MAT 2125 Lecture Notes

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Contents

Important Proofs for Midterm

Theorem 1.3.13 (The Archimedean Property)**.** *The set* $\mathbb{N}_{\geq 1}$ *is not bounded above.*

Proof. Suppose for a contradiction that $\mathbb{N}$ was bounded above. Then by completeness, $a = \sup \mathbb{N}$ exists. Since a is a least upper bound, $a - 1$ is not an upper bound, so there exists $m \in \mathbb{N}$ such that

$$m > a - 1$$

Then since $m \in \mathbb{N}$, we have $m + 1 \in \mathbb{N}$, so

$$m + 1 > a$$

But a is an upper bound, thus a contradiction. $\square$

Proposition 2.2.4 (Uniqueness of Limits)**.** *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence and let* $L_1, L_2 \in \mathbb{R}$. *If*

$$\lim_{n\to\infty} a_n = L_1 \text{ and } \lim_{n\to\infty} a_n = L_2$$

then

$$L_1 = L_2$$

Proof. Suppose for a contradiction $L_1 \neq L_2$. We can assume without loss of generality that $L_1 < L_2$. Define

$$\epsilon = \frac{L_2 - L_1}{2}$$

Since $\lim\limits_{n\to\infty} a_n = L$, there exists n_0 such that $\forall n \geq n_0$

$$L_1 - \epsilon < a_n < L_1 + \epsilon$$

Using the second inequality and the definition of ϵ, we get

$$a_n < L_1 + \epsilon = L_1 + \frac{L_2 - L_1}{2} = L_1 + \frac{L_2}{2} - \frac{L_1}{2} = \frac{L_2 + L_1}{2}$$

Likewise, since $\lim_{n\to\infty} a_n = L_2$, there exists m_0 such that for all $n \geq m_0$,

$$L_2 - \epsilon < a_n < L_2 + \epsilon$$

Then from the first inequality, we get

$$a_n > L_2 - \epsilon = L_2 - \frac{L_2 - L_1}{2} = \frac{L_2 + L_1}{2}$$

So, we get that for all $n \geq \max\{n_0, m_0\}$,

$$a_n > \frac{L_2 + L_1}{2} > a_n$$

Thus, a contradiction. □

Proposition 2.2.8. *Let $(a_n)_{n=1}^{\infty}$ be a sequence which converges to some number $L \in \mathbb{R}$. Then $(a_n)_{n=1}^{\infty}$ is bounded.*

Proof. Since $\lim_{n\to\infty} a_n = L$, set $\epsilon := 1$, there exists n_0 such that for all $n \geq n_0$

$$|a_n - L| < 1$$

So we have that $\forall n \geq n_0$

$$L - 1 < a_n < L + 1$$

Now set

$$M := \max\{a_1, a_2, \ldots, a_{n_0-1}, L + 1\}$$

If $n < n_0$, then it is amongst the set $\{a_1, \ldots, a_{n_0-1}\}$, so M will be the max of this set. Therefore, $\forall n < n_0$, $a_n \leq M$. Then for $n \geq n_0$, by the definition of the limit we know that $a_n < L+1$, so we get that $a_n < L+1 \leq M$. Therefore, for all values of n,the set $\{a_n : n \in \mathbb{N}\}$ is bounded above.

Similarly for the lower bound, take

$$M := \min\{a_1, a_2, \ldots, a_{n_0-1}, L - 1\}$$

If $n < n_0$, then it is in the set $\{a_1, a_2, \ldots, a_{n_0-1}\}$ M' is at most the minimum of this set, so $\forall n < n_0$, $a_n \geq M'$. If $n \geq n_0$, by the definition of the limit we know that for all $n \geq n_0$, $a_n > L - 1$. So M' is at most $L - 1$. Therefore $\forall n \geq n_0$, $a_n > L - 1 \geq M'$. Therefore, the set is bounded below and above, so it is bounded. □

Proposition 2.3.3. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be converging sequences, if*

$$a_n \leq b_n$$

for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Proof. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences with $a_n < b_n$ for all n. Then by the definition of convergence, we have that $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$

$$|a_n - L_a| < \epsilon$$

Similarly for b_n, we have that $\exists m_0$ such that $\forall \epsilon > 0$,

$$|b_n - L_b| < \epsilon$$

Now suppose for a contradiction that $L_a > L_b$, then set $\epsilon := \frac{L_a - L_b}{2}$. So we have

$$L_a - \epsilon < a_n < \epsilon + L_a$$

So,

$$a_n > L_a - \epsilon = L_a - \frac{L_a - L_b}{2} = \frac{L_a + L_b}{2}$$

Similarly for b_n, we have

$$L_b - \epsilon < b_n < L_b + \epsilon$$

$$b_n < L_b + \epsilon = \frac{L_a + L_b}{2}$$

So we have $b_n < \frac{L_b + L_a}{2} < a_n$, but $a_n < b_n$. Thus, a contradiction. □

Theorem 2.3.5 (Squeeze Theorem)**.** *Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ be sequences such that*

(i) $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ converge to the same number L, and

(ii) $a_n \leq b_n \leq c_n$ for all n Then $(b_n)_{n=1}^{\infty}$ also converges to L.

Proof. Let $\epsilon > 0$ be given. Suppose $a_n \leq b_n \leq c_n$ $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge to L, so $\exists n_a, n_c \in \mathbb{N}$ such that for all $n \geq n_a$

$$L - \epsilon < a_n < L + \epsilon$$

and

$$L - \epsilon < c_n < L + \epsilon$$

So

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

Therefore,

$$L - \epsilon < b_n < L + \epsilon$$

By the definition of convergence, $(b_n)_{n=1}^{\infty}$ converges to L. □

Theorem 2.6.1 (Cauchy Convergence Criterion)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then it converges if and only if it is Cauchy.*

Proof. ($\implies$) Assume that $(a_n)_{n=1}^{\infty}$ converges, then there exists n_0 such that for all $\epsilon > 0$, $\forall n \geq n_0$

$$|a_n - L| < \epsilon$$

Now take $\frac{\epsilon}{2}$ in place of ϵ since ϵ is arbitrary, we have

$$|a_n - L| < \frac{\epsilon}{2}$$

Then, for $m, n \geq n_0$, we have

$$a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L|$$

Since $m, n \geq n_0$, by the definition of convergence we have

$$|a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore,

$$|a_m - a_n| < \epsilon$$

as required.

$\square$

Proposition 2.7.3. *For any sequence* $(a_n)_{n=1}^{\infty}$,

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

Proof. If the sequence isn't bounded, then either $\limsup\limits_{n\to\infty} a_n = \infty$ or $\liminf\limits_{n\to\infty} a_n = -\infty$, in either case the result is trivial. So assume that the sequence is bounded. Consider the sets used to define lim sup and lim inf

$$S := \{\beta : \mathbb{R} : \exists n_0 \text{ such that } a_n \leq \beta \quad \forall n \geq n_0\}$$

$$T := \{\alpha : \mathbb{R} : \exists m_0 \text{ such that } a_n \geq \alpha \quad \forall n \geq m_0\}$$

So we have $\alpha \in T$ and $\beta \in S$, then for all $n \geq \max\{n_0, m_0\}$, we have

$$\alpha \leq a_n \leq \beta$$

Thus, we have shown that for every $\alpha \in T$, and every $\beta \in S$, we have $\alpha \leq \beta$. From the definition of lim sup and lim inf, we get that for any eventual lower bound $\alpha \in T$, it is a lower bound for the set of upper bounds S, so

$$\alpha \leq \inf T = \limsup_{n\to\infty} a_n$$

So then $\limsup\limits_{n\to\infty} a_n$ is an upper bound for the set of lower bounds T, so

$$\limsup_{n\to\infty} a_n \geq \sup T = \liminf_{n\to\infty} a_n$$

Therefore,

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

as required.

