

# MAT 2125 Lecture Notes

Last Updated:

April 8, 2023

# Contents

<b>Important Proofs for Midterm</b>	<b>3</b>
<b>1 The Real Numbers <math>\mathbb{R}</math></b>	<b>15</b>
1.1 Fields . . . . .	15
1.2 Ordered Fields . . . . .	15
1.3 Complete Ordered Fields . . . . .	16
<b>2 Completeness of <math>\mathbb{R}</math>, Absolute Value, Sequences</b>	<b>17</b>
<b>3 Convergence of Sequences</b>	<b>18</b>
<b>4 Properties of Convergence, Squeeze Theorem, Monotone Sequences</b>	<b>19</b>
<b>5 Subsequences, Cauchy Sequences</b>	<b>20</b>
<b>6 Limsup and Liminf</b>	<b>21</b>
<b>7 Series</b>	<b>22</b>
7.1 Divergence Test . . . . .	23
7.2 Convergence Tests . . . . .	23
<b>8 Ratio, Root, Alternating Series, and Integral Test, Cauchy Convergence, Topology of <math>\mathbb{R}^d</math></b>	<b>25</b>
8.1 Ratio Test . . . . .	25
8.2 Root Test . . . . .	27
8.3 Alternating Series Test . . . . .	28
8.4 Integral Test . . . . .	29
8.5 Cauchy Convergence Criterion for Series . . . . .	30
8.6 Topology of $\mathbb{R}^d$ . . . . .	30
8.6.1 Norms . . . . .	30
<b>9 <math>\mathbb{R}^d</math></b>	<b>32</b>
9.1 Convergence . . . . .	32

<b>10 Open and Closed Sets in <math>\mathbb{R}^d</math></b>	<b>36</b>
<b>11 Compactness</b>	<b>41</b>
<b>12 Limits of a Function of Continuous Variables</b>	<b>43</b>
<b>13 Continuity</b>	<b>49</b>
13.1 One-Sided Limits . . . . .	49
13.2 Continuity . . . . .	49
13.3 Properties of Continuous Functions . . . . .	52
<b>14 Properties of Continuous Functions Continued</b>	<b>53</b>
14.1 Inverses of Continuous Functions . . . . .	54
<b>15</b>	<b>57</b>

# Important Proofs for Midterm

**Theorem 1.3.13** (The Archimedean Property). *The set  $\mathbb{N}_{\geq 1}$  is not bounded above.*

*Proof.* Suppose for a contradiction that  $\mathbb{N}$  was bounded above. Then by completeness,  $a = \sup \mathbb{N}$  exists. Since  $a$  is a least upper bound,  $a - 1$  is not an upper bound, so there exists  $m \in \mathbb{N}$  such that

$$m > a - 1$$

Then since  $m \in \mathbb{N}$ , we have  $m + 1 \in \mathbb{N}$ , so

$$m + 1 > a$$

But  $a$  is an upper bound, thus a contradiction.  $\square$

**Proposition 2.2.4** (Uniqueness of Limits). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence and let  $L_1, L_2 \in \mathbb{R}$ . If*

$$\lim_{n \rightarrow \infty} a_n = L_1 \text{ and } \lim_{n \rightarrow \infty} a_n = L_2$$

*then*

$$L_1 = L_2$$

*Proof.* Suppose for a contradiction  $L_1 \neq L_2$ . We can assume without loss of generality that  $L_1 < L_2$ . Define

$$\epsilon = \frac{L_2 - L_1}{2}$$

Since  $\lim_{n \rightarrow \infty} a_n = L$ , there exists  $n_0$  such that  $\forall n \geq n_0$

$$L_1 - \epsilon < a_n < L_1 + \epsilon$$

Using the second inequality and the definition of  $\epsilon$ , we get

$$a_n < L_1 + \epsilon = L_1 + \frac{L_2 - L_1}{2} = L_1 + \frac{L_2}{2} - \frac{L_1}{2} = \frac{L_2 + L_1}{2}$$

Likewise, since  $\lim_{n \rightarrow \infty} a_n = L_2$ , there exists  $m_0$  such that for all  $n \geq m_0$ ,

$$L_2 - \epsilon < a_n < L_2 + \epsilon$$

Then from the first inequality, we get

$$a_n > L_2 - \epsilon = L_2 - \frac{L_2 - L_1}{2} = \frac{L_2 + L_1}{2}$$

So, we get that for all  $n \geq \max\{n_0, m_0\}$ ,

$$a_n > \frac{L_2 + L_1}{2} > a_n$$

Thus, a contradiction.  $\square$

**Proposition 2.2.8.** *Let  $(a_n)_{n=1}^{\infty}$  be a sequence which converges to some number  $L \in \mathbb{R}$ . Then  $(a_n)_{n=1}^{\infty}$  is bounded.*

*Proof.* Since  $\lim_{n \rightarrow \infty} a_n = L$ , set  $\epsilon := 1$ , there exists  $n_0$  such that for all  $n \geq n_0$

$$|a_n - L| < 1$$

So we have that  $\forall n \geq n_0$

$$L - 1 < a_n < L + 1$$

Now set

$$M := \max\{a_1, a_2, \dots, a_{n_0-1}, L + 1\}$$

If  $n < n_0$ , then it is amongst the set  $\{a_1, \dots, a_{n_0-1}\}$ , so  $M$  will be the max of this set. Therefore,  $\forall n < n_0$ ,  $a_n \leq M$ . Then for  $n \geq n_0$ , by the definition of the limit we know that  $a_n < L + 1$ , so we get that  $a_n < L + 1 \leq M$ . Therefore, for all values of  $n$ , the set  $\{a_n : n \in \mathbb{N}\}$  is bounded above.

Similarly for the lower bound, take

$$M := \min\{a_1, a_2, \dots, a_{n_0-1}, L - 1\}$$

If  $n < n_0$ , then it is in the set  $\{a_1, a_2, \dots, a_{n_0-1}\}$   $M'$  is at most the minimum of this set, so  $\forall n < n_0$ ,  $a_n \geq M'$ . If  $n \geq n_0$ , by the definition of the limit we know that for all  $n \geq n_0$ ,  $a_n > L - 1$ . So  $M'$  is at most  $L - 1$ . Therefore  $\forall n \geq n_0$ ,  $a_n > L - 1 \geq M'$ . Therefore, the set is bounded below and above, so it is bounded.  $\square$

**Proposition 2.3.3.** *Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  be converging sequences, if*

$$a_n \leq b_n$$

*for all  $n$ , then*

$$\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$$

*Proof.* Suppose that  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are convergent sequences with  $a_n < b_n$  for all  $n$ . Then by the definition of convergence, we have that  $\forall \epsilon > 0, \exists n_0$  such that  $\forall n \geq n_0$

$$|a_n - L_a| < \epsilon$$

Similarly for  $b_n$ , we have that  $\exists m_0$  such that  $\forall \epsilon > 0$ ,

$$|b_n - L_b| < \epsilon$$

Now suppose for a contradiction that  $L_a > L_b$ , then set  $\epsilon := \frac{L_a - L_b}{2}$ . So we have

$$L_a - \epsilon < a_n < \epsilon + L_a$$

So,

$$a_n > L_a - \epsilon = L_a - \frac{L_a - L_b}{2} = \frac{L_a + L_b}{2}$$

Similarly for  $b_n$ , we have

$$L_b - \epsilon < b_n < L_b + \epsilon$$

$$b_n < L_b + \epsilon = \frac{L_a + L_b}{2}$$

So we have  $b_n < \frac{L_b + L_a}{2} < a_n$ , but  $a_n < b_n$ . Thus, a contradiction.  $\square$

**Theorem 2.3.5** (Squeeze Theorem). *Let  $(a_n)_{n=1}^{\infty}$ ,  $(b_n)_{n=1}^{\infty}$ ,  $(c_n)_{n=1}^{\infty}$  be sequences such that*

(i)  $(a_n)_{n=1}^{\infty}$  and  $(c_n)_{n=1}^{\infty}$  converge to the same number  $L$ , and

(ii)  $a_n \leq b_n \leq c_n$  for all  $n$ . Then  $(b_n)_{n=1}^{\infty}$  also converges to  $L$ .

*Proof.* Let  $\epsilon > 0$  be given. Suppose  $a_n \leq b_n \leq c_n$   $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  converge to  $L$ , so  $\exists n_a, n_c \in \mathbb{N}$  such that for all  $n \geq n_a$

$$L - \epsilon < a_n < L + \epsilon$$

and

$$L - \epsilon < c_n < L + \epsilon$$

So

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

Therefore,

$$L - \epsilon < b_n < L + \epsilon$$

By the definition of convergence,  $(b_n)_{n=1}^{\infty}$  converges to  $L$ .  $\square$

**Theorem 2.6.1** (Cauchy Convergence Criterion). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers. Then it converges if and only if it is Cauchy.*

*Proof.* ( $\implies$ ) Assume that  $(a_n)_{n=1}^\infty$  converges, then there exists  $n_0$  such that for all  $\epsilon > 0$ ,  $\forall n \geq n_0$

$$|a_n - L| < \epsilon$$

Now take  $\frac{\epsilon}{2}$  in place of  $\epsilon$  since  $\epsilon$  is arbitrary, we have

$$|a_n - L| < \frac{\epsilon}{2}$$

Then, for  $m, n \geq n_0$ , we have

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L|$$

Since  $m, n \geq n_0$ , by the definition of convergence we have

$$|a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore,

$$|a_m - a_n| < \epsilon$$

as required. □

**Proposition 2.7.3.** *For any sequence  $(a_n)_{n=1}^\infty$ ,*

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

*Proof.* If the sequence isn't bounded, then either  $\limsup_{n \rightarrow \infty} a_n = \infty$  or  $\liminf_{n \rightarrow \infty} a_n = -\infty$ , in either case the result is trivial. So assume that the sequence is bounded. Consider the sets used to define  $\limsup$  and  $\liminf$

$$S := \{\beta : \mathbb{R} : \exists n_0 \text{ such that } a_n \leq \beta \quad \forall n \geq n_0\}$$

$$T := \{\alpha : \mathbb{R} : \exists m_0 \text{ such that } a_n \geq \alpha \quad \forall n \geq m_0\}$$

So we have  $\alpha \in T$  and  $\beta \in S$ , then for all  $n \geq \max\{n_0, m_0\}$ , we have

$$\alpha \leq a_n \leq \beta$$

Thus, we have shown that for every  $\alpha \in T$ , and every  $\beta \in S$ , we have  $\alpha \leq \beta$ . From the definition of  $\limsup$  and  $\liminf$ , we get that for any eventual lower bound  $\alpha \in T$ , it is a lower bound for the set of upper bounds  $S$ , so

$$\alpha \leq \inf T = \limsup_{n \rightarrow \infty} a_n$$

So then  $\limsup_{n \rightarrow \infty} a_n$  is an upper bound for the set of lower bounds  $T$ , so

$$\limsup_{n \rightarrow \infty} a_n \geq \sup T = \liminf_{n \rightarrow \infty} a_n$$

Therefore,

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

as required. □

**Proposition .** *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

*diverges.*

*Proof.* Consider the partial sum of the series

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

Now consider the partial sums which correspond to powers of 2,  $S_{2^N}$  for  $N \in \mathbb{N}$ . So we have the sums  $S_2, S_4, S_8, \dots$ . Now consider the sequence of partial sums

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$\frac{1}{3} > \frac{1}{4}$ , so we have that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

Continuing similarly,

$$S_8 = S_{2^3} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{4}{8} = 1 + \frac{3}{2}$$

$\vdots$

$$S_{2^N} > 1 + \frac{N}{2}$$

So, we have

$$\lim_{N \rightarrow \infty} \left(1 + \frac{N}{2}\right) = \infty$$

But,  $S_{2^N} > 1 + \frac{N}{2}$  for all  $N \in \mathbb{N}$ , so we have that the partial sums diverge. Therefore, the series diverges.  $\square$

