

MAT 2125 Lecture Notes

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Important Proofs for Midterm

Theorem 1.3.13 (The Archimedean Property). *The set $\mathbb{N}_{\geq 1}$ is not bounded above.*

Proof. Suppose for a contradiction that \mathbb{N} was bounded above. Then by completeness, $a = \sup \mathbb{N}$ exists. Since a is a least upper bound, $a - 1$ is not an upper bound, so there exists $m \in \mathbb{N}$ such that

$$m > a - 1$$

Then since $m \in \mathbb{N}$, we have $m + 1 \in \mathbb{N}$, so

$$m + 1 > a$$

But a is an upper bound, thus a contradiction. \square

Proposition 2.2.4 (Uniqueness of Limits). *Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $L_1, L_2 \in \mathbb{R}$. If*

$$\lim_{n \rightarrow \infty} a_n = L_1 \text{ and } \lim_{n \rightarrow \infty} a_n = L_2$$

then

$$L_1 = L_2$$

Proof. Suppose for a contradiction $L_1 \neq L_2$. We can assume without loss of generality that $L_1 < L_2$. Define

$$\epsilon = \frac{L_2 - L_1}{2}$$

Since $\lim_{n \rightarrow \infty} a_n = L$, there exists n_0 such that $\forall n \geq n_0$

$$L_1 - \epsilon < a_n < L_1 + \epsilon$$

Using the second inequality and the definition of ϵ , we get

$$a_n < L_1 + \epsilon = L_1 + \frac{L_2 - L_1}{2} = L_1 + \frac{L_2}{2} - \frac{L_1}{2} = \frac{L_2 + L_1}{2}$$

Likewise, since $\lim_{n \rightarrow \infty} a_n = L_2$, there exists m_0 such that for all $n \geq m_0$,

$$L_2 - \epsilon < a_n < L_2 + \epsilon$$

Then from the first inequality, we get

$$a_n > L_2 - \epsilon = L_2 - \frac{L_2 - L_1}{2} = \frac{L_2 + L_1}{2}$$

So, we get that for all $n \geq \max\{n_0, m_0\}$,

$$a_n > \frac{L_2 + L_1}{2} > a_n$$

Thus, a contradiction. \square

Proposition 2.2.8. *Let $(a_n)_{n=1}^{\infty}$ be a sequence which converges to some number $L \in \mathbb{R}$. Then $(a_n)_{n=1}^{\infty}$ is bounded.*

Proof. Since $\lim_{n \rightarrow \infty} a_n = L$, set $\epsilon := 1$, there exists n_0 such that for all $n \geq n_0$

$$|a_n - L| < 1$$

So we have that $\forall n \geq n_0$

$$L - 1 < a_n < L + 1$$

Now set

$$M := \max\{a_1, a_2, \dots, a_{n_0-1}, L + 1\}$$

If $n < n_0$, then it is amongst the set $\{a_1, \dots, a_{n_0-1}\}$, so M will be the max of this set. Therefore, $\forall n < n_0$, $a_n \leq M$. Then for $n \geq n_0$, by the definition of the limit we know that $a_n < L + 1$, so we get that $a_n < L + 1 \leq M$. Therefore, for all values of n , the set $\{a_n : n \in \mathbb{N}\}$ is bounded above.

Similarly for the lower bound, take

$$M := \min\{a_1, a_2, \dots, a_{n_0-1}, L - 1\}$$

If $n < n_0$, then it is in the set $\{a_1, a_2, \dots, a_{n_0-1}\}$ M' is at most the minimum of this set, so $\forall n < n_0$, $a_n \geq M'$. If $n \geq n_0$, by the definition of the limit we know that for all $n \geq n_0$, $a_n > L - 1$. So M' is at most $L - 1$. Therefore $\forall n \geq n_0$, $a_n > L - 1 \geq M'$. Therefore, the set is bounded below and above, so it is bounded. \square

Proposition 2.3.3. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be converging sequences, if*

$$a_n \leq b_n$$

for all n , then

$$\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$$

Proof. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences with $a_n < b_n$ for all n . Then by the definition of convergence, we have that $\forall \epsilon > 0, \exists n_0$ such that $\forall n \geq n_0$

$$|a_n - L_a| < \epsilon$$

Similarly for b_n , we have that $\exists m_0$ such that $\forall \epsilon > 0$,

$$|b_n - L_b| < \epsilon$$

Now suppose for a contradiction that $L_a > L_b$, then set $\epsilon := \frac{L_a - L_b}{2}$. So we have

$$L_a - \epsilon < a_n < \epsilon + L_a$$

So,

$$a_n > L_a - \epsilon = L_a - \frac{L_a - L_b}{2} = \frac{L_a + L_b}{2}$$

Similarly for b_n , we have

$$L_b - \epsilon < b_n < L_b + \epsilon$$

$$b_n < L_b + \epsilon = \frac{L_a + L_b}{2}$$

So we have $b_n < \frac{L_b + L_a}{2} < a_n$, but $a_n < b_n$. Thus, a contradiction. \square

Theorem 2.3.5 (Squeeze Theorem). *Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ be sequences such that*

(i) $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ converge to the same number L , and

(ii) $a_n \leq b_n \leq c_n$ for all n . Then $(b_n)_{n=1}^{\infty}$ also converges to L .

Proof. Let $\epsilon > 0$ be given. Suppose $a_n \leq b_n \leq c_n$ $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge to L , so $\exists n_a, n_c \in \mathbb{N}$ such that for all $n \geq n_a$

$$L - \epsilon < a_n < L + \epsilon$$

and

$$L - \epsilon < c_n < L + \epsilon$$

So

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

Therefore,

$$L - \epsilon < b_n < L + \epsilon$$

By the definition of convergence, $(b_n)_{n=1}^{\infty}$ converges to L . \square

Theorem 2.6.1 (Cauchy Convergence Criterion). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then it converges if and only if it is Cauchy.*

Proof. (\implies) Assume that $(a_n)_{n=1}^\infty$ converges, then there exists n_0 such that for all $\epsilon > 0$, $\forall n \geq n_0$

$$|a_n - L| < \epsilon$$

Now take $\frac{\epsilon}{2}$ in place of ϵ since ϵ is arbitrary, we have

$$|a_n - L| < \frac{\epsilon}{2}$$

Then, for $m, n \geq n_0$, we have

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L|$$

Since $m, n \geq n_0$, by the definition of convergence we have

$$|a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore,

$$|a_m - a_n| < \epsilon$$

as required. □

Proposition 2.7.3. *For any sequence $(a_n)_{n=1}^\infty$,*

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

Proof. If the sequence isn't bounded, then either $\limsup_{n \rightarrow \infty} a_n = \infty$ or $\liminf_{n \rightarrow \infty} a_n = -\infty$, in either case the result is trivial. So assume that the sequence is bounded. Consider the sets used to define \limsup and \liminf

$$S := \{\beta : \mathbb{R} : \exists n_0 \text{ such that } a_n \leq \beta \quad \forall n \geq n_0\}$$

$$T := \{\alpha : \mathbb{R} : \exists m_0 \text{ such that } a_n \geq \alpha \quad \forall n \geq m_0\}$$

So we have $\alpha \in T$ and $\beta \in S$, then for all $n \geq \max\{n_0, m_0\}$, we have

$$\alpha \leq a_n \leq \beta$$

Thus, we have shown that for every $\alpha \in T$, and every $\beta \in S$, we have $\alpha \leq \beta$. From the definition of \limsup and \liminf , we get that for any eventual lower bound $\alpha \in T$, it is a lower bound for the set of upper bounds S , so

$$\alpha \leq \inf T = \limsup_{n \rightarrow \infty} a_n$$

So then $\limsup_{n \rightarrow \infty} a_n$ is an upper bound for the set of lower bounds T , so

$$\limsup_{n \rightarrow \infty} a_n \geq \sup T = \liminf_{n \rightarrow \infty} a_n$$

Therefore,

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

as required. □

Proposition . *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Proof. Consider the partial sum of the series

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

Now consider the partial sums which correspond to powers of 2, S_{2^N} for $N \in \mathbb{N}$. So we have the sums S_2, S_4, S_8, \dots . Now consider the sequence of partial sums

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$\frac{1}{3} > \frac{1}{4}$, so we have that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

Continuing similarly,

$$S_8 = S_{2^3} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{4}{8} = 1 + \frac{3}{2}$$

\vdots

$$S_{2^N} > 1 + \frac{N}{2}$$

So, we have

$$\lim_{N \rightarrow \infty} \left(1 + \frac{N}{2}\right) = \infty$$

But, $S_{2^N} > 1 + \frac{N}{2}$ for all $N \in \mathbb{N}$, so we have that the partial sums diverge. Therefore, the series diverges. \square

Proposition 3.1.7 (Divergence Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. If the series*

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n \rightarrow \infty} a_n = 0$$

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ converges to L . Set $L := \sum_{n=1}^{\infty} a_n$. Consider the partial sums

$$S_N = \sum_{n=1}^N a_n$$

so $\lim_{n \rightarrow \infty} S_N = L$. We also have that $\lim_{n \rightarrow \infty} S_{N-1} = L$, since

$$\lim_{N \rightarrow \infty} S_{N-1} = \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} a_n = \sum_{n=1}^{\infty-1} a_n = \sum_{n=1}^{\infty} a_n = L$$

Then, we have that

$$S_N - S_{N-1} = \sum_{n=1}^N a_n - \sum_{n=1}^{N-1} a_n = a_N$$

So,

$$\lim_{N \rightarrow \infty} S_N - S_{N-1} = L - L = 0$$

$$\lim_{N \rightarrow \infty} S_N - \lim_{N \rightarrow \infty} S_{N-1} = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n - \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} a_n = \lim_{N \rightarrow \infty} a_N = 0$$

□

Proposition 3.2.1 (Boundedness Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that*

- (i) $a_n \geq 0$ for all n , and
- (ii) *There is a bound $M \in \mathbb{R}$ on the partial sums, so that*

$$\sum_{n=1}^N a_n \leq M$$

for all $N \in \mathbb{N}_{\geq 1}$.

Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $a_n \geq 0$, the partial sums $(S_N)_{N=1}^{\infty}$ satisfy

$$S_N \leq S_{N+1} \text{ for all } N.$$

In other words, $(S_N)_{N=1}^{\infty}$ is an increasing sequence. The second condition ensures that the sequence is bounded above. Therefore, by the Monotone Convergence Criterion, it converges. Therefore, $\sum_{n=1}^{\infty} a_n$ converges. □

Proposition 3.2.2 (Comparison Test). *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers such that*

$$0 \leq a_n \leq b_n \text{ for all } n$$

Then,

(i) *if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$*

(ii) *if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$*

Proof. Since the sequence $\sum_{n=1}^{\infty} b_n$ converges, take $M := \sum_{n=1}^{\infty} b_n$. Then, we have the sequence of partial sums

$$\left(\sum_{n=1}^{\infty} b_n \right)_{n=1}^{\infty}$$

is increasing and converges to M , so M is the supremum of this sequence, therefore

$$\sum_{n=1}^N b_n \leq M$$

for all M . Therefore

$$\sum_{n=1}^N a_n \leq \sum_{n=1}^N b_n \leq M$$

Therefore, by the Boundedness test, $\sum_{n=1}^{\infty} a_n$ converges. (ii) is the contrapositive of (i) so it follows that it holds. \square

Proposition 3.2.3 (Absolute Convergence Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. If the series*

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then so does

$$\sum_{n=1}^{\infty} a_n$$

Proof. Assume $\sum_{n=1}^{\infty} |a_n|$ converges, Write

$$(a_n)_+ = \max\{a_n, 0\}$$

$$(a_n)_- = \max\{-a_n, 0\}$$

So $(a_n)_+$ is all the positive terms from a_n and $(a_n)_-$ is all the negative terms from a_n , but we are negating them so that they are positive, so we have

$$a_n = (a_n)_+ - (a_n)_-$$

Then, we have that

$$0 \leq (a_n)_+ \leq |a_n|$$

So, by the Comparison Test, we have that $|a_n|$ converges so $\sum_{n=1}^{\infty} (a_n)_+$ converges.

