MAT 2125 Midterm Summary Sheet

Fields

Definition: *A field is a set F equipped with two operations, addition and multiplication, and satisfies these axioms*

Field Axioms

Definition: *A* field *is a set F equipped with two operations, addition and multiplication which satisfies the following axioms*

- *(F1)* $\forall a, b \in F \ a + b = b + a$
- *(F2)* $\forall a, b, c \in F \ (a + b) + c = a + (b + c)$
- *(F3)* $\exists 0 \in F$ *s.t* $a + 0 = 0 + a = a \ \forall a \in F$
- *(F4)* $\forall a \in F \ \exists -a \in F$ *s.t* $a + (-a) = (-a) + a = 0$
- *(F5)* $\forall a, b \in F \ a \cdot b = b \cdot a$
- *(F6)* $\forall a, b, c \in F \ (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- *(F7)* $\exists 1 \in F$ *s.t* $1 \neq 0$ *and* $a \cdot 1 = 1 \cdot a = a \ \forall a \in F$
- *(F8)* $\forall a \in F \backslash \{0\} \ \exists a^{-1}$ *s.t* $a \cdot a^{-1} = a^{-1} \cdot a = 1$
- $\forall a, b, c \in F \ (a + b) \cdot c = a \cdot c + b \cdot c$

Definition: *An* ordered field *is a field F equipped with a binary relation $<$ and satisfies these axioms*

Order Axioms

- *(O1) If $a < b$ and $b < c$, then $a < c$*
- *(O2) $\forall a, b \in F$, exactly one of the following is true: $a = b$ or $a < b$ or $b < a$*
- *(O3) $\forall a, b, c \in F$, if $a < b$ then $a + c < b + c$*
- *(O4) $\forall a, b, c \in F$, if $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$*

Definition: *Let F be an ordered field, $S \subseteq F$, $a \in F$. Then a is an upper bound for S if for any $x \in S$*
$$x \leq a$$
a is a lowerbound for S if for any $x \in S$, $a \leq x$. S **is bounded if it bounded above and below.**

Suprema

When a is a least upper bound for S, we write
$$a = \sup S$$

Infima

When a is the greatest lower bound for S, we write
$$a = \inf S$$

Boundedness

Completeness

Definition: *If $S \subseteq F$ is nonempty and bounded above, then $\sup S$ exists.*

The Archimedean Property

Theorem: *The set $\mathbb{N}_{\geq 1}$ is not bounded above.*

Absolute Value and Distance

The *absolute value* of a real number $a \in \mathbb{R}$ is defined by
$$|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Properties:

(i) $|-x| = |x|$

(ii) $-|x| \leq x \leq |x|$

(iii) $|xy| = |x| \cdot |y|$

(iv) $|x + y| \leq |x| + |y|$

(v) $||x| - |y|| \leq |x - y|$

Definition:*Let $x, y \in \mathbb{R}$. The distance between x, y is*
$$d(x, y) := |x - y|$$

Properties:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Sequences

Definition: *A sequence $(a_n)_{n=1}^{\infty}$ is* bounded *if the set $\{a_n : n \in \mathbb{N}\}$ is bounded*

Convergence

Definition: *Let $(a_n)_{n=1}^{\infty}$ be a sequence, and $L \in R$. We say that the sequence converges to L if $\forall \epsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$*
$$|a_n - L| < \epsilon$$

Definition: *We say that $(a_n)_{n=1}^{\infty}$ diverges to ∞ if for every $R > 0$, $\exists n_0 \in \mathbb{N}$, s.t $\forall n \geq n_0$*
$$a_n > R$$

Proposition: If $(a_n)_{n=1}^{\infty}$ converges, then it is bounded.

Limits

Proposition:*(Uniqueness) Let $(a_n)_{n=1}^{\infty}$ be a sequence and $L_1, L_2 \in \mathbb{R}$*
$$\lim_{n\to\infty} a_n = L_1 \wedge \lim_{n\to\infty} a_n = L_2 \implies L_1 = L_2$$

Algebra of Limits

(i) $(a_n + b_n)_{n=1}^{\infty}$ converges to $L_a + L_b$

(ii) $(ca_n)_{n=1}^{\infty}$ converges to cL_n

(iii) $(a_n b_n)_{n=1}^{\infty}$ converges to $L_a L_b$

(iv) $a_n \neq 0 \ \forall n \wedge L_a \neq 0 \implies \left(\frac{1}{a_n}\right)_{n=1}^{\infty} \to \frac{1}{L_a}$

Properties of Limits

Proposition: *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be converging sequences*
$$\forall n \ a_n \leq b_n \implies \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Corollary: *Let $(a_n)_{n=1}^{\infty}$ be a converging sequence*
$$\forall n \ m \leq a_n \leq M \implies m \leq \lim_{n\to\infty} a_n \leq M$$

Squeeze Theorem

Let $(a_n)_{n=1}^{\infty}$,$(b_n)_{n=1}^{\infty}$,$(c_n)_{n=1}^{\infty}$ be sequences such that

(i) $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ converge to same number L

(ii) $a_n \leq b_n \leq c_n \ \forall n$

Then $(b_n)_{n=1}^{\infty}$ also converges to L.

