MAT 2143 Lecture Notes

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Lecture 1

Equivalence Relations

1.1 Review of Equivalence Relations

Set X and a notion of equivalence $\sim$. For all $x, y \in X$, either $x \sim y$ or $x \not\sim y$.
Recall: $X \times X = \{(x, y) : x, y \in \mathbb{R}\}$. Define $R = \{(x, y) : x, y \in \mathbb{R}\ x \sim y\}$.

R is an equivalence relation if

- $x, y \in R\ \forall x \in X$
- $(x, y) \in R \iff (y, x) \in R$
- $(x, y) \in R\ (y, z) \in R \implies (x, z) \in R$

If R is an equivalence relation on X, then we define the equivalence class of $x \in X$ as

$$[x] = \{y \in X : x \sim y\}$$

1.2 Examples of Equivalence Relations

- Take any set X and let $x \sim y$ mean $x = y$
  Reflexive: $x \sim y$? Yes, because $x = x$
  Symmetric: $x \sim y \iff y \sim x$? Yes, because if $x = y$, then $y = x$.
  Transitive: $x \sim y\ y \sim z \implies x \sim z$? Yes, because if $x = y$ and $y = z$, then $x = z$.

- Take $X = \mathbb{R}^2$ and let $(a, b) \sim (c, d)$ mean $a^2 + b^2 = c^2 + d^2$
  Reflexive: $(a, b) \sim (a, b)$? Yes, because $a^2 + b^2 = a^2 + b^2$
  Symmetric: $(a, b) \sim (c, d) \iff (c, d) \sim (a, b)$? Yes, because if $a^2 + b^2 =$

c^2+d^2, then $c^2+d^2=a^2+b^2$.

Transitive: $(a,b)\sim(c,d)$ $(c,d)\sim(e,f) \implies (a,b)\sim(e,f)$? Yes, because if $a^2+b^2=c^2+d^2$ and $c^2+d^2=e^2+f^2$, then $a^2+b^2=e^2+f^2$.

- Take $X=\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$ and let $(a,b)\sim(c,d)$ mean $(ad=bc)$.

  Reflexive: $(a,b)\sim(a,b)$? Yes, because multiplication of $\mathbb{Z}$ is commutative, so $ab=ba$.

  Symmetric: $(a,b)\sim(c,d) \iff (c,d)\sim(a,b)$? Yes,

$$(a,b)\sim(c,d) \implies ad=bc$$

$$cb=da$$

$$(c,d)\sim(a,b)$$

  Transitive: $(a,b)\sim(c,d)$ $(c,d)\sim(e,f) \implies (a,b)\sim(e,f)$? We want $ad=bc,\ cf=de \implies af=be$

  Case 1: $c=0$ Then $bc=0=ad$, $d\in\mathbb{Z}\setminus\{0\}$, so $d\neq 0$, $a=0$
  $cf=0=de$, again $d\neq 0$, so $e=0$.

$$\therefore af=be=0$$

  Case 2: $c\neq 0$ Then $\frac{ad}{c}=b$, $\frac{de}{c}=f$

$$\therefore af=a\cdot\frac{de}{c}=\frac{ad}{c}\cdot e=be$$

Theorem 1.2.1. *Let X be a set with an equivalence relation. Then*

$$[x]\cap[y]\neq\emptyset \implies [x]=[y]$$

So, equivalence classes are disjoint or equal.

Proof. Assume $[x]\cap[y]\neq\emptyset$. So $\exists z\in[x]\cap[y]$

Now let $a \in [x]$

$$\begin{aligned} a &\sim z && \text{(since } z \in [x] \text{ , } z \sim x \sim a) \\ z &\sim y && \text{(since } z \in [y]) \\ a &\sim y && \text{(transitivity)} \\ a &\in [y] \\ \therefore [x] &\subseteq [y] \end{aligned}$$

Now take $b \in [y]$, using the same arguments we get

$$\begin{aligned} b &\sim z && \text{(since } z \in [y] \text{ , } z \sim y \sim b) \\ z &\sim x && \text{(since } z \in [x]) \\ b &\sim x && \text{(transitivity)} \\ b &\in [x] \\ \therefore [y] &\subseteq [x] \end{aligned}$$

□

Observation: If X is some set with an equivalence relation, then every $x \in X$ is in some equivalence class.

Definition 1.2.1 (Partitions)**.** *Say we have some $R_j \subseteq X$ for $j \in \{1, 2, \dots, n\}$, with every $x \in X$ in exactly one R_j, then the R_j form a partition of X.*

Theorem 1.2.2. *Let X be a set with an equivalence relation. Then the equivalence classes form a partition of X.*

Proof. If $z \in X$, then $z \in [z]$, therefore z is in at least one equivalence class. If $z \in [x]$ and $z \in [y]$, then $[x] \cap [y] \neq \emptyset$ therefore $[x] = [y]$ (as shown previously). Therefore z is in at most one equivalence class. □

Theorem 1.2.3. *Let R_j form a partition of X. Say that $x \sim y$ means $x, y \in R_j$ for some j. Then $\sim$ is an equivalence relation on X.*

Proof.

- $x \in X$, so $x \in R_j$ for some j $implies x, x \in R_j \implies x \sim x$
- $x \sim y \iff x, y \in R_j \iff y, x \in R_j \iff y \sim x$

- $$x \sim y \; y \sim z \implies \begin{cases} x, y \in R_i \\ y, z \in R_j \end{cases} \implies y \in R_i, R_j$$
  $$\implies i = j$$
  $$\implies x, z \in R_j$$
  $$\therefore x \sim z$$

$\square$

Example of Finding Equivalence Classes

Take $X = R \times R$, and let $(a, b) \sim (c, d)$ mean $a^2 + b^2 = c^2 + d^2$. Find the equivalence class of $(0, 0)$, $(3, 4)$, (a, b)

$$\begin{aligned} [(0,0)] &= \{(x, y) : (x, y) \sim (0, 0)\} \\ &= \{(x, y) : x^2 + y^2 = 0^2 + 0^2 = 0\} \\ &= \{(x, y) : x = y = 0\} \end{aligned}$$

$$\begin{aligned} [(3,4)] &= \{(x, y) : (x, y) \sim (3, 4)\} \\ &= \{(x, y) : x^2 + y^2 = 3^2 + 4^2 = 25\} \\ &= \{(x, y) : \sqrt{x^2 + y^2} = 5\} \end{aligned}$$

$$\begin{aligned} [(a,b)] &= \{(x, y) : (x, y) \sim (a, b)\} \\ &= \{(x, y) : x^2 + y^2 = a^2 + b^2 = r\} \\ &= \{(x, y) : \sqrt{x^2 + y^2} = r\} \end{aligned}$$

Lecture 2

Well-defined Operations on Equivalence Classes and Number Theory

2.1 Well-defined Operations on Equivalence Classes

Consider a set X, an equivalence relation $\sim$, and an operation $\cdot$. This operation is well-defined on equivalence classes if

$$\left.\begin{aligned} x \sim y \\ w \sim z \end{aligned}\right\} \implies x \cdot w \sim y \cdot z$$

$$\left.\begin{aligned} [x] = [y] \\ [w] = [z] \end{aligned}\right\} \implies [x \cdot w] = [y \cdot z]$$

Example: Let $X = \mathbb{R} \times \mathbb{R}$, $(a,b) \sim (c,d)$ means $a^2 + b^2 = c^2 + d^2$, is addition well-defined on equivalence classes? (*Addition meaning* $(x,y) + (z,y) = (x+z, y+w)$)

$$Let \begin{cases} (a,b) \sim (c,d) \\ (e,f) \sim (g,h) \end{cases} \quad then \begin{cases} a^2 + b^2 = c^2 + d^2 \\ e^2 + f^2 = g^2 + h^2 \end{cases}$$

Now,

$$\begin{cases} (a,b)+(e,f)=(a+e,b+f) \\ (c,d)+(g,h)=(c+g,d+h) \end{cases}$$

Question: Is $(a+e)^2+(b+f)^2=(c+g)^2+(d+h)^2$?

$$(a+e)^2+(b+f)^2=a^2+2ae+e^2+b^2+2bf+f^2$$

$$(c+g)^2+(d+h)^2=c^2+2cg+g^2+d^2+2dh+h^2$$

$a^2+b^2=c^2+d^2$, and $e^2+f^2=g^2+h^2$, so

$$(a+e)^2+(b+f)^2=(c+g)^2+(d+h)^2 \iff 2ae+2bf=2cg+2dh$$

Counterexample: Take

$$(a,b)=(c,d)=(1,2)$$

$$(e,f)=(3,4) \qquad (g,h)=(4,3)$$

So no, addition is not well defined.

Another Example: $X=(\mathbb{Z},\mathbb{Z}\setminus\{0\})$. $(a,b)\sim(c,d)$ means $ad=bc$. Is multiplication well-defined on equivalence classes? (*Multiplication meaning* $(x,y)\cdot(w,z)=(x\cdot w,y\cdot z)$). Let

$$\begin{cases} (a,b)\sim(c,d) \\ (e,f)\sim(g,h) \end{cases} \implies \begin{cases} ad=bc \\ ef=gh \end{cases}$$

Now,

$$\begin{cases} (a,b)\cdot(e,f)=(a\cdot e,b\cdot f) \\ (c,d)\cdot(g,h)=(c\cdot g,d\cdot h) \end{cases}$$

Question: Is $(ae,bf)\sim(cg,dh)$?

$$(ae)(dh)=\text{ad}\cdot\text{eh}$$
$$(bf)(cg)=\text{bc}\cdot\text{fg}$$

We have $(a,b) \sim (c,d)$, so $ad = bc$, and $(e,f) \sim (g,h)$, so $ek = fg$. So yes, multiplication is well-defined on equivalence classes.

2.2 Number Theory

Fact 2.2.1. *Every non-empty set $S \subseteq \mathbb{N}$ has a minimum element d in S*

Proposition 2.2.1. *Let $a, b \in \mathbb{Z}$, $b > 0$, then $\exists!\ q, r \in \mathbb{Z}$ with $a = bq + r$, $0 \leq r < b$*

Proof. (Existence) Let $S = \{a - bx : x \in \mathbb{Z}, a - bx \geq 0\}$. $\emptyset \neq S \subseteq \mathbb{N}$, so S has a minimum element.
Let

$$\begin{cases} r = \min(S) \\ q = \frac{a-r}{b} \end{cases}$$

$r = a - bd$, $d \in \mathbb{Z}$, then $bq + r = b(\frac{a-r}{b}) + r = a - r + r = a$.

