

MAT 2143 Lecture Notes

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Lecture 1

Equivalence Relations

1.1 Review of Equivalence Relations

Set X and a notion of equivalence \sim . For all $x, y \in X$, either $x \sim y$ or $x \not\sim y$.

Recall: $X \times X = \{(x, y) : x, y \in \mathbb{R}\}$. Define $R = \{(x, y) : x, y \in \mathbb{R} \text{ } x \sim y\}$.

R is an **equivalence relation** if

- $x, y \in R \ \forall x \in X$
- $(x, y) \in R \iff (y, x) \in R$
- $(x, y) \in R \ (y, z) \in R \implies (x, z) \in R$

If R is an equivalence relation on X , then we define the **equivalence class** of $x \in X$ as

$$[x] = \{y \in X : x \sim y\}$$

1.2 Examples of Equivalence Relations

- Take any set X and let $x \sim y$ mean $x = y$
Reflexive: $x \sim x$? Yes, because $x = x$
Symmetric: $x \sim y \iff y \sim x$? Yes, because if $x = y$, then $y = x$.
Transitive: $x \sim y \text{ } y \sim z \implies x \sim z$? Yes, because if $x = y$ and $y = z$, then $x = z$.
- Take $X = \mathbb{R}^2$ and let $(a, b) \sim (c, d)$ mean $a^2 + b^2 = c^2 + d^2$
Reflexive: $(a, b) \sim (a, b)$? Yes, because $a^2 + b^2 = a^2 + b^2$
Symmetric: $(a, b) \sim (c, d) \iff (c, d) \sim (a, b)$? Yes, because if $a^2 + b^2 =$

$c^2 + d^2$, then $c^2 + d^2 = a^2 + b^2$.

Transitive: $(a, b) \sim (c, d) \ (c, d) \sim (e, f) \implies (a, b) \sim (e, f)$? Yes, because if $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$, then $a^2 + b^2 = e^2 + f^2$.

- Take $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and let $(a, b) \sim (c, d)$ mean $(ad = bc)$.

Reflexive: $(a, b) \sim (a, b)$? Yes, because multiplication of \mathbb{Z} is commutative, so $ab = ba$.

Symmetric: $(a, b) \sim (c, d) \iff (c, d) \sim (a, b)$? Yes,

$$(a, b) \sim (c, d) \implies ad = bc$$

$$cb = da$$

$$(c, d) \sim (a, b)$$

Transitive: $(a, b) \sim (c, d) \ (c, d) \sim (e, f) \implies (a, b) \sim (e, f)$? We want $ad = bc$, $cf = de \implies af = be$

Case 1: $c = 0$ Then $bc = 0 = ad$, $d \in \mathbb{Z} \setminus \{0\}$, so $d \neq 0$, $a = 0$
 $cf = 0 = de$, again $d \neq 0$, so $e = 0$.

$$\therefore af = be = 0$$

Case 2: $c \neq 0$ Then $\frac{ad}{c} = b$, $\frac{de}{c} = f$

$$\therefore af = a \cdot \frac{de}{c} = \frac{ad}{c} \cdot e = be$$

Theorem 1.2.1. Let X be a set with an equivalence relation. Then

$$[x] \cap [y] \neq \emptyset \implies [x] = [y]$$

So, equivalence classes are disjoint or equal.

Proof. Assume $[x] \cap [y] \neq \emptyset$. So $\exists z \in [x] \cap [y]$

Now let $a \in [x]$

$$\begin{aligned}
 a &\sim z && \text{(since } z \in [x] \text{ , } z \sim x \sim a) \\
 z &\sim y && \text{(since } z \in [y]) \\
 a &\sim y && \text{(transitivity)} \\
 a &\in [y] \\
 \therefore [x] &\subseteq [y]
 \end{aligned}$$

Now take $b \in [y]$, using the same arguments we get

$$\begin{aligned}
 b &\sim z && \text{(since } z \in [y] \text{ , } z \sim y \sim b) \\
 z &\sim x && \text{(since } z \in [x]) \\
 b &\sim x && \text{(transitivity)} \\
 b &\in [x] \\
 \therefore [y] &\subseteq [x]
 \end{aligned}$$

□

Observation: If X is some set with an equivalence relation, then every $x \in X$ is in some equivalence class.

Definition 1.2.1 (Partitions). *Say we have some $R_j \subseteq X$ for $j \in \{1, 2, \dots, n\}$, with every $x \in X$ in exactly one R_j , then the R_j form a partition of X .*

Theorem 1.2.2. *Let X be a set with an equivalence relation. Then the equivalence classes form a partition of X .*

Proof. If $z \in X$, then $z \in [z]$, therefore z is in at least one equivalence class. If $z \in [x]$ and $z \in [y]$, then $[x] \cap [y] \neq \emptyset$ therefore $[x] = [y]$ (as shown previously). Therefore z is in at most one equivalence class. □

Theorem 1.2.3. *Let R_j form a partition of X . Say that $x \sim y$ means $x, y \in R_j$ for some j . Then \sim is an equivalence relation on X .*

Proof.

- $x \in X$, so $x \in R_j$ for some j implies $x, x \in R_j \implies x \sim x$
- $x \sim y \iff x, y \in R_j \iff y, x \in R_j \iff y \sim x$

•

$$\begin{aligned}
 x \sim y \quad y \sim z &\implies \begin{cases} x, y \in R_i \\ y, z \in R_j \end{cases} \implies y \in R_i, R_j \\
 &\implies i = j \\
 &\implies x, z \in R_j \\
 &\therefore x \sim z
 \end{aligned}$$

□

Example of Finding Equivalence Classes

Take $X = R \times R$, and let $(a, b) \sim (c, d)$ mean $a^2 + b^2 = c^2 + d^2$. Find the equivalence class of $(0, 0)$, $(3, 4)$, (a, b)

$$\begin{aligned}
 [(0, 0)] &= \{(x, y) : (x, y) \sim (0, 0)\} \\
 &= \{(x, y) : x^2 + y^2 = 0^2 + 0^2 = 0\} \\
 &= \{(x, y) : x = y = 0\}
 \end{aligned}$$

$$\begin{aligned}
 [(3, 4)] &= \{(x, y) : (x, y) \sim (3, 4)\} \\
 &= \{(x, y) : x^2 + y^2 = 3^2 + 4^2 = 25\} \\
 &= \{(x, y) : \sqrt{x^2 + y^2} = 5\}
 \end{aligned}$$

$$\begin{aligned}
 [(a, b)] &= \{(x, y) : (x, y) \sim (a, b)\} \\
 &= \{(x, y) : x^2 + y^2 = a^2 + b^2 = r\} \\
 &= \{(x, y) : \sqrt{x^2 + y^2} = r\}
 \end{aligned}$$

Lecture 2

Well-defined Operations on Equivalence Classes and Number Theory

2.1 Well-defined Operations on Equivalence Classes

Consider a set X , an equivalence relation \sim , and an operation \cdot . This operation is [well-defined on equivalence classes](#) if

$$\left. \begin{array}{l} x \sim y \\ w \sim z \end{array} \right\} \implies x \cdot w \sim y \cdot z$$
$$\left. \begin{array}{l} [x] = [y] \\ [w] = [z] \end{array} \right\} \implies [x \cdot w] = [y \cdot z]$$

Example: Let $X = \mathbb{R} \times \mathbb{R}$, $(a, b) \sim (c, d)$ means $a^2 + b^2 = c^2 + d^2$, is addition well-defined on equivalence classes? (*Addition meaning $(x, y) + (z, y) = (x + z, y + w)$*)

$$\text{Let } \left\{ \begin{array}{l} (a, b) \sim (c, d) \\ (e, f) \sim (g, h) \end{array} \right. \text{ then } \left\{ \begin{array}{l} a^2 + b^2 = c^2 + d^2 \\ e^2 + f^2 = g^2 + h^2 \end{array} \right.$$

Now,

$$\begin{cases} (a, b) + (e, f) = (a + e, b + f) \\ (c, d) + (g, h) = (c + g, d + h) \end{cases}$$

Question: Is $(a + e)^2 + (b + f)^2 = (c + g)^2 + (d + h)^2$?

$$(a + e)^2 + (b + f)^2 = a^2 + 2ae + e^2 + b^2 + 2bf + f^2$$

$$(c + g)^2 + (d + h)^2 = c^2 + 2cg + g^2 + d^2 + 2dh + h^2$$

$a^2 + b^2 = c^2 + d^2$, and $e^2 + f^2 = g^2 + h^2$, so

$$(a + e)^2 + (b + f)^2 = (c + g)^2 + (d + h)^2 \iff 2ae + 2bf = 2cg + 2dh$$

Counterexample: Take

$$(a, b) = (c, d) = (1, 2)$$

$$(e, f) = (3, 4) \quad (g, h) = (4, 3)$$

So no, addition is not well defined.

