MAT 2384: Ordinary Differentials Lecture Notes

Last Updated:

June 3, 2023

Contents

Chapter 0

Introduction and Basic Terminology

Definition 0.0.1 (Differential Equations)**.** *A differential equation is an equation involving an unknown function y (of one or many variables), derivatives of y, and other known functions of indepedent variables.*

Definition 0.0.2 (Order of Differential Equations)**.** *The* order *of a differential equation is the highest order of a derivative appearing in the equation.*

If the unknown function y is a function of only one variable, $y = f(x)$, we saw that the differential equation is *ordinary.* If y is a function of two or more variables, we say the differential equation is a *partial* differential equation.

Example.

$$x^3 y'' - 3e^x \sin x y' + 3y = \tan x$$

This is an ODE of order 2.

Example.

$$x_1 x_2 \frac{\partial^2 y}{\partial x_1 \partial x_2} - 3e^{x_1} \frac{\partial y}{\partial x_1} = 0$$

This is a PDE of order 2.

Note: In this course, we will only consider ODEs.

Definition 0.0.3. *We say that the function y is a* solution *to a differential equation on an interval I if y is well-defined on I and y satisfies the differential equation.*

Example. Consider the differential equation

$$y'' - 5y' + 4y = 0$$

Show that the function
$$y = Ae^x + Be^{4x}$$
is a solution for the differential equation on $\mathbb{R}$ for any constants A and B.

Solution: We have $y = Ae^x + Be^{4x}$ is well defined on $\mathbb{R}$.
$$y' = Ae^x - 4Be^{4x}$$
$$y'' = Ae^x + 16Be^{4x}$$
So,
$$y'' - 5y' + 4y = Ae^x + 16Be^{4x} - 5Ae^x - 20Be^{4x} + 4Ae^x + 4Be^{4x} = 0$$
Therefore, $y = Ae^x + Be^{4x}$ is a solution to the differential equation for any $A, B \in \mathbb{R}$. This is called the *general solution* to the differential equation.

Remark: The above example shows that a differential equation has infinitely many solutions.

Definition 0.0.4 (Initial Value Problem)**.** *An* intial value problem *(IVP) of order n consists of an ordinary differential equation of order n, and n initial coniditions of the form*
$$y(x_0) = y_0, \quad y'(x_0) = y_1 \ldots$$
$$y^{(n-1)}(x_0) = y_{n-1}$$
Note: $y^{(i)}$ denotes the ith derivative of y.

Example. Consider the IVP of order 3
$$y''' - 3e^x y'' + 6xy' + 2y = x^2$$
$$y(0) = -1 \quad y'(0) = 2 \quad y''(0) = 1$$
Example. Solve the following IVP
$$y'' - 5y + 4y = 0$$
$$y(0) = 1 \quad y'(0) = 2$$
Solution: We saw in the previous example that the general solution to this differential equation is
$$y = Ae^x + Be^{4x}$$
We can use the initial conditions to find the constants A and B.
$$y(0) = 1 \implies 1 = Ae^0 + Be^0 = A + B$$
$$y'(0) = 2 \implies 2 = Ae^0 - 4Be^0 = A + 4B$$
$$A + 4B - A - B = 2 - 1 \implies 3B = 1 \implies B = \frac{1}{3} \quad A = \frac{2}{3}$$

Theorem 0.0.1 (Existence and Uniqueness Theorem for the First Order ODEs)**.** *Consider the IVP:*

$$y' = F(x, y), \quad y(x_0) = y_0$$

- ***Existence:*** *If $F(x, y)$ is continuous in an open rectangular region*

$$R = \left\{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\right\}$$

of the xy-plane that contains the initial point (x_0, y_0), then there exists a solution $y(x)$ to the intial value problem that is defined in some open interval $I = (\alpha, \beta)$ containg x_0.

- ***Uniqueness:*** *If the partial derivative $\frac{\partial F}{\partial y}$ of the function $F(x, y)$ is continuous in the recnagular region R, then the solution $y(x)$ is unique.*

Note: We will always suppose this condition is satisfied in this course.

