f(n))$$

From the previous proposition, then

$$g(n) = \Theta(f(n))$$

$\square$

Lecture 14

Recursivity

Lemma 14.0.1. *Let $F_n = F_{n-1} + F_{n-2}$ denote the nth term of the Fibonacci sequence with $F_0 = 0$ and $F_1 = 1$. And let $\alpha = \frac{\sqrt{5}+1}{2}$ (golden ratio). Then $\forall n \geq 3$,*

$$F_n > \alpha^{n-2}$$

Proof. By induction,

$$\textbf{\textit{Note:}} \ \alpha^2 = \left(\frac{\sqrt{5}+1}{2}\right)^2 = \frac{5+2\sqrt{5}+1}{4} = \frac{\sqrt{5}+3}{2} = \frac{\sqrt{5}+1}{2} + 1 = \alpha + 1$$

- **Base case:** $n = 3$. Then $F_3 = F_2 + F_1 = 1 + 1 = 2$. We have

$$3 > \sqrt{5}$$
$$4 > \sqrt{5} + 1$$
$$2 > \frac{\sqrt{5}+1}{2}$$
$$F_3 > \frac{\sqrt{5}+1}{2}^2 = \alpha^{3-2}$$

  For $n = 4$, $F_4 = F_3 + F_2 = 2 + 1 = 3$. We have

$$2 > \alpha$$
$$2 + 1 > \alpha + 1 \qquad\qquad \text{(From Note)}$$
$$3 > \alpha^2$$
$$F_4 > \alpha^2 = \alpha^{4-2}$$

- **Induction Hypothesis:** Let $k \geq 4$ be an integer. Assume $F_i > \alpha^{i-2}$ for all $3 \leq i \leq k$.

- **Induction Step:**

$$
\begin{aligned}
F_{k+1} &= F_k + F_{k-1} && \text{(By def.)} \\
&> \alpha^{k-2} + \alpha^{(k-1)-2} && \text{(By IH)} \\
&= \alpha^{k-2} + \alpha^{k-3} \\
&= \alpha^{k-3}(\alpha^1 + 1) \\
&= \alpha^{k-3}(\alpha^2) && \text{(From Note)} \\
&= \alpha^{k-1} \\
&= \alpha^{(k+1)-2}
\end{aligned}
$$

$\square$

Theorem 14.0.1 (Lamé). *Let $a, b \in \mathbb{Z}$ such that $a \geq b > 0$. Euclid's algorithm takes $O(\log(b))$ steps.*

Proof. Let $a, b \in \mathbb{Z}$ such that $a \geq b > 0$. Euclid's algorithm performs the following devisions:

$$
\begin{aligned}
a &= q \cdot b + r && 0 \leq r < b \\
r_0 &= q_1 \cdot r_1 + r_2 && 0 \leq r_2 < r_1 \\
r_1 &= q_2 \cdot r_2 + r_3 && 0 \leq r_3 < r_2 \\
&\;\;\vdots && \;\;\vdots \\
r_{n-2} &= q_{n-1} \cdot r_{n-1} + r_n && 0 \leq r_n < r_{n-1} \\
r_{n-1} &= q_n \cdot r_n + r_{n+1} && 0 \leq r_{n+1} < r_n
\end{aligned}
$$

We have

- $r_n = \gcd(a, b)$

- $q_i \geq 1 \; 1 \leq i \leq n - 1$

- $q_n \geq 2$

- n is the number of divisons performed by Euclid's algorithm

Therefore, $r_n = \gcd(a, b) \geq 1 = F_2$

$$
\begin{aligned}
r_{n-1} &= q_n \cdot r_n \geq 2 \cdot 1 = 2 = F_3 \\
r_{n-2} &= q_{n-1} \cdot r_{n-1} + r_n \geq 1 \cdot F_3 + F_2 = F_4 \\
r_{n-3} &= q_{n-2} \cdot r_{n-2} + r_{n-1} \geq 1 \cdot F_4 + F_3 = F_5 \\
&\;\;\vdots \\
r_2 &= q_3 \cdot r_3 + r_4 \geq 1 \cdot F_{n-1} + F_{n-2} = F_n \\
b = r_1 &= q_2 \cdot r_2 + r_3 \geq 1 \cdot F_n + F_{n-1} = F_{n+1}
\end{aligned}
$$

