# MAT 2125 Lecture Notes

Last Updated:

April 12, 2023

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# Important Proofs for Midterm

**Theorem 1.3.13** (The Archimedean Property)**.** *The set* $\mathbb{N}_{\geq 1}$ *is not bounded above.*

*Proof.* Suppose for a contradiction that $\mathbb{N}$ was bounded above. Then by completeness, $a = \sup \mathbb{N}$ exists. Since $a$ is a least upper bound, $a - 1$ is not an upper bound, so there exists $m \in \mathbb{N}$ such that

$$m > a - 1$$

Then since $m \in \mathbb{N}$, we have $m + 1 \in \mathbb{N}$, so

$$m + 1 > a$$

But $a$ is an upper bound, thus a contradiction. □

**Proposition 2.2.4** (Uniqueness of Limits)**.** *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence and let* $L_1, L_2 \in \mathbb{R}$*. If*

$$\lim_{n\to\infty} a_n = L_1 \text{ and } \lim_{n\to\infty} a_n = L_2$$

*then*

$$L_1 = L_2$$

*Proof.* Suppose for a contradiction $L_1 \neq L_2$. We can assume without loss of generality that $L_1 < L_2$. Define

$$\epsilon = \frac{L_2 - L_1}{2}$$

Since $\lim_{n\to\infty} a_n = L$, there exists $n_0$ such that $\forall n \geq n_0$

$$L_1 - \epsilon < a_n < L_1 + \epsilon$$

Using the second inequality and the definition of $\epsilon$, we get

$$a_n < L_1 + \epsilon = L_1 + \frac{L_2 - L_1}{2} = L_1 + \frac{L_2}{2} - \frac{L_1}{2} = \frac{L_2 + L_1}{2}$$

Likewise, since $\lim_{n\to\infty} a_n = L_2$, there exists $m_0$ such that for all $n \geq m_0$,

$$L_2 - \epsilon < a_n < L_2 + \epsilon$$

Then from the first inequality, we get

$$a_n > L_2 - \epsilon = L_2 - \frac{L_2 - L_1}{2} = \frac{L_2 + L_1}{2}$$

So, we get that for all $n \geq \max\{n_0, m_0\}$,

$$a_n > \frac{L_2 + L_1}{2} > a_n$$

Thus, a contradiction. □

**Proposition 2.2.8.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence which converges to some number $L \in \mathbb{R}$. Then $(a_n)_{n=1}^{\infty}$ is bounded.*

*Proof.* Since $\lim_{n\to\infty} a_n = L$, set $\epsilon \coloneqq 1$, there exists $n_0$ such that for all $n \geq n_0$

$$|a_n - L| < 1$$

So we have that $\forall n \geq n_0$

$$L - 1 < a_n < L + 1$$

Now set

$$M \coloneqq \max\{a_1, a_2, \ldots, a_{n_0-1}, L + 1\}$$

If $n < n_0$, then it is amongst the set $\{a_1, \ldots, a_{n_0-1}\}$, so $M$ will be the max of this set. Therefore, $\forall n < n_0$, $a_n \leq M$. Then for $n \geq n_0$, by the definition of the limit we know that $a_n < L + 1$, so we get that $a_n < L + 1 \leq M$. Therefore, for all values of $n$,the set $\{a_n : n \in \mathbb{N}\}$ is bounded above.

Similarly for the lower bound, take

$$M \coloneqq \min\{a_1, a_2, \ldots, a_{n_0-1}, L - 1\}$$

If $n < n_0$, then it is in the set $\{a_1, a_2, \ldots, a_{n_0-1}\}$ $M'$ is at most the minimum of this set, so $\forall n < n_0$, $a_n \geq M'$. If $n \geq n_0$, by the definition of the limit we know that for all $n \geq n_0$, $a_n > L - 1$. So $M'$ is at most $L - 1$. Therefore $\forall n \geq n_0$, $a_n > L - 1 \geq M'$. Therefore, the set is bounded below and above, so it is bounded. □

**Proposition 2.3.3.** *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be converging sequences, if*

$$a_n \leq b_n$$

*for all $n$, then*

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

*Proof.* Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences with $a_n < b_n$ for all $n$. Then by the definition of convergence, we have that $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$

$$|a_n - L_a| < \epsilon$$

Similarly for $b_n$, we have that $\exists m_0$ such that $\forall \epsilon > 0$,

$$|b_n - L_b| < \epsilon$$

Now suppose for a contradiction that $L_a > L_b$, then set $\epsilon := \frac{L_a - L_b}{2}$. So we have

$$L_a - \epsilon < a_n < \epsilon + L_a$$

So,

$$a_n > L_a - \epsilon = L_a - \frac{L_a - L_b}{2} = \frac{L_a + L_b}{2}$$

Similarly for $b_n$, we have

$$L_b - \epsilon < b_n < L_b + \epsilon$$

$$b_n < L_b + \epsilon = \frac{L_a + L_b}{2}$$

So we have $b_n < \frac{L_b + L_a}{2} < a_n$, but $a_n < b_n$. Thus, a contradiction. □

**Theorem 2.3.5** (Squeeze Theorem)**.** *Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ be sequences such that*

*(i) $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ converge to the same number $L$, and*

*(ii) $a_n \leq b_n \leq c_n$ for all $n$ Then $(b_n)_{n=1}^{\infty}$ also converges to $L$.*

*Proof.* Let $\epsilon > 0$ be given. Suppose $a_n \leq b_n \leq c_n$ $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge to $L$, so $\exists n_a, n_c \in \mathbb{N}$ such that for all $n \geq n_a$

$$L - \epsilon < a_n < L + \epsilon$$

and

$$L - \epsilon < c_n < L + \epsilon$$

So

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

Therefore,

$$L - \epsilon < b_n < L + \epsilon$$

By the definition of convergence, $(b_n)_{n=1}^{\infty}$ converges to $L$. □

**Theorem 2.6.1** (Cauchy Convergence Criterion)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then it converges if and only if it is Cauchy.*

*Proof.* ( $\implies$ ) Assume that $(a_n)_{n=1}^{\infty}$ converges, then there exists $n_0$ such that for all $\epsilon > 0$, $\forall n \geq n_0$

$$|a_n - L| < \epsilon$$

Now take $\frac{\epsilon}{2}$ in place of $\epsilon$ since $\epsilon$ is arbitrary, we have

$$|a_n - L| < \frac{\epsilon}{2}$$

Then, for $m, n \geq n_0$, we have

$$a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L|$$

Since $m, n \geq n_0$, by the definition of convergence we have

$$|a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore,

$$|a_m - a_n| < \epsilon$$

as required.

$\square$

**Proposition 2.7.3.** *For any sequence* $(a_n)_{n=1}^{\infty}$,

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

*Proof.* If the sequence isn't bounded, then either $\limsup_{n\to\infty} a_n = \infty$ or $\liminf_{n\to\infty} a_n = -\infty$, in either case the result is trivial. So assume that the sequence is bounded. Consider the sets used to define lim sup and lim inf

$$S \coloneqq \{\beta : \mathbb{R} : \exists n_0 \text{ such that } a_n \leq \beta \ \ \forall n \geq n_0\}$$

$$T \coloneqq \{\alpha : \mathbb{R} : \exists m_0 \text{ such that } a_n \geq \alpha \ \ \forall n \geq m_0\}$$

So we have $\alpha \in T$ and $\beta \in S$, then for all $n \geq \max\{n_0, m_0\}$, we have

$$\alpha \leq a_n \leq \beta$$

Thus, we have shown that for every $\alpha \in T$, and every $\beta \in S$, we have $\alpha \leq \beta$. From the definition of lim sup and lim inf, we get that for any eventual lower bound $\alpha \in T$, it is a lower bound for the set of upper bounds $S$, so

$$\alpha \leq \inf T = \limsup_{n\to\infty} a_n$$

So then $\limsup_{n\to\infty} a_n$ is an upper bound for the set of lower bounds $T$, so

$$\limsup_{n\to\infty} a_n \geq \sup T = \liminf_{n\to\infty} a_n$$

Therefore,

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

as required.

$\square$

**Proposition .** *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

*diverges.*

*Proof.* Consider the partial sum of the series

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

Now consider the partial sums which correspond to powers of 2, $S_{2^N}$ for $N \in \mathbb{N}$. So we have the sums $S_2, S_4, S_8, \ldots$. Now consider the sequence of partial sums

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$\frac{1}{3} > \frac{1}{4}$, so we have that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

Continuing similarly,

$$\begin{aligned} S_8 = S_{2^3} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{4}{8} = 1 + \frac{3}{2} \\ &\vdots \\ S_{2^N} &> 1 + \frac{N}{2} \end{aligned}$$

So, we have

$$\lim_{N \to \infty} \left(1 + \frac{N}{2}\right) = \infty$$

But, $S_{2^N} > 1 + \frac{N}{2}$ for all $N \in \mathbb{N}$, so we have that the partial sums diverge. Therefore, the series diverges. $\square$

**Proposition 3.1.7** (Divergence Test)**.** *Let* $(a_n)_{n=1}^{\infty}$ *bea sequence of real numbers. If the series*

$$\sum_{n=1}^{\infty} a_n$$

*converges, then*

$$\lim_{n \to \infty} a_n = 0$$

*Proof.* Suppose $\sum\limits_{n=1}^{\infty} a_n$ converges to $L$. Set $L \coloneqq \sum\limits_{n=1}^{\infty} a_n$. Consider the partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

so $\lim\limits_{n\to\infty} S_N = L$. We also have that $\lim\limits_{n\to\infty} S_{N-1} = L$, since

$$\lim_{N\to\infty} S_{N-1} = \lim_{N\to\infty} \sum_{n=1}^{N-1} a_n = \sum_{n=1}^{\infty-1} a_n = \sum_{n=1}^{\infty} a_n = L$$

Then, we have that

$$S_N - S_{N-1} = \sum_{n=1}^{N} a_n - \sum_{n=1}^{N-1} a_n = a_N$$

So,

$$\lim_{N\to\infty} S_N - S_{N-1} = L - L = 0$$

$$\lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = \lim_{N\to\infty} \sum_{n=1}^{N} a_n - \lim_{N\to\infty} \sum_{n=1}^{N-1} a_N = \lim_{N\to\infty} a_n = 0$$

□

**Proposition 3.2.1** (Boundedness Test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that*

*(i) $a_n \geq 0$ for all $n$, and*

*(ii) There is a bound $M \in \mathbb{R}$ on the partial sums, so that*

$$\sum_{n=1}^{N} a_n \leq M$$

*for all $N \in N_{\geq 1}$.*

*Then $\sum\limits_{n=1}^{\infty} a_n$ converges.*

*Proof.* Since $a_n \geq 0$, the partial sums $(S_N)_{N=1}^{\infty}$ satisfy

$$S_N \leq S_{N+1} \text{ for all } N.$$

In other words, $(S_N)_{N=1}^{\infty}$ is an increasing sequence. The second condition ensures that the sequence is bounded above. Therefore, by the Monotone Convergence Criterion, it converges. Therefore, $\sum\limits_{n=1}^{\infty} a_n$ converges. □

**Proposition 3.2.2** (Comparison Test)**.** *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers such that*

$$0 \leq a_n \leq b_n \text{ for all } n$$

*Then,*

*(i) if $\sum\limits_{n=1}^{\infty} b_n$ converges, then so does $\sum\limits_{n=1}^{\infty} a_n$*

*(ii) i1f $\sum\limits_{n=1}^{\infty} a_n$ diverges, then so does $\sum\limits_{n=1}^{\infty} a_n$*

*Proof.* Since the sequence $\sum\limits_{n=1}^{\infty}$ converges, take $M := \sum\limits_{n=1}^{\infty}$. Then, we have the sequence of partial sums

$$\left(\sum_{n=1}^{\infty} b_n\right)_{n=1}^{\infty}$$

is increasing and converges to $M$, so $M$ is the supremum of this sequence, therefore

$$\sum_{N=1}^{N} b_n \leq M$$

for all $M$. Therefore

$$\sum_{N=1}^{N} a_n \leq \sum_{N=1}^{N} b_n \leq M$$

Therefore, by the Boundedness test, $\sum\limits_{n=1}^{\infty} a_n$ converges. $(ii)$ is the contrapositive of $(i)$ so it follows that it holds. ☐

**Proposition 3.2.3** (Absolute Convergence Test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. If the series*

$$\sum_{n=1}^{\infty} |a_n|$$

*converges, then so does*

$$\sum_{n=1}^{\infty} a_n$$

*Proof.* Assume $\sum\limits_{n=1}^{\infty} |a_n|$ converges, Write

$$(a_n)_+ = \max\{a_n, 0\}$$

$$(a_n)_- = \max\{-a_n, 0\}$$

So $(a_n)_+$ is all the positive terms from $a_n$ and $(a_n)_-$ is all the negative terms from $a_n$, but we are negating them so that they are positive, so we have

$$a_n = (a_n)_+ - (a_n)_-$$

Then, we have that

$$0 \leq (a_n)_+ \leq |a_n|$$

So, by the Comparison Test, we have that $|a_n|$ converges so $\sum\limits_{n=1}^{\infty} (a_n)_+$ converges. Similarily,

$$0 \leq (a_n)_- \leq |a_n|$$

Therefore by the Comparison Test, $\sum\limits_{n=1}^{\infty} (a_n)_-$ converges. So by linearity,

$$\sum_{n=1} \infty a_n = \sum_{n=1} \infty (a_n)_+ - \sum_{n=1} \infty (a_n)_-$$

converges. □

**Proposition 4.2.3.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$, with*

$$a_n = (a_n^{(1)}, \ldots, a_n^{(d)}) \text{ for each } n \in \mathbb{N}$$

*and let $L = (L_1, \ldots, L_d) \in \mathbb{R}^d$. Then*

$$\lim_{n \to \infty} a_n = L$$

*if and only if, for each $i = 1, \ldots, d$,*

$$\lim_{n \to \infty} a_n^{(i)} = L_i$$

*Proof.* ( $\implies$ ) Assume that $\lim\limits_{n \to \infty} a_n = L$. Then, for each $i = 1, \ldots, d$, we have that $|x_i|^2 \leq \sum_{i=1}^{d} x_i^2 = ||x||_2^2$, therefore

$$|x_i| \leq ||x||_2$$

Using this fact, we then have each component of $||a_n - L||_2$ is less than or equal to it. So

$$|a_n^{(i)} - L_i| \leq ||a_n - L||_2$$

$$-||a_n - L||_2 \leq a_n^{(i)} - L_i \leq ||a_n - L||_2$$

Since $\lim\limits_{n \to \infty} a_n = L$, we have $\lim\limits_{n \to \infty} a_n - L = 0$. By the Squeeze theorem, it follows that

$$\lim_{n \to \infty} a_n^{(i)} - L_i = 0 \implies \lim_{n \to \infty} a_n^{(i)} = L$$

( $\impliedby$ ) Suppose for each $i = 1, \ldots, d$, we have

$$\lim_{n \to \infty} a_n^{(i)} = L_i$$

Then, from the definition of $||\cdot||_2$, we have

$$||a_n - L||_2^2 = (a_n^{(1)} - L_1)^2 + \cdots + (a_n^{(d)} - L_d)^2$$

Now taking limits of both sides

$$\lim_{n\to\infty} ||a_n - L||_2^2 = \lim_{n\to\infty} (a_n^{(1)} - L_1)^2 + \cdots + \lim_{n\to\infty} (a_n^{(d)} - L_d)^2$$

Now we'll prove exercise 2.2.5 which states that if $(a_n)_{n=1}^{\infty}$ is a sequence of non-negative real number converging to $L \geq 0$, then $\lim_{n\to\infty} \sqrt{a_n}$ converges to $\sqrt{L}$. To prove this we will consider two cases where $L = 0$, and $L > 0$.

