MAT 2125 Lecture Notes

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$\mathbb{R}^d$

Recall: $||(x_1, \ldots, x_d)|| \coloneqq \sqrt{x_1^2 + \cdots + x_d^2}$. This is a norm. i.e.

$$||a + b|| \leq ||a||_2 + ||b||_2 \ \forall a, b \in \mathbb{R}^d$$

$$||ca|| = |c| \cdot ||a||_2, \ c \in \mathbb{R}, \ a \in \mathbb{R}^d$$

$$||a||_2 > 0, \ \forall a \in \mathbb{R}^d \setminus \{(0, \ldots, 0)\}$$

$$||(0, \ldots, 0)||_2 = 0$$

Other examples of norms:

- $||(x_1, \ldots, x_d)||_1 \coloneqq |x_1| + \cdots + |x_d|$
- $||(x_1, \ldots, x_d)||_\infty \coloneqq max\{|x_1|, \ldots, |x_d|\}$

Exercise: For $a \in \mathbb{R}^d$,

$$||a||_\infty \leq ||a||_2 \leq ||a||_1 \leq d||a||_\infty (\leq d||a||_2)$$

Interesting Fact: There are other norms. but they are all equivalent in the sense that if $|| \cdot ||, || \cdot ||'$ are norms on $\mathbb{R}^d$, then $\exists v, R > 0$ such that

$$r||a|| \leq ||a||' \leq R||a||$$

9.1 Convergence

Definition 9.1.1. *Let $(a_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}^d$ and let $L \in \mathbb{R}^d$, we say $(a_n)_{n=1}^\infty$* ***converges*** *to L, and write $\lim_{n\to\infty} = L$ or $a_n \to \infty$, if*

$$\lim_{n\to\infty} ||a_n - L||_2 = 0$$

Note: We could define convergence instead using some other norm, say $||\cdot||_1$.

If $||a_n - L||_2 \to 0$, then $||a_n - L||_1 \leq d||a_n - L||_2 \to 0$ If $||a_n - L||_1 \to 0$, then $||a_n - L||_2 \leq d||a_n - L||_1 \to 0$

in general, if $||\cdot||$ is any norm, then since $||\cdot||$ and $||\cdot||_2$ are equivalent.

$$||a_n - L||_2 \to 0 \iff ||a_n - L|| \to 0$$

Example: Say $a_n = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, then

$$||a_n - L||_2 = \sqrt{1/n^2, + \cdots + 1/n^2} = \sqrt{\frac{d}{n^2}} = \frac{\sqrt{d}}{n} \to 0$$

$$\therefore a_n \to L$$

Given a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$, we write $a_1 = (a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(d)})$ where $a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(d)} \in \mathbb{R}$. Similarily,

$$a_1 = (a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(d)})$$
$$a_2 = (a_2^{(1)}, a_2^{(2)}, \ldots, a_2^{(d)})$$
$$a_3 = (a_3^{(1)}, a_3^{(2)}, \ldots, a_3^{(d)})$$
$$\vdots$$
$$L = (L^{(1)}, L^{(2)}, \ldots, L^{(d)}) \in \mathbb{R}^d$$

We get d sequences in $\mathbb{R}$, and d possible limit points $L^{(1)}, \ldots, L^{(d)} \in \mathbb{R}$

Proposition 9.1.1. *Given $(a_n)_{n=1}^{\infty}$ and L as above, $a_n \to L$ as $d \to \infty \iff a_n^{(i)} \to L^{(i)}$ in $\mathbb{R}$ as $n \to \infty$, for $i = 1, \ldots, d$.*

Proof. $\implies$: Suppose $a_n \to L$, i.e.

$$||a_n - L||_2 \to 0$$

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$

$$||x||_2^2 = x_1^2 + \cdots x_d^2 \geq x_i^2 = |x_i|^2 \therefore |x_i \leq ||x||_2$$

Applying this to (*), we get

$$0 \leq |a_n^i - L^{(i)}| \leq ||a_n - L||_2 \to 0$$

So by the squeeze theorem,

$$|a_n^{(i)} - L^{(i)}| \to 0$$

$\implies$: Suppose $a_n^{(i)} \to L^{(i)}$ for $i = 1, \ldots, d$.

$$||a_n - L||_2^2 = (a_n^{(i)} - L^i)^2 + \cdots + (a_n^d - L^(d)) \to 0$$

By algebra of limits,

$$\therefore ||a_n - L||_2 \to 0$$

$\square$

Example: $a^n = ((-1)^n, \frac{1}{n}) \in \mathbb{R}^2$. Does (a_n) converge? No, since $(-1)^n$ does not converge.