Sp $b \geq F_{n+1} > \alpha^{(n+1)-2}$ by the previous lemma.

$$b > \alpha^{n-1}$$
$$\log(b) > \log(\alpha^{n-1})$$
$$\log(b) > n - 1 \log(\alpha)$$
$$\log(b) > (n-1) \log\left(\frac{\sqrt{5}+1}{2}\right)$$
$$\log(b) > (n-1)\frac{2}{5}$$
$$\frac{5\log(b)}{2} + 1 > n$$

Therefore the number of steps $n < 1 + \frac{5}{2}\log(b) < \log(b) + \frac{5}{2}\log(b) = \frac{7}{2}\log(b)$ $\forall b > 3$. So $n = O(\log(b))$. $\qquad\square$

The Fibonnaci recurrence is an example of a linear homogenous recurrence of order k.
$$a_n = c_1 \cdot a_n n - 1 + c_2 \cdot a_{n-2} + \ldots + c_k \cdot a_{n-k}$$

In general, a solution of the form
$$a_n = r^n$$

will work for some $r \in \mathbb{R}$
$$r^n = c_1 \cdot r^{n-1} + c_2 \cdot r^{n-2} + \ldots + c_k \cdot r^{n-k}$$

Divide by r^{n-k},
$$r^k = c_1 \cdot r^{k-1} + c_2 \cdot r^{n-2} + \ldots + c_k$$
$$r^k - c_1 \cdot r^{k-1} + c_2 \cdot r^{n-2} + \ldots + c_k = 0$$

This is known as the characteristic equation. The solutions to this equation are called the *characteristic roots.*

Example: $a_n = 1 \cdot a_{n-1} + 2 \cdot a_{n-2}$. Characteristic Equation:
$$r^2 - 1 \cdot r - 2 = 0 \implies (r+1)(r-1) = 0$$

So we have the roots $r_1 = -1$ and $r_2 = 1$. So,
$$(-1)^n = 1 \cdot (-1)^{n-1} + 2(-1)^{n-2}$$
$$2^n = 1 \cdot 2^{n-1} + 2 \cdot 2^{n-2}$$

Moreover, for all $\alpha, \beta \in \mathbb{R}$, we have
$$(\alpha(-1)^n + \beta \cdot 2^n) = 1 \cdot \left(\alpha \cdot (-1)^{n-1} + \beta \cdot 2^{n-1}\right) + 2 \cdot \left(\alpha(-1)^{n-2} + \beta \cdot 2^{n-2}\right)$$

Any linear combintion works.

Example: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Charactereristic Equation:

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

So,

$$F_n = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

is a solution for any $\alpha, \beta \in \mathbb{R}$. Now we can find α, β to match the base cases.

$$F_0 = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^0 + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^0 = 0 \implies \alpha + \beta = 0$$

$$F_1 = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^1 + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^1 = 1$$

So, $\beta = -\alpha$. Then,

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) - \alpha \left(\frac{1 - \sqrt{5}}{2} \right) = 0$$

$$\alpha \left(\left(\frac{1 + \sqrt{5}}{2} \right) - \left(\frac{1 - \sqrt{5}}{2} \right) \right) = 1$$

$$\alpha \sqrt{5} = 1$$

$$\alpha = \frac{1}{\sqrt{5}}$$

Sos $\beta = -\alpha = -\frac{1}{\sqrt{5}}$. Therefore,

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Lecture 15

Recursivity Continued

Let us consider the special case where some characteristic roots are repeated. We only focus on the case of order $k = 2$.