- **Case 1, L = 0**: Suppose $(a_n)_{n=1}^{\infty} \to 0$, then from the definition of convergence we have that $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$,

  $$|a_n - 0| < \epsilon$$

  Since $\epsilon$ is abritrary, we'll replace $\epsilon$ with $\epsilon^2$, so

  $$|a_n - 0| < \epsilon^2$$

  Then we get

  $$|a_n - 0| = |a_n| < \epsilon^2 \implies \sqrt{|a_n|} < \epsilon$$

  Therefore, $\sqrt{a_n} \to 0$ by the definition of convergence.

- **Case 2,** $L > 0$: Suppose $(a_n)_{n=1}^{\infty} \to L > 0$. Let $\epsilon > 0$ be given, then there exists $n_0$ such that for all $n \geq n_0$,

  $$|a_n - L| < \epsilon$$

  We much such that $|\sqrt{a_n} - \sqrt{L}| < \epsilon$

  $$|\sqrt{a_n} - \sqrt{L}| \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} = \frac{|(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})|}{\sqrt{a_n} + \sqrt{L}}$$

  Since $\sqrt{a_n} + \sqrt{L}$ is positive because $a_n, L \geq 0$, then $\sqrt{a_n} + \sqrt{L} = |\sqrt{a_n} + \sqrt{L}|$, then using the fact that $|a| \cdot |b| = |a \cdot b|$, we get

  $$\frac{|a_n - \sqrt{L}\sqrt{a_n} + \sqrt{L}\sqrt{a_n} + L|}{\sqrt{a_n} + \sqrt{L}} = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}}$$

  Now if we replace $\epsilon$ with $\frac{\epsilon}{\sqrt{L}}$, we get

  $$|\sqrt{a_n} - \sqrt{L}| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon}{\sqrt{L}} \implies |\sqrt{a_n} - \sqrt{L}| < \epsilon$$

  Therefore, $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$

Now going back to the original proof,

$$\lim_{n\to\infty} ||a_n - L||_2^2 = 0$$

So from exercise 2.2.5 we have

$$\lim_{n\to\infty} \sqrt{||a_n - L||_2^2} = \sqrt{0}$$

Therefore,

$$\lim_{n\to\infty} ||a_n - L||_2 = 0$$

as required. □

**Theorem 4.2.2** (Cauchy Covergence $\mathbb{R}^d$)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence $\mathbb{R}^d$. Then it converges if it converges if and only if it is cauchy.*

*Proof.* Suppose $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^d$ that converges to $L \in \mathbb{R}^d$. Let $\epsilon > 0$ be given, then there exists $n_0$ such that $\forall m, n \geq n_0$,

$$||a_n - L||_2 < \epsilon$$

$$||a_m - L||_2 < \epsilon$$

Since $\epsilon$ arbitrary we can replace $\epsilon$ with $\frac{\epsilon}{2}$, so

$$||a_n - L||_2 = \frac{\epsilon}{2} \text{ and } ||a_m - L||_2 = \frac{\epsilon}{2}$$

So,

$$\begin{aligned} ||a_m - a_n||_2 = ||a_m - L + L - a_n||_2 &\leq ||a_m - L||_2 + ||L - a_n||_2 \\ = ||a_m - L||_2 + ||a_n - L||_2 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

**Proposition 4.3.4.** *Given $a \in \mathbb{R}^d$, and $r > 0$, the open ball $B(a, r)$ is an open set.*

**Note: This is example 4.3.4 from the professors notes.**

*Proof.* Recall the definition of an open set is that for any $x$ in the set, we can define an open ball (or epsilon neighborhood) around $x$ such that the ball is contained in the set. So we want an open ball $B(x; \epsilon)$ such that $B(x; \epsilon) \subseteq B(a; r)$. To see this, let $x \in B(a; r)$, so that $||x - a||_2 < r$. Define

$$\epsilon := r - ||a - x||_2 > 0$$

Now take some element $y \in B(x; \epsilon)$, then we want to show that tis element is contained in $B(a; r)$. So, $y \in B(x; \epsilon)$, so that $||y - x||_2 < \epsilon$. Then,

$$\begin{aligned} ||y - a||_2 = ||y - x + x - a||_2 &\leq ||y - x||_2 + ||x - a||_2 \\ &< \epsilon + ||a - x||_2 = r \end{aligned}$$

Therefore,

$$||y - a||_2 < r$$

So $y \in B(a; r)$ as required, so $B(a; r)$ is an open set. $\square$

**Proposition 4.3.5.** *(i) The sets $\emptyset$, $\mathbb{R}^d$ are open*

*(ii) For any finite collection of open sets, $U_1, \ldots, U_m \subseteq \mathbb{R}^d$, their intersection is*

$$U_1 \cap \cdots \cap U_m$$

*is open*

*(iii) For any arbitrary collection of open sets $\{U_\alpha : \alpha \in I\}$, their union,*

$$\bigcup_{\alpha \in I} U_\alpha$$

*is open.*

*Proof.* (i) $(i)$ and $(ii)$ are Exercise 4.3.1

(ii) Will add them later!

(iii) Set

$$U \coloneqq \bigcup_{\alpha \in I} U_\alpha$$

Since $U_\alpha$ is open, there is some $\epsilon > 0$ such that

$$B(x; \epsilon) \subseteq U_\alpha$$

Then since $U$ is the union of all the $U_\alpha$, we have that $U_\alpha \subseteq U$ so it follows that

$$B(x; \epsilon) \subseteq U$$

as required.

$\square$

**Theorem 4.4.5** (Heine-Borel Theorem)**.** *Let $K$ be a subset of $\mathbb{R}^d$. Then $K$ is compact if and only if $K$ is closed and bounded.*

*Proof.* ( $\implies$ ) Suppose that $K$. To see that $K$ is closed, suppose for a contradiction that it is not closed. By proposition 4.3.9, $F$ is closed if and only if for every sequence $(a_n)_{n=1}^\infty$ in $F$, if $(a_n)_{n=1}^\infty$ converges then

$$\lim_{n\to\infty} a_n \in F$$

So, if $K$ is not closed, then it follows that there exists some sequence $(a_n)_{n=1}^\infty$ such that

$$\lim_{n\to\infty} a_n \notin K$$

Then by proposition 2.5.4, if a sequence $(a_n)_{n=1}^{\infty}$ converges, then all subsequences of the sequence converge to the same point, and hence no subsequence converges to a point in $K$. This contradicts the fact that $K$ is compact. Similarly for the boundess of $K$, suppose for a contradiction that $K$ was not bounded. Then for any $n \in \mathbb{N}$, there exists $a_n \in K$ such that $||a_n||_2 \geq n$. So the sequence $(a_n)_{n=1}^{\infty}$ is unbounded, as well as all subsequences $(a_{n_k})_{k=1}^{\infty}$. Therefore, no subsequence converge since they are all unbounded. This contradicts the fact that $K$ is compact.

( $\Longleftarrow$ ) Assume $K$ is closed and bounded. Since $K$ is bounded, any sequence $(a_n)_{n=1}^{\infty}$ in $K$ is bounded. Then by the Bolzano-Weierstrass theorem, $(a_n)_{n=1}^{\infty}$ is bounded so there exists a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. Since $K$ is closed, from proposition 4.3.9 we have that

$$\lim_{k \to \infty} a_{n_k} \in K$$

Therefore every sequence $(a_n)_{n=1}^{\infty}$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to some $L \in K$, so $K$ is compact. $\square$

# Important Proofs for Final

**Theorem 1.3.13** (The Archimedean Property). *The set $\mathbb{N}_{\geq 1}$ is not bounded above.*

*Proof.* Suppose for a contradiction that $\mathbb{N}$ was bounded above. Then by completeness, $a = \sup \mathbb{N}$ exists. Since $a$ is a least upper bound, $a-1$ is not an upper bound, so there exists $m \in \mathbb{N}$ such that

$$m > a - 1$$

Then since $m \in \mathbb{N}$, we have $m + 1 \in \mathbb{N}$, so

$$m + 1 > a$$

But $a$ is an upper bound, thus a contradiction. $\square$

**Proposition 2.2.4** (Uniqueness of Limits). *Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $L_1, L_2 \in \mathbb{R}$. If*

$$\lim_{n\to\infty} a_n = L_1 \text{ and } \lim_{n\to\infty} a_n = L_2$$

*then*

$$L_1 = L_2$$

*Proof.* Suppose for a contradiction $L_1 \neq L_2$. We can assume without loss of generality that $L_1 < L_2$. Define

$$\epsilon = \frac{L_2 - L_1}{2}$$

Since $\lim_{n\to\infty} a_n = L$, there exists $n_0$ such that $\forall n \geq n_0$

$$L_1 - \epsilon < a_n < L_1 + \epsilon$$

Using the second inequality and the definition of $\epsilon$, we get

$$a_n < L_1 + \epsilon = L_1 + \frac{L_2 - L_1}{2} = L_1 + \frac{L_2}{2} - \frac{L_1}{2} = \frac{L_2 + L_1}{2}$$

Likewise, since $\lim_{n\to\infty} a_n = L_2$, there exists $m_0$ such that for all $n \geq m_0$,

$$L_2 - \epsilon < a_n < L_2 + \epsilon$$

Then from the first inequality, we get

$$a_n > L_2 - \epsilon = L_2 - \frac{L_2 - L_1}{2} = \frac{L_2 + L_1}{2}$$

So, we get that for all $n \geq \max\{n_0, m_0\}$,

$$a_n > \frac{L_2 + L_1}{2} > a_n$$

Thus, a contradiction. $\square$

**Proposition 2.2.8.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence which converges to some number $L \in \mathbb{R}$. Then $(a_n)_{n=1}^{\infty}$ is bounded.*

*Proof.* Since $\lim_{n\to\infty} a_n = L$, set $\epsilon := 1$, there exists $n_0$ such that for all $n \geq n_0$

$$|a_n - L| < 1$$

So we have that $\forall n \geq n_0$

$$L - 1 < a_n < L + 1$$

Now set

$$M := \max\{a_1, a_2, \dots, a_{n_0-1}, L + 1\}$$

If $n < n_0$, then it is amongst the set $\{a_1, \dots, a_{n_0-1}\}$, so $M$ will be the max of this set. Therefore, $\forall n < n_0$, $a_n \leq M$. Then for $n \geq n_0$, by the definition of the limit we know that $a_n < L + 1$, so we get that $a_n < L + 1 \leq M$. Therefore, for all values of $n$,the set $\{a_n : n \in \mathbb{N}\}$ is bounded above.

Similarly for the lower bound, take

$$M := \min\{a_1, a_2, \dots, a_{n_0-1}, L - 1\}$$

If $n < n_0$, then it is in the set $\{a_1, a_2, \dots, a_{n_0-1}\}$ $M'$ is at most the minimum of this set, so $\forall n < n_0$, $a_n \geq M'$. If $n \geq n_0$, by the definition of the limit we know that for all $n \geq n_0$, $a_n > L - 1$. So $M'$ is at most $L - 1$. Therefore $\forall n \geq n_0$, $a_n > L - 1 \geq M'$. Therefore, the set is bounded below and above, so it is bounded. $\square$

**Proposition 3.2.1** (Boundedness Test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that*

*(i) $a_n \geq 0$ for all $n$, and*

*(ii) There is a bound $M \in \mathbb{R}$ on the partial sums, so that*

$$\sum_{n=1}^{N} a_n \leq M$$

*for all $N \in N_{\geq 1}$.*

*Then* $\sum_{n=1}^{\infty} a_n$ *converges.*

*Proof.* Since $a_n \geq 0$, the partial sums $(S_N)_{N=1}^{\infty}$ satisfy

$$S_N \leq S_{N+1} \text{ for all } N.$$

In other words, $(S_N)_{N=1}^{\infty}$ is an increasing sequence. The second condition ensures that the sequence is bounded above. Therefore, by the Monotone Convergence Criterion, it converges. Therefore, $\sum_{n=1}^{\infty} a_n$ converges. □

**Proposition 4.2.3.** *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}^d$*, with*

$$a_n = (a_n^{(1)}, \dots, a_n^{(d)}) \text{ for each } n \in \mathbb{N}$$

*and let* $L = (L_1, \dots, L_d) \in \mathbb{R}^d$*. Then*

$$\lim_{n\to\infty} a_n = L$$

*if and only if, for each* $i = 1, \dots, d$*,*

$$\lim_{n\to\infty} a_n^{(i)} = L_i$$

*Proof.* ( $\Longrightarrow$ ) Assume that $\lim_{n\to\infty} a_n = L$. Then, for each $i = 1, \dots, d$, we have that $|x_i|^2 \leq \sum_{i=1}^{d} x_i^2 = ||x||_2^2$, therefore

$$|x_i| \leq ||x||_2$$

Using this fact, we then have each component of $||a_n - L||_2$ is less than or equal to it. So

$$|a_n^{(i)} - L_i| \leq ||a_n - L||_2$$
$$-||a_n - L||_2 \leq a_n^{(i)} - L_i \leq ||a_n - L||_2$$

Since $\lim_{n\to\infty} a_n = L$, we have $\lim_{n\to\infty} a_n - L = 0$. By the Squeeze theorem, it follows that

$$\lim_{n\to\infty} a_n^{(i)} - L_i = 0 \implies \lim_{n\to\infty} a_n^{(i)} = L$$

( $\Longleftarrow$ ) Suppose for each $i = 1, \dots, d$, we have

$$\lim_{n\to\infty} a_n^{(i)} = L_i$$

Then, from the definition of $|| \cdot ||_2$, we have

$$||a_n - L||_2^2 = (a_n^{(1)} - L_1)^2 + \cdots + (a_n^{(d)} - L_d)^2$$

Now taking limits of both sides

$$\lim_{n\to\infty} ||a_n - L||_2^2 = \lim_{n\to\infty} (a_n^{(1)} - L_1)^2 + \cdots + \lim_{n\to\infty} (a_n^{(d)} - L_d)^2$$

Now we'll prove exercise 2.2.5 which states that if $(a_n)_{n=1}^{\infty}$ is a sequence of non-negative real number converging to $L \geq 0$, then $\lim_{n\to\infty} \sqrt{a_n}$ converges to $\sqrt{L}$. To prove this we will consider two cases where $L = 0$, and $L > 0$.