Monotone Convergence Criterion

Let $(a_n)_{n=1}^{\infty}$ be a monotone sequence, it converges if and only if it is bounded. If it is increasing, then
$$(a_n)_{n=1}^{\infty} \to \sup\{a_n : n \in \mathbb{N}_{\geq 1}\}$$
If it is decreasing, $(a_n)_{n=1}^{\infty} \to \inf\{a_n : n \in \mathbb{N}_{\geq 1}\}$

Subsequences

Proposition:*If $(a_n)_{n=1}^{\infty}$ converges to L, then any subsequence also converges to L*

Proposition: *Every sequence contains a monotone subsequence*

Bolzano-WeierStrass Theorem: *Every bounded sequence has a convergent subsequence*

Cauchy Convergence Criterion: *$(a_n)_{n=1}^{\infty}$ converges $\iff$ it is Cauchy*

Limit Superior and Limit Inferior

Definition: *Let $(a_n)_{n=1}^\infty$ be a sequence. The limit superior and inferior of $(a_n)_{n=1}^\infty$ is*

$$\limsup_{n\to\infty} a_n := \inf\{\beta \in \mathbb{R}\ \exists n_0\ s.t\ a_n \le \beta\ \forall n \ge n_0\}$$

$$\liminf_{n\to\infty} a_n := \sup\{\beta \in \mathbb{R}\ \exists n_0\ s.t\ a_n \ge \beta\ \forall n \ge n_0\}$$

Propositions and Theorems

Proposition: *For any sequence $(a_n)_{n=1}^\infty$*

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$$

Proposition: *Let $(a_n)_{n=1}^\infty$ be a bounded sequence. Then*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

Theorem: *Let $(a_n)_{n=1}^\infty$ be a sequence of real numbers. Then $(a_n)_{n=1}^\infty$ converges if and only if*

$$-\infty < \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n < \infty$$

Then

$$\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$$

Series

Definition: *Let $(a_n)_{n=1}^\infty$ be a sequence.*

$$S_N := \sum_{n=1}^{N} a_n = a_1 + \cdots + a_N$$

called the N^{th} partial sum. $\sum_{n=1}^N a_n$ converges to L if $(S_N)_{N=1}^\infty$ to L. Then

$$\sum_{n=1}^\infty a_n = L$$

Propositions and Theorems

Proposition: *Let $(a_n)_{n=1}^\infty, (a_n)_{n=1}^\infty$ be sequences s.t $a_n \le b_n\ \forall n$. If both series converge then*

$$\sum_{n=1}^\infty a_n \le \sum_{n=1}^\infty b_n$$

Proposition: *Let $(a_n)_{n=1}^\infty$ be a sequence and $m \in \mathbb{N}_{\ge 1}$. Then*

$$\sum_{n=1}^\infty a_n \text{ converges} \iff \sum_{n=m}^\infty a_n \text{ converges}$$

Proposition: *Let $(a_n)_{n=1}^\infty$ be a sequence.*

$$\sum_{n=1}^\infty a_n \text{ converges} \implies \lim_{n\to\infty} a_n = 0$$

Convergence Tests

Boundedness Test: *If*

(i) $a_n \ge 0\ \forall n$

(ii) *There is a bound $M \in \mathbb{R}$ on the partials sums so that $\forall N \in \mathbb{N}$,*

$$\sum_{n=1}^N a_n \le M$$

Then $\sum_{n=1}^\infty$ converges.

Comparison Test: *$(b_n)_{n=1}^\infty$ s.t $0 \le a_n \le b_n\ \forall n$, then*

(i) if $\sum_{n=1}^\infty b_n$ converges, then so does $\sum_{n=1}^\infty a_n$

(ii) if $\sum_{n=1}^\infty a_n$ diverges, then so does $\sum_{n=1}^\infty a_n$

Absolute Convergence Test:

$$\sum_{n=1}^\infty |a_n| \text{ converges} \implies \sum_{n=1}^\infty a_n \text{ converges}$$

Ratio Test: *If*

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

Then $\sum_{n=1}^\infty a_n$ converges absolutely. No conclusion can be made if $= 1$, diverges otherwise.

Root Test: *If*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$$

Then $\sum_{n=1}^\infty a_n$ converges (absolutely). If $= 1$ the test in inconclusive, diverges otherwise.