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$$

Charactereristic Equation:

$$1 \cdot r^2 - c_1 \cdot r - c_2 = 0$$

Since the roots are repeated, we have

$$(r - t)^2 = 0 \rightarrow r^2 - 2rt + t^2 = 0$$

For some $t \in \mathbb{R}$. So $c_1 = 2t$, $c_2 = -t^2 = \frac{-c_1^2}{4}$. And the repeated root is $t = \frac{c}{2}$.

For any $\alpha, \beta \in \mathbb{R}$, the general solution is

$$a_n = \alpha \left(\frac{c_1}{2} \right)^n + \left(\beta \cdot n \cdot \frac{c_1}{2} \right)^n$$

Indeed we have

$$\left(\alpha \left(\frac{c_1}{2} \right)^n + \beta \cdot n \cdot \left(\frac{c_1}{2} \right)^n \right) = c_1 \cdot \left(\alpha \left(\frac{c_1}{2} \right)^{n-1} + \left(\beta \cdot (n-1) \cdot \frac{c_1}{2} \right)^{n-1} \right) + c_2 \cdot \left(\alpha \left(\frac{c_1}{2} \right)^{n-1} + \left(\beta \cdot (n-1) \cdot \frac{c_1}{2} \right)^{n-2} \right)$$

Example: $a_0 = 1$, $a_1 = 6$, $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$. Charactereristic Equation

$$r^2 - 6r + 9 = 0 \rightarrow (r - 3)^2 = 0$$

So

$$a_n = \alpha \cdot 3^n + \beta \cdot n \cdot 3^n$$

For some $\alpha, \beta \in \mathbb{R}$. We have

$$a_n = \alpha \cdot 3^0 + \beta \cdot 0 \cdot 3^0 = 1$$

$$a_n = \alpha \cdot 3^1 + \beta \cdot 1 \cdot 3^1 = 6$$

$$\alpha = 1$$

$$3\alpha + 3\beta = 6$$

So $\alpha = 1$, and $\beta = 1$, so

$$a_n = 1 \cdot 3^n + 1 \cdot n \cdot 3^n = (n+1) \cdot 3^n$$

Example: $a_0 = 1$, $a_1 = 1$, $a_n = 4a_{n-1} - 4 \cdot a_{n-2}$ for $n \geq 2$. Charactereristic Equation:

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

So

$$a_n = \alpha \cdot 2^n \beta \cdot n \cdot 2^n$$

for some $\alpha, \beta \in \mathbb{R}$. We have

$$a_0 = \alpha \cdot 2^0 + \beta \cdot 0 \cdot 2^0 = 0$$

$$a_1 = \alpha \cdot 2^1 + \beta \cdot 1 \cdot 2^1 = 1$$

Then $\alpha = 0$, $2\alpha + 2\beta = 1$. So $alpha = 0$, $\beta = \frac{1}{2}$. So

$$a_n = 0\dot{c}2^n + \frac{1}{2} \cdot n \cdot 2^n = n \cdot 2^{n-1}$$

Example: Let S be the set defined recursively by

- $3 \in S$

- If $x, y \in S$, then $x + y \in S$

So we can take $x = 3$, $y = 3$, then $3 + 3 \in S$ and so on.

Conjecture: Let $E = \{3, 6, 9, \ldots\}$ and $S = E$.

Proof. We will prove $S \subseteq E$ and $E \subseteq S$.

$S \subseteq E$ By induction,

- **Base Case:** $3 \in S$ and $3 \in E$
- **Inductive Hypothesis:** Let $x, y \in S$. Assume $x \in E$, and $y \in E$.
- **Induction Step:** We have $x + y \in S$ by definition. We want to show $x + y \in E$. Since $x, y \in E$ (from the induction hypthesis), then $x = 3k$, $y = 3l$ For some $k, l \in \mathbb{Z}$ with $k, l \geq 1$. So $x + y = 3k + 3l = 3(k + l) \in E$.

