# CSI 2101 Lecture Notes

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# Contents

# Definitions, Theorems, Lemmas, and Corollaries

**Definition 4.1.1.** *Let $a$ and $b$ be two integers such that $a \neq 0$. We say that $a$ divides $b$ if there exists $c$ such that $b = ac$. If $a$ divides be we say $a$ is a factor or divisor of $b$. We also can say $b$ is a multiple of $a$.*

**Theorem 4.1.1.** *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$.*

1. *If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$*
2. *If $a \mid b$, then $a \mid bc$ for every integer*
3. *If $a \mid b$ and $b \mid c$, then $a \mid c$*

**Corollary 4.1.1.** *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $a \mid b$ and $a \mid c$, $a \mid (mb + nc)$ for all integers $m$ and $n$*

**Theorem 4.1.2** (The Division Algorithm)**.** *Let $a, d \in \mathbb{Z}$ with $d > 0$. There exists a unique $q$ and $r$ such that*

$$0 \leq r \leq d$$

*and*

$$a = dq + r$$

*We write*

$$q = a \ div \ d$$

$$r = a \ mod \ a$$

**Definition 4.1.2.** *Let $a, b, m \in \mathbb{Z}$ with $m \geq 2$. We say $a$ is congruent to $b$ modulo $m$ if $m \mid (a - b)$. We write $a \equiv b \ (mod \ m)$*

**Theorem 4.1.3.** *Let $a, b, c, d, m \in \mathbb{Z}$ with $m \geq 2$. If $a \equiv b \ (mod \ m)$ and $c \equiv d \ (mod \ m)$, then*

1. *$a + c \equiv b + d \ (mod \ m)$*
2. *$ac \equiv bd \ (mod \ m)$*

**Definition 5.1.1.** *A positive integer $p$ is prime if it admits exactly two divisors.*

**Theorem 5.1.1** (Fundamental Theorem of Arithmetic)**.** *All integers greater than 1 can be written as a product of prime numbers. This representation is unique if we write the prime numbers in non-decreasing order.*

**Theorem 5.1.2.** *Let $n > 1$ be an integer. If $n$ is not prime, then $n$ has a prime divisor $p$ such that $p \leq \sqrt{n}$.*

**Corollary 6.0.1.** *Let*

$$a = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$

$$a = p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_k^{b_k}$$

*Where $p_i$ is prime, $a_i \geq 0$ and $b_i \geq 0$, $1 \leq i \leq k$. Then*

$$gcd(a, b) = p_1^{\min(a_1,b_1)} \cdot p_2^{\min(a_2,b_2)} \cdot \ldots \cdot p_k^{\min(a_k,b_k)}$$

$$lcm(a, b) = p_1^{\max(a_1,b_1)} \cdot p_2^{\max(a_2,b_2)} \cdot \ldots \cdot p_k^{\max(a_k,b_k)}$$

$$gcd(a, b) \cdot lcm(a, b) = ab$$

**Lemma 6.0.1.** *Let $a, b, q, r$ be integers such that*

$$a = b \cdot q + r$$

*Then*

$$gcd(a, b) = gcd(b, r)$$

**Definition 6.0.1** (Euclidean Algorithm)**.**

$$x = a$$

$$y = b$$

*while $y \neq 0$*

$$r = x \mod y$$

$$x = y$$

$$y = r$$

*return $x$*

**Theorem 6.0.1** (Bézout)**.** *Let $a, b \in \mathbb{Z}$ be positive integers. There exists $s, t \in \mathbb{Z}$ such that*

$$s \cdot a + t \cdot b = gcd(a, b)$$

**Lemma 6.0.1.** *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $gcd(a, b) = 1$ and $a \mid (bc)$, then $a \mid c$.*

**Lemma 8.0.1.** *Let $a, b, c \in \mathbb{Z}$, with $a \neq 0$. If $gcd(a, b) = 1$, and $a \mid (bc)$, then $a|c$.*

**Theorem 8.0.1.** *Let $a, b, c, m \in \mathbb{Z}$, with $m \geq 2$. Assume $ac \equiv bc \pmod{m}$ and $gcd(c, m) = 1$. Then $a \equiv b \pmod{m}$.*

**Lemma 8.0.2.** *Let $p$ be a prime number and $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. If $p \mid (a_1 \cdot a_2 \cdot \cdots \cdot a_n)$, then there exists $1 \leq i \leq n$ such that $p \mid a_i$.*

**Theorem 8.0.2.** *Let $m \in \mathbb{Z}$ with $m \geq 2$ and let $a \in \mathbb{Z}_m$. The multiplicative inverse of $a \pmod{n}$ exists if and only if $gcd(a, m) = 1$. When it exists, the inverse of $a \pmod{n}$ is unique.*

# Lecture 1

# Logic and Proof Techniques

TBC.

# Lecture 2

# Proof Examples

TBC.

# Lecture 3

# Proof by Induction and More Examples

# Lecture 4

# Intro to Number Theory

## 4.1 Divisibility

**Definition 4.1.1.** *Let $a$ and $b$ be two integers such that $a \neq 0$. We say that $a$ divides $b$ if there exists $c$ such that $b = ac$. If $a$ divides be we say $a$ is a factor or divisor of $b$. We also can say $b$ is a multiple of $a$.*

**Theorem 4.1.1.** *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$.*

1. *If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$*
2. *If $a \mid b$, then $a \mid bc$ for every integer*
3. *If $a \mid b$ and $b \mid c$, then $a \mid c$*

*Proof.*

1. We have to prove if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Assume that $a \mid b$ and $a \mid c$, then for some $k, l \in \mathbb{Z}$

$$b = k \cdot a$$

$$c = l \cdot a$$

Thus, we have

$$b + c = k \cdot a + l \cdot a = a(k + l)$$

So $a \mid (b + c)$

2. We have to prove if $a \mid b$, $a \mid bc$ for every $c$. Let $a, b \in \mathbb{Z}$ with $a \neq 0$. Assume that $a \mid b$. Then for some $k \in \mathbb{Z}$,

$$b = k \cdot a$$

Let $c \in \mathbb{Z}$, so

$$bc = k \cdot a \cdot c = a \cdot (kc)$$

Therefore, $a \mid bc$

3. We have to prove if $a \mid b$ and $b \mid c$, then $a \mid c$. Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Assume $a \mid b$ and $b \mid c$. Then we have for some $k, l \in \mathbb{Z}$

$$b = k \cdot a$$

$$c = l \cdot b$$

So,

$$c = l \cdot b = l \cdot (k \cdot a) = (lk)a$$

Therefore $a \mid c$

□

**Corollary 4.1.1.** *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $a \mid b$ and $a \mid c$, $a \mid (mb + nc)$ for all integers $m$ and $n$*

*Proof.* Let $a, bc \in \mathbb{Z}$ with $a \neq 0$. Assume $a \mid b$ $a \mid c$. By the previous theorem (part 2), we have $a \mid mb$ and $a \mid nc$. Therefore, by the previous theorem (part 1), $a \mid (mb + nc)$ □

**Theorem 4.1.2** (The Division Algorithm)**.** *Let $a, d \in \mathbb{Z}$ with $d > 0$. There exists a unique $q$ and $r$ such that*

$$0 \leq r \leq d$$

*and*

$$a = dq + r$$

*We write*

$$q = a \ div \ d$$

$$r = a \ mod \ a$$

**Definition 4.1.2.** *Let $a, b, m \in \mathbb{Z}$ with $m \geq 2$. We say $a$ is congruent to $b$ modulo $m$ if $m \mid (a - b)$. We write $a \equiv b \ (mod \ m)$*

**Example:** Prove or disprove. We have $a \equiv b \ (mod \ m)$ if and only if $b \equiv a \ (mod \ m)$

$$\begin{aligned}
& a \equiv b \ (mod \ m) & \\
\iff & m \mid (a - b) & \text{(by definition)} \\
\iff & a - b = km & (k \in \mathbb{Z}) \\
\iff & b - a = -km & \\
\iff & m \mid (b - a) & \\
\iff & b \equiv a \ (mod \ b) & \text{(by definition)}
\end{aligned}$$

**Theorem 4.1.3.** *Let $a, b, c, d, m \in \mathbb{Z}$ with $m \geq 2$. If $a \equiv b \ (mod \ m)$ and $c \equiv d \ (mod \ m)$, then*

1. $a + c \equiv b + d \ (mod \ m)$

2. $ac \equiv bd \ (mod\ m)$

*Proof.* 1. We have to prove $a + c \equiv b + d \ (mod\ m)$. Since $a \equiv b$ and $c \equiv d$, we have

$$m \mid (a - b)$$

$$m \mid (c - c)$$

By theorem 4.1.1 (part 1),we have

$$m \mid ((a - b) + (c - d)$$

$$m \mid ((a + c) - (b + d))$$

Therefore,

$$a + c \equiv b + d \ (mod\ m)$$

2. We have to prove $ac \equiv cd \ (mod\ m)$

Since $a \equiv b$ and $c \equiv d$, we have $m \mid (a - b)$ and $m \mid (c - d)$. By Corollary 4.1.1, we have

$$m \mid (c(a - b) + b(c - d)$$

$$m \mid (ac - bc + bc - bd)$$

$$m \mid (ac - bd)$$

Therefore $ac \equiv bd$.