Alternating Series Test: Suppose

(i) $(a_n)_{n=1}^\infty$ is a decreasing sequence

(ii) $\lim_{n\to\infty} = 0$

Then

$$\sum_{n=1}^\infty (-1)^{n+1} a_n$$

converges. Moreover,

$$\sum_{n=1}^{2N} (-1)^{n+1} a_n \le \sum_{n=1}^{\infty} (-1)^{n+1} a_n \le \sum_{n=1}^{2N-1} a_n$$

Integral Test: *Let $f : [1, \infty) \to \mathbb{R}$ be a function. Supoose that*

(i) $f(x) \ge 0\ \forall x \in [1, \infty)$

(ii) f is decreasing: $f(x) \ge f(y)$ whenever $x \le y$

Then

$$\sum_{n=1}^\infty f(n) \text{ converges} \iff \int_1^\infty f(x)dx \text{ converges}$$

More Series

Proposition: *Let $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$ be sequence and $c \in \mathbb{R}$. Suppose their series' converges, then*

(a) $\sum_{n=1}^\infty (a_n + b_n)$ converges and

$$\sum_{n=1}^\infty (a_n + b_n) = \sum_{n=1}^\infty a_n + \sum_{n=1}^\infty b_n$$

(b) $\sum_{n=1}^\infty c a_n$ converges and

$$\sum_{n=1}^\infty c a_n = c \sum_{n=1}^\infty a_n$$

Cauchy Convergence Criterion For Series

Let $(a_n)_{n=1}^\infty$ be a sequence of real numbers. Then $\sum_{n=1}^\infty a_n$ converges if and only if for ever $\epsilon > 0$, $\exists N_0$ s.t $\forall N \ge M \ge N_0$

$$\left|\sum_{n=M}^N a_n\right| < \epsilon$$

Norms

Definition: *A norm on $\mathbb{R}^d$ is a function $||\cdot|| : \mathbb{R}^d \to [0, \infty)$ satisfying the following*

Properties of Norms

(i) *$||a|| = 0$ if and only if $a = (0, \dots, 0)$*

(ii) $||ca|| = |c| \cdot ||a||$

(iii) $||a + b|| \le ||a|| + ||b||$

Euclidean Norm

$$||(a_1, \dots, a_d)||_2 := \sqrt{a_1^2 + \cdots + a_2^2}$$

Proposition: *Let $a, b \in \mathbb{R}^d$ and let $||\cdot||$ denote the Euclidean norm*

(i) (Cauchy-Schwarz Inequality)

$$|a \cdot b| \le ||a||_2 \cdot ||b||_2$$

(ii) (Triangle Inequality)

$$||a + b||_2 \le ||a||_2 + ||b||_2$$

(iii) Therefore, $||\cdot||_2$ is a norm in $\mathbb{R}^d$.

Convergence in $\mathbb{R}^d$

Definition: *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$ and let $L \in \mathbb{R}^d$. We say that $(a_n)_{n=1}^{\infty}$ converges to L if $\lim_{n\to\infty} ||a_n - L||_2 = 0$. In this case we write*

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

Proposition: *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$ with $a_n = (a_n^{(1)}, \ldots, a_n^{(d)})$. Let $L = (L_1, \ldots, L_d) \in \mathbb{R}^d$.* Then

$$\lim_{n\to\infty} a_n = L \iff \lim_{n\to\infty} a_n^{(i)} = L_i \text{ for each } i$$

Cauchy Convergence

Definition: *A sequence in $\mathbb{R}^d$ is* Cauchy *if for every $\epsilon > 0$, $\exists n_0$ s.t $\forall m, n \geq n_0$*

$$||a_m - a_n||_2 < \epsilon$$

Cauchy Convergence Criterion: *A sequence in $\mathbb{R}^d$ converges if and only if it is Cauchy.*

Definition: *A subset S of $\mathbb{R}^d$ is bounded if $\exists M > 0$ s.t*

$$||x||_2 \leq M$$

Bolzano-Weierstrass Theorem for $\mathbb{R}^d$

Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^d$. Then it has a subsequence which converges

Open and Closed Sets

Definition: *Let $a \in \mathbb{R}^d$ and let $r > 0$. The* open ball *centered at a with a radius r is*

$$B(a; r) := \{x \in \mathbb{R}^d : ||x - a||_2 < r\}$$

Definition: *Let $A \subseteq \mathbb{R}^d$ be a set*

(i) A is open *if for every $x \in A$, there exists $\epsilon > 0$ such that*

$$B(x; \epsilon) \subseteq A$$

(ii) A is closed *if its complement*

$$\mathbb{R}^d \setminus A = \{x \in \mathbb{R}^d : x \notin A\}$$

is open

It is not the case that a set is either open or closed

Proposition:

(i) The sets $\emptyset$ and $\mathbb{R}^d$ are open

(ii) For any finite collection of open sets, their union is open

(iii) For any finite collection of open sets, their intersection is open

Types of Points

Definition: *Let $A \subseteq \mathbb{R}^d$ be a set and let $a \in \mathbb{R}^d$*