$E \subseteq S$ By induction,

- **Base Case:** $3 \in E$ and $3 \in S$

- **Inductive Hypothesis:** Let $m \in E$. Assume that $m \in S$
- **Induction Step:** We want to prove that $m + 3 \in S$. By definition, $3 \in S$. By the induction hypthesis, $m \in S$. From the definition of S, $m + 3 \in S$.

$\square$

Example: Find a rexursvie definition for the set
$$E = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$$

How do we get $\frac{k+1}{k+2}$ from $\frac{k}{k+1}$?

If $x = \frac{k}{k+1}$, then we have
$$kx + x = k \implies x = k(1 - x)$$
$$\frac{x}{1 - x} = k$$
$$\frac{k+1}{k+2} = \frac{\frac{x}{1-x} + 1}{\frac{x}{1-x} + 2} = \frac{\frac{x}{1-x} + \frac{1-x}{1-x}}{\frac{x}{1-x} + \frac{2-2x}{1-x}}$$
$$= \frac{\frac{1}{1-x}}{\frac{2-x}{1-x}}$$
$$= \frac{1}{2 - x}$$

Conjecture:

- $0 \in S$

- If $x \in S$, then $x + \frac{1}{2-x} \in S$

Proof. We want to prove $E = S$, so we will prove $E \subseteq S$ and $S \subseteq E$.

$E \subseteq S$ By induction,

- **Base Case:** $0 \in E$, and $0 \in S$.
- **Inductive Hypothesis:** Let $x \in E$. Assume $x \in S$.
- **Induction Step:** Since $x \in E$, $x = \frac{k}{k+1}$, for an integer $k \geq 0$. We want to show
$$\frac{k+1}{k+2} \in S$$
We know $x \in S$ by the induction hypothesis. By the definition of S,
$$\frac{1}{2 - x} \in S$$
So,
$$\frac{1}{2 - x} = \frac{1}{2 - \frac{k}{k+1}} = \frac{k+1}{2(k+1) - k} = \frac{k+1}{k+2} \in S$$

$S \subseteq E$ By induction,

- **Base Case:** $0 \in S$, and $0 \in E$.
- **Inductive Hypothesis:** Let $x \in S$. That is, assume

$$x = \frac{k}{k+1}$$

for some integer $k \geq 0$

- **Induction Step:** We want to prove that

$$\frac{1}{2-x} \in E$$

We have

$$
\begin{aligned}
\frac{1}{2-x} &= \frac{1}{2 - \frac{k}{k+1}} && \text{(By the IH.)} \\
&= \frac{k+1}{2(k+1) - k} \\
&= \frac{k+1}{(k+1)+1} \in E
\end{aligned}
$$

$\square$

15.1   K-ary Trees

Definition 15.1.1. *A complete k-ary tree with height h and root r is defined recursively by*

- *An isolated node r is a complete k-ary tree with height 0 and root r*

- *Let $h \geq 0$. Let T_i be a complete k-ary tree with height h and root r_i with $1 \leq i \leq k$, and let r be an isolated node. The graph obtained by adding the edges $\{r, r_i\}$ is a complete tree with heigh $h+1$ and root r.*

Lecture 16

K-Ary Trees

Example: Ternary Trees

Lecture 17

Lecture 18

Graphs

A *graph* G is made of a non-empty set V of *vertices* (nodes) together with a set E of *edges*. Each edge in E is an unordered pair $u, v \subseteq V$ with $u \neq v$. We write $G = (V, E)$. Graphs without loops and parallel edges are said to be *simple*.

<div align="center">

Important Note: All graphs this semester are simple.

</div>

We say that u is *adjacent* to v (u and v are neighbors) if $\{u, v\}$ is an edge. An edge e is said to be *incident* to u if one of the two endpoints of e is u. The *degree* of a vertex $u \in V$ is the number of edges incident u.