- **Case 1, L = 0**: Suppose $(a_n)_{n=1}^{\infty} \to 0$, then from the definition of convergence we have that $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$,

$$|a_n - 0| < \epsilon$$

  Since $\epsilon$ is abritrary, we'll replace $\epsilon$ with $\epsilon^2$, so

$$|a_n - 0| < \epsilon^2$$

  Then we get

$$|a_n - 0| = |a_n| < \epsilon^2 \implies \sqrt{|a_n|} < \epsilon$$

  Therefore, $\sqrt{a_n} \to 0$ by the definition of convergence.

- **Case 2,** $L > 0$: Suppose $(a_n)_{n=1}^{\infty} \to L > 0$. Let $\epsilon > 0$ be given, then there exists $n_0$ such that for all $n \geq n_0$,

$$|a_n - L| < \epsilon$$

  We much such that $|\sqrt{a_n} - \sqrt{L}| < \epsilon$

$$|\sqrt{a_n} - \sqrt{L}| \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} = \frac{|(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})|}{\sqrt{a_n} + \sqrt{L}}$$

  Since $\sqrt{a_n} + \sqrt{L}$ is positive because $a_n, L \geq 0$, then $\sqrt{a_n} + \sqrt{L} = |\sqrt{a_n} + \sqrt{L}|$, then using the fact that $|a| \cdot |b| = |a \cdot b|$, we get

$$\frac{|a_n - \sqrt{L}\sqrt{a_n} + \sqrt{L}\sqrt{a_n} + L|}{\sqrt{a_n} + \sqrt{L}} = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}}$$

  Now if we replace $\epsilon$ with $\frac{\epsilon}{\sqrt{L}}$, we get

$$|\sqrt{a_n} - \sqrt{L}| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon}{\sqrt{L}} \implies |\sqrt{a_n} - \sqrt{L}| < \epsilon$$

  Therefore, $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$

Now going back to the original proof,

$$\lim_{n\to\infty} ||a_n - L||_2^2 = 0$$

So from exercise 2.2.5 we have

$$\lim_{n\to\infty} \sqrt{||a_n - L||_2^2} = \sqrt{0}$$

Therefore,

$$\lim_{n\to\infty} ||a_n - L||_2 = 0$$

as required. $\square$

**Proposition 4.3.4.** *Given $a \in \mathbb{R}^d$, and $r > 0$, the open ball $B(a, r)$ is an open set.*

**Note: This is example 4.3.4 from the professors notes.**

*Proof.* Recall the definition of an open set is that for any $x$ in the set, we can define an open ball (or epsilon neighborhood) around $x$ such that the ball is contained in the set. So we want an open ball $B(x; \epsilon)$ such that $B(x; \epsilon) \subseteq B(a; r)$. To see this, let $x \in B(a; r)$, so that $||x - a||_2 < r$. Define

$$\epsilon := r - ||a - x||_2 > 0$$

Now take some element $y \in B(x; \epsilon)$, then we want to show that tis element is contained in $B(a; r)$. So, $y \in B(x; \epsilon)$, so that $||y - x||_2 < \epsilon$. Then,

$$\begin{aligned} ||y - a||_2 = ||y - x + x - a||_2 &\leq ||y - x||_2 + ||x - a||_2 \\ &< \epsilon + ||a - x||_2 = r \end{aligned}$$

Therefore,

$$||y - a||_2 < r$$

So $y \in B(a; r)$ as required, so $B(a; r)$ is an open set. $\square$

**Proposition 4.3.9.** *Let $F \subseteq \mathbb{R}^d$. Then $F$ is closed if and only if for every sequence $(a_n)_{n=1}^{\infty}$ in $F$, if $(a_n)_{n=1}^{\infty}$ converges then*

$$\lim_{n \to \infty} a_n \in F$$

*Proof.* ( $\Longrightarrow$ ) Assume that $F$ is closes, so $\mathbb{R}^d \setminus F$ is open. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $F$ which converges. Suppose for a contradiction that $L := \lim_{n \to \infty} a_n$ is not in $F$. So $L \in \mathbb{R}^d \setminus F$. Then by openness, there exists $\epsilon > 0$ such that $B(L; \epsilon) \subseteq \mathbb{R}^d \setminus F$. Using the definition of convergence, we have that there exists $n$ such that

$$||a_n - L||_2 < \epsilon$$

But, this means that

$$a_n \in B(L; \epsilon) \subseteq \mathbb{R}^d \setminus F$$

which contradicts that $a_n \in F$. Therefore, $L \in F$ as required.

( $\Longleftarrow$ ) Assume that every sequence in $F$ that converges, the limit is in $F$. We want to show that $\mathbb{R}^d \setminus F$ is open. Take a point $x \in \mathbb{R}^d \setminus F$, and suppose for a contradiction that there is no $\epsilon > 0$ such that $B(x; \epsilon) \subseteq \mathbb{R}^d \setminus F$. We can take $\epsilon := \frac{1}{n}$. So for each $n \in \mathbb{N}_{\geq 1}$, $B(x; \frac{1}{n})$ is not contained in $\mathbb{R}^d \setminus F$. So, there must be a point in $B(x; \frac{1}{n})$ that is in $F$. Let $a_n$ be such a point. Then since it is in $B(x; \frac{1}{n})$, we have that

$$||a_n - x||_2 < \frac{1}{n}$$

$x \in \mathbb{R}^d \setminus F$ so $x \neq a_n$, so

$$0 < ||a_n - x||_2 < \frac{1}{n}$$

So by the squeeze theorem,

$$\lim_{n\to\infty} ||a_n - x||_2 = 0$$

So,

$$\lim_{n\to\infty} a_n = x$$

By our hypothesis, every sequence which converges, the limit is in $F$. So $x \in F$, but by our assumption $x \in \mathbb{R}^d \setminus F$, which is a contradiction. Therefore, $\mathbb{R}^d \setminus F$ is open as required. $\square$

**Proposition 5.2.5.** *Let $X \subseteq \mathbb{R}^d$ and let $Y \subseteq \mathbb{R}^m$. Let $f : X \mapsto Y$ and $g : Y \mapsto \mathbb{R}^n$ be functions. Let $a \in X$. Suppose that $f$ is continuous at $a$ and $g$ is continuous at $f(a)$. Then $g \circ f$ is continuous at $a$.*

*Proof.* Let $\epsilon > 0$ be given. Since $g$ is continuous at $f(a)$, there exists $\eta > 0$ such that for all $y \in Y$, if $||y - f(a)||_2 < \eta$, then $||g(y) - g(f(a))||_2 < \epsilon$ (by the definition of continuity). Since $f$ is continuous at $a$, we can find $\delta$ such that for all $x \in X$, if $||x - a||_2 < \delta$, then $||f(x) - f(a)||_2 < \eta$. Now since these inequalities hold for all $x \in X$ and $y \in Y$. We can take $y = f(x)$. So if $x \in X$ and $||x - a||_2 < \delta$, then $||f(x) - f(a)||_2 < \eta$, and so

$$||g(f(x)) - g(f(a))||_2 < \epsilon$$

as required. $\square$

**Theorem 5.3.2.** *Let $K \subseteq \mathbb{R}^d$ be compact and let $f : K \mapsto \mathbb{R}^m$ be a continuous function. Then its image, $f(K)$ is also compact.*

*Proof.* Let $(y_n)_{n=1}^{\infty}$ be a sequence in $f(K)$, we need to find a subsequence that converges to a point in $f(K)$. Recall that $f(K) = \{f(k) : k \in K\}$. So, we can find $x_n \in K$ for each $n$ such that $y_n = f(x_n)$. Since $K$ is compact, there is a subsequence $(x_{n_k})_{n=1}^{\infty}$ which converges to a point $a \in K$. $f$ is continuous so

$$\lim_{x\to a} f(x) = f(a)$$

therefore by the Sequential Characterization of Limits, $\lim_{k\to\infty} f(x_{n_k}) = f(a)$. So, the subsequence $(y_{n_k})_{n=1}^{\infty}$ converges to $f(a) \in f(K)$ as required. $\square$

**Corollary 5.3.3** (Extreme Value Theorem)**.** *Let $K \subset \mathbb{R}^d$ be compact and nonempty, and let $f : K \mapsto \mathbb{R}$ be a continuous function. Then there exists $x_{\min}, x_{\max} \in K$ such that for all $x \in K$,*

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

*In otherwords, the image of $f$ is bounbded above and below, and it attains its boundes.*

*Proof.* By the previous theorem, $f(K)$ is compact. By the Heine-Borel theorem, $f(K)$ is closed and bounded. Since it is bounded, by completeness we have $\sup f(K)$ and $\inf f(K)$ exist. We can construct a sequence $(a_n)_{n=1}^{\infty}$ which converges to $\sup f(K)$. Let $L := \sup f(K)$. For any given $\epsilon > 0$, $L - \epsilon < L$ so it is not an upperbound. Therefore we have some element $a \in f(K)$ such that $L - \epsilon < a \leq L$. Set $\epsilon := \frac{1}{n}$ for $n \in \mathbb{N}_{\geq 1}$. So we can define the seqeunce members of $(a_n)_{n=1}^{\infty}$ as

$$L - 1 < a_1 \leq L$$

$$L - \frac{1}{2} < a_2 \leq L$$

$$L - \frac{1}{3} < a_3 \leq L$$

$$L - \frac{1}{n} < a_n \leq L$$

This holds for all $n \in \mathbb{N}_{\geq 1}$ since if $L - \frac{1}{n}$ is an upper bound, then it contradicts that $L = \sup f(K)$. Then by the Squeeze theorem, we have that this sequence converges to $L$. So since $f(K)$ is closed, $L \in f(K)$. Similarly, we can construct a sequence that converges to $\inf f(K)$. Therefore, by boundedness we have

$$\inf f(K) \leq f(K) \leq \sup f(K)$$

Since $f(K)$ is closed, there exists $x_{\min} \in K$ such that $f(x_{\min}) = \inf f(K)$. Similarly, there exists $x_{\max} \in K$ such that $f(x_{\max}) = \sup f(K)$.

**Note:** It follows that $\sup f(K) \in f(K)$ because $f(K)$ is closed, but I wanted to prove it in an intuitive way to be thorough.

□

**Theorem 5.4.4.** *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \mapsto \mathbb{R}$ be an injective continuous functiion. Then $f^{-1} : f(I) \mapsto \mathbb{R}$ is continuous.*

*Proof.* From Lemma 5.4.3, since $f$ is injective then $f$ is either strictly increasing or strictly decreasing. Assume without loss of generality that $f$ is strictly increasing. Let $b \in f(I)$, we want to show that $\lim\limits_{y \to b} f^{-1}(y) = f^{-1}(b)$. Let $\epsilon > 0$ be given. Set $a := f^{-1}(b)$, so $f(a) = b$. Consider 2 cases where $a \in I^{\circ}$ and $a \in \partial I$. When $a \in \partial I$ the argument follows very similarly so we will suppose that $a \in I^{\circ}$. Since $\epsilon$ is arbitrary, we can assume $a - \epsilon, a + \epsilon \in I$. Then we have

$$f(a - \epsilon) < f(a) < f(a + \epsilon)$$

Then set

$$\delta := \min\{f(a) - f(a - \epsilon), f(a + \epsilon) - f(a)\} > 0$$

We want to prove that if $y \in f(I)$ and $|y - b| < \delta$, then $|f^{-1}(y) - f^{-1}(b)| < \epsilon$. Suppose for a contradiction that this is false, then

$$f^{-1}(y) \leq f^{-1}(b) \text{ or } f^{-1}(y) \geq f^{-1}(b) + \epsilon$$

Set $x \coloneqq f^{-1}(y)$ so $f(x) = y$. So we have

$$x \leq a - \epsilon \text{ or } x \geq a + \epsilon$$

In the first case since $f$ is strictly increasing we get

$$y = f(x) \leq f(a - \epsilon) \leq f(a) - \delta = b - \delta$$

So $y = b - \delta$. In the second case we get

$$y = f(x) \geq f(a + \epsilon) \geq f(a) + \delta = \beta + \delta$$

In both cases, this contradicts that $|y - b| < \delta$ so we must have that our claim was true. $\square$

**Proposition 6.1.5.** *Let $X \subseteq \mathbb{R}$, let $f : X \mapsto \mathbb{R}$ be a function, let $a \in X$ be a non-isolated point. If $f$ is differentiable at $a$ then $f$ is continuous at $a$.*

*Proof.* We want to show that $\lim\limits_{x \to a} f(x) - f(a)$. We have

$$\begin{aligned}
\lim_{x \to a} f(x) - f(a) &= \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} (x - a) \right) \\
&= \left( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left( \lim_{x \to a} x - a \right) \\
&= f'(a) \cdot 0 = 0 \implies \lim_{x \to a} f(x) = f(a)
\end{aligned}$$

We can use algebra of limits since both limits are well defined. $\square$

**Theorem 6.3.2.** *Let $X \subseteq \mathbb{R}$, let $f : X \mapsto \mathbb{R}$ and let $a \in X$ be an interior point. If $f$ has a local maximum or local minimum at $a$ and $f$ is differentiable at $a$, then $f'(a) = 0$.*