**Proposition 3.1.7** (Divergence Test). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers. If the series*

$$\sum_{n=1}^{\infty} a_n$$

*converges, then*

$$\lim_{n \rightarrow \infty} a_n = 0$$



*Proof.* Suppose  $\sum_{n=1}^{\infty} a_n$  converges to  $L$ . Set  $L := \sum_{n=1}^{\infty} a_n$ . Consider the partial sums

$$S_N = \sum_{n=1}^N a_n$$

so  $\lim_{n \rightarrow \infty} S_N = L$ . We also have that  $\lim_{n \rightarrow \infty} S_{N-1} = L$ , since

$$\lim_{N \rightarrow \infty} S_{N-1} = \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} a_n = \sum_{n=1}^{\infty-1} a_n = \sum_{n=1}^{\infty} a_n = L$$

Then, we have that

$$S_N - S_{N-1} = \sum_{n=1}^N a_n - \sum_{n=1}^{N-1} a_n = a_N$$

So,

$$\lim_{N \rightarrow \infty} S_N - S_{N-1} = L - L = 0$$

$$\lim_{N \rightarrow \infty} S_N - \lim_{N \rightarrow \infty} S_{N-1} = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n - \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} a_n = \lim_{N \rightarrow \infty} a_N = 0$$

□

**Proposition 3.2.1** (Boundedness Test). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers. Suppose that*

- (i)  $a_n \geq 0$  for all  $n$ , and
- (ii) *There is a bound  $M \in \mathbb{R}$  on the partial sums, so that*

$$\sum_{n=1}^N a_n \leq M$$

*for all  $N \in \mathbb{N}_{\geq 1}$ .*

*Then  $\sum_{n=1}^{\infty} a_n$  converges.*

*Proof.* Since  $a_n \geq 0$ , the partial sums  $(S_N)_{N=1}^{\infty}$  satisfy

$$S_N \leq S_{N+1} \text{ for all } N.$$

In other words,  $(S_N)_{N=1}^{\infty}$  is an increasing sequence. The second condition ensures that the sequence is bounded above. Therefore, by the Monotone Convergence Criterion, it converges. Therefore,  $\sum_{n=1}^{\infty} a_n$  converges. □

**Proposition 3.2.2** (Comparison Test). *Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  be sequences of real numbers such that*

$$0 \leq a_n \leq b_n \text{ for all } n$$

*Then,*

(i) *if  $\sum_{n=1}^{\infty} b_n$  converges, then so does  $\sum_{n=1}^{\infty} a_n$*

(ii) *if  $\sum_{n=1}^{\infty} a_n$  diverges, then so does  $\sum_{n=1}^{\infty} b_n$*

*Proof.* Since the sequence  $\sum_{n=1}^{\infty} b_n$  converges, take  $M := \sum_{n=1}^{\infty} b_n$ . Then, we have the sequence of partial sums

$$\left( \sum_{n=1}^{\infty} b_n \right)_{n=1}^{\infty}$$

is increasing and converges to  $M$ , so  $M$  is the supremum of this sequence, therefore

$$\sum_{n=1}^N b_n \leq M$$

for all  $M$ . Therefore

$$\sum_{n=1}^N a_n \leq \sum_{n=1}^N b_n \leq M$$

Therefore, by the Boundedness test,  $\sum_{n=1}^{\infty} a_n$  converges. (ii) is the contrapositive of (i) so it follows that it holds.  $\square$

**Proposition 3.2.3** (Absolute Convergence Test). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers. If the series*

$$\sum_{n=1}^{\infty} |a_n|$$

*converges, then so does*

$$\sum_{n=1}^{\infty} a_n$$

*Proof.* Assume  $\sum_{n=1}^{\infty} |a_n|$  converges, Write

$$(a_n)_+ = \max\{a_n, 0\}$$

$$(a_n)_- = \max\{-a_n, 0\}$$

So  $(a_n)_+$  is all the positive terms from  $a_n$  and  $(a_n)_-$  is all the negative terms from  $a_n$ , but we are negating them so that they are positive, so we have

$$a_n = (a_n)_+ - (a_n)_-$$

Then, we have that

$$0 \leq (a_n)_+ \leq |a_n|$$

So, by the Comparison Test, we have that  $|a_n|$  converges so  $\sum_{n=1}^{\infty} (a_n)_+$  converges.

Similarly,

$$0 \leq (a_n)_- \leq |a_n|$$

Therefore by the Comparison Test,  $\sum_{n=1}^{\infty} (a_n)_-$  converges. So by linearity,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n)_+ - \sum_{n=1}^{\infty} (a_n)_-$$

converges. □

**Proposition 4.2.3.** Let  $(a_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{R}^d$ , with

$$a_n = (a_n^{(1)}, \dots, a_n^{(d)}) \text{ for each } n \in \mathbb{N}$$

and let  $L = (L_1, \dots, L_d) \in \mathbb{R}^d$ . Then

$$\lim_{n \rightarrow \infty} a_n = L$$

if and only if, for each  $i = 1, \dots, d$ ,

$$\lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

*Proof.* ( $\implies$ ) Assume that  $\lim_{n \rightarrow \infty} a_n = L$ . Then, for each  $i = 1, \dots, d$ , we have that  $|x_i|^2 \leq \sum_{i=1}^d x_i^2 = \|x\|_2^2$ , therefore

$$|x_i| \leq \|x\|_2$$

Using this fact, we then have each component of  $\|a_n - L\|_2$  is less than or equal to it. So

$$\begin{aligned} |a_n^{(i)} - L_i| &\leq \|a_n - L\|_2 \\ -\|a_n - L\|_2 &\leq a_n^{(i)} - L_i \leq \|a_n - L\|_2 \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} a_n = L$ , we have  $\lim_{n \rightarrow \infty} a_n - L = 0$ . By the Squeeze theorem, it follows that

$$\lim_{n \rightarrow \infty} a_n^{(i)} - L_i = 0 \implies \lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

( $\impliedby$ ) Suppose for each  $i = 1, \dots, d$ , we have

$$\lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

Then, from the definition of  $\|\cdot\|_2$ , we have

$$\|a_n - L\|_2^2 = (a_n^{(1)} - L_1)^2 + \cdots + (a_n^{(d)} - L_d)^2$$

Now taking limits of both sides

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2^2 = \lim_{n \rightarrow \infty} (a_n^{(1)} - L_1)^2 + \cdots + \lim_{n \rightarrow \infty} (a_n^{(d)} - L_d)^2$$

Now we'll prove exercise 2.2.5 which states that if  $(a_n)_{n=1}^\infty$  is a sequence of non-negative real number converging to  $L \geq 0$ , then  $\lim_{n \rightarrow \infty} \sqrt{a_n}$  converges to  $\sqrt{L}$ . To prove this we will consider two cases where  $L = 0$ , and  $L > 0$ .

- **Case 1,  $L = 0$ :** Suppose  $(a_n)_{n=1}^\infty \rightarrow 0$ , then from the definition of convergence we have that  $\forall \epsilon > 0, \exists n_0$  such that  $\forall n \geq n_0$ ,

$$|a_n - 0| < \epsilon$$

Since  $\epsilon$  is arbitrary, we'll replace  $\epsilon$  with  $\epsilon^2$ , so

$$|a_n - 0| < \epsilon^2$$

Then we get

$$|a_n - 0| = |a_n| < \epsilon^2 \implies \sqrt{|a_n|} < \epsilon$$

Therefore,  $\sqrt{a_n} \rightarrow 0$  by the definition of convergence.

- **Case 2,  $L > 0$ :** Suppose  $(a_n)_{n=1}^\infty \rightarrow L > 0$ . Let  $\epsilon > 0$  be given, then there exists  $n_0$  such that for all  $n \geq n_0$ ,

$$|a_n - L| < \epsilon$$

We much such that  $|\sqrt{a_n} - \sqrt{L}| < \epsilon$

$$|\sqrt{a_n} - \sqrt{L}| \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} = \frac{|(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})|}{\sqrt{a_n} + \sqrt{L}}$$

Since  $\sqrt{a_n} + \sqrt{L}$  is positive because  $a_n, L \geq 0$ , then  $\sqrt{a_n} + \sqrt{L} = |\sqrt{a_n} + \sqrt{L}|$ , then using the fact that  $|a| \cdot |b| = |a \cdot b|$ , we get

$$\frac{|a_n - \sqrt{L}\sqrt{a_n} + \sqrt{L}\sqrt{a_n} + L|}{\sqrt{a_n} + \sqrt{L}} = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}}$$

Now if we replace  $\epsilon$  with  $\frac{\epsilon}{\sqrt{L}}$ , we get

$$|\sqrt{a_n} - \sqrt{L}| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon}{\sqrt{L}} \implies |\sqrt{a_n} - \sqrt{L}| < \epsilon$$

Therefore,  $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{L}$

Now going back to the original proof,

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2^2 = 0$$

So from exercise 2.2.5 we have

$$\lim_{n \rightarrow \infty} \sqrt{\|a_n - L\|_2^2} = \sqrt{0}$$

Therefore,

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2 = 0$$

as required.  $\square$

**Theorem 4.2.2** (Cauchy Covergence  $\mathbb{R}^d$ ). *Let  $(a_n)_{n=1}^\infty$  be a sequence  $\mathbb{R}^d$ . Then it converges if it converges if and only if it is cauchy.*

*Proof.* Suppose  $(a_n)_{n=1}^\infty$  is a sequence in  $\mathbb{R}^d$  that converges to  $L \in \mathbb{R}^d$ . Let  $\epsilon > 0$  be given, then there exists  $n_0$  such that  $\forall m, n \geq n_0$ ,

$$\|a_n - L\|_2 < \epsilon$$

$$\|a_m - L\|_2 < \epsilon$$

Since  $\epsilon$  arbitrary we can replace  $\epsilon$  with  $\frac{\epsilon}{2}$ , so

$$\|a_n - L\|_2 = \frac{\epsilon}{2} \text{ and } \|a_m - L\|_2 = \frac{\epsilon}{2}$$

So,

$$\begin{aligned} \|a_m - a_n\|_2 &= \|a_m - L + L - a_n\|_2 \leq \|a_m - L\|_2 + \|L - a_n\|_2 \\ &= \|a_m - L\|_2 + \|a_n - L\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\square$

**Proposition 4.3.4.** *Given  $a \in \mathbb{R}^d$ , and  $r > 0$ , the open ball  $B(a, r)$  is an open set.*

**Note: This is example 4.3.4 from the professors notes.**

*Proof.* Recall the definition of an open set is that for any  $x$  in the set, we can define an open ball (or epsilon neighborhood) around  $x$  such that the ball is contained in the set. So we want an open ball  $B(x; \epsilon)$  such that  $B(x; \epsilon) \subseteq B(a; r)$ . To see this, let  $x \in B(a; r)$ , so that  $\|x - a\|_2 < r$ . Define

$$\epsilon := r - \|a - x\|_2 > 0$$

Now take some element  $y \in B(x; \epsilon)$ , then we want to show that tis element is contained in  $B(a; r)$ . So,  $y \in B(x; \epsilon)$ , so that  $\|y - x\|_2 < \epsilon$ . Then,

$$\begin{aligned} \|y - a\|_2 &= \|y - x + x - a\|_2 \leq \|y - x\|_2 + \|x - a\|_2 \\ &< \epsilon + \|a - x\|_2 = r \end{aligned}$$

Therefore,

$$\|y - a\|_2 < r$$

So  $y \in B(a; r)$  as required, so  $B(a; r)$  is an open set.  $\square$

**Proposition 4.3.5.** (i) The sets  $\emptyset, \mathbb{R}^d$  are open

(ii) For any finite collection of open sets,  $U_1, \dots, U_m \subseteq \mathbb{R}^d$ , their intersection is

$$U_1 \cap \dots \cap U_m$$

is open

(iii) For any arbitrary collection of open sets  $\{U_\alpha : \alpha \in I\}$ , their union,

$$\bigcup_{\alpha \in I} U_\alpha$$

is open.

*Proof.* (i) (i) and (ii) are Exercise 4.3.1

(ii) Will add them later!