Similarly,

$$0 \leq (a_n)_- \leq |a_n|$$

Therefore by the Comparison Test, $\sum_{n=1}^{\infty} (a_n)_-$ converges. So by linearity,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n)_+ - \sum_{n=1}^{\infty} (a_n)_-$$

converges. □

Proposition 4.2.3. Let $(a_n)_{n=1}^{\infty}$ be a sequence in \mathbb{R}^d , with

$$a_n = (a_n^{(1)}, \dots, a_n^{(d)}) \text{ for each } n \in \mathbb{N}$$

and let $L = (L_1, \dots, L_d) \in \mathbb{R}^d$. Then

$$\lim_{n \rightarrow \infty} a_n = L$$

if and only if, for each $i = 1, \dots, d$,

$$\lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

Proof. (\implies) Assume that $\lim_{n \rightarrow \infty} a_n = L$. Then, for each $i = 1, \dots, d$, we have that $|x_i|^2 \leq \sum_{i=1}^d x_i^2 = \|x\|_2^2$, therefore

$$|x_i| \leq \|x\|_2$$

Using this fact, we then have each component of $\|a_n - L\|_2$ is less than or equal to it. So

$$\begin{aligned} |a_n^{(i)} - L_i| &\leq \|a_n - L\|_2 \\ -\|a_n - L\|_2 &\leq a_n^{(i)} - L_i \leq \|a_n - L\|_2 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} a_n = L$, we have $\lim_{n \rightarrow \infty} a_n - L = 0$. By the Squeeze theorem, it follows that

$$\lim_{n \rightarrow \infty} a_n^{(i)} - L_i = 0 \implies \lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

(\impliedby) Suppose for each $i = 1, \dots, d$, we have

$$\lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

Then, from the definition of $\|\cdot\|_2$, we have

$$\|a_n - L\|_2^2 = (a_n^{(1)} - L_1)^2 + \cdots + (a_n^{(d)} - L_d)^2$$

Now taking limits of both sides

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2^2 = \lim_{n \rightarrow \infty} (a_n^{(1)} - L_1)^2 + \cdots + \lim_{n \rightarrow \infty} (a_n^{(d)} - L_d)^2$$

Now we'll prove exercise 2.2.5 which states that if $(a_n)_{n=1}^\infty$ is a sequence of non-negative real number converging to $L \geq 0$, then $\lim_{n \rightarrow \infty} \sqrt{a_n}$ converges to \sqrt{L} . To prove this we will consider two cases where $L = 0$, and $L > 0$.

- **Case 1, $L = 0$:** Suppose $(a_n)_{n=1}^\infty \rightarrow 0$, then from the definition of convergence we have that $\forall \epsilon > 0, \exists n_0$ such that $\forall n \geq n_0$,

$$|a_n - 0| < \epsilon$$

Since ϵ is arbitrary, we'll replace ϵ with ϵ^2 , so

$$|a_n - 0| < \epsilon^2$$

Then we get

$$|a_n - 0| = |a_n| < \epsilon^2 \implies \sqrt{|a_n|} < \epsilon$$

Therefore, $\sqrt{a_n} \rightarrow 0$ by the definition of convergence.

- **Case 2, $L > 0$:** Suppose $(a_n)_{n=1}^\infty \rightarrow L > 0$. Let $\epsilon > 0$ be given, then there exists n_0 such that for all $n \geq n_0$,

$$|a_n - L| < \epsilon$$

We much such that $|\sqrt{a_n} - \sqrt{L}| < \epsilon$

$$|\sqrt{a_n} - \sqrt{L}| \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} = \frac{|(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})|}{\sqrt{a_n} + \sqrt{L}}$$

Since $\sqrt{a_n} + \sqrt{L}$ is positive because $a_n, L \geq 0$, then $\sqrt{a_n} + \sqrt{L} = |\sqrt{a_n} + \sqrt{L}|$, then using the fact that $|a| \cdot |b| = |a \cdot b|$, we get

$$\frac{|a_n - \sqrt{L}\sqrt{a_n} + \sqrt{L}\sqrt{a_n} + L|}{\sqrt{a_n} + \sqrt{L}} = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}}$$

Now if we replace ϵ with $\frac{\epsilon}{\sqrt{L}}$, we get

$$|\sqrt{a_n} - \sqrt{L}| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon}{\sqrt{L}} \implies |\sqrt{a_n} - \sqrt{L}| < \epsilon$$

Therefore, $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{L}$

Now going back to the original proof,

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2^2 = 0$$

So from exercise 2.2.5 we have

$$\lim_{n \rightarrow \infty} \sqrt{\|a_n - L\|_2^2} = \sqrt{0}$$

Therefore,

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2 = 0$$

as required. \square

Theorem 4.2.2 (Cauchy Covergence \mathbb{R}^d). *Let $(a_n)_{n=1}^\infty$ be a sequence \mathbb{R}^d . Then it converges if and only if it is cauchy.*

Proof. Suppose $(a_n)_{n=1}^\infty$ is a sequence in \mathbb{R}^d that converges to $L \in \mathbb{R}^d$. Let $\epsilon > 0$ be given, then there exists n_0 such that $\forall m, n \geq n_0$,

$$\|a_n - L\|_2 < \epsilon$$

$$\|a_m - L\|_2 < \epsilon$$

Since ϵ arbitrary we can replace ϵ with $\frac{\epsilon}{2}$, so

$$\|a_n - L\|_2 = \frac{\epsilon}{2} \text{ and } \|a_m - L\|_2 = \frac{\epsilon}{2}$$

So,

$$\begin{aligned} \|a_m - a_n\|_2 &= \|a_m - L + L - a_n\|_2 \leq \|a_m - L\|_2 + \|L - a_n\|_2 \\ &= \|a_m - L\|_2 + \|a_n - L\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

\square

Proposition 4.3.4. *Given $a \in \mathbb{R}^d$, and $r > 0$, the open ball $B(a, r)$ is an open set.*

Note: This is example 4.3.4 from the professors notes.

Proof. Recall the definition of an open set is that for any x in the set, we can define an open ball (or epsilon neighborhood) around x such that the ball is contained in the set. So we want an open ball $B(x; \epsilon)$ such that $B(x; \epsilon) \subseteq B(a; r)$. To see this, let $x \in B(a; r)$, so that $\|x - a\|_2 < r$. Define

$$\epsilon := r - \|a - x\|_2 > 0$$

Now take some element $y \in B(x; \epsilon)$, then we want to show that tis element is contained in $B(a; r)$. So, $y \in B(x; \epsilon)$, so that $\|y - x\|_2 < \epsilon$. Then,

$$\begin{aligned} \|y - a\|_2 &= \|y - x + x - a\|_2 \leq \|y - x\|_2 + \|x - a\|_2 \\ &< \epsilon + \|a - x\|_2 = r \end{aligned}$$

Therefore,

$$\|y - a\|_2 < r$$

So $y \in B(a; r)$ as required, so $B(a; r)$ is an open set. \square

Proposition 4.3.5. (i) The sets \emptyset, \mathbb{R}^d are open

(ii) For any finite collection of open sets, $U_1, \dots, U_m \subseteq \mathbb{R}^d$, their intersection is

$$U_1 \cap \dots \cap U_m$$

is open

(iii) For any arbitrary collection of open sets $\{U_\alpha : \alpha \in I\}$, their union,

$$\bigcup_{\alpha \in I} U_\alpha$$

is open.

Proof. (i) (i) and (ii) are Exercise 4.3.1

(ii) Will add them later!

(iii) Set

$$U := \bigcup_{\alpha \in I} U_\alpha$$

Since U_α is open, there is some $\epsilon > 0$ such that

$$B(x; \epsilon) \subseteq U_\alpha$$

Then since U is the union of all the U_α , we have that $U_\alpha \subseteq U$ so it follows that

$$B(x; \epsilon) \subseteq U$$

as required. \square

Theorem 4.4.5 (Heine-Borel Theorem). Let K be a subset of \mathbb{R}^d . Then K is compact if and only if K is closed and bounded.

Proof. (\implies) Suppose that K . To see that K is closed, suppose for a contradiction that it is not closed. By proposition 4.3.9, F is closed if and only if for every sequence $(a_n)_{n=1}^\infty$ in F , if $(a_n)_{n=1}^\infty$ converges then

$$\lim_{n \rightarrow \infty} a_n \in F$$

So, if K is not closed, then it follows that there exists some sequence $(a_n)_{n=1}^\infty$ such that

$$\lim_{n \rightarrow \infty} a_n \notin K$$

Then by proposition 2.5.4, if a sequence $(a_n)_{n=1}^\infty$ converges, then all subsequences of the sequence converge to the same point, and hence no subsequence converges to a point in K . This contradicts the fact that K is compact. Similarly for the boundness of K , suppose for a contradiction that K was not bounded. Then for any $n \in \mathbb{N}$, there exists $a_n \in K$ such that $\|a_n\|_2 \geq n$. So the sequence $(a_n)_{n=1}^\infty$ is unbounded, as well as all subsequences $(a_{n_k})_{k=1}^\infty$. Therefore, no subsequence converge since they are all unbounded. This contradicts the fact that K is compact.

(\Leftarrow) Assume K is closed and bounded. Since K is bounded, any sequence $(a_n)_{n=1}^\infty$ in K is bounded. Then by the Bolzano-Weierstrass theorem, $(a_n)_{n=1}^\infty$ is bounded so there exists a convergent subsequence $(a_{n_k})_{k=1}^\infty$ that converges to some $L \in \mathbb{R}^d$. Since K is closed, from proposition 4.3.9 we have that

$$\lim_{k \rightarrow \infty} a_{n_k} \in K$$

Therefore every sequence $(a_n)_{n=1}^\infty$ has a subsequence $(a_{n_k})_{k=1}^\infty$ that converges to some $L \in K$, so K is compact. \square

Important Proofs for Final

Theorem 1.3.13 (The Archimedean Property). *The set $\mathbb{N}_{\geq 1}$ is not bounded above.*

Proof. Suppose for a contradiction that \mathbb{N} was bounded above. Then by completeness, $a = \sup \mathbb{N}$ exists. Since a is a least upper bound, $a - 1$ is not an upper bound, so there exists $m \in \mathbb{N}$ such that

$$m > a - 1$$

Then since $m \in \mathbb{N}$, we have $m + 1 \in \mathbb{N}$, so

$$m + 1 > a$$

But a is an upper bound, thus a contradiction. \square

Proposition 2.2.4 (Uniqueness of Limits). *Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $L_1, L_2 \in \mathbb{R}$. If*

$$\lim_{n \rightarrow \infty} a_n = L_1 \text{ and } \lim_{n \rightarrow \infty} a_n = L_2$$

then

$$L_1 = L_2$$

Proof. Suppose for a contradiction $L_1 \neq L_2$. We can assume without loss of generality that $L_1 < L_2$. Define

$$\epsilon = \frac{L_2 - L_1}{2}$$

Since $\lim_{n \rightarrow \infty} a_n = L_1$, there exists n_0 such that $\forall n \geq n_0$

$$L_1 - \epsilon < a_n < L_1 + \epsilon$$

Using the second inequality and the definition of ϵ , we get

$$a_n < L_1 + \epsilon = L_1 + \frac{L_2 - L_1}{2} = L_1 + \frac{L_2}{2} - \frac{L_1}{2} = \frac{L_2 + L_1}{2}$$

Likewise, since $\lim_{n \rightarrow \infty} a_n = L_2$, there exists m_0 such that for all $n \geq m_0$,

$$L_2 - \epsilon < a_n < L_2 + \epsilon$$

Then from the first inequality, we get

$$a_n > L_2 - \epsilon = L_2 - \frac{L_2 - L_1}{2} = \frac{L_2 + L_1}{2}$$

So, we get that for all $n \geq \max\{n_0, m_0\}$,

$$a_n > \frac{L_2 + L_1}{2} > a_n$$

Thus, a contradiction. \square

Proposition 2.2.8. *Let $(a_n)_{n=1}^{\infty}$ be a sequence which converges to some number $L \in \mathbb{R}$. Then $(a_n)_{n=1}^{\infty}$ is bounded.*

Proof. Since $\lim_{n \rightarrow \infty} a_n = L$, set $\epsilon := 1$, there exists n_0 such that for all $n \geq n_0$

$$|a_n - L| < 1$$

So we have that $\forall n \geq n_0$

$$L - 1 < a_n < L + 1$$

Now set

$$M := \max\{a_1, a_2, \dots, a_{n_0-1}, L + 1\}$$

If $n < n_0$, then it is amongst the set $\{a_1, \dots, a_{n_0-1}\}$, so M will be the max of this set. Therefore, $\forall n < n_0$, $a_n \leq M$. Then for $n \geq n_0$, by the definition of the limit we know that $a_n < L + 1$, so we get that $a_n < L + 1 \leq M$. Therefore, for all values of n , the set $\{a_n : n \in \mathbb{N}\}$ is bounded above.

Similarly for the lower bound, take

$$M := \min\{a_1, a_2, \dots, a_{n_0-1}, L - 1\}$$

If $n < n_0$, then it is in the set $\{a_1, a_2, \dots, a_{n_0-1}\}$ M' is at most the minimum of this set, so $\forall n < n_0$, $a_n \geq M'$. If $n \geq n_0$, by the definition of the limit we know that for all $n \geq n_0$, $a_n > L - 1$. So M' is at most $L - 1$. Therefore $\forall n \geq n_0$, $a_n > L - 1 \geq M'$. Therefore, the set is bounded below and above, so it is bounded. \square

Proposition 3.2.1 (Boundedness Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that*

(i) $a_n \geq 0$ for all n , and

(ii) *There is a bound $M \in \mathbb{R}$ on the partial sums, so that*

$$\sum_{n=1}^N a_n \leq M$$

for all $N \in \mathbb{N}_{\geq 1}$.

Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $a_n \geq 0$, the partial sums $(S_N)_{N=1}^{\infty}$ satisfy

$$S_N \leq S_{N+1} \text{ for all } N.$$

In other words, $(S_N)_{N=1}^{\infty}$ is an increasing sequence. The second condition ensures that the sequence is bounded above. Therefore, by the Monotone Convergence Criterion, it converges. Therefore, $\sum_{n=1}^{\infty} a_n$ converges. \square

Proposition 4.2.3. Let $(a_n)_{n=1}^{\infty}$ be a sequence in \mathbb{R}^d , with

$$a_n = (a_n^{(1)}, \dots, a_n^{(d)}) \text{ for each } n \in \mathbb{N}$$

and let $L = (L_1, \dots, L_d) \in \mathbb{R}^d$. Then

$$\lim_{n \rightarrow \infty} a_n = L$$

if and only if, for each $i = 1, \dots, d$,

$$\lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

Proof. (\implies) Assume that $\lim_{n \rightarrow \infty} a_n = L$. Then, for each $i = 1, \dots, d$, we have that $|x_i|^2 \leq \sum_{i=1}^d x_i^2 = \|x\|_2^2$, therefore

$$|x_i| \leq \|x\|_2$$

Using this fact, we then have each component of $\|a_n - L\|_2$ is less than or equal to it. So

$$\begin{aligned} |a_n^{(i)} - L_i| &\leq \|a_n - L\|_2 \\ -\|a_n - L\|_2 &\leq a_n^{(i)} - L_i \leq \|a_n - L\|_2 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} a_n = L$, we have $\lim_{n \rightarrow \infty} a_n - L = 0$. By the Squeeze theorem, it follows that

$$\lim_{n \rightarrow \infty} a_n^{(i)} - L_i = 0 \implies \lim_{n \rightarrow \infty} a_n^{(i)} = L$$

(\impliedby) Suppose for each $i = 1, \dots, d$, we have

$$\lim_{n \rightarrow \infty} a_n^{(i)} = L_i$$

Then, from the definition of $\|\cdot\|_2$, we have

$$\|a_n - L\|_2^2 = (a_n^{(1)} - L_1)^2 + \dots + (a_n^{(d)} - L_d)^2$$

Now taking limits of both sides

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2^2 = \lim_{n \rightarrow \infty} (a_n^{(1)} - L_1)^2 + \dots + \lim_{n \rightarrow \infty} (a_n^{(d)} - L_d)^2$$

Now we'll prove exercise 2.2.5 which states that if $(a_n)_{n=1}^{\infty}$ is a sequence of non-negative real number converging to $L \geq 0$, then $\lim_{n \rightarrow \infty} \sqrt{a_n}$ converges to \sqrt{L} . To prove this we will consider two cases where $L = 0$, and $L > 0$.

- **Case 1, $L = 0$:** Suppose $(a_n)_{n=1}^{\infty} \rightarrow 0$, then from the definition of convergence we have that $\forall \epsilon > 0, \exists n_0$ such that $\forall n \geq n_0$,

$$|a_n - 0| < \epsilon$$

Since ϵ is arbitrary, we'll replace ϵ with ϵ^2 , so

$$|a_n - 0| < \epsilon^2$$

Then we get

$$|a_n - 0| = |a_n| < \epsilon^2 \implies \sqrt{|a_n|} < \epsilon$$

Therefore, $\sqrt{a_n} \rightarrow 0$ by the definition of convergence.

- **Case 2, $L > 0$:** Suppose $(a_n)_{n=1}^{\infty} \rightarrow L > 0$. Let $\epsilon > 0$ be given, then there exists n_0 such that for all $n \geq n_0$,

$$|a_n - L| < \epsilon$$

We much such that $|\sqrt{a_n} - \sqrt{L}| < \epsilon$

$$|\sqrt{a_n} - \sqrt{L}| \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} = \frac{(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})}{\sqrt{a_n} + \sqrt{L}}$$

Since $\sqrt{a_n} + \sqrt{L}$ is positive because $a_n, L \geq 0$, then $\sqrt{a_n} + \sqrt{L} = |\sqrt{a_n} + \sqrt{L}|$, then using the fact that $|a| \cdot |b| = |a \cdot b|$, we get

$$\frac{|a_n - \sqrt{L}\sqrt{a_n} + \sqrt{L}\sqrt{a_n} + L|}{\sqrt{a_n} + \sqrt{L}} = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}}$$

Now if we replace ϵ with $\frac{\epsilon}{\sqrt{L}}$, we get

$$|\sqrt{a_n} - \sqrt{L}| < \frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon}{\sqrt{L}} \implies |\sqrt{a_n} - \sqrt{L}| < \epsilon$$

Therefore, $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{L}$

Now going back to the original proof,

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2^2 = 0$$

So from exercise 2.2.5 we have

$$\lim_{n \rightarrow \infty} \sqrt{\|a_n - L\|_2^2} = \sqrt{0}$$

Therefore,

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2 = 0$$

as required. □

Proposition 4.3.4. *Given $a \in \mathbb{R}^d$, and $r > 0$, the open ball $B(a, r)$ is an open set.*

Note: This is example 4.3.4 from the professors notes.

Proof. Recall the definition of an open set is that for any x in the set, we can define an open ball (or epsilon neighborhood) around x such that the ball is contained in the set. So we want an open ball $B(x; \epsilon)$ such that $B(x; \epsilon) \subseteq B(a; r)$. To see this, let $x \in B(a; r)$, so that $\|x - a\|_2 < r$. Define

$$\epsilon := r - \|x - a\|_2 > 0$$

Now take some element $y \in B(x; \epsilon)$, then we want to show that this element is contained in $B(a; r)$. So, $y \in B(x; \epsilon)$, so that $\|y - x\|_2 < \epsilon$. Then,

$$\begin{aligned} \|y - a\|_2 &= \|y - x + x - a\|_2 \leq \|y - x\|_2 + \|x - a\|_2 \\ &< \epsilon + \|x - a\|_2 = r \end{aligned}$$

Therefore,

$$\|y - a\|_2 < r$$

So $y \in B(a; r)$ as required, so $B(a; r)$ is an open set. \square

Proposition 4.3.9. *Let $F \subseteq \mathbb{R}^d$. Then F is closed if and only if for every sequence $(a_n)_{n=1}^\infty$ in F , if $(a_n)_{n=1}^\infty$ converges then*

$$\lim_{n \rightarrow \infty} a_n \in F$$

Proof. (\implies) Assume that F is closed, so $\mathbb{R}^d \setminus F$ is open. Let $(a_n)_{n=1}^\infty$ be a sequence in F which converges. Suppose for a contradiction that $L := \lim_{n \rightarrow \infty} a_n$ is not in F . So $L \in \mathbb{R}^d \setminus F$. Then by openness, there exists $\epsilon > 0$ such that $B(L; \epsilon) \subseteq \mathbb{R}^d \setminus F$. Using the definition of convergence, we have that there exists n such that

$$\|a_n - L\|_2 < \epsilon$$

But, this means that

$$a_n \in B(L; \epsilon) \subseteq \mathbb{R}^d \setminus F$$

which contradicts that $a_n \in F$. Therefore, $L \in F$ as required.

(\impliedby) Assume that every sequence in F that converges, the limit is in F . We want to show that $\mathbb{R}^d \setminus F$ is open. Take a point $x \in \mathbb{R}^d \setminus F$, and suppose for a contradiction that there is no $\epsilon > 0$ such that $B(x; \epsilon) \subseteq \mathbb{R}^d \setminus F$. We can take $\epsilon := \frac{1}{n}$. So for each $n \in \mathbb{N}_{\geq 1}$, $B(x; \frac{1}{n})$ is not contained in $\mathbb{R}^d \setminus F$. So, there must be a point in $B(x; \frac{1}{n})$ that is in F . Let a_n be such a point. Then since it is in $B(x; \frac{1}{n})$, we have that

$$\|a_n - x\|_2 < \frac{1}{n}$$

$x \in \mathbb{R}^d \setminus F$ so $x \neq a_n$, so

$$0 < \|a_n - x\|_2 < \frac{1}{n}$$

So by the squeeze theorem,

$$\lim_{n \rightarrow \infty} \|a_n - x\|_2 = 0$$

So,

$$\lim_{n \rightarrow \infty} a_n = x$$

By our hypothesis, every sequence which converges, the limit is in F . So $x \in F$, but by our assumption $x \in \mathbb{R}^d \setminus F$, which is a contradiction. Therefore, $\mathbb{R}^d \setminus F$ is open as required. \square

Proposition 5.2.5. *Let $X \subseteq \mathbb{R}^d$ and let $Y \subseteq \mathbb{R}^m$. Let $f : X \mapsto Y$ and $g : Y \mapsto \mathbb{R}^n$ be functions. Let $a \in X$. Suppose that f is continuous at a and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a .*

Proof. Let $\epsilon > 0$ be given. Since g is continuous at $f(a)$, there exists $\eta > 0$ such that for all $y \in Y$, if $\|y - f(a)\|_2 < \eta$, then $\|g(y) - g(f(a))\|_2 < \epsilon$ (by the definition of continuity). Since f is continuous at a , we can find δ such that for all $x \in X$, if $\|x - a\|_2 < \delta$, then $\|f(x) - f(a)\|_2 < \eta$. Now since these inequalities hold for all $x \in X$ and $y \in Y$. We can take $y = f(x)$. So if $x \in X$ and $\|x - a\|_2 < \delta$, then $\|f(x) - f(a)\|_2 < \eta$, and so

$$\|g(f(x)) - g(f(a))\|_2 < \epsilon$$

as required. \square

Theorem 5.3.2. *Let $K \subseteq \mathbb{R}^d$ be compact and let $f : K \mapsto \mathbb{R}^m$ be a continuous function. Then its image, $f(K)$ is also compact.*

Proof. Let $(y_n)_{n=1}^\infty$ be a sequence in $f(K)$, we need to find a subsequence that converges to a point in $f(K)$. Recall that $f(K) = \{f(k) : k \in K\}$. So, we can find $x_n \in K$ for each n such that $y_n = f(x_n)$. Since K is compact, there is a subsequence $(x_{n_k})_{n=1}^\infty$ which converges to a point $a \in K$. f is continuous so

$$\lim_{x \rightarrow a} f(x) = f(a)$$

therefore by the Sequential Characterization of Limits, $\lim_{k \rightarrow \infty} f(x_{n_k}) = f(a)$. So, the subsequence $(y_{n_k})_{n=1}^\infty$ converges to $f(a) \in f(K)$ as required. \square

Corollary 5.3.3 (Extreme Value Theorem). *Let $K \subset \mathbb{R}^d$ be compact and nonempty, and let $f : K \mapsto \mathbb{R}$ be a continuous function. Then there exists $x_{\min}, x_{\max} \in K$ such that for all $x \in K$,*

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

In otherwords, the image of f is bounded above and below, and it attains its bounds.

Proof. By the previous theorem, $f(K)$ is compact. By the Heine-Borel theorem, $f(K)$ is closed and bounded. Since it is bounded, by completeness we have $\sup f(K)$ and $\inf f(K)$ exist. We can construct a sequence $(a_n)_{n=1}^{\infty}$ which converges to $\sup f(K)$. Let $L := \sup f(K)$. For any given $\epsilon > 0$, $L - \epsilon < L$ so it is not an upperbound. Therefore we have some element $a \in f(K)$ such that $L - \epsilon < a \leq L$. Set $\epsilon := \frac{1}{n}$ for $n \in \mathbb{N}_{\geq 1}$. So we can define the sequence members of $(a_n)_{n=1}^{\infty}$ as

$$L - 1 < a_1 \leq L$$

$$L - \frac{1}{2} < a_2 \leq L$$

$$L - \frac{1}{3} < a_3 \leq L$$

$$L - \frac{1}{n} < a_n \leq L$$

This holds for all $n \in \mathbb{N}_{\geq 1}$ since if $L - \frac{1}{n}$ is an upper bound, then it contradicts that $L = \sup f(K)$. Then by the Squeeze theorem, we have that this sequence converges to L . So since $f(K)$ is closed, $L \in f(K)$. Similarly, we can construct a sequence that converges to $\inf f(K)$. Therefore, by boundedness we have

$$\inf f(K) \leq f(K) \leq \sup f(K)$$

Since $f(K)$ is closed, there exists $x_{\min} \in K$ such that $f(x_{\min}) = \inf f(K)$. Similarly, there exists $x_{\max} \in K$ such that $f(x_{\max}) = \sup f(K)$.

Note: It follows that $\sup f(K) \in f(K)$ because $f(K)$ is closed, but I wanted to prove it in an intuitive way to be thorough.

□

Theorem 5.4.4. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \mapsto \mathbb{R}$ be an injective continuous function. Then $f^{-1} : f(I) \mapsto \mathbb{R}$ is continuous.