□

## 4.2 Arithmetic Modulo m

Let $m \geq 2$ be an integer and

$$\mathbb{Z}_m = \{0, 1, 2, \ldots, m - 1\}$$

We define

$$a +_m b = (a + b) \ (mod\ m)$$

$$a \cdot_m b = (a \cdot b) \ (mod\ m)$$

in $\mathbb{Z}_m$, this is arithmetic modulo m. TBC

# Lecture 5

# Prime Numbers and GCD

## 5.1 Prime Numbers

**Definition 5.1.1.** *A positive integer p is prime if it admits exactly two divisors.*

**Theorem 5.1.1** (Fundamental Theorem of Arithmetic)**.** *All integers greater than 1 can be written as a product of prime numbers. This representation is unique if we write the prime numbers in non-decreasing order.*

*Proof.* **(Existence)** By induction,

- **Base Case:** Take $n = 2$. We have $2 = 2$, the product of 1 prime number.
- **Induction Hypothesis:** Let $k \geq 2$ be an integer. Suppose that all numbers $2, 3, 4, \ldots, k-1, k$ can be written as a product of primes.
- **Induction Step:** Consider $k+1$. If $k+1$ is prime, then we're done. If not, then $K+1 = d \cdot e$ for integers $1 < d < k+1$ and $1 < e < k+1$ By the induction hypothesis, $d$ and $e$ can be written as products of prime. So $k+1 = d \cdot e$ can be written as a product of primes.

**(Uniqueness)** to be seen later. □

**Theorem 5.1.2.** *Let $n > 1$ be an integer. If $n$ is not prime, then $n$ has a prime divisor $p$ such that $p \leq \sqrt{n}$.*

*Proof.* Let $n > 1$, if $n$ is not prime, then $n = a \cdot b$ for two integers $1 < a < n$ and $1 < b < n$. We will show that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ by contradiction. Assume $a > \sqrt{n}$ and $b > \sqrt{n}$. Then $n = a \cdot b > \sqrt{n} \cdot \sqrt{n} = n$. This is a contradiction so $a \leq \sqrt{n}$.

Assume without loss of generality that $a \leq \sqrt{n}$. If $a$ is prime, we're done. If not, then by the fundamental theorem of arithmetic, $a$ is divisible by a prime number $p$ □

**Theorem 5.1.3.** *There exists an infinite number of prime numbers.*

*Proof.* By contradiction, suppose there exists a finite number of prime numbers, say $k$ prime numbers, and we order them

$$p_1 < p_2 < p_3 < \cdots < p_k$$

Consider the number

$$Q = p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1 \in \mathbb{Z}$$

Since $Q > p_k$, then $Q$ is not prime by our assumption. By Theorem 5.1.2, Q is divisible by a prime number. So $p_i \mid Q$ for some $1 \leq i \leq k$. We also have that

$$p_i \mid (p_1 \cdot p_2 \cdot \ldots \cdot p_i \cdot \ldots \cdot p_k)$$

By Corollary 4.1.1, we get

$$p_i \mid (Q - p_1 \cdot p_2 \cdot \ldots \cdot p_k)$$

$p_i \mid 1$ Therefore $p_i = 1$, this is a contradiction since we assumed $p_k$ is the largest prime but $Q > p_k$ is prime. $\square$

# Lecture 6

# Euclidean Algoirthm and Bézout's Theorem

**Corollary 6.0.1.** *Let*

$$a = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$
$$a = p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_k^{b_k}$$

*Where $p_i$ is prime, $a_i \geq 0$ and $b_i \geq 0$, $1 \leq i \leq k$. Then*

$$gcd(a, b) = p_1^{\min(a_1,b_1)} \cdot p_2^{\min(a_2,b_2)} \cdot \ldots \cdot p_k^{\min(a_k,b_k)}$$
$$lcm(a, b) = p_1^{\max(a_1,b_1)} \cdot p_2^{\max(a_2,b_2)} \cdot \ldots \cdot p_k^{\max(a_k,b_k)}$$
$$gcd(a, b) \cdot lcm(a, b) = ab$$

**Example:**

$$24 = 2^3 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$
$$gcd(24, 36) = 2^2 \cdot 3^1 = 12$$
$$lcm(24, 36) = 2^3 \cdot 3^2 = 72$$
$$12 \cdot 72 = 864 = 24 \cdot 36$$

**Lemma 6.0.1.** *Let $a, b, q, r$ be integers such that*

$$a = b \cdot q + r$$

*Then*

$$gcd(a, b) = gcd(b, r)$$

*Proof.* Let $a, b, q, r$ be integers such that

$$a = bq + r$$

Let $d \in \mathbb{Z}$. We will prove that

$$d \mid a \wedge d \mid b \iff d \mid b \wedge d \mid r$$

($\implies$) Let $d \in \mathbb{Z}$. Assume $d \mid a$ and $d \mid b$. Then $d \mid (1 \cdot a + (-q) \cdot b)$, by Corollary 4.1.1. Then $a = bq + r \implies r = a - bq$, so $d \mid (1 \cdot a + (-q) \cdot b) \implies d \mid r$.

($\impliedby$) Let $d \in \mathbb{Z}$. Assume $d \mid b$ and $d \mid r$. Then $d \mid (q \cdot b + 1 \cdot r)$ by Corollary 4.1.1. Then $d \mid a$, therefore $d \mid a$ and $d \mid b$ ☐

**Example:** $gcd(414, 662)$, $662 = 1 \cdot 414 + 248$

$$\begin{aligned} 662 &= 1 \cdot 414 + 248 \\ 414 &= 1 \cdot 248 + 166 \\ 248 &= 1 \cdot 166 + 82 \\ 166 &= 2 \cdot 82 + 2 \\ 82 &= 41 \cdot 2 + 0 \end{aligned}$$

The last none-zero remainder of this sequence is the $gcd$ of 414 and 662 by the previous lemma. (can someone find which lemma this is!)

**Definition 6.0.1** (Euclidean Algorithm)**.**

$$x = a$$

$$y = b$$

*while* $y \neq 0$

$$r = x \mod y$$

$$x = y$$

$$y = r$$

*return* $x$

This algorithm returns the $gcd$ of $a$ and $b$.

**Example:**$gcd(465, 144)$

$$\begin{aligned} 465 &= 3 \cdot 144 + 33 \\ 144 &= 4 \cdot 33 + 12 \\ 33 &= 2 \cdot 12 + 9 \\ 12 &= 1 \cdot 9 + 3 \\ 9 &= 3 \cdot 3 + 0 \end{aligned}$$

Therefore $gcd(465, 144) = 3$.

**Note: When you show the trace of Euclid's algorithm, you must include the last line with a remainder of 0.**

**Theorem 6.0.1** (Bézout)**.** *Let $a, b \in \mathbb{Z}$ be positive integers. There exists $s, t \in \mathbb{Z}$ such that*

$$s \cdot a + t \cdot b = gcd(a, b)$$

*Proof.* Let $a, b \in \mathbb{N} \setminus \{0\}$. Run Euclidian algorithm, and assume without loss of generality $b \leq a$.

$$\begin{aligned}
a &= q \cdot b + r \\
r_0 &= q_1 \cdot r_1 + r_2 \\
r_1 &= q_2 \cdot r^2 + r_3 \\
r_2 &= q_3 \cdot r_3 + r_4 \\
&\vdots \\
r_{n-3} &= q_{n-2} \cdot r_{n-2} + r_{n-1} \\
r_{n-2} &= q_{n-1} \cdot r_{n-1} + r_n \\
r_{n-1} &= q_n \cdot r_n + 0
\end{aligned}$$

Then, we have

$$\begin{aligned}
gcd(a, b) &= r_n \\
&= r_{n-2} - q_{n-1} \cdot r_{n-1} \\
&= r_{n-2} - q_{n-1}(r_{n-3} - q_{n-2} r_{n-2}) \\
&= r_{n-2} - q_{n-1}(r_{n-3} - q_{n-2} r_{n-2}) \\
&= -q_{n-1} \cdot + (1 + q_{n-2} q_{n-1}) \cdot r_{n-2} \\
&\vdots \\
&= s \cdot r_0 + t \cdot r_1 \\
&= s \cdot a + t \cdot tb
\end{aligned}$$

So we read the trace of Euclid's algorithm backward while keeping $gcd(a, b)$ on the same side of the equality. □

**TBC.**

**Lemma 6.0.2.** *Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $gcd(a, b) = 1$ and $a \mid (bc)$, then $a \mid c$.*

*Proof.* Assume $gcd(a, b) = 1$ and $a \mid (bc)$. By Bézout, there exist $s, t \in \mathbb{Z}$ such that

$$s \cdot a + t \cdot b = gcd(a, b) = 1$$

$$s \cdot a \cdot c + t \cdot b \cdot c = c \tag{*}$$

Since $a \mid a$ and $a \mid (bc)$, we have

$$a \mid (s \cdot c \cdot a + t \cdot b \cdot c$$

By Corollary 4.1.1. Then from (*), this means

$$a \mid c$$

$\square$

# Lecture 7

# Applications of Bézout's Theorem

TBC.

