MAT 2143 Lecture Notes

Last updated:

February 15, 2023

Contents

Lecture 1

Equivalence Relations

1.1 Review of Equivalence Relations

Set X and a notion of equivalence $\sim$. For all $x, y \in X$, either $x \sim y$ or $x \not\sim y$.
Recall: $X \times X = \{(x, y) : x, y \in \mathbb{R}\}$. Define $R = \{(x, y) : x, y \in \mathbb{R}\ x \sim y\}$.

R is an equivalence relation if

- $x, y \in R\ \forall x \in X$
- $(x, y) \in R \iff (y, x) \in R$
- $(x, y) \in R\ (y, z) \in R \implies (x, z) \in R$

If R is an equivalence relation on X, then we define the equivalence class of $x \in X$ as

$$[x] = \{y \in X : x \sim y\}$$

1.2 Examples of Equivalence Relations

- Take any set X and let $x \sim y$ mean $x = y$
  Reflexive: $x \sim y$? Yes, because $x = x$
  Symmetric: $x \sim y \iff y \sim x$? Yes, because if $x = y$, then $y = x$.
  Transitive: $x \sim y\ y \sim z \implies x \sim z$? Yes, because if $x = y$ and $y = z$, then $x = z$.

- Take $X = \mathbb{R}^2$ and let $(a, b) \sim (c, d)$ mean $a^2 + b^2 = c^2 + d^2$
  Reflexive: $(a, b) \sim (a, b)$? Yes, because $a^2 + b^2 = a^2 + b^2$
  Symmetric: $(a, b) \sim (c, d) \iff (c, d) \sim (a, b)$? Yes, because if $a^2 + b^2 =$

$c^2 + d^2$, then $c^2 + d^2 = a^2 + b^2$.
Transitive: $(a, b) \sim (c, d)$ $(c, d) \sim (e, f) \implies (a, b) \sim (e, f)$? Yes, because if $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$, then $a^2 + b^2 = e^2 + f^2$.

- Take $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and let $(a, b) \sim (c, d)$ mean $(ad = bc)$.
  Reflexive: $(a, b) \sim (a, b)$? Yes, because multiplication of $\mathbb{Z}$ is commutative, so $ab = ba$.
  Symmetric: $(a, b) \sim (c, d) \iff (c, d) \sim (a, b)$? Yes,

  $$(a, b) \sim (c, d) \implies ad = bc$$

  $$cb = da$$

  $$(c, d) \sim (a, b)$$

  Transitive: $(a, b) \sim (c, d)$ $(c, d) \sim (e, f) \implies (a, b) \sim (e, f)$? We want $ad = bc$, $cf = de \implies af = be$
  Case 1: $c = 0$ Then $bc = 0 = ad$, $d \in \mathbb{Z} \setminus \{0\}$, so $d \neq 0$, $a = 0$
  $cf = 0 = de$, again $d \neq 0$, so $e = 0$.

  $$\therefore af = be = 0$$

  Case 2: $c \neq 0$ Then $\frac{ad}{c} = b$, $\frac{de}{c} = f$

  $$\therefore af = a \cdot \frac{de}{c} = \frac{ad}{c} \cdot e = be$$

Theorem 1.2.1. *Let X be a set with an equivalence relation. Then*

$$[x] \cap [y] \neq \emptyset \implies [x] = [y]$$

So, equivalence classes are disjoint or equal.

Proof. Assume $[x] \cap [y] \neq \emptyset$. So $\exists z \in [x] \cap [y]$

Now let $a \in [x]$

$$\begin{aligned} a &\sim z && \text{(since } z \in [x] \text{ , } z \sim x \sim a) \\ z &\sim y && \text{(since } z \in [y]) \\ a &\sim y && \text{(transitivity)} \\ a &\in [y] \\ \therefore [x] &\subseteq [y] \end{aligned}$$

Now take $b \in [y]$, using the same arguments we get

$$\begin{aligned} b &\sim z && \text{(since } z \in [y] \text{ , } z \sim y \sim b) \\ z &\sim x && \text{(since } z \in [x]) \\ b &\sim x && \text{(transitivity)} \\ b &\in [x] \\ \therefore [y] &\subseteq [x] \end{aligned}$$

□

Observation: If X is some set with an equivalence relation, then every $x \in X$ is in some equivalence class.

Definition 1.2.1 (Partitions)**.** *Say we have some $R_j \subseteq X$ for $j \in \{1, 2, \ldots, n\}$, with every $x \in X$ in exactly one R_j, then the R_j form a partition of X.*

Theorem 1.2.2. *Let X be a set with an equivalence relation. Then the equivalence classes form a partition of X.*

Proof. If $z \in X$, then $z \in [z]$, therefore z is in at least one equivalence class.
If $z \in [x]$ and $z \in [y]$, then $[x] \cap [y] \neq \emptyset$ therefore $[x] = [y]$ (as shown previously).
Therefore z is in at most one equivalence class. □

Theorem 1.2.3. *Let R_j form a partition of X. Say that $x \sim y$ means $x, y \in R_j$ for some j. Then $\sim$ is an equivalence relation on X.*

Proof.

- $x \in X$, so $x \in R_j$ for some j $implies x, x \in R_j \implies x \sim x$
- $x \sim y \iff x, y \in R_j \iff y, x \in R_j \iff y \sim x$

- $$x \sim y \; y \sim z \implies \begin{cases} x, y \in R_i \\ y, z \in R_j \end{cases} \implies y \in R_i, R_j$$
  $$\implies i = j$$
  $$\implies x, z \in R_j$$
  $$\therefore x \sim z$$

$\square$

Example of Finding Equivalence Classes
Take $X = R \times R$, and let $(a, b) \sim (c, d)$ mean $a^2 + b^2 = c^2 + d^2$. Find the equivalence class of $(0, 0)$, $(3, 4)$, (a, b)

$$\begin{aligned}[(0,0)] &= \{(x,y) : (x,y) \sim (0,0)\} \\ &= \{(x,y) : x^2 + y^2 = 0^2 + 0^2 = 0\} \\ &= \{(x,y) : x = y = 0\}\end{aligned}$$

$$\begin{aligned}[(3,4)] &= \{(x,y) : (x,y) \sim (3,4)\} \\ &= \{(x,y) : x^2 + y^2 = 3^2 + 4^2 = 25\} \\ &= \{(x,y) : \sqrt{x^2 + y^2} = 5\}\end{aligned}$$

$$\begin{aligned}[(a,b)] &= \{(x,y) : (x,y) \sim (a,b)\} \\ &= \{(x,y) : x^2 + y^2 = a^2 + b^2 = r\} \\ &= \{(x,y) : \sqrt{x^2 + y^2} = r\}\end{aligned}$$

Lecture 2

Well-defined Operations on Equivalence Classes and Number Theory

2.1 Well-defined Operations on Equivalence Classes

Consider a set X, an equivalence relation $\sim$, and an operation $\cdot$. This operation is well-defined on equivalence classes if

$$\left.\begin{aligned} x \sim y \\ w \sim z \end{aligned}\right\} \implies x \cdot w \sim y \cdot z$$

$$\left.\begin{aligned} [x] = [y] \\ [w] = [z] \end{aligned}\right\} \implies [x \cdot w] = [y \cdot z]$$