Proof. From Lemma 5.4.3, since f is injective then f is either strictly increasing or strictly decreasing. Assume without loss of generality that f is strictly increasing. Let $b \in f(I)$, we want to show that $\lim_{y \rightarrow b} f^{-1}(y) = f^{-1}(b)$. Let $\epsilon > 0$

be given. Set $a := f^{-1}(b)$, so $f(a) = b$. Consider 2 cases where $a \in I^{\circ}$ and $a \in \partial I$. When $a \in \partial I$ the argument follows very similarly so we will suppose that $a \in I^{\circ}$. Since ϵ is arbitrary, we can assume $a - \epsilon, a + \epsilon \in I$. Then we have

$$f(a - \epsilon) < f(a) < f(a + \epsilon)$$

Then set

$$\delta := \min\{f(a) - f(a - \epsilon), f(a + \epsilon) - f(a)\} > 0$$

We want to prove that if $y \in f(I)$ and $|y - b| < \delta$, then $|f^{-1}(y) - f^{-1}(b)| < \epsilon$. Suppose for a contradiction that this is false, then

$$f^{-1}(y) \leq f^{-1}(b) \text{ or } f^{-1}(y) \geq f^{-1}(b) + \epsilon$$

Set $x := f^{-1}(y)$ so $f(x) = y$. So we have

$$x \leq a - \epsilon \text{ or } x \geq a + \epsilon$$

In the first case since f is strictly increasing we get

$$y = f(x) \leq f(a - \epsilon) \leq f(a) - \delta = b - \delta$$

So $y = b - \delta$. In the second case we get

$$y = f(x) \geq f(a + \epsilon) \geq f(a) + \delta = b + \delta$$

In both cases, this contradicts that $|y - b| < \delta$ so we must have that our claim was true. \square

Proposition 6.1.5. *Let $X \subseteq \mathbb{R}$, let $f : X \mapsto \mathbb{R}$ be a function, let $a \in X$ be a non-isolated point. If f is differentiable at a then f is continuous at a .*

Proof. We want to show that $\lim_{x \rightarrow a} f(x) = f(a)$. We have

$$\begin{aligned} \lim_{x \rightarrow a} f(x) - f(a) &= \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} (x - a) \right) \\ &= \left(\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \right) \left(\lim_{x \rightarrow a} x - a \right) \\ &= f'(a) \cdot 0 = 0 \implies \lim_{x \rightarrow a} f(x) = f(a) \end{aligned}$$

We can use algebra of limits since both limits are well defined. \square

Theorem 6.3.2. *Let $X \subseteq \mathbb{R}$, let $f : X \mapsto \mathbb{R}$ and let $a \in X$ be an interior point. If f has a local maximum or local minimum at a and f is differentiable at a , then $f'(a) = 0$.*

Proof. Assume that f is a local minimum at a . Let $r > 0$ be such that $(a - r, a + r) \subseteq X$ and

$$f(a) \leq f(x) \quad \forall x \in (a - r, a + r)$$

In other words, $f(a)$ is the local minimum in the r neighborhood of a . Since f is differentiable, the limit

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists. By the Sequential Characterization of Limits, for any sequence $(x_n)_{n=1}^{\infty}$ in X converging to a , we have

$$f'(a) = \lim_{n \rightarrow \infty} \frac{f(x_n) - f(a)}{x_n - a}$$

We want to construct two sequences that converge to a from the left and right, so from the right consider the sequence with members $x_n \in (a, a + r)$

$$x_n := a + \frac{r}{n + 1}$$

Then $x_n - a > 0$ and $f(x_n) - f(a) \leq 0$ since $f(a) \leq f(x) \forall x \in (a - r, a + r)$. So this implies that

$$f'(a) = \lim_{n \rightarrow \infty} \frac{f(x_n) - f(a)}{x_n - a} \geq 0$$

Similarly, we can construct a sequence from the left with members $x_n \in (a - r, a)$, take

$$x_n := a - \frac{r}{n+1}$$

So $x_n - a < 0$, thus

$$f'(a) = \lim_{n \rightarrow \infty} \frac{f(x_n) - f(a)}{x_n - a} \leq 0$$

Then by combining these inequalities, we get that $f'(a) = 0$, as required. \square

Lecture 1

The Real Numbers \mathbb{R}

Summary: \mathbb{R} is a complete ordered field.

1.1 Fields

Definition 1.1.1. A field is a set F together with operations $+$, \cdot satisfying

- (F1) $a + b = b + a \ \forall a, b \in F$ (Commutativity)
- (F2) $(a + b) + c = a + (b + c) \ \forall a, b, c \in F$ (Associativity)
- (F3) $\exists 0 \in F$ s.t. $0 + a = a \ \forall a \in F$ (Additive Identity)
- (F4) $\exists -a \in F$ s.t. $a + (-a) = 0 \ \forall a \in F$ (Additive Inverse)
- (F5) $a \cdot b = b \cdot a \ \forall a, b \in F$ (Commutativity)
- (F6) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \ \forall a, b, c \in F$ (Associativity)
- (F7) $\exists 1 \in F$ s.t. $1 \cdot a = a \ \forall a \in F$ (Multiplicative Identity)
- (F8) $\forall a \in F \setminus \{0\} \ \exists a^{-1} \in F$ s.t. $a^{-1} \cdot a = 1$ (Multiplicative Inverse)
- (F9) $a \cdot (b + c) = a \cdot b + a \cdot c \ \forall a, b, c \in F$ (Distributivity)

1.2 Ordered Fields

Definition 1.2.1. An ordered field is a field F along with a relation $<$ satisfying

- (O1) $\forall a, b, c \in F$, if $a < b$ and $b < c$ then $a < c$ (Transitivity)
- (O2) $\forall a, b \in F$ exactly one of the following is true,

$$a < b \text{ or } a = b \text{ or } b < a$$

- (O3) $\forall a, b, c \in F$, if $a < b$, then $a + c < b + c$
- $\forall a, b, c \in F$, If $a < b$ and $0 < c$, then $ac < bc$

1.3 Complete Ordered Fields

Definition 1.3.1. Let F be an ordered field. Let $S \subseteq F$. An upper bound of S is some $M \in F$ s.t $\forall x \in S$

$$x \leq M$$

Lecture 2

Completeness of \mathbb{R} , Absolute Value, Sequences

TBC.

Lecture 3

Convergence of Sequences

TBC.

Lecture 4

Properties of Convergence, Squeeze Theorem, Monotone Sequences

TBC.

Lecture 5

Subsequences, Cauchy Sequences

TBC.

Lecture 6

Limsup and Liminf

TBC.

Lecture 7

Series

Recall:

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n$$

$\sum_{n=1}^{\infty} a_n$ "diverges" if above limit does not exist.

Proposition 7.0.1. Suppose $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ converges. Then

(i)

$$\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(ii)

$$\sum_{n=1}^{\infty} cb_n = c \sum_{n=1}^{\infty} b_n \quad \forall c \in \mathbb{R}$$

This says

$$V := \{(a_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} a_n \text{ converges}\}$$

is a vector space over \mathbb{R} .

Note:

$$\left(\sum_{n=1}^N a_n \right) \left(\sum_{n=1}^N b_n \right) \neq \sum_{n=1}^{\infty} a_n b_n$$

Proof. **Exercise.**

□

Proposition 7.0.2. $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=k}^{\infty} a_k$ converges.

Proof. **Exercise.**

□

Example: TBC.

Proposition 7.0.3. *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$.*

7.1 Divergence Test

Proposition 7.1.1 (Divergence Test). *If $a_n \not\rightarrow 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof. $\sum_{n=1}^{\infty} a_n$ converges $\implies S_n \rightarrow L$ for some L , where

$$S_n := \sum_{n=1}^N a_n$$

So,

$$a_n = S_n - S_{n-1}$$

By the Algebra of Limits,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} \\ &= L - L = 0 \end{aligned}$$

□

Example: TBC.

7.2 Convergence Tests

Proposition 7.2.1 (Boundedness Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence, if*

(i) $a_n \geq 0$

(ii) *There is an upper bound on the partial sums*

$$\exists M > 0 \text{ s.t. } \sum_{n=1}^N a_n \leq M$$

Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Let

$$S_N := \sum_{n=1}^N a_n$$

Then

$$\begin{aligned} S_{N+1} &= S_N + a_{N+1} \\ &\geq S_n \end{aligned}$$

So by the Monotone Convergence Criterion, $(S_N)_{N=1}^{\infty}$ converges \iff it is bounded above. By (ii), it is bounded. \square

Proposition 7.2.2 (Comparison Test). *TBC.*

Lecture 8

Ratio, Root, Alternating Series, and Integral Test, Cauchy Convergence, Topology of \mathbb{R}^d

8.1 Ratio Test

Proposition 8.1.1 (Ratio Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of nonzero elements.*

(i) *If*

$$\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) *If*

$$\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

(i) Let

$$q = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

where $q < 1$, $\exists r \in (q, 1)$. By the definition of \limsup ,

$$\left| \frac{a_{n+1}}{a_n} \right| \leq r \quad \forall n \geq n_0$$

for some $n_0 \in \mathbb{N}$.

$$\begin{aligned} \left| \frac{a_{n_0+1}}{a_{n_0}} \right| &\leq r \\ |a_{n_0+1}| &\leq r|a_{n_0}| \\ |a_{n_0+2}| &\leq r|a_{n_0+1}| \leq r^2|a_{n_0}| \end{aligned}$$

By induction, we have

$$0 \leq |a_{n_0+k}| \leq r^k |a_{n_0}|$$

By the comparison test,

$$\sum_{k=1}^{\infty} |a_{n_0+k}|$$

converges since

$$\sum_{k=1}^{\infty} r^k |a_{n_0}|$$

is a geometric sequence and $0 \leq r < 1$.

$$\sum_{n=1}^{\infty} |a_n|$$

Converges, thus $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) Let

$$q = \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

so $\exists n_0$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| > 1$$

for all $n \geq n_0$. Then

$$|a_{n_0+k}| \geq |a_{n_0}| \quad \forall k \geq 0$$

So $a_n \not\rightarrow 0$ as $n \rightarrow \infty$, thus by the divergence test. $\sum_{n=1}^{\infty} a_n$ diverges.

□

Note: The ratio test does not tell us anything when the limit is 1.

For example:

$$\sum_{n \geq 1} \frac{1}{n}$$

diverges, but

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \rightarrow 1$$

But on the other hand,

$$\sum_{n \geq 1} \frac{1}{n+1}$$

converges, and

$$\frac{\frac{1}{n+1(n+2)}}{\frac{1}{n(n+1)}} = \frac{n+1}{n+2} \rightarrow 1$$

8.2 Root Test

Proposition 8.2.1 (Root Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.*

(i) *If*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) *If*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

(i) Let

$$q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$$

Then there exists $r \in (q, 1)$ such that $\exists n_0$

$$\sqrt[n]{|a_n|} \leq r$$

for all $n \geq n_0$. Then,

$$0 \leq |a_n| \leq r^n$$

Therefore $\sum_{n=n_0}^{\infty} r^n$ converges since $0 < r < 1$. Then by the comparison test, $\sum_{n=n_0}^{\infty} |a_n|$ converges.

(ii) Let

$$q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$$

From exercise 2.7.3, there are infinitely many

$$\sqrt[n]{|a_n|} \geq 1 \implies |a_n| \geq 1$$

Thus $a_n \not\rightarrow 0$ as $n \rightarrow \infty$, thus by the divergence test, $\sum_{n=1}^{\infty} a_n$ diverges.

□

Examples:

•

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

$$\sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

•

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$$

$$\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \frac{1}{2n+1} \rightarrow \frac{1}{2}$$

•

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\sqrt[n]{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} = \frac{1}{e^{\frac{\ln n}{n}}} \rightarrow 1$$

8.3 Alternating Series Test

Proposition 8.3.1 (Alternating Series Test). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose*

(i) $(a_n)_{n=1}^{\infty}$ *is decreasing*

(ii) $\lim_{n \rightarrow \infty} a_n = 0$

Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges. Moreover, for any N ,

$$\sum_{n=1}^{2N} (-1)^{n+1} a_n \leq \sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq \sum_{n=1}^{2N-1} (-1)^{n+1} a_n$$

Proof. $a_n \geq 0 \forall n$, since $a_n \rightarrow 0$ as $n \rightarrow \infty$, and $(a_n)_{n=1}^\infty$ is decreasing. Let

$$S_N := \sum_{n=1}^N (-1)^{n+1} a_n$$

If N is even,

$$S_{N+2} = S_N + a_{N+1} - a_{N+2} \geq S_N$$

Since

$$a_{N+2} \leq a_{N+1}$$

So $(S_{2N})_{N=1}^\infty$ is an increasing sequence and $(S_{2N-1})_{N=1}^\infty$ is a decreasing sequence. So by the monotone convergence criterion, both sequence converge.

$$S_{2N-1} + a_{2N} = S_{2N}$$

$$a_{2N} \xrightarrow{N \rightarrow \infty} 0$$

So,

$$\lim_{n \rightarrow \infty} S_{2N-1} = \lim_{N \rightarrow \infty} S_{2N} = L \implies \lim_{N \rightarrow \infty} S_N = L$$

$(S_{2N})_{N=1}^\infty$ is increasing, so

$$L = \sup\{(S_{2N})_{N=1}^\infty\} \implies S_{2N} \leq L$$

Similarly, $(S_{2N-1})_{N=1}^\infty$ is decreasing, so

$$L = \inf\{(S_{2N-1})_{N=1}^\infty\} \implies S_{2N-1} \geq L$$

□

8.4 Integral Test

Proposition 8.4.1. *Let $f : [1, \infty) \rightarrow \mathbb{R}$. Suppose that*

(i) $f(x) \geq 0 \forall x \in [1, \infty)$

(ii) f is decreasing

Then

$\sum_{n=1}^\infty f(n)$ converges \iff the improper integral $\int_1^\infty f(x) dx$ converges.