Another Example: $X = (\mathbb{Z}, \mathbb{Z} \setminus \{0\})$. $(a, b) \sim (c, d)$ means $ad = bc$. Is multiplication well-defined on equivalence classes? (*Multiplication meaning* $(x, y) \cdot (w, z) = (x \cdot w, y \cdot z)$). Let

$$\begin{cases} (a, b) \sim (c, d) \\ (e, f) \sim (g, h) \end{cases} \implies \begin{cases} ad = bc \\ ef = gh \end{cases}$$

Now,

$$\begin{cases} (a, b) \cdot (e, f) = (a \cdot e, b \cdot f) \\ (c, d) \cdot (g, h) = (c \cdot g, d \cdot h) \end{cases}$$

Question: Is $(ae, bf) \sim (cg, dh)$?

$$(ae)(dh) = \text{ad} \cdot \text{eh}$$

$$(bf)(cg) = \text{bc} \cdot \text{fg}$$

We have $(a, b) \sim (c, d)$, so $ad = bc$, and $(e, f) \sim (g, h)$, so $ek = fg$. So yes, multiplication is well-defined on equivalence classes.

2.2 Number Theory

Fact 2.2.1. *Every non-empty set $S \subseteq \mathbb{N}$ has a minimum element d in S*

Proposition 2.2.1. *Let $a, b \in \mathbb{Z}$, $b > 0$, then $\exists! q, r \in \mathbb{Z}$ with $a = bq + r$, $0 \leq r < b$*

Proof. (Existence) Let $S = \{a - bx : x \in \mathbb{Z}, a - bx \geq 0\}$. $\emptyset \neq S \subseteq \mathbb{N}$, so S has a minimum element.

Let

$$\begin{cases} r = \min(S) \\ q = \frac{a-r}{b} \end{cases}$$

$r = a - bd$, $d \in \mathbb{Z}$, then $bq + r = b\left(\frac{a-r}{b}\right) + r = a - r + r = a$.

If $b \leq r$, then $0 \leq r - b < r$, which contradicts the minimality of r .

(Uniqueness) Say $a = bq + r = bp + s$, $0 \leq r, s < b$. Then

$$b(q - p) = s - r$$

So $s - r$ is a multiple of b , but $0 \leq r, s < b$, so it must be that $r - s = 0$, therefore $r = s$. \square

Lecture 3

Number Theory Cont. and Integers Modulo n

3.1 More Number Theory

Definition 3.1.1. $m \mid n$ means $\exists x \in \mathbb{Z}$ with $n = mx$

Definition 3.1.2. Let $a, b \in \mathbb{Z}$. If d is a positive integer with $d \mid a$ and $d \mid b$, if $c \mid a$ and $c \mid b$, then $c \mid d$, then d is a *gcd* of a and b .

Theorem 3.1.1. For every $a, b \in \mathbb{Z}$, $\exists ! \text{ gcd } d$. Furthermore, $\exists x, y \in \mathbb{Z}$, $d = ax + by$. Furthermore, d is the largest common divisor of a, b

Proof. Let $S = \{ax + by : x, y \in \mathbb{Z}, ax + by > 0\}$. $S \subseteq \mathbb{N}$, so $\exists !$ minimum element d in S .

Write

$$a = dq + r \quad (0 \leq r < d)$$

$$a = (ax + by)q + r \quad (\text{some } x, y \in \mathbb{Z})$$

$$r = a(1 - qx) + b(-qy)$$

$$r = ax' + by' \quad (x' = 1 - qx, y' = -qy)$$

$$0 \leq r = ax' + by' < d$$

So. either $r = 0$ or $r \in S$ but not both, but $r < d$ which is the minimum of the set. Therefore $r \notin S$. So $r = 0$ and $d \mid a$. Same argument with

$$b = dq + r \implies d \mid b.$$

Now suppose $c \mid a$ and $c \mid b$, then $a = a'c$ and $b = b'c$, $a', b' \in \mathbb{Z}$.

$$d = ax + by = a'cx + b'cy = c(a'x + b'y)$$

So, $c \mid d$. □

Corollary 3.1.1. *If $\gcd(a, b) = 1$, then $\exists x, y$ such that $ax + by = 1$.*

Proof. Same as the previous proof, in the case that $\gcd(a, b) = 1$. □

Corollary 3.1.2. *If $\gcd(a, b) = d$, then $\{ax + by : x, y \in \mathbb{Z}\} = d \cdot \mathbb{Z}$, $\forall n \in \mathbb{Z}$.*

Proof. No proof was provided in the notes I guess. :P □

Definition 3.1.3 (Least Common Multiple). *Let $a, b \in \mathbb{Z}$. If m is a positive integer with*

- $a \mid m$ and $b \mid m$
- if $a \mid n$ and $b \mid n$, then $m \mid n$

*then m is a **lcm** of a, b .*

Theorem 3.1.2. *For every a, b , $\exists ! \text{lcm } m$.*

Definition 3.1.4. $p \in \mathbb{Z}$ $p > 1$

- p is irreducible if the only positive divisors of p are 1 and p
- p is prime if whenever $p \mid ab$, then $p \mid a$ or $p \mid b$

Proposition 3.1.1. p is prime $\implies p$ is irreducible

Proof. Say p is not irreducible, $p = ab$, and $1 < a, b < p$. Then $p \nmid a$ and $p \nmid b$. □

Proposition 3.1.2. p is irreducible $\implies p$ is prime

Proof. $p \mid ab \implies ab = mp$ for some $m \in \mathbb{Z}$. Say $p \nmid a$, since p is irreducible $\gcd(a, p) = 1$. So $\exists s, t$ such that $as + pt = 1$.

$$b = b(as + pt) = abs + bpt = mps + bpt = (ms + bt)p$$

Therefore b is a multiple of p , so $p \mid b$. □

3.2 Prime Factorization

Theorem 3.2.1. $n \in \mathbb{Z} \ n > 1$

$$\exists! \begin{cases} p_1 p_2 \dots p_s, & \text{distinct primes} \\ e_1 e_2 \dots e_s, & \text{positive integers} \end{cases}$$

With

$$n = p_1^{e_1} \cdot p_2^{e_2} \dots p_s^{e_s}$$

Proof. Proof was omitted. □

Prime Factorization Gives GCD:

Example:

$$a = 2 \cdot 5 \cdot 7^{10} \cdot 13 = 2^{\boxed{1}} \cdot 3^{\boxed{0}} \cdot 5^1 \cdot 7^{10} \cdot 13^1 \cdot 17^{\boxed{0}}$$

$$b = 2 \cdot 3^2 \cdot 7^2 \cdot 17 = 2^1 \cdot 3^2 \cdot 5^{\boxed{0}} \cdot 7^{\boxed{2}} \cdot 13^{\boxed{0}} \cdot 17^1$$

$$\gcd(a, b) = 2^1 \cdot 7^2$$

$$a = p_1^{e_1} \cdot p_2^{e_2} \dots p_s^{e_s}$$

$$b = q_1^{F_1} \cdot q_2^{F_2} \dots q_s^{F_s}$$

$$\forall \text{ prime } p, \text{ define } g(p) = \min \begin{cases} e_i & \text{if } p = p_i \\ f_j & \text{if } p = q_j \\ 0 \end{cases}$$

Then,

$$\gcd(a, b) = \prod_{\text{prime } p} p^{g(p)}$$

Prime Factorization Gives LCM:

Example:

$$a = 2 \cdot 5 \cdot 7^{10} \cdot 13 = 2^1 \cdot 3^0 \cdot 5^{\boxed{1}} \cdot 7^{\boxed{10}} \cdot 13^{\boxed{1}} \cdot 17^0$$

$$b = 2 \cdot 3^2 \cdot 7^2 \cdot 17 = 2^{\boxed{1}} \cdot 3^{\boxed{2}} \cdot 5^0 \cdot 7^2 \cdot 13^0 \cdot 17^{\boxed{1}}$$

$$\text{lcm}(a, b) = 2^1 \cdot 3^2 \cdot 5^1 \cdot 7^{10} \cdot 13^1 \cdot 17^1$$

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

$$b = q_1^{F_1} \cdot q_2^{F_2} \cdots q_s^{F_s}$$

$$\forall \text{ prime } p, \text{ define } l(p) = \min \begin{cases} e_i & \text{if } p = p_i \\ f_j & \text{if } p = q_j \\ 0 & \end{cases}$$

Then,

$$\gcd(a, b) = \prod_{\text{prime } p} p^{l(p)}$$

3.2.1 Summary

- Definition of $\gcd(a, b)$
- $d = \gcd(a, b)$ exists $\implies d$ is a divisor of a and b and $d = ax + by$ for some $x, y \in \mathbb{Z}$
- also, $\{ax + by : x, y \in \mathbb{Z}\} = d\mathbb{Z}$
- Definition of $\text{lcm}(a, b)$
- $m = \text{lcm}(a, b)$ exists and is unique, m is the smallest common multiple.
- Prime Factorization exists and is unique.
- Prime factorization of a and b gives $\gcd(a, b)$ and $\text{lcm}(a, b)$
- $\gcd(a, b) \cdot \text{lcm}(a, b) = |ab|$

3.3 Integers Modulo n

Let $n \in \mathbb{Z}$ with $n \geq 2$, $a \equiv b \pmod{n}$ means $n \mid (a - b)$. So

$$a \equiv b \pmod{n} \iff n \mid (a - b)$$

$$\iff a - b = kn, \text{ for some } k \in \mathbb{Z}$$

$$\iff \frac{a - b}{n} \in \mathbb{Z}$$

Proposition 3.3.1. *Congruence modulo n is an equivalence relation.*

Proof.