(iii) Set

$$U := \bigcup_{\alpha \in I} U_\alpha$$

Since  $U_\alpha$  is open, there is some  $\epsilon > 0$  such that

$$B(x; \epsilon) \subseteq U_\alpha$$

Then since  $U$  is the union of all the  $U_\alpha$ , we have that  $U_\alpha \subseteq U$  so it follows that

$$B(x; \epsilon) \subseteq U$$

as required.  $\square$

**Theorem 4.4.5** (Heine-Borel Theorem). Let  $K$  be a subset of  $\mathbb{R}^d$ . Then  $K$  is compact if and only if  $K$  is closed and bounded.

*Proof.* ( $\implies$ ) Suppose that  $K$ . To see that  $K$  is closed, suppose for a contradiction that it is not closed. By proposition 4.3.9,  $F$  is closed if and only if for every sequence  $(a_n)_{n=1}^\infty$  in  $F$ , if  $(a_n)_{n=1}^\infty$  converges then

$$\lim_{n \rightarrow \infty} a_n \in F$$

So, if  $K$  is not closed, then it follows that there exists some sequence  $(a_n)_{n=1}^\infty$  such that

$$\lim_{n \rightarrow \infty} a_n \notin K$$

Then by proposition 2.5.4, if a sequence  $(a_n)_{n=1}^\infty$  converges, then all subsequences of the sequence converge to the same point, and hence no subsequence converges to a point in  $K$ . This contradicts the fact that  $K$  is compact. Similarly for the boundness of  $K$ , suppose for a contradiction that  $K$  was not bounded. Then for any  $n \in \mathbb{N}$ , there exists  $a_n \in K$  such that  $\|a_n\|_2 \geq n$ . So the sequence  $(a_n)_{n=1}^\infty$  is unbounded, as well as all subsequences  $(a_{n_k})_{k=1}^\infty$ . Therefore, no subsequence converge since they are all unbounded. This contradicts the fact that  $K$  is compact.

(  $\Leftarrow$  ) Assume  $K$  is closed and bounded. Since  $K$  is bounded, any sequence  $(a_n)_{n=1}^\infty$  in  $K$  is bounded. Then by the Bolzano-Weierstrass theorem,  $(a_n)_{n=1}^\infty$  is bounded so there exists a convergent subsequence  $(a_{n_k})_{k=1}^\infty$  that converges to some  $L \in \mathbb{R}^d$ . Since  $K$  is closed, from proposition 4.3.9 we have that

$$\lim_{k \rightarrow \infty} a_{n_k} \in K$$

Therefore every sequence  $(a_n)_{n=1}^\infty$  has a subsequence  $(a_{n_k})_{k=1}^\infty$  that converges to some  $L \in K$ , so  $K$  is compact.  $\square$

# Lecture 1

## The Real Numbers $\mathbb{R}$

**Summary:**  $\mathbb{R}$  is a complete ordered field.

### 1.1 Fields

**Definition 1.1.1.** A field is a set  $F$  together with operations  $+$ ,  $\cdot$  satisfying

- (F1)  $a + b = b + a \ \forall a, b \in F$  (Commutativity)
- (F2)  $(a + b) + c = a + (b + c) \ \forall a, b, c \in F$  (Associativity)
- (F3)  $\exists 0 \in F$  s.t.  $0 + a = a \ \forall a \in F$  (Additive Identity)
- (F4)  $\exists -a \in F$  s.t.  $a + (-a) = 0 \ \forall a \in F$  (Additive Inverse)
- (F5)  $a \cdot b = b \cdot a \ \forall a, b \in F$  (Commutativity)
- (F6)  $(a \cdot b) \cdot c = a \cdot (b \cdot c) \ \forall a, b, c \in F$  (Associativity)
- (F7)  $\exists 1 \in F$  s.t.  $1 \cdot a = a \ \forall a \in F$  (Multiplicative Identity)
- (F8)  $\forall a \in F \setminus \{0\} \ \exists a^{-1} \in F$  s.t.  $a^{-1} \cdot a = 1$  (Multiplicative Inverse)
- (F9)  $a \cdot (b + c) = a \cdot b + a \cdot c \ \forall a, b, c \in F$  (Distributivity)

### 1.2 Ordered Fields

**Definition 1.2.1.** An ordered field is a field  $F$  along with a relation  $<$  satisfying

- (O1)  $\forall a, b, c \in F$ , if  $a < b$  and  $b < c$  then  $a < c$  (Transitivity)
- (O2)  $\forall a, b \in F$  exactly one of the following is true,

$$a < b \text{ or } a = b \text{ or } b < a$$



- (O3)  $\forall a, b, c \in F$ , if  $a < b$ , then  $a + c < b + c$
- $\forall a, b, c \in F$ , If  $a < b$  and  $0 < c$ , then  $ac < bc$

### 1.3 Complete Ordered Fields

**Definition 1.3.1.** Let  $F$  be an ordered field. Let  $S \subseteq F$ . An upper bound of  $S$  is some  $M \in F$  s.t  $\forall x \in S$

$$x \leq M$$

## Lecture 2

# Completeness of $\mathbb{R}$ , Absolute Value, Sequences

TBC.

## Lecture 3

# Convergence of Sequences

TBC.

## Lecture 4

# Properties of Convergence, Squeeze Theorem, Monotone Sequences

TBC.

## Lecture 5

# Subsequences, Cauchy Sequences

TBC.

## Lecture 6

# Limsup and Liminf

TBC.

# Lecture 7

## Series

Recall:

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n$$

$\sum_{n=1}^{\infty} a_n$  "diverges" if above limit does not exist.

**Proposition 7.0.1.** Suppose  $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$  converges. Then

(i)

$$\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(ii)

$$\sum_{n=1}^{\infty} cb_n = c \sum_{n=1}^{\infty} b_n \quad \forall c \in \mathbb{R}$$

This says

$$V := \{(a_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} a_n \text{ converges}\}$$

is a vector space over  $\mathbb{R}$ .

**Note:**

$$\left( \sum_{n=1}^N a_n \right) \left( \sum_{n=1}^N b_n \right) \neq \sum_{n=1}^{\infty} a_n b_n$$

*Proof.* **Exercise.**

□

**Proposition 7.0.2.**  $\sum_{n=1}^{\infty} a_n$  converges iff  $\sum_{n=k}^{\infty} a_k$  converges.

*Proof.* **Exercise.**

□

**Example:** TBC.

**Proposition 7.0.3.** *If  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 0$ .*

## 7.1 Divergence Test

**Proposition 7.1.1** (Divergence Test). *If  $a_n \not\rightarrow 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.*

*Proof.*  $\sum_{n=1}^{\infty} a_n$  converges  $\implies S_n \rightarrow L$  for some  $L$ , where

$$S_n := \sum_{n=1}^N a_n$$

So,

$$a_n = S_n - S_{n-1}$$

By the Algebra of Limits,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} \\ &= L - L = 0 \end{aligned}$$

□

**Example:** TBC.

## 7.2 Convergence Tests

**Proposition 7.2.1** (Boundedness Test). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence, if*

(i)  $a_n \geq 0$

(ii) *There is an upper bound on the partial sums*

$$\exists M > 0 \text{ s.t. } \sum_{n=1}^N a_n \leq M$$

*Then  $\sum_{n=1}^{\infty} a_n$  converges.*

*Proof.* Let

$$S_N := \sum_{n=1}^N a_n$$



Then

$$\begin{aligned} S_{N+1} &= S_N + a_{N+1} \\ &\geq S_n \end{aligned}$$

So by the Monotone Convergence Criterion,  $(S_N)_{N=1}^{\infty}$  converges  $\iff$  it is bounded above. By (ii), it is bounded.  $\square$

**Proposition 7.2.2** (Comparison Test). *TBC.*

## Lecture 8

# Ratio, Root, Alternating Series, and Integral Test, Cauchy Convergence, Topology of $\mathbb{R}^d$

### 8.1 Ratio Test

**Proposition 8.1.1** (Ratio Test). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of nonzero elements.*

(i) *If*

$$\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

*then  $\sum_{n=1}^{\infty} a_n$  converges absolutely.*

(ii) *If*

$$\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

*then  $\sum_{n=1}^{\infty} a_n$  diverges.*

*Proof.*

(i) Let

$$q = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

where  $q < 1$ ,  $\exists r \in (q, 1)$ . By the definition of  $\limsup$ ,

$$\left| \frac{a_{n+1}}{a_n} \right| \leq r \quad \forall n \geq n_0$$

for some  $n_0 \in \mathbb{N}$ .

$$\begin{aligned} \left| \frac{a_{n_0+1}}{a_{n_0}} \right| &\leq r \\ |a_{n_0+1}| &\leq r|a_{n_0}| \\ |a_{n_0+2}| &\leq r|a_{n_0+1}| \leq r^2|a_{n_0}| \end{aligned}$$

By induction, we have

$$0 \leq |a_{n_0+k}| \leq r^k |a_{n_0}|$$

By the comparison test,

$$\sum_{k=1}^{\infty} |a_{n_0+k}|$$

converges since

$$\sum_{k=1}^{\infty} r^k |a_{n_0}|$$

is a geometric sequence and  $0 \leq r < 1$ .

$$\sum_{n=1}^{\infty} |a_n|$$

Converges, thus  $\sum_{n=1}^{\infty} a_n$  converges absolutely.

(ii) Let

$$q = \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

so  $\exists n_0$  such that

$$\left| \frac{a_{n+1}}{a_n} \right| > 1$$

for all  $n \geq n_0$ . Then

$$|a_{n_0+k}| \geq |a_{n_0}| \quad \forall k \geq 0$$

So  $a_n \not\rightarrow 0$  as  $n \rightarrow \infty$ , thus by the divergence test.  $\sum_{n=1}^{\infty} a_n$  diverges.

□

**Note: The ratio test does not tell us anything when the limit is 1.**

**For example:**

$$\sum_{n \geq 1} \frac{1}{n}$$

diverges, but

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \rightarrow 1$$

But on the other hand,

$$\sum_{n \geq 1} \frac{1}{n+1}$$

converges, and

$$\frac{\frac{1}{n+1(n+2)}}{\frac{1}{n(n+1)}} = \frac{n+1}{n+2} \rightarrow 1$$

## 8.2 Root Test

**Proposition 8.2.1** (Root Test). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers.*

(i) *If*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$$

*then  $\sum_{n=1}^{\infty} a_n$  converges absolutely.*

(ii) *If*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$$

*then  $\sum_{n=1}^{\infty} a_n$  diverges.*

*Proof.*

(i) Let

$$q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$$

Then there exists  $r \in (q, 1)$  such that  $\exists n_0$

$$\sqrt[n]{|a_n|} \leq r$$

for all  $n \geq n_0$ . Then,

$$0 \leq |a_n| \leq r^n$$

Therefore  $\sum_{n=n_0}^{\infty} r^n$  converges since  $0 < r < 1$ . Then by the comparison test,  $\sum_{n=n_0}^{\infty} |a_n|$  converges.

(ii) Let

$$q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$$

From exercise 2.7.3, there are infinitely many

$$\sqrt[n]{|a_n|} \geq 1 \implies |a_n| \geq 1$$

Thus  $a_n \not\rightarrow 0$  as  $n \rightarrow \infty$ , thus by the divergence test,  $\sum_{n=1}^{\infty} a_n$  diverges.