Definition 9.1.2. *A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$ is **Cauchy** if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}_{\geq 1}$ such that*

$$||a_n - a_m||_2 < \epsilon \ \forall m, n \geq n_0$$

Theorem 9.1.1 (Cauchy Convergence Criterion for $\mathbb{R}^d$)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$. It converges $\iff$ it is Cauchy.*

Proof. $\implies$: Suppose $a_n \to L \in \mathbb{R}^d$. To show it is Cauchy, let $\epsilon > 0$. $||a_n - L||_2 \to 0$, so $\exists n_0 \in \mathbb{N}_{\geq 1}$ such that

$$||a_n - L||_2 < \frac{\epsilon}{2} \ \forall n \geq n_0)$$

Then if $m, n \geq n_0$,

$$\begin{aligned} ||a_m - a_n||_2 &= ||a_m - L + L - a_n||_2 \\ &\leq ||a_m - L||_2 + ||L - a_n||_2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\therefore (a_n)_{n=1}^{\infty}$ is Cauchy. $\implies$: Suppose $(a_n)_{n=1}^{\infty}$ is Cauchy, write

$$a_n = (a_n^{(1)}, \ldots, a_n^{(d)}$$

For any $m, n \in \mathbb{N}_{\geq 1}$,

$$|a_n^{(i)} - a_m^{(i)} \leq ||a_n - a_m||_2$$

$\therefore (a_n^{(i)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. So by the Cauchy Convergence Criterion, $\exists L^{(i)} \in \mathbb{R}$, such that $a_n^{(i)} \to L^{(i)}$. By the previous proposition,

$$a_n \to (L^{(1)}, \ldots, L^{(d)})$$

$\square$

Definition 9.1.3. *$S \subseteq \mathbb{R}^d$ is **bounded** if $\exists M > 0$ such that*

$$||x|| \leq M \quad \forall x \in S$$

A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$ is bound if $\{a_n : n \in \mathbb{N}_{\geq 1}\}$ is a bounded set.

Theorem 9.1.2 (Bolzano-Weierstrass for $\mathbb{R}^d$)**.** *If $(a_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^d$, then it has a subsequence $(a_{n_k})_{n=1}^{\infty}$ that converges.*

Proof. Write $a_n = (a_n^{(1)}, \ldots, a_n^{(d)})$.
Naive Attempt: TBC. $\square$

Lecture 10

Open and Closed Sets in $\mathbb{R}^d$

Roughly, an open set is one that we draw with dotted lines. The line represents a "boundary" that is the not in the set. This is not a rigorous definition.

Definition 10.0.1 (Open Ball). *Let $a \in \mathbb{R}^d$, $r > 0$. The* ***open ball*** *of radius r centered at a is*

$$B(a; r) := \{x \in \mathbb{R}^d : ||x - a||_2 < r\}$$

Relation to Convergence: If $a_n \to L$, then this means that $||a_n - L||_2 < \epsilon$ for all n large. So, $a_n \in B(L; \epsilon)$

Definition 10.0.2 (Open Sets). *A set $U \subseteq \mathbb{R}^d$ is* ***open*** *if $\forall a \in U$, $\exists r > 0$, such that $B(a; r) \subseteq U$*

Idea: If $a \in U$, then a is not on the boundary but it is truly "inside" the set, so we can fit a ball containing a in the set.

Definition 10.0.3 (Closed Sets). *A set $k \in \mathbb{R}^d$ is* ***closed*** *if its complement $\mathbb{R}^d \setminus k$ is open.*

Example: $U \subseteq (0, 1)$. Is this open? Yes.

Proof. Let $a \in U$. We let $r := \min\{|a - 0|, |a - 1|\}$ (We do this so that r is at most the distance to the closest bound, i.e. if a is closer to 0, then the radius r cannot be $|a - 1|$)then

$$B(a; r) = (a - r, a + r) \subseteq (0, 1) = U$$

$\square$

Example: $U := [0, 1]$. Is this open? No.