$\square$

Proposition . *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Proof. Consider the partial sum of the series

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

Now consider the partial sums which correspond to powers of 2, S_{2^N} for $N \in \mathbb{N}$. So we have the sums $S_2, S_4, S_8, \ldots$. Now consider the sequence of partial sums

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$\frac{1}{3} > \frac{1}{4}$, so we have that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

Continuing similarly,

$$\begin{aligned} S_8 = S_{2^3} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{4}{8} = 1 + \frac{3}{2} \\ &\vdots \\ S_{2^N} &> 1 + \frac{N}{2} \end{aligned}$$

So, we have

$$\lim_{N\to\infty} \left(1 + \frac{N}{2}\right) = \infty$$

But, $S_{2^N} > 1 + \frac{N}{2}$ for all $N \in \mathbb{N}$, so we have that the partial sums diverge. Therefore, the series diverges. $\square$

Proposition 3.1.7 (Divergence Test)**.** *Let* $(a_n)_{n=1}^{\infty}$ *bea sequence of real numbers. If the series*

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n\to\infty} a_n = 0$$

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ converges to L. Set $L \coloneqq \sum_{n=1}^{\infty} a_n$. Consider the partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

so $\lim_{n\to\infty} S_N = L$. We also have that $\lim_{n\to\infty} S_{N-1} = L$, since

$$\lim_{N\to\infty} S_{N-1} = \lim_{N\to\infty} \sum_{n=1}^{N-1} a_n = \sum_{n=1}^{\infty-1} a_n = \sum_{n=1}^{\infty} a_n = L$$

Then, we have that

$$S_N - S_{N-1} = \sum_{n=1}^{N} a_n - \sum_{n=1}^{N-1} a_n = a_N$$

So,

$$\lim_{N\to\infty} S_N - S_{N-1} = L - L = 0$$

$$\lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = \lim_{N\to\infty} \sum_{n=1}^{N} a_n - \lim_{N\to\infty} \sum_{n=1}^{N-1} a_N = \lim_{N\to\infty} a_n = 0$$

$\square$

Proposition 3.2.1 (Boundedness Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that*

(i) $a_n \geq 0$ for all n, and

(ii) There is a bound $M \in \mathbb{R}$ on the partial sums, so that

$$\sum_{n=1}^{N} a_n \leq M$$

for all $N \in N_{\geq 1}$.

Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $a_n \geq 0$, the partial sums $(S_N)_{N=1}^{\infty}$ satisfy

$$S_N \leq S_{N+1} \text{ for all } N.$$

In other words, $(S_N)_{N=1}^{\infty}$ is an increasing sequence. The second condition ensures that the sequence is bounded above. Therefore, by the Monotone Convergence Criterion, it converges. Therefore, $\sum_{n=1}^{\infty} a_n$ converges. $\square$

Proposition 3.2.2 (Comparison Test)**.** *Let* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be sequences of real numbers such that*

$$0 \leq a_n \leq b_n \text{ for all } n$$

Then,

(i) if $\sum_{n=1}^{\infty} b_n$ *converges, then so does* $\sum_{n=1}^{\infty} a_n$

(ii) if $\sum_{n=1}^{\infty} a_n$ *diverges, then so does* $\sum_{n=1}^{\infty} a_n$

Proof. Since the sequence $\sum_{n=1}^{\infty}$ converges, take $M \coloneqq \sum_{n=1}^{\infty}$. Then, we have the sequence of partial sums

$$\left(\sum_{n=1}^{\infty} b_n\right)_{n=1}^{\infty}$$

is increasing and converges to M, so M is the supremum of this sequence, therefore

$$\sum_{N=1}^{N} b_n \leq M$$

for all M. Therefore

$$\sum_{N=1}^{N} a_n \leq \sum_{N=1}^{N} b_n \leq M$$

Therefore, by the Boundedness test, $\sum_{n=1}^{\infty} a_n$ converges. (ii) is the contrapositive of (i) so it follows that it holds. □

Proposition 4.2.3. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}^d$*, with*

$$a_n = (a_n^{(1)}, \ldots, a_n^{(d)}) \text{ for each } n \in \mathbb{N}$$

and let $L = (L_1, \ldots, L_d) \in \mathbb{R}^d$*. Then*

$$\lim_{n\to\infty} a_n = L$$

if and only if, for each $i = 1, \ldots, d$*,*

$$\lim_{n\to\infty} a_n^{(i)} = L_i$$

Proof. ($\implies$) Assume that $\lim_{n\to\infty} a_n = L$. Then, for each $i = 1, \ldots, d$, we have that $|x_i|^2 \leq \sum_{i=1}^{d} x_i^2 = ||x||_2^2$, therefore

$$|x_i| \leq ||x||_2$$

Using this fact, we then have each component of $||a_n - L||_2$ is less than or equal to it. So

$$|a_n^{(i)} - L_i| \leq ||a_n - L||_2$$
$$-||a_n - L||_2 \leq a_n^{(i)} - L_i \leq ||a_n - L||_2$$

Since $\lim_{n\to\infty} a_n = L$, we have $\lim_{n\to\infty} a_n - L = 0$. By the Squeeze theorem, it follows that

$$\lim_{n\to\infty} a_n^{(i)} - L_i = 0 \implies \lim_{n\to\infty} a_n^{(i)} = L$$

($\Longleftarrow$) Suppose for each $i = 1, \ldots, d$, we have

$$\lim_{n\to\infty} a_n^{(i)} = L_i$$

Then, from the definition of $||\cdot||_2$, we have

$$||a_n - L||_2^2 = (a_n^{(1)} - L_1)^2 + \cdots + (a_n^{(d)} - L_d)^2$$

Now taking limits of both sides

$$\lim_{n\to\infty} ||a_n - L||_2^2 = \lim_{n\to\infty} (a_n^{(1)} - L_1)^2 + \cdots + \lim_{n\to\infty} (a_n^{(d)} - L_d)^2$$

Now we'll prove exercise 2.2.5 which states that if $(a_n)_{n=1}^{\infty}$ is a sequence of non-negative real number converging to $L \geq 0$, then $\lim_{n\to\infty} \sqrt{a_n}$ converges to $\sqrt{L}$. To prove this we will consider two cases where $L = 0$, and $L > 0$.

- **Case 1, L = 0**: Suppose $(a_n)_{n=1}^{\infty} \to 0$, then from the definition of convergence we have that $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$,

  $$|a_n - 0| < \epsilon$$

  Since ϵ is abritrary, we'll replace ϵ with ϵ^2, so

  $$|a_n - 0| < \epsilon^2$$

  Then we get

  $$|a_n - 0| = |a_n| < \epsilon^2 \implies \sqrt{|a_n|} < \epsilon$$

  Therefore, $\sqrt{a_n} \to 0$ by the definition of convergence.

- **Case 2,** $L > 0$: Suppose $(a_n)_{n=1}^{\infty} \to L > 0$. Let $\epsilon > 0$ be given, then there exists n_0 such that for all $n \geq n_0$,

  $$|a_n - L| < \epsilon$$

  We much such that $|\sqrt{a_n} - \sqrt{L}| < \epsilon$

  $$|\sqrt{a_n} - \sqrt{L}| \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} = \frac{|(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})|}{\sqrt{a_n} + \sqrt{L}}$$

Since $\sqrt{a_n} + \sqrt{L}$ is positive because $a_n, L \geq 0$, then $\sqrt{a_n} + \sqrt{L} = |\sqrt{a_n} + \sqrt{L}|$, then using the fact that $|a| \cdot |b| = |a \cdot b|$, we get

$$\frac{|a_n - \sqrt{L}\sqrt{a_n} + \sqrt{L}\sqrt{a_n} + L|}{\sqrt{a_n} + \sqrt{L}} = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}}$$

Now if we replace ϵ with $\frac{\epsilon}{\sqrt{L}}$, we get

$$|\sqrt{a_n} - \sqrt{L}| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon}{\sqrt{L}} \implies |\sqrt{a_n} - \sqrt{L}| < \epsilon$$

Therefore, $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$

Now going back to the original proof,

$$\lim_{n\to\infty} ||a_n - L||_2^2 = 0$$

So from exercise 2.2.5 we have

$$\lim_{n\to\infty} \sqrt{||a_n - L||_2^2} = \sqrt{0}$$

Therefore,

$$\lim_{n\to\infty} ||a_n - L||_2 = 0$$

as required. □

Theorem 4.2.2 (Cauchy Covergence $\mathbb{R}^d$)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence $\mathbb{R}^d$. Then it converges if it converges if and only if it is cauchy.*

Proof. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^d$ that converges to $L \in \mathbb{R}^d$. Let $\epsilon > 0$ be given, then there exists n_0 such that $\forall m, n \geq n_0$,

$$||a_n - L||_2 < \epsilon$$

$$||a_m - L||_2 < \epsilon$$

Since ϵ arbitrary we can replace ϵ with $\frac{\epsilon}{2}$, so

$$||a_n - L||_2 = \frac{\epsilon}{2} \text{ and } ||a_m - L||_2 = \frac{\epsilon}{2}$$

So,

$$\begin{aligned} ||a_m - a_n||_2 = ||a_m - L + L - a_n||_2 \leq ||a_m - L||_2 + ||L - a_n||_2 \\ = ||a_m - L||_2 + ||a_n - L||_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

Proposition 4.3.4. *Given $a \in \mathbb{R}^d$, and $r > 0$, the open ball $B(a, r)$ is an open set.*

Note: This is example 4.3.4 from the professors notes.