If $b \leq r$, then $0 \leq r - b < r$, which contradicts the minimality of r.

(Uniqueness) Say $a = bq + r = bp + s$, $0 \leq r, s < b$. Then

$$b(q - p) = s - r$$

So $s - r$ is a multiple of b, but $0 \leq r, s < b$, so it must be that $r - s = 0$, therefore $r = s$. $\square$

Lecture 3

Number Theory Cont. and Integers Modulo n

3.1 More Number Theory

Definition 3.1.1. *$m \mid n$ means $\exists x \in \mathbb{Z}$ with $n = mx$*

Definition 3.1.2. *Let $a, b \in \mathbb{Z}$. If d is a positive integer with $d \mid a$ and $d \mid b$, if $c \mid a$ and $c \mid b$, then $c \mid d$, then d is a gcd of a and b.*

Theorem 3.1.1. *For every $a, b \in \mathbb{Z}$, $\exists\,!$ gcd d. Furthermore, $\exists x, y \in \mathbb{Z}$, $d = ax + by$. Furthermore, d is the largest common divisor of a,b*

Proof. Let $S = \{ax + by : x, y \in \mathbb{Z}, ax + by > 0\}$. $S \subseteq \mathbb{N}$, so $\exists\,!$ minimum element d in S.
Write

$$\begin{aligned} a &= dq + r && (0 \leq r < d) \\ a &= (ax + by)q + r && (\text{some } x, y \in \mathbb{Z}) \\ r &= a(1 - qx) + b(-qy) \\ r &= ax' + by' && (x' = 1 - qx,\ y' = -qy) \\ 0 \leq r = ax' + by' &< d \end{aligned}$$

So. either $r = 0$ or $r \in S$ but not both, but $r < d$ which is the minimum of the set. Therefore $r \notin S$. So $r = 0$ and $d \mid a$. Same argument with

$b = dq + r \implies d \mid b$.
Now suppose $c \mid a$ and $c \mid b$, then $a = a'c$ and $b = b'c$, $a', b' \in \mathbb{Z}$.

$$d = ax + by = a'cx + b'cy = c(a'x + b'y$$

So, $c \mid d$. □

Corollary 3.1.1. *If $gcd(a, b) = 1$, then $\exists x, y$ such that $ax + by = 1$.*

Proof. Same as the previous proof, in the case that $gcd(a, b) = 1$. □

Corollary 3.1.2. *If $gcd(a, b) = d$, then $\{ax + by : x, y \in \mathbb{Z}\} = d \cdot n$, $\forall n \in \mathbb{Z}$.*

Proof. No proof was provided in the notes I guess. :P □

Definition 3.1.3 (Least Common Multiple)**.** *Let $a, b \in \mathbb{Z}$. If m is a positive integer with*

- *$a \mid m$ and $b \mid m$*
- *if $a \mid n$ and $b \mid n$, then $m \mid n$*

then m is a lcm of a, b.

Theorem 3.1.2. *For every a, b, $\exists$! lcm m.*

Definition 3.1.4. *$p \in \mathbb{Z}$ $p > 1$*

- *p is irreducible if the only positive divisors of p are 1 and p*
- *p is prime if whenever $p \mid ab$, then $p|a$ or $p|b$*

Proposition 3.1.1. *p is prime $\implies$ p is irreducible*

Proof. Say p is not irreducible, $p = ab$, and $1 < a, b < p$. Then $p \nmid a$ and $p \nmid b$. □

Proposition 3.1.2. *p is irreducible $\implies$ p is prime*

Proof. $p \mid ab \implies ab = mp$ for some $m \in \mathbb{Z}$. Say $p \nmid a$, since p is irreducible $gcd(a, p) = 1$. So $\exists s, t$ such that $as + pt = 1$.

$$b = b(as + pt) = abs + bpt \qquad = mps + bpt = (ms + bt)p$$

Therefore b is a multiple of p, so $p \mid b$. □

3.2 Prime Factorization

Theorem 3.2.1. $n \in \mathbb{Z}$ $n > 1$

$$\exists! \begin{cases} p_1 p_2 \ldots p_s, & \textit{distinct primes} \\ e_1 e_2 \cdots e_s, & \textit{positive integers} \end{cases}$$

With

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

Proof. Proof was omitted. □

Prime Factorization Gives GCD:

Example:

$$a = 2 \cdot 5 \cdot 7^{10} \cdot 13 = 2^{1} \cdot 3^{0} \cdot 5^1 \cdot 7^{10} \cdot 13^1 \cdot 17^{0}$$

$$b = 2 \cdot 3^2 \cdot 7^2 \cdot 17 = 2^1 \cdot 3^2 \cdot 5^{0} \cdot 7^{2} \cdot 13^{0} \cdot 17^1$$

$$gcd(a, b) = 2^1 \cdot 7^2$$

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

$$b = q_1^{F_1} \cdot q_2^{F_2} \cdots q_s^{F_s}$$

$$\forall \text{ prime } p, \text{ define } g(p) = min \begin{cases} e_i & \text{if } p = p_i \\ f_j & \text{if } p = q_j \cdot \\ 0 \end{cases}$$

Then,

$$gcd(a, b) = \prod_{prime\ p} p^{g(p)}$$

Prime Factorization Gives LCM:

Example:

$$a = 2 \cdot 5 \cdot 7^{10} \cdot 13 = 2^1 \cdot 3^0 \cdot 5^{1} \cdot 7^{10} \cdot 13^{1} \cdot 17^0$$

$$b = 2 \cdot 3^2 \cdot 7^2 \cdot 17 = 2^{1} \cdot 3^{2} \cdot 5^0 \cdot 7^2 \cdot 13^0 \cdot 17^{1}$$

$$lcm(a, b) = 2^1 \cdot 3^2 \cdot 5^1 \cdot 7^{10} \cdot 13^1 \cdot 17^1$$

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

$$b = q_1^{F_1} \cdot q_2^{F_2} \cdots q_s^{F_s}$$

$\forall$ prime p, define $l(p) = min \begin{cases} e_i & \text{if } p = p_i \\ f_j & \text{if } p = q_j \\ 0 & \end{cases}.$

Then,

$$gcd(a,b) = \prod_{prime\ \ p} p^{l(p)}$$

3.2.1 Summary

- Definition of $gcd(a,b)$
- $d = gcd(a,b)$ exists $\implies$ d is a divisor of a and b and $d = ax + by$ for some $x, y \in \mathbb{Z}$
- also, $\{ax + by : x, y \in \mathbb{Z}\} = d\mathbb{Z}$
- Definition of $lcm(a,b)$
- $m = lcm(a,b)$ exists and is unique, m is the smallest common multiple.
- Prime Factorization exists and is unique.
- Prime factorization of a and b gives $gcd(a,b)$ and $lcm(a,b)$
- $gcd(a,b) \cdot lcm(a,b) = |ab|$

3.3 Integers Modulo n

Let $n \in \mathbb{Z}$ with $n \geq 2$, $a \equiv b \pmod{n}$ means $n \mid (a - b)$. So

$$\begin{aligned} a \equiv b \quad (\text{mod } ()n) &\iff n \mid (a-b) \\ &\iff a - b = kn, \text{ for some } k \in \mathbb{Z} \\ &\iff \frac{a-b}{n} \in \mathbb{Z} \end{aligned}$$

Proposition 3.3.1. *Congruence modulo n is an equivalence relation.*

Proof.

- **Reflexivity:** Show $a \equiv a \pmod{n}\ \forall a \in \mathbb{Z}$.

$$\frac{a-a}{n} = 0 \in \mathbb{Z}$$

  So $a \equiv a \pmod{n}$.

- **Symmetric:** Show $a \equiv b \iff b \equiv a\ \forall a, b \in \mathbb{Z}$

$$a \equiv b \iff \frac{a-b}{n} \in \mathbb{Z} \iff -\frac{a-b}{n} = \frac{b-a}{n} \in \mathbb{Z} \iff b \equiv a$$

- **Transitivity:** Show $a \equiv b \wedge b \equiv c \implies a \equiv c\ \forall a, b, c \in \mathbb{Z}$

$$\begin{aligned} a \equiv b \equiv c &\implies \frac{a-b}{n} \in \mathbb{Z} \wedge \frac{b-c}{n} \in \mathbb{Z} \\ &\implies \frac{a-b}{n} + \frac{b-c}{n} = \frac{a-c}{n} \in \mathbb{Z} \implies a \equiv c \end{aligned}$$

□

Example: Define $\mathbb{Z}_n = \{[k]_n : k \in \mathbb{Z}\}$. Consider $n = 5$.

- $[2] = \{\ldots, -8, -3, 2, 7, 12, 17, \ldots\}$
- $[0] = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$
- $[7] = \{\ldots, -3, 2, 7, 12, 17, 22, \ldots\}$

Note: $\mathbb{Z}_5$ is a set containing 5 elements, and each element of $\mathbb{Z}_5$ is a subset of $\mathbb{Z}$. Also, recall that equivalence classes are disjoint or equal, so $[2] = [7]$

$$\begin{aligned} \mathbb{Z}_5 &= \{[0], [1], [2], [3], [4]\} \\ &= \{[-2], [-1], [0], [1], [2]\} \end{aligned}$$

Examples:

(a) $[3]_5 = [8]_5$ $3 \equiv 8 \pmod{5}$ $5|(8-3)$

(b) $[3]_9 = [-24]_9 \ 3 \equiv -24 \pmod 9 \ 9|(3-(-24))$

All representations in an equivalence classare equivalent (modulo n).

Question: Is addition and multiplication on $\mathbb{Z}_n$ well defined? We want

$$\begin{cases} [a] + [b] = [a+b] \\ [a] \cdot [b] = [a \cdot b] \end{cases}$$

We'll see this in the lecture.