Example: Let $X = \mathbb{R} \times \mathbb{R}$, $(a, b) \sim (c, d)$ means $a^2 + b^2 = c^2 + d^2$, is addition well-defined on equivalence classes? (*Addition meaning* $(x, y) + (z, y) = (x + z, y + w)$)

$$Let \begin{cases} (a, b) \sim (c, d) \\ (e, f) \sim (g, h) \end{cases} \quad then \begin{cases} a^2 + b^2 = c^2 + d^2 \\ e^2 + f^2 = g^2 + h^2 \end{cases}$$

Now,

$$\begin{cases} (a,b)+(e,f)=(a+e,b+f) \\ (c,d)+(g,h)=(c+g,d+h) \end{cases}$$

Question: Is $(a+e)^2+(b+f)^2=(c+g)^2+(d+h)^2$?

$$(a+e)^2+(b+f)^2=a^2+2ae+e^2+b^2+2bf+f^2$$

$$(c+g)^2+(d+h)^2=c^2+2cg+g^2+d^2+2dh+h^2$$

$a^2+b^2=c^2+d^2$, and $e^2+f^2=g^2+h^2$, so

$$(a+e)^2+(b+f)^2=(c+g)^2+(d+h)^2 \iff 2ae+2bf=2cg+2dh$$

Counterexample: Take

$$(a,b)=(c,d)=(1,2)$$

$$(e,f)=(3,4) \quad (g,h)=(4,3)$$

So no, addition is not well defined.

Another Example: TBC.

2.2 Number Theory

Fact 2.2.1. *Every non-empty set $S \subseteq \mathbb{N}$ has a minimum element d in S*

Proposition 2.2.1. *Let $a,b \in \mathbb{Z}$, $b>0$, then $\exists!\ q,r \in \mathbb{Z}$ with $a=bq+r$, $0 \le r < b$*

Proof. (Existence) Let $S=\{a-bx : x \in \mathbb{Z}, a-bx \ge 0\}$. $\emptyset \ne S \subseteq \mathbb{N}$, so S has a minimum element.
Let

$$\begin{cases} r=min(S) \\ q=\frac{a-r}{b} \end{cases}$$

$r=a-bd$, $d \in \mathbb{Z}$, then $bq+r=b(\frac{a-r}{b})+r=a-r+r=a$.

If $b \le r$, then $0 \le r-b < r$, which contradicts the minimality of r.

(Uniqueness) Say $a = bq + r = bp + s$, $0 \leq r, s < b$. Then

$$b(q - p) = s - r$$

So $s - r$ is a multiple of b, but $0 \leq r, s < b$, so it must be that $r - s = 0$, therefore $r = s$. $\square$

Lecture 3

Number Theory Cont. and Integers Modulo n

3.1 More Number Theory

Definition 3.1.1. *$m \mid n$ means $\exists x \in \mathbb{Z}$ with $n = mx$*

Definition 3.1.2. *Let $a, b \in \mathbb{Z}$. If d is a positive integer with $d \mid a$ and $d \mid b$, if $c \mid a$ and $c \mid b$, then $c \mid d$, then d is a gcd of a and b.*

Theorem 3.1.1. *For every $a, b \in \mathbb{Z}$, $\exists\,!$ gcd d. Furthermore, $\exists x, y \in \mathbb{Z}$, $d = ax + by$. Furthermore, d is the largest common divisor of a,b*

Proof. Let $S = \{ax + by : x, y \in \mathbb{Z}, ax + by > 0\}$. $S \subseteq \mathbb{N}$, so $\exists\,!$ minimum element d in S.
Write

$$
\begin{aligned}
a &= dq + r && (0 \leq r < d) \\
a &= (ax + by)q + r && (\text{some } x, y \in \mathbb{Z}) \\
r &= a(1 - qx) + b(-qy) \\
r &= ax' + by' && (x' = 1 - qx,\ y' = -qy) \\
0 \leq r = ax' + by' &< d
\end{aligned}
$$

So. either $r = 0$ or $r \in S$ but not both, but $r < d$ which is the minimum of the set. Therefore $r \notin S$. So $r = 0$ and $d \mid a$. Same argument with

$b = dq + r \implies d \mid b$.
Now suppose $c \mid a$ and $c \mid b$, then $a = a'c$ and $b = b'c$, $a', b' \in \mathbb{Z}$.

$$d = ax + by = a'cx + b'cy = c(a'x + b'y$$

So, $c \mid d$. □

Corollary 3.1.1. *If $gcd(a, b) = 1$, then $\exists x, y$ such that $ax + by = 1$.*

Proof. Same as the previous proof, in the case that $gcd(a, b) = 1$. □

Corollary 3.1.2. *If $gcd(a, b) = d$, then $\{ax + by : x, y \in \mathbb{Z}\} = d \cdot n$, $\forall n \in \mathbb{Z}$.*

Proof. No proof was provided in the notes I guess. :P □

Definition 3.1.3 (Least Common Multiple)**.** *Let $a, b \in \mathbb{Z}$. If m is a positive integer with*

- *$a \mid m$ and $b \mid m$*
- *if $a \mid n$ and $b \mid n$, then $m \mid n$*

then m is a lcm of a, b.

Theorem 3.1.2. *For every a, b, $\exists\ !$ lcm m.*

Definition 3.1.4. $p \in \mathbb{Z}$ $p > 1$

- *p is irreducible if the only positive divisors of p are 1 and p*
- *p is prime if whenever $p \mid ab$, then $p|a$ or $p|b$*

Proposition 3.1.1. *p is prime $\implies$ p is irreducible*

Proof. Say p is not irreducible, $p = ab$, and $1 < a, b < p$. Then $p \nmid a$ and $p \nmid b$. □

Proposition 3.1.2. *p is irreducible $\implies$ p is prime*

Proof. $p \mid ab \implies ab = mp$ for some $m \in \mathbb{Z}$. Say $p \nmid a$, since p is irreducible $gcd(a, p) = 1$. So $\exists s, t$ such that $as + pt = 1$.

$$b = b(as + pt) = abs + bpt \qquad = mps + bpt = (ms + bt)p$$

Therefore b is a multiple of p, so $p \mid b$. □

3.2 Prime Factorization

Theorem 3.2.1. $n \in \mathbb{Z}$ $n > 1$

$$\exists! \begin{cases} p_1 p_2 \dots p_s, & \textit{distinct primes} \\ e_1 e_2 \cdots e_s, & \textit{positive integers} \end{cases}$$

With

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

Proof. Proof was omitted. $\square$

Prime Factorization Gives GCD:

Example:

$$a = 2 \cdot 5 \cdot 7^{10} \cdot 13 = 2^{1} \cdot 3^{0} \cdot 5^1 \cdot 7^{10} \cdot 13^1 \cdot 17^{0}$$

$$b = 2 \cdot 3^2 \cdot 7^2 \cdot 17 = 2^1 \cdot 3^2 \cdot 5^{0} \cdot 7^{2} \cdot 13^{0} \cdot 17^1$$

$$gcd(a, b) = 2^1 \cdot 7^2$$

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

$$b = q_1^{F_1} \cdot q_2^{F_2} \cdots q_s^{F_s}$$

$$\forall \text{ prime } p, \text{ define } g(p) = min \begin{cases} e_i & \text{if } p = p_i \\ f_j & \text{if } p = q_j \, . \\ 0 \end{cases}$$

Then,

$$gcd(a, b) = \prod_{prime\ p} p^{g(p)}$$

Prime Factorization Gives LCM:

Example:

$$a = 2 \cdot 5 \cdot 7^{10} \cdot 13 = 2^1 \cdot 3^0 \cdot 5^{1} \cdot 7^{10} \cdot 13^{1} \cdot 17^0$$

$$b = 2 \cdot 3^2 \cdot 7^2 \cdot 17 = 2^{1} \cdot 3^{2} \cdot 5^0 \cdot 7^2 \cdot 13^0 \cdot 17^{1}$$

$$lcm(a, b) = 2^1 \cdot 3^2 \cdot 5^1 \cdot 7^{10} \cdot 13^1 \cdot 17^1$$

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

$$b = q_1^{F_1} \cdot q_2^{F_2} \cdots q_s^{F_s}$$

$$\forall \text{ prime } p, \text{ define } l(p) = min \begin{cases} e_i & \text{if } p = p_i \\ f_j & \text{if } p = q_j \\ 0 & \end{cases}.$$

Then,

$$gcd(a, b) = \prod_{prime\ p} p^{l(p)}$$

3.3 Integers Modulo n

TBC.