□

**Examples:**

•

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

$$\sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

•

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$$

$$\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \frac{1}{2n+1} \rightarrow \frac{1}{2}$$

•

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\sqrt[n]{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} = \frac{1}{e^{\frac{\ln n}{n}}} \rightarrow 1$$

## 8.3 Alternating Series Test

**Proposition 8.3.1** (Alternating Series Test). *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers. Suppose*

(i)  $(a_n)_{n=1}^{\infty}$  *is decreasing*

(ii)  $\lim_{n \rightarrow \infty} a_n = 0$

*Then*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

*converges. Moreover, for any  $N$ ,*

$$\sum_{n=1}^{2N} (-1)^{n+1} a_n \leq \sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq \sum_{n=1}^{2N-1} (-1)^{n+1} a_n$$

*Proof.*  $a_n \geq 0 \forall n$ , since  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ , and  $(a_n)_{n=1}^\infty$  is decreasing. Let

$$S_N := \sum_{n=1}^N (-1)^{n+1} a_n$$

If  $N$  is even,

$$S_{N+2} = S_N + a_{N+1} - a_{N+2} \geq S_N$$

Since

$$a_{N+2} \leq a_{N+1}$$

So  $(S_{2N})_{N=1}^\infty$  is an increasing sequence and  $(S_{2N-1})_{N=1}^\infty$  is a decreasing sequence. So by the monotone convergence criterion, both sequence converge.

$$S_{2N-1} + a_{2N} = S_{2N}$$

$$a_{2N} \xrightarrow{N \rightarrow \infty} 0$$

So,

$$\lim_{n \rightarrow \infty} S_{2N-1} = \lim_{N \rightarrow \infty} S_{2N} = L \implies \lim_{N \rightarrow \infty} S_N = L$$

$(S_{2N})_{N=1}^\infty$  is increasing, so

$$L = \sup\{(S_{2N})_{N=1}^\infty\} \implies S_{2N} \leq L$$

Similarly,  $(S_{2N-1})_{N=1}^\infty$  is decreasing, so

$$L = \inf\{(S_{2N-1})_{N=1}^\infty\} \implies S_{2N-1} \geq L$$

□

## 8.4 Integral Test

**Proposition 8.4.1.** *Let  $f : [1, \infty) \rightarrow \mathbb{R}$ . Suppose that*

(i)  $f(x) \geq 0 \forall x \in [1, \infty)$

(ii)  $f$  is decreasing

*Then*

$\sum_{n=1}^\infty f(n)$  converges  $\iff$  the improper integral  $\int_1^\infty f(x) dx$  converges.

## 8.5 Cauchy Convergence Criterion for Series

**Proposition 8.5.1.** *Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \forall \epsilon > 0 \exists N_0 \text{ such that } N \leq M \leq N_0, |\sum_{n=M}^N a_n| < \epsilon.$$

*Proof.* Let

$$S_N := \sum_{n=1}^N a_n$$

Then  $\sum_{n=1}^{\infty} a_n$  converges  $\iff (S_N)_{N=1}^{\infty}$  converges. Cauchy convergence criterion for sequences says that

$$\begin{aligned} (S_N)_{N=1}^{\infty} \text{ converges} &\iff \text{it is cauchy.} \\ &\iff \forall \epsilon > 0 \exists N_0 \text{ s, t } |S_N - S_M| < \epsilon \forall N, M \geq N_0 \end{aligned}$$

□

## 8.6 Topology of $\mathbb{R}^d$

### 8.6.1 Norms

**Definition 8.6.1.** *A norm on  $\mathbb{R}^d$  is a function  $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$  satisfying the following properties:*

- (i)  $\|a\| = 0 \iff a = (0, \dots, 0)$
- (ii)  $\|ca\| = |c| \cdot \|a\| \forall c \in \mathbb{R}, a \in \mathbb{R}^d$
- (iii)  $\|a + b\| \leq \|a\| + \|b\| \forall a, b \in \mathbb{R}^d$

The euclidean norm of  $\mathbb{R}^d$  is given by

$$\|a, \dots, a_d\|_2 = \sqrt{\sum_{i=1}^d a_i^2}$$

The dot product on  $\mathbb{R}^d$  is given by

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = \sum_{i=1}^d a_i b_i$$

We also have the  $l_1$ -norm

$$\|(a_1, \dots, a_n)\|_1 = \sum_{i=1}^d |a_i|$$

And the  $l_{\infty}$ -norm

$$\|(a_1, \dots, a_d)\|_{\infty} = \max\{|a_1|, \dots, |a_d|\}$$

**Proposition 8.6.1.** Let  $a, b \in \mathbb{R}^d$ , write  $\|\cdot\|$  for the euclidean norm on  $\mathbb{R}^d$ . Then

(i) **Cauchy Schwarz Inequality:**

$$|a \cdot b| \leq \|a\| \cdot \|b\|$$

(ii) **Triangle Inequality:**

$$\|a + b\| \leq \|a\| + \|b\|$$

(iii)  $\|\cdot\|$  is a norm on  $\mathbb{R}^d$

*Proof.*

(i) Consider the quadratic function

$$\begin{aligned} P(t) &= \|a + tb\|^2 \\ &= (a + tb) \cdot (a + tb) \\ &= a \cdot a + 2a \cdot tb + tb \cdot tb \\ &= \|a\|^2 + 2t(a \cdot b) + t^2\|b\|^2 \end{aligned}$$

The discriminant of  $P(t)$  is less than or equal to 0,

$$\begin{aligned} (2a \cdot b)^2 - 4\|a\|^2\|b\|^2 &\leq 0 \\ (a \cdot b)^2 &\leq \|a\|^2\|b\|^2 \\ |a \cdot b| &\leq \|a\| \cdot \|b\| \end{aligned}$$

(ii)

$$\begin{aligned} \|a + b\|^2 &= \|a\|^2 + 2a \cdot b + \|b\|^2 \\ &\leq \|a\|^2 + 2\|a\| \cdot \|b\| + \|b\|^2 \\ &\leq \|a\|^2 + 2\|a\| \cdot \|b\| + \|b\|^2 \\ &= (\|a\| + \|b\|)^2 \end{aligned}$$

$$\|a + b\|^2 \leq (\|a\| + \|b\|)^2 \implies \|a + b\| \leq \|a\| + \|b\|$$

(iii) **Exercise.** We want to prove  $\|a\| = 0 \iff a = 0$  and  $\|ca\| = |c| \cdot \|a\|$ .

□



# Lecture 9

$\mathbb{R}^d$

**Recall:**  $\|(x_1, \dots, x_d)\| := \sqrt{x_1^2 + \dots + x_d^2}$ . This is a norm. i.e.

$$\|a + b\| \leq \|a\|_2 + \|b\|_2 \quad \forall a, b \in \mathbb{R}^d$$

$$\|ca\| = |c| \cdot \|a\|_2, \quad c \in \mathbb{R}, \quad a \in \mathbb{R}^d$$

$$\|a\|_2 > 0, \quad \forall a \in \mathbb{R}^d \setminus \{(0, \dots, 0)\}$$

$$\|(0, \dots, 0)\|_2 = 0$$

Other examples of norms:

- $\|(x_1, \dots, x_d)\|_1 := |x_1| + \dots + |x_d|$
- $\|(x_1, \dots, x_d)\|_\infty := \max\{|x_1|, \dots, |x_d|\}$

**Exercise:** For  $a \in \mathbb{R}^d$ ,

$$\|a\|_\infty \leq \|a\|_2 \leq \|a\|_1 \leq d\|a\|_\infty (\leq d\|a\|_2)$$

**Interesting Fact:** There are other norms. but they are all equivalent in the sense that if  $\|\cdot\|, \|\cdot\|'$  are norms on  $\mathbb{R}^d$ , then  $\exists v, R > 0$  such that

$$r\|a\| \leq \|a\|' \leq R\|a\|$$

## 9.1 Convergence

**Definition 9.1.1.** Let  $(a_n)_{n=1}^\infty$  be a sequence in  $\mathbb{R}^d$  and let  $L \in \mathbb{R}^d$ , we say  $(a_n)_{n=1}^\infty$  **converges** to  $L$ , and write  $\lim_{n \rightarrow \infty} a_n = L$  or  $a_n \rightarrow \infty$ , if

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2 = 0$$

*Note: We could define convergence instead using some other norm, say  $\|\cdot\|_1$ .*

If  $\|a_n - L\|_2 \rightarrow 0$ , then  $\|a_n - L\|_1 \leq d\|a_n - L\|_2 \rightarrow 0$ . If  $\|a_n - L\|_1 \rightarrow 0$ , then  $\|a_n - L\|_2 \leq d\|a_n - L\|_1 \rightarrow 0$ .

in general, if  $\|\cdot\|$  is any norm, then since  $\|\cdot\|$  and  $\|\cdot\|_2$  are equivalent.

$$\|a_n - L\|_2 \rightarrow 0 \iff \|a_n - L\| \rightarrow 0$$

**Example:** Say  $a_n = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ , then

$$\|a_n - L\|_2 = \sqrt{1/n^2 + \dots + 1/n^2} = \sqrt{\frac{d}{n^2}} = \frac{\sqrt{d}}{n} \rightarrow 0$$

$$\therefore a_n \rightarrow L$$

Given a sequence  $(a_n)_{n=1}^\infty$  in  $\mathbb{R}^d$ , we write  $a_1 = (a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)})$  where  $a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)} \in \mathbb{R}$ . Similarly,

$$a_1 = (a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)})$$

$$a_2 = (a_2^{(1)}, a_2^{(2)}, \dots, a_2^{(d)})$$

$$a_3 = (a_3^{(1)}, a_3^{(2)}, \dots, a_3^{(d)})$$

$$\vdots$$

$$L = (L^{(1)}, L^{(2)}, \dots, L^{(d)}) \in \mathbb{R}^d$$

We get  $d$  sequences in  $\mathbb{R}$ , and  $d$  possible limit points  $L^{(1)}, \dots, L^{(d)} \in \mathbb{R}$

**Proposition 9.1.1.** *Given  $(a_n)_{n=1}^\infty$  and  $L$  as above,  $a_n \rightarrow L$  as  $n \rightarrow \infty \iff a_n^{(i)} \rightarrow L^{(i)}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , for  $i = 1, \dots, d$ .*

*Proof.*  $\implies$ : Suppose  $a_n \rightarrow L$ , i.e.

$$\|a_n - L\|_2 \rightarrow 0$$

For  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$\|x\|_2^2 = x_1^2 + \dots + x_d^2 \geq x_i^2 = |x_i|^2 \therefore |x_i| \leq \|x\|_2$$

Applying this to (\*), we get

$$0 \leq |a_n^{(i)} - L^{(i)}| \leq \|a_n - L\|_2 \rightarrow 0$$

So by the squeeze theorem,

$$|a_n^{(i)} - L^{(i)}| \rightarrow 0$$

$\implies$  : Suppose  $a_n^{(i)} \rightarrow L^{(i)}$  for  $i = 1, \dots, d$ .

$$\|a_n - L\|_2^2 = (a_n^{(1)} - L^{(1)})^2 + \dots + (a_n^{(d)} - L^{(d)})^2 \rightarrow 0$$

By algebra of limits,

$$\therefore \|a_n - L\|_2 \rightarrow 0$$

□

**Example:**  $a^n = ((-1)^n, \frac{1}{n}) \in \mathbb{R}^2$ . Does  $(a_n)$  converge? No, since  $(-1)^n$  does not converge.

**Definition 9.1.2.** A sequence  $(a_n)_{n=1}^\infty$  in  $\mathbb{R}^d$  is **Cauchy** if  $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}_{\geq 1}$  such that

$$\|a_n - a_m\|_2 < \epsilon \quad \forall m, n \geq n_0$$

**Theorem 9.1.1** (Cauchy Convergence Criterion for  $\mathbb{R}^d$ ). Let  $(a_n)_{n=1}^\infty$  be a sequence in  $\mathbb{R}^d$ . It converges  $\iff$  it is Cauchy.

*Proof.*  $\implies$  : Suppose  $a_n \rightarrow L \in \mathbb{R}^d$ . To show it is Cauchy, let  $\epsilon > 0$ .  $\|a_n - L\|_2 \rightarrow 0$ , so  $\exists n_0 \in \mathbb{N}_{\geq 1}$  such that

$$\|a_n - L\|_2 < \frac{\epsilon}{2} \quad \forall n \geq n_0$$

Then if  $m, n \geq n_0$ ,

$$\begin{aligned} \|a_m - a_n\|_2 &= \|a_m - L + L - a_n\|_2 \\ &\leq \|a_m - L\|_2 + \|L - a_n\|_2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\therefore (a_n)_{n=1}^\infty$  is Cauchy.  $\implies$  : Suppose  $(a_n)_{n=1}^\infty$  is Cauchy, write

$$a_n = (a_n^{(1)}, \dots, a_n^{(d)})$$

For any  $m, n \in \mathbb{N}_{\geq 1}$ ,

$$|a_n^{(i)} - a_m^{(i)}| \leq \|a_n - a_m\|_2$$

$\therefore (a_n^{(i)})_{n=1}^\infty$  is Cauchy in  $\mathbb{R}$ . So by the Cauchy Convergence Criterion,  $\exists L^{(i)} \in \mathbb{R}$ , such that  $a_n^{(i)} \rightarrow L^{(i)}$ . By the previous proposition,

$$a_n \rightarrow (L^{(1)}, \dots, L^{(d)})$$

□

**Definition 9.1.3.**  $S \subseteq \mathbb{R}^d$  is **bounded** if  $\exists M > 0$  such that

$$\|x\| \leq M \quad \forall x \in S$$

A sequence  $(a_n)_{n=1}^\infty$  in  $\mathbb{R}^d$  is bound if  $\{a_n : n \in \mathbb{N}_{\geq 1}\}$  is a bounded set.