8.5 Cauchy Convergence Criterion for Series

Proposition 8.5.1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \forall \epsilon > 0 \exists N_0 \text{ such that } N \leq M \leq N_0, |\sum_{n=M}^N a_n| < \epsilon.$$

Proof. Let

$$S_N := \sum_{n=1}^N a_n$$

Then $\sum_{n=1}^{\infty} a_n$ converges $\iff (S_N)_{N=1}^{\infty}$ converges. Cauchy convergence criterion for sequences says that

$$\begin{aligned} (S_N)_{N=1}^{\infty} \text{ converges} &\iff \text{it is cauchy.} \\ &\iff \forall \epsilon > 0 \exists N_0 \text{ s, t } |S_N - S_M| < \epsilon \forall N, M \geq N_0 \end{aligned}$$

□

8.6 Topology of \mathbb{R}^d

8.6.1 Norms

Definition 8.6.1. *A norm on \mathbb{R}^d is a function $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ satisfying the following properties:*

- (i) $\|a\| = 0 \iff a = (0, \dots, 0)$
- (ii) $\|ca\| = |c| \cdot \|a\| \forall c \in \mathbb{R}, a \in \mathbb{R}^d$
- (iii) $\|a + b\| \leq \|a\| + \|b\| \forall a, b \in \mathbb{R}^d$

The euclidean norm of \mathbb{R}^d is given by

$$\|a, \dots, a_d\|_2 = \sqrt{\sum_{i=1}^d a_i^2}$$

The dot product on \mathbb{R}^d is given by

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = \sum_{i=1}^d a_i b_i$$

We also have the l_1 -norm

$$\|(a_1, \dots, a_n)\|_1 = \sum_{i=1}^d |a_i|$$

And the l_{∞} -norm

$$\|(a_1, \dots, a_d)\|_{\infty} = \max\{|a_1|, \dots, |a_d|\}$$

Proposition 8.6.1. Let $a, b \in \mathbb{R}^d$, write $\|\cdot\|$ for the euclidean norm on \mathbb{R}^d . Then

(i) **Cauchy Schwarz Inequality:**

$$|a \cdot b| \leq \|a\| \cdot \|b\|$$

(ii) **Triangle Inequality:**

$$\|a + b\| \leq \|a\| + \|b\|$$

(iii) $\|\cdot\|$ is a norm on \mathbb{R}^d

Proof.

(i) Consider the quadratic function

$$\begin{aligned} P(t) &= \|a + tb\|^2 \\ &= (a + tb) \cdot (a + tb) \\ &= a \cdot a + 2a \cdot tb + tb \cdot tb \\ &= \|a\|^2 + 2t(a \cdot b) + t^2\|b\|^2 \end{aligned}$$

The discriminant of $P(t)$ is less than or equal to 0,

$$\begin{aligned} (2a \cdot b)^2 - 4\|a\|^2\|b\|^2 &\leq 0 \\ (a \cdot b)^2 &\leq \|a\|^2\|b\|^2 \\ |a \cdot b| &\leq \|a\| \cdot \|b\| \end{aligned}$$

(ii)

$$\begin{aligned} \|a + b\|^2 &= \|a\|^2 + 2a \cdot b + \|b\|^2 \\ &\leq \|a\|^2 + 2\|a\| \cdot \|b\| + \|b\|^2 \\ &\leq \|a\|^2 + 2\|a\| \cdot \|b\| + \|b\|^2 \\ &= (\|a\| + \|b\|)^2 \end{aligned}$$

$$\|a + b\|^2 \leq (\|a\| + \|b\|)^2 \implies \|a + b\| \leq \|a\| + \|b\|$$

(iii) **Exercise.** We want to prove $\|a\| = 0 \iff a = 0$ and $\|ca\| = |c| \cdot \|a\|$.

□

Lecture 9

\mathbb{R}^d

Recall: $\|(x_1, \dots, x_d)\| := \sqrt{x_1^2 + \dots + x_d^2}$. This is a norm. i.e.

$$\|a + b\| \leq \|a\|_2 + \|b\|_2 \quad \forall a, b \in \mathbb{R}^d$$

$$\|ca\| = |c| \cdot \|a\|_2, \quad c \in \mathbb{R}, \quad a \in \mathbb{R}^d$$

$$\|a\|_2 > 0, \quad \forall a \in \mathbb{R}^d \setminus \{(0, \dots, 0)\}$$

$$\|(0, \dots, 0)\|_2 = 0$$

Other examples of norms:

- $\|(x_1, \dots, x_d)\|_1 := |x_1| + \dots + |x_d|$
- $\|(x_1, \dots, x_d)\|_\infty := \max\{|x_1|, \dots, |x_d|\}$

Exercise: For $a \in \mathbb{R}^d$,

$$\|a\|_\infty \leq \|a\|_2 \leq \|a\|_1 \leq d\|a\|_\infty (\leq d\|a\|_2)$$

Interesting Fact: There are other norms. but they are all equivalent in the sense that if $\|\cdot\|, \|\cdot\|'$ are norms on \mathbb{R}^d , then $\exists v, R > 0$ such that

$$r\|a\| \leq \|a\|' \leq R\|a\|$$

9.1 Convergence

Definition 9.1.1. Let $(a_n)_{n=1}^\infty$ be a sequence in \mathbb{R}^d and let $L \in \mathbb{R}^d$, we say $(a_n)_{n=1}^\infty$ **converges** to L , and write $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow \infty$, if

$$\lim_{n \rightarrow \infty} \|a_n - L\|_2 = 0$$

Note: We could define convergence instead using some other norm, say $\|\cdot\|_1$.

If $\|a_n - L\|_2 \rightarrow 0$, then $\|a_n - L\|_1 \leq d\|a_n - L\|_2 \rightarrow 0$. If $\|a_n - L\|_1 \rightarrow 0$, then $\|a_n - L\|_2 \leq d\|a_n - L\|_1 \rightarrow 0$.

in general, if $\|\cdot\|$ is any norm, then since $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent.

$$\|a_n - L\|_2 \rightarrow 0 \iff \|a_n - L\| \rightarrow 0$$

Example: Say $a_n = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then

$$\|a_n - L\|_2 = \sqrt{1/n^2 + \dots + 1/n^2} = \sqrt{\frac{d}{n^2}} = \frac{\sqrt{d}}{n} \rightarrow 0$$

$$\therefore a_n \rightarrow L$$

Given a sequence $(a_n)_{n=1}^\infty$ in \mathbb{R}^d , we write $a_1 = (a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)})$ where $a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)} \in \mathbb{R}$. Similarly,

$$a_1 = (a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(d)})$$

$$a_2 = (a_2^{(1)}, a_2^{(2)}, \dots, a_2^{(d)})$$

$$a_3 = (a_3^{(1)}, a_3^{(2)}, \dots, a_3^{(d)})$$

$$\vdots$$

$$L = (L^{(1)}, L^{(2)}, \dots, L^{(d)}) \in \mathbb{R}^d$$

We get d sequences in \mathbb{R} , and d possible limit points $L^{(1)}, \dots, L^{(d)} \in \mathbb{R}$

Proposition 9.1.1. *Given $(a_n)_{n=1}^\infty$ and L as above, $a_n \rightarrow L$ as $n \rightarrow \infty \iff a_n^{(i)} \rightarrow L^{(i)}$ in \mathbb{R} as $n \rightarrow \infty$, for $i = 1, \dots, d$.*

Proof. \implies : Suppose $a_n \rightarrow L$, i.e.

$$\|a_n - L\|_2 \rightarrow 0$$

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$\|x\|_2^2 = x_1^2 + \dots + x_d^2 \geq x_i^2 = |x_i|^2 \therefore |x_i| \leq \|x\|_2$$

Applying this to (*), we get

$$0 \leq |a_n^{(i)} - L^{(i)}| \leq \|a_n - L\|_2 \rightarrow 0$$

So by the squeeze theorem,

$$|a_n^{(i)} - L^{(i)}| \rightarrow 0$$

\implies : Suppose $a_n^{(i)} \rightarrow L^{(i)}$ for $i = 1, \dots, d$.

$$\|a_n - L\|_2^2 = (a_n^{(1)} - L^{(1)})^2 + \dots + (a_n^{(d)} - L^{(d)})^2 \rightarrow 0$$

By algebra of limits,

$$\therefore \|a_n - L\|_2 \rightarrow 0$$

□

Example: $a^n = ((-1)^n, \frac{1}{n}) \in \mathbb{R}^2$. Does (a_n) converge? No, since $(-1)^n$ does not converge.

Definition 9.1.2. A sequence $(a_n)_{n=1}^\infty$ in \mathbb{R}^d is **Cauchy** if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}_{\geq 1}$ such that

$$\|a_n - a_m\|_2 < \epsilon \quad \forall m, n \geq n_0$$

Theorem 9.1.1 (Cauchy Convergence Criterion for \mathbb{R}^d). Let $(a_n)_{n=1}^\infty$ be a sequence in \mathbb{R}^d . It converges \iff it is Cauchy.

Proof. \implies : Suppose $a_n \rightarrow L \in \mathbb{R}^d$. To show it is Cauchy, let $\epsilon > 0$. $\|a_n - L\|_2 \rightarrow 0$, so $\exists n_0 \in \mathbb{N}_{\geq 1}$ such that

$$\|a_n - L\|_2 < \frac{\epsilon}{2} \quad \forall n \geq n_0$$

Then if $m, n \geq n_0$,

$$\begin{aligned} \|a_m - a_n\|_2 &= \|a_m - L + L - a_n\|_2 \\ &\leq \|a_m - L\|_2 + \|L - a_n\|_2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\therefore (a_n)_{n=1}^\infty$ is Cauchy. \implies : Suppose $(a_n)_{n=1}^\infty$ is Cauchy, write

$$a_n = (a_n^{(1)}, \dots, a_n^{(d)})$$

For any $m, n \in \mathbb{N}_{\geq 1}$,

$$|a_n^{(i)} - a_m^{(i)}| \leq \|a_n - a_m\|_2$$

$\therefore (a_n^{(i)})_{n=1}^\infty$ is Cauchy in \mathbb{R} . So by the Cauchy Convergence Criterion, $\exists L^{(i)} \in \mathbb{R}$, such that $a_n^{(i)} \rightarrow L^{(i)}$. By the previous proposition,

$$a_n \rightarrow (L^{(1)}, \dots, L^{(d)})$$

□

Definition 9.1.3. $S \subseteq \mathbb{R}^d$ is **bounded** if $\exists M > 0$ such that

$$\|x\| \leq M \quad \forall x \in S$$

A sequence $(a_n)_{n=1}^\infty$ in \mathbb{R}^d is bound if $\{a_n : n \in \mathbb{N}_{\geq 1}\}$ is a bounded set.

Theorem 9.1.2 (Bolzano-Weierstrass for \mathbb{R}^d). *If $(a_n)_{n=1}^\infty$ is a bounded sequence in \mathbb{R}^d , then it has a subsequence $(a_{n_k})_{n=1}^\infty$ that converges.*

Proof. Write $a_n = (a_n^{(1)}, \dots, a_n^{(d)})$.

We will prove it by induction on d . For $d = 1$, this is the Bolzano-Weierstrass theorem for \mathbb{R} . For $d > 1$ write

$$b_n := (a_n^{(1)}, \dots, a_n^{(d-1)}) \in \mathbb{R}^{d-1}$$

By the induction hypothesis, b_n has a subsequence $(b_{n_k})_{k=1}^\infty$ that converges. Let $L \in \mathbb{R}^{d-1}$ be the limit of this subsequence. Then $L \in \mathbb{R}^{d-1}$ is the limit of b_n . $(a_{n_k}^{(d)})_{k=1}^\infty$ is a bounded sequence in \mathbb{R} , so it has a subsequence $(a_{n_{k_j}}^{(d)})_{j=1}^\infty$ that converges. Let $L^{(d)} \in \mathbb{R}$ be the limit of this subsequence. Then $L = (L^{(1)}, \dots, L^{(d-1)}, L^{(d)})$ is the limit of a_n . \square

Lecture 10

Open and Closed Sets in \mathbb{R}^d

Roughly, an open set is one that we draw with dotted lines. The line represents a "boundary" that is not in the set. This is not a rigorous definition.

Definition 10.0.1 (Open Ball). Let $a \in \mathbb{R}^d$, $r > 0$. The **open ball** of radius r centered at a is

$$B(a; r) := \{x \in \mathbb{R}^d : \|x - a\|_2 < r\}$$

Relation to Convergence: If $a_n \rightarrow L$, then this means that $\|a_n - L\|_2 < \epsilon$ for all n large. So, $a_n \in B(L; \epsilon)$

Definition 10.0.2 (Open Sets). A set $U \subseteq \mathbb{R}^d$ is **open** if $\forall a \in U, \exists r > 0$, such that $B(a; r) \subseteq U$

Idea: If $a \in U$, then a is not on the boundary but it is truly "inside" the set, so we can fit a ball containing a in the set.