# Lecture 8

# GCD and Modulo n, Multiplicative Inverses in Modulo n

**Lemma 8.0.1.** *Let $a, b, c \in \mathbb{Z}$, with $a \neq 0$. If $gcd(a, b) = 1$, and $a \mid (bc)$, then $a|c$.*

*Proof.* Seen last week. $\square$

**Theorem 8.0.1.** *Let $a, b, c, m \in \mathbb{Z}$, with $m \geq 2$. Assume $ac \equiv bc \ (mod\ m)$ and $gcd(c, m) = 1$. Then $a \equiv b \ (mod\ m)$.*

*Proof.* Let $a, b, c, m \in \mathbb{Z}$ with $m \geq 2$. Assume $ac \equiv bc \ (mod\ m)$ and $gcd(c, m) = 1$.

$$
\begin{aligned}
m &\mid (ac - bc) && \text{(def of mod)} \\
m &\mid (c(a - b)) \\
m &\mid (a - b) && \text{(by previous lemma)} \\
a &\equiv b \ (mod\ m) && \text{(def of mod)}
\end{aligned}
$$

$\square$

**Lemma 8.0.2.** *Let $p$ be a prime number and $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. If $p \mid (a_1 \cdot a_2 \cdot \cdots \cdot a_n)$, then there exists $1 \leq i \leq n$ such that $p \mid a_i$.*

*Proof.* By induction on $n$.

- **Base Case:** $n = 1$. Let $p$ be a prime number, if $p \mid a_1$, then $p \mid a_1$
- **Induction Hypothesis:** Let $k \geq 1$ be an integer. Suppose that for all integers $a_1, a_2, \ldots, a_k$

$$p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k) \implies \exists 1 \leq i \leq k \ s.t \ p \mid a_i$$

If $p \mid a_{k+1}$, then we're done. If not, then

$$gcd(p, a_{k+1}) = 1$$

So $p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k)$ by the previous lemma. By the induction hypthesis, ther exists $1 \leq i \leq k$ such that $p \mid a_i$.

**Induction Step:** Suppose

$$p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k \cdot a_{k+1})$$

$\square$

**Theorem 8.0.2.** *Let $m \in \mathbb{Z}$ with $m \geq 2$ and let $a \in \mathbb{Z}_m$. The multiplicative inverse of $a$ $(mod\ n)$ exists if and only if $gcd(a, m) = 1$. When it exists, the inverse of $a$ $(mod\ n)$ is unique.*

*Proof.* Let $m \in \mathbb{Z}$ with $m \geq 2$ and $a \in \mathbb{Z}_m$

**( $\Longrightarrow$ ):** Assume the multiplicative inverse of $a$ $(mod\ n)$ exists. Let $\bar{a}$ be this inverse. By definition,

$$\begin{aligned} a \cdot \bar{a} &\equiv 1\ (mod\ m) \\ m &\mid (a \cdot \bar{a} - 1) && \text{(def. of modulo)} \end{aligned}$$

Then, $a \cdot \bar{a} - 1 = k \cdot m$ for some $m \in \mathbb{Z}$. Let $d = gcd(a, m)$ Then $d|a$ and $d|m$. By a result seen in class,

$$\begin{aligned} &d \mid (\bar{a} \cdot a + (-k)m) \\ &d \mid 1 \end{aligned}$$

So, $d = 1$

**( $\Longleftarrow$ ):** Assume $gcd(a, m) = 1$. By Bézout, there exists $s, t \in \mathbb{Z}$ such that

$$s \cdot a + t \cdot m = gcd(a, m) = 1$$

$$\begin{aligned} s \cdot a + t \cdot m &\equiv 1\ (mod\ m) \\ s \cdot a + t \cdot 0 &\equiv 1\ (mod\ m) \\ s \cdot a &\equiv 1\ (mod\ m) \end{aligned}$$

So, we can take $\bar{a} \equiv s$ $(mod\ m)$

**(Uniqueness):** Consider two arbitrary multiplicative inverses of $a$ $(mod\ m)$. Denote them by, $s, s' \in \mathbb{Z}_m$. So by definition

$$sa \equiv 1\ (mod\ m) \text{ and } s'a \equiv 1\ (mod\ m)$$

Then $gcd(a, m) = 1$ by the previous proof, also we have

$$
\begin{aligned}
sa &\equiv s'a \ (mod\ m) & \\
m &\mid (sa - s'a) & \text{(def. of modulo)} \\
m &\mid (a(s - s')) & \\
m &\mid (s - s') & \text{(since } gcd(a, m) = 1\text{)} \\
s &\equiv s' \ (mod\ m) & \text{(def. of modulo)}
\end{aligned}
$$

Therefore, $s$ and $s'$ are the same in $\mathbb{Z}_m$. □

**Example:** Find the multiplicative inverse of 101 (mod 4620).

**Euclid:**

$$
\begin{aligned}
4620 &= 45 \cdot 101 + 75 \\
101 &= 1 \cdot 75 + 26 \\
75 &= 2 \cdot 26 + 23 \\
26 &= 1 \cdot 23 + 3 \\
23 &= 7 \cdot 3 + 2 \\
3 &= 1 \cdot 2 + 1 \\
2 &= 2 \cdot 1 + 0
\end{aligned}
$$

**Bézout:**

$$
\begin{aligned}
1 &= 3 - 1 \cdot 2 \\
1 &= 3 - 1 \cdot (23 - 7 \cdot 3) \\
1 &= 3 - 1 \cdot 23 + 7 \cdot 3 \\
1 &= 8 \cdot 3 - 1 \cdot 23 \\
1 &= -1 \cdot 23 + 8 \cdot 3 \\
1 &= -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) \\
1 &= -1 \cdot 23 + 8 \cdot 26 - 8 \cdot 23 \\
1 &= -9 \cdot 23 + 8 \cdot 26 \\
1 &= 8 \cdot 26 - 9 \cdot 23 \\
1 &= 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) \\
1 &= 8 \cdot 26 - 9 \cdot 75 + 18 \cdot 26 \\
1 &= -9 \cdot 75 + 26 \cdot 26 \\
1 &= -9 \cdot 75 + 26 \cdot (101 - 1 \cdot 75) \\
1 &= -9 \cdot 75 + 26 \cdot 101 - 26 \cdot 75 \\
1 &= 26 \cdot 101 - 35 \cdot 75 \\
1 &= 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101) \\
1 &= 26 \cdot 101 - 35 \cdot 4620 + 1575 \cdot 21 \\
1 &= -35 \cdot 4620 + 1601 \cdot 101
\end{aligned}
$$

So,

$$
\begin{aligned}
-35 \cdot 4620 + 1601 \cdot 101 &\equiv 1 \ (mod\ 4620) \\
-35 \cdot 0 + 1601 \cdot 101 &\equiv 1 \ (mod\ 4620) \\
1601 \cdot 101 &\equiv 1 \ (mod\ 4620) \\
101 &\equiv 1601 \ (mod\ 4620)
\end{aligned}
$$

Therefore, the inverse of 101 in $\mathbb{Z}_{4620}$ is 1601.

**Example:** Find the multiplicative inverses in $\mathbb{Z}_{10}$.

- $\bar{0}$ does not exist since $gcd(0, 10) = 10 \neq 1$
- $\bar{1} \equiv 1 \ (mod\ 10)$
- $\bar{2}$ does not exist since $gcd(2, 10) = 2 \neq 1$
- $\bar{3} \equiv 7 \ (mod\ 10)$
- $\bar{4}$ does not exist since $gcd(4, 10) = 2 \neq 1$
- $\bar{5}$ does not exist since $gcd(5, 10) = 5 \neq 1$
- $\bar{6}$ does not exist since $gcd(6, 10) = 2 \neq 1$
- $\bar{7} \equiv 3 \ (mod\ 10)$
- $\bar{8}$ does not exist since $gcd(8, 10) = 2 \neq 1$
- $\bar{9} \equiv 9 \ (mod\ 10)$

**This concludes the material for midterm 1.**

# Lecture 9

# Solving Congruences

**Definition 9.0.1** (Linear Congruence)**.** $ax \equiv b$ *(mod m)*

**Example:**

$$3x \equiv 5 \ (mod\ 7)$$
$$x \equiv 0 \ (mod\ 7)$$
$$x - 0 = 7k$$

**Question:** What is the multiplicative inverse of 3 (mod 7) So we have $3x \equiv 5$ (mod 7).

$$15x \equiv 25 \ (mod\ 7)$$
$$x \equiv 4 \ (mod\ 7)$$
$$3 \cdot 4 = 12 \equiv 5 \ (mod\ 7)$$

## 9.1 Linear Congruence System

Find $x$ such that

$$x \equiv a_1 \ (mod\ m_1)$$
$$x \equiv a_2 \ (mod\ m_n)$$
$$\vdots$$
$$x \equiv a_n \ (mod\ m_n)$$

**Example:**

$$x \equiv 2 \ (mod\ 3)$$
$$x \equiv 3 \ (mod\ 5)$$
$$x \equiv 5 \ (mod\ 7)$$

Try $x = 68$

$$68 \equiv 2 \ (mod\ 3)$$

$$68 \equiv 3 \ (mod\ 5)$$
$$68 \equiv 5 \ (mod\ 7)$$

So, $x = 68$ is a solution to the system.