Theorem 18.0.1 (Handshaking Lemma)**.** *Let $G = (V, E)$ be a graph.*

$$\sum_{u \in V} deg(u) = 2|E|$$

Proof. Look at an abritrary edge $u, v \in E$. Each edge is counted twice. $\qquad\square$

Theorem 18.0.2. *Let $G = (V, E)$ be a graph. Then G has an even number of vertices with an odd degree.*

Proof. By contradiction. Let V_{even} denote the set of vertices of G with an even degree, and V_{odd} denote the set of vertices of G with an odd degree. So

$$V_{even} \cap V_{odd} = \emptyset$$

$$V_{even} \cup V_{odd} = V$$

For a contradiction, assume $|V_{odd}|$ is odd. Then

$$
\begin{aligned}
2|E| &= \sum_{u \in V} \deg(u) && \text{(Handshaking Lemma)} \\
&= \sum_{u \in V_{even}} \deg(u) + \sum_{u \in V_{odd}} \deg(u) \\
&= 2k + 2l + 1 \\
&= 2(k + l) + 1
\end{aligned}
$$

But $2|E|$ is even, so this is a contradiction.

□

Example: Can you find a graph with 5 verices with degrees 1,2,3,3,3? Yes, since by the previous theorem, we have an even number of vertices with an odd degree.

Example: Can you find a graph with 5 vertices with degrees 1,2,2,3,3? No, since by the previous theorem, we have an odd number of vertices with odd degree.

Definition 18.0.1. *A* path *in a graph* $G = (V, E)$ *is a sequence of vertices* $v_0, v_1, \ldots, v_n$ *such that* $\{v_i, v_{i+1}\} \in E$ *for all* $0 \leq i \leq n - 1$*. A path can also be described as a sequence of the* $n - 1$ *edges. The vertices* v_0 *and* v_n *are the* endpoints *of the path and* n *is its* length

Lecture 20

Lecture 21

Spanning Trees Bipartite Graphs

Theorem 21.0.1. *Every connected graph graph $G = (V, E)$ has a spanning tree.*

Proof. Let $G = (V, E)$ be a connected graph. By induction on the $m = |E|$.

- **Base Case:** $m = 0$. For G to be connected graph, it must contain a single vertext v. Then v itself is a spanning tree.

- **Induction Hypothesis:** Let $k \geq 0$ be an integer. Assume that all connected graphs with k edges have a spanning tree.

- **Induction Step:** Let G be a connected graph with $k + 1$ edges. We consider two cases.

  - **Case 1:** G is a tree, then it is its own spanning tree.
  - **Case 2:** If G is not a tree, since G is connected, then G has a cycle. Remove an edge $e = \{a, b\}$ from this cycle. We get a graph G' that is connected. Indeed, if a path uses e, we can reroute it along the other edges of the cycle. So G' is connected and it has k edges. By the induction hypothesis, G' has a spanning tree T. T covers all vertice of G', so it covers all vertices of G, so T is a spanning tree of G.

$\square$

This gives us a way to build a spanning tree; If G is a tree, then it is its own spanning tree. Otherwise, find a cycle, remove an edge from this cycle, and recursively find a spanning tree.

Corollary 21.0.1. *Every graph with n vertices and m edges has at least $n - m$ connected components.*

Proof. Let G be a graph with n vertices and m edges. By the previous theorem, every connected component of G has a spanning tree. Let F be the union of these spanning trees. Then F is a forest with n vertices, and $m' \leq m$. Moreover, the number of connected components in F is the same as in G. So G has $n - m' \geq n - m$ connected components. $\square$

We say that two sets S and T *partition* a set E if

- $S \neq \emptyset$

- $T \neq \emptyset$

- $S \cup T = E$

- $S \cap T =$

We say that a graph $G = (V, E)$ is *bipartite* if V can be partitioned into two sets A and B such that each edge has one endpoint in A and one endpoint in B.