Definition 10.0.3 (Closed Sets). A set $k \in \mathbb{R}^d$ is **closed** if its complement $\mathbb{R}^d \setminus k$ is open.

Example: $U \subseteq (0, 1)$. Is this open? Yes.

Proof. Let $a \in U$. We let $r := \min\{|a - 0|, |a - 1|\}$ (We do this so that r is at most the distance to the closest bound, i.e. if a is closer to 0, then the radius r cannot be $|a - 1|$) then

$$B(a; r) = (a - r, a + r) \subseteq (0, 1) = U$$

□

Example: $U := [0, 1]$. Is this open? No.

Proof. Let $a := 0 \in U$. The for any $r > 0$, $\exists z \in B(a; r) = (-r, r)$ s.t $z < 0$, so $z \notin U$. Therefore $B(a; r) \not\subseteq U$ □

Is U closed? This is the same as asking if $\mathbb{R} \setminus U = (-\infty, 0) \cup (1, \infty)$ is open. This is open.

Proof. Let $a \in (-\infty, 0) \cup (1, \infty)$.

- **Case 1:** $a \in (-\infty, 0)$. Set $r := |a|$, so

$$B(a; r) = (a - r, a + r) = (2a, 0) \subseteq U$$

- **Case 2:** $a \in (1, \infty)$ similar.

□

Therefore $U = [0, 1]$ is closed.

Example: Is $U := (0, 1]$ open? No, for any $r > 0$

$$B(1; r) \not\subseteq U$$

Therefore, it is not open.

Note: Sets are not always open or closed. Most sets are neither open nor closed.

This set U is one such example U is not closed since $\mathbb{R} \setminus U = (-\infty, 0] \cup (1, \infty)$
 $0 \in \mathbb{R} \setminus U$ but $\forall r > 0, B(0; r) \not\subseteq \mathbb{R} \setminus U$

Example: For any $a \in \mathbb{R}^d, r > 0$ $B(a; r)$ is an open set.

Proof. Let $x \in B(a; r)$, so $\|x - a\|_2 < r$. Set

$$r_0 := r - \|x - a\|_2 > 0$$

Claim: $B(x; r_0) \subseteq B(a; r)$ To see this, let $y \in B(x; r_0)$ so $\|y - x\|_2 < r_0$. So,

$$\begin{aligned} \|y - a\|_2 &\leq \|y - x\|_2 + \|x - a\|_2 && (\triangle\text{-inequality}) \\ &< r_0 + \|x - a\|_2 \\ &= r \end{aligned}$$

□

Proposition 10.0.1. (i) \emptyset, \mathbb{R}^d are both open in \mathbb{R}^d

(ii) If $U_1, U_2, \dots, U_n \subseteq \mathbb{R}^d$ are all open, then so is $U_1 \cap U_2 \cap \dots \cap U_n$.

(iii) If $U_\alpha \subseteq \mathbb{R}^d$ is an open set for all $\alpha \in I$, (I is some index set) then

$$\bigcup_{\alpha \in I} U_\alpha$$

is open.

Proof. (i),(ii) are exercises.

(iii): Set

$$V := \bigcup_{\alpha \in I} U_{\alpha}$$

Let $a \in V$. This means $\exists \alpha \in I$ such that $a \in U_{\alpha}$. U_{α} is open so $\exists r > 0$ s.t $B(a; r) \subseteq U_{\alpha}$. $U_{\alpha} \subseteq \bigcup_{\alpha \in I} U_{\alpha} = V$ So $B(a; r) \subseteq V$ as required. \square

Example: For any $n \in \mathbb{N}_{\geq 1}$.

$$\left(\frac{-1}{n}, \frac{1}{n} \right) = B(0; \frac{1}{n})$$

is open in \mathbb{R} . The intersection of these open sets is

$$\bigcap_{n=1}^{\infty} \left(\frac{1}{n}, \frac{-1}{n} \right) = \{0\}$$

which is not open. This shows that openness is not preserved by infinite intersections.

Example: Let

$$U := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$$

U is open but not closed.

$$U = V \cap W$$

where

$$V := \{(x, y) : x > 0\} \quad W := \{(x, y) : y > 0\}$$

To show V is open, let $a = (x, y) \in V$. Set $r := x > 0$. Then if $(w, z) \in B(a; r)$. Then

$$|w - z| \leq \|(w, z) - a\|_2 < r = x$$

$$\therefore w > x - x = 0$$

So $(w, z) \in U$. Similarly, W is open. Therefore U is open.

Not Closed: Exercise.

Proposition 10.0.2. Let $K \subseteq \mathbb{R}^d$. K is closed \iff for any subsequence $(a_n)_{n=1}^{\infty}$ in K , If it converges, then

$$\lim_{n \rightarrow \infty} a_n \in K$$

Proof. (\implies) Suppose K is closed. Let $(a_n)_{n=1}^\infty$ be a sequence in K s.t

$$L := \lim_{n \rightarrow \infty} a_n$$

exists. Suppose for a contradiction $L \notin K$. This means $L \in \mathbb{R}^d \setminus K$, which is open. So $\exists r > 0$ such that

$$B(L; r) \subseteq \mathbb{R}^d \setminus K$$

Since $a_n \rightarrow L$, we must have $a_n \in B(L; r)$ for some n (in fact, for all n sufficiently large). So $a_n \in B(L; r) \subseteq \mathbb{R}^d \setminus K$. Therefore $a_n \notin K$, which is a contradiction.

(\impliedby) Suppose K is not closed, and we'll prove $\exists (a_n)_{n=1}^\infty$ in K such that $a_n \rightarrow L \notin K$. Since K is not closed, $\mathbb{R}^d \setminus K$ is not open. So $\exists L \in \mathbb{R}^d \setminus K$ such that $\forall r > 0$

$$B(L; r) \not\subseteq \mathbb{R}^d \setminus K$$

For each $n \in \mathbb{N}_{\geq 1}$, we can find $a_n \in B(L; \frac{1}{n})$ such that $a_n \notin \mathbb{R}^d \setminus K$. So $a_n \in K$. This gives a sequence $(a_n)_{n=1}^\infty$ in K and

$$\|a_n - L\|_2 < \frac{1}{n} \rightarrow 0$$

Therefore by the Squeeze Theorem,

$$\|a_n - L\|_2 \rightarrow 0 \implies a_n \rightarrow L$$

$L \in \mathbb{R}^d \setminus K$, so $L \notin K$. □

Definition 10.0.4. Let $A \subseteq \mathbb{R}^d$ and let $a \in \mathbb{R}^d$, a is:

- (i) an **interior point** if $\exists r > 0$ s.t $B(a; r) \subseteq A$
- (ii) an **accumulation point** if \exists a sequence $(a_n)_{n=1}^\infty$ in A s.t $a_n \rightarrow a$
- (iii) a **boundary point** if it is an accumulation point and it is not an interior point.

$$A^\circ := \{\text{All interior points}\}$$

$$\bar{A} := \{\text{All accumulation points}\}$$

$$\partial A := \{\text{All boundary points}\} = \bar{A} \setminus A^\circ$$

Note: The set of interior points, accumulation points, and boundary points are referred to as the **interior** of A , the **closure** of A , and the **boundary** of A respectively

Example: $A := (0, 1] \cup \{2\}$

$$A^\circ = (0, 1)$$

$$\bar{A} = [0, 1] \cup \{2\}$$

$$\partial A = \{0, 1, 2\}$$

Example: $A := \mathbb{Q}$

Since any open interval contains irrational numbers, we have

$$A^\circ = \text{set}$$

Proposition from chapter 2,

$$\bar{A} = \mathbb{R}$$

$$\partial A = \mathbb{R}$$

Lecture 11

Compactness

Definition 11.0.1. A set $A \subseteq \mathbb{R}^d$ is (sequentially) compact if every sequence $(a_n)_{n=1}^\infty$ in A has a subsequence $(a_{n_k})_{k=1}^\infty$ that converges to a point in A .

Example 1: Is $[0,1]$ compact? Yes.

Proof. Recall: Bolzano-Weierstrass theorem states bounded sequence has a convergent subsequence.

Therefore, every sequence $(a_n)_{n=1}^\infty$ in $[0,1]$ has a subsequence $(a_{n_k})_{k=1}^\infty$ that converges. So

$$0 \leq a_{n_k} \leq 1 \implies 0 \leq \lim_{k \rightarrow \infty} a_{n_k} \leq 1 \\ \therefore L \in [0,1]$$

□

Example 2: Is $(0,1)$ compact? No.

Proof. By counter example, let $a_n := \frac{1}{n+1}$, so $a_n \rightarrow 0$. Therefore for all subsequences of a_n , $a_{n_k} \rightarrow 0$. So there exists no subsequence which converges to a point in $(0,1)$. □

Example 3: Is $[0, \infty)$ compact? No.

Proof. The Bolzano-Weierstrass theorem does not apply since $[0, \infty)$ is unbounded. Set $a_n := n$, then $a_n \rightarrow \infty$, so it has no bounded subsequence and therefore no convergent subsequences. □

Theorem 11.0.1 (Heine-Borel). Let $A \subseteq \mathbb{R}^d$. A is compact $\iff A$ is closed and not bounded.

Proof. (\implies) Similar to example 1. Assume A is closed and bounded. Let $(a_n)_{n=1}^\infty$ be a sequence in A . The sequence is bounded since A is, so by the Bolzano-Weierstrass theorem for \mathbb{R}^d , it has a subsequence $(a_{n_k})_{k=1}^\infty$ that converges to some $L \in \mathbb{R}^d$. $a_{n_k} \in A \forall k$ and $a_{n_k} \rightarrow L$ and A is closed, so by the

sequential characterization of closedness, $L \in A$, therefore A is compact.

(\Leftarrow) Assume A is compact. To show A is closed, assume for a contradiction that A is not closed. Therefore there exists a sequence $(a_n)_{n=1}^\infty$ in A such that $a_n \rightarrow L \notin A$. Then for any subsequence $(a_{n_k})_{k=1}^\infty$, we have

$$a_{n_k} \rightarrow L \notin A$$

This contradicts that A is compact, therefore A is closed.

To show A is bounded, assume for a contradiction that A is not bounded. Then $\forall n \in \mathbb{N}_{\geq 1}$, there exists $a_n \in A$ such that $\|a_n\|_2 \geq n$. This gives a sequence $(a_n)_{n=1}^\infty$. Since A is compact, it has a subsequence $(a_{n_k})_{k=1}^\infty$ that converges. But

$$\|a_{n_k}\|_2 \geq n_k \rightarrow \infty$$

So $(a_{n_k})_{k=1}^\infty$ is unbounded, which is a contradiction. \square

Proposition 11.0.1.

(i) If $k_1, \dots, k_n \subseteq \mathbb{R}^d$ are compact, then $\bigcup_{i=1}^n k_i$ is compact.

(ii) If $k_1, \dots, k_n \subseteq \mathbb{R}^d$ are compact, then $\bigcap_{i=1}^n k_i$ is compact.

Proof. Exercise:

(i) Assume $A := \bigcup_{i=1}^n k_i$. Let $(a_n)_{n=1}^\infty$ be a sequence in A . Then there exists $i \in \{1, \dots, n\}$ such that $a_n \in k_i$. Since k_i is compact, it has a subsequence $(a_{n_k})_{k=1}^\infty$ that converges to some $L \in \mathbb{R}^d$. $L \in k_i$ and $k_i \subseteq A$, so $L \in A$. Therefore A is compact.

(ii) Assume $A := \bigcap_{i=1}^n k_i$. Let $(a_n)_{n=1}^\infty$ be a sequence in A . Then $a_n \in k_i$ $\forall i \in \{1, \dots, n\}$. Since k_i is compact, it has a subsequence $(a_{n_k})_{k=1}^\infty$ that converges to some $L \in \mathbb{R}^d$. $L \in k_i$ $\forall i \in \{1, \dots, n\}$ and $k_i \subseteq A$, so $L \in A$. Therefore A is compact. \square

Definition 11.0.2. $A \subseteq \mathbb{R}^d$ is **compact** if for any collection

$$U_\alpha : \alpha \in I$$

of open sets such that

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha$$

There exists finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Lecture 12

Limits of a Function of Continuous Variables

A sequence is a function $\mathbb{N} \rightarrow \mathbb{R}$. Here, we'll consider function that are going from $\mathbb{R} \rightarrow \mathbb{R}$ (or $\mathbb{R}^d \rightarrow \mathbb{R}^m$).

Definition 12.0.1. Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X . $f : X \rightarrow \mathbb{R}^m$, $L \in \mathbb{R}^m$. We say the limit of f as X approaches a is L if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X$$

$$x \in B(a; \delta) \wedge x \neq a \implies \|f(x) - L\|_2 < \epsilon$$

The idea is like the definition of convergence of a sequence, except we replace $n \geq n_0$ (which captures "n is sufficiently large") with $x \in B(a; \delta)$, $x \neq a$ (which captures "x is close to, but not equal to a"). In other words, the definition says that if x is close to (but not equal to) a then $f(x)$ is close to L .

Why "not equal to"?: Often we consider the limit as x approaches a when $f(a)$ is not defined. Other times we compare the limit to $f(a)$. So we do not want to use $f(a)$ in the definition of the limit.