Proof. Recall the definition of an open set is that for any x in the set, we can define an open ball (or epsilon neighborhood) around x such that the ball is contained in the set. So we want an open ball $B(x;\epsilon)$ such that $B(x;\epsilon) \subseteq B(a;r)$. To see this, let $x \in B(a;r)$, so that $||x-a||_2 < r$. Define

$$\epsilon := r - ||a-x||_2 > 0$$

Now take some element $y \in B(x;\epsilon)$, then we want to show that tis element is contained in $B(a;r)$. So, $y \in B(x;\epsilon)$, so that $||y-x||_2 < \epsilon$. Then,

$$\begin{aligned} ||y-a||_2 = ||y-x+x-a||_2 &\leq ||y-x||_2 + ||x-a||_2 \\ &< \epsilon + ||a-x||_2 = r \end{aligned}$$

Therefore,

$$||y-a||_2 < r$$

So $y \in B(a;r)$ as required, so $B(a;r)$ is an open set. □

Proposition 4.3.5. *(i) The sets $\emptyset$, $\mathbb{R}^d$ are open*

(ii) For any finite collection of open sets, $U_1, \ldots, U_m \subseteq \mathbb{R}^d$, their intersection is

$$U_1 \cap \cdots \cap U_m$$

is open

(iii) For any arbitrary collection of open sets $\{U_\alpha : \alpha \in I\}$, their union,

$$\bigcup_{\alpha \in I} U_\alpha$$

is open.

Proof. (i) (i) and (ii) are Exercise 4.3.1

(ii) Will add them later!

(iii) Set

$$U := \bigcup_{\alpha \in I} U_\alpha$$

Since U_α is open, there is some $\epsilon > 0$ such that

$$B(x;\epsilon) \subseteq U_\alpha$$

Then since U is the union of all the U_α, we have that $U_\alpha \subseteq U$ so it follows that

$$B(x;\epsilon) \subseteq U$$

as required.

□

Theorem 4.4.5 (Heine-Borel Theorem)**.** *Let K be a subset of $\mathbb{R}^d$. Then K is compact if and only if K is closed and bounded.*

Proof. ($\Longrightarrow$) Suppose that K. To see that K is closed, suppose for a contradiction that it is not closed. By proposition 4.3.9, F is closed if and only if for every sequence $(a_n)_{n=1}^{\infty}$ in F, if $(a_n)_{n=1}^{\infty}$ converges then

$$\lim_{n\to\infty} a_n \in F$$

So, if K is not closed, then it follows that there exists some sequence $(a_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} a_n \notin K$$

Then by proposition 2.5.4, if a sequence $(a_n)_{n=1}^{\infty}$ converges, then all subsequences of the sequence converge to the same point, and hence no subsequence converges to a point in K. This contradicts the fact that K is compact. Similarly for the boundess of K, suppose for a contradiction that K was not bounded. Then for any $n \in \mathbb{N}$, there exists $a_n \in K$ such that $||a_n||_2 \geq n$. So the sequence $(a_n)_{n=1}^{\infty}$ is unbounded, as well as all subsequences $(a_{n_k})_{k=1}^{\infty}$. Therefore, no subsequence converge since they are all unbounded. This contradicts the fact that K is compact.

($\Longleftarrow$) Assume K is closed and bounded. Since K is bounded, any sequence $(a_n)_{n=1}^{\infty}$ in K is bounded. Then by the Bolzano-Weierstrass theorem, $(a_n)_{n=1}^{\infty}$ is bounded so there exists a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. Since K is closed, from proposition 4.3.9 we have that

$$\lim_{k\to\infty} a_{n_k} \in K$$

Therefore every sequence $(a_n)_{n=1}^{\infty}$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to some $L \in K$, so K is compact. $\square$

Lecture 1

The Real Numbers $\mathbb{R}$

Summary: $\mathbb{R}$ is a complete ordered field.

1.1 Fields

Definition 1.1.1. *A* field *is a set F together with operations $+, \cdot$ satisfying*

- *(F1)* $a + b = b + a \ \forall a, b \in F$ *(Commutativity)*
- *(F2)* $(a + b) + c = a + (b + c) \ \forall a, b, c \in F$ *(Associativity)*
- *(F3)* $\exists 0 \in F$ *s.t* $0 + a = a \ \forall a \in F$ *(Additive Identity)*
- *(F4)* $\exists - a \in F$ *s.t* $a + (-a) = 0 \ \forall a \in F$ *(Additive Inverse)*
- *(F5)* $a \cdot b = b \cdot a \ \forall a, b \in F$ *(Commutativity)*
- *(F6)* $(a \cdot b) \cdot c = a \cdot (b \cdot c) \ \forall a, b, c \in F$ *(Associativity)*
- *(F7)* $\exists 1 \in F$ *s.t* $1 \cdot a = a \ \forall a \in F$ *(Multiplicative Identity)*
- *(F8)* $\forall a \in F \setminus \{0\} \ \exists a^{-1} \in F$ *s.t* $a^{-1} \cdot a = 1$ *(Multiplicative Inverse)*
- *(F9)* $a \cdot (b + c) = a \cdot b + a \cdot c \ \forall a, b, c \in F$ *(Distributivity)*

1.2 Ordered Fields

Definition 1.2.1. *An* ordered field *is a field F along with a relation $<$ satistfying*

- *(O1)* $\forall a, b, c \in F$*, if* $a < b$ *and* $b < c$ *then* $a < c$ *(Transitivity)*
- *(O2)* $\forall a, b \in F$ *exactly one of the following is true,*

$$a < b \text{ or } a = b \text{ or } b < a$$

- *(O3)* $\forall a, b, c \in F$, *if* $a < b$, *then* $a + c < b + c$
- $\forall a, b, c \in F$, *If* $a < b$ *and* $0 < c$, *then* $ac < bc$

1.3 Complete Ordered Fields

Definition 1.3.1. *Let* F *be an ordered field. Let* $S \subseteq F$. *An upper bound or* S *is some* $M \in F$ *s.t* $\forall x \in S$

$$x \leq M$$

Lecture 2

Completeness of $\mathbb{R}$, Absolute Value, Sequences

TBC.

Lecture 3

Convergence of Sequences

TBC.

Lecture 4

Properties of Convergence, Squeeze Theorem, Monotone Sequences

TBC.

Lecture 5

Subsequences, Cauchy Sequences

TBC.

Lecture 6

Limsup and Liminf

TBC.

Lecture 7

Series

Recall:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

$\sum\limits_{n=1}^{\infty} a_n$ "diverges" if above limit does not exisit.

Proposition 7.0.1. *Suppose $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ converges. Then*

(i)

$$\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(ii)

$$\sum_{n=1}^{\infty} cb_n = c\sum_{n=1}^{\infty} b_n \ \forall c \in \mathbb{R}$$

This says

$$V := \{(a_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} a_n \ \textit{converges}\}$$

is a vector space over $\mathbb{R}$.

Note:

$$\left(\sum_{n=1}^{N} a_n\right)\left(\sum_{n=1}^{N} b_n\right) \neq \sum_{n=1}^{\infty} a_n b_n$$

Proof. **Exercise.** □

Proposition 7.0.2. $\sum\limits_{n=1}^{\infty} a_n$ *converges* iff $\sum_{n=k}^{\infty} a_k$ *converges.*

Proof. **Exercise.** ∎

Example: TBC.

Proposition 7.0.3. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

7.1 Divergence Test

Proposition 7.1.1 (Divergence Test)**.** *If* $a_n \not\to 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. $\sum_{n=1}^{\infty} a_n$ converges $\implies S_n \to L$ for some L, where

$$S_n \coloneqq \sum_{n=1}^{N} a_n$$

So,

$$a_n = S_n - S_{n-1}$$

By the Algebra of Limits,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= L - L = 0\end{aligned}$$

∎

Example: TBC.