(i) a is an interior point *of A if $\exists \epsilon > 0$ s.t $B(a; \epsilon) \subseteq A$*

$$A^\circ := \{x \in \mathbb{R}^d : \ x \text{ is an interior point}\}$$

(ii) a in an accumulation point *of A if there is a sequence in A s.t $a = \lim_{n\to\infty} a_n$. The closure of A is*

$$\bar{A} := \{x \in \mathbb{R}^d : x \text{ is an accumulation point}\}$$

(iii) a is a boundary point *of A if it is an accumulation point of A and it is not an interior point. The boundary of A is*

$$\partial A := \{x \in \mathbb{R}^d : \ x \text{ is a boundary point}\} = \bar{A} \setminus A^\circ$$

(iv) a is an isolated point *of A if $\exists \epsilon > 0$ s.t $B(a; \epsilon) \cap A = \{a\}$*

(v) a is a limit point *of A if it an accumulation point and not an isolated point*

Compactness

Definition: *Let $K \subseteq \mathbb{R}^d$ be a set.* We say that K is compact if every sequence in K has a subsequence that converges to a point in K.

Heine-Borel Theorem

Let $K \subseteq \mathbb{R}^d$

$$K \text{ is compact} \iff K \text{ is closed and bounded}$$

Proposition:

(i) For any finite collection of compact sets, their union is compact

(ii) For any arbitrary collection of compact sets, their intersection is compact

Continuous Functions

Definition: We write

$$\lim_{x\to a} f(x) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in X \setminus \{a\}$ and $||x - a||_2 < \delta$, then $||f(x) - L||_2 < \epsilon$.

Proposition: Let $X \subseteq \mathbb{R}^d$, $Y \subseteq Y^m$, $f : X \mapsto Y$, $g : Y \mapsto \mathbb{R}^n$. Suppose that f is continuous at a and g is continuous at $f(a)$, then $g \circ f : X \mapsto \mathbb{R}^n$ is continuous at a.

Properties of Continuous Functions

Proposition: Let $X \subseteq \mathbb{R}^d$ and let $a \in \mathbb{R}^d$ be a limit point. Let $f, g : X \mapsto \mathbb{R}^m$ and $\gamma : X \mapsto \mathbb{R}$ be functions which are all continuous at a. Let $c \in \mathbb{R}$. Then

(i) $f + g$ is continuous at a

(ii) $c \cdot f$ is continuous at a

(iii) $\gamma \cdot f$ is continuous at a

(iv) if $\gamma(x) \neq 0$ for all $x \in X$, then $\frac{1}{\gamma}$ is continuous at a.

Sequential Characterization of Limits

Let $X \subseteq \mathbb{R}^d$ and let $a \in \mathbb{R}^d$ be a limit point. Let $f : X \mapsto \mathbb{R}^m$ and let $L \in \mathbb{R}^m$. Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $(a_n)_{n=1}^{\infty}$ in $X \setminus a$ which converges to a, we have

$$\lim_{n\to\infty} f(a_n) = L$$

Algebra of Limits

Let $X \subseteq \mathbb{R}^d$ and let$a \in \mathbb{R}^d$ be a limit point. Let $f, g : X \mapsto \mathbb{R}^m$ and $\gamma : X \mapsto \mathbb{R}$ be functions which all have limits at a. Let $c \in \mathbb{R}$, then

(i) $\lim_{x\to a} (f(x) + g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$

(ii) $\lim_{x\to a} (cf(x)) = c\left(\lim_{x\to a} f(x)\right)$

(iii) $\lim_{x\to a} (\gamma(x)f(x)) = \left(\lim_{x\to a} \gamma(x)\right)\left(\lim_{x\to a} f(x)\right)$

(iv) If $\gamma(x) \neq 0$ for all $x \in X$ and $\lim_{x\to a} \gamma(x) \neq 0$, then $\lim_{x\to a} \frac{1}{\gamma(x)} = \frac{1}{\lim_{x\to a} \gamma(x)}$

Proposition (Squeeze Theorem). Let $X \in \mathbb{R}^d$ and let $a \in \mathbb{R}^d$ be a limit point. Let $f, g, h : X \mapsto \mathbb{R}$ with

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in X$$

Then if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Continuity

Definition. Let $X \subseteq \mathbb{R}^d$ and let $a \in X$ be a point which is not isolated. Let $f : X \mapsto \mathbb{R}^m$. We say f is continuous at a if

$$\lim_{x\to a} f(x) = f(a)$$

Properties of Continuous Functions

Definition. Let $X \subseteq \mathbb{R}^d$ and let $f : X \mapsto \mathbb{R}^m$ be a function. We say f is continuous on X if f is continuous at a for every $a \in X$.