- **Reflexivity:** Show $a \equiv a \pmod{n} \forall a \in \mathbb{Z}$.

$$\frac{a - a}{n} = 0 \in \mathbb{Z}$$

So $a \equiv a \pmod{n}$.

- **Symmetric:** Show $a \equiv b \iff b \equiv a \forall a, b \in \mathbb{Z}$

$$a \equiv b \iff \frac{a - b}{n} \in \mathbb{Z} \iff -\frac{a - b}{n} = \frac{b - a}{n} \in \mathbb{Z} \iff b \equiv a$$

- **Transitivity:** Show $a \equiv b \wedge b \equiv c \implies a \equiv c \forall a, b, c \in \mathbb{Z}$

$$\begin{aligned} a \equiv b \equiv c &\implies \frac{a - b}{n} \in \mathbb{Z} \wedge \frac{b - c}{n} \in \mathbb{Z} \\ &\implies \frac{a - b}{n} + \frac{b - c}{n} = \frac{a - c}{n} \in \mathbb{Z} \implies a \equiv c \end{aligned}$$

□

Example: Define $\mathbb{Z}_n = \{[k]_n : k \in \mathbb{Z}\}$. Consider $n = 5$.

- $[2] = \{\dots, -8, -3, 2, 7, 12, 17, \dots\}$
- $[0] = \{\dots, -10, -5, 0, 5, 10, \dots\}$
- $[7] = \{\dots, -3, 2, 7, 12, 17, 22, \dots\}$

Note: \mathbb{Z}_5 is a set containing 5 elements, and each element of \mathbb{Z}_5 is a subset of \mathbb{Z} . Also, recall that equivalence classes are disjoint or equal, so $[2] = [7]$

$$\begin{aligned} \mathbb{Z}_5 &= \{[0], [1], [2], [3], [4]\} \\ &= \{[-2], [-1], [0], [1], [2]\} \end{aligned}$$

Examples:

(a) $[3]_5 = [8]_5 \quad 3 \equiv 8 \pmod{5} \quad 5|(8 - 3)$

$$(b) \quad [3]_9 = [-24]_9 \quad 3 \equiv -24 \pmod{9} \quad 9|(3 - (-24))$$

All representations in an equivalence class are equivalent (modulo n).

Question: Is addition and multiplication on \mathbb{Z}_n well defined? We want

$$\begin{cases} [a] + [b] = [a + b] \\ [a] \cdot [b] = [a \cdot b] \end{cases}$$

We'll see this in the lecture.

Lecture 4

Operations on \mathbb{Z}_n , Symmetries, and Groups

4.1 Arithmetic Modulo n

Question: Is addition and multiplication on \mathbb{Z}_n well defined? We want

$$\begin{cases} [a] + [b] = [a + b] \\ [a] \cdot [b] = [a \cdot b] \end{cases}$$

Proposition 4.1.1. *Let $n \in \mathbb{Z}$ $n \geq 2$. Suppose*

$$\begin{aligned} a &\equiv a' \pmod{n} \\ b &\equiv b' \pmod{n} \end{aligned}$$

then

$$\begin{aligned} a + b &\equiv a' + b' \pmod{n} \\ ab &\equiv a'b' \pmod{n} \end{aligned}$$

So, addition and multiplication on integers are well defined on congruence classes.

Proof.

$$a \equiv a' \pmod{n} \iff \frac{a - a'}{n} \in \mathbb{Z} \iff a' = a + sn \text{ for some } s \in \mathbb{Z}$$

$$b \equiv b' \pmod{n} \iff \frac{b - b'}{n} \in \mathbb{Z} \iff b' = b + tn \text{ for some } t \in \mathbb{Z}$$

Then,

$$\frac{(a + b) - (a' + b')}{n} = \frac{a - a'}{n} + \frac{b - b'}{n} \in \mathbb{Z}$$

So, $a + b \equiv a' + b' \pmod{n}$. Also,

$$\begin{aligned} \frac{ab - a'b'}{n} &= \frac{ab - a'b + a'b - a'b'}{n} \\ &= \left(\frac{a - a'}{n} \right) b + \left(\frac{b - b'}{n} \right) a' \in \mathbb{Z} \end{aligned}$$

So $ab \equiv a'b' \pmod{n}$. □

4.1.1 Properties of Arithmetic Modulo n

- **Commutative:** $a + b \equiv b + a \pmod{n}$
- **Commutative:** $ab \equiv ba \pmod{n}$
- **Associative:** $(a + b) + c \equiv a + (b + c) \pmod{n}$
- **Associative:** $(ab)c \equiv a(bc) \pmod{n}$
- **Distributive:** $a(b + c) \equiv ab + ac \pmod{n}$
- **Identity for +:** $a + 0 \equiv a \pmod{n}$
- **Identity for ·:** $a \cdot 1 \equiv a \pmod{n}$
- **Additive Inverse:** $a + (-a) \equiv 0 \pmod{n}$
- **Multiplicative Inverse?**

4.1.2 Multiplicative Inverses

Proposition 4.1.2. Let $a \in \mathbb{Z}_n$, $\exists b \in \mathbb{Z}_n$ with $ab \equiv 1 \pmod{n} \iff \gcd(a, n) = 1$.

Proof. Suppose such b exists, then

$$\begin{aligned} ab - 1 &= rn \text{ for some } r \in \mathbb{Z} \\ ab + (-r)n &= 1 \\ \therefore \gcd(a, n) &= 1 \end{aligned}$$

Suppose $\gcd(a, n) = 1$, then

$$\begin{aligned} as + nt &= 1 \text{ for some } s, t \in \mathbb{Z} \\ as - 1 &= (-t)n \\ as &\equiv 1 \pmod{n} \end{aligned}$$

So we can choose $b = s$. □

Example: Addition and multiplication in \mathbb{Z}_6 .

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

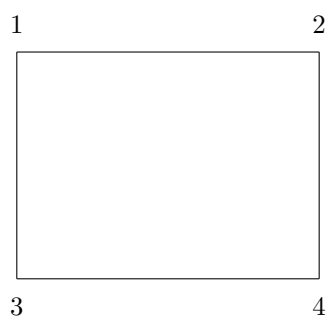
The table being symmetric implies that addition is commutative, 0-row and 0-column implies that 0 is the identity for addition, every row has a 0 implies that the additive inverse exists for every element.

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

The table being symmetric implies that multiplication is commutative, 1-row and 1-column is the header implies that 1 is the identity for multiplication, some rows not having 1 implies that some elements have no multiplicative inverse.

4.2 Symmetries

Consider the symmetries of a rectangle.



The notation for functions is

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ f(1) & f(2) & f(3) & f(4) \end{pmatrix}$$

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

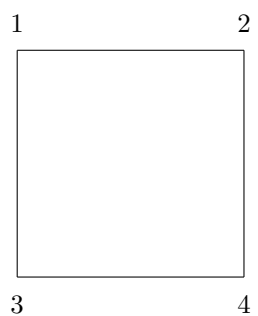
$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Claim: $\{\epsilon, \rho, \alpha, \beta\}$ are *all* the symmetries of a rectangle.

Proof. DGD Question - Will add later.

□

Consider the symmetries of a square.



$$\begin{array}{ccc}
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} & \epsilon & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} & 90^\circ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} & 180^\circ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} & 270^\circ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}
\end{array}$$

4.2.1 Properties of Symmetries

$S = \{\alpha, \beta, \dots\}$ symmetries of some object, with the operation composition.