Lecture 4

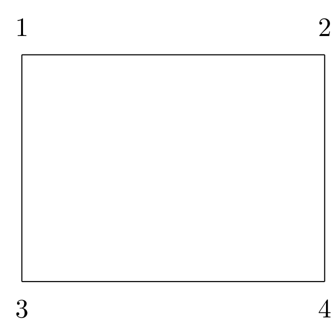
Operations on $\mathbb{Z}_n$, Symmetries, and Groups

4.1 Arithmetic Modulo n

TBC.

4.2 Symmetries

Consider the symmetries of a rectangle.



The notation for functions is

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ f(1) & f(2) & f(3) & f(4) \end{pmatrix}$$

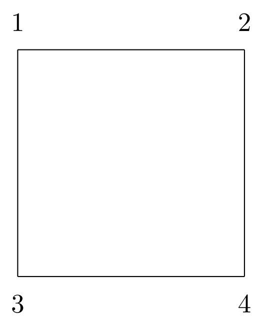
$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Claim: $\{\epsilon, \rho, \alpha, \beta\}$ are *all* the symmetries of a rectangle.

Proof. DGD Question - Will add later. $\square$

Consider the symmetries of a square.



$$\begin{array}{ccc}
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} & \epsilon & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} & 90^\circ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} & 180^\circ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} & 270^\circ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}
\end{array}$$

4.2.1 Properties of Symmetries

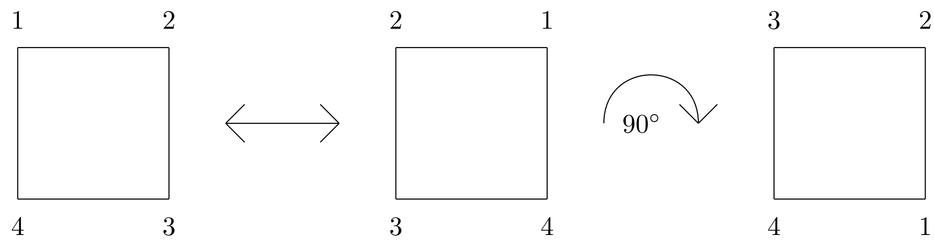
$S = \{\alpha, \beta, \dots\}$ symmetries of some objection, with the operation composition.
Properties:

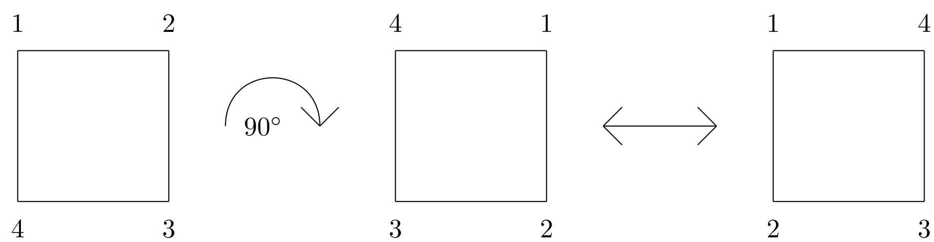
- $\alpha \circ \beta$ is a symmetry $\forall \alpha, \beta \in S$
- $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ $\forall \alpha, \beta, \gamma \in S$

- $\exists \epsilon \in S$ such that $\epsilon \circ \alpha = \alpha \circ \epsilon = \alpha \ \forall \alpha \in S$
- $\forall \alpha \in S$, $\exists \beta \in S$, such that $\alpha \circ \beta = \beta circ \alpha = \epsilon \ \forall \alpha, \beta \in S$

Note: we often write $\alpha\beta$ instead of $\alpha \circ \beta$

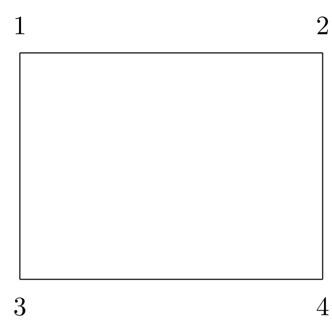
Example: $S =$ symmetries of some object, is $gh = hg \ \forall g, h \in S$?. **Answer:** For a rectangle, yes. But for a square, no.





These symmetries do not compute, so $gh \neq hg$.

4.2.2 Generating Sets



$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Check: $\alpha\beta = \rho$, $\alpha^2 = \epsilon$. So $\forall g \in S$, g can be written in terms of α, β.

We say that $\{\alpha, \beta\}$ generates S

4.3 Groups

Let S be some set with some operation $\cdot$. Then $(S, \cdot)$ is a group if

- **Closure:** $ab \in S \ \forall a, b \in S$
- **Associativity:** $(ab)c = a(bc) \ \forall a, b, c \in S$
- **Identity:** $\exists \epsilon \in S$ such that $x\epsilon = \epsilon x = x \ \forall x \in S$
- **Inverses:** $\forall x \in S$, $\exists y \in S$ such that $xy = yx = \epsilon$

Examples:

- Symmetries of an object form a group.
- $(\mathbb{R}, +)$ forms a group.
- $(\mathbb{R} \setminus \{0\}, \cdot)$ forms a group.
- $(\mathbb{Z}, +)$ forms a group.
- $(\mathbb{Z}, \cdot)$ does not form a group since inverses are typically not integers.
- $(\mathbb{Z}_n, +)$ forms a group.
- $(\mathbb{Z}_n \setminus \{0\}, \cdot)$ forms a group.

Lecture 5

More Examples of Groups, Groups of Units of $\mathbb{Z}_n$

TBC.

Lecture 6

Basic Properties of Groups, Products of Groups, Isomorphisms

Examples were left out I may come back to finish

6.1 Basic Properties of Groups

Proposition 6.1.1. *In every group, the identity is unique.*

Proof. Suppose a, b are identities, so

$$\left.\begin{aligned} \text{ax} = \text{xa} = \text{x} \\ \text{bx} = \text{xb} = \text{x} \end{aligned}\right\} \forall x$$

Because b is an identity, we have $a = ab$, and since a is an identity, we have $ab = b$. So

$$a = ab = b$$

$$\therefore a = b$$

$\square$

Proposition 6.1.2. *In every group, the equation $ax = b$ has a unique solution x for all a, b*

Proof. There was no proof :(☐

Proposition 6.1.3. *In every group,* $ab = ac \implies b = c$

Proof. Again, no proof :(☐

Note: For matricies it is not the same, $AB = AC \not\Longrightarrow B = C$

Proposition 6.1.4. *In every group,* $(ab)^{-1} = b^{-1}a^{-1}$

Proof.