Chapter 1

Ordinary Differential Equations of First Order

The goal of this chapter is to solve ODE's of order 1.

Definition 1.0.1. *The* standard form *of an ODE of order 1 is an expression of the form*

$$y' = f(x, y)$$

We can rewrite y' *as* $\frac{dy}{dx}$ *and we have the* differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Example. Consider the differential equation

$$2xy' + 3y = 2y' + \sin x$$

The standard form is

$$2xy' - 2y' = \sin x - 3y \implies y' = \frac{\sin x - 3y}{2x - 2}$$

The differential form is

$$\begin{aligned} 2x\frac{dy}{dx} + 3y = 2\frac{dy}{dx} + \sin x \\ \implies 2xdy + 3ydx = 2dy + \sin xdx \\ \implies (3y - \sin x)dx(2x - 2)dy = 0 \end{aligned}$$

1.1 Seperable First Order Ordinary Differential Equations

Definition 1.1.1. *A first order ODE is called* seperable *if it can be written in the form*

$$F(x)dx = G(y)dy$$

1.1.1 Solving Seperable ODE's

To solve a seperable ODE,

1. Write $y' = \frac{dy}{dx}$
2. Seperate the ODE to write it in the form
$$F(x)dx = G(y)dy$$
3. Take integrals of both sides
4. If an initial condition is given, solve for the constant of integration C.

Example. Solve the IVP

$$(y^2 + 1)y' = \frac{x}{y} \quad y(1) = 1$$

Solution: We can write $y' = \frac{dy}{dx}$ and we get

$$(y^2 + 1)\frac{dy}{dx} = \frac{x}{y} \implies (y^2 + 1)ydy = xdx$$

Taking integrals on both sides, we have

$$\int y^3 + ydy = \int xdx \implies \frac{y^4}{4} + \frac{y^2}{2} = \frac{x^2}{2} + C$$

Using our initial condition, we have $y = 1$ when $x = 1$, then

$$\frac{1}{4} + \frac{1}{2} = \frac{1}{2} + C$$

Therefore $C = \frac{1}{2}$ and the solution to the IVP is

$$\frac{y^4}{4} + \frac{y^2}{2} = \frac{x}{2} + \frac{1}{4}$$

This is called the *implicit solution* since we could not explcitly solve for y in terms of x.

Example. Solve the IVP

$$e^x y' = (x + 1)y^2 \quad y(0) = -\frac{1}{2}$$

Solution:

$$\begin{aligned} e^x \frac{dy}{dx} &= (x + 1)y^2 \\ \implies \frac{1}{y^2}dy &= \frac{x + 1}{e^x}dx \\ \implies \int \frac{1}{y^2}dy &= \int (x + 1)e^{-x}dx \end{aligned}$$

We can use integration by parts to solve the right hand side integral. Let $u = x + 1$ and $dv = e^{-x}dx$, $u' = 1$, and $v = -e^{-x}$. Then

$$\begin{aligned}
\int (x+1)e^{-x}dx &= uv - \int u'vdx \\
&= -(x+1)e^{-x} - \int -e^{-x}dx \\
&= -(x+1)e^{-x} - e^{-x} + C
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{y^{-2+1}}{-2+1} &= -(x+1)e^{-x} - e^{-x} + C \\
-\frac{1}{y} &= -(x+1)e^{-x} - e^{-x} + C
\end{aligned}$$

Setting $y = -\frac{1}{2}$ and $x = 0$, we have

$$2 = -2 + C \implies C = 4$$

Therefore the implicit solution is

$$-\frac{1}{y} = -(x+1)e^{-x} - e^{-x} - 4$$

We can rewrite this as an explicit solution as

$$y = \frac{1}{(x+2)e^{-x} - 4}$$

1.2 First Order ODE's With Homogeneous Coefficients

Definition 1.2.1. *A function $F(x,y)$ of two variables is called* homogeneous *of degree k if*

$$F(\lambda x, \lambda y) = \lambda^k \cdot F(x,y)$$

This type of ODEs can be made seperable after a suitable change of variables of the unknown function.