Theorem. Let $K \subseteq \mathbb{R}^d$ be compact and let $f : K \mapsto \mathbb{R}^m$ be a continuous function. Then its image $f(K)$ is also compact.

Extreme Value Theorem

Let $K \subset \mathbb{R}^d$ be compact and nonempty, and let $f : K \mapsto \mathbb{R}$ be a continuous function. Then there exists $x_{\min}, x_{\max} \in K$ such that for all $x \in K$,

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

In other words, the image of f is bounded and attains its bounds.

Intermediate Value Theorem

Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function. Let $y \in \mathbb{R}$ be any value between $f(a)$ and $f(b)$. Then there exists $z \in [a, b]$ such that $f(z) = y$.

Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function. Then $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$

More on Continuous Functions

When f is bijective, it follows that is has an inverse.

Definition. Let $X \subseteq \mathbb{R}$ and let $f : X \mapsto \mathbb{R}$ be a function.

(i) We say f is weakly increasing if for $x, y \in X$
$$x \leq y \implies f(x) \leq f(y)$$

(ii) We say f is strictly increasing if for $x, y \in X$,
$$x < y \implies f(x) < f(y)$$

Similarly for weakly and strictly decreasing.

Lemma. Let $a < b$ and let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function. The following equivalent.

(i) f is either strictly increasing or strictly decreasing.

(ii) f is injective

Theorem. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \mapsto \mathbb{R}$ be an injective continuous function. Then $f^{-1} : f(I) \mapsto \mathbb{R}$ is continuous.

Uniform Continuity

Let $X \subseteq \mathbb{R}^d$ and $f : X \mapsto \mathbb{R}^m$ be a function. We say that f is uniformly continuous on X if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$, if $||x - y||_2 < \delta$, then $||f(x) - f(y)||_2 < \epsilon$

Theorem. Let $K \subseteq \mathbb{R}^d$ be compact and let $f : K \mapsto \mathbb{R}^m$ be continuous. Then f is uniformly continuous.

Infinite Limits and Limits at Infinity

Definition. Let $A \subseteq \mathbb{R}^d$ and $f : A \mapsto \mathbb{R}^m$

- If $m = 1$ and $a \in \mathbb{R}^d$ is a limit point of A, then we write $\lim_{x \to a} f(x) = \infty$ if for every $R > 0$, there exists $\delta > 0$ such that if $x \in A \setminus \{a\}$ and $||x - a||_2 < \delta$ then
$$f(x) > R$$
- Similarly,If $m = 1$ and $a \in \mathbb{R}^d$ is a limit point of A, then we write $\lim_{x \to a} f(x) = -\infty$ if for every $R > 0$, there exists $\delta > 0$ such that if $x \in A \setminus \{a\}$ and $||x - a||_2 < \delta$ then
$$f(x) < -R$$
- If $d = 1$, A is not bounded above, and $L \in \mathbb{R}^m$, we write $\lim_{x \to \infty} f(x) = L$ if for every $\epsilon > 0$ there exists $R > 0$ such that if $x \in A$ and $x > R$, then
$$||f(x) - L||_2 < \epsilon$$
- Similarly, If $d = 1$, A is not bounded above, and $L \in \mathbb{R}^m$, we write $\lim_{x \to -\infty} f(x) = L$ if for every $\epsilon > 0$ there exists $R > 0$ such that if $x \in A$ and $x < -R$, then
$$||f(x) - L||_2 < \epsilon$$
- If A is not bounded and $L \in \mathbb{R}^m$, we write $\lim_{||x||_2 \to \infty} f(x) = L$ if for every $\epsilon > 0$ there exists $R > 0$ such that if $x \in A$ and $||x||_2 > R$, then
$$||f(x) - L||_2 < \epsilon$$

Differentiation

The Derivative

Let $X \subseteq \mathbb{R}$, $f : X \mapsto \mathbb{R}$ be a function, let $a \in X$ be a non-isolated point. We write
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Proposition: If f is differentiable at a then f is continuous at a.

Computation Rules for Derivatives

Let $X \subseteq \mathbb{R}$, let $f, g : X \mapsto \mathbb{R}$ be functions, let $a \in X$ be a non-isolated point. Suppose that f and g are both differentiable at a, and let $c \in \mathbb{R}$. Then

(i) **Linearity:** $(cf)'(a) = c(f'(a))$ and $(f + g)'(a) = f'(a) + g'(a)$

(ii) **Product:** $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

More on Computing Derivatives

Chain Rule

Let $X, Y \subseteq \mathbb{R}$, let $f : X \mapsto \mathbb{R}$ and $g : Y \mapsto \mathbb{R}$ be functions, let $a \in X$ be a non-isolated point. Suppose that $f(X) \subseteq Y$ and that $f(a)$ is a non-isolated point of Y. Suppose also that f is differentiable at a and g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, and
$$(g \circ f)'(a) = g'(f(a))f'(a)$$