**Theorem 9.1.2** (Bolzano-Weierstrass for  $\mathbb{R}^d$ ). *If  $(a_n)_{n=1}^\infty$  is a bounded sequence in  $\mathbb{R}^d$ , then it has a subsequence  $(a_{n_k})_{n=1}^\infty$  that converges.*

*Proof.* Write  $a_n = (a_n^{(1)}, \dots, a_n^{(d)})$ .

We will prove it by induction on  $d$ . For  $d = 1$ , this is the Bolzano-Weierstrass theorem for  $\mathbb{R}$ . For  $d > 1$  write

$$b_n := (a_n^{(1)}, \dots, a_n^{(d-1)}) \in \mathbb{R}^{d-1}$$

By the induction hypothesis,  $b_n$  has a subsequence  $(b_{n_k})_{k=1}^\infty$  that converges. Let  $L \in \mathbb{R}^{d-1}$  be the limit of this subsequence. Then  $L \in \mathbb{R}^{d-1}$  is the limit of  $b_n$ .  $(a_{n_k}^{(d)})_{k=1}^\infty$  is a bounded sequence in  $\mathbb{R}$ , so it has a subsequence  $(a_{n_{k_j}}^{(d)})_{j=1}^\infty$  that converges. Let  $L^{(d)} \in \mathbb{R}$  be the limit of this subsequence. Then  $L = (L^{(1)}, \dots, L^{(d-1)}, L^{(d)})$  is the limit of  $a_n$ .  $\square$

## Lecture 10

# Open and Closed Sets in $\mathbb{R}^d$

Roughly, an open set is one that we draw with dotted lines. The line represents a "boundary" that is not in the set. This is not a rigorous definition.

**Definition 10.0.1** (Open Ball). *Let  $a \in \mathbb{R}^d$ ,  $r > 0$ . The **open ball** of radius  $r$  centered at  $a$  is*

$$B(a; r) := \{x \in \mathbb{R}^d : \|x - a\|_2 < r\}$$

**Relation to Convergence:** If  $a_n \rightarrow L$ , then this means that  $\|a_n - L\|_2 < \epsilon$  for all  $n$  large. So,  $a_n \in B(L; \epsilon)$

**Definition 10.0.2** (Open Sets). *A set  $U \subseteq \mathbb{R}^d$  is **open** if  $\forall a \in U, \exists r > 0$ , such that  $B(a; r) \subseteq U$*

**Idea:** If  $a \in U$ , then  $a$  is not on the boundary but it is truly "inside" the set, so we can fit a ball containing  $a$  in the set.

**Definition 10.0.3** (Closed Sets). *A set  $k \in \mathbb{R}^d$  is **closed** if its complement  $\mathbb{R}^d \setminus k$  is open.*

**Example:**  $U \subseteq (0, 1)$ . Is this open? Yes.

*Proof.* Let  $a \in U$ . We let  $r := \min\{|a - 0|, |a - 1|\}$  (We do this so that  $r$  is at most the distance to the closest bound, i.e. if  $a$  is closer to 0, then the radius  $r$  cannot be  $|a - 1|$ ) then

$$B(a; r) = (a - r, a + r) \subseteq (0, 1) = U$$

□

**Example:**  $U := [0, 1]$ . Is this open? No.

*Proof.* Let  $a := 0 \in U$ . The for any  $r > 0$ ,  $\exists z \in B(a; r) = (-r, r)$  s.t  $z < 0$ , so  $z \notin U$ . Therefore  $B(a; r) \not\subseteq U$  □

Is  $U$  closed? This is the same as asking if  $\mathbb{R} \setminus U = (-\infty, 0) \cup (1, \infty)$  is open. This is open.

*Proof.* Let  $a \in (-\infty, 0) \cup (1, \infty)$ .

- **Case 1:**  $a \in (-\infty, 0)$ . Set  $r := |a|$ , so

$$B(a; r) = (a - r, a + r) = (2a, 0) \subseteq U$$

- **Case 2:**  $a \in (1, \infty)$  similar.

□

Therefore  $U = [0, 1]$  is closed.

**Example:** Is  $U := (0, 1]$  open? No, for any  $r > 0$

$$B(1; r) \not\subseteq U$$

Therefore, it is not open.

**Note:** Sets are not always open or closed. Most sets are neither open nor closed.

This set  $U$  is one such example  $U$  is not closed since  $\mathbb{R} \setminus U = (-\infty, 0] \cup (1, \infty)$   
 $0 \in \mathbb{R} \setminus U$  but  $\forall r > 0, B(0; r) \not\subseteq \mathbb{R} \setminus U$

**Example:** For any  $a \in \mathbb{R}^d, r > 0$   $B(a; r)$  is an open set.

*Proof.* Let  $x \in B(a; r)$ , so  $\|x - a\|_2 < r$ . Set

$$r_0 := r - \|x - a\|_2 > 0$$

**Claim:**  $B(x; r_0) \subseteq B(a; r)$  To see this, let  $y \in B(x; r_0)$  so  $\|y - x\|_2 < r_0$ . So,

$$\begin{aligned} \|y - a\|_2 &\leq \|y - x\|_2 + \|x - a\|_2 && (\triangle\text{-inequality}) \\ &< r_0 + \|x - a\|_2 \\ &= r \end{aligned}$$

□

**Proposition 10.0.1.** (i)  $\emptyset, \mathbb{R}^d$  are both open in  $\mathbb{R}^d$

(ii) If  $U_1, U_2, \dots, U_n \subseteq \mathbb{R}^d$  are all open, then so is  $U_1 \cap U_2 \cap \dots \cap U_n$ .

(iii) If  $U_\alpha \subseteq \mathbb{R}^d$  is an open set for all  $\alpha \in I$ , ( $I$  is some index set) then

$$\bigcup_{\alpha \in I} U_\alpha$$

is open.

*Proof.* (i),(ii) are exercises.

**(iii):** Set

$$V := \bigcup_{\alpha \in I} U_\alpha$$

Let  $a \in V$ . This means  $\exists \alpha \in I$  such that  $a \in U_\alpha$ .  $U_\alpha$  is open so  $\exists r > 0$  s.t  $B(a; r) \subseteq U_\alpha$ .  $U_\alpha \subseteq \bigcup_{\alpha \in I} U_\alpha = V$  So  $B(a; r) \subseteq V$  as required.  $\square$

**Example:** For any  $n \in \mathbb{N}_{\geq 1}$ .

$$\left( \frac{-1}{n}, \frac{1}{n} \right) = B(0; \frac{1}{n})$$

is open in  $\mathbb{R}$ . The intersection of these open sets is

$$\bigcap_{n=1}^{\infty} \left( \frac{1}{n}, \frac{-1}{n} \right) = \{0\}$$

which is not open. This shows that openness is not preserved by infinite intersections.

**Example:** Let

$$U := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$$

$U$  is open but not closed.

$$U = V \cap W$$

where

$$V := \{(x, y) : x > 0\} \quad W := \{(x, y) : y > 0\}$$

To show  $V$  is open, let  $a = (x, y) \in V$ . Set  $r := x > 0$ . Then if  $(w, z) \in B(a; r)$ . Then

$$|w - z| \leq \|(w, z) - a\|_2 < r = x$$

$$\therefore w > x - x = 0$$

So  $(w, z) \in U$ . Similarly,  $W$  is open. Therefore  $U$  is open.

**Not Closed:** Exercise.

**Proposition 10.0.2.** Let  $K \subseteq \mathbb{R}^d$ .  $K$  is closed  $\iff$  for any subsequence  $(a_n)_{n=1}^{\infty}$  in  $K$ , If it converges, then

$$\lim_{n \rightarrow \infty} a_n \in K$$

*Proof.* (  $\implies$  ) Suppose  $K$  is closed. Let  $(a_n)_{n=1}^\infty$  be a sequence in  $K$  s.t

$$L := \lim_{n \rightarrow \infty} a_n$$

exists. Suppose for a contradiction  $L \notin K$ . This means  $L \in \mathbb{R}^d \setminus K$ , which is open. So  $\exists r > 0$  such that

$$B(L; r) \subseteq \mathbb{R}^d \setminus K$$

Since  $a_n \rightarrow L$ , we must have  $a_n \in B(L; r)$  for some  $n$  (in fact, for all  $n$  sufficiently large). So  $a_n \in B(L; r) \subseteq \mathbb{R}^d \setminus K$ . Therefore  $a_n \notin K$ , which is a contradiction.

(  $\impliedby$  ) Suppose  $K$  is not closed, and we'll prove  $\exists (a_n)_{n=1}^\infty$  in  $K$  such that  $a_n \rightarrow L \notin K$ . Since  $K$  is not closed,  $\mathbb{R}^d \setminus K$  is not open. So  $\exists L \in \mathbb{R}^d \setminus K$  such that  $\forall r > 0$

$$B(L; r) \not\subseteq \mathbb{R}^d \setminus K$$

For each  $n \in \mathbb{N}_{\geq 1}$ , we can find  $a_n \in B(L; \frac{1}{n})$  such that  $a_n \notin \mathbb{R}^d \setminus K$ . So  $a_n \in K$ . This gives a sequence  $(a_n)_{n=1}^\infty$  in  $K$  and

$$\|a_n - L\|_2 < \frac{1}{n} \rightarrow 0$$

Therefore by the Squeeze Theorem,

$$\|a_n - L\|_2 \rightarrow 0 \implies a_n \rightarrow L$$

$L \in \mathbb{R}^d \setminus K$ , so  $L \notin K$ . □

**Definition 10.0.4.** Let  $A \subseteq \mathbb{R}^d$  and let  $a \in \mathbb{R}^d$ ,  $a$  is:

(i) an **interior point** if  $\exists r > 0$  s.t  $B(a; r) \subseteq A$

(ii) an **accumulation point** if  $\exists$  a sequence  $(a_n)_{n=1}^\infty$  in  $A$  s.t  $a_n \rightarrow a$

(iii) a **boundary point** if it is an accumulation point and it is not an interior point.

$$A^\circ := \{\text{All interior points}\}$$

$$\bar{A} := \{\text{All accumulation points}\}$$

$$\partial A := \{\text{All boundary points}\} = \bar{A} \setminus A^\circ$$

**Note:** The set of interior points, accumulation points, and boundary points are referred to as the **interior** of  $A$ , the **closure** of  $A$ , and the **boundary** of  $A$  respectively

**Example:**  $A := (0, 1] \cup \{2\}$

$$A^\circ = (0, 1)$$

$$\bar{A} = [0, 1] \cup \{2\}$$



$$\partial A = \{0, 1, 2\}$$

**Example:**  $A := \mathbb{Q}$

Since any open interval contains irrational numbers, we have

$$A^\circ = \text{set}$$

Proposition from chapter 2,

$$\bar{A} = \mathbb{R}$$

$$\partial A = \mathbb{R}$$

# Lecture 11

## Compactness

**Definition 11.0.1.** A set  $A \subseteq \mathbb{R}^d$  is (sequentially) compact if every sequence  $(a_n)_{n=1}^\infty$  in  $A$  has a subsequence  $(a_{n_k})_{k=1}^\infty$  that converges to a point in  $A$ .