Example.

$$F(x,y) = 3x^2y - 2xy^2 + y^3$$

We can check if its homogeneous by the definition,

$$\begin{aligned}
F(\lambda x, \lambda y) &= 3(\lambda x)^2(\lambda y) - 2(\lambda x)(\lambda y)^2 + (\lambda y^3) \\
&= 3\lambda^3x^2y - 2\lambda^3xy^2 + \lambda^3y^3 \\
&= \lambda^3(3x^2y - 2xy^2 + y^3) \\
&= \lambda^3F(x,y)
\end{aligned}$$

Therefore, $F(x,y)$ is homogeneous of degree 3. We can tell quickly if a polynomial is homogeneous is by looking at the exponents of each term. If the sum of the exponents of each term is the same, then the polynomial is homogeneous, with order being the sum of the exponents in each term (i.e x^2y has exponents 2,1, xy^2 has exponents 1,2, and y^3 has exponents 3, each sum to 3).

Definition 1.2.2. *A first order ODE given in differential form*

$$M(x,y)dx + N(x,y)dy = 0$$

is called of homogeneous coefficients *if both $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree.*

Example.

$$(3x^2 + 2y^2 + 2xy)dx - 4xydy = 0$$

Both terms are homogeneous of degree 2, therefore this is a differential equation of homogeneous coefficients.

Theorem 1.2.1. *A first order ODE of homogeneous coefficients can be made seperable by changing the function using one of the following substitutions:*

- *Set $u := \frac{y}{x}$ or*
- $u := \frac{x}{y}$

Example. Solve the following IVP

$$(x^2 - y^2)dx + 2xydy = 0 \quad y(1) = 2$$

Solution: This is a first order ODE with homogeneous coefficients. Let

$$u := \frac{y}{x} \implies y = xu$$

$$\frac{dy}{dx} = 1 \cdot u + x \cdot \frac{du}{dx} \implies dy = udx + xdu$$

So, we have

$$(x^2 - y^2)dx + 2xydy = 0 \implies (x^2 - x^2u^2)dx + 2x(xu)(udx + xdu) = 0$$

Simplyfing, we get

$$\begin{aligned} x^2dx - x^2u^2dx + 2x^2u^2dx + 2x^3udu &= 0 \\ dx - u^2dx + 2u^2dx + 2xudu &= 0 \\ (1+u^2)dx + 2xudu &= 0 \\ (1+u^2)dx &= -2xudu \\ -\frac{1}{x}dx &= \frac{2u}{1+u^2}du \end{aligned}$$

Now that it's seperable, we can integrate both sides,

$$\begin{aligned} -\int \frac{1}{x} dx &= \int \frac{2u}{1+u^2} du \\ -\ln(x) &= \ln(1+u^2) + C \end{aligned}$$

Now using our initial condition, we have $y(1) = 2$. But, our differential equation is a function of u not y, so we must calculate u using our initial condition. So, $u(1) = \frac{y(1)}{1} = 2$. So,

$$-\ln 1 = \ln 5 + C \implies C = -\ln 5$$

Therefore, our solution is

$$\begin{aligned} \ln x &= \ln(1+u^2) - \ln 5 \\ \ln\left(\frac{5}{x}\right) &= \ln(1+u^2) \\ \frac{5}{x} &= 1 + u^2 \\ u^2 &= \frac{5}{x} - 1 \\ \frac{y^2}{x^2} &= \frac{5}{x} - 1 \\ y &= \sqrt{5x - x^2} \end{aligned}$$

We take the positive square root since if we took the negative square root, then $y(1) = -2$ which is not our initial condition.