7.2 Convergence Tests

Proposition 7.2.1 (Boundedness Test)**.** *Let* $(a_n)_{n=1}^{\infty}$ *bea sequence, if*

(i) $a_n \geq 0$

(ii) There is an upper bound on the parital sums

$$\exists M > 0 \; s.t \; \sum_{n=1}^{N} a_n \leq M$$

Then $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let

$$S_N \coloneqq \sum_{n=1}^{N} a_n$$

Then

$$\begin{aligned}
S_{N+1} &= S_N + a_{N+1} \\
&\geq S_n
\end{aligned}$$

So by the Monotone Convergence Criterion, $(S_N)_{N=1}^{\infty}$ converges $\iff$ it is bounded avove. By (ii), it is bounded. $\square$

Proposition 7.2.2 (Comparison Test)**.** *TBC.*

Lecture 8

Ratio, Root, Alternating Series, and Integral Test, Cauchy Convergence, Topology of $\mathbb{R}^d$

8.1 Ratio Test

Proposition 8.1.1 (Ratio Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of nonzero elements.*

(i) If

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n} > 1\right| > 1$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

(i) Let

$$q = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

where $q < 1$, $\exists r \in (q, 1)$. By the definition of $\limsup$,

$$\left|\frac{a_{n+1}}{a_n}\right| \leq r \ \forall n \geq n_0$$

for some $n_0 \in \mathbb{N}$.

$$\begin{aligned}
&\left|\frac{a_{n_0+1}}{a_{n_0}}\right| \leq r \\
&|a_{n_0+1}| \leq r|a_{n_0}| \\
&|a_{n_0+2}| \leq r|a_{n_0+1}| \leq r^2|a_{n_0}|
\end{aligned}$$

By induction, we have

$$0 \leq |a_{n_0+k} \leq r^k|a_{n_0}|$$

By the comparison test,

$$\sum_{k=1}^{\infty} |a_{n_0+k}|$$

converges since

$$\sum_{k=1}^{\infty} r^k|a_{n_0}|$$

is a geometric sequence and $0 \leq r < 1$.

$$\sum_{n=1}^{\infty} |a_n|$$

Converges, thus $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

(ii) Let

$$q = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$$

so $\exists n_0$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1$$

for all $n \geq n_0$. Then

$$|a_{n_0+k} \geq |a_{n_0}| \ \forall k \geq 0$$

So $a_n \not\to 0$ as $n \to 0$, thus by the divergence test. $\sum\limits_{n=1}^{\infty} a_n$ diverges.

$\square$

Note: The ratio test does not tell us anything when the limit is 1.

For example:

$$\sum_{n\geq 1}\frac{1}{n}$$

diverges, but

$$\frac{\frac{1}{n+1}}{\frac{1}{n}}=\frac{n}{n+1}\to 1$$

But on the other hand,

$$\sum_{n\geq 1}\frac{1}{n+1}$$

converges, and

$$\frac{\frac{1}{n+1(n+2)}}{\frac{1}{n(n+1)}}=\frac{n+1}{n+2}\to 1$$

8.2 Root Test

Proposition 8.2.1 (Root Test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.*

(i) If

$$\limsup_{n\to\infty}\sqrt[n]{|a_n|}<1$$

then $\sum_{n=1}^{\infty}a_n$ converges absolutely.

(ii) If

$$\limsup_{n\to\infty}\sqrt[n]{|a_n|}>1$$

then $\sum_{n=1}^{\infty}a_n$ diverges.

Proof.

(i) Let

$$q=\limsup_{n\to\infty}\sqrt[n]{|a_n|}<1$$

Then there exists $r\in(q,1)$ such that $\exists n_0$

$$\sqrt[n]{|a_n|}\leq r$$

for all $n\geq n_0$. Then,

$$0\leq|a_n|\leq r^n$$

Therefore $\sum_{n=n_0}^{\infty}r^n$ converges since $0<r<1$. Then by the comparison test, $\sum_{n=n_0}^{\infty}|a_n|$ converges.

(ii) Let

$$q = \limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$$

From exercise 2.7.3, there are infinitely many

$$\sqrt[n]{|a_n|} \geq 1 \implies |a_n| \geq 1$$

Thus $a_n \not\to 0$ as $n \to \infty$, thus by the divergence test, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

$\square$

Examples:

- $$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$
  $$\sqrt[n]{\left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

- $$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)_n$$
  $$\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \frac{1}{2^{n+1}} \to \frac{1}{2}$$

- $$\sum_{n=1}^{\infty} \frac{1}{n}_n$$
  $$\sqrt[n]{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} = \frac{1}{e^{\frac{\ln n}{n}}} \to 0$$

8.3 Alternating Series Test

Proposition 8.3.1 (Alternating Series Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose*

(i) $(a_n)_{n=1}^{\infty}$ *is decreasing*

(ii) $\lim_{n\to\infty} a_n = 0$

Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges. Moreover, for any N,

$$\sum_{n=1}^{2N} (-1)^{n+1} a_n \leq \sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq \sum_{n=1}^{2N-1} (-1)^{n+1} a_n$$

Proof. $a_n \geq 0\ \forall n$, since $a_n \to 0$ as $n \to \infty$, and $(a_n)_{n=1}^{\infty}$ is decreasing. Let

$$S_N := \sum_{n=1}^{N} (-1)^{n+1} a_n$$

If N is even,

$$S_{N+2} = S_N + a_{N+1} - a_{N+2} \geq S_N$$

Since

$$a_{N+2} \leq a_{N+1}$$

So $(S_{2N})_{N=1}^{\infty}$ is an increasing sequence and $(S_{2N-1})_{N=1}^{\infty}$ is a decreasing sequence. So by the monotone convergence criterion, both sequence converge.

$$S_{2N-1} + a_{2N} = S_{2N}$$

$$a_{2N} \xrightarrow{N \to \infty} 0$$

So,

$$\lim_{n \to \infty} S_{2N-1} = \lim_{N \to \infty} S_{2N} = L \implies \lim_{N \to \infty} S_N = L$$

$(S_{2N})_{N=1}^{\infty}$ is increasing, so

$$L = \sup\{(S_{2N})_{N=1}^{\infty}\} \implies S_{2N} \leq L$$

Similarily, $(S_{2N-1})_{N=1}^{\infty}$ is decreasing, so

$$L = \inf\{(S_{2N_1})_{n=1}^{\infty}\} \implies S_{2N-1} \geq L$$

$\square$

8.4 Integral Test

Proposition 8.4.1. *Let $f : [1, \infty) \to \mathbb{R}$. Suppose that*

(i) $f(x) \geq 0\ \forall x \in [1, \infty)$

(ii) f *is decreasing*

Then

$\sum_{n=1}^{\infty} f(n)$ *converges* $\iff$ *the improper integral* $\int_1^{\infty} f(x)\ dx$ *converges.*

8.5 Cauchy Convergence Criterion for Series

Proposition 8.5.1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then*

$\sum_{n=1}^{\infty} a_n$ *converges* $\iff \forall \epsilon > 0\ \exists N_0$ *such that* $N \leq M \leq N_0$, $|\sum_{n=M}^{N} a_n|\epsilon$.

Proof. Let

$$S_N \coloneqq \sum_{n=1}^{N} a_n$$

Then $\sum_{n=1}^{\infty} a_n$ converges $\iff (S_N)_{N=1}^{\infty}$ converges. Cauchy convergence criterion for sequences says that

$$(S_N)_{n=1}^{\infty} \text{ converges} \iff \text{it is cauchy.}$$

$$\iff \forall \epsilon > 0\ \exists N_0\ s,t\ |S_N - S_M| < \epsilon\ \forall N, M \geq N_0$$

□

8.6 Topology of $\mathbb{R}^d$

8.6.1 Norms

Definition 8.6.1. *A norm on $\mathbb{R}^d$ is a function $||\cdot|| : \mathbb{R}^d \to [0, \infty)$ satisfying the following properties:*

(i) $||a|| = 0 \iff a = (0, \ldots, 0)$

(ii) $||ca|| = |c| \cdot ||a||\ \forall c \in \mathbb{R}, a \in \mathbb{R}^d$

(iii) $||a + b|| \leq ||a|| + ||b||\ \forall a, b \in \mathbb{R}^d$

The euclidean norm of $\mathbb{R}^d$ is given by

$$||a, \ldots, a_d||_2 = \sqrt{\sum_{i=1}^{d} a_i^2}$$

The dot prdouct on $\mathbb{R}^d$ is given by

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum_{i=1}^{d} a_i b_i$$

We also have the l_1-norm

$$||(a_1, \ldots, a_n)||_1 = \sum_{i=1}^{d} |a_i|$$

And the l_∞-norm

$$||(a_1, \ldots, a_d)||_\infty = \max\{|a_1|, \ldots, |a_d|\}$$

Proposition 8.6.1. *Let $a, b \in \mathbb{R}^d$, write $||\cdot||$ for the euclidean norm on $\mathbb{R}^d$. Then*

(i) ***Cauchy Schwarz Inequality:***

$$|a \cdot b| \leq ||a|| \cdot ||b||$$

(ii) ***Triangle Inequality:***

$$||a + b|| \leq ||a|| + ||b||$$

(iii) $||\cdot||$ is a norm on $\mathbb{R}^d$

Proof.