Inverse Rule

Let $X \subseteq \mathbb{R}$ be an interval, let $f : X \mapsto \mathbb{R}$ be a continuous injective function. Let $a \in X$. If f is differentiable at a and $f'(0) \neq 0$ then $f^{-1}(a) \neq 0$ then $f^{-1} : f(X) \mapsto \mathbb{R}$ is differentiable at $f(a)$ and
$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

Optimizing Differentiable Functions

Definition. Let $X \subseteq \mathbb{R}$, let $f : X \mapsto \mathbb{R}$, and let $a \in X$ be an interior point

(i) a is a local minimum of f if there exists $r > 0$ such that $(a - r, a + r) \subseteq X$ and
$$f(a) \leq f(x) \text{ for all } x \in (a - r, a + r)$$

(ii) a is a local minimum of f if there exists $r > 0$ such that $(a - r, a + r) \subseteq X$ and
$$f(a) \geq f(x) \text{ for all } x \in (a - r, a + r)$$

Theorem. Let $X \subseteq \mathbb{R}$, let $f : X \mapsto \mathbb{R}$ and let $a \in X$ be an interior point. If f has a local maximum or local minimum at a and f is differentiable at a, then $f'(a) = 0$

The Mean Value Theorem

Theorem (Rolle's Theorem). Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function that is differentiable on (a, b). If $f(a) = f(b)$ then there exists $x_0 \in (a, b)$ such that
$$f'(x_0) = 0$$

Theorem (Cauchy's Mean Value Theorem). Suppose that $f, g : [a, b] \mapsto \mathbb{R}$ are continuous functions that are differentiable on (a, b). Then there exists $x_0 \in (a, b)$ such that
$$(f(b) - f(a))g'(x_0) = (g(b) - g(a))f'(x_0)$$

Corollary (Mean Value Theorem). Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function that is differentiable on (a, b). There there exists $x_0 \in (a, b)$ such that
$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Darboux Sums

A *partition* of an interval $[a, b]$ is a finite set $\{t_0, t_1, \ldots, t_n\}$ such that

$$a = t_0 < t_1 < \cdots < t_n = b$$

A partition breaks up the interval $[a, b]$ into n subintervals

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n]$$

Let $= \{t_0, t_1, \ldots, t_n\}$ be a partition and let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function. For $i = 1, \ldots, n$, define

$$m_1(P, f) := \inf f([t_{i-1}, t_i]) = \inf\{f(t) : t \in [t_{n-1}, t_1\}$$

$$M_1(P, f) := \sup f([t_{i-1}, t_i]) = \sup\{f(t) : t \in [t_{n-1}, t_1\}$$

Darboux Sum

Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition and let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function. The lower Darboux sum of f for P is

$$L(P, f) := \sum_{i=1}^{n} m_i(P, f)(t_i - t_{i-1})$$

The upper Darboux sum of f for P is

$$U(P, f) := \sum_{n=1}^{n} M_i(P, f)(t_i - t_{i-1}$$

Definition. Let P, P' be partitions. We say that P' refines P if for all $X \in P'$, there exists $Y \in P$ such that $X \subseteq Y$.

Lemma. Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function and let P, P' be partitions of $[a, b]$ such that P' refines P, then

$$L(P, f) \leq L(P', f) \text{ and } U(P', f) \leq U(P, f)$$

To understand this conceptually, consider f restricted to the interval $[a, b]$, then take $c \in [a, b]$

$$\inf f|_{[a,b]}(b - a) = \inf f|_{[a,b]} \cdot (b - c) + \inf f|_{[a,b]}(c - a)$$

you can see this by factoring out $\inf f|_{[a,b]}$ and you will have the left side of the equality, then we have

$$\inf f|_{[a,b]}(b - a) \leq \inf f|_{[c,b]}(b - c) + \inf f|_{[a,c]}(c - a)$$

To understand this, consider the infimum of $[a, b]$. We have that either the infimum either occurs in $f|_[c, b]$, so $\inf f|_{[c,b]} = \inf f|_{[a,c]}$ or it occurs only in $f|_{[a,b]}$, so in this case $\inf f|_{[a,c]} < \inf f|_{[b,c]}$, and thus $\inf f|_{[a,c]} \leq \inf f|_{[b,c]}$. The same argument can be applied to the supremum to get $\sup f|_{[a,c]} \geq \sup f|_{[b,c]}$. So as the number of intervals increases, the lower Darboux sum increases and the upper Darboux sum decreases.

The Riemman Integral

Corollary. Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function and let P, P' be partitions of $[a, b]$. Then

$$L(P, f) \leq U(P', f)$$

Definition. A bounded function $f : [a, b] \mapsto \mathbb{R}$ is (Riemann) integrable if for all partitions P of $[a, b]$

$$\sup\{L(P, f)\} = \inf\{U(P, f)\}$$

Then we set

$$\int_a^b f(t)dt = \sup\{L(P, f)\} = \inf\{U(P, f)\}$$

Proposition. Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function. Then f is integral if and only if for every $\epsilon > 0$, there exists a partition P such that

$$U(P, f) - L(P, f) < \epsilon$$

Theorem. If $f : [a, b] \mapsto \mathbb{R}$ is continuous then f is integrable.