Proof. Let $a := 0 \in U$. The for any $r > 0$, $\exists z \in B(a; r) = (-r, r)$ s.t $z < 0$, so $z \notin U$. Therefore $B(a; r) \subseteq U$ $\square$

Is U closed? This is the same as asking if $\mathbb{R} \setminus U = (-\infty, 0) \cup (1, \infty)$ is open. This is open.

Proof. Let $a \in (-\infty, 0) \cup (1, \infty)$.

- **Case 1:** $a \in (-\infty, 0)$. Set $r := |a|$, so

$$B(a; r) = (a - r, a + r) = (2a, 0) \subseteq U$$

- **Case 2:** $a \in (1, \infty)$ similar.

$\square$

Therefore $U = [0, 1]$ is closed.

Example: Is $U \coloneqq (0, 1]$ open? No, for any $r > 0$

$$B(1; r) \nsubseteq U$$

Therefore, it is not open.

Note: *Sets are not always open or closed. Most sets are neither open nor closed.*

This set U is one such example U is not closed since $\mathbb{R} \setminus U = (-\infty, 0] \cup (1 \infty)$ $0 \in \mathbb{R}\ U$ but $\forall r > 0$, $B(0; r) \nsubseteq R \setminus U$

Example: For any $a \in \mathbb{R}^d$, $r > 0$ $B(a; r)$ is an open set.

Proof. Let $x \in B(a; r)$, so $||x - a||_2 < r$. Set

$$r_0 \coloneqq r - ||x - a||_2 > 0$$

Claim: $B(x; r_0) \subseteq B(a; r)$ To see this, let $y \in B(x; r_0)$ so $|y - x||_2 < r_0$. So,

$$\begin{aligned} ||y - a||_2 &\leq ||y - x||_2 + ||x - a||_2 && (\triangle\text{-inequality}) \\ &< r_0 + ||x - a||_2 \\ &= r \end{aligned}$$

$\square$

Proposition 10.0.1. *(i) $\emptyset$, $\mathbb{R}^d$ are both open in $\mathbb{R}^d$*

(ii) If $U_1, U_2, \dots, U_n \subseteq \mathbb{R}^d$ are all open, then so is $U_1 \cap U_2 \cap \dots \cap U_n$.

(iii) If $U_a \subseteq \mathbb{R}^d$ is an open set for all $\alpha \in I$, (I is some index set) then

$$\bigcup_{a \in I} U_a$$

is open.

Proof. *(i),(ii) are exercises.*

(iii): Set

$$V := \bigcup_{\alpha \in I} U_a$$

Let $a \in V$. This means $\exists \alpha \in I$ such that $a \in U_\alpha$. U_α is open so $\exists r > 0$ s.t $B(a;r) \subseteq U_\alpha$. $U_\alpha \leq \bigcup_{\alpha \in I} U_\alpha = V$ So $B(a;r) \subseteq V$ as required. □

Example: For any $n \in \mathbb{N}_{\geq 1}$.

$$\left(\frac{-1}{n}, \frac{1}{n}\right) = B(0; \frac{1}{n})$$

is open in $\mathbb{R}$. The intersection of these open sets is

$$\bigcap_{n=1}^{\infty} \left(\frac{1}{n}, \frac{-1}{n}\right) = \{0\}$$

which is not open. This shows that openess is not preserved by infinite intersections.

Example: Let

$$U := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$$

U is open but not closed.

$$U = V \cap W$$

where

$$V := \{(x, y) : x > 0\} \qquad W := \{(x, y) : y > 0\}$$

To show V is open, let $a = (x, y) \in V$. Set $r := x > 0$. Then if $(w, z) \in B(a;r)$. Then

$$|w - z| \leq ||(w, z) - a||_2 < r = x$$

$$\therefore w > x - x = 0$$

So $(w, z) \in U$. Similarly, W is open. Therefore U is open.

Not Closed: Exercise.

Proposition 10.0.2. *Let $K \subseteq \mathbb{R}^d$. K is closed $\iff$ for any subsequence $(a_n)_{n=1}^{\infty}$ in K, If it converges, then*

$$lim_{n \to \infty} a_n \in K$$

Proof. ($\implies$) Suppose K is closed. Let $(a_n)_{n=1}^{\infty}$ be a sequence in K s.t

$$L := lim_{n\to\infty} a_n$$

exists. Suppose for a contradction $L \notin K$. This means $L \in \mathbb{R}^d \setminus K$, which is open. So $\exists r > 0$ such that

$$B(L; r) \subseteq \mathbb{R}^d \setminus K$$

Since $a_n \to L$, we must have $a_n \in B(L; a)$ for some n (in fact, for all n sufficiently large. So $a_n \in B(L; r) \subseteq \mathbb{R}^d \setminus K$. Therefore $a_n \notin K$, which is a contradiction.