(i) Consider the equadratic function

$$\begin{aligned} P(t) &= ||a + tb||^2 \\ &= (a + tb) \cdot (a + tb) \\ &= a \cdot a + 2a \cdot tb + tb \cdot tb \\ &= ||a||^2 + 2t(a \cdot b) + t^2||b||^2 \end{aligned}$$

The discriminant of $P(t)$ is less than or equal to 0,

$$\begin{aligned} (2a \cdot b)^2 - 4||a||^2||b||^2 &\leq 0 \\ (a \cdot b)^2 &\leq ||a||^2||b||^2 \\ |a \cdot b| &\leq ||a|| \cdot ||b|| \end{aligned}$$

(ii)

$$\begin{aligned} ||a + b||^2 &= ||a||^2 + 2a \cdot b + ||b||^2 \\ &\leq ||a||^2 + 2|a \cdot b|| + ||b||^2 \\ &\leq ||a||^2 + 2||a|| \cdot ||b|| + ||b||^2 \\ &= (||a|| + ||b||)^2 \end{aligned}$$

$$||a + b||^2 \leq (||a|| + ||b||)^2 \implies ||a + b|| \leq ||a|| + ||b||$$

(iii) **Exercise.** We want to prove $||a|| = 0 \iff a = 0$ and $||ca|| = |c| \cdot ||a||$.

$\square$

Lecture 9

$\mathbb{R}^d$

Recall: $||(x_1, \dots, x_d)|| \coloneqq \sqrt{x_1^2 + \cdots + x_d^2}$. This is a norm. i.e.

$$||a+b|| \leq ||a||_2 + ||b||_2 \ \forall a, b \in \mathbb{R}^d$$

$$||ca|| = |c| \cdot ||a||_2, \ c \in \mathbb{R}, \ a \in \mathbb{R}^d$$

$$||a||_2 > 0, \ \forall a \in \mathbb{R}^d \setminus \{(0, \dots, 0)\}$$

$$||(0, \dots, 0)||_2 = 0$$

Other examples of norms:

- $||(x_1, \dots, x_d)||_1 \coloneqq |x_1| + \cdots + |x_d|$
- $||(x_1, \dots, x_d)||_\infty \coloneqq max\{|x_1|, \dots, |x_d|\}$

Exercise: For $a \in \mathbb{R}^d$,

$$||a||_\infty \leq ||a||_2 \leq ||a||_1 \leq d||a||_\infty (\leq d||a||_2)$$

Interesting Fact: There are other norms. but they are all equivalent in the sense that if $|| \cdot ||, || \cdot ||'$ are norms on $\mathbb{R}^d$, then $\exists v, R > 0$ such that

$$r||a|| \leq ||a||' \leq R||a||$$

9.1 Convergence

Definition 9.1.1. *Let $(a_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}^d$ and let $L \in \mathbb{R}^d$, we say $(a_n)_{n=1}^\infty$ **converges** to L, and write $\lim_{n\to\infty} = L$ or $a_n \to \infty$, if*

$$\lim_{n\to\infty} ||a_n - L||_2 = 0$$

Note: We could define convergence instead using some other norm, say $||\cdot||_1$.

If $||a_n - L||_2 \to 0$, then $||a_n - L||_1 \leq d||a_n - L||_2 \to 0$ If $||a_n - L||_1 \to 0$, then $||a_n - L||_2 \leq d||a_n - L||_1 \to 0$

in general, if $||\cdot||$ is any norm, then since $||\cdot||$ and $||\cdot||_2$ are equivalent.

$$||a_n - L||_2 \to 0 \iff ||a_n - L|| \to 0$$

Example: Say $a_n = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$, then

$$||a_n - L||_2 = \sqrt{1/n^2, + \cdots + 1/n^2} = \sqrt{\frac{d}{n^2}} = \frac{\sqrt{d}}{n} \to 0$$

$$\therefore a_n \to L$$

Given a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$, we write $a_1 = (a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)})$ where $a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)} \in \mathbb{R}$. Similarily,

$$a_1 = (a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)})$$

$$a_2 = (a_2^{(1)}, a_2^{(2)}, \dots, a_2^{(d)})$$

$$a_3 = (a_3^{(1)}, a_3^{(2)}, \dots, a_3^{(d)})$$

$$\vdots$$

$$L = (L^{(1)}, L^{(2)}, \dots, L^{(d)}) \in \mathbb{R}^d$$

We get d sequences in $\mathbb{R}$, and d possible limit points $L^{(1)}, \dots, L^{(d)} \in \mathbb{R}$

Proposition 9.1.1. *Given $(a_n)_{n=1}^{\infty}$ and L as above, $a_n \to L$ as $d \to \infty \iff a_n^{(i)} \to L^{(i)}$ in $\mathbb{R}$ as $n \to \infty$, for $i = 1, \dots, d$.*

Proof. $\implies$: Suppose $a_n \to L$, i.e.

$$||a_n - L||_2 \to 0$$

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$||x||_2^2 = x_1^2 + \cdots x_d^2 \geq x_i^2 = |x_i|^2 \therefore |x_i \leq ||x||_2$$

Applying this to (*), we get

$$0 \leq |a_n^i - L^{(i)}| \leq ||a_n - L||_2 \to 0$$

So by the squeeze theorem,

$$|a_n^{(i)} - L^{(i)}| \to 0$$

$\implies$: Suppose $a_n^{(i)} \to L^{(i)}$ for $i = 1, \dots, d$.

$$||a_n - L||_2^2 = (a_n^{(i)} - L^i)^2 + \cdots + (a_n^d - L^(d)) \to 0$$

By algebra of limits,

$$\therefore ||a_n - L||_2 \to 0$$

$\square$

Example: $a^n = ((-1)^n, \frac{1}{n}) \in \mathbb{R}^2$. Does (a_n) converge? No, since $(-1)^n$ does not converge.

Definition 9.1.2. *A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$ is **Cauchy** if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}_{\geq 1}$ such that*

$$||a_n - a_m||_2 < \epsilon \ \forall m, n \geq n_0$$

Theorem 9.1.1 (Cauchy Convergence Criterion for $\mathbb{R}^d$)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$. It converges $\iff$ it is Cauchy.*

Proof. $\implies$: Suppose $a_n \to L \in \mathbb{R}^d$. To show it is Cauchy, let $\epsilon > 0$. $||a_n - L||_2 \to 0$, so $\exists n_0 \in \mathbb{N}_{\geq 1}$ such that

$$||a_n - L||_2 < \frac{\epsilon}{2} \ \forall n \geq n_0)$$

Then if $m, n \geq n_0$,

$$\begin{aligned} ||a_m - a_n||_2 &= ||a_m - L + L - a_n||_2 \\ &\leq ||a_m - L||_2 + ||L - a_n||_2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\therefore (a_n)_{n=1}^{\infty}$ is Cauchy. $\implies$: Suppose $(a_n)_{n=1}^{\infty}$ is Cauchy, write

$$a_n = (a_n^{(1)}, \dots, a_n^{(d)}$$

For any $m, n \in \mathbb{N}_{\geq 1}$,

$$|a_n^{(i)} - a_m^{(i)} \leq ||a_n - a_m||_2$$

$\therefore (a_n^{(i)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. So by the Cauchy Convergence Criterion, $\exists L^{(i)} \in \mathbb{R}$, such that $a_n^{(i)} \to L^{(i)}$. By the previous proposition,

$$a_n \to (L^{(1)}, \dots, L^{(d)})$$

$\square$

Definition 9.1.3. *$S \subseteq \mathbb{R}^d$ is **bounded** if $\exists M > 0$ such that*

$$||x|| \leq M \quad \forall x \in S$$

A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$ is bound if $\{a_n : n \in \mathbb{N}_{\geq 1}\}$ is a bounded set.