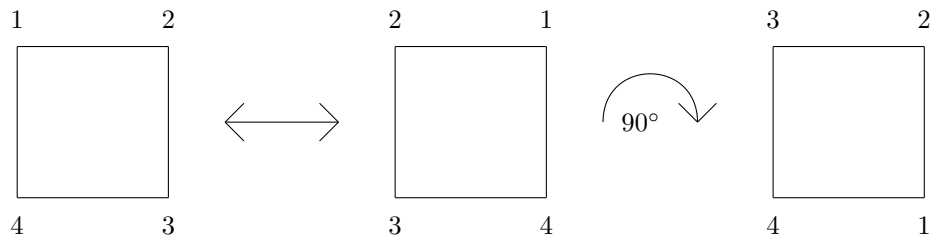
Properties:

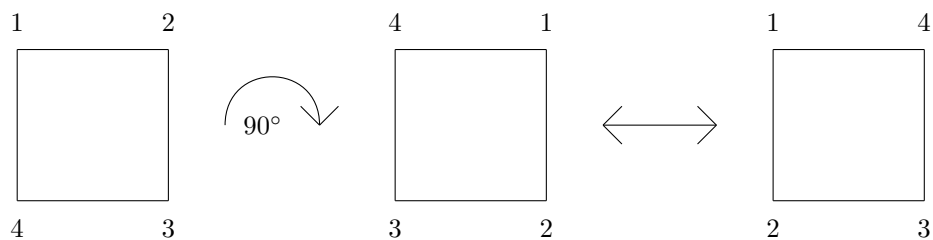
- $\alpha \circ \beta$ is a symmetry $\forall \alpha, \beta \in S$
- $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) \quad \forall \alpha, \beta, \gamma \in S$
- $\exists \epsilon \in S$ such that $\epsilon \circ \alpha = \alpha \circ \epsilon = \alpha \quad \forall \alpha \in S$
- $\forall \alpha \in S, \exists \beta \in S$, such that $\alpha \circ \beta = \beta \circ \alpha = \epsilon \quad \forall \alpha, \beta \in S$

Note: we often write $\alpha\beta$ instead of $\alpha \circ \beta$

Example: $S =$ symmetries of some object, is $gh = hg \quad \forall g, h \in S$? **Answer:**

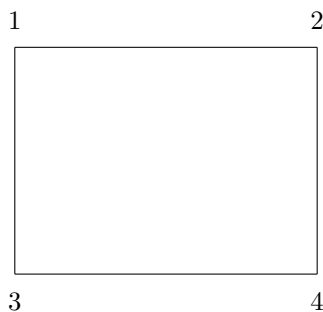
For a rectangle, yes. But for a square, no.





These symmetries do not commute, so $gh \neq hg$.

4.2.2 Generating Sets



$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Check: $\alpha\beta = \rho$, $\alpha^2 = \epsilon$. So $\forall g \in S$, g can be written in terms of α, β .

We say that $\{\alpha, \beta\}$ **generates** S

4.3 Groups

Let S be some set with some operation \cdot . Then (S, \cdot) is a **group** if

- **Closure:** $ab \in S \forall a, b \in S$
- **Associativity:** $(ab)c = a(bc) \forall a, b, c \in S$
- **Identity:** $\exists e \in S$ such that $xe = ex = x \forall x \in S$

- **Inverses:** $\forall x \in S, \exists y \in S$ such that $xy = yx = \epsilon$

Examples:

- Symmetries of an object form a group.
- $(\mathbb{R}, +)$ forms a group.
- $(\mathbb{R} \setminus \{0\}, \cdot)$ forms a group.
- $(\mathbb{Z}, +)$ forms a group.
- (\mathbb{Z}, \cdot) does not form a group since inverses are typically not integers.
- $(\mathbb{Z}_n, +)$ forms a group.
- $(\mathbb{Z}_n \setminus \{0\}, \cdot)$ forms a group.

Lecture 5

More Examples of Groups

This lecture was not well organizing so I am not gonna type it out.

Lecture 6

Basic Properties of Groups, Products of Groups, Isomorphisms

Examples were left out I may come back to finish

6.1 Basic Properties of Groups

Proposition 6.1.1. *In every group, the identity is unique.*

Proof. Suppose a, b are identities, so

$$\left. \begin{array}{l} ax = xa = x \\ bx = xb = x \end{array} \right\} \forall x$$

Because b is an identity, we have $a = ab$, and since a is an identity, we have $ab = b$. So

$$a = ab = b$$

$$\therefore a = b$$

□

Proposition 6.1.2. *In every group, the equation $ax = b$ has a unique solution x for all a, b*

Proof. There was no proof :(

□

Proposition 6.1.3. *In every group, $ab = ac \implies b = c$*

Proof. Again, no proof :(

□

Note: For matrices it is not the same, $AB = AC \not\Rightarrow B = C$

Proposition 6.1.4. *In every group, $(ab)^{-1} = b^{-1}a^{-1}$*

Proof.

$$\begin{aligned} (ab)(b^{-1}a^{-1}) &= a(bb^{-1})a^{-1} & (b^{-1}a^{-1})(ab) &= b^{-1}(a^{-1}a)b \\ &= a\epsilon a^{-1} & &= b^{-1}\epsilon b \\ &= aa^{-1} & &= b^{-1}b \\ &= \epsilon & &= \epsilon \end{aligned}$$

□

Proposition 6.1.5. *In every group, $(a^{-1})^{-1} = a$*

Proof. Since a^{-1} is the inverse of a , we have

$$aa^{-1} = a^{-1}a = \epsilon$$

but then,

$$a^{-1}a = aa^{-1} = \epsilon$$

So a is the inverse of a^{-1}

□

Proposition 6.1.6. *In every group, if $xy = x$, for some x, y , then $y = \epsilon$. So if y behaves as the identity just once, then y is the identity.*

Proof. No proof again :P.

□

Proposition 6.1.7. *In every group, if $xy = \epsilon$, for some x, y , then $y = x^{-1}$. So if y behaves like x^{-1} on one side, then y is x^{-1}*

Proof. No proof D:

□

Proposition 6.1.8. *In every group, the Cayley table has exactly one row and column that matches the headers, and no other row or column matches the header even once.*

Proof. Start by taking G to be some group, then let $x, y \in G$. And let H be a subgroup of G .

just kidding no proof. □

Proposition 6.1.9. *In every group, every row and column of the Cayley table contains each element exactly once.*

Proof. Why does the prof include a spot for the proof. □

6.1.1 Small Groups

- Say G has one element $G = \{x\}$

Closure: $x \cdot x = x$

Identity: $x = \epsilon$

Inverse: $x^{-1} = x$

\cdot	x
x	x

- Say G has two elements, it must have an identity so $G = \{\epsilon, x\}$ If $xx = x$, $x = \epsilon$, this is a contradiction, So $xx = \epsilon$

\cdot	ϵ	x
ϵ	ϵ	x
x	x	ϵ

- Say G has three elements. $G = \{\epsilon, x, y\}$

\cdot	ϵ	x	y
ϵ	ϵ	x	y
x	x	y	ϵ
y	y	ϵ	x

$$x\epsilon = x \implies xy \neq x$$

$$\epsilon y = y \implies xy \neq y$$

So $xy = \epsilon$

$$x\epsilon = x \implies xx \neq x$$

$$xy = \epsilon \implies xx \neq \epsilon$$

So $xx = y$

- Say G has 4 elements. Assignment Question!

6.2 Products of Groups

G, H are groups, define

$$G \times H = \{(g, h) : g \in G, h \in H\}$$

$$(x, a) \cdot (y, b) = (x \cdot y, a \cdot b)$$

$$G_1 \times G_2 \times \cdots \times G_k = \{(g_1, g_2, \dots, g_k) : g_j \in G_j\}$$

To reiterate, operations are done by component according to the operations of the group. i.e Suppose we have a group $G = (A, +)$ and $H = (B, \cdot)$ and $g, a \in G, h, b \in H$.

$$(g, h) \times (a, b) = (g + a, h \cdot b)$$

Proposition 6.2.1. *The product of groups is a group.*

Proof. Exercise. □

6.3 Isomorphisms

Suppose $\phi : G \rightarrow H$ is a bijection between two groups with the property

$$\phi(xy) = \phi(x)\phi(y)$$

Then ϕ is an [isomorphism](#) of $G \cong H$. So

$$G : x \cdot y = z \implies H : \phi(x) \cdot \phi(y) = \phi(z)$$

\cdot	y
x	z

\cdot	y'
x'	z'

$$x' = \phi(x) \qquad y' = \phi(y) \qquad z' = x'y' = \phi(z)$$

Start with G 's Cayley table, change the names (symbols, consistently) and permute the rows and columns. This gives H 's Cayley table.

Example:

$$\mathbb{Z}_2 = \{0, 1\} \qquad \mathbb{Z}^\times = \{-1, 1\} \qquad G = (\{\mathbb{Z}^+, \mathbb{Z}^-\}, \cdot)$$

$+$	0	1
0	0	1
1	1	0

\cdot	1	-1
1	1	-1
-1	-1	1

\cdot	$+$	$-$
$+$	$+$	$-$
$-$	$-$	$+$

$$\mathbb{Z}_2 \cong \mathbb{Z}^\times \cong G$$

Proposition 6.3.1. *All groups with two elements are isomorphic*

Proof. If G has two elements, then its Cayley table looks like

\cdot	ϵ	x
ϵ	ϵ	x
x	x	ϵ

Except they may use different symbols and have reordered rows/columns, so they are all isomorphic. \square

Lecture 7

Automorphisms, Subgroups

7.1 Automorphisms

Example: Let $H = \text{symmetries of a rectangle}$ and $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(x, y) : x \in \mathbb{Z}_2, y \in \mathbb{Z}_2\}$

$$H = \{\epsilon, \alpha, \beta, \rho\} \text{ with composition}$$

$$K = \{00, 01, 10, 11\} \text{ with addition in } \mathbb{Z}_2$$

H	ϵ	α	β	ρ	G	00	01	10	11
ϵ	ϵ	α	β	ρ	00	00	01	10	11
α	α	ϵ	ρ	β	01	01	00	11	10
β	β	ρ	ϵ	α	10	10	11	00	01
ρ	ρ	β	α	ϵ	11	11	10	01	00

$$\phi \begin{cases} \epsilon \rightarrow 00 \\ \alpha \rightarrow 01 \\ \beta \rightarrow 10 \\ \rho \rightarrow 11 \end{cases} \quad \text{or} \quad \phi \begin{cases} \epsilon \rightarrow 00 \\ \alpha \rightarrow 01 \\ \beta \rightarrow 11 \\ \rho \rightarrow 10 \end{cases}$$

In fact, all we need for the isomorphism is $\epsilon \rightarrow 00$, we can have $\alpha, \beta, \rho \rightarrow 01, 10, 11$ in any order.

