MAT 3172 Lecture Notes

Last Updated:

May 9, 2023

Contents

1 Set Theory **2**
1.1 Review of Set Theory 2
1.1.1 Properties of Sets 4
1.2 Indicator Function 5
1.2.1 Properties of Indicator Functions: 6
1.3 Limsup and Liminf of Sets 6
1.4 Fields and Algebras 11
1.4.1 Properties of Probability Measures 12
1.4.2 Expectations 13

Chapter 1

Set Theory

1.1 Review of Set Theory

Let Ω be an abstract set representing the sample space of a random experiment. The power set of Ω by $\mathcal{P}(\Omega)$ is defined to be the set of all subsets of Ω. Elements of Ω are outcomes and its subsets are events. Therefore,

$$\mathcal{P}(\Omega) = \{A : A \subseteq \Omega\}$$

For $A, B \in \mathcal{P}(\Omega)$, we define

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$\bar{A} = A^c = \{x : x \notin A\}$$

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

In terms of events $A \cup B$ occurs if and only if at least one of the two events A and B occurs. Also, $A \cap B$ occurs if both A and B occurs. The empty set is denoted by $\emptyset$.

Examples of Sample Spaces: When flipping a coin, we have two outcomes, so

$$\Omega = \{H, T\}$$

If we flip a coin and role a dice,

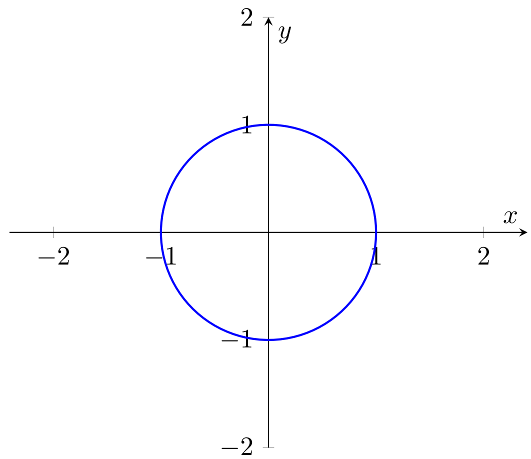
$$\Omega = \{1H, 2H, \ldots, 6H, 1T, \ldots 6T\}$$

If we flip a coin until we observe a head,

$$\Omega = \{H, TH, TTH, TTTH, \ldots\}$$

Here, there are infinite outcomes so the sample space is infinite, but it is countable since we can list all the possibilities.

If we pick a choose are sample space to be the points with distance one from the origin, we have the points in the unit circle,



The sample space is defined by

$$\Omega = \{(x, y) : d((x, y), (0, 0)) \leq 1\} = \{(x, y) : x^2 + y^2 \leq 1\}$$

In this example, the sample space omega is infinite as well, but it is uncountable.

Examples of Events: An event is a subset of the sample space. For example, in the case of rolling a dice and flipping a coin, we have

$$\Omega = \{1H, 2H, \ldots, 6H, 1T, \ldots 6T\}$$

And we can define an event E as

$$E = \{\text{Coin is heads and the dice is even}\} = \{2H, 4H, 6H\} \subset \Omega$$

If we flip a coin until the first head appears,

$$\Omega = \{H, TH, TTH, \ldots\}$$

And we can define an event E as

$$E = \{\text{First head appears before the 5th trial}\} = \{H, TH, TTH, TTTH, TTTTH\} \subset \Omega$$

Examples of Power Sets: Consider the sample space obtained by rolling a dice,

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

And let $E = \{2, 4, 6\}$ be the event we roll an even number. Then, the power set of E is

$$\mathcal{P}(E) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

The cardinality of the power set is

$$|\mathcal{P}(E)| = 2^{|E|} = 8$$

Examples of Set Operations:

$$\Omega = \{1, 2, \ldots, 6\}$$

$$A = \{1, 2, 3\} \ B = \{1, 2, 3\}$$

$$A \cup B = \{1, 2, 3, 4, 6\}$$

$$A \cap B = \{2\}$$

$$A^c = \{1, 3, 5\}$$

$$A \setminus B = \{x : x \in A, x \notin B\} = \{4, 6\}$$

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

Example of Empty Set: Is $\{\{\}\} = \{\}$? **No**. $\{\{\}\}$ is a set with one element, which is the empty set, so $\{\{\}\} = \{\emptyset\}$.