Theorem 9.1.2 (Bolzano-Weierstrass for $\mathbb{R}^d$). *If $(a_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^d$, then it has a subsequence $(a_{n_k})_{n=1}^{\infty}$ that converges.*

Proof. Write $a_n = (a_n^{(1)}, \ldots, a_n^{(d)})$.
We will prove it by indunction on d. For $d = 1$, this is the Bolzano-Weierstrass theorem for $\mathbb{R}$ For $d > 1$ write

$$b_n \coloneqq (a_n^{(1)}, \ldots, a_n^{(d-1)}) \in \mathbb{R}^{d-1}$$

By the induction hypothesis, b_n has a subsequence $(b_{n_k})_{k=1}^{\infty}$ that converges. Let $L \in \mathbb{R}^{d-1}$ be the limit of this subsequence. Then $L \in \mathbb{R}^{d-1}$ is the limit of b_n. $(a_{n_k}^{(d)})_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, so it has a subsequence $(a_{n_{k_j}}^{(d)})_{j=1}^{\infty}$ that converges. Let $L^{(d)} \in \mathbb{R}$ be the limit of this subsequence. Then $L = (L^{(1)}, \ldots, L^{(d-1)}, L^{(d)})$ is the limit of a_n. $\square$

Lecture 10

Open and Closed Sets in $\mathbb{R}^d$

Roughly, an open set is one that we draw with dotted lines. The line represents a "boundary" that is the not in the set. This is not a rigorous definition.

Definition 10.0.1 (Open Ball)**.** *Let $a \in \mathbb{R}^d$, $r > 0$. The **open ball** of radius r centered at a is*

$$B(a; r) := \{x \in \mathbb{R}^d : ||x - a||_2 < r\}$$

Relation to Convergence: If $a_n \to L$, then this means that $||a_n - L||_2 < \epsilon$ for all n large. So, $a_n \in B(L; \epsilon)$

Definition 10.0.2 (Open Sets)**.** *A set $U \subseteq \mathbb{R}^d$ is **open** if $\forall a \in U$, $\exists r > 0$, such that $B(a; r) \subseteq U$*

Idea: If $a \in U$, then a is not on the boundary but it is truly "inside" the set, so we can fit a ball containing a in the set.

Definition 10.0.3 (Closed Sets)**.** *A set $k \in \mathbb{R}^d$ is **closed** if its complement $\mathbb{R}^d \setminus k$ is open.*

Example: $U \subseteq (0, 1)$. Is this open? Yes.

Proof. Let $a \in U$. We let $r := \min\{|a - 0|, |a - 1|\}$ (We do this so that r is at most the distance to the closest bound, i.e. if a is closer to 0, then the radius r cannot be $|a - 1|$)then

$$B(a; r) = (a - r, a + r) \subseteq (0, 1) = U$$

□

Example: $U := [0, 1]$. Is this open? No.

Proof. Let $a := 0 \in U$. The for any $r > 0$, $\exists z \in B(a; r) = (-r, r)$ s.t $z < 0$, so $z \notin U$. Therefore $B(a; r) \subseteq U$ □

Is U closed? This is the same as asking if $\mathbb{R} \setminus U = (-\infty, 0) \cup (1, \infty)$ is open. This is open.

Proof. Let $a \in (-\infty, 0) \cup (1, \infty)$.

- **Case 1:** $a \in (-\infty, 0)$. Set $r := |a|$, so

$$B(a; r) = (a - r, a + r) = (2a, 0) \subseteq U$$

- **Case 2:** $a \in (1, \infty)$ similar.

$\square$

Therefore $U = [0, 1]$ is closed.

Example: Is $U \coloneqq (0, 1]$ open? No, for any $r > 0$

$$B(1; r) \not\subseteq U$$

Therefore, it is not open.

Note: *Sets are not always open or closed. Most sets are neither open nor closed.*

This set U is one such example U is not closed since $\mathbb{R} \setminus U = (-\infty, 0] \cup (1\infty)$ $0 \in \mathbb{R}\ U$ but $\forall r > 0$, $B(0; r) \not\subseteq R \setminus U$

Example: For any $a \in \mathbb{R}^d$, $r > 0$ $B(a; r)$ is an open set.

Proof. Let $x \in B(a; r)$, so $||x - a||_2 < r$. Set

$$r_0 \coloneqq r - ||x - a||_2 > 0$$

Claim: $B(x; r_0) \subseteq B(a; r)$ To see this, let $y \in B(x; r_0)$ so $|y - x||_2 < r_0$. So,

$$\begin{aligned} ||y - a||_2 &\leq ||y - x||_2 + ||x - a||_2 && (\triangle\text{-inequality}) \\ &< r_0 + ||x - a||_2 \\ &= r \end{aligned}$$

$\square$

Proposition 10.0.1. *(i) $\emptyset$, $\mathbb{R}^d$ are both open in $\mathbb{R}^d$*

(ii) If $U_1, U_2, \ldots, U_n \subseteq \mathbb{R}^d$ are all open, then so is $U_1 \cap U_2 \cap \cdots \cap U_n$.

(iii) If $U_a \subseteq \mathbb{R}^d$ is an open set for all $\alpha \in I$, (I is some index set) then

$$\bigcup_{a \in I} U_a$$

is open.

Proof. *(i),(ii) are exercises.*

(iii): Set

$$V := \bigcup_{\alpha \in I} U_a$$

Let $a \in V$. This means $\exists \alpha \in I$ such that $a \in U_\alpha$. U_α is open so $\exists r > 0$ s.t $B(a; r) \subseteq U_\alpha$. $U_\alpha \leq \bigcup_{\alpha \in I} U_\alpha = V$ So $B(a; r) \subseteq V$ as required. □

Example: For any $n \in \mathbb{N}_{\geq 1}$.

$$\left(\frac{-1}{n}, \frac{1}{n}\right) = B(0; \frac{1}{n})$$

is open in $\mathbb{R}$. The intersection of these open sets is

$$\bigcap_{n=1}^{\infty} \left(\frac{1}{n}, \frac{-1}{n}\right) = \{0\}$$

which is not open. This shows that openess is not preserved by infinite intersections.

Example: Let

$$U := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$$

U is open but not closed.

$$U = V \cap W$$

where

$$V := \{(x, y) : x > 0\} \qquad W := \{(x, y) : y > 0\}$$

To show V is open, let $a = (x, y) \in V$. Set $r := x > 0$. Then if $(w, z) \in B(a; r)$. Then

$$|w - z| \leq ||(w, z) - a||_2 < r = x$$

$$\therefore w > x - x = 0$$

So $(w, z) \in U$. Similarly, W is open. Therefore U is open.

Not Closed: Exercise.

Proposition 10.0.2. *Let $K \subseteq \mathbb{R}^d$. K is closed $\iff$ for any subsequence $(a_n)_{n=1}^{\infty}$ in K, If it converges, then*

$$lim_{n \to \infty} a_n \in K$$

Proof. ($\implies$) Suppose K is closed. Let $(a_n)_{n=1}^{\infty}$ be a sequence in K s.t

$$L := lim_{n\to\infty} a_n$$

exists. Suppose for a contradction $L \notin K$. This means $L \in \mathbb{R}^d \setminus K$, which is open. So $\exists r > 0$ such that

$$B(L;r) \subseteq \mathbb{R}^d \setminus K$$

Since $a_n \to L$, we must have $a_n \in B(L;a)$ for some n (in fact, for all n sufficiently large. So $a_n \in B(L;r) \subseteq \mathbb{R}^d \setminus K$. Therefore $a_n \notin K$, which is a contradiction.