Lecture 4

Operations on $\mathbb{Z}_n$, Symmetries, and Groups

4.1 Arithmetic Modulo n

Question: Is addition and multiplication on $\mathbb{Z}_n$ well defined? We want

$$\begin{cases} [a] + [b] = [a + b] \\ [a] \cdot [b] = [a \cdot b] \end{cases}$$

Proposition 4.1.1. *Let $n \in \mathbb{Z}$ $n \geq 2$. Suppose*

$$\begin{aligned} a &\equiv a' \pmod{n} \\ b &\equiv b' \pmod{n} \end{aligned}$$

then

$$\begin{aligned} a + b &\equiv a' + b' \pmod{n} \\ ab &\equiv a'b' \pmod{n} \end{aligned}$$

So, addition and multiplication on integers are well well defined on congruence classes.

Proof.

$$a \equiv a' \pmod n \iff \frac{a-a'}{n} \in \mathbb{Z} \iff a' = a + sn \text{ for some } s \in \mathbb{Z}$$

$$b \equiv b' \pmod n \iff \frac{b-b'}{n} \in \mathbb{Z} \iff b' = b + tn \text{ for some } t \in \mathbb{Z}$$

Then,

$$\frac{(a+b)-(a'+b')}{n} = \frac{a-a'}{n} + \frac{b-b'}{n} \in \mathbb{Z}$$

So, $a + b \equiv a' + b' \pmod n$. Also,

$$\begin{aligned}\frac{ab-a'b'}{n} &= \frac{ab-a'b+a'b-a'b'}{n}\\ &= \left(\frac{a-a'}{n}\right)b + \left(\frac{b-b'}{n}\right)a' \in \mathbb{Z}\end{aligned}$$

So $ab \equiv a'b' \pmod n$. ◻

4.1.1 Properties of Arithmetic Modulo n

- **Commutative:** $a + b \equiv b + a \pmod n$
- **Commutative:** $ab \equiv ba \pmod n$
- **Associative:** $(a + b) + c \equiv a + (b + c) \pmod n$
- **Associative:** $(ab)c \equiv a(bc) \pmod n$
- **Distributive:** $a(b + c) \equiv ab + ac \pmod n$
- **Identity for +:** $a + 0 \equiv a \pmod n$
- **Identity for ·:** $a \cdot 1 \equiv a \pmod n$
- **Additive Inverse:** $a + (-a) \equiv 0 \pmod n$
- **Multiplicative Inverse?**

4.1.2 Multiplicative Inverses

Proposition 4.1.2. *Let* $a \in \mathbb{Z}_n$, $\exists b \in \mathbb{Z}_n$ *with* $ab \equiv 1 \pmod n \iff gcd(a, n) = 1$.

Proof. Suppose such b exists, then

$$\begin{aligned} ab - 1 &= rn \text{ for some } r \in \mathbb{Z} \\ ab + (-r)n &= 1 \\ \therefore gcd(a, n) &= 1 \end{aligned}$$

Suppose $gcd(a, n) = 1$, then

$$\begin{aligned} as + nt &= 1 \text{ for some } s, t \in \mathbb{Z} \\ as - 1 &= (-t)n \\ as &\equiv 1 \pmod{n} \end{aligned}$$

So we can choose $b = s$. ☐

Example: Addition and multiplication in $\mathbb{Z}_6$.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

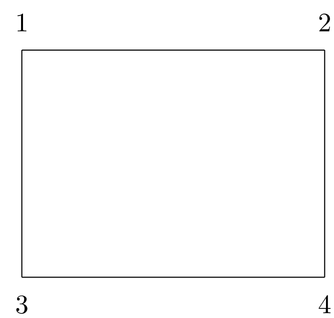
The table being symmetric implies that addition is commutative, 0-row and 0-column implies that 0 is the identity for addition, every row has a 0 imples that the addiive inverse exists for every element.

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

The table being symmetric implies that multiplication is commutative, 1-row and 1-column is the header implies that 1 is the identity for multiplication, some rows not having 1 implies that some elements have no multiplicative inverse.

4.2 Symmetries

Consider the symmetries of a rectangle.



The notation for functions is

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ f(1) & f(2) & f(3) & f(4) \end{pmatrix}$$

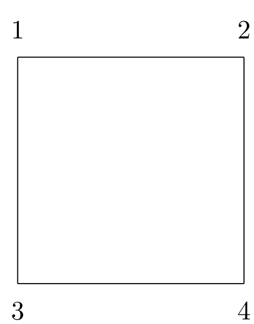
$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Claim: $\{\epsilon, \rho, \alpha, \beta\}$ are *all* the symmetries of a rectangle.

Proof. DGD Question - Will add later. $\square$

Consider the symmetries of a square.



$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	ϵ	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$	$90°$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	$180°$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$	$270°$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

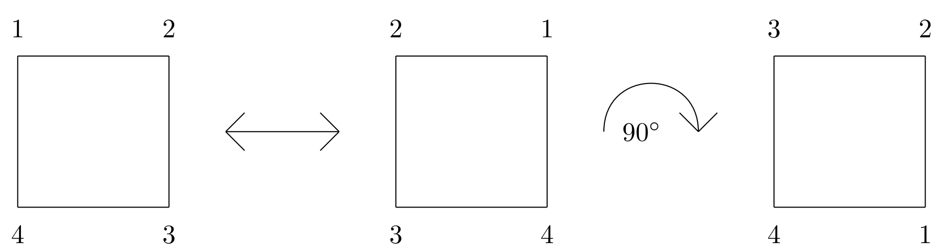
4.2.1 Properties of Symmetries

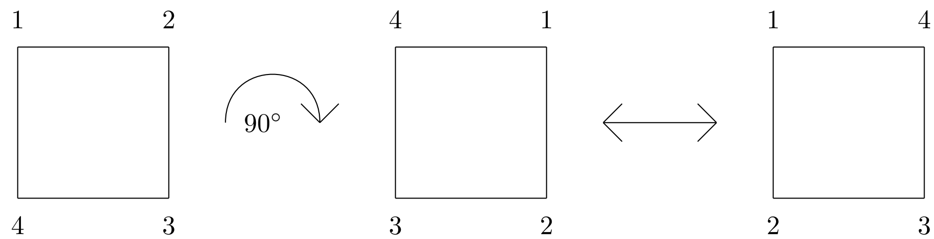
$S = \{\alpha, \beta, \ldots\}$ symmetries of some objection, with the operation composition.
Properties:

- $\alpha \circ \beta$ is a symmetry $\forall \alpha, \beta \in S$
- $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ $\forall \alpha, \beta, \gamma \in S$
- $\exists \epsilon \in S$ such that $\epsilon \circ \alpha = \alpha \circ \epsilon = \alpha$ $\forall \alpha \in S$
- $\forall \alpha \in S$, $\exists \beta \in S$, such that $\alpha \circ \beta = \beta circ \alpha = \epsilon$ $\forall \alpha, \beta \in S$

Note: we often write $\alpha\beta$ instead of $\alpha \circ \beta$

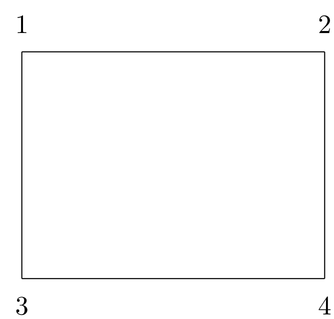
Example: $S =$ symmetries of some object, is $gh = hg$ $\forall g, h \in S$?. **Answer:** For a rectangle, yes. But for a square, no.





These symmetries do not compute, so $gh \neq hg$.

4.2.2 Generating Sets



$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Check: $\alpha\beta = \rho$, $\alpha^2 = \epsilon$. So $\forall g \in S$, g can be written in terms of α, β.

We say that $\{\alpha, \beta\}$ generates S

4.3 Groups

Let S be some set with some operation $\cdot$. Then $(S, \cdot)$ is a group if

- **Closure:** $ab \in S \ \forall a, b \in S$
- **Associativity:** $(ab)c = a(bc) \ \forall a, b, c \in S$
- **Identity:** $\exists \epsilon \in S$ such that $x\epsilon = \epsilon x = x \ \forall x \in S$

- **Inverses:** $\forall x \in S$, $\exists y \in S$ such that $xy = yx = \epsilon$

Examples:

- Symmetries of an object form a group.
- $(\mathbb{R}, +)$ forms a group.
- $(\mathbb{R} \setminus \{0\}, \cdot)$ forms a group.
- $(\mathbb{Z}, +)$ forms a group.
- $(\mathbb{Z}, \cdot)$ does not form a group since inverses are typically not integers.
- $(\mathbb{Z}_n, +)$ forms a group.
- $(\mathbb{Z}_n \setminus \{0\}, \cdot)$ forms a group.

Lecture 5

More Examples of Groups

This lecture was not well organizing so I am not gonna type it out.

Lecture 6

Basic Properties of Groups, Products of Groups, Isomorphisms

Examples were left out I may come back to finish

6.1 Basic Properties of Groups

Proposition 6.1.1. *In every group, the identity is unique.*

Proof. Suppose a, b are identities, so

$$\left.\begin{array}{l}\text{ax} = \text{xa} = \text{x} \\ \text{bx} = \text{xb} = \text{x}\end{array}\right\} \forall x$$

Because b is an identity, we have $a = ab$, and since a is an identity, we have $ab = b$. So

$$a = ab = b$$

$$\therefore a = b$$

$\square$

Proposition 6.1.2. *In every group, the equation $ax = b$ has a unique solution x for all a, b*

Proof. There was no proof :(☐

Proposition 6.1.3. *In every group, $ab = ac \implies b = c$*

Proof. Again, no proof :(☐

Note: For matricies it is not the same, $AB = AC \not\Longrightarrow B = C$

Proposition 6.1.4. *In every group, $(ab)^{-1} = b^{-1}a^{-1}$*

Proof.

$$\begin{aligned}(ab)(b^{-1}a^{-1}) &= a(bb^{-1})a^{-1} \\ &= a\epsilon a^{-1} \\ &= aa^{-1} \\ &= \epsilon\end{aligned} \qquad \begin{aligned}(b^{-1}a^{-1})(ab) &= b^{-1}(a^{-1}a)b \\ &= b^{-1}\epsilon b \\ &= b^{-1}b \\ &= \epsilon\end{aligned}$$