$$\begin{aligned} (ab)(b^{-1}a^{-1}) &= a(bb^{-1})a^{-1} \\ &= a\epsilon a^{-1} \\ &= aa^{-1} \\ &= \epsilon \end{aligned} \qquad \begin{aligned} (b^{-1}a^{-1})(ab) &= b^{-1}(a^{-1}a)b \\ &= b^{-1}\epsilon b \\ &= b^{-1}b \\ &= \epsilon \end{aligned}$$

☐

Proposition 6.1.5. *In every group,* $(a^{-1})^{-1} = a$

Proof. Since a^{-1} is the inverse of a, we have

$$aa^{-1} = a^{-1}a = \epsilon$$

but then,

$$a^{-1}a = aa^{-1} = \epsilon$$

So a is the inverse of a^{-1} ☐

Proposition 6.1.6. *In every group, if* $xy = x$*, for some* x, y*, then* $y = \epsilon$*. So if* y *behaves as the identity just once, then* y *is the identity.*

Proof. No proof again :P. ☐

Proposition 6.1.7. *In every group, if* $xy = \epsilon$*, for some* x, y*, then* $y = x^{-1}$*. So if* y *behaves like* x^{-1} *on one side, then* y *is* x^{-1}

Proof. No proof D: ☐

Proposition 6.1.8. *In every group, the Cayley table has exactly one row and column that matches the headers, and no other row or column mathes the header even once.*

Proof. Start by taking G to be some group, then let $x, y \in G$. And let H be a subgroup of G.

just kidding no proof. $\square$

Proposition 6.1.9. *In every group, every row and column of the Cayley table contains each element exactly once.*

Proof. Why does the prof include a spot for the proof. $\square$

6.1.1 Small Groups

- Say G has one element $G = \{x\}$
  Closure: $x \cdot x = x$
  Identity: $x = \epsilon$
  Inverse: $x^{-1} = x$

$$\begin{array}{c|c} \cdot & x \\ \hline x & x \end{array}$$

- Say G has two elements, it must have an identity so $G = \{\epsilon, x\}$ If $xx = x$, $x = \epsilon$, this is a contradiction, So $xx = \epsilon$

$$\begin{array}{c|cc} \cdot & \epsilon & x \\ \hline \epsilon & \epsilon & x \\ x & x & \epsilon \end{array}$$

- Say G has three elements. $G = \{\epsilon, x, y\}$

$$\begin{array}{c|ccc} \cdot & \epsilon & x & y \\ \hline \epsilon & \epsilon & x & y \\ x & x & y & \epsilon \\ y & y & \epsilon & x \end{array}$$

$$x\epsilon = x \implies xy \neq x$$

$$\epsilon y = y \implies xy = \neq y$$

So $xy = \epsilon$

$$x\epsilon = x \implies xx \neq x$$
$$xy = \epsilon \implies xx \neq \epsilon$$

So $xx = y$

- Say G has 4 elements. Assignment Question!

6.2 Products of Groups

G, H are groups, define

$$G \times H = \{(g, h) : g \in G, h \in H\}$$

$$(x, a) \cdot (y, b) = (x \cdot y, a \cdot b)$$

$$G_1 \times G_2 \times \cdots G_k = \{(g_1, g_2, \ldots, g_n : g_j \in G_j\}$$

To reiterate, operations are done by component according to the operations of the group. i.e Suppose we have a group $G = (A, +)$ and $H = (B, \cdot)$ and $g, a \in G$, $h, b \in H$.

$$(g, h) \times (a, b) = (g + a, h \cdot b)$$

Proposition 6.2.1. *The product of groups is a group.*

Proof. Exercise. □

Example:

6.3 Isomorphisms

Suppose $\phi : G \to H$ is a bijection between two groups with the property

$$\phi(xy) = \phi(x)\phi(y)$$

Then ϕ is an isomorphism of $G \cong H$. So

$$G : x \cdot y = z \implies H : \phi(x) \cdot \phi(y) = \phi(z)$$

$\cdot$	y
x	z

$\cdot$	y'
x'	z'

$$x' = \phi(x) \qquad y' = \phi(y) \qquad z' = x'y' = \phi(z)$$

Start with G's Cayley table, change the names (symbols, consistently) and permute the rows and columns. This gives H's Cayley table.

Example:

$\mathbb{Z}_2 = \{0, 1\}$

$+$	0	1
0	0	1
1	1	0

$\mathbb{Z}^\times = \{-1, 1\}$

$\cdot$	1	-1
1	1	-1
-1	-1	1

$G = (\{\mathbb{Z}^+, \mathbb{Z}^-\}, \cdot)$

$\cdot$	$+$	$-$
$+$	$+$	$-$
$-$	$-$	$+$

$$\mathbb{Z}_2 \cong \mathbb{Z}^\times \cong G$$

Proposition 6.3.1. *All groups with two elements are isomorphic*

Proof. If G has two elements, then its Cayley table looks like

$\cdot$	ϵ	x
ϵ	ϵ	x
x	x	ϵ

Except they may use different symbols and have reordered rows/columns, so they are all isomorphic. $\square$

Lecture 7

Automorphisms, Subgroups

7.1 Automorphisms

Example: Let $H =$ *symmetries of a rectangle* and $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(x, y) : x \in \mathbb{Z}_2,\ y \in \mathbb{Z}_2\}$

$$H = \{\epsilon, \alpha, \beta, \rho\} \textit{ with composition}$$

$$K = \{00, 01, 10, 11\} \textit{ with addition in } \mathbb{Z}_2$$

H	ϵ	α	β	ρ
ϵ	ϵ	α	β	ρ
α	α	ϵ	ρ	β
β	β	ρ	ϵ	α
ρ	ρ	β	α	ϵ

G	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

$$\phi \begin{cases} \epsilon \to 00 \\ \alpha \to 01 \\ \beta \to 10 \\ \rho \to 11 \end{cases} \qquad \textit{or} \qquad \phi \begin{cases} \epsilon \to 00 \\ \alpha \to 01 \\ \beta \to 11 \\ \rho \to 10 \end{cases}$$

In fact, all we need for the isomorphism is $\epsilon \to 00$, we can have $\alpha, \beta, \rho \to 01, 10, 11$ in any order.

An automorphism of G is an isomorphism $G \to G$, this is a symmetry group of G. The set of all automorphisms of G is a group we call $aut(G)$, the automorphism group of G.

H	ϵ	α	β	ρ
ϵ	ϵ	α	β	ρ
α	α	ϵ	ρ	β
β	β	ρ	ϵ	α
ρ	ρ	β	α	ϵ

H	ϵ	α	ρ	β
ϵ	ϵ	α	ρ	β
α	α	ϵ	β	ρ
ρ	ρ	β	ϵ	α
β	β	ρ	α	ϵ

Let ϕ be any bijection $\{\epsilon, \alpha, \beta, \rho\} \to \{\epsilon, \alpha, \rho\, \beta\}$ with $\phi(\epsilon) = \epsilon$. Then ϕ is an automorphism of H.

Exercise: Let $G =$ *the symmetries of an equilateral triangle*. Show that

$$aut(H) \cong G$$

7.2 Quaternions

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$\pm$ and 1 operate as expected. And

$$i^2 = j^2 = k^2 = ijk = -1$$

Q_8	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	j	j
$-i$	$-i$	i	1	1	$-k$	k	$-j$	j
j	j	$-j$	$-k$	k	-1	1	$-i$	i
$-j$	$-j$	j	k	$-k$	1	-1	i	$-i$
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

- **Closure:** Yes, Q_8 is closed.