1.1.1 Properties of Sets

- $A \subset A$, $\emptyset \subset A$
- $A \subset B$ and $B \subset A$ implies $A = B$
- $A \subset C$ and $B \subset C$ implies $A \cup B \subset C$ and $A \cap B \subset C$.
- $A \subset B$ if and only if $B^c \subset A^c$
- $(A^c)^c = A$, ${}^c = \Omega$, $\Omega^c = \emptyset$
- $A \cup B = B \cup A$, $A \cap B = B \cap A$
- $A \cup A = A$, $A \cap \Omega = A$, $A \cup A^c = \Omega$, $A \cap A^c = \emptyset$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

Example: We have

$$\bigcup_{n=1}^{\infty}\left[0,\frac{n}{n+1}\right) = [0,1)$$

To show that these sets are equal, consider the limit of the sequence $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$. As n becomes large, the limit approaches 1. So, the union of all these sets will contain elements that become arbitrarly close to 1 but do not reach 1, so we have $[0,1)$.

$$\bigcap_{n=1}^{\infty}\left(0,\frac{1}{n}\right) = \emptyset$$

To show that these sets are equal, consider the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$, As this sequence approaches 0, the intersection of all these sets will contain elements that become arbitrarly close to 0 but do not reach 0, so we have the set $(0,0)$ which is empty. Therefore, when taking the intersection of all these sents with $\left(0,\frac{1}{n}\right)$ which become arbitrarly small, we have the empty set.

Example: Prove that $A\Delta B = A^c \Delta B^c$.

Proof. Note that $A\setminus B = A\cap B^c$.

$$\begin{aligned}
A^c\Delta B^c &= (A^c \cup B^c)\setminus(A^c\cap B^c)\\
&= (A\cap B)^c \cap ((A\cup B)^c)^c\\
&= (A\cap B)^c \cap (A\cup B) = (A\cup B)\setminus(A\cap B)\\
&= A\Delta B
\end{aligned}$$

□

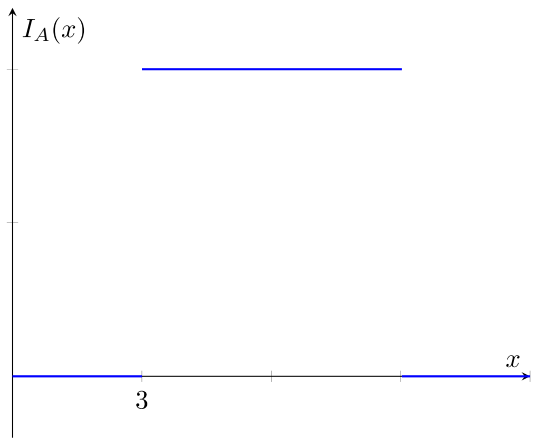
1.2 Indicator Function

Let $A\subset\Omega$. The indicator function of A is defined as

$$I(x\in A) = I_A(x) = \begin{cases} 1 & x\in A\\ 0 & x\notin A\end{cases}$$

Example: $A = [1,3]$

$$I_A(x) = I_{[1,3]}(x)$$



1.2.1 Properties of Indicator Functions:

- $I_{A\cup B} = \max(I_A, I_B)$
- $I_{A\cap B} = I_A \cdot I_B$
- $I_{A\Delta B} = I_A + I_B \pmod 2$
- $A \subset B$ if and only if $I_A \leq I_B$
- $I_{\cup_i A_i} \leq \sum_i I_{A_i}$

1.3 Limsup and Liminf of Sets

Exercise: Prove that

$$I_{\cup_{i=1}^{\infty} A_i} = 1 - \prod_{i=1}^{\infty}(1 - I_{A_i})$$

and

$$I_{A\Delta B} = (I_A - I_B)^2$$

Definition 1.3.1. *Let $\{A_n\}$ be a sequence of events. Then*

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

and

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

Example: Suppose we have a sequence of events $A_1, A_2, A_3, \ldots$, we can define

$$B_n = \bigcap_{m=n}^{\infty} A_m$$

So its sequence members are

$$B_n = A_n \cap A_{n+1} \cap \cdots$$
$$B_{n+1} = A_{n+1} \cap A_{n+2} \cap \cdots$$
$$B_{n+2} = A_{n+2} \cap A_{n+3} \cap \cdots$$