($\impliedby$) Suppose K is not closed, and we'll prove $\exists (a_n)_{n=1}^{\infty}$ in K such that $a_n \to L \notin K$. Since K is not closed, $\mathbb{R}^d \setminus K$ is not open. So $\exists L \in \mathbb{R}^d \setminus K$ such that $\forall r > 0$

$$B(L;r) \nsubseteq \mathbb{R}^d \setminus K$$

For each $n \in \mathbb{N}_{\geq 1}$, we can fine $a_n \in B(L; \frac{1}{n})$ such that $a_n \notin \mathbb{R}^d \setminus K$. So $a_n \in K$. This gives a sequence $(a_n)_{n=1}^{\infty}$ in K and

$$||a_n - L||_2 < \frac{1}{n} \to 0$$

Therefore by the Squeeze Theorem,

$$||a_n - L||_2 \to 0 \implies a_n \to L$$

$L \in \mathbb{R}^d \setminus K$, so $L \notin K$. $\square$

Definition 10.0.4. *Let $A \subseteq \mathbb{R}^d$ and let $a \in \mathbb{R}^d$, a is:*

*(i) an **interior point** if $\exists r > 0$ s.t $B(a;r) \subseteq A$*

*(ii) an **accumulation point** if $\exists$ a sequence $(a_n)_{n=1}^{\infty}$ in A s.t $a_n \to a$*

*(iii) a **boundary point** if it is an accumulation point and it is not an interior point.*

$$A^\circ := \{\textit{All interior points}\}$$
$$\bar{A} := \{\textit{All accumulation points}\}$$
$$\partial A := \{\textit{All boundary points}\} = \bar{A} \setminus A^\circ$$

Note: *The set of interior points, accumulation points, and boundary points are referred to as the **interior** of A, the **closure** of A, and the **boundary** of A respectively*

Example: $A := (0,1] \cup \{2\}$

$$A^\circ = (0,1)$$
$$\bar{A} = [0,1] \cup \{2\}$$

$$\partial A = \{0, 1, 2\}$$

Example: $A \coloneqq \mathbb{Q}$

Since any open interval contains irrational numbers, we have

$$A^\circ = set$$

Proposition from chapter 2,

$$\bar{A} = \mathbb{R}$$

$$\partial A = \mathbb{R}$$

Lecture 11

Compactness

Definition 11.0.1. *A set $A \subseteq \mathbb{R}^d$ is (sequentially) compact if every sequence $(a_n)_{n=1}^{\infty}$ in A has a subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to a point in A.*

Example 1: Is [0,1] compact? Yes.

Proof. **Recall:** Bolzano-Weierstrass theorem states bounded sequence has a convergent subsequence.

Therefore, every sequence $(a_n)_{n=1}^{\infty}$ in $[0,1]$ has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges. So

$$0 \leq a_{n_k} \leq 1 \implies 0 \leq \lim_{k\to\infty} a_{n_k} \leq 1$$

$$\therefore L \in [0,1]$$

□

Example 2: Is (0,1) compact? No.

Proof. By counter example, let $a_n := \frac{1}{n+1}$, so $a_n \to 0$. Therefore for all subsequences of a_n, $a_{n_k} \to 0$. So there exists no subsequence which converges to a point in (0,1). □

Example 3: Is $[0,\infty)$ compact? No.

Proof. The Bolzano-Weierstrass theorem does not apply since $[0,\infty)$ is unbounded. Set $a_n := n$, then $a_n \to \infty$, so it has no bounded subsequence and therefore no convergent subsequences. □

Theorem 11.0.1 (Heine-Borel)**.** *Let $A \subseteq \mathbb{R}^d$. A is compact $\iff$ A is closed and not bounded.*

Proof. ($\implies$) Similar to example 1. Assume A is closed and bounded. Let $(a_n)_{n=1}^{\infty}$ be a sequence in A. The sequence is bounded since A is, so by the Bolzano-Weierstrass theorem for $\mathbb{R}^d$, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. $a_{n_k} \in A \; \forall k$ and $a_{n_k} \to L$ and A is closed, so by the sequential characterization of closedness, $L \in A$, therefore A is compact.

($\Longleftarrow$) Assume A is compact. To show A is closed, assume for a contradction that A is not closed. Therefore there exists a sequence $(a_n)_{n=1}^{\infty}$ in A such that $a_n \to L \notin A$. Then for any subsequence $(a_{nk})_{k=1}^{\infty}$, we have

$$a_n \to L \notin A$$

This contradicts that A is compact, therefore A is closed.

Tow show A is bounded, assume for a contradiction that A is not bounded. Then $\forall n \in \mathbb{N}_{\geq 1}$, there exists $a_n \in A$ such that $||a_n||_2 \geq n$. This gives a sequence $(a_n)_{n=1}^{\infty}$. Since A is compact, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges. But

$$||a_{n_k}||_2 \geq n_k \to \infty$$

So $(a_{nk})_{k=1}^{\infty}$ is unbounded, which is a contradiction. $\square$

Proposition 11.0.1.

(i) If $k_1, \ldots, k_n \subseteq \mathbb{R}^d$ are compact, then $\bigcup_{i=1}^{n} k_i$ is compact.

(ii) If $k_1, \ldots, k_n \subseteq \mathbb{R}^d$ are compact, then $\bigcap_{i=1}^{n} k_i$ is compact.

Proof. **Exercise:**

(i) Assume $A \coloneqq \bigcup_{i=1}^{n} k_i$. Let $(a_n)_{n=1}^{\infty}$ be a sequence in A. Then there exists $i \in \{1, \ldots, n\}$ such that $a_n \in k_i$. Since k_i is compact, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. $L \in k_i$ and $k_i \subseteq A$, so $L \in A$. Therefore A is compact.

(ii) Assume $A \coloneqq \bigcap_{i=1}^{n} k_i$. Let $(a_n)_{n=1}^{\infty}$ be a sequence in A. Then $a_n \in k_i$ $\forall i \in \{1, \ldots, n\}$. Since k_i is compact, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. $L \in k_i$ $\forall i \in \{1, \ldots, n\}$ and $k_i \subseteq A$, so $L \in A$. Therefore A is compact.

$\square$

Definition 11.0.2. *$A \subseteq \mathbb{R}^d$ is **compact** if for any collection*

$$U_\alpha : \alpha \in I$$

of open sets such that

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha$$

There exists finitely many indeces $\alpha_1, \ldots, \alpha_n$ such that

$$A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

Lecture 12

Limits of a Function of Continous Variables

A sequence is a function $\mathbb{N} \to \mathbb{R}$. Here, we'll consider function that are going from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{R}^d \to \mathbb{R}^m$).

Definition 12.0.1. *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X. $f : X \to \mathbb{R}^m$, $L \in \mathbb{R}^m$. We say the limit of f as X approaches a is L if*

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t \ \forall x \in X$$

$$x \in B(a; \delta) \wedge x \neq a \implies ||f(x) - L||_2 < \epsilon$$

The idea is like the definition of convergence of a sequence, except we replace $n \geq n_0$ (which captures "n is sufficiently large") with $x \in B(a; \delta)$, $x \neq a$ (which captures "x is close to, but not equal to a). In other words, the definition says that if x is close to (but not equal to) a then $f(x)$ is close to L.

Why "not equal to"?: Often we consider the limit as x approaches a when $f(a)$ is not defined. Other times we compare the limite to $f(a)$. So we do not want to use $f(a)$ in the definition of the limit.

Notation: We write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \ as \ x \to d$$

to mean that the limit of f is L as x approaches a.

Example: $f : \mathbb{R} \to \mathbb{R}$. $f(x) := 3x - 2$. Let $a \in \mathbb{R}$. Claim

$$\lim_{x \to a} f(x) = 3a - 2$$

Proof. Let $\epsilon > 0$. Consider

$$\begin{aligned}|f(x) - (3a-2)| &= |3x - 2 - 3a + 2| \\ &= 3|x - a|\end{aligned}$$

We want this $< \epsilon$, set $\delta \coloneqq \frac{\epsilon}{3}$. Then if $x \in B(a;\delta) = (a-\delta, a+\delta)$ (i.e $|x-a| < \delta$) then

$$|f(x) - (3a-2)| = 3|x-a| < 3\delta = \frac{3\epsilon}{3} = \epsilon$$

$\square$

Example: $g : \mathbb{R} \to \mathbb{R}$. $g(x) \coloneqq x^2$. Claim:

$$\lim_{x \to a} g(x) = a^2$$

Proof. Let $\epsilon > 0$ be given.

$$\begin{aligned}|g(x) - a^2| &= |x^2 - a^2| \\ &= |x-a||x+a|\end{aligned}$$

What happens if x is close to a? Intuitively, $|x+a|$ is close to $|a+a|$ and $|x-a|$ is small.