- **Identity:** 1 is the identity for Q_8.
- **Inverse:** Every column has the identity (1), so an inverse exists for every element in Q_8.
- **Associativity:** Consider the set of matrices M_8 with entries in $\mathbb{C}$

$$M_8 = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right\}$$

  And the function $\phi : M_8 \to Q_8$

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

  $\phi : M_8 \to Q_8$ is a bijection

$$\phi(ab) = \phi(a)\phi(b)$$

  Because M_8 is a set of matrices, it is closed, associative, has an identity and has inverses, therefore M_8 is a group. Q_8 is isomorphic to M_8, so it follows that it is also a group.

Therefore, Q_8 is closed, has identity, has inverses and is associative.

7.3 Subgroups

Consider the following

- G is a group with operation $\cdot$
- H is a subset of G
- H is a group with the same operation $\cdot$

Then H is a subgroup of G. We denote subgroups as $H \leq G$ or $H < G$

$$(\mathbb{Z}, +) < (\mathbb{Q}, +) < (\mathbb{R}, +) < (\mathbb{C}, +)$$

Example:

$$(\mathbb{Z}_3, +) \not\leq (\mathbb{Z}_5, +)$$

This is the case because

$$\mathbb{Z}_3 = \{0, 1, 2\} = \{[0], [1], [2]\} \not\subseteq \{[0], [1], [2], [3], [4]\} = \{0, 1, 2, 3, 4\} = \mathbb{Z}_5$$

These sets are equivalence classes **not** numbers so they are not subsets of each other.

7.3.1 Subgroup Test

Proposition 7.3.1. *Suppose H is a subset of G, if $H \neq \emptyset$*

$$x, y \in H \implies xy \in H$$

$$x \in H \implies x^{-1} \in H$$

then H is a subgroup.

Proof. Show that H is a group

- **Closure:** Is given.
- **Associative:** G is associative so any subset "inherits" associativity.
- **Identity:** Let ϵ_g be the identity in G. $\epsilon_a \cdot a = a \ \forall a \in H$. $\exists a \in H$, so $a^{-1} \in H$, since H is a subset of G, $a, a^{-1} \in G$, therefore $a \cdot a^{-1} = \epsilon_g \in H$.
- **Inverse:** Given

□

Proposition 7.3.2. *H a subgroup of $G \implies \epsilon_g \in H$ and so $\epsilon_H \in G$*

Proof. $H \neq \emptyset$, so let $x \in H$, then $x^{-1} \in H$, then $x \cdot x^{-1} = \epsilon_G \in H$. Furthermore

$$\epsilon_G \cdot h = h \cdot \epsilon_G = h \ \forall h \in H$$

Since $H \subseteq G$,and H has a unique identity, then $\epsilon_G = \epsilon_H$ □

7.3.2 Alternative Versions of Subgroup Test

Suppose H is a subset of G, if

- $H \neq \emptyset$

- $x, y \in H \implies xy \in H$
- $x \in H \implies x^{-1} \in H$

then H is a subgroup.

Lecture 8

Lattices and Cyclic Groups

Recall: H is a subgroup of G if

- $H \subseteq G$
- They have the same operation (Cayley table of H is obtained by deleting rows/columns from G
- H is a group

Subgroup Test: If $H \subseteq G$ with the same operation and H is not empty,

$$x, y \in H \implies xy \in H$$

$$x \in H \implies x^{-1} \in H$$

then H is a subgroup of G.

8.1 Find all subgroups of $(\mathbb{Z}, +)$

Say H is a subgroup of $\mathbb{Z}$, $H \neq \{0\}$. Let n be the smallest positive integer in H, then

$$\{\dots, -n, 0, n, 2n, 3n, \dots\} \subseteq H$$

$$n\mathbb{Z} = \{nK : k \in Z\} \subseteq H$$

Suppose $x \in H \setminus n\mathbb{Z}$, then write $x = qn + r$ for $0 \leq r < n$. By closure, we have

$$x - qn = r \in H$$

But, this contradicts the minimality of n unless $r = 0$, but if $r = 0$, then $x \in n\mathbb{Z}$. Therefore, $H = n\mathbb{Z}$

Subgroups of $\mathbb{Z}$: $n\mathbb{Z}\ \forall n \in \mathbb{Z}$, $(n = 0 \implies H = \{0\})$

8.2 Symmetries of a Square

Lemma 8.2.1. *There are at most eight symmetries of a square.*

Proof. Let γ be a symmetry, γ maps corners to corners.

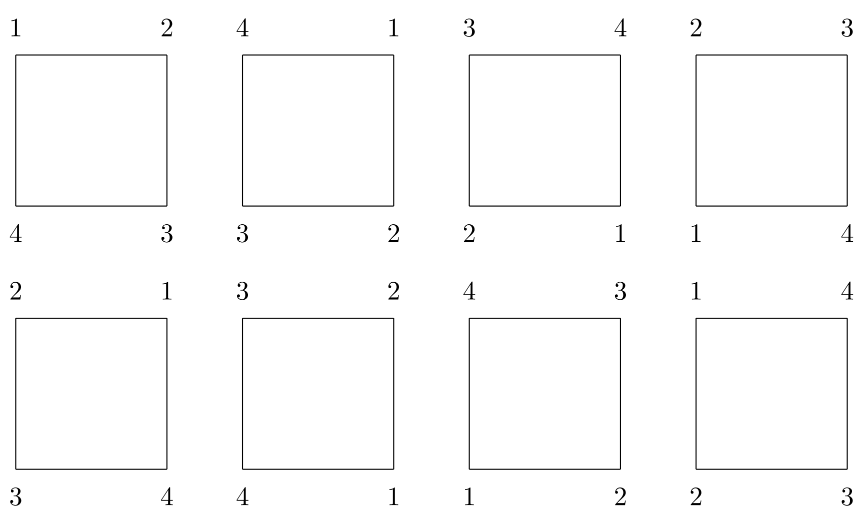
- $\gamma(1)$ has at most four possibilities, then $\gamma(2)$ must be one of the corners adjacent to $\gamma(1)$
- $\gamma(2)$ has at most two possibilities, then $\gamma(4)$ must be the other corner adjacent to $\gamma(1)$
- $\gamma(4)$ has at most one possibility, then $\gamma(3)$ must be $\{1,2,3,4\}\backslash\{\gamma(1),\gamma(2),\gamma(3),\gamma(4)\}$
- $\gamma(3)$ has at most one possibility

So, we have $4 \cdot 2 \cdot 1 \cdot 1 = 8$ possibilities. $\square$

Question: Do all possibilities work?

Lemma 8.2.2. *There are at least eight symmetries of a square*

Proof. Consider the symmetries of a square.



$\square$

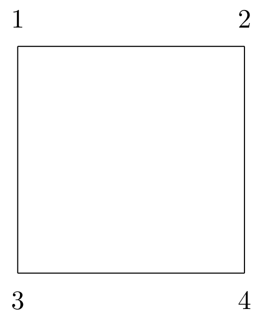
Let

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Proposition 8.2.1.

$$\rho\mu = \mu\rho^{-1} = \mu\rho^3$$

Proof. Consider the square and function μ, ρ.