**Example 1:** Is  $[0,1]$  compact? Yes.

*Proof. Recall:* Bolzano-Weierstrass theorem states bounded sequence has a convergent subsequence.

Therefore, every sequence  $(a_n)_{n=1}^\infty$  in  $[0,1]$  has a subsequence  $(a_{n_k})_{k=1}^\infty$  that converges. So

$$0 \leq a_{n_k} \leq 1 \implies 0 \leq \lim_{k \rightarrow \infty} a_{n_k} \leq 1$$
$$\therefore L \in [0,1]$$

□

**Example 2:** Is  $(0,1)$  compact? No.

*Proof.* By counter example, let  $a_n := \frac{1}{n+1}$ , so  $a_n \rightarrow 0$ . Therefore for all subsequences of  $a_n$ ,  $a_{n_k} \rightarrow 0$ . So there exists no subsequence which converges to a point in  $(0,1)$ . □

**Example 3:** Is  $[0, \infty)$  compact? No.

*Proof.* The Bolzano-Weierstrass theorem does not apply since  $[0, \infty)$  is unbounded. Set  $a_n := n$ , then  $a_n \rightarrow \infty$ , so it has no bounded subsequence and therefore no convergent subsequences. □

**Theorem 11.0.1** (Heine-Borel). Let  $A \subseteq \mathbb{R}^d$ .  $A$  is compact  $\iff A$  is closed and not bounded.

*Proof.* (  $\implies$  ) Similar to example 1. Assume  $A$  is closed and bounded. Let  $(a_n)_{n=1}^\infty$  be a sequence in  $A$ . The sequence is bounded since  $A$  is, so by the Bolzano-Weierstrass theorem for  $\mathbb{R}^d$ , it has a subsequence  $(a_{n_k})_{k=1}^\infty$  that converges to some  $L \in \mathbb{R}^d$ .  $a_{n_k} \in A \forall k$  and  $a_{n_k} \rightarrow L$  and  $A$  is closed, so by the

sequential characterization of closedness,  $L \in A$ , therefore  $A$  is compact.

( $\Leftarrow$ ) Assume  $A$  is compact. To show  $A$  is closed, assume for a contradiction that  $A$  is not closed. Therefore there exists a sequence  $(a_n)_{n=1}^\infty$  in  $A$  such that  $a_n \rightarrow L \notin A$ . Then for any subsequence  $(a_{n_k})_{k=1}^\infty$ , we have

$$a_{n_k} \rightarrow L \notin A$$

This contradicts that  $A$  is compact, therefore  $A$  is closed.

To show  $A$  is bounded, assume for a contradiction that  $A$  is not bounded. Then  $\forall n \in \mathbb{N}_{\geq 1}$ , there exists  $a_n \in A$  such that  $\|a_n\|_2 \geq n$ . This gives a sequence  $(a_n)_{n=1}^\infty$ . Since  $A$  is compact, it has a subsequence  $(a_{n_k})_{k=1}^\infty$  that converges. But

$$\|a_{n_k}\|_2 \geq n_k \rightarrow \infty$$

So  $(a_{n_k})_{k=1}^\infty$  is unbounded, which is a contradiction.  $\square$

**Proposition 11.0.1.**

(i) If  $k_1, \dots, k_n \subseteq \mathbb{R}^d$  are compact, then  $\bigcup_{i=1}^n k_i$  is compact.

(ii) If  $k_1, \dots, k_n \subseteq \mathbb{R}^d$  are compact, then  $\bigcap_{i=1}^n k_i$  is compact.

*Proof. Exercise:*

(i) Assume  $A := \bigcup_{i=1}^n k_i$ . Let  $(a_n)_{n=1}^\infty$  be a sequence in  $A$ . Then there exists  $i \in \{1, \dots, n\}$  such that  $a_n \in k_i$ . Since  $k_i$  is compact, it has a subsequence  $(a_{n_k})_{k=1}^\infty$  that converges to some  $L \in \mathbb{R}^d$ .  $L \in k_i$  and  $k_i \subseteq A$ , so  $L \in A$ . Therefore  $A$  is compact.

(ii) Assume  $A := \bigcap_{i=1}^n k_i$ . Let  $(a_n)_{n=1}^\infty$  be a sequence in  $A$ . Then  $a_n \in k_i$   $\forall i \in \{1, \dots, n\}$ . Since  $k_i$  is compact, it has a subsequence  $(a_{n_k})_{k=1}^\infty$  that converges to some  $L \in \mathbb{R}^d$ .  $L \in k_i$   $\forall i \in \{1, \dots, n\}$  and  $k_i \subseteq A$ , so  $L \in A$ . Therefore  $A$  is compact.  $\square$

**Definition 11.0.2.**  $A \subseteq \mathbb{R}^d$  is **compact** if for any collection

$$U_\alpha : \alpha \in I$$

of open sets such that

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha$$

There exists finitely many indices  $\alpha_1, \dots, \alpha_n$  such that

$$A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

## Lecture 12

# Limits of a Function of Continuous Variables

A sequence is a function  $\mathbb{N} \rightarrow \mathbb{R}$ . Here, we'll consider function that are going from  $\mathbb{R} \rightarrow \mathbb{R}$  (or  $\mathbb{R}^d \rightarrow \mathbb{R}^m$ ).

**Definition 12.0.1.** Let  $X \subseteq \mathbb{R}^d$ ,  $a \in \mathbb{R}^d$  a limit point of  $X$ .  $f : X \rightarrow \mathbb{R}^m$ ,  $L \in \mathbb{R}^m$ . We say the limit of  $f$  as  $X$  approaches  $a$  is  $L$  if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X$$

$$x \in B(a; \delta) \wedge x \neq a \implies \|f(x) - L\|_2 < \epsilon$$

The idea is like the definition of convergence of a sequence, except we replace  $n \geq n_0$  (which captures "n is sufficiently large") with  $x \in B(a; \delta)$ ,  $x \neq a$  (which captures "x is close to, but not equal to a"). In other words, the definition says that if  $x$  is close to (but not equal to)  $a$  then  $f(x)$  is close to  $L$ .

**Why "not equal to"?:** Often we consider the limit as  $x$  approaches  $a$  when  $f(a)$  is not defined. Other times we compare the limit to  $f(a)$ . So we do not want to use  $f(a)$  in the definition of the limit.

**Notation:** We write

$$\lim_{x \rightarrow a} f(x) = L$$

or

$$f(x) \rightarrow L \text{ as } x \rightarrow a$$

to mean that the limit of  $f$  is  $L$  as  $x$  approaches  $a$ .

**Example:**  $f : \mathbb{R} \rightarrow \mathbb{R}$ .  $f(x) := 3x - 2$ . Let  $a \in \mathbb{R}$ . Claim

$$\lim_{x \rightarrow a} f(x) = 3a - 2$$

*Proof.* Let  $\epsilon > 0$ . Consider

$$\begin{aligned} |f(x) - (3a - 2)| &= |3x - 2 - 3a + 2| \\ &= 3|x - a| \end{aligned}$$

We want this  $< \epsilon$ , set  $\delta := \frac{\epsilon}{3}$ . Then if  $x \in B(a; \delta) = (a - \delta, a + \delta)$  (i.e.  $|x - a| < \delta$ ) then

$$|f(x) - (3a - 2)| = 3|x - a| < 3\delta = \frac{3\epsilon}{3} = \epsilon$$

□

**Example:**  $g : \mathbb{R} \rightarrow \mathbb{R}$ .  $g(x) := x^2$ . Claim:

$$\lim_{x \rightarrow a} g(x) = a^2$$

*Proof.* Let  $\epsilon > 0$  be given.

$$\begin{aligned} |g(x) - a^2| &= |x^2 - a^2| \\ &= |x - a||x + a| \end{aligned}$$

What happens if  $x$  is close to  $a$ ? Intuitively,  $|x + a|$  is close to  $|a + a|$  and  $|x - a|$  is small.

$$\begin{aligned} |x + a| &= |x - a + a + a| \leq |x - a| + |a + a| \\ &< 2|a| + \delta && (\text{if } |x - a| < \delta) \\ &\leq 2|a| + 1 && (\delta \leq 1) \end{aligned}$$

Then,

$$\begin{aligned} |x^2 - a^2| &= |x - a||x + a| \leq |x - a|(2|a| + 1) \\ &< \delta(2|a| + 1) && (\text{if } |x - a| < \delta) \\ &\leq \epsilon && (\text{if } \delta \leq \frac{\epsilon}{2|a|+1}) \end{aligned}$$

**Important:** Do not define  $\delta$  in terms of  $x$  or  $\delta$ ! We can use  $a$  here since  $a$  is constant

So we set  $\delta := \min\{1, \frac{\epsilon}{2|a|+1}\}$  Then  $\delta \leq \frac{\epsilon}{2|a|+1}$  and  $\delta \leq 1$ . So if  $|x - a| < \delta$ . Then from the work above,  $|x^2 - a^2| < \epsilon$  as required. □

**Note:** In proofs where we have  $\delta - \epsilon$ , we often use

$$\delta := \min\{\dots\}$$

In proofs where we have  $n_0 - \epsilon$ , we often use

$$n_0 := \max\{\dots\}$$

**Proposition 12.0.1** (Uniqueness of Limits). *Let  $f : X \rightarrow \mathbb{R}^m$  ( $X \subseteq \mathbb{R}^d$ ),  $a \in \mathbb{R}^d$  a limit point of  $X$ ,  $L, L' \in \mathbb{R}^m$ . If the limit of  $f$  as  $x \rightarrow a$  is  $L$  and the limit of  $f$  as  $x \rightarrow a$  is  $L'$ , then  $L = L'$*

*Proof.* By contradiction. Suppose  $L \neq L'$ . So

$$\|L - L'\|_2 > 0$$

Set

$$\epsilon := \frac{\|L - L'\|_2}{2} > 0$$

Since  $f(x) \rightarrow L$  as  $x \rightarrow a$ ,  $\exists \delta > 0$  such that if  $x \in X \cap B(a; \delta) \setminus \{a\}$ , then

$$\|f(x) - L\|_2 < \epsilon$$

Since  $f(x) \rightarrow L'$  as  $x \rightarrow a$ ,  $\exists \delta' > 0$  such that

$$x \in X \cap B(a; \delta') \setminus \{a\} \implies \|f(x) - L'\|_2 < \epsilon$$

Let  $\delta_0 \neq \min\{\delta, \delta'\}$ . Let

$$x \in X \cap B(a; \delta_0) \setminus \{a\}$$

Then

$$x \in X \cap B(a; \delta) \setminus \{a\}$$

So,

$$\begin{aligned} \|f(x) - L\|_2 &< \epsilon \\ \|f(x) - L'\|_2 &< \epsilon \end{aligned}$$

So,

$$\begin{aligned} \|L - L'\|_2 &\leq \|L - f(x)\|_2 + \|f(x) - L'\|_2 < \epsilon + \epsilon \\ &= \|L - L'\|_2 \end{aligned}$$

And thus, a contradiction. □

**Proposition 12.0.2** (Sequential Characterization of Limits). *Let  $X \subseteq \mathbb{R}^d$ ,  $a \in \mathbb{R}^d$ , a limit point of  $X$ .  $f : X \rightarrow \mathbb{R}^m$ ,  $L \in \mathbb{R}^m$ .*

*$\lim_{x \rightarrow a} f(x) = L \iff$  for every sequence  $(x_n)_{n=1}^\infty$  in  $X$  such that  $x_n \rightarrow a$ , we have*

$$\lim_{n \rightarrow \infty} f(x_n) = L$$

*Proof.* ( $\implies$ ) Suppose  $\lim_{x \rightarrow a} f(x) = L$ . Let  $(x_n)_{n=1}^\infty$  be a sequence in  $X \setminus \{a\}$  such that  $x_n \rightarrow a$ . We must show that  $f(x_n) \rightarrow L$ .