☐

Proposition 6.1.5. *In every group, $(a^{-1})^{-1} = a$*

Proof. Since a^{-1} is the inverse of a, we have

$$aa^{-1} = a^{-1}a = \epsilon$$

but then,

$$a^{-1}a = aa^{-1} = \epsilon$$

So a is the inverse of a^{-1} ☐

Proposition 6.1.6. *In every group, if $xy = x$, for some x, y, then $y = \epsilon$. So if y behaves as the identity just once, then y is the identity.*

Proof. No proof again :P. ☐

Proposition 6.1.7. *In every group, if $xy = \epsilon$, for some x, y, then $y = x^{-1}$. So if y behaves like x^{-1} on one side, then y is x^{-1}*

Proof. No proof D: ☐

Proposition 6.1.8. *In every group, the Cayley table has exactly one row and column that matches the headers, and no other row or column mathes the header even once.*

Proof. Start by taking G to be some group, then let $x, y \in G$. And let H be a subgroup of G.

just kidding no proof. $\square$

Proposition 6.1.9. *In every group, every row and column of the Cayley table contains each element exactly once.*

Proof. Why does the prof include a spot for the proof. $\square$

6.1.1 Small Groups

- Say G has one element $G = \{x\}$
  Closure: $x \cdot x = x$
  Identity: $x = \epsilon$
  Inverse: $x^{-1} = x$

$\cdot$	x
x	x

- Say G has two elements, it must have an identity so $G = \{\epsilon, x\}$ If $xx = x$, $x = \epsilon$, this is a contradiction, So $xx = \epsilon$

$\cdot$	ϵ	x
ϵ	ϵ	x
x	x	ϵ

- Say G has three elements. $G = \{\epsilon, x, y\}$

$\cdot$	ϵ	x	y
ϵ	ϵ	x	y
x	x	y	ϵ
y	y	ϵ	x

$$x\epsilon = x \implies xy \neq x$$
$$\epsilon y = y \implies xy =\neq y$$

So $xy = \epsilon$

$$x\epsilon = x \implies xx \neq x$$
$$xy = \epsilon \implies xx \neq \epsilon$$

So $xx = y$

- Say G has 4 elements. Assignment Question!

6.2 Products of Groups

G, H are groups, define

$$G \times H = \{(g, h) : g \in G, h \in H\}$$

$$(x, a) \cdot (y, b) = (x \cdot y, a \cdot b)$$

$$G_1 \times G_2 \times \cdots G_k = \{(g_1, g_2, \ldots, g_n : g_j \in G_j\}$$

To reiterate, operations are done by component according to the operations of the group. i.e Suppose we have a group $G = (A, +)$ and $H = (B, \cdot)$ and $g, a \in G$, $h, b \in H$.

$$(g, h) \times (a, b) = (g + a, h \cdot b)$$

Proposition 6.2.1. *The product of groups is a group.*

Proof. Exercise. □

6.3 Isomorphisms

Suppose $\phi : G \to H$ is a bijection between two groups with the property

$$\phi(xy) = \phi(x)\phi(y)$$

Then ϕ is an isomorphism of $G \cong H$. So

$$G : x \cdot y = z \implies H : \phi(x) \cdot \phi(y) = \phi(z)$$

$\cdot$	y
x	z

$\cdot$	y'
x’	z'

$$x' = \phi(x) \qquad y' = \phi(y) \qquad z' = x'y' = \phi(z)$$

Start with G's Cayley table, change the names (symbols, consistently) and permute the rows and columns. This gives H's Cayley table.

Example:

$\mathbb{Z}_2 = \{0, 1\}$

$+$	0	1
0	0	1
1	1	0

$\mathbb{Z}^\times = \{-1, 1\}$

$\cdot$	1	-1
1	1	-1
-1	-1	1

$G = (\{\mathbb{Z}^+, \mathbb{Z}^-\}, \cdot)$

$\cdot$	$+$	$-$
$+$	$+$	$-$
$-$	$-$	$+$

$$\mathbb{Z}_2 \cong \mathbb{Z}^\times \cong G$$

Proposition 6.3.1. *All groups with two elements are isomorphic*

Proof. If G has two elements, then its Cayley table looks like

$\cdot$	ϵ	x
ϵ	ϵ	x
x	x	ϵ

Except they may use different symbols and have reordered rows/columns, so they are all isomorphic. $\square$

Lecture 7

Automorphisms, Subgroups

7.1 Automorphisms

Example: Let $H =$ *symmetries of a rectangle* and $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(x, y) : x \in \mathbb{Z}_2,\ y \in \mathbb{Z}_2\}$

$$H = \{\epsilon, \alpha, \beta, \rho\} \textit{ with composition}$$

$$K = \{00, 01, 10, 11\} \textit{ with addition in } \mathbb{Z}_2$$

H	ϵ	α	β	ρ
ϵ	ϵ	α	β	ρ
α	α	ϵ	ρ	β
β	β	ρ	ϵ	α
ρ	ρ	β	α	ϵ

G	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

$$\phi \begin{cases} \epsilon \to 00 \\ \alpha \to 01 \\ \beta \to 10 \\ \rho \to 11 \end{cases} \qquad \textit{or} \qquad \phi \begin{cases} \epsilon \to 00 \\ \alpha \to 01 \\ \beta \to 11 \\ \rho \to 10 \end{cases}$$

In fact, all we need for the isomorphism is $\epsilon \to 00$, we can have $\alpha, \beta, \rho \to 01, 10, 11$ in any order.

An automorphism of G is an isomorphism $G \to G$, this is a symmetry group of G. The set of all automorphisms of G is a group we call $aut(G)$, the automorphism group of G.

H	ϵ	α	β	ρ
ϵ	ϵ	α	β	ρ
α	α	ϵ	ρ	β
β	β	ρ	ϵ	α
ρ	ρ	β	α	ϵ

H	ϵ	α	ρ	β
ϵ	ϵ	α	ρ	β
α	α	ϵ	β	ρ
ρ	ρ	β	ϵ	α
β	β	ρ	α	ϵ

Let ϕ be any bijection $\{\epsilon, \alpha, \beta, \rho\} \to \{\epsilon, \alpha, \rho\, \beta\}$ with $\phi(\epsilon) = \epsilon$. Then ϕ is an automorphism of H.

Exercise: Let $G =$ *the symmetries of an equilateral triangle.* Show that

$$aut(H) \cong G$$

7.2 Quaternions

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$\pm$ and 1 operate as expected. And

$$i^2 = j^2 = k^2 = ijk = -1$$

Q_8	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	j	j
$-i$	$-i$	i	1	1	$-k$	k	$-j$	j
j	j	$-j$	$-k$	k	-1	1	$-i$	i
$-j$	$-j$	j	k	$-k$	1	-1	i	$-i$
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

- **Closure:** Yes, Q_8 is closed.

- **Identity:** 1 is the identity for Q_8.
- **Inverse:** Every column has the identity (1), so an inverse exists for every element in Q_8.
- **Associativity:** Consider the set of matrices M_8 with entries in $\mathbb{C}$

$$M_8 = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right\}$$

  And the function $\phi : M_8 \to Q_8$

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

  $\phi : M_8 \to Q_8$ is a bijection

$$\phi(ab) = \phi(a)\phi(b)$$

  Because M_8 is a set of matrices, it is closed, associative, has an identity and has inverses, therefore M_8 is a group. Q_8 is isomorphic to M_8, so it follows that it is also a group.

Therefore, Q_8 is closed, has identity, has inverses and is associative.

7.3 Subgroups

Consider the following

- G is a group with operation $\cdot$
- H is a subset of G
- H is a group with the same operation $\cdot$

Then H is a subgroup of G. We denote subgroups as $H \leq G$ or $H < G$

$$(\mathbb{Z}, +) < (\mathbb{Q}, +) < (\mathbb{R}, +) < (\mathbb{C}, +)$$

Example:

$$(\mathbb{Z}_3, +) \not\leq (\mathbb{Z}_5, +)$$

This is the case because

$$\mathbb{Z}_3 = \{0,1,2\} = \{[0],[1],[2]\} \not\subseteq \{[0],[1],[2],[3],[4]\} = \{0,1,2,3,4\} = \mathbb{Z}_5$$

These sets are equivalence classes **not** numbers so they are not subsets of each other.

7.3.1 Subgroup Test

Proposition 7.3.1. *Suppose H is a subset of G, if $H \neq \emptyset$*

$$x, y \in H \implies xy \in H$$

$$x \in H \implies x^{-1} \in H$$

then H is a subgroup.

Proof. Show that H is a group

- **Closure:** Is given.
- **Associative:** G is associative so any subset "inherits" associativity.
- **Identity:** Let ϵ_g be the identity in G. $\epsilon_a \cdot a = a \ \forall a \in H$. $\exists a \in H$, so $a^{-1} \in H$, since H is a subset of G, $a, a^{-1} \in G$, therefore $a \cdot a^{-1} = \epsilon_g \in H$.
- **Inverse:** Given

□

Proposition 7.3.2. *H a subgroup of $G \implies \epsilon_g \in H$ and so $\epsilon_H \in G$*

Proof. $H \neq \emptyset$, so let $x \in H$, then $x^{-1} \in H$, then $x \cdot x^{-1} = \epsilon_G \in H$. Furthermore

$$\epsilon_G \cdot h = h \cdot \epsilon_G = h \ \forall h \in H$$

Since $H \subseteq G$,and H has a unique identity, then $\epsilon_G = \epsilon_H$ □

7.3.2 Alternative Versions of Subgroup Test

Suppose H is a subset of G, if

- $H \neq \emptyset$

- $x, y \in H \implies xy \in H$
- $x \in H \implies x^{-1} \in H$

then H is a subgroup.

Lecture 8

Lattices and Cyclic Groups

Recall: H is a subgroup of G if

- $H \subseteq G$
- They have the same operation (Cayley table of H is obtained by deleting rows/columns from G
- H is a group

Subgroup Test: If $H \subseteq G$ with the same operation and H is not empty,

$$x, y \in H \implies xy \in H$$

$$x \in H \implies x^{-1} \in H$$

then H is a subgroup of G.