$$\begin{aligned}|x+a| = |x - a + a + a| &\leq |x-a| + |a+a| \\ &< 2|a| + \delta && (\text{if } |x-a| < \delta) \\ &\leq 2|a| + 1 && (\delta \leq 1)\end{aligned}$$

Then,

$$\begin{aligned}|x^2 - a^2| = |x-a||x+a| &\leq |x-a|(2|a|+1) \\ &< \delta(2|a|+1) && (\text{if } |x-a| < \delta) \\ &\leq \epsilon && (\text{if } \delta \leq \tfrac{\epsilon}{2|a|+1})\end{aligned}$$

Important: Do not define δ in terms of x or δ! We can use a here since a is constant

So we set $\delta \coloneqq \min\{1, \frac{\epsilon}{2|a|+1}\}$ Then $\delta \leq \frac{\epsilon}{2|a|+1}$ and $\delta \leq 1$. So if $|x-a| < \delta$. Then from the work above, $|x^2 - a^2| < \epsilon$ as required. $\square$

Note: In proofs where we have $\delta - \epsilon$, we often use

$$\delta \coloneqq \min\{\dots\}$$

In proofs where we have $n_0 - \epsilon$, we often use

$$n_0 \coloneqq \max\{\dots\}$$

Proposition 12.0.1 (Uniqueness of Limits)**.** *Let $f : X \to \mathbb{R}^m$ ($X \subseteq \mathbb{R}^d$), $a \in \mathbb{R}^d$ a limit point of X, $L, L' \in \mathbb{R}^m$. If the limit of f as $x \to a$ is L and the limit of f as $x \to a$ is L', then $L = L'$*

Proof. By contradction. Suppose $L \neq L'$. So

$$||L - L'||_2 > 0$$

Set

$$\epsilon := \frac{||L - L||_2}{2} > 0$$

Since $f(x) \to L$ as $x \to a$, $\exists \delta > 0$ such that if $x \in X \cap B(a; \delta) \setminus \{a\}$, then

$$||f(x) - L||_2 < \epsilon$$

Since $f(x) \to L'$ as $x \to a$, $\exists \delta' > 0$ such that

$$x \in X \cap B(a; \delta') \setminus \{a\} \implies ||f(x) - L'||_2 < \epsilon$$

Let $\delta_0 \neq \min\{\delta, \delta'\}$. Let

$$x \in X \cap B(a; \delta_0) \setminus \{a\}$$

Then

$$x \in X \cap B(a; \delta) \setminus \{a\}$$

So,

$$||f(x) - L||_2 < \epsilon$$
$$||f(x) - L'||_2 < \epsilon$$

So,

$$\begin{aligned} ||L - L'||_2 \leq ||L - f(x)||_2 + ||f(x) - L'|| &< \epsilon + \epsilon \\ &= ||L - L'||_2 \end{aligned}$$

And thus, a contradction. ∎

Proposition 12.0.2 (Sequential Characterization of Limits)**.** *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$, a limit point of X. $f : X \to \mathbb{R}^m$, $L \in \mathbb{R}^m$.*

$\lim_{x \to a} f(x) = L \iff$ *for every sequence $(x_n)_{n=1}^\infty$ in X such that $x_n \to a$, we have*

$$\lim_{n \to \infty} f(x_n) = L$$

Proof. ($\implies$) Suppose $\lim_{x \to a} f(x) = L$. Let $(x_n)_{n=1}^\infty$ be a sequence in $X \setminus \{a\}$ such that $x_n \to a$. We must show that $f(x_n) \to L$.

Let $\epsilon > 0$ be given. Since $f(x) \to L$ as $x \to a$, $\exists \delta$ such that

$$x \in X \cap B(a; \delta) \setminus \{a\} \implies ||f(x) - L||_2 < \epsilon$$

Since $x_n \to$, using δ in place of ϵ, $\exists n_0$ such that $\forall n \geq n_0$, $||x_n a||_2 < \delta$. i.e. $x_n \in B(a, \delta)$. Also $x_n \in X \setminus \{a\}$ Therefore,

$$||f(x) - L||_2 < \epsilon$$

($\Longleftarrow$) Suppose $\forall$ sequences $(x_n)_{n=1}^{\infty}$ ins $X \setminus \{a\}$ converging to a, $f(x) \to L$, and for a contradction, suppose

$$f(x) \not\to L$$

We negate "$f(x) \to L$" to get that $\exists \epsilon > 0$ such that $\forall \gamma > 0$, $\exists x \in X \cap B(a; \delta) \setminus \{a\}$ such that $||f(x) - L||_2 \geq \epsilon$.

This gives a sequence $(x_{n n})_{n=1}^{\infty}$ in $X \setminus \{a\}$, $||x_n - a||_2 \leq \frac{1}{n}$ $\forall n$, so by the squeeze theorem

$$||x_n - a||_2 \to 0$$

Since $||f(x_n) - L||_2 \geq \epsilon$, $f(x_n) \not\to L$. This is a contradction. $\square$

Note: if $\lim_{n\to\infty} f(x_n) = L$ for *some* sequence $(x_n)_{n=1}^{\infty}$ in $X \setminus \{a\}$ convering to a, it *does not* follow that $\lim_{x\to a} f(x) = L$

Example:

$$f(x) := \begin{cases} 0 \text{ if } x = \frac{1}{n},\, n \in \mathbb{N}_{\geq 1} \\ 1 \text{ otherwise} \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist but $\lim_{n\to\infty} f(\frac{1}{n}) = 0$

Proposition 12.0.3 (Algebra of Limits)**.** *Let $x \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X, $f : X \mathbb{R}^m$,$g : X \to \mathbb{R}^m$, $L, K \in \mathbb{R}^m$. Suppose* $\lim_{x\to a} f(x) = L$, *$lim_{x\to a} g(x) = K$*

(i)

$$\lim_{x\to 0} f(x) + g(x) = L + K$$

(ii)

$$\lim_{cf(x)} = cL$$

(iii) If $m = 1$,

$$lim_{x\to a} f(x)g(x) = LK$$

(iv) If $m = 1$, $g(x) \neq 0$ $\forall x \in X$, $K \neq 0$. Then

$$lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{K}$$

Proof. (i) Use Sequential Characterization: Let $(x_n)_{n=1}^{\infty}$ be in $X \setminus \{a\}$ such that $x \to a$. Then $f(x_n) \to L$ and $g(x_n) \to K$ So by algebra of limits for sequences,

$$f(x_n) + g(x_n) = L + K$$

$$\therefore f(x) + g(x) \to L + K$$

(ii) **Exercise.**

(iii) **Exercise.**

(iv) **Exercise.**

$\square$

Theorem 12.0.1 (Squeeze Theorem)**.** *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X, $f, g, h : X \to \mathbb{R}$*

$$f(x) \leq g(x) \leq h(x)\ \forall x \in X$$

and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then

$$\lim_{x \to a} g(x) = L$$

Proof. **Exercise** $\square$

If $f : X \to \mathbb{R}_m$, We can define functions

$$f_1, \ldots, f_m : X \to \mathbb{R}$$

by

$$(f_1(x), \ldots, f_m(x)) = f(x)$$

$f_1, \ldots, f_m$ are callF ed the *component functions* of f.

Proposition 12.0.4. *Let $X \in \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X, $f : X \to \mathbb{R}^m$, $f_1, \ldots, f_m$ its component functions. $L = (L_1, \ldots, L_m) \in \mathbb{R}^m$. Then*

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a} f_i(x) = L_i\ \forall 1 \leq i \leq m$$

Proof. **Exercise.** $\square$

Definition 12.0.2. *Let $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $f : X \to \mathbb{R}^d$.*

- *If a is a limit point of $X \cap (a, \infty)$ then we write $\lim_{x \to a^+} f(x) = L$ to mean that*

  $$\lim_{x \to a} g(x) = L$$

  where

  $$g = f\big|_{X \cap (a, \infty)}$$

- *If a is a limit point of $X \cap (-\infty, a)$ then we write $\lim_{x \to a^+} f(x) = L$ to mean that*

  $$\lim_{x \to a} g(x) = L$$

  where

  $$g = f\ |_{X \cap (-\infty, a)}$$

Example:

$$f(x) \coloneqq \begin{cases} -1, x < 0 \\ 0, x = 0 \\ 1, x > 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = 1 \neq \lim_{x \to 0^-} f(x) = -1$$