Example. Solve the IVP

$$(2x + y)dx - xdy = 0 \quad y(1) = -2 \quad x > 0$$

Solution: This is a first order ODE with homogeneous coefficients. Let

$$u = \frac{y}{x} \implies y = xu$$

$$dy = udx + xdu$$

Substituting into our differential equation, we get

$$\begin{aligned} (2x + xu)dx - x(udx + xdu) &= 0 \\ (2 + u)dx - (udx + xdu) &= 0 \\ 2dx + udx - udx - xdu &= 0 \\ 2dx = xdu &\implies \frac{2}{x}dx = du \end{aligned}$$

This differential equation in u is sperable, so we can integrate

$$\int \frac{2}{x} dx = \int du$$
$$2 \ln x = u + C$$

Using your initial condition, $y(1) = -2$. so $u(1) = \frac{y(1)}{1} = -2$. Therefore,

$$2 \ln 1 = -2 + c \implies C = 2$$

Now solving for y,

$$u = 2 \ln x - 2$$
$$\frac{y}{x} = 2 \ln x - 2$$
$$y = x(2 \ln x - 2)$$

This is our explicit solution to the initial value problem.

1.3 Exact First Order ODEs

Definition 1.3.1. *Given a function $F(x, y)$ of two variables, the differential of $F(x, y)$ denoted by dF is defined by*

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

Example. Let

$$F(x, y) = 2x^2y^3 + \sin(x + 2y)$$

Then

$$dF = (4xy^3 + \cos(x + 2y))dx + (6x^2y^2 + 2\cos(x + 2y))dy$$

Remark:

$$dF = 0 \iff \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \iff \frac{\partial F}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial y} = 0$$

So, $F(x, y) = C$ is a constant function. Therefore,

$$dF = 0 \iff F(x, y) = C$$

Definition 1.3.2. *A first order ODE*

$$M(x, y)dx + N(x, y)dy = 0$$

is called exact *if there exists a continuous function $F(x, y)$ such that*

$$\frac{\partial F}{\partial x} = M(x, y) \text{ and } \frac{\partial F}{\partial x} = N(x, y)$$

So if $M(x, y)dx + N(x, y)dy = 0$ is exact, then

$$dF = 0 \implies F(x, y) = C$$

In summary, if $M(x,y)dx + N(x,y)dy = 0$ is exact, then find $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial x} = N(x,y)$$

Then, the (implicit) solution to the ODE is $F(x,y) = C$. Furthermore, since $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$, then

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial x \partial y}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial y \partial x}$$

So by the Clairaut-Schwarz Theorem, the ODE is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem 1.3.1 (Condition for Exactness)**.** *The first order ODE $M(x,y)dx + N(x,y)dy = 0$ (with M, N continuous) is exact if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

1.3.1 Steps to Solving Exact ODEs

1. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Look for a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \frac{\partial F}{\partial y} = N$$

3. The general solution to the ODE is $F(x,y) = C$.
4. If an intial condition is given, use it to find C.

Example. Solve the following IVP

$$(6x - 2y^2 + 2xy^3)dx + (3x^2y^2 - 4xy)dy = 0, \quad y(1) = -2$$

Solution. We first check exactness.

$$\frac{\partial M}{\partial y} = -4y + 6xy^2 = 6xy^2 - 4y$$

$$\frac{\partial N}{\partial x} = 6xy^2 - 4y$$

Therefore, this ODE is exact. Now we need to find a function $F(x, y)$ satisfying the partial derivatives. We can do this by integrating N with respect to y, so we have

$$\frac{\partial F}{\partial y} = 3x^2y^2 - 4xy$$

$$F(x, y) = \int 3x^2y^2 - 4xydy = 3x^2 \int y^2dy - 4x \int ydy = x^2y^3 - 2xy^2 + h(x)$$