### 9.1.1 Substitution Method

$$x \equiv 2 \ (mod\ 3)$$
$$x = 3 \cdot t + 2$$

For some $t \in \mathbb{Z}$

$$x \equiv \ (mod\ 5)$$
$$3t + 2 \equiv 3 \ (mod\ 5)$$
$$3t \equiv 1 \ (mod\ 5)$$

Multiply $3x$ by the multiplicative inverse of 3 in $\mathbb{Z}_5$.

$$2 \cdot 3t \equiv 2 \cdot 1 \ (mod\ 5)$$
$$t \equiv 2 \ (mod\ 5)$$
$$t = 5u + 2 \ (mod\ 5)$$

For an $u \in \mathbb{Z}$

$$\left.\begin{aligned} x &= 3t + 2 \\ t &= 5u + 2 \end{aligned}\right\}$$
$$\implies x = ?$$
$$x = 3(5u + 2) + 2 = 15u + 8$$

$$\begin{aligned} 15u + 8 &\equiv 5 \ (mod\ 7) \\ 15u &\equiv -3 \ (mod\ 7) \\ 15u &\equiv 4 \ (mod\ 7) \\ 15u - 14u &\equiv 4 \ (mod\ 7) \\ u &\equiv 4 \ (mod\ 7) \end{aligned}$$

So $u = 7v + 4$ for some $v \in \mathbb{Z}$. Thus,

$$\begin{aligned} x &= 15u + 8 \\ &= 15(7v + 4) + 8 \\ &= 105v + 68 \end{aligned}$$

So,

$$105v + 68 \equiv 2 \ (mod\ 3)$$
$$105v + 68 \equiv 3 \ (mod\ 5)$$
$$105v + 68 \equiv 5 \ (mod\ 7)$$

**Example:**

$$x \equiv 1 \ (mod\ 4)$$
$$x \equiv 3 \ (mod\ 5)$$

Then $x = 4t + 1$ for some $t \in \mathbb{Z}$. Then from the second equation, we get

$$4t + 1 \equiv 3 \ (mod\ 5)$$
$$4t + 1 - 1 \equiv 3 - 1 \ (mod\ 5)$$
$$4t \equiv 2 \ (mod\ 5)$$
$$4 \cdot 4t \equiv 4 \cdot 2 \ (mod\ 5)$$
$$16t \equiv 8 \ (mod\ 5)$$
$$16t \equiv 8 \ (mod\ 5)$$
$$16t - 15t \equiv 8 - 5 \ (mod\ 5)$$
$$t \equiv 3 \ (mod\ 5)$$

Thus, $t = 5u + 3$ for some $u \in \mathbb{Z}$. So $x = 20u + 13$ is a solution to the system.

$$20u + 13 \equiv 1 \ (mod\ 4)$$
$$20u + 13 \equiv 3 \ (mod\ 5)$$

**Question:** Are there systems that admit no solution? Consider

$$x \equiv 2 \ (mod\ 4)$$

$$x \equiv 3 \ (mod\ 6)$$

So $x = 4t + 2$ for some $t \in \mathbb{Z}$

$$4t + 2 \equiv 3 \ (mod\ 6)$$

$$4t \equiv 1 \ (mod\ 6)$$

But, 4 does not have a multiplicative inverse in $\mathbb{Z}_6$ since $gcd(4, 6) \neq 1$.

**Theorem 9.1.1** (Chinese Remainder Theorem)**.** *Let $m_1, m_2, \ldots, m_r \in \mathbb{Z}$ be pairwise co-prime integers such that $m_i \geq 2$ for $1 \leq i \leq r$*

**Definition 9.1.1** (Pairwise Co-prime). $gcd(m_i, m_j) = 1$

Let $a_1, a_2, \ldots, a_r \in \mathbb{Z}$, then the system

$$\begin{aligned} x &\equiv a_1 \ (mod\ m_1) \\ x &\equiv a_2 \ (mod\ m_2) \\ &\vdots \\ x &\equiv a_r \ (mod\ m_r) \end{aligned}$$

admits a unique solution $(mod\ m_1 \cdot m_2 \cdots m_r)$.In other words, the solution exists and is unique in $\mathbb{Z}_{m_1 \cdot m_2 \cdots m_r}$

Consider the system

$$\begin{aligned} x &\equiv 2 \ (mod\ 3) \\ x &\equiv 3 \ (mod\ 5) \\ x &\equiv 5 \ (mod\ 7) \end{aligned}$$

So we have $\mathbb{Z}_{3 \cdot 5 \cdot 7} = \mathbb{Z}_{105}$, $68 \in \mathbb{Z}_{105}$ and $x = 105u + 68$.

# Lecture 10

# Fermat's Theorem

**Theorem 10.0.1** (Fermat's Theorem)**.** *Let $p, a \in \mathbb{Z}$ such that $p$ is prime, then*

1. $$a^p \equiv a \ (mod\ p)$$

2. *If $gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \ (mod\ p)$*

**Example:**

$$1534^{2016} \pmod{2017}$$

2017 is prime and $1534 < 2017$, so $gcd(1534, 2017) = 1$ and $1534^{2016} \equiv 1$ ( mod 2017)

*Proof.* For (2), we need to usthe following property

$$1 \cdot a, 2 \cdot a, 3 \cdot a, \ldots, (p-1) \cdot a$$

are all different $\pmod{p}$.Consider $s, e \in \{1, 2, \ldots, p-1\}$ such that

$$ra \equiv sa \pmod{p}$$

Since $\gcd(a, p) = 1$, we can divide both sides by $a$ to get

$$r \equiv s \pmod{p}$$

Then since $r, s < p$, then $r = s$‘’

□

## Lecture 12

# Intro to Cryptography

# Lecture 13

# Asympotic Notation

## 13.1 Big-O Notation

The *O-notation* describes an asympotic upper bound.

**Definition 13.1.1.** *Let*

$$f : \mathbb{N} \to \mathbb{R}^+$$
$$g : \mathbb{N} \to \mathbb{R}^+$$

*be two functions. We say that $f$ is* O(g) *if there exists a real number $c > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$,*

$$f(n) \leq c \cdot g(n)$$

**Notation:**

$$f(n) \leq c \cdot g(n)$$
$$f = O(g)$$
$$\exists c \exists k \forall n (n \geq k \implies f(n) \leq c \cdot g(n))$$

**Domain:** $k, n \in \mathbb{N}$, $c \in \mathbb{R}^+ \setminus \{0\}$

**Example:** $13x^3 + 12x^2 + 5 = O(x^3)$. We have

$$13x^3 + 12x^2 + 5 \leq 13x^3 + 12x^3 + 5x^2 = 30x^3$$

Take $c = 30$ and $k = 1$. So

$$13x^3 + 12x^2 + 5 \leq 30 \cdot x^3$$

for all $x \geq 1$. Therefore $13x^3 + 12x^2 + 5 = O(x^3)$.

**Example:** $x^2 = O\left(\frac{1}{2}x^2 - 10x\right)$. We have

$$x^2 \leq 2\left(\frac{1}{2}x^2 - 10x\right)$$

Now we want

$$x^2 \geq 40x$$

so that that $x^2 - 40x$ is positive. So

$$x > 40$$

Then,

$$\begin{aligned} x^2 &= 2x^2 - x^2 \\ &\leq 2x^2 - 40x \\ &= 4\left(\frac{1}{2}x^2 - 10x\right) \end{aligned}$$

So take $c = 4$ and $k = 40$. Then $x^2 = O\left(\frac{1}{2}x^2 - 10x\right)$ for all $x \geq 40$.

**Proposition 13.1.1.** *Let* $a > 0$ *and* $b > 0$*. be two rael numbers. We have*

$$log^a(x) = O(x^b)$$

*Proof.* Let $a > 0$ and $b > 0$ be two real numbers. We'll use that fact that $\forall x \geq 0$, we have $x \leq e^x$. From which, we have $log(x) \leq x$. Let $x$ be an integer. We have, by the previous property,

$$\begin{aligned} \log(x^{\frac{b}{a}}) &\leq x^{\frac{b}{a}} \\ \frac{b}{a}\log(x) &\leq x^{\frac{b}{a}} \\ \left(\frac{b}{a}\right)^a \log^a(x) &\leq x^b \\ \log^a(x) &\leq \left(\frac{a}{b}\right)^a x^b \end{aligned}$$

So we take $c = \left(\frac{a}{b}\right)^a$ and $k = 1$. □

## 13.2 Big-Omega Notation

The $\Omega$-*notation* describes an asympotic lower bound.

**Definition 13.2.1.** *Let*

$$f : \mathbb{N} \to \mathbb{R}^+$$

$$g : \mathbb{N} \to \mathbb{R}^+$$

*be two functions. We say that* $f$ *is* $\Omega(g)$ *if there exists a real number* $c > 0$ *and* $k \in \mathbb{N}$ *such that for all* $n \geq k$*,*

$$f(n) \geq c \cdot g(n)$$

**Notation:**

$$f(n) = \Omega(g(n))$$
$$f = \Omega(g)$$
$$\exists c \exists k \forall n (n \geq k \implies f(n) \geq c \cdot g(n))$$

**Domain:** $k, n \in \mathbb{N}$, $c \in \mathbb{R}^+ \setminus \{0\}$

**Example:** $13x^3 + 12x^2 + 5 = \Omega(x^3)$.

$$13x^3 + 12x^2 + 5 \geq 13x^3$$

Take $c = 13$ and $k = 0$. So $13x^3 + 12x^2 + 5 = \Omega(x^3)$.

**Example:** $x^2 = \Omega\left(\frac{1}{2}x^2 - 10x\right)$.

$$\begin{aligned} x^2 &\geq \frac{1}{2}x^2 \\ &\geq \frac{1}{2}x^2 - 10x \\ &= 1 \cdot \left(\frac{1}{2}x^2 - 10x\right) \end{aligned}$$

Take $c = 1$ and $k = 0$. So $x^2 = \Omega\left(\frac{1}{2}x^2 - 10x\right)$ $\forall x \geq k$.

**Proposition 13.2.1.** *Let $f(n)$ and $g(n)$ be two functions.*

$$f(n) = O(g(n)) \iff g(n) - \Omega(f(n))$$

*Proof.* ( $\implies$ ) Let $f(n)$ and $g(n)$ be two functions. Assume $f(n) = O(g(n))$. Then there exists $c > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$, we have $f(n) \leq c \cdot g(n)$. So,

$$f(n) \leq c \cdot g(n)$$

given that $n \geq g(n)$, then

$$g(n) \geq \frac{1}{c} f(n)$$

( $\impliedby$ ) The proof follows the same. $\square$

## 13.3 Big-Theta Notation

The $\Theta$-*notation* describes an asympotic upper and lower bound.