These sets are getting smaller because surely the intersection of more sets will be smaller since $|A \cap B| \leq \min(|A|, |B|)$. So as we take more intersections, the sets become smaller and smaller. Now we can look at the union of these sets,

$$\bigcup_{n=1}^{\infty} B_n = \liminf A_n$$

If instead we take B_n to be the union of all the sets,

$$B_n = \bigcup_{m=n}^{\infty} A_m$$

So the sequence members are

$$B_n = A_n \cup A_{n+1} \cup \cdots$$
$$B_{n+1} = A_{n+1} \cup A_{n+2} \cup \cdots$$
$$B_{n+2} = A_{n+2} \cup A_{n+3} \cup \cdots$$

These sets are getting larger since we are taking the union of more sets, then we can look at the intersection of these sets,

$$\bigcap_{m=n}^{\infty} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_n = \limsup A_n$$

Lemma 1.3.1. *We have*

$$\limsup A_n = \left\{ \omega : \sum_{i=1}^{\infty} I_{A_i}(\omega) = \infty \right\}$$

and

$$\liminf A_n = \left\{ \omega : \sum_{i=1}^{\infty} I_{A_i^c}(\omega) < \infty \right\}$$

We can also express this as

$$\limsup_{n\to\infty} A_n = \left\{ x \in X : \limsup_{n\to\infty} I_{A_n}(x) = 1 \right\}$$

and

$$\liminf_{n\to\infty} A_n = \left\{ x \in X : \liminf_{n\to\infty} I_{A_n}(x) = 1 \right\}$$

Proof. If $\omega \in \limsup A_n$, then $\omega \in \bigcup_{m=n}^{\infty} A_m$ for all integers n. Therefore, for any integer n there exists an integer k_n such that $\omega \in A_{k_n}$, since

$$\sum_{i=1}^{\infty} A_{A_i}(\omega) \geq \sum_{i=1}^{\infty} I_{A_{k_i}}(\omega) = \infty$$

Conversely, for any integer n, by definition of the limit superior,

$$\sum_{i=n}^{\infty} I_{A_i}(\omega) = \infty$$

This implies that $\omega \in \bigcup_{j=n}^{\infty} A_j$ for all integers n. Therefore,

$$\omega \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \limsup A_n$$

Then, we can notice that

$$\omega \liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

implies that there exists an integer n_0 such that

$$\omega \in \bigcap_{k=n_0}^{\infty} A_k$$

Therefore,

$$\sum_{n=1}^{\infty} I_{A_n^c}(\omega) = \sum_{n=1}^{n_0-1} I_{A_n^c}(\omega) \leq n_0 < \infty$$

□

Note: For this reason, sometimes we write $\limsup A_n = A_n$ infinitely often. If $\liminf A_n = \limsup A_n$, then

$$\lim A_n = \liminf A_n = \limsup A_n$$

Remark: The proof of the lemma above can be simplified by noticing the fact that

$$(\limsup A_n)^c = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c = \liminf A_n^c$$

$$(\liminf A_n)^c = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m^c = \limsup A_n^c$$

To summarize, with the limit superior we have infinitely many cases where $I_{A_i}(\omega) = 1$. So what it means for $\omega \in \limsup A_n$ is that ω is in infinitely many of the A_i's. For the limit inferior, it means that ω is in all but finitely many of the A_i's.

Example: Consider $\omega \in A_i$ when i is odd, so

$$\omega \in A_1, \omega \notin A_2, \omega \in A_3, \omega \notin A_4, \omega \in A_5, \ldots$$

ω is in infinitely many (but countable) number of the A_i's. So $\omega \in \limsup A_n$. Now consider $\omega \in A_i$ when $i \geq 10$, so

$$\omega \notin A_1, \omega \notin A_2, \omega \notin A_3, \ldots, \omega \notin A_9, \omega \in A_{10}, \omega \in A_{11}, \omega \in A_{12}, \ldots$$

So, ω is not in finitely many of the A_i's, therefore $\omega \in \liminf A_n$.

Example: Consider sample space of flipping a coin and infinite number of times, and the event $E = \{HTTHT\}$, so the event that we get $HTTHT$ in that order. Because this outcome is possible, it will occur an infinite number of times in the sequence of events, so E is in the limit superior, and the probability that E occurs infinitely often is 1.