We add $h(x)$ since when integrating with respect to y, we are treating x as a constant so $h(x)$ is constant with respect to y. So we have

$$F(x, y) = x^2y^3 - 2xy^2 + h(x)$$

Now we can use the first equation to solve for $h(x)$,

$$\frac{\partial F}{\partial x} = 2xy^3 - 2y^2 + h'(x)$$

This equation is equal to M, so we can plug M in and get

$$M = 6x - 2y^2 + 2xy^3 = 2xy^3 - 2y^2 + h'(x) \implies h'(x) = 6x$$

Now we can solve for $h(x)$ by taking the integral,

$$h(x) = \int 6xdx = 3x^2 + C_1$$

Now, we get

$$F(x, y) = x^2y^3 - 2xy^2 + 3x^2 + C_1$$

So the general solution to the ODE is

$$x^2y^3 - 2xy^2 + 3x^2 + C_1 = C_2 \implies x^2y^3 - 2xy^2 + 3x^3 = C$$

Now using the intial condition $y(1) = -2$, then

$$1^2(-2)^3 - 2(1)(-2)^2 + 3(1)^2 = C \implies C = -13$$

Therefore, the solution to the IVP is

$$x^2y^3 - 2xy^2 + 3x^3 = -13$$

Example. Solve the IVP

$$(2x\cos(y) - 3x^2y + ye^{xy})dx + (-x^2\sin(y) + xe^{xy} - x^3)dy = 0, \quad y(0) = 1$$

Solution. We first check exactness,

$$\frac{\partial M}{\partial y} = -2x\sin(y) - 3x^2 + e^{xy} + ye^{xy}$$

$$\frac{\partial N}{\partial x} = -2x\sin(y) + ye^{xy} + e^{xy} - 3x^2$$

Therefore this ODE is exact, so we look for $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N$$

$$\begin{aligned}
F(x, y) &= \int 2x\cos(y) - 3x^2 y + ye^{xy} dx \\
&= 2\cos(y)\int xdx - 3y\int x^2 dx + y\int e^{xy} dx \\
&= x^2 - x^3 y + y\frac{e^{xy}}{y} + h(y)
\end{aligned}$$

So we have

$$F(x, y) = x^2\cos(y) - x^3 y + e^{xy} + h(y)$$

Then,

$$\frac{\partial F}{\partial y} = -x^2\sin(y) - x^3 + xe^{xy} + h'(y) = N \implies h'(y) = 0$$

So $h(y)$ is a constant, say $h(y) = K$, then our general solution for $F(x, y)$ is

$$F(x, y) = x^2\cos(y) - x^3 y + e^{xy} + k \implies x^2\cos(y) - x^3 y + e^{xy} = C$$

Using the condition, $y(0) = 1$, we get

$$(0)^2\cos(1) - (0)^3(1) + e^{0\cdot 1} = C \implies C = 1$$

Therefore the (implicit) solution to the IVP is

$$x^2\cos(y) - x^3 y + e^{xy} = 1$$

1.4 First Order ODEs With an Integrating Factor

Definition 1.4.1 (Integrating Factor)**.** *We say that the function $\mu(x, y)$ is an* integrating factor *of the first-order ODE*

$$M(x, y)dx + N(x, y)dy = 0$$

if the new ODE

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact.

In general, finding an integrating factor is not easy. However, there are some special cases where we can find an integrating factor easily.

Theorem 1.4.1. *For the ODE*

$$M(x,y)dx + N(x,y)dy = 0$$

1. *If*

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$$

for some function g of y only, then an integration factor exists given by

$$\mu(y) = \exp\left(-\int g(y)dy\right)$$

2. *If*

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(x)$$

for some function f of x only, then an integration factor exists given by

$$\mu(x) = \exp\left(\int f(x)dx\right)$$

Example. Solve the IVP

$$(y^4 + xy)dx + (xy^3 - x^2 + 2y^3e^y)dy = 0, \quad y(0) = 1$$

Solution. It's clear this ODE is not exact, so we need to find an integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4y^3 + x - y^3 + 2x = 3y^3 + 3x$$

If we dvidie by M, we get

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3(y^3+x)}{y^4+xy} = \frac{3(y^3+x)}{y(y^3+x)} = \frac{3}{y}$$

Therefore, we have our integrating factor

$$\mu(y) = \exp\left(-\int \frac{3}{y}dy\right) = \exp\left(-3\int \frac{1}{y}dy\right) = \exp\left(\ln(y^{-3})\right) = y^{-3}$$