An **automorphism** of G is an isomorphism $G \rightarrow G$, this is a symmetry group of G . The set of all automorphisms of G is a group we call $\text{aut}(G)$, the **automorphism group** of G .

H	ϵ	α	β	ρ
ϵ	ϵ	α	β	ρ
α	α	ϵ	ρ	β
β	β	ρ	ϵ	α
ρ	ρ	β	α	ϵ

H	ϵ	α	ρ	β
ϵ	ϵ	α	ρ	β
α	α	ϵ	β	ρ
ρ	ρ	β	ϵ	α
β	β	ρ	α	ϵ

Let ϕ be any bijection $\{\epsilon, \alpha, \beta, \rho\} \rightarrow \{\epsilon, \alpha, \rho, \beta\}$ with $\phi(\epsilon) = \epsilon$. Then ϕ is an automorphism of H .

Exercise: Let $G = \text{the symmetries of an equilateral triangle}$. Show that

$$\text{aut}(H) \cong G$$

7.2 Quaternions

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

\pm and 1 operate as expected. And

$$i^2 = j^2 = k^2 = ijk = -1$$

Q_8	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	j	$-j$
$-i$	$-i$	i	1	-1	$-k$	k	$-j$	j
j	j	$-j$	$-k$	k	-1	1	$-i$	i
$-j$	$-j$	j	k	$-k$	1	-1	i	$-i$
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

- **Closure:** Yes, Q_8 is closed.

- **Identity:** 1 is the identity for Q_8 .
- **Inverse:** Every column has the identity (1), so an inverse exists for every element in Q_8 .
- **Associativity:** Consider the set of matrices M_8 with entries in \mathbb{C}

$$M_8 = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right\}$$

And the function $\phi : M_8 \rightarrow Q_8$

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$\phi : M_8 \rightarrow Q_8$ is a bijection

$$\phi(ab) = \phi(a)\phi(b)$$

Because M_8 is a set of matrices, it is closed, associative, has an identity and has inverses, therefore M_8 is a group. Q_8 is isomorphic to M_8 , so it follows that it is also a group.

Therefore, Q_8 is closed, has identity, has inverses and is associative.

7.3 Subgroups

Consider the following

- G is a group with operation \cdot
- H is a subset of G
- H is a group with the same operation \cdot

Then H is a **subgroup** of G . We denote subgroups as $H \leq G$ or $H < G$

$$(\mathbb{Z}, +) < (\mathbb{Q}, +) < (\mathbb{R}, +) < (\mathbb{C}, +)$$

Example:

$$(\mathbb{Z}_3, +) \not\leq (\mathbb{Z}_5, +)$$

This is the case because

$$\mathbb{Z}_3 = \{0, 1, 2\} = \{[0], [1], [2]\} \not\subseteq \{[0], [1], [2], [3], [4]\} = \{0, 1, 2, 3, 4\} = \mathbb{Z}_5$$

These sets are equivalence classes **not** numbers so they are not subsets of each other.

7.3.1 Subgroup Test

Proposition 7.3.1. *Suppose H is a subset of G , if $H \neq \emptyset$*

$$x, y \in H \implies xy \in H$$

$$x \in H \implies x^{-1} \in H$$

then H is a subgroup.

Proof. Show that H is a group

- **Closure:** Is given.
- **Associative:** G is associative so any subset "inherits" associativity.
- **Identity:** Let ϵ_g be the identity in G . $\epsilon_a \cdot a = a \forall a \in H$. $\exists a \in H$, so $a^{-1} \in H$, since H is a subset of G , $a, a^{-1} \in G$, therefore $a \cdot a^{-1} = \epsilon_g \in H$.
- **Inverse:** Given

□

Proposition 7.3.2. *H a subgroup of $G \implies \epsilon_g \in H$ and so $\epsilon_H \in G$*

Proof. $H \neq \emptyset$, so let $x \in H$, then $x^{-1} \in H$, then $x \cdot x^{-1} = \epsilon_G \in H$. Furthermore

$$\epsilon_G \cdot h = h \cdot \epsilon_G = h \forall h \in H$$

Since $H \subseteq G$, and H has a unique identity, then $\epsilon_G = \epsilon_H$

□

7.3.2 Alternative Versions of Subgroup Test

Suppose H is a subset of G , if

- $H \neq \emptyset$

- $x, y \in H \implies xy \in H$

- $x \in H \implies x^{-1} \in H$

then H is a subgroup.

Lecture 8

Lattices and Cyclic Groups

Recall: H is a subgroup of G if

- $H \subseteq G$
- They have the same operation (Cayley table of H is obtained by deleting rows/columns from G)
- H is a group

Subgroup Test: If $H \subseteq G$ with the same operation and H is not empty,

$$x, y \in H \implies xy \in H$$

$$x \in H \implies x^{-1} \in H$$

then H is a subgroup of G .

8.1 Find all subgroups of $(\mathbb{Z}, +)$

Say H is a subgroup of \mathbb{Z} , $H \neq \{0\}$. Let n be the smallest positive integer in H , then

$$\{\dots, -n, 0, n, 2n, 3n, \dots\} \subseteq H$$

$$n\mathbb{Z} = \{nK : K \in \mathbb{Z}\} \subseteq H$$

Suppose $x \in H \setminus n\mathbb{Z}$, then write $x = qn + r$ for $0 \leq r < n$. By closure, we have

$$x - qn = r \in H$$

But, this contradicts the minimality of n unless $r = 0$, but if $r = 0$, then $x \in n\mathbb{Z}$.
Therefore, $H = n\mathbb{Z}$

Subgroups of \mathbb{Z} : $n\mathbb{Z} \forall n \in \mathbb{Z}, (n = 0 \implies H = \{0\})$

8.2 Symmetries of a Square

Lemma 8.2.1. *There are at most eight symmetries of a square.*

Proof. Let γ be a symmetry, γ maps corners to corners.

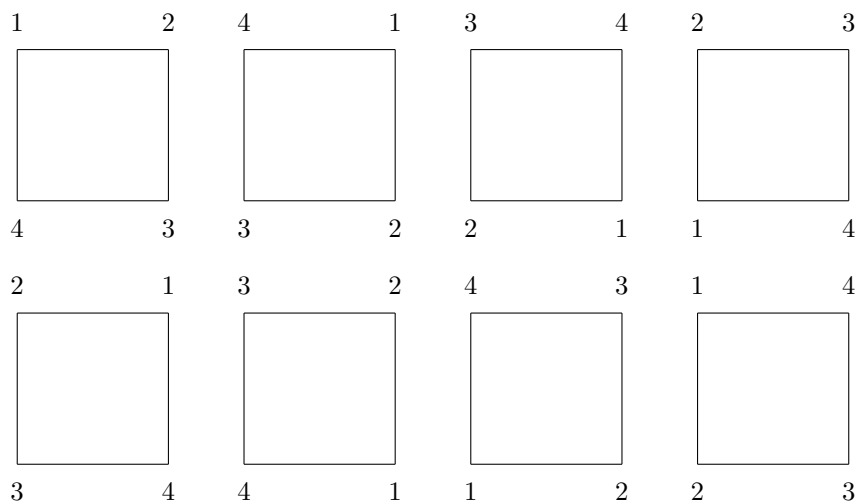
- $\gamma(1)$ has at most four possibilities, then $\gamma(2)$ must be one of the corners adjacent to $\gamma(1)$
- $\gamma(2)$ has at most two possibilities, then $\gamma(4)$ must be the other corner adjacent to $\gamma(1)$
- $\gamma(4)$ has at most one possibility, then $\gamma(3)$ must be $\{1, 2, 3, 4\} \setminus \{\gamma(1), \gamma(2), \gamma(3), \gamma(4)\}$
- $\gamma(3)$ has at most one possibility

So, we have $4 \cdot 2 \cdot 1 \cdot 1 = 8$ possibilities. □

Question: Do all possibilities work?

Lemma 8.2.2. *There are at least eight symmetries of a square*

Proof. Consider the symmetries of a square.