8.1 Find all subgroups of $(\mathbb{Z}, +)$

Say H is a subgroup of $\mathbb{Z}$, $H \neq \{0\}$. Let n be the smallest positive integer in H, then

$$\{\dots, -n, 0, n, 2n, 3n, \dots\} \subseteq H$$

$$n\mathbb{Z} = \{nK : k \in Z\} \subseteq H$$

Suppose $x \in H \setminus n\mathbb{Z}$, then write $x = qn + r$ for $0 \leq r < n$. By closure, we have

$$x - qn = r \in H$$

But, this contradicts the minimality of n unless $r = 0$, but if $r = 0$, then $x \in n\mathbb{Z}$. Therefore, $H = n\mathbb{Z}$

Subgroups of $\mathbb{Z}$: $n\mathbb{Z}\ \forall n \in \mathbb{Z}$, $(n = 0 \implies H = \{0\})$

8.2 Symmetries of a Square

Lemma 8.2.1. *There are at most eight symmetries of a square.*

Proof. Let γ be a symmetry, γ maps corners to corners.

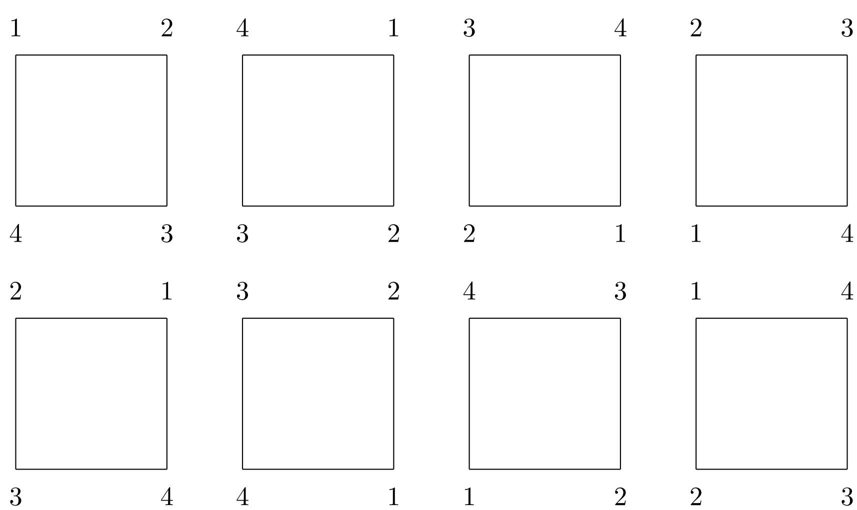
- $\gamma(1)$ has at most four possibilities, then $\gamma(2)$ must be one of the corners adjacent to $\gamma(1)$
- $\gamma(2)$ has at most two possibilities, then $\gamma(4)$ must be the other corner adjacent to $\gamma(1)$
- $\gamma(4)$ has at most one possibility, then $\gamma(3)$ must be $\{1, 2, 3, 4\} \backslash \{\gamma(1), \gamma(2), \gamma(3), \gamma(4)\}$
- $\gamma(3)$ has at most one possibility

So, we have $4 \cdot 2 \cdot 1 \cdot 1 = 8$ possibilities. $\square$

Question: Do all possibilities work?

Lemma 8.2.2. *There are at least eight symmetries of a square*

Proof. Consider the symmetries of a square.



$\square$

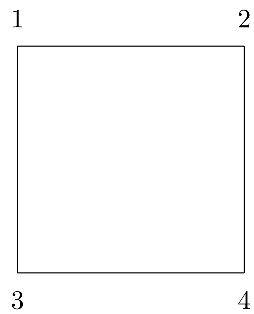
Let

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Proposition 8.2.1.

$$\rho\mu = \mu\rho^{-1} = \mu\rho^3$$

Proof. Consider the square and function μ, ρ.



$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\rho\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad \mu\rho^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

The 2 functions are equal. $\square$

Example:

$$\rho^2\mu\rho\mu\rho^3 = \mu\rho^6\rho\mu\rho^3 = \mu\rho^7\mu\rho^3 = \mu\mu\rho^{21}\rho^3 = \mu\rho^{24} = \epsilon$$

$$\mu\rho\mu\rho^2\mu\rho = \mu\mu\rho^3\rho^2\mu\rho = \mu^2\rho^5\mu\rho = \rho\mu\rho = \mu\rho^3\rho = \mu\rho^3\rho = \mu$$

Corollary 8.2.1.

$$
\begin{aligned}
G &=< \mu, \rho : \mu^2 = \epsilon, \rho^4 = \epsilon, \rho\mu = \mu\rho^3 > \\
&= \{\mu^i \rho^j : 0 \leq i \leq 1, 0 \leq j \leq 3\} \\
&= \{\rho^i \mu^i : 0 \leq i \leq 1, 0 \leq j \leq 3\}
\end{aligned}
$$

Proof. Any sequence of μ's and ρ's can be written as $\mu^s \rho^t$ using $\rho\mu = \mu\rho^3$ ($\rho^t \mu^s$ using $\mu\rho = \rho^3\mu$) reduce powers on μ and ρ using $\mu^2 = \epsilon$ $\rho^4 = \epsilon$

$$G = \{\mu^i \rho^j : 0 \leq i \leq 1,\ 0 \leq j \leq 3\}$$

These are all distinct since $|G| = 8$ so the relations $\mu^2 = \epsilon$ $\rho^4 = \epsilon$ $\rho\mu = \mu\rho^3$ are sufficient to characterize G $\square$

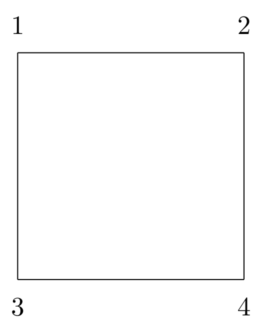
Compare:

$$F =< \alpha, \beta : \alpha^2 = \epsilon, \beta^4 = \epsilon >$$

$\alpha\beta, \alpha\beta\alpha, \alpha\beta\alpha\beta, \ldots$ are all distinct

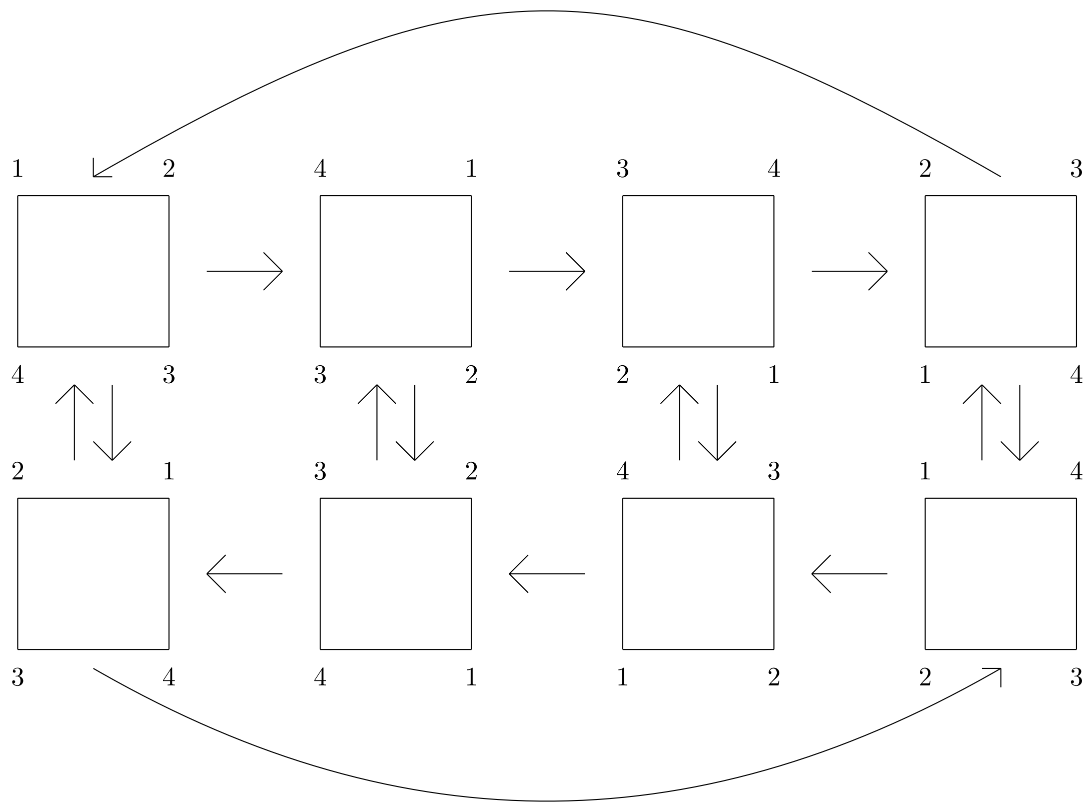
$$\alpha^3 \beta^7 \alpha\beta = \alpha\beta^3 \alpha\beta$$

$$|F| = \infty$$



$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$
\begin{aligned}
G &= \{\mu^i \rho^i : 0 \leq i \leq 1, 0 \leq j \leq 3\} \\
&= \{\rho^i \mu^i : 0 \leq i \leq 1, 0 \leq j \leq 3\}
\end{aligned}
$$



Elements of G are symmetries of a square, they also permute G itself! Buy they are not symmetries of G.

$$\mu \cdot \rho = \mu\rho$$
$$\mu\mu\mu\rho = \mu(\mu)\mu(\rho) \neq \mu(\mu\rho) = \mu\mu\rho$$