Lemma 1.3.2. *Let $\{A_n\}$ be a sequence of events, then*

1. *If $A_n \subset A_{n+1}$ for any integer n, then*

$$\lim A_n = \bigcup_{n=1}^{\infty} A_n$$

2. *If $A_{n+1} \subset A_n$ for any integer n, then*

$$\lim A_n = \bigcap_{n=1}^{\infty} A_n$$

Proof. We can prove (1) and (2) similarly as follows, not that in this case,

$$\bigcap_{m=n}^{\infty} A_m = \bigcap_{m=1}^{\infty} A_m$$

for all integers n. If $A_n \subset A_{n+1}$ for any integer n, then we have that

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

So the set is getting bigger, now consider the limit superior

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \left(\bigcup_{m=1}^{\infty} A_m \right) \cap \left(\bigcup_{m=2}^{\infty} A_m \right) \cap \left(\bigcup_{m=3}^{\infty} A_m \right) \cap \cdots$$

These sets are equal in size since if $A_1 \subset A_2$, then $A_1 \cup A_2 = A_2$. Therefore, we get that the intersection of these sets is $\bigcup\limits_{m=1}^{\infty} A_m$, and thus

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \bigcup_{n=1}^{\infty} A_n$$

Furthermore,

$$\begin{aligned}
\liminf_{n\to\infty} A_n &= \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \\
&= \left(\bigcap_{m=1}^{\infty} A_m\right) \cup \left(\bigcap_{m=2}^{\infty} A_m\right) \cup \left(\bigcap_{m=3}^{\infty} A_m\right) \cup \cdots \\
&= A_1 \cup A_2 \cup A_3 \cup \cdots \\
&= \bigcup_{n=1}^{\infty} A_n
\end{aligned}$$

Therefore

$$\limsup A_n = \liminf A_n \implies \limsup A_n = \liminf A_n = \lim A_n$$

The proof for (2) follows the same. $\square$

Example:

$$\lim_{n\to\infty} \left[0, 1 - \frac{1}{n}\right] = \lim_{n\to\infty} \left[0, 1 - \frac{1}{n}\right) = [0, 1)$$

To see this, we have that

$$A_1 = \{0\}, A_2 = \left[0, \frac{1}{2}\right], A_3 = \left[0, \frac{2}{3}\right], A_4 = \left[0, \frac{3}{4}\right], \ldots$$

So the set A_n is increasing, therefore

$$\limsup A_n = \liminf A_n = \lim A_n = \bigcup_{n=1}^{\infty} A_n = [0, 1]$$

Example:

$$\lim_{n\to\infty} \left[0, 1 + \frac{1}{n}\right] = \lim_{n\to\infty} \left[0, 1 + \frac{1}{n}\right) = [0, 1)$$

Similarly,

$$A_1 = [0, 2], A_2 = \left[0, 1 + \frac{1}{2}\right], A_3 = \left[0, 1 + \frac{1}{3}\right], A_4 = \left[0, 1 + \frac{1}{4}\right], \ldots$$

The set A_n is decreasing, therefore

$$\limsup A_n = \liminf A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left[0, 1 + \frac{1}{n}\right] = [0, 1]$$

Example: Let $B, C \subset \Omega$ and define the sequence

$$A_n = \begin{cases} B & \text{if } n \text{ is odd} \\ C & \text{if } n \text{ is even} \end{cases}$$

Then we have

$$\bigcup_{m=n}^{\infty} = B \cup C \implies \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = B \cup C = \limsup A_n$$

Similarly for the limit inferior,

$$\bigcap_{m=n}^{\infty} A_m = B \cap C \implies \bigcup_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = B \cup C = \liminf A_n$$

Therefore, we have

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = B \cup C \text{ and } \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m = B \cap C$$

If $B \cap C \neq B \cup C$, then $B \cap C = \liminf A_n \neq \limsup A_n = B \cup C$.

1.4 Fields and Algebras

Definition 1.4.1 (Fields (Algebras))**.** *A* field *(or* algebra*) is a class of subsets of Ω (called events) that contain Ω and are closed under finite union, finite intersection, and complementation. In otherwords, a family of subsets of Ω (say $\mathcal{A}$) is a field if*

- $\Omega \in \mathcal{A}$
- *If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$*
- *IF $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$*

Remarks. If $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$. This is true because

$$(A^c \cup B^c)^c = A \cap B$$

Definition 1.4.2 (σ-field)**.** *A σ-field (or σ-algebra) is a field that is closed under countable union (which implies that it is closed under countable intersection).*

Example. Let Ω be a set and $A, B \subset \Omega$. Then,

$$\mathcal{A} = \{\Omega, \emptyset, A, A^c, B, B^c, A \cup B, A \cap B, A \cap B^c, A^c \cap B, A^c \cup B^c, A^c \cap B^c\}$$

Examples of σ-fields.