($\impliedby$) Suppose K is not closed, and we'll prove $\exists (a_n)_{n=1}^{\infty}$ in K such that $a_n \to L \notin K$. Since K is not closed, $\mathbb{R}^d \setminus K$ is not open. So $\exists L \in \mathbb{R}^d \setminus K$ such that $\forall r > 0$

$$B(L; r) \nsubseteq \mathbb{R}^d \setminus K$$

For each $n \in \mathbb{N}_{\geq 1}$, we can fine $a_n \in B(L; \frac{1}{n})$ such that $a_n \notin \mathbb{R}^d \setminus K$. So $a_n \in K$. This gives a sequence $(a_n)_{n=1}^{\infty}$ in K and

$$||a_n - L||_2 < \frac{1}{n} \to 0$$

Therefore by the Squeeze Theorem,

$$||a_n - L||_2 \to 0 \implies a_n \to L$$

$L \in \mathbb{R}^d \setminus K$, so $L \notin K$. ☐

Definition 10.0.4. *Let $A \subseteq \mathbb{R}^d$ and let $a \in \mathbb{R}^d$, a is:*

(i) an ***interior point*** *if $\exists r > 0$ s.t $B(a; r) \subseteq A$*

(ii) an ***accumulation point*** *if $\exists$ a sequence $(a_n)_{n=1}^{\infty}$ in A s.t $a_n \to a$*

(iii) a ***boundary point*** *if it is an accumulation point and it is not an interior point.*

$$A^\circ := \{\textit{All interior points}\}$$

$$\bar{A} := \{\textit{All accumulation points}\}$$

$$\partial A := \{\textit{All boundary points}\} = \bar{A} \setminus A^\circ$$

Note: *The set of interior points, accumulation points, and boundary points are referred to as the* ***interior*** *of A, the* ***closure*** *of A, and the* ***boundary*** *of A respectively*

Example: $A := (0, 1] \cup \{2\}$

$$A^\circ = (0, 1)$$

$$\bar{A} = [0, 1] \cup \{2\}$$

$$\partial A = \{0, 1, 2\}$$

Example: $A \coloneqq \mathbb{Q}$

Since any open interval contains irrational numbers, we have

$$A^\circ = set$$

Proposition from chapter 2,

$$\bar{A} = \mathbb{R}$$

$$\partial A = \mathbb{R}$$

Lecture 11

Compactness

Definition 11.0.1. *A set $A \subseteq \mathbb{R}^d$ is (sequentially) compact if every sequence $(a_n)_{n=1}^{\infty}$ in A has a subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to a point in A.*

Example TBC

Lecture 12

Limits of a Function of Continous Variables

A sequence is a function $\mathbb{N} \to \mathbb{R}$. Here, we'll consider function that are going from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{R}^d \to \mathbb{R}^m$).

Definition 12.0.1. *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X. $f : X \to \mathbb{R}^m$, $L \in \mathbb{R}^m$. We say the limit of f as X approaches a is L if*

$$\forall \epsilon > 0, \exists \delta > 0 \; s.t \; \forall x \in X$$

$$x \in B(a; \delta) \wedge x \neq a \implies ||f(x) - L||_2 < \epsilon$$

The idea is like the definition of convergence of a sequence, except we replace $n \geq n_0$ (which captures "n is sufficiently large") with $x \in B(a; \delta)$, $x \neq a$ (which captures "x is close to, but not equal to a). In other words, the definition says that if x is close to (but not equal to) a then $f(x)$ is close to L.

Why "not equal to"?: Often we consider the limit as x approaches a when $f(a)$ is not defined. Other times we compare the limite to $f(a)$. So we do not want to use $f(a)$ in the definition of the limit.

Notation: We write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \; as \; x \to d$$

to mean that the limit of f is L as x approaches a.