**Definition 13.3.1.** *Let*

$$f : \mathbb{N} \to \mathbb{R}^+$$
$$g : \mathbb{N} \to \mathbb{R}^+$$

*be two functions. We say that $f$ is $\Theta(g)$ if there exists a real number $c_1 > 0$, $c_2 > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$. In otherwords,*

$$f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$

**Notation:**

$$f(n) = \Theta(g(n))$$

$$f = \Theta(g)$$

**Proposition 13.3.1.** *Let $f(n)$ and $g(n)$ be two functions. $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $g(n) = \Omega(f(n))$.*

*Proof.*

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$

By the definition of theta, so

$$g(n) = \Omega(f(n)) \text{ and } g(n) = O(f(n))$$

From the previous proposition, then

$$g(n) = \Theta(f(n))$$

□

# Lecture 14

# Recursivity

**Lemma 14.0.1.** *Let $F_n = F_{n-1} + F_{n-2}$ denote the $n$th term of the Fibonacci sequence with $F_0 = 0$ and $F_1 = 1$. And let $\alpha = \frac{\sqrt{5}+1}{2}$ (golden ratio). Then $\forall n \geq 3$,*

$$F_n > \alpha^{n-2}$$

*Proof.* By induction,

$$\textbf{\textit{Note:}}\ \alpha^2 = \left(\tfrac{\sqrt{5}+1}{2}\right)^2 = \tfrac{5+2\sqrt{5}+1}{4} = \tfrac{\sqrt{5}+3}{2} = \tfrac{\sqrt{5}+1}{2} + 1 = \alpha + 1$$

- **Base case:** $n = 3$. Then $F_3 = F_2 + F_1 = 1 + 1 = 2$. We have

$$\begin{aligned} 3 &> \sqrt{5} \\ 4 &> \sqrt{5} + 1 \\ 2 &> \frac{\sqrt{5}+1}{2} \\ F_3 &> \frac{\sqrt{5}+1}{2}^{2} = \alpha^{3-2} \end{aligned}$$

  For $n = 4$, $F_4 = F_3 + F_2 = 2 + 1 = 3$. We have

$$\begin{aligned} 2 &> \alpha \\ 2 + 1 &> \alpha + 1 && \text{(From Note)} \\ 3 &> \alpha^2 \\ F_4 &> \alpha^2 = \alpha^{4-2} \end{aligned}$$

- **Induction Hypothesis:** Let $k \geq 4$ be an integer. Assume $F_i > \alpha^{i-2}$ for all $3 \leq i \leq k$.

- **Induction Step:**

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} && \text{(By def.)} \\ &> \alpha^{k-2} + \alpha^{(k-1)-2} && \text{(By IH)} \\ &= \alpha^{k-2} + \alpha^{k-3} \\ &= \alpha^{k-3}(\alpha^1 + 1) \\ &= \alpha^{k-3}(\alpha^2) && \text{(From Note)} \\ &= \alpha^{k-1} \\ &= \alpha^{(k+1)-2} \end{aligned}$$

□

**Theorem 14.0.1** (Lamé)**.** *Let $a, b \in \mathbb{Z}$ such that $a \geq b > 0$. Euclid's algorithm takes $O(\log(b))$ steps.*

*Proof.* Let $a, b \in \mathbb{Z}$ such that $a \geq b > 0$. Euclid's algorithm performs the following devisions:

$$\begin{aligned} a &= q \cdot b + r && 0 \leq r < b \\ r_0 &= q_1 \cdot r_1 + r_2 && 0 \leq r_2 < r_1 \\ r_1 &= q_2 \cdot r_2 + r_3 && 0 \leq r_3 < r_2 \\ &\vdots && \vdots \\ r_{n-2} &= q_{n-1} \cdot r_{n-1} + r_n && 0 \leq r_n < r_{n-1} \\ r_{n-1} &= q_n \cdot r_n + r_{n+1} && 0 \leq r_{n+1} < r_n \end{aligned}$$

We have

- $r_n = \gcd(a, b)$
- $q_i \geq 1 \; 1 \leq i \leq n-1$
- $q_n \geq 2$
- $n$ is the number of divisons performed by Euclid's algorithm

Therefore, $r_n = \gcd(a, b) \geq 1 = F_2$

$$\begin{aligned} r_{n-1} &= q_n \cdot r_n \geq 2 \cdot 1 = 2 = F_3 \\ r_{n-2} &= q_{n-1} \cdot r_{n-1} + r_n \geq 1 \cdot F_3 + F_2 = F_4 \\ r_{n-3} &= q_{n-2} \cdot r_{n-2} + r_{n-1} \geq 1 \cdot F_4 + F_3 = F_5 \\ &\vdots \\ r_2 &= q_3 \cdot r_3 + r_4 \geq 1 \cdot F_{n-1} + F_{n-2} = F_n \\ b = r_1 &= q_2 \cdot r_2 + r_3 \geq 1 \cdot F_n + F_{n-1} = F_{n+1} \end{aligned}$$

Sp $b \geq F_{n+1} > \alpha^{(n+1)-2}$ by the previous lemma.

$$\begin{aligned}
b &> \alpha^{n-1} \\
\log(b) &> \log(\alpha^{n-1}) \\
\log(b) &> n - 1 \log(\alpha) \\
\log(b) &> (n-1) \log\left(\frac{\sqrt{5}+1}{2}\right) \\
\log(b) &> (n-1)\frac{2}{5} \\
\frac{5\log(b)}{2} + 1 &> n
\end{aligned}$$

Therefore the number of steps $n < 1 + \frac{5}{2}\log(b) < \log(b) + \frac{5}{2}\log(b) = \frac{7}{2}\log(b)$ $\forall b > 3$. So $n = O(\log(b))$. $\square$

The Fibonnaci recurrence is an example of a linear homogenous recurrence of order $k$.

$$a_n = c_1 \cdot a_n n - 1 + c_2 \cdot a_{n-2} + \ldots + c_k \cdot a_{n-k}$$

In general, a solution of the form

$$a_n = r^n$$

will work for some $r \in \mathbb{R}$

$$r^n = c_1 \cdot r^{n-1} + c_2 \cdot r^{n-2} + \ldots + c_k \cdot r^{n-k}$$

Divide by $r^{n-k}$,

$$r^k = c_1 \cdot r^{k-1} + c_2 \cdot r^{n-2} + \ldots + c_k$$
$$r^k - c_1 \cdot r^{k-1} + c_2 \cdot r^{n-2} + \ldots + c_k = 0$$

This is known as the characteristic equation. The solutions to this equation are called the *characteristic roots.*

**Example:** $a_n = 1 \cdot a_{n-1} + 2 \cdot a_{n-2}$. Characteristic Equation:

$$r^2 - 1 \cdot r - 2 = 0 \implies (r+1)(r-1) = 0$$

So we have the roots $r_1 = -1$ and $r_2 = 1$. So,

$$(-1)^n = 1 \cdot (-1)^{n-1} + 2(-1)^{n-2}$$

$$2^n = 1 \cdot 2^{n-1} + 2 \cdot 2^{n-2}$$

Moreover, for all $\alpha, \beta \in \mathbb{R}$, we have

$$(\alpha(-1)^n + \beta \cdot 2^n) = 1 \cdot \left(\alpha \cdot (-1)^{n-1} + \beta \cdot 2^{n-1}\right) + 2 \cdot \left(\alpha(-1)^{n-2} + \beta \cdot 2^{n-2}\right)$$

Any linear combintion works.

**Example:** $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Charactereristic Equation:

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

So,

$$F_n = \alpha \left( \frac{1+\sqrt{5}}{2} \right)^n + \beta \left( \frac{1-\sqrt{5}}{2} \right)^n$$

is a solution for any $\alpha, \beta \in \mathbb{R}$. Now we can find $\alpha, \beta$ to match the base cases.

$$F_0 = \alpha \left( \frac{1+\sqrt{5}}{2} \right)^0 + \beta \left( \frac{1-\sqrt{5}}{2} \right)^0 = 0 \implies \alpha + \beta = 0$$

$$F_1 = \alpha \left( \frac{1+\sqrt{5}}{2} \right)^1 + \beta \left( \frac{1-\sqrt{5}}{2} \right)^1 = 1$$

So, $\beta = -\alpha$. Then,

$$\alpha \left( \frac{1+\sqrt{5}}{2} \right) - \alpha \left( \frac{1-\sqrt{5}}{2} \right) = 0$$

$$\alpha \left( \left( \frac{1+\sqrt{5}}{2} \right) - \left( \frac{1-\sqrt{5}}{2} \right) \right) = 1$$

$$\alpha\sqrt{5} = 1$$

$$\alpha = \frac{1}{\sqrt{5}}$$

Sos $\beta = -\alpha = -\frac{1}{\sqrt{5}}$. Therefore,

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

# Lecture 15

# Recursivity Continued

Let us consider the special case where some characteristic roots are repeated. We only focus on the case of order $k = 2$.

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$$

Charactereristic Equation:

$$1 \cdot r^2 - c_1 \cdot r - c_2 = 0$$

Since the roots are repeated, we have

$$(r - t)^2 = 0 \rightarrow r^2 - 2rt + t^2 = 0$$

For some $t \in \mathbb{R}$. So $c_1 = 2t$, $c_2 = -t^2 = \frac{-c_1^2}{4}$. And the repeated root is $t = \frac{c}{2}$.

