MAT 2384: Numerical Methods Lecture Notes

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Contents

Chapter 1

Iterative Methods to Solve The Equation $f(x) = 0$

Given a continous function f, the goal of this chapter is to estimate the solution of the equation $f(x) = 0$ in a certain interval I numerically.

Theorem 1.0.1 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continous function. Let $y \in \mathbb{R}$ be any value between $f(a)$ and $f(b)$. Then there exists $z \in [a, b]$ such that $f(z) = y$.*

Bolzano's Theorem is a special case of the Intermediate Value Theorem, which states

Theorem 1.0.2 (Bolzano's Theorem)**.** *If a continuous function defined on an interval I is sometimes positive and sometimes negative, then it must be 0 at some point. So there exists $x_0 \in I$ such that $f(x_0) = 0$.*

Proof. Without loss of generality, assume $f(a) \leq f(b)$. Let $y \in [f(a), f(b)]$. Set

$$S := \{x \in [a, b] : f(x) \leq y_0\}$$

S is a subset of $[a, b]$ so it is bounded, $a \in S$ since $f(a) \leq y_0$. Therefore $S \neq \emptyset$. Thus by completeness, there exists $x_0 := \sup S \in [a, b]$. We want $f(x_0) = y_0$. Consider the cases where $f(x_0) = y_0$, $f(x_0) < y_0$, and $f(x_0) \geq y_0$.

- **Case 1:** $f(x_0) = y_0$ This case is trivial since this is the result we want.
- **Case 2:** $f(x_0) < y_0$ Set $\epsilon := y_0 - f(x_0)$. Since f is continous at x_0, $\exists \delta > 0$ such that

  $$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

  Since $f(x_0) < y_0 \leq f(b)$, we can find $x > x_0$ such that $x \in [a, b]$ and $|x - x_0| < \delta$. Then $f(x) < f(x_0) + \epsilon = y_0$. So $x \in S$ by the definition of S, but $x > x_0$ which contradicts the fact that $x_0 = \sup S$.

- **Case 3:** $f(x_0) > y_0$ Set $\epsilon := f(x_0) - y_0$. Since f is continous at x_0, $\exists\delta > 0$ such that if $x \in [a, b]$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. So $f(x) > f(x_0) - \epsilon = y_0$ and $x_0 > a$. We can assume that $x - \delta > a$ since δ can be arbitrarly small, and we claim $x_0 - \delta$ is an upper bound for S. To prove this, if $x > x - \delta$, then either $|x - x_0| < \delta$, in which case $f(x) > f(x_0) - \epsilon = y_0$, or $x > x_0$ then $x \neq S$ since x_0 is an upper bound for S. Therefore, if $x > x_0 - \delta$, then $x \neq S$, thus proving the claim. This contradicts that x_0 is the supremum of S.

□

Example. Prove that the equation

$$2x^3 + 2x - 4 = 0$$

has a unique root in $[0, 1]$.

Proof. Set $f(x) \coloneqq 3x^2 + 2x - 4$, this function is continuous since it is a polynomial.We have $f(0) = -4 < 0$ and $f(1) = 1 > 0$, so by the intermediate value theorem, there exists $c \in [0, 1]$ such that $f(c) = 0$. It follows that c is unique since the polynomial is injective by virtue of x^3 and x being injective. □

1.1 Fixed-Point Iteration

Definition 1.1.1. *We say that the value $x = r$ is a* fixed point *for a function $g(x)$ if $g(r) = r$.*

Example. $g(x) = \frac{5-x^2}{4}$. $r = 1$ is a fixed-point for g since $g(1) = 1$.

Graphically, fixed-point of $g(x)$ correspond to the intersection of the graph of $g(x)$ and the line $y = x$. Given an equation $f(x) = 0$, we can write it under the form

$$g(x) = x$$

by isolating one x in the equation.

Example. $3x^3 + 2x - 5 = 0$. We can write this as

$$x = \frac{5 - 3x^3}{2}$$

Set $g(x) \coloneqq \frac{5-3x^3}{2}$. Then $g(x) = x$. Finding a root for $f(x) = 0$ is equivalent to finding a fixed-point for $g(x)$.