□

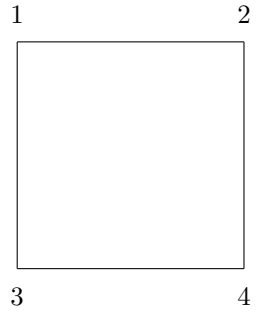
Let

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Proposition 8.2.1.

$$\rho\mu = \mu\rho^{-1} = \mu\rho^3$$

Proof. Consider the square and function μ, ρ .



$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\begin{aligned} \rho\mu &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} & \mu\rho^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} & & = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \end{aligned}$$

The 2 functions are equal.

□

Example:

$$\rho^2\mu\rho\mu\rho^3 = \mu\rho^6\rho\mu\rho^3 = \mu\rho^7\mu\rho^3 = \mu\mu\rho^{21}\rho^3 = \mu\rho^{24} = \epsilon$$

$$\mu\rho\mu\rho^2\mu\rho = \mu\mu\rho^3\rho^2\mu\rho = \mu^2\rho^5\mu\rho = \rho\mu\rho = \mu\rho^3\rho = \mu\rho^3\rho = \mu$$

Corollary 8.2.1.

$$\begin{aligned} G &= \langle \mu, \rho : \mu^2 = \epsilon, \rho^4 = \epsilon, \rho\mu = \mu\rho^3 \rangle \\ &= \{\mu^i \rho^j : 0 \leq i \leq 1, 0 \leq j \leq 3\} \\ &= \{\rho^i \mu^j : 0 \leq i \leq 1, 0 \leq j \leq 3\} \end{aligned}$$

Proof. Any sequence of μ 's and ρ 's can be written as $\mu^s \rho^t$ using $\rho\mu = \mu\rho^3$ ($\rho^t \mu^s$ using $\mu\rho = \rho^3 \mu$) reduce powers on μ and ρ using $\mu^2 = \epsilon$ $\rho^4 = \epsilon$

$$G = \{\mu^i \rho^j : 0 \leq i \leq 1, 0 \leq j \leq 3\}$$

These are all distinct since $|G| = 8$ so the relations $\mu^2 = \epsilon$ $\rho^4 = \epsilon$ $\rho\mu = \mu\rho^3$ are sufficient to characterize G \square

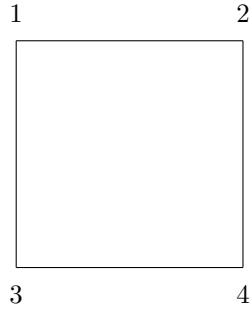
Compare:

$$F = \langle \alpha, \beta : \alpha^2 = \epsilon, \beta^4 = \epsilon \rangle$$

$\alpha\beta, \alpha\beta\alpha, \alpha\beta\alpha\beta, \dots$ are all distinct

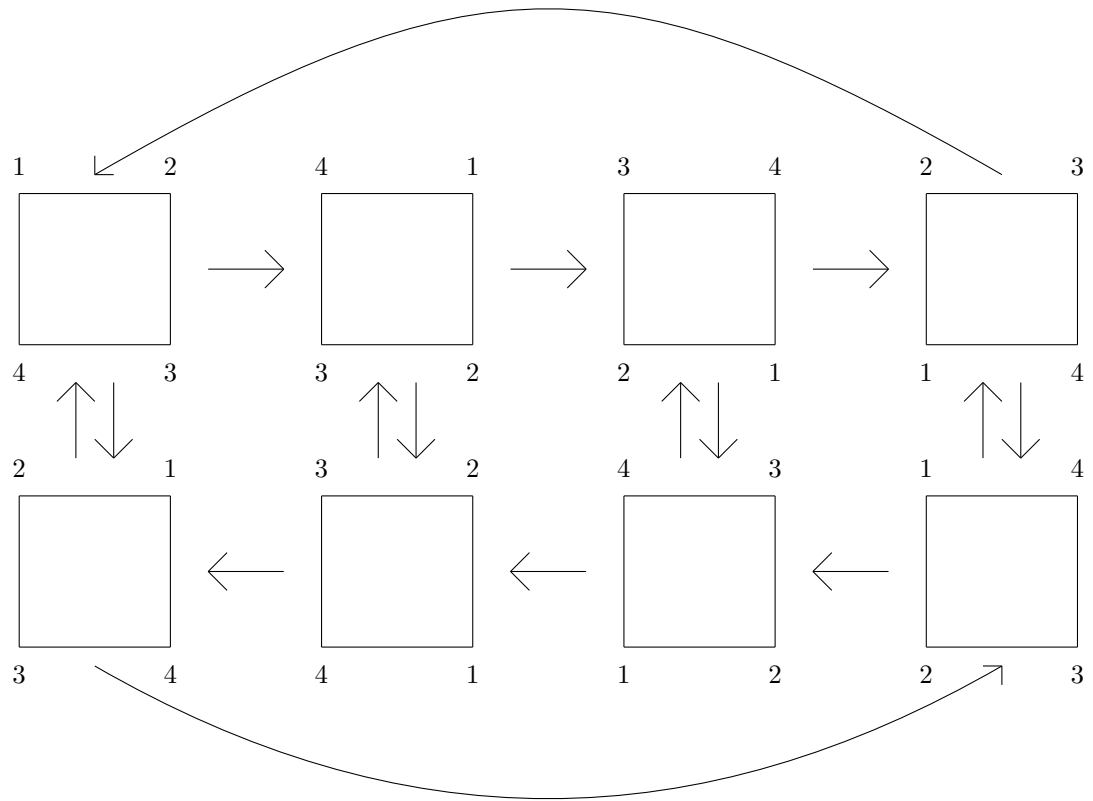
$$\alpha^3 \beta^7 \alpha \beta = \alpha \beta^3 \alpha \beta$$

$$|F| = \infty$$



$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\begin{aligned} G &= \{\mu^i \rho^j : 0 \leq i \leq 1, 0 \leq j \leq 3\} \\ &= \{\rho^i \mu^j : 0 \leq i \leq 1, 0 \leq j \leq 3\} \end{aligned}$$



Elements of G are symmetries of a square, they also permute G itself! But they are not symmetries of G .

$$\mu \cdot \rho = \mu\rho$$

$$\mu\mu\mu\rho = \mu(\mu)\mu(\rho) \neq \mu(\mu\rho) = \mu\mu\rho$$

8.2.1 Subgroups of Symmetries of a Square

$$G = \text{Sym}(\square) = \{\mu^i \rho^j : 0 \leq i \leq 1, 0 \leq j \leq 3\}$$

$\langle \epsilon \rangle = \{\epsilon\}$	$\langle \mu, \rho \rangle = G = \langle \mu, \rho^3 \rangle$
$\langle \mu \rangle = \{\epsilon, \mu\}$	$\langle \mu, \rho^2 \rangle = \{\epsilon, \mu, \rho^2, \mu\rho^2\}$
$\langle \rho \rangle = \{\epsilon, \rho, \rho^2, \rho^3\}$	
$\langle \rho^2 \rangle = \{\epsilon, \rho^2\}$	$\langle \mu, \mu\rho \rangle = \{\epsilon, \mu, \rho, \dots\} = G$
$\langle \rho^3 \rangle = \{\epsilon, \rho^3, \rho^2, \rho\}$	$\langle \mu, \mu\rho^3 \rangle = \{\epsilon, \mu, \rho^3, \dots\} = G$
$\langle \mu\rho \rangle = \{\epsilon, \mu\rho\}$	$\langle \mu, \mu\rho^2 \rangle = \{\epsilon, \mu, \rho^2, \mu\rho^2\}$
$\langle \mu\rho^2 \rangle = \{\epsilon, \mu\rho^2\}$	
$\langle \mu\rho^3 \rangle = \{\epsilon, \mu\rho^3\}$	$\langle \rho, \mu\rho \rangle = \{\epsilon, \mu, \rho, \dots\} = G$
	$\langle \rho, \mu\rho^3 \rangle = \{\epsilon, \mu, \rho, \dots\} = G$
	$\langle \rho, \mu\rho^2 \rangle = \{\epsilon, \mu, \rho, \dots\} = G$
	$\langle \rho^2, \mu\rho^2 \rangle = \{\epsilon, \rho^2, \mu\rho, \mu\rho^3\} = G$
	$\langle \rho^2, \mu\rho^3 \rangle = \{\epsilon, \rho^2, \mu\rho^3, \mu\rho\} = G$
	$\langle \rho^2, \mu\rho^2 \rangle = \{\epsilon, \rho^2, \mu\rho^2, \mu\} = G$
	$\langle \mu\rho, \mu\rho^2 \rangle = \{\epsilon, \rho, \mu, \dots\} = G$
	$\langle \mu\rho, \mu\rho^3 \rangle = \{\epsilon, \mu\rho, \mu\rho^3, \rho^2\} = G$
	$\langle \mu\rho^2, \mu\rho^3 \rangle = \{\epsilon, \rho, \mu, \dots\} = G$

Lecture 9

Cyclic Groups

9.1 Cyclic Groups

G is **cyclic** if $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ for some $g \in G$. g is a **generator** of G , there could be other generators. For addition,

$$G\langle g \rangle = \{kg : k \in \mathbb{Z}\}$$

The **order of g** is the smallest positive integer n with $g^n = \epsilon$, written as $|g|$. For addition, it's the smallest positive integer n with $ng = \epsilon$.