$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\begin{aligned} \rho\mu &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \end{aligned} \qquad \begin{aligned} \mu\rho^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \end{aligned}$$

The 2 functions are equal. $\square$

Example:

$$\rho^2\mu\rho\mu\rho^3 = \mu\rho^6\rho\mu\rho^3 = \mu\rho^7\mu\rho^3 = \mu\mu\rho^{21}\rho^3 = \mu\rho^{24} = \epsilon$$

$$\mu\rho\mu\rho^2\mu\rho = \mu\mu\rho^3\rho^2\mu\rho = \mu^2\rho^5\mu\rho = \rho\mu\rho = \mu\rho^3\rho = \mu\rho^3\rho = \mu$$

Corollary 8.2.1.

$$\begin{aligned} G &=< \mu, \rho : \mu^2 = \epsilon, \rho^4 = \epsilon, \rho\mu = \mu\rho^3 > \\ &= \{\mu^i \rho^j : 0 \le i \le 1, 0 \le j \le 3\} \\ &= \{\rho^i \mu^i : 0 \le i \le 1, 0 \le j \le 3\} \end{aligned}$$

Proof. Any sequence of μ's and ρ's can be written as $\mu^s \rho^t$ using $\rho\mu = \mu\rho^3$ ($\rho^t \mu^s$ using $\mu\rho = \rho^3\mu$) reduce powers on μ and ρ using $\mu^2 = \epsilon$ $\rho^4 = \epsilon$

$$G = \{\mu^i \rho^j : 0 \le i \le 1,\ 0 \le j \le 3\}$$

These are all distinct since $|G| = 8$ so the relations $\mu^2 = \epsilon$ $\rho^4 = \epsilon$ $\rho\mu = \mu\rho^3$ are sufficient to characterize G $\square$

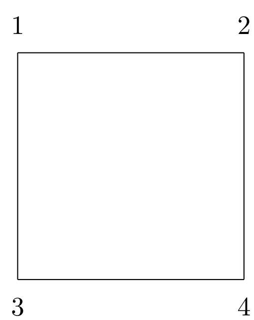
Compare:

$$F =< \alpha, \beta : \alpha^2 = \epsilon, \beta^4 = \epsilon >$$

$\alpha\beta, \alpha\beta\alpha, \alpha\beta\alpha\beta, \ldots$ are all distinct

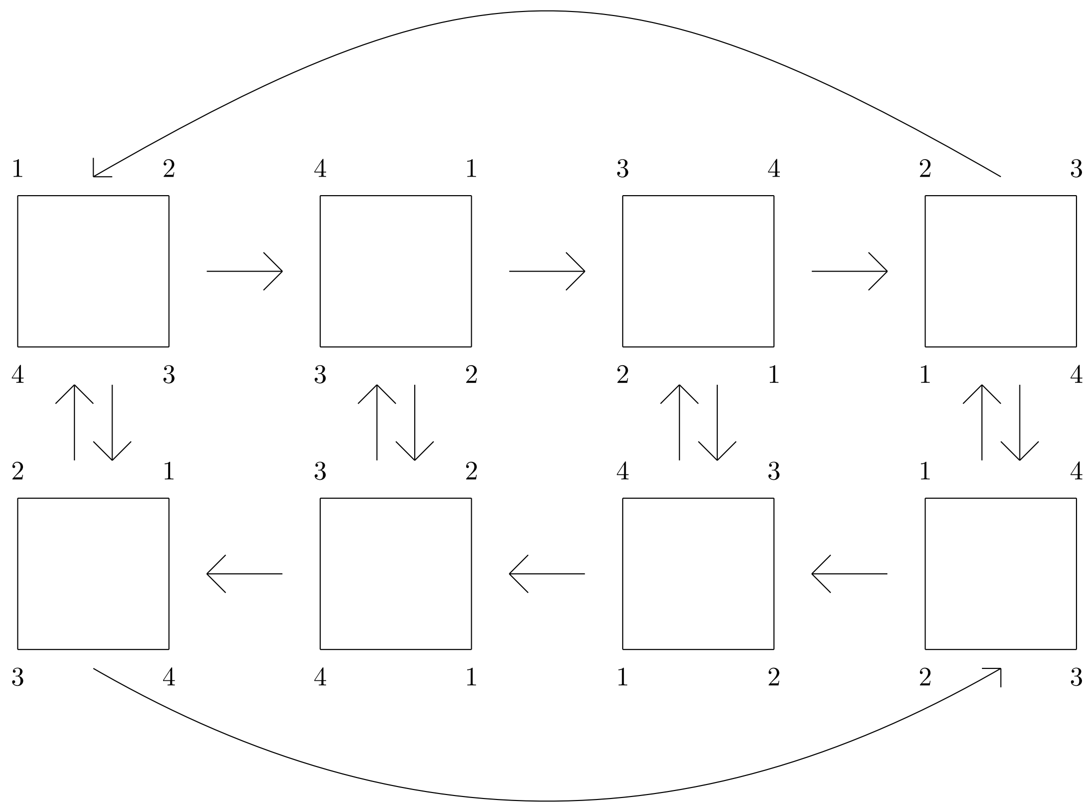
$$\alpha^3 \beta^7 \alpha\beta = \alpha\beta^3\alpha\beta$$

$$|F| = \infty$$



$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\begin{aligned} G &= \{\mu^i \rho^i : 0 \le i \le 1, 0 \le j \le 3\} \\ &= \{\rho^i \mu^i : 0 \le i \le 1, 0 \le j \le 3\} \end{aligned}$$



Elements of G are symmetries of a square, they also permute G itself! Buy they are not symmetries of G.

$$\mu \cdot \rho = \mu\rho$$
$$\mu\mu\mu\rho = \mu(\mu)\mu(\rho) \neq \mu(\mu\rho) = \mu\mu\rho$$

8.2.1 Subgroups of Symmetries of a Square

$$G = Sym(\square) = \{\mu^i\rho^i : 0 \leq i \leq 1, 0 \leq j \leq 3\}$$

$$< \epsilon > = \{\epsilon\}$$
$$< \mu > = \{\epsilon, \mu\}$$
$$< \rho > = \{\epsilon, \rho, \rho^2, \rho^3\}$$
$$< \rho^2 > = \{\epsilon, \rho^2\}$$
$$< \rho^3 > = \{\epsilon, \rho^3, \rho^2, \rho\}$$
$$< \mu\rho > = \{\epsilon, \mu\rho\}$$
$$< \mu\rho^2 > = \{\epsilon, \mu\rho^2\}$$
$$< \mu\rho^3 > = \{\epsilon, \mu\rho^3\}$$

$$< \mu, \rho > = G =< \mu, \rho^3 >$$
$$< \mu, \rho^2 > = \{\epsilon, \mu, \rho^2, \mu\rho^2\}$$

$$< \mu, \mu\rho > = \{\epsilon, \mu, \rho, \dots\} = G$$
$$< \mu, \mu\rho^3 > = \{\epsilon, \mu, \rho^3, \dots\} = G$$
$$< \mu, \mu\rho^2 > = \{\epsilon, \mu, \rho^2, \mu\rho^2\}$$

$$< \rho, \mu\rho > = \{\epsilon, \mu, \rho, \dots\} = G$$
$$< \rho, \mu\rho^3 > = \{\epsilon, \mu, \rho, \dots\} = G$$
$$< \rho, \mu\rho^2 > = \{\epsilon, \mu, \rho, \dots\} = G$$
$$< \rho^2, \mu\rho^2 > = \{\epsilon, \rho^2, \mu\rho, \mu\rho^3\} = G$$
$$< \rho^2, \mu\rho^3 > = \{\epsilon, \rho^2, \mu\rho^3, \mu\rho\} = G$$
$$< \rho^2, \mu\rho^2 > = \{\epsilon, \rho^2, \mu\rho^2, \mu\} = G$$
$$< \mu\rho, \mu\rho^2 > = \{\epsilon, \rho, \mu, \dots\} = G$$
$$< \mu\rho, \mu\rho^3 > = \{\epsilon, \mu\rho, \mu\rho^3, \rho^2\} = G$$
$$< \mu\rho^2, \mu\rho^3 > = \{\epsilon, \rho, \mu, \dots\} = G$$

Lecture 9

Cyclic Groups

9.1 Cyclic Groups

G is cyclic if $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ for some $g \in G$. g is a generator of G, there could be other generators. For addition,

$$G\langle g \rangle = \{kg : k \in \mathbb{Z}\}$$

The order of g is the smallest positive integer n with $g^n = \epsilon$, written as $|g|$. For addition, it's the smallest positive integer n with $ng = \epsilon$.