1.1.1 Steps to Solving Using Fixed-Point Iteration

Start with a first estimation x_0 (will be given) of the root, and form the following sequence (known as the *iteration sequence*)

$$x_0, x_1 = g(x_0), x_2 = g(x_1), \ldots, x_n = g(x_{n-1})$$

If this sequence converges to a value a, then we can prove that a is a fixed-point for g, hence a root for $f(x) = 0$.

Theorem 1.1.1. *Assume that the function g has a fixed-point s on an interval I, if*

(i) $g(x)$ *is continuous on* I

(ii) $g'(x)$ *is continuous on* I

(iii) $|g'(x)| < 1$ *for all* $x \in I$

Then the iteration sequence converges.

The steps for solving are as follows

1. Start with $f(x) = 0$
2. Rewrite $f(x) = 0$ under the form $x = g(x)$
3. Verify that the sequence $x_0, x_1 = g(x_0), x_2 = g(x_1), \ldots, x_n = g(x_{n-1})$ converges using the above theorem (or otherwise)
4. Compute terms of the above sequence and stop when you reach the required accuracy

Example. Consider the equation

$$x^3 + 12x - 3 = 0$$

1. Prove that the equation has a unique root in $[-1.9, 1.9]$
2. Use the Fixed-Point iteration method to estimate the value of the root to 6 decimal points starting with $x_0 = 1.8$

Solution: Using the steps, we have

1. Set $f(x) := x^3 + 12x - 3$. Since $f(x)$ is a polynomial, it is continuous, so by the intermediate value theorem, we have there exists $c \in [-1.9, 1.9]$ such that $f(c) = 0$. $f(x)$ is injective since x^3 and x are injective, so c is unique.
2. Set $g(x) := \frac{3-x^3}{12}$.

3. Checking the conditions of the theorem, $g(x)$ is continuous since it is a polynomial, $g'(x) = -\frac{x^2}{4}$ is continuous since it is a polynomial. Then

$$|g'(x)| = \frac{x^2}{4} \le \frac{1.9^2}{4} = 0.902 < 1$$

Therefore, the sequence converges.

4. We have to calculate the terms of the iteration sequence,

$$\begin{aligned}
x_0 &= 1.8 \\
x_1 &= g(x_0) = \frac{3 - 1.8^2}{12} = -0.236000 \\
x_2 &= g(x_1) = \frac{3 - (0.236)^2}{12} = 0.251095 \\
x_3 &= g(x_2) = \frac{3 - (0.251095)^2}{12} = 0.24861 \\
x_4 &= g(x_3) = \frac{3 - (0.24861)^2}{12} = 0.248718 \\
x_5 &= g(x_4) = \frac{3 - (0.248718)^2}{12} = 0.248718
\end{aligned}$$

We stop when 2 consecutive terms agree on the first 6 decimal points. So the root is 0.248718 correct to 6 decimal points.

1.2 Newton's Method

Newton's method is a technique for solving equations of the form $f(x) = 0$ by successive approximation. The idea is to pick an initial guess x_0 such that $f(x_0)$ is reasonably close to 0. We then find the equation of the line tangent to $y = f(x)$ at $x = x_0$, and determine where this tangent line intersects the x axis at the new point x_1. So,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We then find the equation of the line tangent to $y = f(x)$ at $x = x_1$, and repeat this process, so we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example. Using Newton's method, estimate the value of the root of the equation

$$x^3 + 12x - 3 = 0$$

on $[0, 2]$. Start by showing that the equation has a unique root on $[0, 2]$, then approximate (to 6 decimal places) with the starting point $x_0 = 1.8$.

Solution. We have $f(x) = x^3 + 12x - 3$, and

$$f(0) = -3 \text{ and } f(2) = 29$$

Therefore by the intermediate value theorem, there exists $c \in [0, 2]$ such that $f(c) = 0$. $f(x)$ is injective since $f'(x) = 2x^2 + 12$ is striclty increasing on $[0, 2]$, so c is unique. Now using Newton's method,

$$x_0 = 1.8$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.675138$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.270469$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.248748$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.248718$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.248718$$

Therefore, the our root is 0.248718 correct to 6 decimal places.