*Proof.* Assume that $f$ is a local minimum at $a$. Let $r > 0$ be such that $(a - r, a + r) \subseteq X$ and

$$f(a) \leq f(x) \quad \forall x \in (a - r, a + r)$$

In other words, $f(a)$ is the local minimum in the $r$ neighborhood of $a$. Since $f$ is differentiable, the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. By the Sequential Characterization of Limits, for any sequence $(x_n)_{n=1}^{\infty}$ in $X$ converging to $a$, we have

$$f'(a) = \lim_{n \to \infty} \frac{f(x_n) - f(a)}{x_n - a}$$

We want to construct two sequences that converge to $a$ from the left and right, so from the right consider the sequence with members $x_n \in (a, a + r)$

$$x_n \coloneqq a + \frac{r}{n + 1}$$

Then $x_n - a > 0$ and $f(x_n) - f(a) \leq 0$ since $f(a) \leq f(x)$ $\forall x \in (a-r, a+r)$. So this implies that

$$f'(a) = \lim_{n\to\infty} \frac{f(x_n) = f(a)}{x_n - a} \geq 0$$

Similarily, we can construct a sequence from the left with members $x_n \in (a - r, a)$, take

$$x_n \coloneqq a - \frac{r}{n+1}$$

So $x_n - a < 0$, thus

$$f'(a) = \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} \leq 0$$

Then by combining these inequalities, we get that $f'(a) = 0$, as required. □

**Theorem 6.4.1** (Rolle's Theorem)**.** *Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function that is differentiable on $(a, b)$. If $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that*

$$f'(x_0) = 0$$

*Proof.* If $f$ is constant then it follows that $f'(x_0) = 0$ for all $x_0 \in (a, b)$. Otherwise, by the Extreme Value theorem there exists $x_{\min}, x_{\max}$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

Since $f$ is not constant, we know that we have either (or both)

$$f(x_{\min}) < f(a) = f(b) \text{ or } f(x_{\max}) > f(a) = f(b)$$

□

**Theorem 7.1.11.** *If $f : [a, b] \mapsto \mathbb{R}$ is continuous, then $f$ is integrable.*

*Proof.* Since $f$ is continuous, then by definition there exists $\delta$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. By proposition 5.3.6, we have that $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$. Let $P = \{t_0, \ldots, t_{n-1}\}$ be a partion such that

$$|t_i - t_{i-1}| < \delta$$

for all $i$. Since $f$ is uniformly continuous, then for $x, y \in [t_{i-1}, t_i]$ if $|x - y| < \delta$, then $|f(x) - f(y)|$. So, the maximum distance from $|f(x) - f(y)|$ cannot exceed $\epsilon$. In otherwords,

$$M_i(P, f) - m_i(P, f) < \epsilon$$

By the extreme value theorem, we can find a maximum and minimum $f(x_{\min})$, $f(x_{\max})$ for any interval $[t_{i-1}, t_i]$. From the partitions we have that

$$f(x_{\min}) = m_i(P, f) \text{ and } f(x_{\max}) = M_i(P, f)$$

Now if we take the sums of these intervals, we get

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{n} (M_i(P, f) - m_i(P, f))(t_i - t_{i-1}) \\ &< \sum_{i=1}^{n} \epsilon(t_i - t_{i-1}) \\ &= \epsilon \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= \epsilon(t_n - t_0) \end{aligned}$$

Since $\epsilon$, is arbitrary, we can replace $\epsilon$ with $\frac{\epsilon}{b-a}$. Then we get that

$$U(P, f) - L(P, f) < \frac{\epsilon}{b-a}(t_n - t_0) = \epsilon$$

Then from proposition 7.1.10, which states that $f$ is integrable if and only if for every $\epsilon >$, there exists a partition $P$ such that

$$U(P, f) - L(P, f) < \epsilon$$

we have that $f$ is integrable. $\square$

**Proposition .** *Let $X$ be a set, let $(f_n : X \mapsto \mathbb{R}^m)$ be a sequence of functions, and let $f : X \mapsto \mathbb{R}^m$. If $f_n$ converges uniformly to $f$, then $f$ converges pointwise to $f$.*

*Proof.* Suppose $f_n$ converge uniformaly to $f$, then $\forall \epsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$,

$$||f_n(x) - f(x)||_2 < \epsilon \quad \forall x \in X$$

Therefore, from the definition of the limit, we have that

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \in X$$

as required. $\square$

**Theorem 8.2.6** (The Weierstrass $M$-test)**.** *Let $X$ be a set, let $(f_n : X \to \mathbb{R})_{n=1}^{\infty}$ be a sequence of functions, and let $(M_n)_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Suppose that the follow hold*

*(i)* $|f_n(x) \leq M_n$ *for all* $x \in X$, *and*

*(ii)* $\sum_{n=1}^{\infty} M_n$ *converges*

*Then* $\sum_{n=1}^{\infty} f_n$ *converges uniformly.*

*Proof.* For each $x \in X$, by the comparison test we have that the series

$$\sum_{n=1}^{\infty} |f_n(x)|$$

converges since

$$|f_n(x)| \leq M_n \implies \sum_{n=1}^{\infty} |f_n(x)| \leq \sum_{n=1}^{\infty} M_n$$

Therefore, $\sum\limits_{n=1}^{\infty} f_n(x)$ converges absolutely. Set

$$g(x) \coloneqq \sum_{n=1}^{\infty} f_n(x)$$

We want to show that this sequence converges, that is, for any $\epsilon > 0$, $\exists n_0$ such that $\forall x \in X$

$$|\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{N_0} f_n(x| < \epsilon$$

Since $\sum\limits_{n=1}^{\infty} M_n$ converges, so by the Cauchy convergence criterion, for all $\epsilon > 0$, there exists $N_0$ such that

$$\sum_{n=1}^{\infty} M_n - \sum_{n=1}^{N_0} M_n| < \epsilon$$

So, take

$$g \coloneqq \sum_{n=1}^{\infty} f_n(x)$$

We want to show that $g = \mathrm{u} - \sum_{n=1}^{\infty} f_n(x)$. Now, take $N \geq N_0$, then

$$\begin{aligned}
|g(x) - \sum_{n=1}^{N} f_n(x)| &= |\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{N} f_n(x)| \\
&= |\sum_{n=N+1}^{\infty} f_n(x)| \\
&\leq \sum_{n=N+1}^{\infty} |f_n(x)| \\
&\leq \sum_{n=N+1}^{\infty} M_n \\
&< \epsilon
\end{aligned}$$

as required. $\square$

**Theorem 8.3.1.** *Let $X \subseteq \mathbb{R}^d$ be a set and $a \in X$. Let $(f_n : X \to \mathbb{R}^m)_{n=1}^{\infty}$ be a sequence of functions which converges unifromly to $f : X \to \mathbb{R}^m$. If each $f_n$ is continuous at $a$, then so is $f$. Hence, if each $f_n$ is continuous, then so is $f$.*

*Proof.* Since $f_n$ converges uniformly to $f$, then for any $\epsilon > 0$, there exists $n_0$ such that

$$||f_n(x) - f(x)||_2 < \epsilon$$

Now take any $n \geq n_0$, we want to show that there exists $\delta > 0$ such that if

$$||x - a||_2 < \delta$$

then

$$||f(x) - f(a)|| < \epsilon$$

Since $f_n$ is continuous, then for all $x \in X$, we have that when $||x - a||_2 < \delta$, then

$$||f_n(x) - f_n(a)||_2 < \epsilon$$

Now take $x \in X$ such that $||x - a||_2 < \delta$, then

$$\begin{aligned} ||f(x) - f(a)||_2 &= ||f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)||_2 \\ &\leq ||f(x) - f_n(x)||_2 + ||f_n(x) - f_n(a)||_2 + ||f_n(a) - f(a)||_2 \end{aligned}$$

Then since $f_n$ converges uniformly to $f$, we have that

$$||f(x) - f_n(x)||_2 < \epsilon \text{ and } ||f_n(a) - f(a)||_2 < \epsilon$$

Therefore,

$$||f(x) - f_n(x)||_2 + ||f_n(x) - f_n(a)||_2 + ||f_n(a) - f(a)||_2 = \epsilon + \epsilon + \epsilon$$

Since $\epsilon$ is arbitrary, we can replace it with $\frac{\epsilon}{3}$, and we get

$$||f(x) - f(a)||_2 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

as required. □

**Theorem 9.1.15.** *Let $\sum_{n=0}^{\infty} a_n(x - c)^n$ be a power series, and define*

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

*(this value is 0 when $\limsup$ is $\infty$ and $\infty$ when $\limsup$ is 0). Then for $b \in \mathbb{R}$,*

*(i) if $|b - c| < R$, then $\sum_{n=0}^{\infty} a_n(x - c)^n$ converges, while*

*(ii) if $|b-c| \geq R$, then $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ diverges.*

*Proof.* Let $b \in \mathbb{R}$ with $b \neq 0$, since the case where $b = c$ is trivial. We have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(b-c)^n|} = \limsup_{n\to\infty} |b-c| \sqrt[n]{|a_n|} = |b-c| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|b-c|}{R}$$

Hence by the root test, if $\frac{|b-c|}{R} < 1$ (i.e $|b-c| < R$), the series converges, while if $\frac{|b-c|}{R} \geq 1$ $|b-c| > R$, the series diverges as required $\square$

# Lecture 1

# The Real Numbers $\mathbb{R}$

**Summary:** $\mathbb{R}$ is a complete ordered field.

## 1.1 Fields

**Definition 1.1.1.** *A* field *is a set $F$ together with operations $+, \cdot$ satisfying*

- *(F1)* $a + b = b + a \; \forall a, b \in F$ *(Commutativity)*
- *(F2)* $(a + b) + c = a + (b + c) \; \forall a, b, c \in F$ *(Associativity)*
- *(F3)* $\exists 0 \in F$ *s.t* $0 + a = a \; \forall a \in F$ *(Additive Identity)*
- *(F4)* $\exists - a \in F$ *s.t* $a + (-a) = 0 \; \forall a \in F$ *(Additive Inverse)*
- *(F5)* $a \cdot b = b \cdot a \; \forall a, b \in F$ *(Commutativity)*
- *(F6)* $(a \cdot b) \cdot c = a \cdot (b \cdot c) \; \forall a, b, c \in F$ *(Associativity)*
- *(F7)* $\exists 1 \in F$ *s.t* $1 \cdot a = a \; \forall a \in F$ *(Multiplicative Identity)*
- *(F8)* $\forall a \in F \setminus \{0\} \; \exists a^{-1} \in F$ *s.t* $a^{-1} \cdot a = 1$ *(Multiplicative Inverse)*
- *(F9)* $a \cdot (b + c) = a \cdot b + a \cdot c \; \forall a, b, c \in F$ *(Distributivity)*

## 1.2 Ordered Fields

**Definition 1.2.1.** *An* ordered field *is a field $F$ along with a relation $<$ satistfying*

- *(O1)* $\forall a, b, c \in F$*, if* $a < b$ *and* $b < c$ *then* $a < c$ *(Transitivity)*
- *(O2)* $\forall a, b \in F$ *exactly one of the following is true,*

$$a < b \text{ or } a = b \text{ or } b < a$$

- *(O3)* $\forall a, b, c \in F$, *if* $a < b$, *then* $a + c < b + c$
- $\forall a, b, c \in F$, *If* $a < b$ *and* $0 < c$, *then* $ac < bc$

## 1.3 Complete Ordered Fields

**Definition 1.3.1.** *Let* $F$ *be an ordered field. Let* $S \subseteq F$. *An upper bound or* $S$ *is some* $M \in F$ *s.t* $\forall x \in S$

$$x \leq M$$

# Lecture 2

# Completeness of $\mathbb{R}$,<br>Absolute Value, Sequences

TBC.

# Lecture 3

# Convergence of Sequences

TBC.

## Lecture 4

# Properties of Convergence, Squeeze Theorem, Monotone Sequences

TBC.

# Lecture 5

# Subsequences, Cauchy Sequences

TBC.

# Lecture 6

# Limsup and Liminf

TBC.

# Lecture 7

# Series

**Recall:**

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

$\sum\limits_{n=1}^{\infty} a_n$ "diverges" if above limit does not exisit.

**Proposition 7.0.1.** *Suppose $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ converges. Then*

*(i)*

$$\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

*(ii)*

$$\sum_{n=1}^{\infty} cb_n = c\sum_{n=1}^{\infty} b_n \ \forall c \in \mathbb{R}$$

*This says*

$$V \coloneqq \{(a_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} a_n \ converges\}$$

*is a vector space over $\mathbb{R}$.*

**Note:**

$$\left(\sum_{n=1}^{N} a_n\right)\left(\sum_{n=1}^{N} b_n\right) \neq \sum_{n=1}^{\infty} a_n b_n$$

*Proof.* **Exercise.** □

**Proposition 7.0.2.** $\sum\limits_{n=1}^{\infty} a_n$ *converges $iff$ $\sum_{n=k}^{\infty} a_k$ converges.*

*Proof.* **Exercise.** □

**Example:** TBC.

**Proposition 7.0.3.** *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

## 7.1 Divergence Test

**Proposition 7.1.1** (Divergence Test)**.** *If* $a_n \not\to 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

*Proof.* $\sum_{n=1}^{\infty} a_n$ converges $\implies S_n \to L$ for some $L$, where

$$S_n \coloneqq \sum_{n=1}^{N} a_n$$

So,

$$a_n = S_n - S_{n-1}$$

By the Algebra of Limits,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= L - L = 0\end{aligned}$$

□

**Example:** TBC.

## 7.2 Convergence Tests

**Proposition 7.2.1** (Boundedness Test)**.** *Let* $(a_n)_{n=1}^{\infty}$ *bea sequence, if*

*(i)* $a_n \geq 0$

*(ii) There is an upper bound on the parital sums*

$$\exists M > 0 \ s.t \ \sum_{n=1}^{N} a_n \leq M$$

*Then* $\sum_{n=1}^{\infty} a_n$ *converges.*

*Proof.* Let

$$S_N \coloneqq \sum_{n=1}^{N} a_n$$

Then

$$\begin{aligned} S_{N+1} &= S_N + a_{N+1} \\ &\geq S_n \end{aligned}$$

So by the Monotone Convergence Criterion, $(S_N)_{N=1}^{\infty}$ converges $\iff$ it is bounded avove. By $(ii)$, it is bounded. $\square$

**Proposition 7.2.2** (Comparison Test)**.** *TBC.*

# Lecture 8

# Ratio, Root, Alternating Series, and Integral Test, Cauchy Convergence, Topology of $\mathbb{R}^d$

## 8.1 Ratio Test

**Proposition 8.1.1** (Ratio Test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of nonzero elements.*

*(i) If*

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

*then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

*(ii) If*

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n} > 1\right| > 1$$

*then $\sum_{n=1}^{\infty} a_n$ diverges.*

*Proof.*

(i) Let

$$q = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

where $q < 1$, $\exists r \in (q, 1)$. By the definition of lim sup,

$$\left|\frac{a_{n+1}}{a_n}\right| \leq r \ \forall n \geq n_0$$

for some $n_0 \in \mathbb{N}$.

$$\left|\frac{a_{n_0+1}}{a_{n_0}}\right| \leq r$$
$$|a_{n_0+1}| \leq r|a_{n_0}|$$
$$|a_{n_0+2}| \leq r|a_{n_0+1}| \leq r^2|a_{n_0}|$$

By induction, we have

$$0 \leq |a_{n_0+k} \leq r^k|a_{n_0}|$$

By the comparison test,

$$\sum_{k=1}^{\infty} |a_{n_0+k}|$$

converges since

$$\sum_{k=1}^{\infty} r^k|a_{n_0}|$$

is a geometric sequence and $0 \leq r < 1$.

$$\sum_{n=1}^{\infty} |a_n|$$

Converges, thus $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) Let

$$q = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$$

so $\exists n_0$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1$$

for all $n \geq n_0$. Then

$$|a_{n_0+k} \geq |a_{n_0}| \ \forall k \geq 0$$

So $a_n \not\to 0$ as $n \to 0$, thus by the divergence test. $\sum_{n=1}^{\infty} a_n$ diverges.