Proposition 9.1.1. *G is cyclic $\implies$ G is abelian*

Proof. Since G is cyclic, then $G = \langle g \rangle$ for some $g \in G$, take $x, y \in G$. Then $x = g^s$ and $y = g^t$. So

$$xy = g^s g^t = g^{s+t} = g^{t+s} = g^t g^s = yx$$

□

However, G being abelian $\not\Longrightarrow$ G is cyclic. **Examples:** Are the following in cyclic? Find generators, and all orders

Q_8:

- $\langle 1 \rangle = \{1\}$ Order 1
- $\langle -1 \rangle =$ {-1,1} Order 2

- $\langle i\rangle = \{i, -1, -i, 1\}$ Order 4
- $\langle -i\rangle = \{-i, -1, i, 1\}$ Order 4
- $\langle \pm j\rangle = \{\pm j, -1, \mp j, 1\}$ Order 4
- $\langle \pm k\rangle = \{\pm k, -1, \mp k, 1\}$ Order 4

Not cyclic

$\mathbb{Z}$:

- $\langle 1\rangle = \{k \cdot 1 : k \in \mathbb{Z}\} = \mathbb{Z}$ Order is ∞, so no finite order

Are there other generators? Consider -1

- $\langle -1\rangle = \{k \cdot (-1) : k \in \mathbb{Z}\} = \mathbb{Z}$

$\mathbb{Z}_5$:

- $\langle 1\rangle = \{1, 2, 3, 4, 5 = 0\}$ Order 5
- $\langle 2\rangle = \{2, 4, 6 = 1, 3, 5 = 0\}$ Order 5
- $\langle -2\rangle = \langle 3\rangle = \{3, 1, 4, 2, 5 = 0\}$ Order 5
- $\langle -1\rangle = \langle 4\rangle = \{4, 3, 2, 1, 5 = 0\}$ Order 5
- $\langle 0\rangle = \{0\}$

Therefore $\langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 4\rangle$ generate the group, so it is cyclic. **$\mathbb{Z}_9^\times$:**

- $\langle 1\rangle = \{1\}$ Order 1
- $\langle 2\rangle = \{2, 4, 8, 16 = 7, 14 = 5, 10 = 1\}$ Order 6
- $\langle -4\rangle = \langle 4\rangle = \{4, 7, 1\}$ Order 3
- $\langle -2\rangle = \langle 5\rangle = \{5, 7, 8, 4, 2, 1\}$ Order 6
- $\langle -1\rangle = \langle 8\rangle = \{8, 1\}$ Order 2

Therefore $\langle 2\rangle$ and $\langle 5\rangle$ generate the group, so it is cyclic.

$\mathbb{Z}_8^\times$:

- TBC

$\mathbb{Q}$**:**

- $\langle 1 \rangle = \mathbb{Z}$
- $\langle 0 \rangle = \{0\}$
- $\langle q \rangle = q\mathbb{Z}, q \in \mathbb{Q}$

Therefore not cyclic.

$\mathbb{R}$**:**

- $\langle 1 \rangle = \mathbb{Z}$
- $\langle 0 \rangle = \{0\}$
- $\langle r \rangle = r\mathbb{Z}$

Therefore not cyclic.

Note: $q\mathbb{Z} \cong \mathbb{Z}$ *and* $r\mathbb{Z} \cong \mathbb{Z}$

$\mathbb{Z}_2 \times \mathbb{Z}_4$**:**

- $\langle 00 \rangle = \{00\}$
- $\langle 01 \rangle = \{01, 02, 03, 00\}$
- $\langle 02 \rangle = \{02, 00\}$
- $\langle 03 \rangle = \{03, 02, 01, 00\}$
- $\langle 00 \rangle = \{00\}$

TBC $\mathbb{Z}_2 \times \mathbb{Z}_3$**:**

- $\langle 00 \rangle = \{00\}$
- $\langle 01 \rangle = \{01, 02, 00\}$
- $\langle 02 \rangle = \{02, 01, 00\}$
- $\langle 10 \rangle = \{10, 00\}$
- $\langle 11 \rangle = \{11, 02, 10, 01, 12, 00\}$
- $\langle 12 \rangle = \{12, 01, 10, 02, 11, 00\}$

Proposition 9.1.2. *G is cyclic $\implies$ all subgroups of G are cyclic*

Proof. Let $G = \langle a \rangle = \{a^i : i \in \mathbb{Z}\}$. Let H be a sub group of G. $H = \{a^i : \textit{some } i \in \mathbb{Z}\}$, could be $H = \{a^0\} = \{\epsilon\}$. Let

$$n = min\{k : a^k \in H, k > 0\}$$

$$\langle a^n \rangle = \{(a^n)^k : k \in \mathbb{Z}\} = \{a^{kn} : k \in \mathbb{Z}\} = \{a^k : k \in n\mathbb{Z}\}$$

$$\langle a^n \rangle \leq H \leq G$$

□

TBC

$$\cdots$$

but: Let $G = \langle g \rangle = \{g, g^2, g^3, \ldots, g^n = \epsilon\}$. Then $|G| = |g| = n$

Proposition 9.1.3. *Suppose $|a| = n < \infty$, then*

$$a^j = \epsilon \iff n|j$$

In otherwords,

$$\{j : a^j = \epsilon\} = n\mathbb{Z}$$

Furthermore,

$$a^s = a^t \iff n|s - t$$

Example: $|a| = 5$

$$a^5 = a^{10} = a^{-15} = a^{1005} = \cdots = \epsilon$$

$a^j \neq \epsilon$ when j is not a multiple of 5.