We multiply the original ODE with $\mu(y) = y^{-3}$

$$y^{-3}(y^4 + xy)dx + y^{-3}(xy^3 - x^2 + 2y^3e^y)dy = (y + xy^2)dx + (x - x^2y^{-3}2e^y)$$

Now we can check the exactness of this ODE,

$$\frac{\partial M}{\partial y} = 1 - 2xy^{-3} \text{ and } \frac{\partial N}{\partial x} = 1 - 2xy^{-3}$$

Therefore, this ODE is exact, so we look for $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N$$

The first equation is simpler so we will start with that,

$$\begin{aligned} F(x, y) &= \int y + xy^{-2} dx \\ &= y \int dx + y^{-2} \int x dx \\ &= xy + \frac{x^2 y^{-2}}{2} \end{aligned}$$

Now we derive with respect to y and use the second equation,

$$\frac{\partial F}{\partial y} = x - x^2 y^{-3} + h'(y) = N = x - x^2 y^{-3} + 2e^y \implies h'(y) = 2e^y$$

Then

$$h(y) = \int h'(y) dy = \int 2e^y dy = 2e^y + k$$

So we get the function

$$F(x, y) = xy + \frac{x^2 y^{-2}}{2} + 2e^y + k$$

Then setting $F(x, y)$ equal to a constant to get our (implicit) general solution,

$$xy + \frac{x^2 y^{-2}}{2} + 2e^y = C$$

Using the initial condition, $y(0) = 1$, we get

$$(0)(1) + \frac{(0)^2 (1)^{-2}}{2} + 2e^1 = C \implies C = 2e$$

Therefore, the (implicit) solution to the IVP is

$$xy + \frac{x^2 y^{-2}}{2} + 2e^y = 2e$$

Example. Solve the following IVP

$$(x^2 + 4xy + 3y^2)dx + (x^2 + 2xy)dy = 0, \quad y(1) = 1, \ x > 0$$

Solution. We can see that this function is of homogeneous coefficients, but we will solve it using the integrating factor. First we calculate the partial derivatives,

$$\frac{\partial M}{\partial y} = 4x + 6y$$

$$\frac{\partial N}{\partial x} = 2x + 2y$$

we see that these are not equal so this ODE is not exact, then we calculate the difference,

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x + 4y$$

Then to obtain a function of only x, we divide by N,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x+4y}{x^2+2xy} = \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$$

An integrating factor exists and is given by

$$\mu(x) = \exp\left(\int (f(x)dx)\right) = \exp(2\ln x) = x^2$$

Now we can multiply the original ODE by $\mu(x) = x^2$,

$$(x^2 + 4x^3y + 3x^2y^2)dx + (x^4 + 2x^3y)dy = 0$$

Now we can check the exactness of this ODE, we'll denote the new ODE by M^* and N^*,

$$\frac{\partial M^*}{\partial y} = 4x^3 + 6x^2y$$

$$\frac{\partial N^*}{\partial x} = 4x^3 + 6x^2y$$

Therefore, this ODE is exact, so we look for $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M^* \text{ and } \frac{\partial F}{\partial y} = N^*$$

The second equation is simpler so we'll start with this one,

$$F(x,y) = \int x^4 + 2x^3y dy = x^4y + x^3y^2 + h(x)$$

Now we derive with respect to x and use the first equation,

$$\frac{\partial F}{\partial x} = 4x^3 + 3x^2y^2 + h'(x)$$

Then, $M^* = x^4 + 4x^3y + 3x^2y^2$, which gives us $h'(x) = x^4$. Now

$$h(x) = \int x^4 dx = \frac{x^5}{5} + k$$

So we get the function

$$F(x,y) = x^4y + x^3y^2 + \frac{x^5}{5} + k$$

The general solution is given by setting $F(x, y)$ equal to a constant,

$$x^4y + x^3y^2 + \frac{x^5}{5} = C$$

Then using our initial value $y(1) = 1$,

$$1 + 1 + \frac{1}{5} = \frac{11}{5}$$

Thus, the solution to the IVP is

$$x^4y + x^3y^2 + \frac{x^5}{5} = \frac{11}{5}$$

Example. Solve the following IVP with initial condition $y(0) = 1$,

$$(3xy - 2y^2 \sin x + 4y)dx + (3x^2 + 8x + 6y \cos x)dy = 0$$

Solution. With sin and cos in our function, its certainly not of homogeneous coefficients, we check for exactness,