For any $\alpha, \beta \in \mathbb{R}$, the general solution is

$$a_n = \alpha \left(\frac{c_1}{2}\right)^n + \left(\beta \cdot n \cdot \frac{c_1}{2}\right)^n$$

Indeed we have

$$\left(\alpha \left(\frac{c_1}{2}\right)^n + \beta \cdot n \cdot \left(\frac{c_1}{2}\right)^n\right) = c_1 \cdot \left(\alpha \left(\frac{c_1}{2}\right)^{n-1} + \left(\beta \cdot (n-1) \cdot \frac{c_1}{2}\right)^{n-1}\right) + c_2 \cdot \left(\alpha \left(\frac{c_1}{2}\right)^{n-1} + \left(\beta \cdot (n-1) \cdot \frac{c_1}{2}\right)^{n-2}\right.$$

**Example:** $a_0 = 1$, $a_1 = 6$, $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$. Charactereristic Equation

$$r^2 - 6r + 9 = 0 \rightarrow (r - 3)^2 = 0$$

So

$$a_n = \alpha \cdot 3^n + \beta \cdot n \cdot 3^n$$

For some $\alpha, \beta \in \mathbb{R}$. We have

$$a_n = \alpha \cdot 3^0 + \beta \cdot 0 \cdot 3^0 = 1$$

$$a_n = \alpha \cdot 3^1 + \beta \cdot 1 \cdot 3^1 = 6$$

$$\alpha = 1$$

$$3\alpha + 3\beta = 6$$

So $\alpha = 1$, and $\beta = 1$, so

$$a_n = 1 \cdot 3^n + 1 \cdot n \cdot 3^n = (n+1) \cdot 3^n$$

**Example:** $a_0 = 1$, $a_1 = 1$, $a_n = 4a_{n-1} - 4 \cdot a_{n-2}$ for $n \geq 2$. Charactereristic Equation:

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

So

$$a_n = \alpha \cdot 2^n \beta \cdot n \cdot 2^n$$

for some $\alpha, \beta \in \mathbb{R}$. We have

$$a_0 = \alpha \cdot 2^0 + \beta \cdot 0 \cdot 2^0 = 0$$

$$a_1 = \alpha \cdot 2^1 + \beta \cdot 1 \cdot 2^1 = 1$$

Then $\alpha = 0$, $2\alpha + 2\beta = 1$. So $alpha = 0$, $\beta = \frac{1}{2}$. So

$$a_n = 0c\dot{2}^n + \frac{1}{2} \cdot n \cdot 2^n = n \cdot 2^{n-1}$$

**Example:** Let $S$ be the set defined recursively by

- $3 \in S$
- If $x, y \in S$, then $x + y \in S$

So we can take $x = 3$, $y = 3$, then $3 + 3 \in S$ and so on.

**Conjecture:** Let $E = \{3, 6, 9, \ldots\}$ and $S = E$.

*Proof.* We will prove $S \subseteq E$ and $E \subseteq S$.

$S \subseteq E$ By induction,

- **Base Case:** $3 \in S$ and $3 \in E$
- **Inductive Hypothesis:** Let $x, y \in S$. Assume $x \in E$, and $y \in E$.
- **Induction Step:** We have $x + y \in S$ by definition. We want to show $x + y \in E$. Since $x, y \in E$ (from the induction hypthesis), then $x = 3k$, $y = 3l$ For some $k, l \in \mathbb{Z}$ with $k, l \geq 1$. So $x + y = 3k + 3l = 3(k + l) \in E$.

$E \subseteq S$ By induction,

- **Base Case:** $3 \in E$ and $3 \in S$

- **Inductive Hypothesis:** Let $m \in E$. Assume that $m \in S$
- **Induction Step:** We want to prove that $m + 3 \in S$. By definition, $3 \in S$. By the induction hypthesis, $m \in S$. From the definition of $S$, $m + 3 \in S$.

$\square$

**Example:** Find a rexursvie definition for the set

$$E = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$$

How do we get $\frac{k+1}{k+2}$ from $\frac{k}{k+1}$?

If $x = \frac{k}{k+1}$, then we have

$$kx + x = k \implies x = k(1 - x)$$

$$\frac{x}{1-x} = k$$

$$\frac{k+1}{k+2} = \frac{\frac{x}{1-x} + 1}{\frac{x}{1-x} + 2} = \frac{\frac{x}{1-x} + \frac{1-x}{1-x}}{\frac{x}{1-x} + \frac{2-2x}{1-x}}$$

$$= \frac{\frac{1}{1-x}}{\frac{2-x}{1-x}}$$

$$= \frac{1}{2-x}$$

**Conjecture:**

- $0 \in S$
- If $x \in S$, then $x + \frac{1}{2-x} \in S$

*Proof.* We want to prove $E = S$, so we will prove $E \subseteq S$ and $S \subseteq E$.

$E \subseteq S$ By induction,

- **Base Case:** $0 \in E$, and $0 \in S$.
- **Inductive Hypothesis:** Let $x \in E$. Assume $x \in S$.
- **Induction Step:** Since $x \in E$, $x = \frac{k}{k+1}$, for an integer $k \geq 0$. We want to show

$$\frac{k+1}{k+2} \in S$$

We know $x \in S$ by the induction hypothesis. By the definition of $S$,

$$\frac{1}{2-x} \in S$$

So,

$$\frac{1}{2-x} = \frac{1}{2 - \frac{k}{k+1}} = \frac{k+1}{2(k+1) - k} = \frac{k+1}{k+2} \in S$$

$S \subseteq E$ By induction,

- **Base Case:** $0 \in S$, and $0 \in E$.
- **Inductive Hypothesis:** Let $x \in S$. That is, assume

$$x = \frac{k}{k+1}$$

  for some integer $k \geq 0$
- **Induction Step:** We want to prove that

$$\frac{1}{2-x} \in E$$

  We have

$$\begin{aligned} \frac{1}{2-x} &= \frac{1}{2-\frac{k}{k+1}} && \text{(By the IH.)} \\ &= \frac{k+1}{2(k+1)-k} \\ &= \frac{k+1}{(k+1)+1} \in E \end{aligned}$$

□

## 15.1 K-ary Trees

**Definition 15.1.1.** *A complete $k$-ary tree with height $h$ and root $r$ is defined recursively by*

- *An isolated node $r$ is a complete $k$-ary tree with height $0$ and root $r$*
- *Let $h \geq 0$. Let $T_i$ be a complete $k$-ary tree with height $h$ and root $r_i$ with $1 \leq i \leq k$, and let $r$ be an isolated node. The graph obtained by adding the edges $\{r, r_i\}$ is a complete tree with heigh $h+1$ and root $r$.*

# Lecture 16

# K-Ary Trees

**Example: Ternary Trees**

# Lecture 17

# Lecture 18

# Graphs

**Definition 18.0.1.** *A* graph *$G$ is made of a non-empty set $V$ of* vertices *(nodes) together with a set $E$ of* edges. *Each edge in $E$ is an unordered pair $u, v \subseteq V$ with $u \neq v$. We write $G = (V, E)$. Graphs without loops and parallel edges are said to be* simple.

**Important Note: All graphs this semester are simple.**

We say that $u$ is *adjacent* to $v$ ($u$ and $v$ are neighbors) if $\{u, v\}$ is an edge. An edge $e$ is said to be *incident* to $u$ if one of the two endpoints of $e$ is $u$. The *degree* of a vertex $u \in V$ is the number of edges incident $u$.

**Theorem 18.0.1** (Handshaking Lemma)**.** *Let $G = (V, E)$ be a graph.*

$$\sum_{u \in V} deg(u) = 2|E|$$

*Proof.* Look at an abritrary edge $u, v \in E$. Each edge is counted twice. $\square$

**Theorem 18.0.2.** *Let $G = (V, E)$ be a graph. Then $G$ has an even number of vertices with an odd degree.*

*Proof.* By contradiction. Let $V_{even}$ denote the set of vertices of $G$ with an even degree, and $V_{odd}$ denote the set of vertices of $G$ with an odd degree. So

$$V_{even} \cap V_{odd} = \emptyset$$

$$V_{even} \cup V_{odd} = V$$

For a contradiction, assume $|V_{odd}|$ is odd. Then

$$\begin{aligned}
2|E| &= \sum_{u \in V} \deg(u) && \text{(Handshaking Lemma)} \\
&= \sum_{u \in V_{even}} \deg(u) + \sum_{u \in V_{odd}} \deg(u) \\
&= 2k + 2l + 1 \\
&= 2(k + l) + 1
\end{aligned}$$

But $2|E|$ is even, so this is a contradiction.

$\square$

**Example:** Can you find a graph with 5 verices with degrees 1,2,3,3,3? Yes, since by the previous theorem, we have an even number of vertices with an odd degree.

**Example:** Can you find a graph with 5 vertices with degrees 1,2,2,3,3? No, since by the previous theorem, we have an odd number of vertices with odd degree.

**Definition 18.0.2.** *A* path *in a graph* $G = (V, E)$ *is a sequence of vertices* $v_0, v_1, \ldots, v_n$ *such that* $\{v_i, v_{i+1}\} \in E$ *for all* $0 \leq i \leq n-1$. *A path can also be described as a sequence of the* $n-1$ *edges. The vertices* $v_0$ *and* $v_n$ *are the* endpoints *of the path and* $n$ *is its* length.