8.2.1 Subgroups of Symmetries of a Square

$$G = Sym(\square) = \{\mu^i\rho^i : 0 \leq i \leq 1, 0 \leq j \leq 3\}$$

$$
\begin{aligned}
\langle \epsilon \rangle &= \{\epsilon\} \\
\langle \mu \rangle &= \{\epsilon, \mu\} \\
\langle \rho \rangle &= \{\epsilon, \rho, \rho^2, \rho^3\} \\
\langle \rho^2 \rangle &= \{\epsilon, \rho^2\} \\
\langle \rho^3 \rangle &= \{\epsilon, \rho^3, \rho^2, \rho\} \\
\langle \mu\rho \rangle &= \{\epsilon, \mu\rho\} \\
\langle \mu\rho^2 \rangle &= \{\epsilon, \mu\rho^2\} \\
\langle \mu\rho^3 \rangle &= \{\epsilon, \mu\rho^3\}
\end{aligned}
$$

$$
\begin{aligned}
\langle \mu, \rho \rangle &= G = \langle \mu, \rho^3 \rangle \\
\langle \mu, \rho^2 \rangle &= \{\epsilon, \mu, \rho^2, \mu\rho^2\} \\
\\
\langle \mu, \mu\rho \rangle &= \{\epsilon, \mu, \rho, \dots\} = G \\
\langle \mu, \mu\rho^3 \rangle &= \{\epsilon, \mu, \rho^3, \dots\} = G \\
\langle \mu, \mu\rho^2 \rangle &= \{\epsilon, \mu, \rho^2, \mu\rho^2\} \\
\\
\langle \rho, \mu\rho \rangle &= \{\epsilon, \mu, \rho, \dots\} = G \\
\langle \rho, \mu\rho^3 \rangle &= \{\epsilon, \mu, \rho, \dots\} = G \\
\langle \rho, \mu\rho^2 \rangle &= \{\epsilon, \mu, \rho, \dots\} = G \\
\langle \rho^2, \mu\rho^2 \rangle &= \{\epsilon, \rho^2, \mu\rho, \mu\rho^3\} = G \\
\langle \rho^2, \mu\rho^3 \rangle &= \{\epsilon, \rho^2, \mu\rho^3, \mu\rho\} = G \\
\langle \rho^2, \mu\rho^2 \rangle &= \{\epsilon, \rho^2, \mu\rho^2, \mu\} = G \\
\langle \mu\rho, \mu\rho^2 \rangle &= \{\epsilon, \rho, \mu, \dots\} = G \\
\langle \mu\rho, \mu\rho^3 \rangle &= \{\epsilon, \mu\rho, \mu\rho^3, \rho^2\} = G \\
\langle \mu\rho^2, \mu\rho^3 \rangle &= \{\epsilon, \rho, \mu, \dots\} = G
\end{aligned}
$$

Lecture 9

Cyclic Groups

9.1 Cyclic Groups

G is cyclic if $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ for some $g \in G$. g is a generator of G, there could be other generators. For addition,

$$G\langle g \rangle = \{kg : k \in \mathbb{Z}\}$$

The order of g is the smallest positive integer n with $g^n = \epsilon$, written as $|g|$. For addition, it's the smallest positive integer n with $ng = \epsilon$.

Proposition 9.1.1. *G is cyclic $\implies$ G is abelian*

Proof. Since G is cyclic, then $G = \langle g \rangle$ for some $g \in G$, take $x, y \in G$. Then $x = g^s$ and $y = g^t$. So

$$xy = g^s g^t = g^{s+t} = g^{t+s} = g^t g^s = yx$$

□

However, G being abelian $\not\Longrightarrow$ G is cyclic. **Examples:** Are the following in cyclic? Find generators, and all orders

Q_8:

- $\langle 1 \rangle = \{1\}$ Order 1
- $\langle -1 \rangle =$ {-1,1} Order 2

- $\langle i \rangle = \{i, -1, -i, 1\}$ Order 4
- $\langle -i \rangle = \{-i, -1, i, 1\}$ Order 4
- $\langle \pm j \rangle = \{\pm j, -1, \mp j, 1\}$ Order 4
- $\langle \pm k \rangle = \{\pm k, -1, \mp k, 1\}$ Order 4

Not cyclic

$\mathbb{Z}$:

- $\langle 1 \rangle = \{k \cdot 1 : k \in \mathbb{Z}\} = \mathbb{Z}$ Order is ∞, so no finite order

Are there other generators? Consider -1

- $\langle -1 \rangle = \{k \cdot (-1) : k \in \mathbb{Z}\} = \mathbb{Z}$

$\mathbb{Z}_5$:

- $\langle 1 \rangle = \{1, 2, 3, 4, 5 = 0\}$ Order 5
- $\langle 2 \rangle = \{2, 4, 6 = 1, 3, 5 = 0\}$ Order 5
- $\langle -2 \rangle = \langle 3 \rangle = \{3, 1, 4, 2, 5 = 0\}$ Order 5
- $\langle -1 \rangle = \langle 4 \rangle = \{4, 3, 2, 1, 5 = 0\}$ Order 5
- $\langle 0 \rangle = \{0\}$

Therefore $\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle$ generate the group, so it is cyclic. **$\mathbb{Z}_9^\times$:**

- $\langle 1 \rangle = \{1\}$ Order 1
- $\langle 2 \rangle = \{2, 4, 8, 16 = 7, 14 = 5, 10 = 1\}$ Order 6
- $\langle -4 \rangle = \langle 4 \rangle = \{4, 7, 1\}$ Order 3
- $\langle -2 \rangle = \langle 5 \rangle = \{5, 7, 8, 4, 2, 1\}$ Order 6
- $\langle -1 \rangle = \langle 8 \rangle = \{8, 1\}$ Order 2

Therefore $\langle 2 \rangle$ and $\langle 5 \rangle$ generate the group, so it is cyclic.

$\mathbb{Z}_8^\times$

- $\langle 1 \rangle = \{1\}$ Order 1

- $\langle 3 \rangle = \{3, 1\}$ Order 2
- $\langle 5 \rangle = \{5, 1\}$ Order 2
- $\langle 7 \rangle = \{7, 1\}$ Order 2

Therefore not cyclic.

$\mathbb{Q}$:

- $\langle 1 \rangle = \mathbb{Z}$
- $\langle 0 \rangle = \{0\}$
- $\langle q \rangle = q\mathbb{Z}, q \in \mathbb{Q}$

Therefore not cyclic.

$\mathbb{R}$:

- $\langle 1 \rangle = \mathbb{Z}$
- $\langle 0 \rangle = \{0\}$
- $\langle r \rangle = r\mathbb{Z}$

Therefore not cyclic.

Note: $q\mathbb{Z} \cong \mathbb{Z}$ and $r\mathbb{Z} \cong \mathbb{Z}$

$\mathbb{Z}_2 \times \mathbb{Z}_4$:

- $\langle 00 \rangle = \{00\}$ Order 1
- $\langle 01 \rangle = \{01, 02, 03, 00\}$ Order 4
- $\langle 02 \rangle = \{02, 00\}$ Order 2
- $\langle 03 \rangle = \{03, 02, 01, 00\}$ Order 4
- $\langle 10 \rangle = \{10, 00\}$ Order 2
- $\langle 11 \rangle = \{11, 02, 13, 00\}$ Order 4
- $\langle 12 \rangle = \{12, 00\}$ Order 2
- $\langle 13 \rangle = \{13, 02, 11, 00\}$ Order 4

Therefore not cyclic.

$\mathbb{Z}_2 \times \mathbb{Z}_3$**:**

- $\langle 00 \rangle = \{00\}$ Order 1
- $\langle 01 \rangle = \{01, 02, 00\}$ Order 3
- $\langle 02 \rangle = \{02, 01, 00\}$ Order 3
- $\langle 10 \rangle = \{10, 00\}$ Order 2
- $\langle 11 \rangle = \{11, 02, 10, 01, 12, 00\}$ Order 6
- $\langle 12 \rangle = \{12, 01, 10, 02, 11, 00\}$ Order 6

Therefore cyclic

Proposition 9.1.2. *G is cyclic $\implies$ all subgroups of G are cyclic*

Proof. Let $G = \langle a \rangle = \{a^i : i \in \mathbb{Z}\}$. Let H be a sub group of G.

$$H = \{a^i : some\ i \in \mathbb{Z}\}$$

could be $H = \{a^0\} = \{\epsilon\}$. Let

$$n = min\{k : a^k \in H, k > 0\}$$

$$\langle a^n \rangle = \{(a^n)^k : k \in \mathbb{Z}\} = \{a^{kn} : k \in \mathbb{Z}\} = \{a^k : k \in n\mathbb{Z}\}$$

$$\langle a^n \rangle \leq H \leq G$$

Suppose $a^j \in H$ with $j \notin \mathbb{Z}$ so $\langle a^n \rangle \neq H$. Then

$$j = qn + r \quad 0 \leq r < n \quad r \neq 0$$

So

$$a^r = a^{j-qn} = a^j (a^n)^{-q} \in H$$

This contradicts the minimality of n. Therefore $H = \langle a^n \rangle$. □

Definition 9.1.1 (Order)**.** *The order of an element $g \in G$ is the smallest positive integer n such that $g^n = \epsilon$. We write $|g|$ for the order of g, if no such n exists we say $|g| = \infty$.*

Proposition 9.1.3. *Suppose* $|a| = n < \infty$*, then*

$$a^j = \epsilon \iff n|j$$

In otherwords,

$$\{j : a^j = \epsilon\} = n\mathbb{Z}$$

Furthermore,

$$a^s = a^t \iff n|s - t$$

Example: $|a| = 5$

$$a^5 = a^{10} = a^{-15} = a^{1005} = \cdots = \epsilon$$

$a^j \neq \epsilon$ when j is not a multiple of 5.

Proof. $\impliedby$ if $n|j$ then $j = tn$ for some $t \in \mathbb{Z}$

$$a^j = a^{tn} = (a^n)^t = \epsilon^t = \epsilon$$

$\implies$: if $a^j = \epsilon$, then write $j = qn + r$ for $0 \leq r < n$

$$a^r = a^{j-qn} =^j (a^n)^{-q} = \epsilon(\epsilon^{-q}) = \epsilon$$

but n is the smallest positive integer with $a^n = \epsilon$, so $0 \leq r < n$ implies $r = 0$. Therefore $j = nq$ and $n \mid j$. □

Also,

$$a^s = a^t \iff a^{s-t}\epsilon \iff n|s - t$$

Corollary 9.1.1. $|a| = |b|$ *is equivalent to*

$$a^j = \epsilon \iff b^j = \epsilon$$

Proposition 9.1.4. *Suppose* $a \in G$*,* $|a| = n < \infty$*,* $k \in \mathbb{Z}$*. Then*

$$|a^k| = \frac{n}{gcd(k, m)}$$

Example: $|a| = 12$

- $\langle a^1 \rangle = \{a^1, a^2, a^3, \ldots, a^{12}\}$

- $\langle a^5 \rangle = \{a^5, a^{10}, a^3 \ldots, a^{12}\} = \langle a \rangle$
- $\langle a^4 \rangle = \{a^4, a^8, a^{12} = a^0\}$
- $\langle a^{10} \rangle = \{a^{10}, a^8, a^6, a^4, a^2, a^0\}$