$\square$

**Note: The ratio test does not tell us anything when the limit is 1.**

**For example:**

$$\sum_{n\geq 1}\frac{1}{n}$$

diverges, but

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1$$

But on the other hand,

$$\sum_{n\geq 1}\frac{1}{n+1}$$

converges, and

$$\frac{\frac{1}{n+1(n+2)}}{\frac{1}{n(n+1)}} = \frac{n+1}{n+2} \to 1$$

## 8.2 Root Test

**Proposition 8.2.1** (Root Test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.*

*(i) If*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$$

*then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

*(ii) If*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$$

*then $\sum_{n=1}^{\infty} a_n$ diverges.*

*Proof.*

(i) Let

$$q = \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$$

Then there exists $r \in (q, 1)$ such that $\exists n_0$

$$\sqrt[n]{|a_n|} \leq r$$

for all $n \geq n_0$. Then,

$$0 \leq |a_n| \leq r^n$$

Therefore $\sum_{n=n_0}^{\infty} r^n$ converges since $0 < r < 1$. Then by the comparison test, $\sum_{n=n_0}^{\infty} |a_n|$ converges.

(ii) Let
$$q = \limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$$
From exercise 2.7.3, there are infinitely many
$$\sqrt[n]{|a_n|} \geq 1 \implies |a_n| \geq 1$$
Thus $a_n \not\to 0$ as $n \to \infty$, thus by the divergence test, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

□

**Examples:**

- $$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$
$$\sqrt[n]{\left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

- $$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)_n$$
$$\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \frac{1}{2^{n+1}} \to \frac{1}{2}$$

- $$\sum_{n=1}^{\infty} \frac{1}{n}_n$$
$$\sqrt[n]{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} = \frac{1}{e^{\frac{\ln n}{n}}} \to 0$$

## 8.3 Alternating Series Test

**Proposition 8.3.1** (Alternating Series Test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose*

*(i)* $(a_n)_{n=1}^{\infty}$ *is decreasing*

*(ii)* $\lim_{n\to\infty} a_n = 0$

*Then*
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$
*converges. Moreover, for any $N$,*
$$\sum_{n=1}^{2N} (-1)^{n+1} a_n \leq \sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq \sum_{n=1}^{2N-1} (-1)^{n+1} a_n$$

*Proof.* $a_n \geq 0 \ \forall n$, since $a_n \to 0$ as $n \to \infty$, and $(a_n)_{n=1}^{\infty}$ is decreasing. Let

$$S_N \coloneqq \sum_{n=1}^{N} (-1)^{n+1} a_n$$

If $N$ is even,

$$S_{N+2} = S_N + a_{N+1} - a_{N+2} \geq S_N$$

Since

$$a_{N+2} \leq a_{N+1}$$

So $(S_{2N})_{N=1}^{\infty}$ is an increasing sequence and $(S_{2N-1})_{N=1}^{\infty}$ is a decreasing sequence. So by the monotone convergence criterion, both sequence converge.

$$S_{2N-1} + a_{2N} = S_{2N}$$

$$a_{2N} \xrightarrow{N\to\infty} 0$$

So,

$$\lim_{n\to\infty} S_{2N-1} = \lim_{N\to\infty} S_{2N} = L \implies \lim_{N\to\infty} S_N = L$$

$(S_{2N})_{N=1}^{\infty}$ is increasing, so

$$L = \sup\{(S_{2N})_{N=1}^{\infty}\} \implies S_{2N} \leq L$$

Similarily, $(S_{2N-1})_{N=1}^{\infty}$ is decreasing, so

$$L = \inf\{(S_{2N_1})_{n=1}^{\infty}\} \implies S_{2N-1} \geq L$$

□

## 8.4 Integral Test

**Proposition 8.4.1.** *Let $f : [1, \infty) \to \mathbb{R}$. Suppose that*

*(i)* $f(x) \geq 0 \ \forall x \in [1, \infty)$

*(ii)* $f$ *is decreasing*

*Then*

$\sum_{n=1}^{\infty} f(n)$ *converges* $\iff$ *the improper integral* $\int_1^{\infty} f(x)\ dx$ *converges.*

## 8.5 Cauchy Convergence Criterion for Series

**Proposition 8.5.1.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then*

$\sum_{n=1}^{\infty} a_n$ *converges* $\iff \forall \epsilon > 0\ \exists N_0$ *such that* $N \leq M \leq N_0$, $|\sum_{n=M}^{N} a_n|\epsilon$.

*Proof.* Let

$$S_N \coloneqq \sum_{n=1}^{N} a_n$$

Then $\sum_{n=1}^{\infty} a_n$ converges $\iff (S_N)_{N=1}^{\infty}$ converges. Cauchy convergence criterion for sequences says that

$$(S_N)_{n=1}^{\infty} \text{ converges } \iff \text{ it is cauchy.}$$

$$\iff \forall \epsilon > 0\ \exists N_0\ s,t\ |S_N - S_M| < \epsilon\ \forall N, M \geq N_0$$

□

## 8.6 Topology of $\mathbb{R}^d$

### 8.6.1 Norms

**Definition 8.6.1.** *A norm on $\mathbb{R}^d$ is a function $||\cdot|| : \mathbb{R}^d \to [0, \infty)$ satisfying the following properties:*

*(i)* $||a|| = 0 \iff a = (0, \dots, 0)$

*(ii)* $||ca|| = |c| \cdot ||a||\ \forall c \in \mathbb{R}, a \in \mathbb{R}^d$

*(iii)* $||a + b|| \leq ||a|| + ||b||\ \forall a, b \in \mathbb{R}^d$

The euclidean norm of $\mathbb{R}^d$ is given by

$$||a, \dots, a_d||_2 = \sqrt{\sum_{i=1}^{d} a_i^2}$$

The dot prdouct on $\mathbb{R}^d$ is given by

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = \sum_{i=1}^{d} a_i b_i$$

We also have the $l_1$-norm

$$||(a_1, \dots, a_n)||_1 = \sum_{i=1}^{d} |a_i|$$

And the $l_\infty$-norm

$$||(a_1, \dots, a_d)||_\infty = \max\{|a_1|, \dots, |a_d|\}$$

**Proposition 8.6.1.** *Let $a, b \in \mathbb{R}^d$, write $||\cdot||$ for the euclidean norm on $\mathbb{R}^d$. Then*

*(i)* ***Cauchy Schwarz Inequality:***

$$|a \cdot b| \leq ||a|| \cdot ||b||$$

*(ii)* ***Triangle Inequality:***

$$||a + b|| \leq ||a|| + ||b||$$

*(iii)* $||\cdot||$ *is a norm on* $\mathbb{R}^d$

*Proof.*

(i) Consider the equadratic function

$$\begin{aligned} P(t) &= ||a + tb||^2 \\ &= (a + tb) \cdot (a + tb) \\ &= a \cdot a + 2a \cdot tb + tb \cdot tb \\ &= ||a||^2 + 2t(a \cdot b) + t^2||b||^2 \end{aligned}$$

The discriminant of $P(t)$ is less than or equal to 0,

$$\begin{aligned} (2a \cdot b)^2 - 4||a||^2||b||^2 &\leq 0 \\ (a \cdot b)^2 &\leq ||a||^2||b||^2 \\ |a \cdot b| &\leq ||a|| \cdot ||b|| \end{aligned}$$

(ii)

$$\begin{aligned} ||a + b||^2 &= ||a||^2 + 2a \cdot b + ||b||^2 \\ &\leq ||a||^2 + 2|a \cdot b|| + ||b||^2 \\ &\leq ||a||^2 + 2||a|| \cdot ||b|| + ||b||^2 \\ &= (||a|| + ||b||)^2 \end{aligned}$$

$$||a + b||^2 \leq (||a|| + ||b||)^2 \implies ||a + b|| \leq ||a|| + ||b||$$

(iii) **Exercise.** We want to prove $||a|| = 0 \iff a = 0$ and $||ca|| = |c| \cdot ||a||$.

$\square$

# Lecture 9

# $\mathbb{R}^d$

**Recall:** $||(x_1, \ldots, x_d)|| \coloneqq \sqrt{x_1^2 + \cdots + x_d^2}$. This is a norm. i.e.

$$||a + b|| \leq ||a||_2 + ||b||_2 \ \forall a, b \in \mathbb{R}^d$$

$$||ca|| = |c| \cdot ||a||_2, \ c \in \mathbb{R}, \ a \in \mathbb{R}^d$$

$$||a||_2 > 0, \ \forall a \in \mathbb{R}^d \setminus \{(0, \ldots, 0)\}$$

$$||(0, \ldots, 0)||_2 = 0$$

Other examples of norms:

- $||(x_1, \ldots, x_d)||_1 \coloneqq |x_1| + \cdots + |x_d|$
- $||(x_1, \ldots, x_d)||_\infty \coloneqq max\{|x_1|, \ldots, |x_d|\}$

**Exercise:** For $a \in \mathbb{R}^d$,

$$||a||_\infty \leq ||a||_2 \leq ||a||_1 \leq d||a||_\infty (\leq d||a||_2)$$

**Interesting Fact:** There are other norms. but they are all equivalent in the sense that if $|| \cdot ||, || \cdot ||'$ are norms on $\mathbb{R}^d$, then $\exists v, R > 0$ such that

$$r||a|| \leq ||a||' \leq R||a||$$

## 9.1 Convergence

**Definition 9.1.1.** *Let $(a_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}^d$ and let $L \in \mathbb{R}^d$, we say $(a_n)_{n=1}^\infty$ **converges** to $L$, and write $\lim_{n\to\infty} = L$ or $a_n \to \infty$, if*

$$\lim_{n\to\infty} ||a_n - L||_2 = 0$$

*Note: We could define convergence instead using some other norm, say* $||\cdot||_1$.

If $||a_n - L||_2 \to 0$, then $||a_n - L||_1 \leq d||a_n - L||_2 \to 0$ If $||a_n - L||_1 \to 0$, then $||a_n - L||_2 \leq d||a_n - L||_1 \to 0$

in general, if $||\cdot||$ is any norm, then since $||\cdot||$ and $||\cdot||_2$ are equivalent.

$$||a_n - L||_2 \to 0 \iff ||a_n - L|| \to 0$$

**Example:** Say $a_n = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, then

$$||a_n - L||_2 = \sqrt{1/n^2, + \cdots + 1/n^2} = \sqrt{\frac{d}{n^2}} = \frac{\sqrt{d}}{n} \to 0$$

$$\therefore a_n \to L$$

Given a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$, we write $a_1 = (a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(d)})$ where $a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(d)} \in \mathbb{R}$. Similarily,

$$a_1 = (a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(d)})$$
$$a_2 = (a_2^{(1)}, a_2^{(2)}, \ldots, a_2^{(d)})$$
$$a_3 = (a_3^{(1)}, a_3^{(2)}, \ldots, a_3^{(d)})$$
$$\vdots$$
$$L = (L^{(1)}, L^{(2)}, \ldots, L^{(d)}) \in \mathbb{R}^d$$

We get $d$ sequences in $\mathbb{R}$, and $d$ possible limit points $L^{(1)}, \ldots, L^{(d)} \in \mathbb{R}$

**Proposition 9.1.1.** *Given* $(a_n)_{n=1}^{\infty}$ *and* $L$ *as above,* $a_n \to L$ *as* $d \to \infty \iff a_n^{(i)} \to L^{(i)}$ *in* $\mathbb{R}$ *as* $n \to \infty$*, for* $i = 1, \ldots, d$.

*Proof.* $\implies$: Suppose $a_n \to L$, i.e.

$$||a_n - L||_2 \to 0$$

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$

$$||x||_2^2 = x_1^2 + \cdots x_d^2 \geq x_i^2 = |x_i|^2 \therefore |x_i \leq ||x||_2$$

Applying this to (*), we get

$$0 \leq |a_n^i - L^{(i)}| \leq ||a_n - L||_2 \to 0$$

So by the squeeze theorem,

$$|a_n^{(i)} - L^{(i)}| \to 0$$

$\implies$: Suppose $a_n^{(i)} \to L^{(i)}$ for $i = 1, \ldots, d$.

$$||a_n - L||_2^2 = (a_n^{(i)} - L^i)^2 + \cdots + (a_n^d - L^(d)) \to 0$$

By algebra of limits,

$$\therefore ||a_n - L||_2 \to 0$$

$\square$

**Example:** $a^n = ((-1)^n, \frac{1}{n}) \in \mathbb{R}^2$. Does $(a_n)$ converge? No, since $(-1)^n$ does not converge.