Let  $\epsilon > 0$  be given. Since  $f(x) \rightarrow L$  as  $x \rightarrow a$ ,  $\exists \delta$  such that

$$x \in X \cap B(a; \delta) \setminus \{a\} \implies \|f(x) - L\|_2 < \epsilon$$

Since  $x_n \rightarrow a$ , using  $\delta$  in place of  $\epsilon$ ,  $\exists n_0$  such that  $\forall n \geq n_0$ ,  $\|x_n - a\|_2 < \delta$ . i.e.  $x_n \in B(a, \delta)$ . Also  $x_n \in X \setminus \{a\}$ . Therefore,

$$\|f(x_n) - L\|_2 < \epsilon$$

( $\Leftarrow$ ) Suppose  $\forall$  sequences  $(x_n)_{n=1}^\infty$  in  $X \setminus \{a\}$  converging to  $a$ ,  $f(x_n) \rightarrow L$ , and for a contradiction, suppose

$$f(x) \not\rightarrow L$$

We negate " $f(x) \rightarrow L$ " to get that  $\exists \epsilon > 0$  such that  $\forall \gamma > 0$ ,  $\exists x \in X \cap B(a; \gamma) \setminus \{a\}$  such that  $\|f(x) - L\|_2 \geq \epsilon$ .

This gives a sequence  $(x_n)_{n=1}^\infty$  in  $X \setminus \{a\}$ ,  $\|x_n - a\|_2 \leq \frac{1}{n} \forall n$ , so by the squeeze theorem

$$\|x_n - a\|_2 \rightarrow 0$$

Since  $\|f(x_n) - L\|_2 \geq \epsilon$ ,  $f(x_n) \not\rightarrow L$ . This is a contradiction.  $\square$

**Note:** if  $\lim_{n \rightarrow \infty} f(x_n) = L$  for *some* sequence  $(x_n)_{n=1}^\infty$  in  $X \setminus \{a\}$  converging to  $a$ , it *does not* follow that  $\lim_{x \rightarrow a} f(x) = L$

**Example:**

$$f(x) := \begin{cases} 0 & \text{if } x = \frac{1}{n}, n \in \mathbb{N}_{\geq 1} \\ 1 & \text{otherwise} \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$  does not exist but  $\lim_{n \rightarrow \infty} f(\frac{1}{n}) = 0$

**Proposition 12.0.3** (Algebra of Limits). *Let  $x \subseteq \mathbb{R}^d$ ,  $a \in \mathbb{R}^d$  a limit point of  $X$ ,  $f : X \rightarrow \mathbb{R}^m$ ,  $g : X \rightarrow \mathbb{R}^m$ ,  $L, K \in \mathbb{R}^m$ . Suppose  $\lim_{x \rightarrow a} f(x) = L$ ,  $\lim_{x \rightarrow a} g(x) = K$*

(i)

$$\lim_{x \rightarrow a} f(x) + g(x) = L + K$$

(ii)

$$\lim_{cf(x)} = cL$$

(iii) If  $m = 1$ ,

$$\lim_{x \rightarrow a} f(x)g(x) = LK$$

(iv) If  $m = 1$ ,  $g(x) \neq 0 \forall x \in X$ ,  $K \neq 0$ . Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{K}$$

*Proof.* (i) Use Sequential Characterization: Let  $(x_n)_{n=1}^\infty$  be in  $X \setminus \{a\}$  such that  $x_n \rightarrow a$ . Then  $f(x_n) \rightarrow L$  and  $g(x_n) \rightarrow K$ . So by algebra of limits for sequences,

$$f(x_n) + g(x_n) \rightarrow L + K$$

$$\therefore f(x) + g(x) \rightarrow L + K$$

(ii) **Exercise.**

(iii) **Exercise.**

(iv) **Exercise.**

□

**Theorem 12.0.1** (Squeeze Theorem). *Let  $X \subseteq \mathbb{R}^d$ ,  $a \in \mathbb{R}^d$  a limit point of  $X$ ,  $f, g, h : X \rightarrow \mathbb{R}$*

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in X$$

and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

Then

$$\lim_{x \rightarrow a} g(x) = L$$

*Proof.* **Exercise**

□

If  $f : X \rightarrow \mathbb{R}_m$ , We can define functions

$$f_1, \dots, f_m : X \rightarrow \mathbb{R}$$

by

$$(f_1(x), \dots, f_m(x)) = f(x)$$

$f_1, \dots, f_m$  are called the *component functions* of  $f$ .

**Proposition 12.0.4.** *Let  $X \subseteq \mathbb{R}^d$ ,  $a \in \mathbb{R}^d$  a limit point of  $X$ ,  $f : X \rightarrow \mathbb{R}^m$ ,  $f_1, \dots, f_m$  its component functions.  $L = (L_1, \dots, L_m) \in \mathbb{R}^m$ . Then*

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a} f_i(x) = L_i \quad \forall 1 \leq i \leq m$$

*Proof.* **Exercise.**

□

**Definition 12.0.2.** *Let  $X \subseteq \mathbb{R}$ ,  $a \in \mathbb{R}$ ,  $f : X \rightarrow \mathbb{R}^d$ .*

- *If  $a$  is a limit point of  $X \cap (a, \infty)$  then we write  $\lim_{x \rightarrow a^+} f(x) = L$  to mean that*

$$\lim_{x \rightarrow a} g(x) = L$$

where

$$g = f|_{X \cap (a, \infty)}$$

- *If  $a$  is a limit point of  $X \cap (-\infty, a)$  then we write  $\lim_{x \rightarrow a^-} f(x) = L$  to mean that*

$$\lim_{x \rightarrow a} g(x) = L$$

where

$$g = f|_{X \cap (-\infty, a)}$$



**Example:**

$$f(x) := \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} f(x) = 1 \neq \lim_{x \rightarrow 0^-} f(x) = -1$$

# Lecture 13

## Continuity

### 13.1 One-Sided Limits

**Recall:** For  $X \subseteq \mathbb{R}$ ,  $f : X \rightarrow \mathbb{R}^m$ ,  $a \in \mathbb{R}$

- If  $a$  is a limit point of  $X \cap [a, \infty]$ , then

$$\lim_{x \rightarrow a^+} f(x) := \lim_{x \rightarrow a} f_{X \cap [a, \infty)}(x)$$

$\lim_{x \rightarrow a^-} f(x)$  is defined similarly.

In other words,  $\lim_{x \rightarrow a^+} f(x) = L$  means  $\forall \epsilon > 0, \exists \delta > 0$  such that if  $a < x < a + \delta$  and  $x \in X$ , then

$$\|f(x) - L\|_2 < \epsilon$$

### 13.2 Continuity

**Definition 13.2.1.** Let  $f : X \rightarrow \mathbb{R}^m$  where  $X \subseteq \mathbb{R}^d$ . Let  $a \in X$ .

- If  $a$  is not an isolated point, then we say  $f$  is continuous at  $a$  if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

- If  $a$  is an isolated point, then we always say  $f$  is continuous at  $a$ .

**Definition 13.2.2** ( $\delta - \epsilon$  Characterization of Continuity).  $f$  is continuous at  $a$   $\iff \forall \epsilon > 0, \exists \gamma > 0$  s.t if

$$x \in X \cap B(a; \delta)$$

then

$$\|f(x) - f(a)\|_2 < \epsilon$$

In other words if  $x$  is close to  $a$ , the  $f(x)$  is close to  $f(a)$ . This holds regardless of whether or not  $a$  is isolated.

**Example:**  $f : \mathbb{R} \mapsto \mathbb{R} \ f(x) := x^2$ . For any  $a \in \mathbb{R}$ , we proved last lecture that

$$\lim_{x \rightarrow a} f(x) = a^2 = f(a)$$

Therefore,  $f$  is continuous at  $a$ .

**Example:**

$$f(x) := \begin{cases} 1 & \text{if } x \leq 1 \\ 2 & \text{if } x > 1 \end{cases}$$

For this function,

$$\lim_{x \rightarrow 1^+ f(x)=2 \neq f(x)=1}$$

Therefore,  $f$  is not continuous at 1.

**Example:**

$$f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ L & \text{if } x = 0 \text{ for some } L \in \mathbb{R} \end{cases}$$

No matter how we choose  $L$ ,  $f$  is not continuous at 0.

*Proof.* We want to show  $\lim_{x \rightarrow 0} f(x)$  does not exist. Consider the sequence

$$\left(\frac{1}{2\pi n}\right)_{n=1}^{\infty} \rightarrow 0$$

and  $f\left(\frac{1}{2\pi n} = 0 \rightarrow 0\right)$ . But we also have

$$\left(\frac{1}{2\pi n + \pi/2}\right)_{n=1}^{\infty} \rightarrow 0$$

and  $f\left(\frac{1}{2\pi n + \pi/2}\right) = 1 \rightarrow 1$ . So if  $\lim_{x \rightarrow 0} f(x)$  exists, then  $f(a_n) \rightarrow L$  for a sequence  $(a_n)_{n=1}^{\infty}$  in  $(0, \infty)$  such that  $a_n \rightarrow 0$ . So  $L = 0$  and  $L = 1$ , therefore a contradiction  $\square$

**Example:**

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$f$  is not continuous at any point.

*Proof.* Let  $a \in \mathbb{R}$ . We'll prove  $\lim_{x \rightarrow a} f(x)$  does not exist.

$$\exists ((n)_{n=1}^{\infty} x) \in \mathbb{Q} \text{ s.t } x_n \rightarrow a$$

so  $f(x_n) = 1 \rightarrow 1$

$$\exists ((n)_{n=1}^{\infty} y) \in \mathbb{R} \setminus \mathbb{Q} \text{ s.t } y_n \rightarrow a$$

Since we can find  $z_n \in \mathbb{Q}$  such that  $z_n \rightarrow a + \sqrt{2}$ , so  $y := z_n - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ ,  $y_n \rightarrow a$ . So  $f(y_n) = 0 \rightarrow 0$ . Therefore, the  $\lim_{x \rightarrow a} f(x)$  does not exist.  $\square$

**Example:**

$$f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$f$  is continuous.

*Proof.* Use squeeze theorem, if  $x \geq 0$ , then  $-x \leq f(x) \leq x$ . So

$$-|x| \leq f(x) \leq |x|$$

$$\lim_{x \rightarrow 0} |x| = 0 = \lim_{x \rightarrow 0} -|x|$$

$$\therefore \lim_{x \rightarrow 0} f(x) = 0 = f(0)$$

□

**Proposition 13.2.1.** Let  $X \subseteq \mathbb{R}^d$ ,  $Y \subseteq \mathbb{R}^m$ , and  $f : X \mapsto Y$ ,  $g : Y \rightarrow \mathbb{R}^n$ ,  $a \in X$ . If  $f$  is continuous at  $a$  and  $g$  is continuous at  $f(a)$ , then  $g \circ f$  is continuous at  $a$ .

*Proof.* Use the  $\delta - \epsilon$  characterization. Let  $\epsilon > 0$  be given. Since  $g$  is at  $f(a)$ ,  $\exists \eta > 0$  (using  $\eta$  instead of  $\delta$ ), such that if  $y \in B(f(a); \eta) \cap Y$ , then

$$\|g(x) - g(f(a))\|_2 < \epsilon$$

Since  $f$  is continuous at  $a$ ,  $\exists \delta > 0$  such that if  $x \in B(a; \delta) \cap X$ , then

$$\|f(x) - f(a)\|_2 < \eta$$

In other words,  $f(x) \in B(f(a); \eta)$ . Therefore if  $x \in B(a; \delta) \cap X$ , then since  $f(x) \in B(f(a); \eta)$ ,

$$\|g(x) - g(f(a))\|_2 < \epsilon$$

as required. □

**Proposition 13.2.2.** Let  $X \subseteq \mathbb{R}^d$ ,  $f, g : X \mapsto \mathbb{R}^m$ ,  $a \in X$  such that  $f, g$  are continuous at  $a$ ,

(i)  $f + g : X \mapsto \mathbb{R}^m$  is continuous at  $a$

(ii)  $cf : X \mapsto \mathbb{R}^m$  is continuous at  $a$  for any  $c \in \mathbb{R}$

(iii)  $\gamma f : X \mapsto \mathbb{R}^m$  is continuous at  $a$  if  $\gamma : X \mapsto \mathbb{R}$  is continuous at  $a$ .