Proof. Let $|a^k| = m$, then $\epsilon = (a^k)^m = a^k m$ Therefore, $n|km$ and km is a multiple of $|a|$ by the previous theorem. Let $d = gcd(kn)$ and set

$$\begin{cases} n = n'd \\ k = k'd \end{cases}$$

$$gcd(n', k') = 1$$

Since $n|km$ for some $t \in \mathbb{Z}$ we have,

$$km = tn$$

$$dk'm = tdn'$$

$$k'm = tn'$$

$$m = \frac{tn'}{k'} = \frac{t}{k'} \cdot n'$$

This must be an integer because $gcd(k', n') = 1 \implies k' \mid t$ Smallest $m \iff$ smallest t with $\frac{tn'}{k'}$ positive integer. So □

Corollary 9.1.2. *Suppose $G = \langle a \rangle$, with $|a| = n < \infty$, then the generators of G are $\{a^k : gcd(n,k) = 1\}$*

Proof.

$$|a^k| = \frac{n}{gcd(n,k)} = n \iff gcd(n,k) = 1$$

□

Corollary 9.1.3. *$\mathbb{Z}_n = \langle 1 \rangle$ and $|1| = n$. Generators of $\mathbb{Z}$ with addition are*

$$\{k \cdot 1 : gcd(n,k) = 1\} = \{k : gcd(n,k) = 1\} = \mathbb{Z}_n^\times$$

Corollary 9.1.4. *all nonzero elements of $\mathbb{Z}_n$ are generators of $\mathbb{Z}_n \iff n$ is prime*

Proof. We want $|k| = \frac{n}{gcd(n,k)} = n$ for $k = 1, 2, 3, \ldots, n-1$. So $gcd(n,k) = 1$ □

Lecture 10

Subgroups of Cyclic Groups, Lattices, $\mathbb{T}$

- G cyclic means there exists $g \in G$ with $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$
- The order of an element g is the smallest positive integer n with $g^n = \epsilon$
- Notation: Order of an element g is written $|g|$. Order (=size!) of a group G is written $|G|$. $|g| = \infty$ means $g^k \neq \epsilon \ \forall k \in \mathbb{Z}$
- $\{k : g^k = \epsilon\} = |g|\mathbb{Z}$ so $g^k \epsilon \iff |g|$ divides k
- $|x| = |y|$ is equivalent to $x^k = \epsilon \iff y^k = \epsilon$
- if $|g| = n < \infty$, then

$$G = \langle g \rangle = \{g, g^2, \cdots, g^n = \epsilon\}$$

$$|G| = |g|$$

$$|g^k| = \frac{n}{gcd(n,k)}$$

generators of G are exactly $\{g^k : gcd(n,k) = 1\}$

Corollary 10.0.1. *All nonzero elements of $\mathbb{Z}_n$ are generators of $\mathbb{Z}_n \iff n$ is prime.*

Proof. We want $k = \frac{n}{gcd(n,k)} = n$ for $k = 1, 2, 3, \ldots, n-1$. So $gcd(n,k) = 1$ for $k = 1, 2, 3, \ldots, n-1$. Therefore n is prime. $\square$

Theorem 10.0.1. *G has no subgroups other than $\{\epsilon\}$ and $G \iff G$ is cyclic of prime order $\iff |G|$ is prime.*

Proof. Suppose $g \in G$, then $\langle g \rangle$ is a subgroup of G. Therefore, either $\langle g \rangle = G$ or $\langle g \rangle = \{\epsilon\}$. g is a generator of G So

$$G = \{g, g^2, g^3, \ldots, g^n = \epsilon\}$$

g^k is a generator for $k = 1, 2, \ldots, n-1$ Therefore,

$$\frac{n}{gcd(n,k)} = n$$

So n is prime, therefore G is cyclic of prime order $G \cong \mathbb{Z}_n$ for n prime.

Conversely,

$$G = \{g, g^2, \ldots, g^n = \epsilon\}$$

then $S \neq \emptyset$ and $S \neq \{\epsilon\} \implies \langle S \rangle = G$. So $x \in S$, $x = g^k$ then

$$|x| = |g^k| = \frac{n}{gcd(n,k)}$$

So the only subgroups are $\{\epsilon\}$ and G $\square$

Theorem 10.0.2. *Suppose G, H are both cyclic, $G \cong H \iff |G| = |H|$*

Proof. ($\implies$)an isomorphism is a bijection.

($\impliedby$)$G = \langle a \rangle$ and $H = \langle b \rangle$, then

$$|a| = |G| = |H| = |b|$$

define

$$\phi : G \to H$$

$$\phi(a^k) = b^k$$

We have 2 cases, either the order is infinite.

$$\begin{cases} G = \{\ldots, a^-2, a^{-1}, a^0, a^1, a^2, \ldots\} \\ H = \{\ldots, a^-2, a^{-1}, a^0, a^1, a^2, \ldots\} \end{cases}$$

Or their order is finite

$$\begin{cases} G = \{a, a^2, a^3, \ldots, a^n \epsilon\} \\ H = \{b, b^2, b^3, \ldots, b^n = \epsilon\} \end{cases}$$

In both cases ϕ is a bijection.

$$\phi(a^s a^t) = \phi(a^{s+t}) = b^{s+t} = b^s b^t = \phi(a^s)\phi(a^t)$$

$\square$

Subgroups of $C_n = \langle a \rangle = \{a, a^2, \ldots, a^n\}$

- C_n is cyclic, therefore all subgroups are cyclic
- $|a^k| = \frac{n}{gcd(k,n)}$
- Let $d \mid n$ then $|a^d| = \frac{n}{gcd(d,n) = \frac{n}{d}}$

So for each $d \mid n$, then $\langle a \rangle^d \cong C_{\frac{n}{d}}$ is a subgroup.

No let $k \in \{1, 2, 3 \in, n\}$. Suppose $gcd(k, n) = d$ for some $d \mid n$, then

$$k \in \{d, 2d, 3d, \ldots, \frac{n}{d} d\}$$

So $a^k \in \langle a^d \rangle$. i.e. all elements of order $\frac{n}{d}$ are contained in the subgroup $\langle a^d \rangle$

Conclusion: For all $d \mid n$, there is a unique subgroup of C_n of order $\frac{n}{d}$, generated by a^d.

Example: $n = 2$. $C_{12} = \langle a \rangle = \{a, a^2, a^3, \ldots, a^{11}, a^{12}\}$

- **Order 12:** a^1, a^5, a^7, a^11 $\langle a \rangle = C_{12} = \langle a^5 \rangle = \langle a^7 \rangle = \langle a^11 \rangle$
- **Order 6:** $a^2 a^{10}$ $\langle a^2 \rangle = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}\} = \langle a^{10} \rangle$
- **Order 4:** a^3, a^9 $\langle a^3 \rangle = \{a^3, a^6, a^9, a^{12}\} = \langle a^9 \rangle$
- **Order 3:** a^4, a^8 $\langle a^4 \rangle = \{a^4, a^8, a^{12}\} = \langle a^8 \rangle$
- **Order2:** a^6 $\langle a^6 \rangle = \{a^6, a^{12}\}$
- **Order 1:** a^2 $\langle a^2 \rangle = \{a^{12}\}$

Example: $n = 12$ $\mathbb{Z}_{12} = \{1, 2, 3, \dots, 12\}$

- **Order 12:** $1, 5, 7, 11$ $\langle 1 \rangle = \mathbb{Z}_{12} = \langle 5 \rangle = \langle 7 \rangle = \langle 11 \rangle$
- **Order 6:** $2, 10$ $\langle 2 \rangle = \{2, 4, 6, 8, 10, 12\} = \langle 10 \rangle$
- **Order 4:** $3, 9$ $\langle 3 \rangle = \{3, 6, 9, 12\} = \langle 9 \rangle$
- **Order 3:** $4, 8$ $\langle 4 \rangle = \{4, 8, 12\} = \langle 8 \rangle$
- **Order 2:** 6 $\langle 6 \rangle = \{6, 12\}$
- **Order 1:** 12 $\langle 12 \rangle = \{12\}$

10.0.1 Lattices

Subgroups of $\mathbb{Z}_{12}$

$$\begin{tikzcd}
& \mathbb{Z}_{12} \arrow[dl, dash] \arrow[dr, dash] & & \\
\mathbb{Z}_4 \arrow[dr, dash] & & \mathbb{Z}_6 \arrow[dr, dash] & \\
& \mathbb{Z}_2 \arrow[dr, dash] & & \mathbb{Z}_1 \arrow[dl, dash] \\
& & \mathbb{Z}_1 &
\end{tikzcd}$$

Positive Divisors of 12

$$\begin{tikzcd}
& 12 \arrow[dl, dash] \arrow[dr, dash] & & \\
4 \arrow[dr, dash] & & 6 \arrow[dr, dash] & \\
& 2 \arrow[dr, dash] & & 3 \arrow[dl, dash] \\
& & 1 &
\end{tikzcd}$$

Cyclic groups with subgroups $\cong$ integers with divisibility

10.1 Complex Numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

$$\mathbb{C} = \{re^{i\theta} : r, \theta \in \mathbb{R}\}$$

Lemma 10.1.1.

$$e^{i\theta} = \cos\theta = i \sin\theta$$

$$z = re^{i\theta} = re^{i(\theta + 2k\pi)} = -re^{i(\theta + (2k+1)\theta)}$$

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta$$

$$\begin{cases} |z| = |a + bi| = \sqrt{a^2 + b^2} = r \\ \frac{b}{a} = \tan\theta \end{cases}$$

Refer to the profs notes for the rest of the complex number stuff.