**Definition 9.1.2.** *A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$ is **Cauchy** if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}_{\geq 1}$ such that*

$$||a_n - a_m||_2 < \epsilon \ \forall m, n \geq n_0$$

**Theorem 9.1.1** (Cauchy Convergence Criterion for $\mathbb{R}^d$)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$. It converges $\iff$ it is Cauchy.*

*Proof.* $\implies$: Suppose $a_n \to L \in \mathbb{R}^d$. To show it is Cauchy, let $\epsilon > 0$. $||a_n - L||_2 \to 0$, so $\exists n_0 \in \mathbb{N}_{\geq 1}$ such that

$$||a_n - L||_2 < \frac{\epsilon}{2} \ \forall n \geq n_0)$$

Then if $m, n \geq n_0$,

$$\begin{aligned} ||a_m - a_n||_2 &= ||a_m - L + L - a_n||_2 \\ &\leq ||a_m - L||_2 + ||L - a_n||_2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\therefore (a_n)_{n=1}^{\infty}$ is Cauchy. $\implies$: Suppose $(a_n)_{n=1}^{\infty}$ is Cauchy, write

$$a_n = (a_n^{(1)}, \ldots, a_n^{(d)}$$

For any $m, n \in \mathbb{N}_{\geq 1}$,

$$|a_n^{(i)} - a_m^{(i)} \leq ||a_n - a_m||_2$$

$\therefore (a_n^{(i)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. So by the Cauchy Convergence Criterion, $\exists L^{(i)} \in \mathbb{R}$, such that $a_n^{(i)} \to L^{(i)}$. By the previous proposition,

$$a_n \to (L^{(1)}, \ldots, L^{(d)})$$

$\square$

**Definition 9.1.3.** *$S \subseteq \mathbb{R}^d$ is **bounded** if $\exists M > 0$ such that*

$$||x|| \leq M \quad \forall x \in S$$

A sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^d$ is bound if $\{a_n : n \in \mathbb{N}_{\geq 1}\}$ is a bounded set.

**Theorem 9.1.2** (Bolzano-Weierstrass for $\mathbb{R}^d$)**.** *If $(a_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^d$, then it has a subsequence $(a_{n_k})_{n=1}^{\infty}$ that converges.*

*Proof.* Write $a_n = (a_n^{(1)}, \dots, a_n^{(d)})$.
We will prove it by indunction on $d$. For $d = 1$, this is the Bolzano-Weierstrass theorem for $\mathbb{R}$ For $d > 1$ write

$$b_n \coloneqq (a_n^{(1)}, \dots, a_n^{(d-1)}) \in \mathbb{R}^{d-1}$$

By the induction hypothesis, $b_n$ has a subsequence $(b_{n_k})_{k=1}^{\infty}$ that converges. Let $L \in \mathbb{R}^{d-1}$ be the limit of this subsequence. Then $L \in \mathbb{R}^{d-1}$ is the limit of $b_n$. $(a_{n_k}^{(d)})_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, so it has a subsequence $(a_{n_{k_j}}^{(d)})_{j=1}^{\infty}$ that converges. Let $L^{(d)} \in \mathbb{R}$ be the limit of this subsequence. Then $L = (L^{(1)}, \dots, L^{(d-1)}, L^{(d)})$ is the limit of $a_n$. $\square$

# Lecture 10

# Open and Closed Sets in $\mathbb{R}^d$

Roughly, an open set is one that we draw with dotted lines. The line represents a "boundary" that is the not in the set. This is not a rigorous definition.

**Definition 10.0.1** (Open Ball)**.** *Let $a \in \mathbb{R}^d$, $r > 0$. The* ***open ball*** *of radius $r$ centered at $a$ is*

$$B(a;r) \coloneqq \{x \in \mathbb{R}^d : ||x-a||_2 < r\}$$

**Relation to Convergence:** If $a_n \to L$, then this means that $||a_n - L||_2 < \epsilon$ for all $n$ large. So, $a_n \in B(L;\epsilon)$

**Definition 10.0.2** (Open Sets)**.** *A set $U \subseteq \mathbb{R}^d$ is* ***open*** *if $\forall a \in U$, $\exists r > 0$, such that $B(a;r) \subseteq U$*

**Idea:** If $a \in U$, then $a$ is not on the boundary but it is truly "inside" the set, so we can fit a ball containing a in the set.

**Definition 10.0.3** (Closed Sets)**.** *A set $k \in \mathbb{R}^d$ is* ***closed*** *if its complement $\mathbb{R}^d \setminus k$ is open.*

**Example:** $U \subseteq (0,1)$. Is this open? Yes.

*Proof.* Let $a \in U$. We let $r := \min\{|a-0|, |a-1|\}$ (We do this so that $r$ is at most the distance to the closest bound, i.e. if $a$ is closer to 0, then the radius $r$ cannot be $|a-1|$)then

$$B(a;r) = (a-r, a+r) \subseteq (0,1) = U$$

□

**Example:** $U \coloneqq [0,1]$. Is this open? No.

*Proof.* Let $a \coloneqq 0 \in U$. The for any $r > 0$, $\exists z \in B(a;r) = (-r,r)$ s.t $z < 0$, so $z \notin U$. Therefore $B(a;r) \subseteq U$ □

Is $U$ closed? This is the same as asking if $\mathbb{R} \setminus U = (-\infty, 0) \cup (1, \infty)$ is open. This is open.

*Proof.* Let $a \in (-\infty, 0) \cup (1, \infty)$.

- **Case 1:** $a \in (-\infty, 0)$. Set $r := |a|$, so

$$B(a; r) = (a - r, a + r) = (2a, 0) \subseteq U$$

- **Case 2:** $a \in (1, \infty)$ similar.

$\square$

Therefore $U = [0, 1]$ is closed.

**Example:** Is $U \coloneqq (0, 1]$ open? No, for any $r > 0$

$$B(1; r) \nsubseteq U$$

Therefore, it is not open.

***Note:*** *Sets are not always open or closed. Most sets are neither open nor closed.*

This set $U$ is one such example $U$ is not closed since $\mathbb{R} \setminus U = (-\infty, 0] \cup (1\infty)$ $0 \in \mathbb{R}\ U$ but $\forall r > 0$, $B(0; r) \nsubseteq R \setminus U$

**Example:** For any $a \in \mathbb{R}^d$, $r > 0$ $B(a; r)$ is an open set.

*Proof.* Let $x \in B(a; r)$, so $||x - a||_2 < r$. Set

$$r_0 \coloneqq r - ||x - a||_2 > 0$$

**Claim:** $B(x; r_0) \subseteq B(a; r)$ To see this, let $y \in B(x; r_0)$ so $|y - x||_2 < r_0$. So,

$$\begin{aligned} ||y - a||_2 &\leq ||y - x||_2 + ||x - a||_2 && (\triangle\text{-inequality}) \\ &< r_0 + ||x - a||_2 \\ &= r \end{aligned}$$

$\square$

**Proposition 10.0.1.** *(i) $\emptyset$, $\mathbb{R}^d$ are both open in $\mathbb{R}^d$*

*(ii) If $U_1, U_2, \ldots, U_n \subseteq \mathbb{R}^d$ are all open, then so is $U_1 \cap U_2 \cap \cdots \cap U_n$.*

*(iii) If $U_a \subseteq \mathbb{R}^d$ is an open set for all $\alpha \in I$, (I is some index set) then*

$$\bigcup_{a \in I} U_a$$

*is open.*

*Proof.* *(i),(ii) are exercises.*

**(iii):** Set
$$V := \bigcup_{\alpha \in I} U_a$$
Let $a \in V$. This means $\exists \alpha \in I$ such that $a \in U_\alpha$. $U_\alpha$ is open so $\exists r > 0$ s.t $B(a;r) \subseteq U_\alpha$. $U_\alpha \leq \bigcup_{\alpha \in I} U_\alpha = V$ So $B(a;r) \subseteq V$ as required. $\square$

**Example:** For any $n \in \mathbb{N}_{\geq 1}$.
$$\left(\frac{-1}{n}, \frac{1}{n}\right) = B(0; \frac{1}{n})$$
is open in $\mathbb{R}$. The intersection of these open sets is
$$\bigcap_{n=1}^{\infty} \left(\frac{1}{n}, \frac{-1}{n}\right) = \{0\}$$
which is not open. This shows that openess is not preserved by infinite intersections.

**Example:** Let
$$U := \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$$
$U$ is open but not closed.
$$U = V \cap W$$
where
$$V := \{(x,y) : x > 0\} \qquad W := \{(x,y) : y > 0\}$$
To show $V$ is open, let $a = (x,y) \in V$. Set $r := x > 0$. Then if $(w,z) \in B(a;r)$. Then
$$|w - z| \leq ||(w,z) - a||_2 < r = x$$
$$\therefore w > x - x = 0$$
So $(w,z) \in U$. Similarly, $W$ is open. Therefore $U$ is open.

**Not Closed:** Exercise.

**Proposition 10.0.2.** *Let $K \subseteq \mathbb{R}^d$. $K$ is closed $\iff$ for any subsequence $(a_n)_{n=1}^{\infty}$ in $K$, If it converges, then*
$$lim_{n \to \infty} a_n \in K$$

*Proof.* ($\implies$) Suppose $K$ is closed. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $K$ s.t

$$L := lim_{n\to\infty} a_n$$

exists. Suppose for a contradction $L \notin K$. This means $L \in \mathbb{R}^d \setminus K$, which is open. So $\exists r > 0$ such that

$$B(L; r) \subseteq \mathbb{R}^d \setminus K$$

Since $a_n \to L$, we must have $a_n \in B(L; a)$ for some $n$ (in fact, for all n sufficiently large. So $a_n \in B(L; r) \subseteq \mathbb{R}^d \setminus K$. Therefore $a_n \notin K$, which is a contradiction.

($\impliedby$) Suppose $K$ is not closed, and we'll prove $\exists (a_n)_{n=1}^{\infty}$ in $K$ such that $a_n \to L \notin K$. Since $K$ is not closed, $\mathbb{R}^d \setminus K$ is not open. So $\exists L \in \mathbb{R}^d \setminus K$ such that $\forall r > 0$

$$B(L; r) \not\subseteq \mathbb{R}^d \setminus K$$

For each $n \in \mathbb{N}_{\geq 1}$, we can fine $a_n \in B(L; \frac{1}{n})$ such that $a_n \notin \mathbb{R}^d \setminus K$. So $a_n \in K$. This gives a sequence $(a_n)_{n=1}^{\infty}$ in $K$ and

$$||a_n - L||_2 < \frac{1}{n} \to 0$$

Therefore by the Squeeze Theorem,

$$||a_n - L||_2 \to 0 \implies a_n \to L$$

$L \in \mathbb{R}^d \setminus K$, so $L \notin K$. $\square$

**Definition 10.0.4.** *Let $A \subseteq \mathbb{R}^d$ and let $a \in \mathbb{R}^d$, $a$ is:*

*(i) an **interior point** if $\exists r > 0$ s.t $B(a; r) \subseteq A$*

*(ii) an **accumulation point** if $\exists$ a sequence $(a_n)_{n=1}^{\infty}$ in $A$ s.t $a_n \to a$*

*(iii) a **boundary point** if it is an accumulation point and it is not an interior point.*

$$A^\circ := \{\textit{All interior points}\}$$

$$\bar{A} := \{\textit{All accumulation points}\}$$

$$\partial A := \{\textit{All boundary points}\} = \bar{A} \setminus A^\circ$$

***Note:*** *The set of interior points, accumulation points, and boundary points are referred to as the **interior** of $A$, the **closure** of $A$, and the **boundary** of $A$ respectively*

**Example:** $A := (0, 1] \cup \{2\}$

$$A^\circ = (0, 1)$$

$$\bar{A} = [0, 1] \cup \{2\}$$

$$\partial A = \{0, 1, 2\}$$

**Example:** $A \coloneqq \mathbb{Q}$

Since any open interval contains irrational numbers, we have

$$A^\circ = set$$

Proposition from chapter 2,

$$\bar{A} = \mathbb{R}$$

$$\partial A = \mathbb{R}$$

# Lecture 11

# Compactness

**Definition 11.0.1.** *A set $A \subseteq \mathbb{R}^d$ is (sequentially) compact if every sequence $(a_n)_{n=1}^{\infty}$ in $A$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to a point in $A$.*

**Example 1:** Is [0,1] compact? Yes.

*Proof.* **Recall:** Bolzano-Weierstrass theorem states bounded sequence has a convergent subsequence.

Therefore, every sequence $(a_n)_{n=1}^{\infty}$ in $[0,1]$ has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges. So

$$0 \leq a_{n_k} \leq 1 \implies 0 \leq \lim_{k\to\infty} a_{n_k} \leq 1$$

$$\therefore L \in [0,1]$$

□

**Example 2:** Is (0,1) compact? No.

*Proof.* By counter example, let $a_n \coloneqq \frac{1}{n+1}$, so $a_n \to 0$. Therefore for all subsequences of $a_n$, $a_{n_k} \to 0$. So there exists no subsequence which converges to a point in (0,1). □

**Example 3:** Is $[0,\infty)$ compact? No.

*Proof.* The Bolzano-Weierstrass theorem does not apply since $[0,\infty)$ is unbounded. Set $a_n \coloneqq n$, then $a_n \to \infty$, so it has no bounded subsequence and therefore no convergent subsequences. □

**Theorem 11.0.1** (Heine-Borel)**.** *Let $A \subseteq \mathbb{R}^d$. $A$ is compact $\iff$ $A$ is closed and not bounded.*

*Proof.* ( $\implies$ ) Similar to example 1. Assume $A$ is closed and bounded. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $A$. The sequence is bounded since $A$ is, so by the Bolzano-Weierstrass theorem for $\mathbb{R}^d$, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. $a_{n_k} \in A$ $\forall k$ and $a_{n_k} \to L$ and $A$ is closed, so by the sequential characterization of closedness, $L \in A$, therefore $A$ is compact.

( $\Longleftarrow$ ) Assume $A$ is compact. To show $A$ is closed, assume for a contradction that $A$ is not closed. Therefore there exists a sequence $(a_n)_{n=1}^{\infty}$ in $A$ such that $a_n \to L \notin A$. Then for any subsequence $(a_{nk})_{k=1}^{\infty}$, we have

$$a_n \to L \notin A$$

This contradicts that $A$ is compact, therefore $A$ is closed.

Tow show $A$ is bounded, assume for a contradiction that $A$ is not bounded. Then $\forall n \in \mathbb{N}_{\geq 1}$, there exists $a_n \in A$ such that $||a_n||_2 \geq n$. This gives a sequence $(a_n)_{n=1}^{\infty}$. Since $A$ is compact, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges. But

$$||a_{n_k}||_2 \geq n_k \to \infty$$

So $(a_{nk})_{k=1}^{\infty}$ is unbounded, which is a contradiction. $\square$

**Proposition 11.0.1.**

*(i) If $k_1, \dots, k_n \subseteq \mathbb{R}^d$ are compact, then $\bigcup_{i=1}^{n} k_i$ is compact.*

*(ii) If $k_1, \dots, k_n \subseteq \mathbb{R}^d$ are compact, then $\bigcap_{i=1}^{n} k_i$ is compact.*

*Proof.* **Exercise:**

(i) Assume $A := \bigcup_{i=1}^{n} k_i$. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $A$. Then there exists $i \in \{1, \dots, n\}$ such that $a_n \in k_i$. Since $k_i$ is compact, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. $L \in k_i$ and $k_i \subseteq A$, so $L \in A$. Therefore $A$ is compact.