Properties of the Integral

Proposition (Additive Property). Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function and let $c \in (a, b)$. Then f is integrable if and only if $f|_{[a,c]}$ and $f|_{[c,b]}$ are both integrable. In this case,

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

Proposition (Linearity). Let $f, g : [a, b] \mapsto \mathbb{R}$ be bounded integrable functions and let $c \in \mathbb{R}$. Then $cf + g$ is integrable and

$$\int_a^b cf(t) + g(t)dt = c\int_a^b f(t)dt + \int_a^b g(t)dt$$

Proposition. Let $f, g : [a, b] \mapsto \mathbb{R}$ be integrable. If $f(t) \leq g(t)$ for all $t \in [a, b]$ then

$$\int_a^b f(t)dt \leq \int_a^b$$

Corollary. Let $f : [a, b] \mapsto \mathbb{R}$ be integrable. If $m, M \in \mathbb{R}$ are such that

$$m \leq f(t) \leq M$$

for all $x \in [a, b]$ then

$$m(b - a) \leq \int_a^b f(t)dt \leq M(b - a)$$

The Fundamental Theorem of Calculus

Fundemental Theorem of Calculus

Let $f : [a, b] \mapsto \mathbb{R}$ be an integrable function. Define $F : [a, b] \mapsto \mathbb{R}$ by

$$F(x) := \int_a^x f(t)dt$$

For any $x \in [a, b]$, if f is continuous at x then F is differentiable at x and

$$F'(x) = f(x)$$

Theorem. Let $F : [a, b] \mapsto \mathbb{R}$ be a differentiable function, such that $F' : [a, b] \mapsto \mathbb{R}$ is continuous. Then

$$\int_a^b F'(t)dt = F(b) - F(a)$$

Improper Integrals

Definition. Let $f : (a, b] \mapsto \mathbb{R}$ be a function such that, for every $x \in (a, b]$, $f|_{[x,b]}$ is Riemann integrable. Then we define

$$\int_a^b f(t)dt := \lim_{x \to a^+} \int_x^b f(t)dt$$

provided that this limit exists. Likewise, if $f : [a, b) \mapsto \mathbb{R}$ is such that $f|_{[a,b]}$ is integrable for all $x \in [a, b)$, then

$$\int_a^b f(t)dt := \lim_{x \to b^-} \int_a^x f(t)dt$$

Sequences and Series of Functions

Pointwise Limits

Let X be a set, let $(f_n : X \mapsto \mathbb{R}^m)_{n=1}^{\infty}$ be a sequence of functions, and let $f : X \mapsto \mathbb{R}^m$. We say that $(f_n)_{n=1}^{\infty}$ converges pointwise to f if for every $x \in X$,

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Uniform Convergence

Let X be a set, let $(f_n : X \mapsto \mathbb{R}^m)_{n=1}^{\infty}$ be a sequence of functions, and let $f : X \mapsto \mathbb{R}^m$. We say that $(f_n)_{n=1}^{\infty}$ converges *uniformly* to f if for every $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$ and $x \in X$,

$$||f_n(x) - f(x)||_2 < \epsilon$$

Note: The uniform convergence of f_n to f implies that f_n converges pointwise to f.

Series of Functions

Let X be a set, let $(f_n : X \mapsto \mathbb{R}^m)_{n=1}^{\infty}$ be a sequence of functions, and let $f : X \mapsto \mathbb{R}^m$. We define

$$u - \sum_{n=1}^{\infty} f_n \coloneqq u - \lim_{N\to\infty} \sum_{n=1}^{N} f_n$$

provided that this limit exists. When this limit exists, we say that the series $\sum\limits_{n=1}^{\infty}$ converges uniformly.

The Weierstrass M-test

Let X be a set, let $(f_n : X \mapsto \mathbb{R}^m)_{n=1}^{\infty}$ be a sequence of functions, and let $(M_n)_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Suppose that the following hold:

(i) $|f_n(x)| \leq M_n$ for all $x \in X$ and

(ii) $\sum\limits_{n=1}^{\infty} M_n$ converges

Then $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly

Properties of Uniform Convergence

Theorem. Let X be a set, let $(f_n : X \mapsto \mathbb{R}^m)_{n=1}^{\infty}$ be a sequence of functions which uniformly converge to $f : X \mapsto \mathbb{R}^m$. If each f_n is continuous at a, then so is f. Hence if each f_n is continuous on X, then so is f.