Notation: We write

$$\lim_{x \rightarrow a} f(x) = L$$

or

$$f(x) \rightarrow L \text{ as } x \rightarrow a$$

to mean that the limit of f is L as x approaches a .

Example: $f : \mathbb{R} \rightarrow \mathbb{R}$. $f(x) := 3x - 2$. Let $a \in \mathbb{R}$. Claim

$$\lim_{x \rightarrow a} f(x) = 3a - 2$$

Proof. Let $\epsilon > 0$. Consider

$$\begin{aligned} |f(x) - (3a - 2)| &= |3x - 2 - 3a + 2| \\ &= 3|x - a| \end{aligned}$$

We want this $< \epsilon$, set $\delta := \frac{\epsilon}{3}$. Then if $x \in B(a; \delta) = (a - \delta, a + \delta)$ (i.e. $|x - a| < \delta$) then

$$|f(x) - (3a - 2)| = 3|x - a| < 3\delta = \frac{3\epsilon}{3} = \epsilon$$

□

Example: $g : \mathbb{R} \rightarrow \mathbb{R}$. $g(x) := x^2$. Claim:

$$\lim_{x \rightarrow a} g(x) = a^2$$

Proof. Let $\epsilon > 0$ be given.

$$\begin{aligned} |g(x) - a^2| &= |x^2 - a^2| \\ &= |x - a||x + a| \end{aligned}$$

What happens if x is close to a ? Intuitively, $|x + a|$ is close to $|a + a|$ and $|x - a|$ is small.

$$\begin{aligned} |x + a| &= |x - a + a + a| \leq |x - a| + |a + a| \\ &< 2|a| + \delta && (\text{if } |x - a| < \delta) \\ &\leq 2|a| + 1 && (\delta \leq 1) \end{aligned}$$

Then,

$$\begin{aligned} |x^2 - a^2| &= |x - a||x + a| \leq |x - a|(2|a| + 1) \\ &< \delta(2|a| + 1) && (\text{if } |x - a| < \delta) \\ &\leq \epsilon && (\text{if } \delta \leq \frac{\epsilon}{2|a|+1}) \end{aligned}$$

Important: Do not define δ in terms of x or δ ! We can use a here since a is constant

So we set $\delta := \min\{1, \frac{\epsilon}{2|a|+1}\}$ Then $\delta \leq \frac{\epsilon}{2|a|+1}$ and $\delta \leq 1$. So if $|x - a| < \delta$. Then from the work above, $|x^2 - a^2| < \epsilon$ as required. □

Note: In proofs where we have $\delta - \epsilon$, we often use

$$\delta := \min\{\dots\}$$

In proofs where we have $n_0 - \epsilon$, we often use

$$n_0 := \max\{\dots\}$$

Proposition 12.0.1 (Uniqueness of Limits). *Let $f : X \rightarrow \mathbb{R}^m$ ($X \subseteq \mathbb{R}^d$), $a \in \mathbb{R}^d$ a limit point of X , $L, L' \in \mathbb{R}^m$. If the limit of f as $x \rightarrow a$ is L and the limit of f as $x \rightarrow a$ is L' , then $L = L'$*

Proof. By contradiction. Suppose $L \neq L'$. So

$$\|L - L'\|_2 > 0$$

Set

$$\epsilon := \frac{\|L - L'\|_2}{2} > 0$$

Since $f(x) \rightarrow L$ as $x \rightarrow a$, $\exists \delta > 0$ such that if $x \in X \cap B(a; \delta) \setminus \{a\}$, then

$$\|f(x) - L\|_2 < \epsilon$$

Since $f(x) \rightarrow L'$ as $x \rightarrow a$, $\exists \delta' > 0$ such that

$$x \in X \cap B(a; \delta') \setminus \{a\} \implies \|f(x) - L'\|_2 < \epsilon$$

Let $\delta_0 \neq \min\{\delta, \delta'\}$. Let

$$x \in X \cap B(a; \delta_0) \setminus \{a\}$$

Then

$$x \in X \cap B(a; \delta) \setminus \{a\}$$

So,

$$\begin{aligned} \|f(x) - L\|_2 &< \epsilon \\ \|f(x) - L'\|_2 &< \epsilon \end{aligned}$$

So,

$$\begin{aligned} \|L - L'\|_2 &\leq \|L - f(x)\|_2 + \|f(x) - L'\|_2 < \epsilon + \epsilon \\ &= \|L - L'\|_2 \end{aligned}$$

And thus, a contradiction. □

Proposition 12.0.2 (Sequential Characterization of Limits). *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$, a limit point of X . $f : X \rightarrow \mathbb{R}^m$, $L \in \mathbb{R}^m$.*

$\lim_{x \rightarrow a} f(x) = L \iff$ for every sequence $(x_n)_{n=1}^\infty$ in X such that $x_n \rightarrow a$, we have

$$\lim_{n \rightarrow \infty} f(x_n) = L$$

Proof. (\implies) Suppose $\lim_{x \rightarrow a} f(x) = L$. Let $(x_n)_{n=1}^\infty$ be a sequence in $X \setminus \{a\}$ such that $x_n \rightarrow a$. We must show that $f(x_n) \rightarrow L$.

Let $\epsilon > 0$ be given. Since $f(x) \rightarrow L$ as $x \rightarrow a$, $\exists \delta$ such that

$$x \in X \cap B(a; \delta) \setminus \{a\} \implies \|f(x) - L\|_2 < \epsilon$$

Since $x_n \rightarrow a$, using δ in place of ϵ , $\exists n_0$ such that $\forall n \geq n_0$, $\|x_n - a\|_2 < \delta$. i.e. $x_n \in B(a, \delta)$. Also $x_n \in X \setminus \{a\}$. Therefore,

$$\|f(x_n) - L\|_2 < \epsilon$$

(\Leftarrow) Suppose \forall sequences $(x_n)_{n=1}^\infty$ in $X \setminus \{a\}$ converging to a , $f(x_n) \rightarrow L$, and for a contradiction, suppose

$$f(x) \not\rightarrow L$$

We negate " $f(x) \rightarrow L$ " to get that $\exists \epsilon > 0$ such that $\forall \gamma > 0$, $\exists x \in X \cap B(a; \gamma) \setminus \{a\}$ such that $\|f(x) - L\|_2 \geq \epsilon$.

This gives a sequence $(x_n)_{n=1}^\infty$ in $X \setminus \{a\}$, $\|x_n - a\|_2 \leq \frac{1}{n} \forall n$, so by the squeeze theorem

$$\|x_n - a\|_2 \rightarrow 0$$

Since $\|f(x_n) - L\|_2 \geq \epsilon$, $f(x_n) \not\rightarrow L$. This is a contradiction. \square

Note: if $\lim_{n \rightarrow \infty} f(x_n) = L$ for *some* sequence $(x_n)_{n=1}^\infty$ in $X \setminus \{a\}$ converging to a , it *does not* follow that $\lim_{x \rightarrow a} f(x) = L$

Example:

$$f(x) := \begin{cases} 0 & \text{if } x = \frac{1}{n}, n \in \mathbb{N}_{\geq 1} \\ 1 & \text{otherwise} \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$ does not exist but $\lim_{n \rightarrow \infty} f(\frac{1}{n}) = 0$

Proposition 12.0.3 (Algebra of Limits). *Let $x \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X , $f : X \rightarrow \mathbb{R}^m$, $g : X \rightarrow \mathbb{R}^m$, $L, K \in \mathbb{R}^m$. Suppose $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a} g(x) = K$*

(i)

$$\lim_{x \rightarrow a} f(x) + g(x) = L + K$$

(ii)

$$\lim_{cf(x)} = cL$$

(iii) If $m = 1$,

$$\lim_{x \rightarrow a} f(x)g(x) = LK$$

(iv) If $m = 1$, $g(x) \neq 0 \forall x \in X$, $K \neq 0$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{K}$$

Proof. (i) Use Sequential Characterization: Let $(x_n)_{n=1}^\infty$ be in $X \setminus \{a\}$ such that $x_n \rightarrow a$. Then $f(x_n) \rightarrow L$ and $g(x_n) \rightarrow K$. So by algebra of limits for sequences,

$$f(x_n) + g(x_n) \rightarrow L + K$$

$$\therefore f(x) + g(x) \rightarrow L + K$$

(ii) **Exercise.**

(iii) **Exercise.**

(iv) **Exercise.**

□

Theorem 12.0.1 (Squeeze Theorem). *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X , $f, g, h : X \rightarrow \mathbb{R}$*

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in X$$

and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

Then

$$\lim_{x \rightarrow a} g(x) = L$$

Proof. **Exercise**

□

If $f : X \rightarrow \mathbb{R}_m$, We can define functions

$$f_1, \dots, f_m : X \rightarrow \mathbb{R}$$

by

$$(f_1(x), \dots, f_m(x)) = f(x)$$

f_1, \dots, f_m are called the *component functions* of f .

Proposition 12.0.4. *Let $X \subseteq \mathbb{R}^d$, $a \in \mathbb{R}^d$ a limit point of X , $f : X \rightarrow \mathbb{R}^m$, f_1, \dots, f_m its component functions. $L = (L_1, \dots, L_m) \in \mathbb{R}^m$. Then*

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a} f_i(x) = L_i \quad \forall 1 \leq i \leq m$$

Proof. **Exercise.**

□

Definition 12.0.2. *Let $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $f : X \rightarrow \mathbb{R}^d$.*

- *If a is a limit point of $X \cap (a, \infty)$ then we write $\lim_{x \rightarrow a^+} f(x) = L$ to mean that*

$$\lim_{x \rightarrow a} g(x) = L$$

where

$$g = f|_{X \cap (a, \infty)}$$

- *If a is a limit point of $X \cap (-\infty, a)$ then we write $\lim_{x \rightarrow a^-} f(x) = L$ to mean that*

$$\lim_{x \rightarrow a} g(x) = L$$

where

$$g = f|_{X \cap (-\infty, a)}$$

Example:

$$f(x) := \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} f(x) = 1 \neq \lim_{x \rightarrow 0^-} f(x) = -1$$

Lecture 13

Continuity

13.1 One-Sided Limits

Recall: For $X \subseteq \mathbb{R}$, $f : X \rightarrow \mathbb{R}^m$, $a \in \mathbb{R}$

- If a is a limit point of $X \cap [a, \infty]$, then

$$\lim_{x \rightarrow a^+} f(x) := \lim_{x \rightarrow a} f_{X \cap [a, \infty)}(x)$$

$\lim_{x \rightarrow a^-} f(x)$ is defined similarly.

In other words, $\lim_{x \rightarrow a^+} f(x) = L$ means $\forall \epsilon > 0, \exists \delta > 0$ such that if $a < x < a + \delta$ and $x \in X$, then

$$\|f(x) - L\|_2 < \epsilon$$

13.2 Continuity

Definition 13.2.1. Let $f : X \rightarrow \mathbb{R}^m$ where $X \subseteq \mathbb{R}^d$. Let $a \in X$.

- If a is not an isolated point, then we say f is continuous at a if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

- If a is an isolated point, then we always say f is continuous at a .

Definition 13.2.2 ($\delta - \epsilon$ Characterization of Continuity). f is continuous at a $\iff \forall \epsilon > 0, \exists \gamma > 0$ s.t if

$$x \in X \cap B(a; \delta)$$

then

$$\|f(x) - f(a)\|_2 < \epsilon$$

In other words if x is close to a , the $f(x)$ is close to $f(a)$. This holds regardless of whether or not a is isolated.

Example: $f : \mathbb{R} \mapsto \mathbb{R} \ f(x) := x^2$. For any $a \in \mathbb{R}$, we proved last lecture that

$$\lim_{x \rightarrow a} f(x) = a^2 = f(a)$$

Therefore, f is continuous at a .

Example:

$$f(x) := \begin{cases} 1 & \text{if } x \leq 1 \\ 2 & \text{if } x > 1 \end{cases}$$

For this function,

$$\lim_{x \rightarrow 1^+ f(x)=2 \neq f(x)=1}$$

Therefore, f is not continuous at 1.

Example:

$$f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ L & \text{if } x = 0 \text{ for some } L \in \mathbb{R} \end{cases}$$

No matter how we choose L , f is not continuous at 0.

Proof. We want to show $\lim_{x \rightarrow 0} f(x)$ does not exist. Consider the sequence

$$\left(\frac{1}{2\pi n}\right)_{n=1}^{\infty} \rightarrow 0$$

and $f\left(\frac{1}{2\pi n} = 0 \rightarrow 0\right)$. But we also have

$$\left(\frac{1}{2\pi n + \pi/2}\right)_{n=1}^{\infty} \rightarrow 0$$

and $f\left(\frac{1}{2\pi n + \pi/2}\right) = 1 \rightarrow 1$. So if $\lim_{x \rightarrow 0} f(x)$ exists, then $f(a_n) \rightarrow L$ for a sequence $(a_n)_{n=1}^{\infty}$ in $(0, \infty)$ such that $a_n \rightarrow 0$. So $L = 0$ and $L = 1$, therefore a contradiction \square

Example:

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

f is not continuous at any point.