(ii) Assume $A := \bigcap_{i=1}^{n} k_i$. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $A$. Then $a_n \in k_i$ $\forall i \in \{1, \dots, n\}$. Since $k_i$ is compact, it has a subsequence $(a_{nk})_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}^d$. $L \in k_i$ $\forall i \in \{1, \dots, n\}$ and $k_i \subseteq A$, so $L \in A$. Therefore $A$ is compact.

$\square$

**Definition 11.0.2.** *$A \subseteq \mathbb{R}^d$ is **compact** if for any collection*

$$U_\alpha : \alpha \in I$$

*of open sets such that*

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha$$

*There exists finitely many indeces $\alpha_1, \dots, \alpha_n$ such that*

$$A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

# Lecture 12

# Limits of a Function of Continuous Variables

A sequence is a function $\mathbb{N} \to \mathbb{R}$. Here, we'll consider function that are going from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{R}^d \to \mathbb{R}^m$).

**Definition 12.0.1.** *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of $X$. $f : X \to \mathbb{R}^m$, $L \in \mathbb{R}^m$. We say the limit of $f$ as $X$ approaches $a$ is $L$ if*

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t \ \forall x \in X$$

$$x \in B(a;\delta) \land x \neq a \implies ||f(x) - L||_2 < \epsilon$$

The idea is like the definition of convergence of a sequence, except we replace $n \geq n_0$ (which captures "$n$ is sufficiently large") with $x \in B(a;\delta)$, $x \neq a$ (which captures "$x$ is close to, but not equal to $a$). In other words, the definition says that if $x$ is close to (but not equal to) $a$ then $f(x)$ is close to $L$.

**Why "not equal to"?:** Often we consider the limit as $x$ approaches $a$ when $f(a)$ is not defined. Other times we compare the limite to $f(a)$. So we do not want to use $f(a)$ in the definition of the limit.

**Notation:** We write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \ as \ x \to d$$

to mean that the limit of $f$ is $L$ as $x$ approaches $a$.

**Example:** $f : \mathbb{R} \to \mathbb{R}$. $f(x) \coloneqq 3x - 2$. Let $a \in \mathbb{R}$. Claim

$$\lim_{x \to a} f(x) = 3a - 2$$

*Proof.* Let $\epsilon > 0$. Consider

$$\begin{aligned} |f(x) - (3a - 2)| &= |3x - 2 - 3a + 2| \\ &= 3|x - a| \end{aligned}$$

We want this $< \epsilon$, set $\delta \coloneqq \frac{\epsilon}{3}$. Then if $x \in B(a;\delta) = (a-\delta, a+\delta)$ (i.e $|x-a| < \delta$) then

$$|f(x) - (3a - 2)| = 3|x - a| < 3\delta = \frac{3\epsilon}{3} = \epsilon$$

$\square$

**Example:** $g : \mathbb{R} \to \mathbb{R}$. $g(x) \coloneqq x^2$. Claim:

$$\lim_{x \to a} g(x) = a^2$$

*Proof.* Let $\epsilon > 0$ be given.

$$\begin{aligned} |g(x) - a^2| &= |x^2 - a^2| \\ &= |x - a||x + a| \end{aligned}$$

What happens if $x$ is close to $a$? Intuitively, $|x+a|$ is close to $|a+a|$ and $|x-a|$ is small.

$$\begin{aligned} |x + a| = |x - a + a + a| &\leq |x - a| + |a + a| \\ &< 2|a| + \delta && \text{(if } |x - a| < \delta) \\ &\leq 2|a| + 1 && (\delta \leq 1) \end{aligned}$$

Then,

$$\begin{aligned} |x^2 - a^2| = |x - a||x + a| &\leq |x - a|(2|a| + 1) \\ &< \delta(2|a| + 1) && \text{(if } |x - a| < \delta) \\ &\leq \epsilon && \text{(if } \delta \leq \tfrac{\epsilon}{2|a|+1}) \end{aligned}$$

**Important:** Do not define $\delta$ in terms of $x$ or $\delta$! We can use $a$ here since $a$ is constant

So we set $\delta \coloneqq \min\{1, \frac{\epsilon}{2|a|+1}\}$ Then $\delta \leq \frac{\epsilon}{2|a|+1}$ and $\delta \leq 1$. So if $|x - a| < \delta$. Then from the work above, $|x^2 - a^2| < \epsilon$ as required. $\square$

**Note:** In proofs where we have $\delta - \epsilon$, we often use

$$\delta \coloneqq \min\{\dots\}$$

In proofs where we have $n_0 - \epsilon$, we often use

$$n_0 \coloneqq \max\{\dots\}$$

**Proposition 12.0.1** (Uniqueness of Limits)**.** *Let $f : X \to \mathbb{R}^m$ ($X \subseteq \mathbb{R}^d$), $a \in \mathbb{R}^d$ a limit point of $X$, $L, L' \in \mathbb{R}^m$. If the limit of $f$ as $x \to a$ is $L$ and the limit of $f$ as $x \to a$ is $L'$, then $L = L'$*

*Proof.* By contradction. Suppose $L \neq L'$. So

$$||L - L'||_2 > 0$$

Set

$$\epsilon := \frac{||L - L||_2}{2} > 0$$

Since $f(x) \to L$ as $x \to a$, $\exists \delta > 0$ such that if $x \in X \cap B(a;\delta) \setminus \{a\}$, then

$$||f(x) - L||_2 < \epsilon$$

Since $f(x) \to L'$ as $x \to a$, $\exists \delta' > 0$ such that

$$x \in X \cap B(a;\delta') \setminus \{a\} \implies ||f(x) - L'||_2 < \epsilon$$

Let $\delta_0 \neq \min\{\delta, \delta'\}$. Let

$$x \in X \cap B(a;\delta_0) \setminus \{a\}$$

Then

$$x \in X \cap B(a;\delta) \setminus \{a\}$$

So,

$$\begin{aligned} ||f(x) - L||_2 &< \epsilon \\ ||f(x) - L'||_2 &< \epsilon \end{aligned}$$

So,

$$\begin{aligned} ||L - L'||_2 \leq ||L - f(x)||_2 + ||f(x) - L'|| &< \epsilon + \epsilon \\ &= ||L - L'||_2 \end{aligned}$$

And thus, a contradction. $\square$

**Proposition 12.0.2** (Sequential Characterization of Limits)**.** *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$, a limit point of $X$. $f : X \to \mathbb{R}^m$, $L \in \mathbb{R}^m$.*

$\lim_{x \to a} f(x) = L \iff$ *for every sequence $(x_n)_{n=1}^{\infty}$ in $X$ such that $x_n \to a$, we have*

$$\lim_{n \to \infty} f(x_n) = L$$

*Proof.* ($\implies$) Suppose $\lim_{x \to a} f(x) = L$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $X \setminus \{a\}$ such that $x_n \to a$. We must show that $f(x_n) \to L$.

Let $\epsilon > 0$ be given. Since $f(x) \to L$ as $x \to a$, $\exists \delta$ such that

$$x \in X \cap B(a;\delta) \setminus \{a\} \implies ||f(x) - L||_2 < \epsilon$$

Since $x_n \to$, using $\delta$ in place of $\epsilon$, $\exists n_0$ such that $\forall n \geq n_0$, $||x_n a||_2 < \delta$. i.e. $x_n \in B(a, \delta)$. Also $x_n \in X \setminus \{a\}$ Therefore,

$$||f(x) - L||_2 < \epsilon$$

( $\Longleftarrow$ ) Suppose $\forall$ sequences $(x_n)_{n=1}^{\infty}$ ins $X \setminus \{a\}$ converging to $a$, $f(x) \to L$, and for a contradction, suppose

$$f(x) \not\to L$$

We negate "$f(x) \to L$" to get that $\exists \epsilon > 0$ such that $\forall \gamma > 0$, $\exists x \in X \cap B(a; \delta) \setminus \{a\}$ such that $||f(x) - L||_2 \geq \epsilon$.

This gives a sequence $(x_{nn})_{n=1}^{\infty}$ in $X \setminus \{a\}$, $||x_n - a||_2 \leq \frac{1}{n}$ $\forall n$, so by the squeeze theorem

$$||x_n - a||_2 \to 0$$

Since $||f(x_n) - L||_2 \geq \epsilon$, $f(x_n) \not\to L$. This is a contradction. ☐

**Note:** if $\lim_{n\to\infty} f(x_n) = L$ for *some* sequence $(x_n)_{n=1}^{\infty}$ in $X \setminus \{a\}$ convering to a, it *does not* follow that $\lim_{x\to a} f(x) = L$

**Example:**

$$f(x) := \begin{cases} 0 \text{ if } x = \frac{1}{n},\ n \in \mathbb{N}_{\geq 1} \\ 1 \text{ otherwise} \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist but $\lim_{n\to\infty} f(\frac{1}{n}) = 0$

**Proposition 12.0.3** (Algebra of Limits)**.** *Let $x \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of $X$, $f : X\mathbb{R}^m$ ,$g : X \to \mathbb{R}^m$, $L, K \in \mathbb{R}^m$. Suppose* $\lim_{x\to a} f(x) = L$, $lim_{x\to a} g(x) = K$

*(i)*

$$\lim_{x\to 0} f(x) + g(x) = L + K$$

*(ii)*

$$\lim_{cf(x)} = cL$$

*(iii) If $m = 1$,*

$$lim_{x\to a} f(x)g(x) = LK$$

*(iv) If $m = 1$, $g(x) \neq 0$ $\forall x \in X$, $K \neq 0$. Then*

$$lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{K}$$

*Proof.* (i) Use Sequential Characterization: Let $(x_n)_{n=1}^{\infty}$ be in $X \setminus \{a\}$ such that $x \to a$. Then $f(x_n) \to L$ and $g(x_n) \to K$ So by algebra of limits for sequences,

$$f(x_n) + g(x_n) = L + K$$

$$\therefore f(x) + g(x) \to L + K$$

(ii) **Exercise.**

(iii) **Exercise.**

(iv) **Exercise.**

□

**Theorem 12.0.1** (Squeeze Theorem). *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of $X$, $f, g, h : X \to \mathbb{R}$*

$$f(x) \leq g(x) \leq h(x) \ \forall x \in X$$

*and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

*Then*

$$\lim_{x \to a} g(x) = L$$

*Proof.* **Exercise** □

If $f : X \to \mathbb{R}_m$, We can define functions

$$f_1, \ldots, f_m : X \to \mathbb{R}$$

by

$$(f_1(x), \ldots, f_m(x)) = f(x)$$

$f_1, \ldots, f_m$ are called the *component functions* of $f$.

**Proposition 12.0.4.** *Let $X \in \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of $X$, $f : X \to \mathbb{R}^m$, $f_1, \ldots, f_m$ its component functions. $L = (L_1, \ldots, L_m) \in \mathbb{R}^m$. Then*

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a} f_i(x) = L_i \ \forall 1 \leq i \leq m$$

*Proof.* **Exercise.** □

**Definition 12.0.2.** *Let $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $f : X \to \mathbb{R}^d$.*

- *If $a$ is a limit point of $X \cap (a, \infty)$ then we write $\lim_{x \to a^+} f(x) = L$ to mean that*

  $$\lim_{x \to a} g(x) = L$$

  *where*

  $$g = f\big|_{X \cap (a, \infty)}$$

- *If $a$ is a limit point of $X \cap (-\infty, a)$ then we write $\lim_{x \to a^+} f(x) = L$ to mean that*

  $$\lim_{x \to a} g(x) = L$$

  *where*

  $$g = f \mid_{X \cap (-\infty, a)}$$

**Example:**

$$f(x) \coloneqq \begin{cases} -1, x < 0 \\ 0, x = 0 \\ 1, x > 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = 1 \neq \lim_{x \to 0^-} f(x) = -1$$

# Lecture 13

# Continuity

## 13.1 One-Sided Limits

**Recall:** For $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}^m$, $a \in \mathbb{R}$

- If $a$ is a limit point of $X \cap [a, \infty]$, then

$$\lim_{x \to a^+} f(x) := \lim_{x \to a} f_{X \cap [a,\infty)}(x)$$

  $\lim_{x \to a^-} f(x)$ is defined similarly.

In other words, $\lim_{x \to a^+} f(x) = L$ means $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $a < x < a + \delta$ and $x \in X$, then

$$||f(x) - L||_2 < \epsilon$$

## 13.2 Continuity

**Definition 13.2.1.** *Let $f : X \to \mathbb{R}^m$ where $X \subseteq \mathbb{R}^d$. Let $a \in X$.*

- *If a is not an isolated point, then we say f is continuous at a if*

$$\lim_{x \to a} f(x) = f(a)$$

- *If a is an isolated point, then we always say $f$ is* continuous *at a.*

**Definition 13.2.2** ($\delta - \epsilon$ Characterization of Continuity)**.** *f is continuous at a* $\iff$ $\forall \epsilon > 0$, $\exists \gamma > 0$ *s.t if*

$$x \in X \cap B(a; \delta)$$

*then*

$$||f(x) - f(a)||_2 < \epsilon$$

*In other words if x is close to a, the f(x) is close to f(a). This holds regardless of whether or not a is isolated.*

**Example:** $f : \mathbb{R} \mapsto \mathbb{R}$ $f(x) := x^2$. For any $a \in \mathbb{R}$, we proved last lecture that

$$\lim_{x \to a} f(x) = a^2 = f(a)$$

Therefore, $f$ is continuous at $a$.

**Example:**

$$f(x) := \begin{cases} 1 \text{ if } x \leq 1 \\ 2 \text{ if } x > 1 \end{cases}$$

For this function,

$$\lim_{x \to 1^+ f(x) = 2 \neq f(x) = 1}$$

Therefore, $f$ is not continuous at 1.

**Example:**

$$f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x > 0 \\ L \text{ if } x = 0 \text{ for some } L \in \mathbb{R} \end{cases}$$

No matter how we choose $L$, $f$ is not continuous at 0.