Example: $f : \mathbb{R} \to \mathbb{R}$. $f(x) \coloneqq 3x - 2$. Let $a \in \mathbb{R}$. Claim

$$\lim_{x \to a} f(x) = 3a - 2$$

Proof. Let $\epsilon > 0$. Consider

$$\begin{aligned} |f(x) - (3a-2)| &= |3x - 2 - 3a + 2| \\ &= 3|x-a| \end{aligned}$$

We want this $< \epsilon$, set $\delta \coloneqq \frac{\epsilon}{3}$. Then if $x \in B(a;\delta) = (a-\delta, a+\delta)$ (i.e $|x-a| < \delta$) then

$$|f(x) - (3a-2)| = 3|x-a| < 3\delta = \frac{3\epsilon}{3} = \epsilon$$

$\square$

Example: $g : \mathbb{R} \to \mathbb{R}$. $g(x) \coloneqq x^2$. Claim:

$$\lim_{x \to a} g(x) = a^2$$

Proof. Let $\epsilon > 0$ be given.

$$\begin{aligned} |g(x) - a^2| &= |x^2 - a^2| \\ &= |x-a||x+a| \end{aligned}$$

What happens if x is close to a? Intuitively, $|x+a|$ is close to $|a+a|$ and $|x-a|$ is small.

$$\begin{aligned} |x+a| = |x - a + a + a| &\le |x-a| + |a+a| \\ &< 2|a| + \delta && \text{(if } |x-a| < \delta) \\ &\le 2|a| + 1 && (\delta \le 1) \end{aligned}$$

Then,

$$\begin{aligned} |x^2 - a^2| = |x-a||x+a| &\le |x-a|(2|a|+1) \\ &< \delta(2|a|+1) && \text{(if } |x-a| < \delta) \\ &\le \epsilon && \text{(if } \delta \le \tfrac{\epsilon}{2|a|+1}) \end{aligned}$$

Important: Do not define δ in terms of x or δ! We can use a here since a is constant

So we set $\delta \coloneqq \min\{1, \frac{\epsilon}{2|a|+1}\}$ Then $\delta \le \frac{\epsilon}{2|a|+1}$ and $\delta \le 1$. So if $|x-a| < \delta$. Then from the work above, $|x^2 - a^2| < \epsilon$ as required. $\square$

Note: In proofs where we have $\delta - \epsilon$, we often use

$$\delta \coloneqq \min\{\dots\}$$

In proofs where we have $n_0 - \epsilon$, we often use

$$n_0 \coloneqq \max\{\dots\}$$

Proposition 12.0.1 (Uniqueness of Limits)**.** *Let $f : X \to \mathbb{R}^m$ ($X \subseteq \mathbb{R}^d$), $a \in \mathbb{R}^d$ a limit point of X, $L, L' \in \mathbb{R}^m$. If the limit of f as $x \to a$ is L and the limit of f as $x \to a$ is L', then $L = L'$*

Proof. By contradction. Suppose $L \neq L'$. So

$$||L - L'||_2 > 0$$

Set

$$\epsilon := \frac{||L - L||_2}{2} > 0$$

Since $f(x) \to L$ as $x \to a$, $\exists \delta > 0$ such that if $x \in X \cap B(a;\delta) \setminus \{a\}$, then

$$||f(x) - L||_2 < \epsilon$$

Since $f(x) \to L'$ as $x \to a$, $\exists \delta' > 0$ such that

$$x \in X \cap B(a;\delta') \setminus \{a\} \implies ||f(x) - L'||_2 < \epsilon$$

Let $\delta_0 \neq \min\{\delta, \delta'\}$. Let

$$x \in X \cap B(a;\delta_0) \setminus \{a\}$$

Then

$$x \in X \cap B(a;\delta) \setminus \{a\}$$

So,

$$\begin{aligned} ||f(x) - L||_2 &< \epsilon \\ ||f(x) - L'||_2 &< \epsilon \end{aligned}$$

So,

$$\begin{aligned} ||L - L'||_2 \leq ||L - f(x)||_2 + ||f(x) - L'|| &< \epsilon + \epsilon \\ &= ||L - L'||_2 \end{aligned}$$

And thus, a contradction. $\square$

Proposition 12.0.2 (Sequential Characterization of Limits)**.** *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$, a limit point of X. $f : X \to \mathbb{R}^m$, $L \in \mathbb{R}^m$.*

$\lim_{x \to a} f(x) = L \iff$ *for every sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \to a$, we have*

$$\lim_{n \to \infty} f(x_n) = L$$