MAT 3172 Summary Sheet

Indicator Functions

Definition. Let $A \subset \Omega$. The indicator function of A is defined as

$$I(x \in A) = I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Properties of Indicator Functions

- $I_{A \cup B} = \max(I_A, I_B)$
- $I_{A \cap B} = I_A \cdot I_B$
- $I_{A \Delta B} = I_A + I_B \pmod 2$
- $A \subset B$ if and only if $I_A \leq I_B$
- $I_{\cup_i A_i} \leq \sum_i I_{A_i}$

Set Theoretic Limits

Definition. Let $\{A_n\}$ be a sequence of events,

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

Lemma

$$\limsup A_n = \left\{ \omega : \sum_{i=1}^{\infty} I_{A_i}(\omega) = \infty \right\}$$

$$\liminf A_n = \left\{ \omega : \sum_{i=1}^{\infty} I_{A_i^c}(\omega) < \infty \right\}$$

To summarize, with the limit superior we have infinitely many cases where $I_{A_i}(\omega) = 1$. For the limit inferior, it means that ω is in all but finitely many of the A_i's.

Lemma

Let A_n be a sequence of events. Then

1. If $A_n \subset A_{n+1}$ for any integer n, then
$$\lim A_n = \bigcup_{n=1}^{\infty} A_n$$
2. If $A_{n+1} \subset A_n$ for any integer n, then
$$\lim A_n \bigcap_{n=1}^{\infty} A_n$$

More on Set Theoretic Limits

Lemma

Let $\{A_n\}$ be a sequence of events, then

1. If $A_n \subset A_{n+1}$ for any integer n, then
$$\lim A_n = \bigcup_{n=1}^{\infty} A_n$$
2. If $A_{n+1} \subset A_n$ for any integer n, then
$$\lim A_n = \bigcap_{n=1}^{\infty} A_n$$

Algebras

Definition. An *algebra* (also called a field) $\mathcal{A}$ is a class of subsets of Ω (called events) that satisfies

- $\Omega \subset \mathcal{A}$
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$
- If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$

Definition. A σ-*algebra* (also called a σ-*field*) is an algebra that is closed under countable union.

Probability Measures

Definition. Let Ω be a sample space and $\mathcal{F}$ be a σ-field on Ω. A probability measure P is defined on $\mathcal{F}$ such that

(i) $P(\Omega) = 1$

(ii) If $A_1, A_2, \ldots \in \mathcal{F}$ are pairwise disjoint, then
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Properties of Probability Measures

(i) $P(\Omega) = 1 = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset) \implies P(\emptyset) = 0$

(ii) $(A \backslash B) \cup (A \cap B) = A$ and $(A \backslash B) \cap (A \cap B) = \emptyset$, we have
$$P(A \setminus B) = P(A) - P(A \cap B)$$

(iii) Similarly, $(A \setminus B) \cup B = A \cup B$ and $(A \setminus B) \cap B = \emptyset$, which implies
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(iv) If $A \subset B$ then $A \cup (B \setminus A) = B$. Therefore,
$$P(A) + P(B \setminus A) = P(B)$$
and furthermore, $P(A) \leq P(B)$

Expectations

Definition. Let $X : \Omega \to \mathbb{R}$, an expecation E is an operator with the following properties,

(i) If $X \geq 0$, then $E(X) \geq 0$

(ii) If $c \in \mathbb{R}$ is a constant, then $E(cX) = cE(X)$

(iii) $E(X_1 + X_2) = E(X_1) + E(X_2)$

(iv) $E(1) = 1$

(v) If $X_n(\omega)$ is monotonically increasing and $X_n(\omega) \to X(\omega)$, then
$$\lim_{n \to \infty} E(X_n) = E(X)$$

Finding Probabilities Using Expecations

Definition. For any event A, define

$$P(A) = E(I_A(\omega))$$

For simplicity we write $P(A) = E(I_A)$.

Properties

(i) $E\left(\sum_{i=1}^{n} c_i X_i\right) = \sum_{i=1}^{n} c_i E(X_i)$

(ii) If $X \leq Y \leq Z$, then $E(X) \leq E(Y) \leq E(Z)$

(iii) If $\{A_i\}$ is a sequence of events, then
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

Fatou's Lemma

If $\{A_n\}$ is a family of events, then

1. $$P(\liminf A_n) \leq \liminf P(A_n) \leq \limsup P(A_n) \leq P(\limsup A_n)$$
2. If $\lim A_n = A$, then $\lim P(A_n) = P(A)$

Lemma

If $A_n \subset A_{n+1}$ for any n, then $\lim P(A_n) = P(A)$. If $A_{n+1} \subset A_n$ for any n, then $\lim P(A_n) = P(A)$ where $\lim A_n = A$.