Corollary. Let $X \subseteq \mathbb{R}^d$ and suppose $(f_n : X \mapsto \mathbb{R}^m)_{n=1}^{\infty}$ is a sequence of continuous functions. If $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly, then the function $\sum_{n=1}^{\infty} f_n$ is continuous.

Theorem. Let $(f_n : [a,b] \mapsto \mathbb{R})_{n=1}^{\infty}$ be a sequence of continuous functions which converges uniformly to $f : [a,b] \mapsto \mathbb{R}$. Then

$$\int_a^b f(t)dt = \lim_{n\to\infty} \int_a^b f_n(t)dt$$

Corollary. Let $(f_n : [a,b] \mapsto \mathbb{R})_{n=1}^{\infty}$ be a sequence of continuous. If the series $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly, then

$$\sum_{n=1}^{\infty} \int_a^b f_n(t)dt = \int_a^b \sum_{n=1}^{\infty} f_n(t)dt$$

Theorem. Let $(f_n : [a,b] \mapsto \mathbb{R})_{n=1}^{\infty}$ be a sequence of differentiable functions such that f_n' is continuous for each n. Suppose that the sequence $(f_n')_{n=1}^{\infty}$ converges uniformly to some function $g : [a,b] \mapsto \mathbb{R}$ and that $(f_n)_{n=1}^{\infty}$ converges pointwise to f. Then f is differentiable and $f' = g$.

Series of Functions Continued

Properties of Uniform Convergence

Let $(f_n : [a,b] \mapsto \mathbb{R}^m)_{n=1}^{\infty}$ be a sequence of differentiable functions such that f_n' is continuous for each n, and let $f = \sum\limits_{n=1}^{\infty} f_n$. If the series $\sum\limits_{n=1}^{\infty} f_n'$ converges uniformly, then

$$f' = \sum_{n=1}^{\infty} f_n'$$

Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

where $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers $c \in \mathbb{R}$ and x is a variable. The numbers $a_0, a_1, \ldots$ are the coefficents of the power series, and c is called the center of the power series.

Convergence of a Power Series

Let $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ be a power series. The interval of convergence of this power series is the set

$$\left\{ b \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(b-c)^n \text{ converges} \right\}$$

Theorem. Let $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ be a power series and define

$$R \coloneqq \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$$

(interpreted as 0 if lim sup is ∞ and ∞ if lim sup is 0) Then for $b \in \mathbb{R}$

(i) If $|b-c| < R$, then $\sum\limits_{n=0}^{\infty} a_n(b-c)^n$ converges, while

(ii) if $|b-c| < R$, then $\sum\limits_{n=0}^{\infty} a_n(b-c)^n$ diverges

Note: R is called the *radius of convergence*

Proposition. Let $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ be a power series and let R be its radius of convergence. Let $[a,b]$ be any closed bounded interval contained in $(c-R, c+R)$ (which is $\mathbb{R}$ when $R = \infty$). Then the series converges uniformly on $[a,b]$

Continuity, Integration, and Differentiation

Theorem. Let $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ be a power series with interval of convergence I, and define $f : I \mapsto \mathbb{R}$ by

$$f(x) \coloneqq \sum_{n=0}^{\infty} a_n(b-c)^n$$

Then f is continuous on I and for any $a, b \in I$,

$$\int_a^b f(t)dt = \sum_{n=1}^{\infty} \frac{a_n}{n+1} \left((b-c)^{n+1} - (a-c)^{n+1}\right)$$

Theorem. Let $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ be a power series with radius of convergence $R > 0$, and define $f : (c-R, c+R) \mapsto \mathbb{R}$ by

$$f(x) \coloneqq \sum_{n=0}^{\infty} a_n(x-c)^n$$

Then the power series $\sum_{n=0}^{\infty} na_n(x-c)^{n-1}$ also has a radius of convergence R, and for $x \in (c-R, c+R)$,

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$$

Corollary. Let $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ be a power series with radius of convergence $R > 0$, and define $f : (c-R, c+R) \mapsto \mathbb{R}$ by

$$f(x) \coloneqq \sum_{n=0}^{\infty} a_n(x-c)^n$$

Then for each n,

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Taylor Series

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \mapsto \mathbb{R}$ be a function which is infinitely differentiable. Meaning that $f^{(n)}$ exists for all n. For $c \in I$, the *Taylor Series* of f centered at c is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

For $N \in \mathbb{N}_{\geq 0}$ the N^{th} Taylor polynomial of f is

$$P_N(x) \coloneqq \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Lagrange Remainder Theorem

Let $f : (a,b) \mapsto \mathbb{R}$ be a function which $f', f^{(2)}, \ldots, f^{(N+1)}$ all exist on (a,b), let $c \in (a,b)$, and let $P_N(x)$ be the N^{th} Taylor polynomial of f centered at c. Then for $x \in (a,b)$, there exists z between c and x such that

$$f(x) - P_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1}$$