Proposition 9.1.1. G is cyclic $\implies G$ is abelian

Proof. Since G is cyclic, then $G = \langle g \rangle$ for some $g \in G$, take $x, y \in G$. Then $x = g^s$ and $y = g^t$. So

$$xy = g^s g^t = g^{s+t} = g^{t+s} = g^t g^s = yx$$

□

However, G being abelian $\not\Rightarrow G$ is cyclic. **Examples:** Are the following in cyclic? Find generators, and all orders

Q_8 :

- $\langle 1 \rangle = \{1\}$ Order 1
- $\langle -1 \rangle = \{-1, 1\}$ Order 2

- $\langle i \rangle = \{i, -1, -i, 1\}$ Order 4
- $\langle -i \rangle = \{-i, -1, i, 1\}$ Order 4
- $\langle \pm j \rangle = \{\pm j, -1, \mp j, 1\}$ Order 4
- $\langle \pm k \rangle = \{\pm k, -1, \mp k, 1\}$ Order 4

Not cyclic

\mathbb{Z} :

- $\langle 1 \rangle = \{k \cdot 1 : k \in \mathbb{Z}\} = \mathbb{Z}$ Order is ∞ , so no finite order

Are there other generators? Consider -1

- $\langle -1 \rangle = \{k \cdot (-1) : k \in \mathbb{Z}\} = \mathbb{Z}$

\mathbb{Z}_5 :

- $\langle 1 \rangle = \{1, 2, 3, 4, 5 = 0\}$ Order 5
- $\langle 2 \rangle = \{2, 4, 6 = 1, 3, 5 = 0\}$ Order 5
- $\langle -2 \rangle = \langle 3 \rangle = \{3, 1, 4, 2, 5 = 0\}$ Order 5
- $\langle -1 \rangle = \langle 4 \rangle = \{4, 3, 2, 1, 5 = 0\}$ Order 5
- $\langle 0 \rangle = \{0\}$

Therefore $\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle$ generate the group, so it is cyclic. \mathbb{Z}_9^\times :

- $\langle 1 \rangle = \{1\}$ Order 1
- $\langle 2 \rangle = \{2, 4, 8, 16 = 7, 14 = 5, 10 = 1\}$ Order 6
- $\langle -4 \rangle = \langle 4 \rangle = \{4, 7, 1\}$ Order 3
- $\langle -2 \rangle = \langle 5 \rangle = \{5, 7, 8, 4, 2, 1\}$ Order 6
- $\langle -1 \rangle = \langle 8 \rangle = \{8, 1\}$ Order 2

Therefore $\langle 2 \rangle$ and $\langle 5 \rangle$ generate the group, so it is cyclic.

\mathbb{Z}_8^\times

- $\langle 1 \rangle = \{1\}$ Order 1

- $\langle 3 \rangle = \{3, 1\}$ Order 2
- $\langle 5 \rangle = \{5, 1\}$ Order 2
- $\langle 7 \rangle = \{7, 1\}$ Order 2

Therefore not cyclic.

\mathbb{Q} :

- $\langle 1 \rangle = \mathbb{Z}$
- $\langle 0 \rangle = \{0\}$
- $\langle q \rangle = q\mathbb{Z}, q \in \mathbb{Q}$

Therefore not cyclic.

\mathbb{R} :

- $\langle 1 \rangle = \mathbb{Z}$
- $\langle 0 \rangle = \{0\}$
- $\langle r \rangle = r\mathbb{Z}$

Therefore not cyclic.

Note: $q\mathbb{Z} \cong \mathbb{Z}$ and $r\mathbb{Z} \cong \mathbb{Z}$

$\mathbb{Z}_2 \times \mathbb{Z}_4$:

- $\langle 00 \rangle = \{00\}$ Order 1
- $\langle 01 \rangle = \{01, 02, 03, 00\}$ Order 4
- $\langle 02 \rangle = \{02, 00\}$ Order 2
- $\langle 03 \rangle = \{03, 02, 01, 00\}$ Order 4
- $\langle 10 \rangle = \{10, 00\}$ Order 2
- $\langle 11 \rangle = \{11, 02, 13, 00\}$ Order 4
- $\langle 12 \rangle = \{12, 00\}$ Order 2
- $\langle 13 \rangle = \{13, 02, 11, 00\}$ Order 4

Therefore not cyclic.

$\mathbb{Z}_2 \times \mathbb{Z}_3$:

- $\langle 00 \rangle = \{00\}$ Order 1
- $\langle 01 \rangle = \{01, 02, 00\}$ Order 3
- $\langle 02 \rangle = \{02, 01, 00\}$ Order 3
- $\langle 10 \rangle = \{10, 00\}$ Order 2
- $\langle 11 \rangle = \{11, 02, 10, 01, 12, 00\}$ Order 6
- $\langle 12 \rangle = \{12, 01, 10, 02, 11, 00\}$ Order 6

Therefore cyclic

Proposition 9.1.2. *G is cyclic \implies all subgroups of G are cyclic*

Proof. Let $G = \langle a \rangle = \{a^i : i \in \mathbb{Z}\}$. Let H be a sub group of G .

$$H = \{a^i : \text{some } i \in \mathbb{Z}\}$$

could be $H = \{a^0\} = \{\epsilon\}$. Let

$$n = \min\{k : a^k \in H, k > 0\}$$

$$\langle a^n \rangle = \{(a^n)^k : k \in \mathbb{Z}\} = \{a^{kn} : k \in \mathbb{Z}\} = \{a^k : k \in n\mathbb{Z}\}$$

$$\langle a^n \rangle \leq H \leq G$$

Suppose $a^j \in H$ with $j \notin \mathbb{Z}$ so $\langle a^n \rangle \neq H$. Then

$$j = qn + r \quad 0 \leq r < n \quad r \neq 0$$

So

$$a^r = a^{j-qn} = a^j(a^n)^{-q} \in H$$

This contradicts the minimality of n . Therefore $H = \langle a^n \rangle$. □

Definition 9.1.1 (Order). *The **order** of an element $g \in G$ is the smallest positive integer n such that $g^n = \epsilon$. We write $|g|$ for the order of g , if no such n exists we say $|g| = \infty$.*

Proposition 9.1.3. Suppose $|a| = n < \infty$, then

$$a^j = \epsilon \iff n|j$$

In other words,

$$\{j : a^j = \epsilon\} = n\mathbb{Z}$$

Furthermore,

$$a^s = a^t \iff n|s - t$$

Example: $|a| = 5$

$$a^5 = a^{10} = a^{-15} = a^{1005} = \dots = \epsilon$$

$a^j \neq \epsilon$ when j is not a multiple of 5.

Proof. \Leftarrow if $n|j$ then $j = tn$ for some $t \in \mathbb{Z}$

$$a^j = a^{tn} = (a^n)^t = \epsilon^t = \epsilon$$

\Rightarrow : if $a^j = \epsilon$, then write $j = qn + r$ for $0 \leq r < n$

$$a^r = a^{j-qn} = (a^n)^{-q} = \epsilon(\epsilon^{-q}) = \epsilon$$

but n is the smallest positive integer with $a^n = \epsilon$, so $0 \leq r < n$ implies $r = 0$.
Therefore $j = nq$ and $n | j$. \square

Also,

$$a^s = a^t \iff a^{s-t} = \epsilon \iff n|s - t$$

Corollary 9.1.1. $|a| = |b|$ is equivalent to

$$a^j = \epsilon \iff b^j = \epsilon$$

Proposition 9.1.4. Suppose $a \in G$, $|a| = n < \infty$, $k \in \mathbb{Z}$. Then

$$|a^k| = \frac{n}{\gcd(k, n)}$$

Example: $|a| = 12$

- $\langle a^1 \rangle = \{a^1, a^2, a^3, \dots, a^{12}\}$

- $\langle a^5 \rangle = \{a^5, a^{10}, a^3, \dots, a^{12}\} = \langle a \rangle$
- $\langle a^4 \rangle = \{a^4, a^8, a^{12} = a^0\}$
- $\langle a^{10} \rangle = \{a^{10}, a^8, a^6, a^4, a^2, a^0\}$

Proof. Let $|a^k| = m$, then $\epsilon = (a^k)^m = a^{km}$. Therefore, $n|km$ and km is a multiple of $|a|$ by the previous theorem. Let $d = \gcd(kn)$ and set

$$\begin{cases} n = n'd \\ k = k'd \end{cases}$$

$$\gcd(n', k') = 1$$

Since $n|km$ for some $t \in \mathbb{Z}$ we have,

$$km = tn$$

$$dk'm = tdn'$$

$$k'm = tn'$$

$$m = \frac{tn'}{k'} = \frac{t}{k'} \cdot n'$$

This must be an integer because $\gcd(k', n') = 1 \implies k' \mid t$. Smallest $m \iff$ smallest t with $\frac{tn'}{k'}$ positive integer. So \square

