MAT 3172 Summary Sheet

Indicator Functions

Definition. Let $A \subset \Omega$. The indicator function of A is defined as

$$I(x \in A) = I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Properties of Indicator Functions

- $I_{A \cup B} = \max(I_A, I_B)$
- $I_{A \cap B} = I_A \cdot I_B$
- $I_{A \Delta B} = I_A + I_B \pmod 2$
- $A \subset B$ if and only if $I_A \leq I_B$
- $I_{\cup_i A_i} \leq \sum_i I_{A_i}$

Set Theoretic Limits

Definition. Let $\{A_n\}$ be a sequence of events,

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

Lemma

$$\limsup A_n = \left\{ \omega : \sum_{i=1}^{\infty} I_{A_i}(\omega) = \infty \right\}$$

$$\liminf A_n = \left\{ \omega : \sum_{i=1}^{\infty} I_{A_i^c}(\omega) < \infty \right\}$$

To summarize, with the limit superior we have infinitely many cases where $I_{A_i}(\omega) = 1$. For the limit inferior, it means that ω is in all but finitely many of the A_i's.

Lemma

Let A_n be a sequence of events. Then

1. If $A_n \subset A_{n+1}$ for any integer n, then

$$\lim A_n = \bigcup_{n=1}^{\infty} A_n$$

2. If $A_{n+1} \subset A_n$ for any integer n, then

$$\lim A_n \bigcap_{n=1}^{\infty} A_n$$