Proof. $\impliedby$ if $n|j$ then $j = tn$ for some $t \in \mathbb{Z}$

$$a^j = a^{tn} = (a^n)^t = \epsilon^t = \epsilon$$

$\implies$: if $a^j = \epsilon$, then write $j = qn + r$ for $0 \leq r < n$

$$a^r = a^{j-qn} =^j (a^n)^{-q} = \epsilon(\epsilon^{-q}) = \epsilon$$

but a^n is the smallest positive integer with a^n TBC □

Also,
$$a^s = a^t \iff a^{s-t}\epsilon \iff n|s-t$$

Corollary 9.1.1. $|a| = |b|$ *is equivalent to*
$$a^j = \epsilon \iff b^j = \epsilon$$

Proposition 9.1.4. *Suppose* $a \in G$, $|a| = n < \infty$, $k \in \mathbb{Z}$. *Then*
$$|a^k| = \frac{n}{gcd(k,m)}$$

Example: $|a| = 12$

- $\langle a^1 \rangle = \{a^1, a^2, a^3, \ldots, a^{12}\}$
- $\langle a^5 \rangle = \{a^5, a^{10}, a^3 \ldots, a^{12}\} = \langle a \rangle$
- $\langle a^4 \rangle = \{a^4, a^8, a^{12} = a^0\}$
- $\langle a^{10} \rangle = \{a^{10}, a^8, a^6, a^4, a^2, a^0\}$

Proof. Let $|a^k| = m$, then $\epsilon = (a^k)^m = a^k m$ Therefore, $n|km$ and km is a multiple of $|a|$ by the previous theorem. Let $d = gcd(kn)$ and set
$$\begin{cases} n = n'd \\ k = k'd \end{cases}$$
$$gcd(n', k') = 1$$

Since $n|km$ for some $t \in \mathbb{Z}$ we have,
$$km = tn$$
$$dk'm = tdn'$$
$$k'm = tn'$$
$$m = \frac{tn'}{k'} = \frac{t}{k'} \cdot n'$$
This must be an integer because $gcd(k', n') = 1 \implies k' \mid t$ Smallest $m \iff$ smallest t with $\frac{tn'}{k'}$ positive integer. So $\square$

Corollary 9.1.2. *Suppose* $G = \langle a \rangle$, *with* $|a| = n < \infty$, *then the generators of* G *are* $\{a^k : gcd(n,k) = 1\}$

Proof.

$$|a^k|$$

TBC $\square$

Corollary 9.1.3. *$\mathbb{Z}_n = \langle 1 \rangle$ and $|1| = a$. Generators of $\mathbb{Z}$ with addition are*

$$\{k \cdot 1 : gcd(n,k) = 1\} = \{k : gcd(n,k) = 1\} = \mathbb{Z}_n^{\times}$$

TBC

Corollary 9.1.4. *all nonzero elements of $\mathbb{Z}_n$ are generators of $\mathbb{Z}_n$ $\iff$ n is prime*

Proof. We want $|k| = \frac{n}{gcd(n,k)} = n$ for $k = 1, 2, 3, \ldots, n-1$. So $gcd(n,k) = 1$ $\square$

Lecture 10

- G cyclic means there exists $g \in G$ with $G = \langle g \rangle = \{g^k : k \in \int\}$
- The order of an element g is the smallest positive integer n with $g^n = \epsilon$
- Notation: Order of an element g is written $|g|$. Order (=size!) of a group G is written $|G|$. $|g| = \infty$ means $g^k \neq \epsilon \; \forall k \in \mathbb{Z}$
- $\{k : g^k = \epsilon\} = |g|\mathbb{Z}$ so $g^k \epsilon \iff |g|$ divides k
- $|x| = |y|$ is equivalent to $x^k = \epsilon \iff y^k = \epsilon$
- if $|g| = n < \infty$, then

$$G = \langle g \rangle = \{g, g^2, \cdots, g^n = \epsilon\}$$

$$|G| = |g|$$

$$|g^k| = \frac{n}{gcd(n,k)}$$

generators of G are exactly $\{g^k : gcd(n,k) = 1\}$

Corollary 10.0.1. *All nonzero elements of $\mathbb{Z}_n$ are generators of $\mathbb{Z}_n \iff n$ is prime.*

Proof. We want $k = \frac{n}{gcd(n,k)} = n$ for $k = 1, 2, 3, \ldots, n-1$. So $gcd(n,n)$ TBC $\square$

Theorem 10.0.1. *G has no subgroups other than $\{\epsilon\}$ and $G \iff G$ is cyclic of prime order $\iff |G|$ is prime.*

Proof. Suppose $g \in G$, then $\langle g \rangle$ is a subgroup of G. Therefore, either $\langle g \rangle = G$ or $\langle g \rangle = \{\epsilon\}$. g is a generator of G So

$$G = \{g, g^2, g^3, \ldots, g^n = \epsilon\}$$

g^k is a generator for $k = 1, 2, \ldots, n-1$ Therefore,

$$\frac{n}{gcd(n,k)} = n$$

So n is prime, therefore G is cyclic of prime order $G \cong \mathbb{Z}_n$ for n prime.

Conversely,

$$G = \{g, g^2, \ldots, g^n = \epsilon\}$$

then $S \neq \emptyset$ and $S \neq \{\epsilon\} \implies \langle S \rangle = G$. So $x \in S$, $x = g^k$ then

$$|x| = |g^k| = \frac{n}{gcd(n,k)}$$

So the only subgroups are $\{\epsilon\}$ and G □

Theorem 10.0.2. *Suppose G, H are both cyclic, $G \cong H \iff |G| = |H|$*

Proof. $(\implies)$an isomorphism is a bijection.

$(\impliedby)G = \langle a \rangle$ and $H = \langle b \rangle$, then

$$|a| = |G| = |H| = |b|$$

define

$$\phi : G \to H$$

$$\phi(a^k) = b^k$$

We have 2 cases, either the order is infinite.

$$\begin{cases} G = \{\ldots, a^-2, a^{-1}, a^0, a^1, a^2, \ldots\} \\ H = \{\ldots, a^-2, a^{-1}, a^0, a^1, a^2, \ldots\} \end{cases}$$

Or their order is finite

$$\begin{cases} G = \{a, a^2, a^3, \ldots, a^n \epsilon\} \\ H = \{b, b^2, b^3, \ldots, b^n = \epsilon\} \end{cases}$$

In both cases ϕ is a bijection.
TBC □

Subgroups of $C_n = \langle a \rangle = \{a, a^2, \ldots, a^n\}$

- C_n is cyclic, therefore all subgroups are cyclic
- $|a^k| = \frac{n}{gcd(k,n)}$
- Let $d \mid n$ then $|a^d| = \frac{n}{gcd(d,n) = \frac{n}{d}}$

So for each $d \mid n$, then $\langle a \rangle^d \cong C_{\frac{n}{d}}$ is a subgroup.

No let $k \in \{1, 2, 3 \in, n\}$. Suppose $gcd(k, n) = d$ for some $d \mid n$, then

$$k \in \{d, 2d, 3d, \ldots, \frac{n}{d}d\}$$

So $a^k \in \langle a^d \rangle$. i.e. all elements of order $\frac{n}{d}$ are contained in the subgroup $\langle a^d \rangle$

Conclusion: For all $d \mid n$ TBC.

Example: $n = 2$. $C_{12} = \langle a \rangle = \{a, a^2, a^3, \ldots, a^{11}, a^{12}\}$

- **Order 12:** a^1, a^5, a^7, a^11 $\langle a \rangle = C_{12} = \langle a^5 \rangle = \langle a^7 \rangle = \langle a^1 1 \rangle$
- **Order 6:** $a^2 a^{10}$ $\langle a^2 \rangle = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}\} = \langle a^{10} \rangle$
- **Order 4:** a^3, a^9 $\langle a^3 \rangle = \{a^3, a^6, a^9, a^{12}\} = \langle a^9 \rangle$
- **Order 3:** a^4, a^8 $\langle a^4 \rangle = \{a^4, a^8, a^{12}\} = \langle a^8 \rangle$
-

Example: $n = 12$ $\mathbb{Z}_{12} = \{1, 2, 3, \ldots, 12\}$
TBC.

10.0.1 Lattices

TBC.

Cyclic groups with subgroups $\cong$ integers with divisibility

10.1 Complex Numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

$$\mathbb{C} = \{re^{i\theta} : r, \theta \in \mathbb{R}\}$$

Lemma 10.1.1.

$$e^{i\theta} = \cos\theta = i \sin\theta$$

test