**Definition 18.0.3.** *If there is a path with endpoints* $u, w \in V$, *we say that* $v$ *and* $w$ *are* connected. *If any two vertices of a graph are connected, then we say that the graph is* connected.

# Lecture 20

# More on Graphs

**Definition 20.0.1** (Cycles)**.** *A* cycle *is a sequence of vertices* $v_0, v_1, v_2, \ldots, v_{l-1}, v_0$ *such that* $v_0, v_1, v_2, \ldots, v_{l-1}$ *is a path and* $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{l-2}, v_{l-1}\}, \{v_{l-1}, v_0\}$ *are disstict edges. The* length *of this cycle is* $l$

> **Note:** Cycles of length 0,1,2 are now allowed by this definition.

**Definition 20.0.2** (Walks)**.** *A* walk *is a path where we allow vertices to be repeated. A* closed walk *is a cycle where we allow vertices to be repeated.*

**Definition 20.0.3** (Subgraph)**.** *Let* $G = (V, E)$ *be a graph. A* subgraph $H$ *of* $G$*, denoted by* $H \subseteq G$*, is a graph* $H = (V', E')$*, where* $V' \subseteq V$ *and* $E' \subseteq E$*.*

**Definition 20.0.4** (Connectedness)**.** *A* connected component *of* $G$ *is a subgraph of* $G$ *consisting of*

- *All vertices that are connected to a given vertex.*
- *Together with all edges incident to them.*

**Definition 20.0.5** (Forests)**.** *A* forest *is a graph that has no cycle. A* tree *is a connect forest. A* leaf *in a forest is a vertex of degree 1.*

**Theorem 20.0.1.** *Let* $G = (V, E)$ *be a graph. Let* $n = |V|$ *and* $m = |E|$*. If* $G$ *is a forest, then* $n > m$ *and* $G$ *has* $n - m$ *connected componenets (trees).*

*Proof.* By induction on $m$.

- **Base Case:** $m = 0$. If a forest has no edges, then $n > 0 = m$. Moreover, each vertex is its own connected component, so there are exactly $n = n - 0 = n - m$ connected components.
- **Induction Hypothesis:** Let $k \geq 0$ be an integer. Assume that for all graphs $G$ with $n$ vetices and $k$ edges, if $G$ is a forest, then $n > k$ and $G$ has $n - k$ connected componenets.

- **Induction Step:** Let $G$ be a forest with $n$ vertices and $k+1$ edges. Remove an arbitrary edge $e = \{a, b\}$ from $G$ without modifying the vertices. Removing $e$ from $G$ does not create a cycle, so the resulting graph $G'$ is a forest with $n$ vertices and $k$ edges. By the induction hypothesis, $n > k$ and $G'$ has $n - k$ connected components.

Observe that $a$ and $b$ cannot both belong to the same connected component of $G'$. Otherwise, the path from $a$ to $b$ would create a cycle in $G$ and so $G$ would not be a forest. So, $a$ and $b$ are in two different connected components of $G'$. So,

$$n - k \geq 2 \implies n \geq 2 + k = 1 + (1 + k) > k + 1$$

If we put $e$ back in $G$, this connects the two connected components for $a$ and $b$ together. So $G$ has

$$(n - k) - 1 = n - (k + 1)$$

connected components, as required. $\square$

**Corollary 20.0.1.** *Let $G = (V, E)$ be a tree. Then*

$$n = |V| = |E| + 1 = m + 1$$

*Proof.* A tree is a forest with 1 connected component. By the previous theorem, $n - m = 1$, so $n = m + 1$. $\square$

**Definition 20.0.6** (Spanning Tree)**.** *A* spanning tree *of a connected graph $G$ is a subgraph of $G$ that includes all vertcies of $G$ and that is a tree.*

# Lecture 21

# Spanning Trees Bipartite Graphs

**Theorem 21.0.1.** *Every connected graph graph $G = (V, E)$ has a spanning tree.*

*Proof.* Let $G = (V, E)$ be a connected graph. By induction on the $m = |E|$.

- **Base Case:** $m = 0$. For $G$ to be connected graph, it must contain a single vertext $v$. Then $v$ itself is a spanning tree.
- **Induction Hypothesis:** Let $k \geq 0$ be an integer. Assume that all connected graphs with $k$ edges have a spanning tree.
- **Induction Step:** Let $G$ be a connected graph with $k + 1$ edges. We consider two cases.
  - **Case 1:** $G$ is a tree, then it is its own spanning tree.
  - **Case 2:** If $G$ is not a tree, since $G$ is connected, then $G$ has a cycle. Remove an edge $e = \{a, b\}$ from this cycle. We get a graph $G'$ that is connected. Indeed, if a path uses $e$, we can reroute it along the other edges of the cycle. So $G'$ is connected and it has $k$ edges. By the induction hypothesis, $G'$ has a spanning tree $T$. $T$ covers all vertice of $G'$, so it covers all vertices of $G$, so $T$ is a spanning tree of $G$.

$\square$

This gives us a way to build a spanning tree; If $G$ is a tree, then it is its own spanning tree. Otherwise, find a cycle, remove an edge from this cycle, and recursively find a spanning tree.

**Corollary 21.0.1.** *Every graph with $n$ vertices and $m$ edges has at least $n - m$ connected components.*

*Proof.* Let $G$ be a graph with $n$ vertices and $m$ edges. By the previous theorem, every connected component of $G$ has a spanning tree. Let $F$ be the union of these spanning trees. Then $F$ is a forest with $n$ vertices, and $m' \leq m$. Moreover, the number of connected components in $F$ is the same as in $G$. So $G$ has $n - m' \geq n - m$ connected components. $\square$

We say that two sets $S$ and $T$ *partition* a set $E$ if

- $S \neq \emptyset$
- $T \neq \emptyset$
- $S \cup T = E$
- $S \cap T =$

We say that a graph $G = (V, E)$ is *bipartite* if $V$ can be partitioned into two sets $A$ and $B$ such that each edge has one endpoint in $A$ and one endpoint in $B$.

# Lecture 22

# Bipartite Graphs

**Note: Final Exam Topics**

- All lectures and all exercises
- 2 hour Exam
- 6 long answer questions and **nothing** Examples
- Study notes and exercises!

**Lemma 22.0.1.** *let $G = (V, E)$ be a bipartite graph with partition $(A, B)$. Then*

$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(V)$$

*Proof.* In the proof of the handshaking lemma, we saw that each edge in

$$\sum_{v \in V} \deg(v)$$

is counted twice. Since each edge has one endpoint in $A$ and one endpoint in $B$, the equality follows. □

**Lemma 22.0.2.** *Let $G = (V, E)$ be a graph. If $G$ has a closed walk of odd length, then $G$ has a cycle of odd length.*

*Proof.* By induction on the length $l$ of the closed walk.

- **Base Case:** $l = 1$. Closed walks of length 1 do not exist. So the base case holds trivially.
- **Induction Hypothesis:** Let $K \in \mathbb{Z}$ be an odd integer. Assume that if a graph has a closed walk of odd length at most $K$, then it has a cycle of odd length.

- **Induction Step:** Let $G = (V, E)$ be a graph having a closed walk of length $k + 2$ (which is odd). Let

$$v_0, v_1, v_2, \dots v_{k+1}, v_0$$

be this closed walk. If all $v_i$'s are different then this is a cycle with odd length. Otherwise, two vertices must be repeated, say $v_i$ and $v_j$

$$v_0, v_1, v_2, \dots, v_{i-1}, v_i, \dots v_{j-1}, v_j, \dots, v_k, v_{k+1}$$

Consider the closed walks

$$v_i, v_{i+1}, \dots, v_{j-1}, v_j = v_i$$

$$v_j, v_{j+1}, \dots, v_{k+1}, v_0, v_1, \dots, v_{i-1}, v_i = v_j$$

Since the length of the walk is $k + 2$, we have the length of the first walk is less than $k + 2$ and the second walk as well. Since $k + 2$ is odd, then either the first walk or the second walk has odd length. By the induction hypothesis, one of these walks is a cycle. Therefore $G$ has a cycle of odd length.

□

**Theorem 22.0.1.** *A graph is bipartite if and only if it has no odd cycles.*

*Proof.* ( $\Longrightarrow$ ) Assume $G$ is bipartite with partition $V = A \cup B$. If $G$ has no cycle, then it is bipartite. Otherwise, let

$$v_0, v_1, v_2, \dots, v_{l-1}, v_0$$

be a cycle of length $l \geq 3$. We will show that $l$ is even. Without loss of generality, assume $v_0 \in A$. Since $v_0 \in A$, we must have $v_1 \in B$, then $v_2 \in A$, and so on. So $v_{l-1} \in A$ if $l - 1$ is even and $v_{l-1} \in B$ if $l - 1$ is odd. Since $\{v_{l-1}, v_0\}$ is an edge of the cycle, $v_{l-1}$ must be in $B$. So $l - 1$ is odd, and $l$ is even.

( $\Longleftarrow$ ) Assume $G$ has no cycle of odd length. We want to prove that $G$ is bipartite. Consider two cases where $G$ is connected and where $G$ is not connected.

