MAT 2143 Suggested Exercises Solutions

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3.1 Question 1

Find all $x \in \mathbb{Z}$ satisfying each of the following equations.

(a) $3x \equiv 2 \pmod 7$

(b) $5x + 1 \equiv 13 \pmod{23}$

(c) $5x + 1 \equiv 13 \pmod{26}$

(d) $9x \equiv 3 \pmod 5$

(e) $5x \equiv 1 \pmod 6$

(f) $3x \equiv 1 \pmod 6$

Solution:

(a) $3x \equiv 2 \pmod 7$, we have $3 \cdot 5 \equiv 1 \pmod 7$, so $3^{-1} \equiv 5 \pmod 7$, then

$$x \equiv 2 \cdot 5 \equiv 3 \pmod 7$$

So

$$x \in 3 \cdot \mathbb{Z}$$

(b) $5x + 1 \equiv 13 \pmod{23} \implies 5x \equiv 12 \pmod{23}$. We have $5 \cdot 14 = 70 \equiv 1 \pmod{23}$, so $5^{-1} \equiv 14 \pmod{23}$, then

(c)

$$5x \equiv 12 \implies x \equiv 12 \cdot 14 \equiv 7 \pmod{23}$$

Therefore

$$x \in 7 \cdot \mathbb{Z}$$

The rest of these are easy.

3.2 Question 2

Which of the following multiplaction tables defined on the set $G = \{a, b, c, d\}$ form a group? Support your answer in each case.

(a)

$\circ$	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

(b)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(c)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

(d)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	b	a	d
d	d	d	b	c

Solution:

(a) is *not* a group since there does not exist an identity. a cannot be the identity since $ab = c \neq ba = b$. b cannot be the identity since $bc = c \neq cb = d$. c cannot be the identity since no element combined with c yields that element. d cannot be the identity for the same reason as c.

(b) is a group. Since the corresponding row and column for a matches the headers, a is the identity, and each element is its own inverse. We also have that $(bc)d = (d)d = a$ and $b(cd) = b(b) = a$ so the operation is associative. Therefore (b) is a group.

(c) is a group. a is the identity, and a appears in each column and row so each element has an inverse. We also have that $(bc)d = (d)d = c$ and $b(cd) = b(b) = c$ so the operation is associative. Therefore (c) is a group.

(d) is *not* a group since the corresponding row and column for a matches the headers, but d does not have an inverse.

3.3 Question 3

Write out Cayley tables for groups formed by the symmetries of a rectange and for $(\mathbb{Z}_4, +)$. How many elements are in each group? Are the groups the same? Why or why not?

Solution: The Cayley table for the symmetry group of a rectangle is

$\circ$	ϵ	ρ	α	β
ϵ	ϵ	ρ	α	β
ρ	ρ	ϵ	β	α
α	α	β	ϵ	ρ
β	β	α	ρ	ϵ

Where $\alpha, \beta, \epsilon, \rho$ are defined as we've seen in class. The Cayley table for $(\mathbb{Z}_4, +)$ is

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2