*Proof.* We want to show $\lim_{x \to 0} f(x)$ does not exist. Consider the sequence

$$\left(\frac{1}{2\pi n}\right)_{n=1}^{\infty} \to 0$$

and $f\left(\frac{1}{2\pi n} = 0 \to 0\right)$. But we also have

$$\left(\frac{1}{2\pi n + \pi/2}\right)_{n=1}^{\infty} \to 0$$

and $f\left(\frac{1}{2\pi n + \pi/2}\right) = 1 \to 1$. So if $\lim_{x \to 0} f(x)$ exists, then $f(a_n) \to L$ for a sequence $(a_{n n})_{n=1}^{\infty}$ in $(0, \infty)$ such that $a_n \to 0$. So $L = 0$ and $L = 1$, therefore a contradiction $\square$

**Example:**

$$f(x) := \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

$f$ is not continuous at any point.

*Proof.* Let $a \in \mathbb{R}$. We'll prove $\lim_{x \to a} f(x)$ does not exist.

$$\exists ((_n)_{n=1}^{\infty} x) \in \mathbb{Q} \text{ s.t } x_n \to a$$

so $f(x_n) = 1 \to 1$

$$\exists ((_n)_{n=1}^{\infty} y) \in \mathbb{R} \setminus \mathbb{Q} \text{ s.t } y_n \to a$$

Since we can find $z_n \in \mathbb{Q}$ such that $z_N \to a + \sqrt{2}$, so $y := z_n - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, $y_n \to a$. So $f(y_n) = 0 \to 0$. Therefore, the $\lim_{x \to a} f(x)$ does not exist. $\square$

**Example:**

$$f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

$f$ is continuous.

*Proof.* Use squeeze theorem, if $x \geq 0$, then $-x \leq f(x) \leq x$. So

$$-|x| \leq f(x) \leq |x|$$

$$\lim_{x \to 0} |x| = 0 = \lim_{x \to 0} -|x|$$

$$\therefore \lim_{x \to 0} f(x) = 0 = f(0)$$

□

**Proposition 13.2.1.** *Let $X \subseteq \mathbb{R}^d$, $Y \subseteq \mathbb{R}^m$, and $f : X \mapsto Y$, $g : Y \to \mathbb{R}^n$, $a \in X$. If $f$ is continuous at $a$ and $g$ is continuous at f(a), then $g \circ f$ is continuous at a.*

*Proof.* Use the $\delta - \epsilon$ characterization. Let $\epsilon > 0$ be given. Since $g$ is at $f(a)$, $\exists \eta > 0$ (using $\eta$ instead of $\delta$), such that if $y \in B(f(a); \eta) \cap Y$, then

$$||g(x) - g(f(a))||_2 < \epsilon$$

Since $f$ is continuous at $a$, $\exists \delta > 0$ such that if $x \in B(a; \delta) \cap X$, then

$$||f(x) - f(a)||_2 < \eta$$

In other words, $f(x) \in B(f(a); \eta)$. Therefore if $x \in B(a; \delta) \cap X$, then since $f(x) \in B(f(a); \eta)$,

$$||g(x) - g(f(a))||_2 < \epsilon$$

as required. □

**Proposition 13.2.2.** *Let $X \subseteq \mathbb{R}^d$, $f, g : X \mapsto \mathbb{R}^m$, $a \in X$ such that $f, g$ are continuous at a,*

*(i) $f + g : X \mapsto \mathbb{R}^m$ is continuous at a*

*(ii) $cf : X \mapsto \mathbb{R}^m$ is continuous at a for any $c \in \mathbb{R}$*

*(iii) $\gamma f : X \mapsto \mathbb{R}^m$ is continuous at a if $\gamma : X \mapsto \mathbb{R}$ is continuous at $a$.*

*(iv) If $m = 1$, and $g(x) \neq 0$, $\forall x \in X$, then*

$$\frac{f}{g} : X \mapsto \mathbb{R}$$

*is continuous at a.*

*Proof.* Follows from the algebra of limits. □

If $f : X \mapsto \mathbb{R}^m$, $f = (f_1, \dots, f_m)$ where $f_i : X \mapsto \mathbb{R}$, then $f$ is continuous at $a$ if and only if $f_i$ is continuous at $a$ for all $i \in \{1, \dots, m\}$.

- You are allowed to use the fact that sin, cos, exp, log are all continuous functions log are all continuous log : $[0, \infty) \mapsto \mathbb{R}$ is continuous at all $a \in [0, \infty)$.

## 13.3 Properties of Continuous Functions

$f : X \mapsto \mathbb{R}^m$ is *(globally) continuous* if $f$ is continuous at $a$ for all $a \in X$.

**Theorem 13.3.1.** *Let $K \subseteq \mathbb{R}^d$ be a compact set. If $f : K \mapsto \mathbb{R}^m$ is continuous, then $f(K)$ is compact.*

*Proof.* Let $(a_n)_{n=1}^{\infty}$ be a sequence in $f(K)$. We need to show that $\exists$ a subsequence. We need to show that $\exists$ a subsequence $(a_{n_k})_{k=1}^{\infty}$ s.t $a_{n_k} \to b$ for some $b \in f(K)$. Since $a_n \in f(K)$, $\exists x_n \in K$ s.t $f(x_n) = a_n$ for all $n$. This gives a sequence $(x_n)_{n=1}^{\infty}$ in $K$. Since $K$ is compact, $\exists$ a subsequence $(x_{n_k})_{k=1}^{\infty}$ s.t $x_{n_k} \to x_0$. Since $f$ is continuous at $x_0$, $f(x_{n_k}) \to f(x_0)$. Therefore, $a_{n_k} \to f(x_0)$ and $(a_{n_k})_{n=1}^{\infty}$ is a subsequences of $(a_n)_{n=1}^{\infty}$. This is what we wanted. □

**Corollary 13.3.1** (Extreme Value Theorem)**.** *Let $K \subseteq \mathbb{R}^d$ be compact, $f : K \to \mathbb{R}$ a continuous function. Then $f$ is bounded and it attains its bounds, i.e $\exists x_{\min}, x_{\max} \in K$ such that $\forall x \in K$*

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

*Proof.* From the previous result, $f(K)$ is compact in $\mathbb{R}$. By the Heine-Borel theorem, this means $f(k)$ is closed and bounded. Therefore, $f$ is bounded. Since $f(K)$ is closed, $\sup f(K) \in f(K)$ since there is a sequence $(y_n)_{n=1}^{\infty}$ in $f(K)$ such that $y_n \to \sup f(K)$. So $\exists x_{\max} \in K$ such that

$$f(x_{\max}) = \sup f(k)$$

Therefore,

$$f(x) \leq f(x_{\max}) \; \forall x \in K$$

Similarily, $\exists x_{\min} \in K$ such that $f(x_{\min}) \leq f(x) \; \forall x \in K$. □

# Lecture 14

# Properties of Continuous Functions Continued

**Recall:**

**Corollary 13.3.1** (Extreme Value Theorem). *If $K \subseteq \mathbb{R}^d$ is compact and $f : K \to \mathbb{R}$ is continuous, then $\exists x_{\min}, x_{\max} \in K$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in K$.*

**Theorem 14.0.1** (Intermediate Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $y_0 \in \mathbb{R}$ be any number between $f(a)$ and $f(b)$. Then there exists a number $x_0$ between $a$ and $b$ such that $f(x_0) = y_0$.*

*Proof.* Without loss of generality, assume $f(a) \leq f(b)$. So $x_0 \in [f(a), f(b)]$. Let

$$S \coloneqq \{x \in [a, b] : f(x) \leq x_0\}$$

$S \subseteq [a, b]$, so $S$ is bounded. $a \in S$ since $f(a) \leq x_0$. Therefore $S \neq \emptyset$. So $\exists x_0 \coloneqq \sup S \in [a, b]$. We will show $f(x_0) = y_0$. We will consider the cases, where $f(x_0) = y_0$, $f(x_0) < y_0$, and $f(x_0) > y_0$. If $f(x_0) < y_0$, set $\epsilon \coloneqq y_0 - f(x_0)$. Since $f$ is continuous at $x_0$, $\exists \delta > 0$ s.t if $|x - x_0| < \delta$ and $x \in [a, b]$. Then $|f(x) - f(x_0)| < \epsilon$. Since $f(x_0) < y_0 \leq f(b)$ for $x_0 \neq b$. So we can find $x > x_0$ such that $x \in [a, b]$ and $|x - x_0| < \delta$. Then $f(x) < f(x_0) + \epsilon = y_0$. So $x \in S$ by the definition of $S$, since $S = \{x : f(x) \leq y_0\}$. This is a contradiction since $x > x_0$, but $x_0$ is an upper bound for $S$.

If $f(x_0) > y_0$, set $\epsilon \coloneqq f(x_0) - y_0$. Since $f$ is continuous at $x_0$, $\exists \delta > 0$ such that if $x \in [a, b]$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. So $f(x_0) > y_0 \leq f(a)$. So $x_0 > a$. We may assume that $x_0 - \delta > a$ since $\delta$ is abritrary and can be arbitrarly small.

Claim: $x_0 - \delta$ is an upper bound for $S$. Proof of claim, if $x > x - \delta$, then either $|x - x_0| < \delta$, in which case $f(x) > f(x_0) - \epsilon = y - 0$, or $x > x_0$, in which case $x \notin S$ since $x_0$ is an upper bound for $S$. Therefore, if $x > x_0 - \delta$, then $x \notin S$.

This proves the claim.

The claim contradits that $x_0$ is the least upper bound for $S$. $\square$

**Corollary 14.0.1.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous. Then* $f([a, b]) = [c, d]$ *for some* $c \leq d$.

*Proof.* By extreme value theorem, $\exists x_{\min}, x_{\max}$ such that $f(x) \in [f(x_{\min}), f(x_{\max})]$ $\forall x \in [a, b]$. So we can set $c \coloneqq f(x_{\min})$ and $d \coloneqq f(x_{\max})$ and we have

$$f([a, b]) \subseteq [c, d]$$

for $y_0 \in [c, d] = [f(x_{\min}), f(x_{\max})]$. By the Intermediate Value Theorem, $\exists x_0$ between $x_{\min}$ nad $x_{\max}$ such that $f(x_0) = y_0$. Therefore

$$[c, d] \subseteq f([a, b])$$

Thus,

$$f([a, b]) = [c, d]$$

$\square$

## 14.1 Inverses of Continuous Functions

Let $f : A \mapsto B$, $f$ is

- **Injective** (or one-to-one) If $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- **Surjective** (or onto) If $\forall y \in B$, $\exists x \in A$ such that

$$f(x) = y$$

- **Bijective** (or one-to-one and onto) If $f$ is both injective and surjective.

$$\begin{aligned} f \text{ is bijective} &\iff \text{it is inveritble} \\ &\iff \exists f^{-1} : B \mapsto A \text{ s.t } f \circ f^{-1} = id_B \text{ and } f^{-1} \circ f = id_A \end{aligned}$$

*Proof.* ( $\Longleftarrow$ ) If $f$ is inveritble, then it is

- **Injective** since if $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$
- **Surjective** since given $y \in B$, $x \coloneqq f^{-1}(y)$ satisfies $f(x) = y$.

( $\Longrightarrow$ ) If $f$ is bijective, then for $y \in B$, $\exists x \in A$ such that $f(x) = y$. Also, this $x$ is unique since $f$ is injective. So we can define

$$f^{-1}(y) \coloneqq x$$

and this is an inverse to $f$. $\square$

Given $f : A \mapsto \mathbb{R}$ that is injective, we view $f$ as a function $A \mapsto f(A)$, and this way we force $f$ to be surjective. Thus $f^{-1} : f(A) \mapsto A$ exists.

**Question:** If $f$ is injective and continuous, must $f^{-1}$ be continuous? **No**. Consider the example with $f : [0,1) \cup [2,\infty]$

$$f(x) := \begin{cases} x_1 \text{ if } x < 1 \\ x - 1 \text{ if } x \geq 2 \end{cases}$$

$f^{-1} : [0,\infty) \mapsto [0,1) \cup [2,\infty] \subseteq \mathbb{R}$, so

$$f^{-1}(y) = \begin{cases} y_1 \text{ if } y < 1 \\ y + 1 \text{ if } y \geq 1 \end{cases}$$

$f^{-1}$ is not continuous.

**Definition 14.1.1.** *Let $A \subseteq \mathbb{R}$, $f : A \mapsto \mathbb{R}$. $f$ is*

*(i)* increasing *if $\forall x, y \in A$, $x \leq y$ then $f(x) \leq f(y)$*

*(ii)* strictly increasing *if $\forall x, y \in A$, $x < y$ then $f(x) < f(y)$*

*(iii)* decreasing *if $\forall x, y \in A$, $x \leq y$ then $f(x) \geq f(y)$*

*(iv)* strictly decreasing *if $\forall x, y \in A$, $x < y$ then $f(x) > f(y)$*

**Lemma 14.1.1.** *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \mapsto \mathbb{R}$ be an injective continuous function. Then $f$ is either strictly increasing or strictly decreasing.*

*Proof.* It suffices to consider the case where $I$ is a closed, bounded interval. A general interval $I$ is a union of an increasing sequence of closed bounded intervals, so if $f$ is strictly increasing or strictly decreasing on each of these subintervals, then $f$ is strinctly increasing or decreasing on all of $I$. Consider the case $I = [a,b]$ with $(a < b)$. Without loss of generality, assume $f(a) < f(b)$, in this case we'll show that $f$ is strictly increasing. By contradiction, if $f$ is not strictly increasing, then there exists $x_1 < x_2$ such that

$$f(x_1) \geq f(x_2)$$

We'll break this into 2 cases.

- **Case 1:** $f(x) > f(b)$. In this case,

  $$f(a) < f(b) < f(x_1)$$

  and

  $$f|_{[a,x_1]}$$

  is continuous. So by the Intermediate Value Theorem, $\exists z \in [a, x_1]$ such that $f(z) = f(b)$. But,

  $$z \leq x_1 < x_2 \leq b$$

  So $z \neq b$, which contradicts that $f$ is injective.

- **Case 2:** $f(x_1) \le f(b)$. In this case, $f(x_2) \le f(x_1) \le f(b)$ and $f|_{[x_1,b]}$ is continuous. So by the Intermediate Value Theorem, $\exists z \in [x_1, b]$ such that $f(z) = f(x_1)$. $z \ge x_2 > x_1$ so $z \ne x_1$. But, this contradicts that $f$ is injective.

In both cases, we get a contridction, so our assumption that $f$ is not strictly increasing is false. $\square$

# Lecture 15

Let $I \subset \mathbb{R}$ be an interval, $f : I \mapsto \mathbb{R}$ be a continuous function. Then $f^{-1}$ : $f(I) \mapsto \mathbb{R}$ is continuous.