- The power set $\mathcal{P}(\Omega)$
- $\mathcal{F} = \{\Omega, \emptyset\}$
- The family of subsets or $\mathbb{R}$ which are either countable or their complements are countable.
- Let $\mathcal{B}$ be the smallest σ-field containing all open sets. Then $\mathcal{B}$ is called the Borel σ-field.

Definition 1.4.3 (Probability Measure)**.** *Let Ω be a sample space and $\mathcal{F}$ be a σ-field on Ω. A probability measure P is defined on $\mathcal{F}$ such that*

(i) $P(\Omega) = 1$

(ii) If $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_1)$$

1.4.1 Properties of Probability Measures

(i) Since $P(\Omega) = 1 = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset)$, we have $P(\emptyset) = 0$

(ii) Since $(A \setminus B) \cup (A \cap B) = A$ and $(A \setminus B) \cap (A \cap B) = \emptyset$, we have

$$P(A \setminus B) = P(A) - P(A \cap B)$$

(iii) Similarly, $(A \setminus B) \cup B = A \cup B$ and $(A \setminus B) \cap B = \emptyset$, which implies

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(iv) If $A \subset B$ then $A \cup (B \setminus A) = B$. Therefore,

$$P(A) + P(B \setminus A) = P(B)$$

and furthermore, $P(A) \leq P(B)$

1.4.2 Expectations

Definition 1.4.4. *Let $X : \Omega \to \mathbb{R}$ and assume E be an operator with the following properties,*

1. *If $X \geq 0$ then $E(X) \geq 0$*
2. *If $c \in \mathbb{R}$ is a constant, then $E(cX) = cE(X)$*
3. *$E(X_1 + X_2) = E(X_1) + E(X_2)$*
4. *$E(1) = 1$*
5. *If $X_n(\omega)$ is monotonically increasing and $X_n(\omega) \to X(\omega)$, then*

$$\lim_{n\to\infty} E(X_n) = E(X)$$

Example. Flip 2 coins,

$$\Omega = \{HH, HT, TH, TT\}$$

Define $X : \Omega \to \mathbb{R}$, with

$$X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 0$$

So X is a random variable which represents the number of heads.

Example. Flip a coin until a head appears,

$$\Omega = \{H, TH, TTH, TTTH, \ldots\}$$

With

$$X(H) = 1, X(TH) = 2, X(TTH) = 3, X(TTTH) = 4, \ldots$$

So X is a random variable which represents the number of trials until a head appears.

Example. Define

$$\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$$

and

$$X(x, y) = \sqrt{x^2 + y^2} = \text{ Distance of } (x, y) \text{ from } (0, 0)$$

Example. Let X be a random variable, and

$$E(X) = \lim_{D\to\infty} \frac{\int_{-D}^{D} X(\omega)d\omega}{2D}$$

Check if E satisfies the definition for an expectation.

Solution. We have to check the 5 axioms,

1. Its clear that if $X \geq 0$ then $E(X) \geq 0$ since the integral of a non-negative function is non-negative.

2. $$E(cX) = \lim_{D\to\infty} \frac{\int_{-D}^{D} cX(\omega)d\omega}{2D} = c \lim_{D\to\infty} \frac{\int_{-D}^{D} X(\omega)d\omega}{2D} = cE(x)$$

3. $$\begin{aligned} E(X_1 + X_2) &= \lim_{D\to\infty} \frac{\int_{-D}^{D}(X_1(\omega) + X_2(\omega))d\omega}{2D} \\ &= \lim_{D\to\infty} \left(\frac{\int_{-D}^{D} X_1(\omega)d\omega}{2D} + \frac{\int_{-D}^{D} X_2(\omega)d\omega}{2D} \right) \\ &= \lim_{D\to\infty} \frac{\int_{-D}^{D} X_1(\omega)d\omega}{2D} + \lim_{D\to\infty} \frac{\int_{-D}^{D} X_2(\omega)d\omega}{2D} \\ &= E(X_1) + E(X_2) \end{aligned}$$

4. $$E(1) = \lim_{D\to\infty} \frac{\int_{-D}^{D} 1d\omega}{2D} = \lim_{D\to\infty} \frac{2D}{2D} = 1$$

5. The 5th axiom fails however. Take $\Omega = \mathbb{R}$ and $X_n(\omega) = I_{[-n,-n]}(\omega)$, then

$$\lim_{D\to\infty} \frac{\int_{-D}^{D} X(\omega)d\omega}{2D} = 0$$

   But, $x_n(\omega) \to 1$. So the operator in this example is not a proper form of expectation.