Lecture 11

Subgroups of $\mathbb{T}$, Permutations, Disjoint Cycles

11.1 Subgroups of a Finite Cyclic Subgroup

$G = \langle g \rangle$ with $|g| = |G| = n = md$, choose r with $\gcd(n, r) = d$, then

$$|g^r| = \frac{n}{\gcd(n, r)} = \frac{n}{d} = m$$

Any subgroup of order m can be obtained this way. If

$$H = \langle g^r \rangle$$

is a subgroup of G of order m, then

$$H = \langle g^r, g^{2r}, \ldots, g^{mr} = \epsilon \rangle \text{ and } |g| = n = |nr|$$

Furthermore,

$$(g^{tr})^m = (g^{mr})^t = (\epsilon)^t = \epsilon$$

So H consists of m elements and $x \in H \to x^m = \epsilon$. So

$$\begin{aligned} x^m = (g^k)^m = \epsilon &\iff g^{km} = \epsilon \\ &\iff |g| \text{ divides } km \\ &\iff n \mid km \\ &\iff dm \mid km \\ &\iff k \text{ is a multiple of } d \\ &\iff x \in \{g^d, g^{2d}, \ldots, g^{md}\} \end{aligned}$$

$$\left.\begin{aligned} |g| &= n = md \\ |g^r| &= m \end{aligned}\right\} \implies \langle g^r \rangle = \{x : x^m = \epsilon\}$$

Proposition 11.1.1. *Let $x = (a_1, a_2, \ldots, a_t) \in G_1 \times G_2 \times \cdots \times G_t$, then $|x| = (|a_1|, |a_2|, \ldots, |a_t|)$*

Proof. $x^k = \epsilon$ means $(a_i)^k = \epsilon_i \ \forall i = 1, 2, \ldots t$, k is a multiple of each $|a_i|$ smallest such k is $\operatorname{lcm}(|a_1|, |a_2|, \ldots, |a_t|)$. □

Proposition 11.1.2. *Let $G_1, G_2, \ldots, G_t$ be finite groups.*

$$G_1 \times G_2 \times \cdots \times G_t \text{ cyclic} \iff \begin{cases} \text{each } G_i \text{ is cyclic} \\ \gcd(|G_i|, |G_j|) = 1) \ \forall 1 \le i < j \le t \end{cases}$$

Proof. See assignment 3. □

Subgroups of

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

$$R = \{e^{2\pi i j/n} : 0 \le j < n, n \ge 1\}$$

Consider only reduced fractions $\gcd(i, n) = 1$ so

$$e^{2\pi i 4/12} \to e^{2\pi i 1/3}$$

then $e^{2\pi i j/n}$ generates

$$R_n = \{e^{2\pi i j/n} : 0 \le j < n\}$$

order of $e^{2\pi i j/n}$ is n finite. If $S \subseteq R$, then let $m = $ lcm of all the n, then

- $S \subseteq \langle e^{2\pi i 1/m} \rangle$
- $e^{2\pi i 1/m} \in \langle S \rangle$
- So $\langle S \rangle = \langle e^{2\pi i 1/m} \rangle$

every finite subgroup is cyclic.

Suppose $S \subseteq R$. Choose
TBC

Lecture 12

Lecture 13

Lecture 14

Lecture 15

Lecture 16

Lecture 17

Lecture 18

Lecture 19

Normal Subgroups, Quotient Groups

Recall: The external direct product is defined as

$$H, K \text{ groups } \implies G = H \times K = \{(x, y) : x \in H, y \in K\}$$

The internal direct product is defined on subgroups H, K of G with

$$H \cap K = \{\epsilon\}$$

$$xy = yx \quad \forall x \in H \ y \in K$$

Uniqueness: If $G = HK$ as an internal direct product, then $\forall g \in G$, $\exists! x \in H, y \in K$ such that $g = xy$.

Isomorphisms: If $G = H \times K$ as an external direct product, then

$$G = (H \times \{\epsilon\})(\{\epsilon\} \times K)$$

as an internal direct product. If $G = HK$ as an internal direct product then $G \cong H \times K$ as an external direct product.

Theorem 19.0.1. *If $G = HK$ as internal direct product, then*

$$G \cong H \times K$$

Proof. For $g \in G$, $\exists 1 x \in H, y \in K$ such that $g = xy$. Define

$$\psi : G \mapsto H \times K$$

by $\psi(g) = (x, y)$. So

$$\left.\begin{aligned} g_1 &= x_1 y_1 \\ g_2 &= x_2 y_2 \end{aligned}\right\} \implies g_1 g_2 = x_1 y_1 x_2 y_2 =$$

Claim: ψ is an isomorphism. TBC $\square$

Theorem 19.0.2. *If $G = H \times K$ as an external direct product, then $G \cong MN$ as an internal direct product where*

$$M = H \times \{\epsilon_K\} \text{ and } N = \{\epsilon_H\} \times K$$

Proof.
- M and N are subgroups of G
- $(x, y) \in M \implies y = \epsilon_K$ and $(x, y) \in N \implies x = \epsilon_H$ so

  $$(x, y) \in M \cap N) \implies (x, y) = (\epsilon_H, \epsilon_G)$$

  So $M \cap N = \epsilon_G$
- $$\begin{aligned} (x, \epsilon_K)(\epsilon_H, y) &= (x\epsilon_H, \epsilon_K y) \\ &= (\epsilon_H x, y\epsilon_K) \\ &= (\epsilon_H y)(x, \epsilon_K) \end{aligned}$$

  So $hK = Kh$ if $h \in H$ and $k \in K$
- $(x, y) \in G \implies (x, y) = (x, \epsilon_K)(\epsilon_K, y)$ So $HK = G$

$\square$

Example: Consider $D_6 = \langle \mu, \rho \rangle$. Set $H = \langle \mu \rangle$ and $K = \langle \rho \rangle$. Is $D_6 = HK$ as an inner direct product?

We want to check if $H \cap K = \{\epsilon\}$.

$$H = \{\epsilon, \mu\} \quad K = \{\epsilon, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$$

So we have

$$H \cap K = \{\epsilon\}$$

We also want to show that

$$hk = kh \ \forall h \in H \ k \in K$$

No, since

$$\rho\mu = \mu\rho^{-1} =\neq \rho\mu$$

Is $D_6 = HK$? Yes, since

$$HK = \{hk : h \in H \ k \in K\} = \{\mu^i \rho^j : 0 \leq i \leq 1 \ 0 \leq \rho \leq 5\} = D_6$$

So this is not a direct product.

Example: Set $H = \langle \rho^3 \rangle = \{\epsilon, \rho^3\}$, and

$$K = \langle \mu, \rho^2 \rangle = \{\epsilon, \mu, \rho^2, \mu\rho^2, \rho^4, \mu\rho^4\}$$

Is $D_6 = HK$ an inner direct product?

We have that $\{H \cap K = \{\epsilon\}\}$. We want to show that

$$hk = kh \ \forall h \in H \ k \in K$$

Yes, since $(\rho^3)^{-1} = \rho^3$, so $\mu\rho^3 = \rho^3\mu$, we can show that ρ^3 commutes with every element in K. We can also just check that the generator of H (ρ^3) commutes with the generator of K (μ, ρ^2). Now we want to check if $D_6 = HK$.

$$HK = \{hk : h \in H \ k \in K\} = D_6$$

So this is a direct product.

Notice: $H \cong \mathbb{Z}_2$ and $K \cong D_3$. To prove this, we want to show that

$$K = \langle \alpha, \beta : \alpha^2 = \beta^3 = \epsilon, \alpha\beta = \beta^{-1}\alpha \rangle$$

by mapping elements of K to α, β.

Question: For which m is $D_{2m} \cong \mathbb{Z}_2 \times D_m$?

19.1 Normal Subgroups

Definition 19.1.1. *For $K < G$, we say K is a normal subgroup of G if*

$$gK = Kg \ \forall g \in G$$

We denote a normal subgroup as

$$K \triangleleft G$$

Note: $gx = xg \forall x \in K \implies gK = Kg$ but $gK = Kg \not\Longrightarrow gx = xg$

Facts:

- G abelian $\implies$ every subgroup is normal
- $K < Z(G) \implies K \triangleleft G$
- $[G : K] = 2 \implies K \triangleleft G$

Example:

$$S_3 = \{\epsilon, (23), (13), (12), (123), (132)\}$$

Is $K = \{\epsilon, (12)\}$ normal in S_3?

$$g \in G \implies gK = \{g\epsilon, g(12)\}$$

$$Kg = \{\epsilon g, (12)g\}$$

The following statements are equivalent:

- $K \triangleleft G$
- $gK = Kg \ \forall g \in G$
- $gKg^{-1} = K \ \forall g \in G$
- Define phi_g by $\phi_g(x) = gxg^{-1} \ \forall x \in G$ then ϕ_g maps $KtoK$

Lemma 19.1.1. *Suppose $K \triangleleft G$, then*

$$g_1K = g_2K \iff g_1K = Kg_2 \iff g_1Kg_2^{-1} = K$$

TBC

19.2 Quotient Groups

Definition 19.2.1. *Let $K \triangleleft G$. Define a group G/K, then the elements of the group are the cosets of K in G, so*

$$G/K = \{gK : g \in G\}$$

The operation is on the representatives of the cosets

$$xK \cdot yK = xyK$$

The *order* of G/K is

$$[G : K] = \frac{|G|}{|K|}$$

Question: Is this operation well-defined?

$$x_1K = x_2K \quad y_1K = y_2K$$

$$\begin{cases} x_1K \cdot y_1K = x_1y_1K \\ x_2K \cdot y_2K = x_2y_2K \end{cases}$$

From the coset comparison theorem, is $x_1y_1K = x_2y_2K$, $(x_2y_2)^{-1}x_1y_1 \in K$?

$$(x_2y_2)^{-1}x_1y_1 = y_2^{-1}x_2^{-1}x_1y_1 = y_2^{-1}Ky_1$$

Since $x_1K = x_2K$, $x_2^{-1}x_1 \in K$. So

$$ky_1 = Ky + 1 = y_1K$$

so

$$ky_1 = y_1k' \text{ some } k' \in K$$

Then

$$(x_2y_2)^{-1}x_1y_1 = y_2^{-1}y_1k' = k''k' \in K$$

$$\therefore x_1y_1K =$$

Theorem 19.2.1. *If $K \triangleleft G$, then G/K is a group.*

Proof.

- **closure:** $xK \cdot yK = xyK$

- **Associativity:**

$$aK \cdot (bK \cdot cK) = aK \cdot (bc)K = a(bc)K$$
$$(aK \cdot bK) \cdot cK = (ab)K \cdot cK = (ab)cK$$

$$a(bc) = (ab)c \text{ since } G \text{ is a group}$$

- **Identity:**

$$aK \cdot \epsilon K = a\epsilon K = aK$$

$$\epsilon K \cdot aK = \epsilon aK = aK$$

- **Inverses:**

$$aK \cdot a^{-1}K = aa^{-1}K = \epsilon K = K$$

$$a^{-1}K \cdot aK = a^{-1}1K = \epsilon K = K$$

$\square$