Corollary 9.1.2. Suppose $G = \langle a \rangle$, with $|a| = n < \infty$, then the generators of G are $\{a^k : \gcd(n, k) = 1\}$

Proof.

$$|a^k| = \frac{n}{\gcd(n, k)} = n \iff \gcd(n, k) = 1$$

\square

Corollary 9.1.3. $\mathbb{Z}_n = \langle 1 \rangle$ and $|1| = n$. Generators of \mathbb{Z} with addition are

$$\{k \cdot 1 : \gcd(n, k) = 1\} = \{k : \gcd(n, k) = 1\} = \mathbb{Z}_n^\times$$

Corollary 9.1.4. all nonzero elements of \mathbb{Z}_n are generators of $\mathbb{Z}_n \iff n$ is prime

Proof. We want $|k| = \frac{n}{\gcd(n, k)} = n$ for $k = 1, 2, 3, \dots, n-1$. So $\gcd(n, k) = 1$ \square

Lecture 10

Subgroups of Cyclic Groups, Lattices, \mathbb{T}

- G **cyclic** means there exists $g \in G$ with $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$
- The **order** of an element g is the smallest positive integer n with $g^n = \epsilon$
- Notation: **Order of an element** g is written $|g|$. **Order (=size!) of a group** G is written $|G|$. $|g| = \infty$ means $g^k \neq \epsilon \forall k \in \mathbb{Z}$
- $\{k : g^k = \epsilon\} = |g|\mathbb{Z}$ so $g^k = \epsilon \iff |g| \text{ divides } k$
- $|x| = |y|$ is equivalent to $x^k = \epsilon \iff y^k = \epsilon$
- if $|g| = n < \infty$, then

$$G = \langle g \rangle = \{g, g^2, \dots, g^n = \epsilon\}$$

$$|G| = |g|$$

$$|g^k| = \frac{n}{\gcd(n, k)}$$

generators of G are exactly $\{g^k : \gcd(n, k) = 1\}$

Corollary 10.0.1. *All nonzero elements of \mathbb{Z}_n are generators of $\mathbb{Z}_n \iff n$ is prime.*

Proof. We want $k = \frac{n}{\gcd(n, k)} = n$ for $k = 1, 2, 3, \dots, n-1$. So $\gcd(n, k) = 1$ for $k = 1, 2, 3, \dots, n-1$. Therefore n is prime. \square

Theorem 10.0.1. G has no subgroups other than $\{\epsilon\}$ and $G \iff G$ is cyclic of prime order $\iff |G|$ is prime.

Proof. Suppose $g \in G$, then $\langle g \rangle$ is a subgroup of G . Therefore, either $\langle g \rangle = G$ or $\langle g \rangle = \{\epsilon\}$. g is a generator of G So

$$G = \{g, g^2, g^3, \dots, g^n = \epsilon\}$$

g^k is a generator for $k = 1, 2, \dots, n-1$ Therefore,

$$\frac{n}{\gcd(n, k)} = n$$

So n is prime, therefore G is cyclic of prime order $G \cong \mathbb{Z}_n$ for n prime.

Conversely,

$$G = \{g, g^2, \dots, g^n = \epsilon\}$$

then $S \neq \emptyset$ and $S \neq \{\epsilon\} \implies \langle S \rangle = G$. So $x \in S$, $x = g^k$ then

$$|x| = |g^k| = \frac{n}{\gcd(n, k)}$$

So the only subgroups are $\{\epsilon\}$ and G □

Theorem 10.0.2. Suppose G, H are both cyclic, $G \cong H \iff |G| = |H|$

Proof. (\implies) an isomorphism is a bijection.

(\longleftarrow) $G = \langle a \rangle$ and $H = \langle b \rangle$, then

$$|a| = |G| = |H| = |b|$$

define

$$\phi : G \rightarrow H$$

$$\phi(a^k) = b^k$$

We have 2 cases, either the order is infinite.

$$\begin{cases} G = \{\dots, a^{-2}, a^{-1}, a^0, a^1, a^2, \dots\} \\ H = \{\dots, a^{-2}, a^{-1}, a^0, a^1, a^2, \dots\} \end{cases}$$

Or their order is finite

$$\begin{cases} G = \{a, a^2, a^3, \dots, a^n \epsilon\} \\ H = \{b, b^2, b^3, \dots, b^n = \epsilon\} \end{cases}$$

In both cases ϕ is a bijection.

$$\phi(a^s a^t) = \phi(a^{s+t}) = b^{s+t} = b^s b^t = \phi(a^s) \phi(a^t)$$

□

Subgroups of $C_n = \langle a \rangle = \{a, a^2, \dots, a^n\}$

- C_n is cyclic, therefore all subgroups are cyclic
- $|a^k| = \frac{n}{\gcd(k, n)}$
- Let $d \mid n$ then $|a^d| = \frac{n}{\gcd(d, n) = \frac{n}{d}}$

So for each $d \mid n$, then $\langle a \rangle^d \cong C_{\frac{n}{d}}$ is a subgroup.

No let $k \in \{1, 2, 3 \in n\}$. Suppose $\gcd(k, n) = d$ for some $d \mid n$, then

$$k \in \{d, 2d, 3d, \dots, \frac{n}{d}d\}$$

So $a^k \in \langle a^d \rangle$. i.e. all elements of order $\frac{n}{d}$ are contained in the subgroup $\langle a^d \rangle$

Conclusion: For all $d \mid n$, there is a unique subgroup of C_n of order $\frac{n}{d}$, generated by a^d .

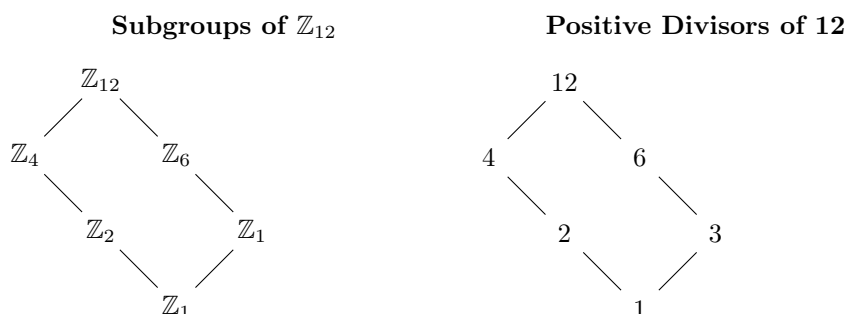
Example: $n = 2$. $C_{12} = \langle a \rangle = \{a, a^2, a^3, \dots, a^{11}, a^{12}\}$

- **Order 12:** $a^1, a^5, a^7, a^{11} \langle a \rangle = C_{12} = \langle a^5 \rangle = \langle a^7 \rangle = \langle a^{11} \rangle$
- **Order 6:** $a^2 a^{10} \langle a^2 \rangle = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}\} = \langle a^{10} \rangle$
- **Order 4:** $a^3, a^9 \langle a^3 \rangle = \{a^3, a^6, a^9, a^{12}\} = \langle a^9 \rangle$
- **Order 3:** $a^4, a^8 \langle a^4 \rangle = \{a^4, a^8, a^{12}\} = \langle a^8 \rangle$
- **Order 2:** $a^6 \langle a^6 \rangle = \{a^6, a^{12}\}$
- **Order 1:** $a^2 \langle a^2 \rangle = \{a^{12}\}$

Example: $n = 12$ $\mathbb{Z}_{12} = \{1, 2, 3, \dots, 12\}$

- **Order 12:** $1, 5, 7, 11$ $\langle 1 \rangle = \mathbb{Z}_{12} = \langle 5 \rangle = \langle 7 \rangle = \langle 11 \rangle$
- **Order 6:** $2, 10$ $\langle 2 \rangle = \{2, 4, 6, 8, 10, 12\} = \langle 10 \rangle$
- **Order 4:** $3, 9$ $\langle 3 \rangle = \{3, 6, 9, 12\} = \langle 9 \rangle$
- **Order 3:** $4, 8$ $\langle 4 \rangle = \{4, 8, 12\} = \langle 8 \rangle$
- **Order 2:** 6 $\langle 6 \rangle = \{6, 12\}$
- **Order 1:** 12 $\langle 12 \rangle = \{12\}$

10.0.1 Lattices



Cyclic groups with subgroups \cong integers with divisibility

10.1 Complex Numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

$$\mathbb{C} = \{re^{i\theta} : r, \theta \in \mathbb{R}\}$$

Lemma 10.1.1.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$z = re^{i\theta} = re^{i(\theta+2k\pi)} = -re^{i(\theta+(2k+1)\theta)}$$

$$z = re^{i\theta} = r \cos \theta + ir \sin \theta$$

$$\begin{cases} |z| = |a + bi| = \sqrt{a^2 + b^2} = r \\ \frac{b}{a} = \tan \theta \end{cases}$$

Refer to the profs notes for the rest of the complex number stuff.