Proof. Let $a \in \mathbb{R}$. We'll prove $\lim_{x \rightarrow a} f(x)$ does not exist.

$$\exists ((n)_{n=1}^{\infty} x) \in \mathbb{Q} \text{ s.t } x_n \rightarrow a$$

so $f(x_n) = 1 \rightarrow 1$

$$\exists ((n)_{n=1}^{\infty} y) \in \mathbb{R} \setminus \mathbb{Q} \text{ s.t } y_n \rightarrow a$$

Since we can find $z_n \in \mathbb{Q}$ such that $z_n \rightarrow a + \sqrt{2}$, so $y := z_n - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, $y_n \rightarrow a$. So $f(y_n) = 0 \rightarrow 0$. Therefore, the $\lim_{x \rightarrow a} f(x)$ does not exist. \square

Example:

$$f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f is continuous.

Proof. Use squeeze theorem, if $x \geq 0$, then $-x \leq f(x) \leq x$. So

$$\begin{aligned} -|x| &\leq f(x) \leq |x| \\ \lim_{x \rightarrow 0} |x| &= 0 = \lim_{x \rightarrow 0} -|x| \\ \therefore \lim_{x \rightarrow 0} f(x) &= 0 = f(0) \end{aligned}$$

□

Proposition 13.2.1. Let $X \subseteq \mathbb{R}^d$, $Y \subseteq \mathbb{R}^m$, and $f : X \mapsto Y$, $g : Y \rightarrow \mathbb{R}^n$, $a \in X$. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a .

Proof. Use the $\delta - \epsilon$ characterization. Let $\epsilon > 0$ be given. Since g is at $f(a)$, $\exists \eta > 0$ (using η instead of δ), such that if $y \in B(f(a); \eta) \cap Y$, then

$$\|g(x) - g(f(a))\|_2 < \epsilon$$

Since f is continuous at a , $\exists \delta > 0$ such that if $x \in B(a; \delta) \cap X$, then

$$\|f(x) - f(a)\|_2 < \eta$$

In other words, $f(x) \in B(f(a); \eta)$. Therefore if $x \in B(a; \delta) \cap X$, then since $f(x) \in B(f(a); \eta)$,

$$\|g(x) - g(f(a))\|_2 < \epsilon$$

as required. □

Proposition 13.2.2. Let $X \subseteq \mathbb{R}^d$, $f, g : X \mapsto \mathbb{R}^m$, $a \in X$ such that f, g are continuous at a ,

- (i) $f + g : X \mapsto \mathbb{R}^m$ is continuous at a
- (ii) $cf : X \mapsto \mathbb{R}^m$ is continuous at a for any $c \in \mathbb{R}$
- (iii) $\gamma f : X \mapsto \mathbb{R}^m$ is continuous at a if $\gamma : X \mapsto \mathbb{R}$ is continuous at a .
- (iv) If $m = 1$, and $g(x) \neq 0, \forall x \in X$, then

$$\frac{f}{g} : X \mapsto \mathbb{R}$$

is continuous at a .

Proof. Follows from the algebra of limits. □

If $f : X \mapsto \mathbb{R}^m$, $f = (f_1, \dots, f_m)$ where $f_i : X \mapsto \mathbb{R}$, then f is continuous at a if and only if f_i is continuous at a for all $i \in \{1, \dots, m\}$.

- You are allowed to use the fact that \sin , \cos , \exp , \log are all continuous functions $\log : [0, \infty) \mapsto \mathbb{R}$ is continuous at all $a \in [0, \infty)$.

13.3 Properties of Continuous Functions

$f : X \mapsto \mathbb{R}^m$ is (globally) continuous if f is continuous at a for all $a \in X$.

Theorem 13.3.1. Let $K \subseteq \mathbb{R}^d$ be a compact set. If $f : K \mapsto \mathbb{R}^m$ is continuous, then $f(K)$ is compact.

Proof. Let $(a_n)_{n=1}^\infty$ be a sequence in $f(K)$. We need to show that \exists a subsequence. We need to show that \exists a subsequence $(a_{n_k})_{k=1}^\infty$ s.t $a_{n_k} \rightarrow b$ for some $b \in f(K)$. Since $a_n \in f(K)$, $\exists x_n \in K$ s.t $f(x_n) = a_n$ for all n . This gives a sequence $(x_n)_{n=1}^\infty$ in K . Since K is compact, \exists a subsequence $(x_{n_k})_{k=1}^\infty$ s.t $x_{n_k} \rightarrow x_0$. Since f is continuous at x_0 , $f(x_{n_k}) \rightarrow f(x_0)$. Therefore, $a_{n_k} \rightarrow f(x_0)$ and $(a_{n_k})_{k=1}^\infty$ is a subsequence of $(a_n)_{n=1}^\infty$. This is what we wanted. \square

Corollary 13.3.1 (Extreme Value Theorem). Let $K \subseteq \mathbb{R}^d$ be compact, $f : K \rightarrow \mathbb{R}$ a continuous function. Then f is bounded and it attains its bounds, i.e $\exists x_{\min}, x_{\max} \in K$ such that $\forall x \in K$

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

Proof. From the previous result, $f(K)$ is compact in \mathbb{R} . By the Heine-Borel theorem, this means $f(K)$ is closed and bounded. Therefore, f is bounded. Since $f(K)$ is closed, $\sup f(K) \in f(K)$ since there is a sequence $(y_n)_{n=1}^\infty$ in $f(K)$ such that $y_n \rightarrow \sup f(K)$. So $\exists x_{\max} \in K$ such that

$$f(x_{\max}) = \sup f(K)$$

Therefore,

$$f(x) \leq f(x_{\max}) \quad \forall x \in K$$

Similarly, $\exists x_{\min} \in K$ such that $f(x_{\min}) \leq f(x) \quad \forall x \in K$. \square

Lecture 14

Properties of Continuous Functions Continued

Recall:

Corollary 13.3.1 (Extreme Value Theorem). *If $K \subseteq \mathbb{R}^d$ is compact and $f : K \rightarrow \mathbb{R}$ is continuous, then $\exists x_{\min}, x_{\max} \in K$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in K$.*

Theorem 14.0.1 (Intermediate Value Theorem). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous, and let $y_0 \in \mathbb{R}$ be any number between $f(a)$ and $f(b)$. Then there exists a number x_0 between a and b such that $f(x_0) = y_0$.*

Proof. Without loss of generality, assume $f(a) \leq f(b)$. So $x_0 \in [f(a), f(b)]$. Let

$$S := \{x \in [a, b] : f(x) \leq x_0\}$$

$S \subseteq [a, b]$, so S is bounded. $a \in S$ since $f(a) \leq x_0$. Therefore $S \neq \emptyset$. So $\exists x_0 := \sup S \in [a, b]$. We will show $f(x_0) = y_0$. We will consider the cases, where $f(x_0) = y_0$, $f(x_0) < y_0$, and $f(x_0) > y_0$. If $f(x_0) < y_0$, set $\epsilon := y_0 - f(x_0)$. Since f is continuous at x_0 , $\exists \delta > 0$ s.t if $|x - x_0| < \delta$ and $x \in [a, b]$. Then $|f(x) - f(x_0)| < \epsilon$. Since $f(x_0) < y_0 \leq f(b)$ for $x_0 \neq b$. So we can find $x > x_0$ such that $x \in [a, b]$ and $|x - x_0| < \delta$. Then $f(x) < f(x_0) + \epsilon = y_0$. So $x \in S$ by the definition of S , since $S = \{x : f(x) \leq y_0\}$. This is a contradiction since $x > x_0$, but x_0 is an upper bound for S .

If $f(x_0) > y_0$, set $\epsilon := f(x_0) - y_0$. Since f is continuous at x_0 , $\exists \delta > 0$ such that if $x \in [a, b]$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. So $f(x_0) > y_0 \leq f(a)$. So $x_0 > a$. We may assume that $x_0 - \delta > a$ since δ is arbitrary and can be arbitrarily small.

Claim: $x_0 - \delta$ is an upper bound for S . Proof of claim, if $x > x_0 - \delta$, then either $|x - x_0| < \delta$, in which case $f(x) > f(x_0) - \epsilon = y_0$, or $x > x_0$, in which case $x \notin S$ since x_0 is an upper bound for S . Therefore, if $x > x_0 - \delta$, then $x \notin S$.

This proves the claim.

The claim contradicts that x_0 is the least upper bound for S . \square

Corollary 14.0.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then $f([a, b]) = [c, d]$ for some $c \leq d$.*

Proof. By extreme value theorem, $\exists x_{\min}, x_{\max}$ such that $f(x) \in [f(x_{\min}), f(x_{\max})]$ $\forall x \in [a, b]$. So we can set $c := f(x_{\min})$ and $d := f(x_{\max})$ and we have

$$f([a, b]) \subseteq [c, d]$$

for $y_0 \in [c, d] = [f(x_{\min}), f(x_{\max})]$. By the Intermediate Value Theorem, $\exists x_0$ between x_{\min} and x_{\max} such that $f(x_0) = y_0$. Therefore

$$[c, d] \subseteq f([a, b])$$

Thus,

$$f([a, b]) = [c, d]$$

\square

14.1 Inverses of Continuous Functions

Let $f : A \mapsto B$, f is

- **Injective** (or one-to-one) If $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- **Surjective** (or onto) If $\forall y \in B$, $\exists x \in A$ such that

$$f(x) = y$$

- **Bijjective** (or one-to-one and onto) If f is both injective and surjective.

f is bijective \iff it is invertible

$$\iff \exists f^{-1} : B \mapsto A \text{ s.t } f \circ f^{-1} = id_B \text{ and } f^{-1} \circ f = id_A$$

Proof. (\Leftarrow) If f is invertible, then it is

- **Injective** since if $f(x) = f(y)$ then $x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$
- **Surjective** since given $y \in B$, $x := f^{-1}(y)$ satisfies $f(x) = y$.

(\Rightarrow) If f is bijective, then for $y \in B$, $\exists x \in A$ such that $f(x) = y$. Also, this x is unique since f is injective. So we can define

$$f^{-1}(y) := x$$

and this is an inverse to f . \square

Given $f : A \mapsto \mathbb{R}$ that is injective, we view f as a function $A \mapsto f(A)$, and this way we force f to be surjective. Thus $f^{-1} : f(A) \mapsto A$ exists.

Question: If f is injective and continuous, must f^{-1} be continuous? **No.** Consider the example with $f : [0, 1) \cup [2, \infty]$

$$f(x) := \begin{cases} x_1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$f^{-1} : [0, \infty) \mapsto [0, 1) \cup [2, \infty] \subseteq \mathbb{R}$, so

$$f^{-1}(y) = \begin{cases} y_1 & \text{if } y < 1 \\ y + 1 & \text{if } y \geq 1 \end{cases}$$

f^{-1} is not continuous.

Definition 14.1.1. Let $A \subseteq \mathbb{R}$, $f : A \mapsto \mathbb{R}$. f is

- (i) increasing if $\forall x, y \in A$, $x \leq y$ then $f(x) \leq f(y)$
- (ii) strictly increasing if $\forall x, y \in A$, $x < y$ then $f(x) < f(y)$
- (iii) decreasing if $\forall x, y \in A$, $x \leq y$ then $f(x) \geq f(y)$
- (iv) strictly decreasing if $\forall x, y \in A$, $x < y$ then $f(x) > f(y)$

Lemma 14.1.1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \mapsto \mathbb{R}$ be an injective continuous function. Then f is either strictly increasing or strictly decreasing.

Proof. It suffices to consider the case where I is a closed, bounded interval. A general interval I is a union of an increasing sequence of closed bounded intervals, so if f is strictly increasing or strictly decreasing on each of these subintervals, then f is strictly increasing or decreasing on all of I . Consider the case $I = [a, b]$ with $(a < b)$. Without loss of generality, assume $f(a) < f(b)$, in this case we'll show that f is strictly increasing. By contradiction, if f is not strictly increasing, then there exists $x_1 < x_2$ such that

$$f(x_1) \geq f(x_2)$$

We'll break this into 2 cases.

- **Case 1:** $f(x) > f(b)$. In this case,

$$f(a) < f(b) < f(x_1)$$

and

$$f|_{[a, x_1]}$$

is continuous. So by the Intermediate Value Theorem, $\exists z \in [a, x_1]$ such that $f(z) = f(b)$. But,

$$z \leq x_1 < x_2 \leq b$$

So $z \neq b$, which contradicts that f is injective.

- **Case 2:** $f(x_1) \leq f(b)$. In this case, $f(x_2) \leq f(x_1) \leq f(b)$ and $f|_{[x_1, b]}$ is continuous. So by the Intermediate Value Theorem, $\exists z \in [x_1, b]$ such that $f(z) = f(x_1)$. $z \geq x_2 > x_1$ so $z \neq x_1$. But, this contradicts that f is injective.

In both cases, we get a contradiction, so our assumption that f is not strictly increasing is false. \square

Lecture 15

Let $I \subset \mathbb{R}$ be an interval, $f : I \mapsto \mathbb{R}$ be a continuous function. Then $f^{-1} : f(I) \mapsto \mathbb{R}$ is continuous.