- **Case 1:** $G$ **is connected.** We need to build $A$ and $B$. Let $v_0$ be an arbitrary vertext. Let

$$A = \{w \in V : \text{ there is a path of even length between } v_0 \text{ and w}\}$$

$$B = \{w \in V : \text{ there is a path of odd length between } v_0 \text{ and } w\}$$

We will show that $A$ and $B$ are disjoint and that $A \cup B = V$, so $A$ and $B$ partition $V$. Since $G$ is connected, every vertex is connected to $v_0$, so every vertex belongs to either $A$ or $B$. We want to show that $A$ and $B$

are disjoint. For a contradiction, suppose $u \in A \cap B$. Since $v \in A$, there is a path of even length

$$v_0, v_1, \dots, v_s = u$$

where $s$ is even. Since $v \in B$, there is a path of odd length between $v_0$ and $v$. The reverse path also has odd length.

$$u = v_s, v_{s+1}, v_{s+2} \dots, v_{s+t} = v_0$$

where $t$ is odd. Now

$$v_0, v_1, \dots, v_s, v_{s+1}, \dots, v_{s+t} = v_0$$

is a closed walk of length $s+t$ where $s+t$ is odd. By the previous lemma, there is a cycle of odd length. This is a contradiction. So $A \cap B = \emptyset$. So $A, B$ is a partition of $V$. It remains to show that all edges have one endpoint in $A$ and one endpoint in $B$. For a contradiction, suppose there is an edge $\{u, w\}$ with both endpoints in $A$. Then there is a path of even length from $u$ to $v_0$ and a path of even length from $v_0$ to $w$. If $u$ and $w$ are different, then this is a cycle of even length then take the edge $\{u, w\}$ we get an odd length. by the previous lemma, there is a cycle of odd length. Thus, a contradiction.

- **Case 2: $G$ is not connected.** Then we can apply case (1) to each connected component $C$ of $G$. For each $C$, we get a partition $(A_c, B_c)$. It remains to take

$$A = \bigcup_{C \in G} A_C \text{ and } B = \bigcup_{C \in G} B_C$$

So $G$ is bipartite.

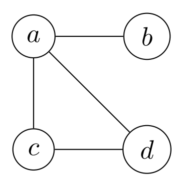
$\square$

# Lecture 23

# Matchings

**Definition 23.0.1** (Matching)**.** *A* matching *in a graph $G = (V, E)$ is a subset of $M \subseteq E$ where no pair of edges share a vertex.*

**Example:**

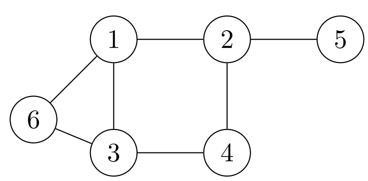


$$M = \{\{a, b\}, \{c, d\}\}$$

$M = \emptyset$ is a matching.

**Definition 23.0.2** (Maximum Matching)**.** *A* maximum matching *if it contains the greatest number of edges possible.*

**Example:**



$$M = \{\{6, 3\}, \{6, 1\}\}$$

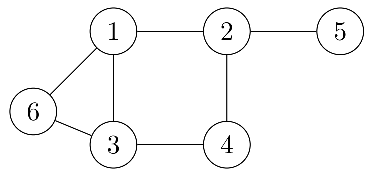
This is not a maximum matching because we could add the edge $\{1, 2\}$ to get a matching with 3 edges. So

$$M = \{\{6, 3\}, \{6, 1\}, \{2, 1\}\}$$

This matching is maximum.

**Definition 23.0.3** (Perfect Matchings)**.** *A matching is* perfect *if it matches* all *vertices.*

**Example:**




$$M = \{\{6, 3\}, \{6, 1\}, \{2, 1\}\}$$

is a perfect matching. In general, if a graph has an odd number of vertices, it cannot have a perfect matching.

**Theorem 23.0.1.** *Let $G = (V, E)$ be a graph whose set of edges is the union of two matchings. Then $G$ is bipartite.*

*Proof.* Let $M_1$ and $M_2$ be the two matchings such that $E = M_1 \cup M_2$. We have to prove that $G$ does not have a cycle of odd length. By the previous theorem, this implies that $G$ is bipartite. For a contradiction, let

$$v_0, v_1, v_2, \ldots, v_{l-1}, v_0$$

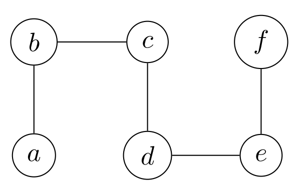
be a cycle of odd length. The edge $\{v_0, v_1\}$ is in $M_1$ or $M_2$. Without loss of generality, assume that $\{v_0, v_1\} \in M_1$. Then $\{v_1, v_2\} \in M_2$ otherwise $M_1$ would not be a matching. Then $\{v_2, v_3\} \in M_1$ and so on. So we have

$$\{v_i, v_{i+1}\} \in M_1 \text{ if } i \text{ is even}$$

$$\{v_i, v_{i+1}\} \in M_2 \text{ if } i \text{ is odd}$$

Then, $\{v_{l-2}, v_{l-1} \in M_2\}$ since $l-2$ is odd, and $\{v_{l-1}, v_0\} \in M_1$ since $l-1$ is even. But, the edge $\{v_0, v_1\}$ in $M_1$ shares an edge with $\{v_{l-1}, v_0\}$ in $M_1$. This is a contradiction since $M_1$ is a matching. $\square$

**Example:**




$$M_1 = \{\{a, b\}, \{c, d\}, \{e, f\}\}$$
$$M_2 = \{\{b, c\}, \{d, e\}\}$$

$M_1 \cup M_2$ is equal to the set of edges. So this graph is bipartite.

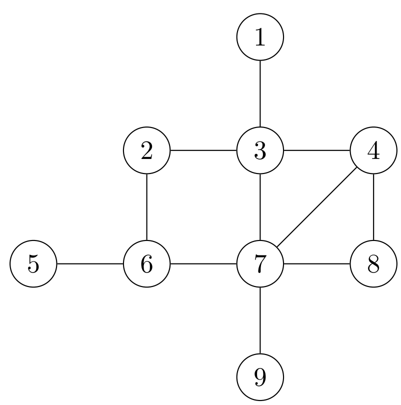
# Lecture 24

# Neighbour Sets

**Definition 24.0.1** (Neighbour Set)**.** *Let $G = (V, E)$ be a graph. Let $S \subseteq V$. The* neighbour set *of $S$, denoted $N(S)$ is the set of vertices having at least one neighbour in $S$.*

$$N(S) = \{v \in V | \{v, s\} \text{ is an edge for some } s \in S\}$$

**Example:**




$$N(\{3, 4\}) = \{1, 2, 3, 4, 7, 8\}$$
$$N(\{1\}) = \{3\}$$
$$N(\{2, 3, 4\}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

**Theorem 24.0.1** (Hall's Theorem)**.** *Let $G = (V, E)$ be a bipartite graph with partition $(A, B)$. There exists a matching that matches all vertices in $A$ if and only for every $S \subseteq A$. We have $|N(S)| \geq |S|$.*

*Proof.* ($\implies$) Suppose all vertices in $A$ can be matched. Let $S \subseteq A$ be a subset. We need to show that $|N(S)| \geq |S|$. For every $s \in S$, let $v$ be its partner in the matching. All these $v$'s are different. So there are $|S|$ of them. They are all neighbours of vertices in $S$. So they are in $N(S)$. Therefore, $|N(S)| \geq |S|$.

( $\Longleftarrow$ ) Assume that for every subset $S \subseteq A$, we have $|N(S)| \geq |S|$. We want to prove that all vertices in $A$ can be matched. By indunction,

- **Base Case:** $n = |A| = 1$. Then $|N(A)| \geq |A| = 1$. So the unique vertex in $A$ has a neighbour in $B$ and it can be matched.

- **Induction Hypothesis:** Let $K \geq 1$ be an integer. Assume that for all bipartite graphs $G$ with partition $(A, B)$ where $|A| \leq K$, if $|N(S)| \geq |S|$ for all $S \subseteq A$, then all vertices in $A$ can be matched.

- **Induction Step:** Let $G$ be a bipartite graph with partition $(A, B)$ where $|A| = k+1$. Consider two cases where for every $X \subset A$, we have $|N(X)| \geq |X| + 1$, and there exists a subset $X \subset A$ such that

$$|N(X) < |X| + 1 \implies |N(X)| = |X|$$

  - **Case 1:** Let $a \in A$ and match it with an arbitrary neighbour $b \in B$. Remove $a$ and $b$ from $G$. Now for every proper subset $X \subset A$, we have $|N(X)| \geq |X|$. By the induction hypothesis, all vertices in $A \setminus \{a\}$ can be matched. So all vertices in $A$ can be matched.
  - **Case 2:** Let $X \subset A$ be a proper subset such that $|N(X)| = |X|$. Observe that all subsets $X' \subseteq X$ are subsets of $A$, and $|N(X')| \geq |X'|$. So by the induction hypothesis, all vertices in $X$ can be matched. Remove all vertices in $X$ and $N(X)$ from $G$. Suppose we can show that for all subsets $Y \subseteq A \setminus X$, $Y$ has at least $|Y|$ neighbours in $B \setminus N(X)$, then we apply the induction hypothesis to match all vertices in $A \setminus X$. Combining the two matchings will give us a matching that matches all vertices in $A$. To prove our supposition, suppose for a contradiction that there exists a subset $Y \subseteq A \setminus X$ that has less than $|Y|$ neighbours in $B \setminus N(X)$. Then the vertices in $X \cup Y$ have less than $|N(X)| + |Y|$ neighbours in $G$. Then

$$\begin{aligned} |N(X \cup Y)| &< |N(X)| + |Y| \\ &= |X| + |Y| \\ &= |X \cup Y| \\ &\leq |N(X \cup Y)| \end{aligned}$$

    Which is a contradiction.

$\square$