(iv) If  $m = 1$ , and  $g(x) \neq 0, \forall x \in X$ , then

$$\frac{f}{g} : X \mapsto \mathbb{R}$$

is continuous at  $a$ .

*Proof.* Follows from the algebra of limits. □

If  $f : X \mapsto \mathbb{R}^m$ ,  $f = (f_1, \dots, f_m)$  where  $f_i : X \mapsto \mathbb{R}$ , then  $f$  is continuous at  $a$  if and only if  $f_i$  is continuous at  $a$  for all  $i \in \{1, \dots, m\}$ .

- You are allowed to use the fact that  $\sin$ ,  $\cos$ ,  $\exp$ ,  $\log$  are all continuous functions  $\log : [0, \infty) \mapsto \mathbb{R}$  is continuous at all  $a \in [0, \infty)$ .

### 13.3 Properties of Continuous Functions

$f : X \mapsto \mathbb{R}^m$  is (globally) continuous if  $f$  is continuous at  $a$  for all  $a \in X$ .

**Theorem 13.3.1.** *Let  $K \subseteq \mathbb{R}^d$  be a compact set. If  $f : K \mapsto \mathbb{R}^m$  is continuous, then  $f(K)$  is compact.*

*Proof.* Let  $(a_n)_{n=1}^\infty$  be a sequence in  $f(K)$ . We need to show that  $\exists$  a subsequence. We need to show that  $\exists$  a subsequence  $(a_{n_k})_{k=1}^\infty$  s.t  $a_{n_k} \rightarrow b$  for some  $b \in f(K)$ . Since  $a_n \in f(K)$ ,  $\exists x_n \in K$  s.t  $f(x_n) = a_n$  for all  $n$ . This gives a sequence  $(x_n)_{n=1}^\infty$  in  $K$ . Since  $K$  is compact,  $\exists$  a subsequence  $(x_{n_k})_{k=1}^\infty$  s.t  $x_{n_k} \rightarrow x_0$ . Since  $f$  is continuous at  $x_0$ ,  $f(x_{n_k}) \rightarrow f(x_0)$ . Therefore,  $a_{n_k} \rightarrow f(x_0)$  and  $(a_{n_k})_{k=1}^\infty$  is a subsequence of  $(a_n)_{n=1}^\infty$ . This is what we wanted.  $\square$

**Corollary 13.3.1** (Extreme Value Theorem). *Let  $K \subseteq \mathbb{R}^d$  be compact,  $f : K \rightarrow \mathbb{R}$  a continuous function. Then  $f$  is bounded and it attains its bounds, i.e  $\exists x_{\min}, x_{\max} \in K$  such that  $\forall x \in K$*

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

*Proof.* From the previous result,  $f(K)$  is compact in  $\mathbb{R}$ . By the Heine-Borel theorem, this means  $f(K)$  is closed and bounded. Therefore,  $f$  is bounded. Since  $f(K)$  is closed,  $\sup f(K) \in f(K)$  since there is a sequence  $(y_n)_{n=1}^\infty$  in  $f(K)$  such that  $y_n \rightarrow \sup f(K)$ . So  $\exists x_{\max} \in K$  such that

$$f(x_{\max}) = \sup f(K)$$

Therefore,

$$f(x) \leq f(x_{\max}) \quad \forall x \in K$$

Similarly,  $\exists x_{\min} \in K$  such that  $f(x_{\min}) \leq f(x) \quad \forall x \in K$ .  $\square$

## Lecture 14

# Properties of Continuous Functions Continued

**Recall:**

**Corollary 13.3.1** (Extreme Value Theorem). *If  $K \subseteq \mathbb{R}^d$  is compact and  $f : K \rightarrow \mathbb{R}$  is continuous, then  $\exists x_{\min}, x_{\max} \in K$  such that  $f(x_{\min}) \leq f(x) \leq f(x_{\max})$  for all  $x \in K$ .*

**Theorem 14.0.1** (Intermediate Value Theorem). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and let  $y_0 \in \mathbb{R}$  be any number between  $f(a)$  and  $f(b)$ . Then there exists a number  $x_0$  between  $a$  and  $b$  such that  $f(x_0) = y_0$ .*

*Proof.* Without loss of generality, assume  $f(a) \leq f(b)$ . So  $x_0 \in [f(a), f(b)]$ . Let

$$S := \{x \in [a, b] : f(x) \leq x_0\}$$

$S \subseteq [a, b]$ , so  $S$  is bounded.  $a \in S$  since  $f(a) \leq x_0$ . Therefore  $S \neq \emptyset$ . So  $\exists x_0 := \sup S \in [a, b]$ . We will show  $f(x_0) = y_0$ . We will consider the cases, where  $f(x_0) = y_0$ ,  $f(x_0) < y_0$ , and  $f(x_0) > y_0$ . If  $f(x_0) < y_0$ , set  $\epsilon := y_0 - f(x_0)$ . Since  $f$  is continuous at  $x_0$ ,  $\exists \delta > 0$  s.t if  $|x - x_0| < \delta$  and  $x \in [a, b]$ . Then  $|f(x) - f(x_0)| < \epsilon$ . Since  $f(x_0) < y_0 \leq f(b)$  for  $x_0 \neq b$ . So we can find  $x > x_0$  such that  $x \in [a, b]$  and  $|x - x_0| < \delta$ . Then  $f(x) < f(x_0) + \epsilon = y_0$ . So  $x \in S$  by the definition of  $S$ , since  $S = \{x : f(x) \leq y_0\}$ . This is a contradiction since  $x > x_0$ , but  $x_0$  is an upper bound for  $S$ .

If  $f(x_0) > y_0$ , set  $\epsilon := f(x_0) - y_0$ . Since  $f$  is continuous at  $x_0$ ,  $\exists \delta > 0$  such that if  $x \in [a, b]$  and  $|x - x_0| < \delta$ , then  $|f(x) - f(x_0)| < \epsilon$ . So  $f(x_0) > y_0 \leq f(a)$ . So  $x_0 > a$ . We may assume that  $x_0 - \delta > a$  since  $\delta$  is arbitrary and can be arbitrarily small.

Claim:  $x_0 - \delta$  is an upper bound for  $S$ . Proof of claim, if  $x > x_0 - \delta$ , then either  $|x - x_0| < \delta$ , in which case  $f(x) > f(x_0) - \epsilon = y_0$ , or  $x > x_0$ , in which case  $x \notin S$  since  $x_0$  is an upper bound for  $S$ . Therefore, if  $x > x_0 - \delta$ , then  $x \notin S$ .

This proves the claim.

The claim contradicts that  $x_0$  is the least upper bound for  $S$ .  $\square$

**Corollary 14.0.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f([a, b]) = [c, d]$  for some  $c \leq d$ .*

*Proof.* By extreme value theorem,  $\exists x_{\min}, x_{\max}$  such that  $f(x) \in [f(x_{\min}), f(x_{\max})]$   $\forall x \in [a, b]$ . So we can set  $c := f(x_{\min})$  and  $d := f(x_{\max})$  and we have

$$f([a, b]) \subseteq [c, d]$$

for  $y_0 \in [c, d] = [f(x_{\min}), f(x_{\max})]$ . By the Intermediate Value Theorem,  $\exists x_0$  between  $x_{\min}$  and  $x_{\max}$  such that  $f(x_0) = y_0$ . Therefore

$$[c, d] \subseteq f([a, b])$$

Thus,

$$f([a, b]) = [c, d]$$

$\square$

## 14.1 Inverses of Continuous Functions

Let  $f : A \mapsto B$ ,  $f$  is

- **Injective** (or one-to-one) If  $\forall x, y \in A$ , if  $f(x) = f(y)$ , then  $x = y$ .
- **Surjective** (or onto) If  $\forall y \in B$ ,  $\exists x \in A$  such that

$$f(x) = y$$

- **Bijjective** (or one-to-one and onto) If  $f$  is both injective and surjective.

$f$  is bijective  $\iff$  it is invertible

$$\iff \exists f^{-1} : B \mapsto A \text{ s.t } f \circ f^{-1} = id_B \text{ and } f^{-1} \circ f = id_A$$

*Proof.* (  $\Leftarrow$  ) If  $f$  is invertible, then it is

- **Injective** since if  $f(x) = f(y)$  then  $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$
- **Surjective** since given  $y \in B$ ,  $x := f^{-1}(y)$  satisfies  $f(x) = y$ .

(  $\Rightarrow$  ) If  $f$  is bijective, then for  $y \in B$ ,  $\exists x \in A$  such that  $f(x) = y$ . Also, this  $x$  is unique since  $f$  is injective. So we can define

$$f^{-1}(y) := x$$

and this is an inverse to  $f$ .  $\square$

Given  $f : A \mapsto \mathbb{R}$  that is injective, we view  $f$  as a function  $A \mapsto f(A)$ , and this way we force  $f$  to be surjective. Thus  $f^{-1} : f(A) \mapsto A$  exists.

**Question:** If  $f$  is injective and continuous, must  $f^{-1}$  be continuous? **No.** Consider the example with  $f : [0, 1) \cup [2, \infty]$

$$f(x) := \begin{cases} x_1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$f^{-1} : [0, \infty) \mapsto [0, 1) \cup [2, \infty] \subseteq \mathbb{R}$ , so

$$f^{-1}(y) = \begin{cases} y_1 & \text{if } y < 1 \\ y + 1 & \text{if } y \geq 1 \end{cases}$$

$f^{-1}$  is not continuous.

**Definition 14.1.1.** Let  $A \subseteq \mathbb{R}$ ,  $f : A \mapsto \mathbb{R}$ .  $f$  is

- (i) increasing if  $\forall x, y \in A$ ,  $x \leq y$  then  $f(x) \leq f(y)$
- (ii) strictly increasing if  $\forall x, y \in A$ ,  $x < y$  then  $f(x) < f(y)$
- (iii) decreasing if  $\forall x, y \in A$ ,  $x \leq y$  then  $f(x) \geq f(y)$
- (iv) strictly decreasing if  $\forall x, y \in A$ ,  $x < y$  then  $f(x) > f(y)$

**Lemma 14.1.1.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \mapsto \mathbb{R}$  be an injective continuous function. Then  $f$  is either strictly increasing or strictly decreasing.

*Proof.* It suffices to consider the case where  $I$  is a closed, bounded interval. A general interval  $I$  is a union of an increasing sequence of closed bounded intervals, so if  $f$  is strictly increasing or strictly decreasing on each of these subintervals, then  $f$  is strictly increasing or decreasing on all of  $I$ . Consider the case  $I = [a, b]$  with  $(a < b)$ . Without loss of generality, assume  $f(a) < f(b)$ , in this case we'll show that  $f$  is strictly increasing. By contradiction, if  $f$  is not strictly increasing, then there exists  $x_1 < x_2$  such that

$$f(x_1) \geq f(x_2)$$

We'll break this into 2 cases.

- **Case 1:**  $f(x) > f(b)$ . In this case,

$$f(a) < f(b) < f(x_1)$$

and

$$f|_{[a, x_1]}$$

is continuous. So by the Intermediate Value Theorem,  $\exists z \in [a, x_1]$  such that  $f(z) = f(b)$ . But,

$$z \leq x_1 < x_2 \leq b$$

So  $z \neq b$ , which contradicts that  $f$  is injective.



- **Case 2:**  $f(x_1) \leq f(b)$ . In this case,  $f(x_2) \leq f(x_1) \leq f(b)$  and  $f|_{[x_1, b]}$  is continuous. So by the Intermediate Value Theorem,  $\exists z \in [x_1, b]$  such that  $f(z) = f(x_1)$ .  $z \geq x_2 > x_1$  so  $z \neq x_1$ . But, this contradicts that  $f$  is injective.

In both cases, we get a contradiction, so our assumption that  $f$  is not strictly increasing is false.  $\square$

# Lecture 15

Let  $I \subset \mathbb{R}$  be an interval,  $f : I \mapsto \mathbb{R}$  be a continuous function. Then  $f^{-1} : f(I) \mapsto \mathbb{R}$  is continuous.