$$\frac{\partial M}{\partial y} = 3x - 4y \sin x + 4$$

$$\frac{\partial N}{\partial x} = 6x + 8 - 6y \sin x$$

We see that these are not equal so this ODE is not exact, then we calculate the difference of the partial derivatives

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x + 2y \sin xx - 4$$

Then divide by M to find a function of y,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-3x + 2y \sin x - 4}{3xy - 2y^2 \sin x + 4y} = \frac{-3x + 2y \sin x - 4}{-y(-3x + 2y \sin x - 4) = -\frac{1}{y}}$$

An integrating factor exists and is given by

$$\mu(y) = \exp\left(-\int -\frac{1}{y}\right) = e^{\ln y} = y$$

Multiplying the original ODE by $\mu(y) = y$,

$$(3xy^2 - 2y^3 \sin x + 4y^2)dx + (3x^2y + 8xy + 6y^2 \cos x)dy = 0$$

Now checking exactness of our new ODE,

$$\frac{\partial M^*}{\partial y} = 6xy - 6y^2 \sin x + 8y$$

$$\frac{\partial N^*}{\partial y} = 6xy - 6y^2 \sin x + 8y$$

This ODE is exact and we can now solve for $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M^* \text{ and } \frac{\partial F}{\partial y} = N^*$$

Both of the equations are similar in complexity so we'll start with the first one,

$$F(x,y) = \int 3xy^2 - 2y^3 \sin x + 4ydx = \frac{3x^2y^2}{2} = 2y^2 \cos x + 4xy^2 + h(y)$$

Then using the second equation to solve for y,

$$\frac{\partial F}{\partial y} = 3x^2y + 6y^2 \cos x + 8xy + h'(y)$$

Then, $N^* = 3x^2y + 8xy + 6y^2 \cos x$, which gives us $h'(y) = 0$. So we get $h(x) = k$, and the function

$$F(x,y) = \frac{3x^2y^2}{2} + 2y^3 \cos x + 4xy^2 + k$$

The general solution is

$$\frac{3x^2y^2}{2} + 2y^3 \cos x + 4xy^2 = C$$

Using our inital condition $y(0) = 1$, we get

$$0 + 2 + 0 = C$$

The solution to the IVP is

$$\frac{3x^2y^2}{2} + 2y^3 \cos x + 4xy^2 = 2$$

Example. Find the general solution of the following ODE,

$$(e^{x+y}ye^y)dx + (xe^y - 1)dy = 0$$

Solution. We have exponential functions so it is certainly not of homogeneous coefficients, we first check for exactness,

$$\frac{\partial M}{\partial y} = e^{x+y} + e^y + ye^y$$

$$\frac{\partial N}{\partial x} = e^y$$

We see that these are not equal so this ODE is not exact, then we calculate the difference of the partial derivatives

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^y$$

This function is exactly M, so we will divide by M to find a function of y,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = 1 = g(y)$$

An integrating factor exists and is given by

$$\mu(y) = \exp\left(-\int g(y)dy\right) = \exp\left(-\int 1dy\right) = e^{-y}$$

Multiplying the original ODE by $\mu(y) = e^{-y}$,

$$(e^x + y)dx + (x - e^{-y})dy = 0$$

Now checking exactness of our new ODE,

$$\frac{\partial M^*}{\partial y} = 1$$

$$\frac{\partial N^*}{\partial y} = 1$$

This ODE is exact and we can now solve for $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M^* \text{ and } \frac{\partial F}{\partial y} = N^*$$

Both of the equations are similar in complexity so we'll start with the first one,

$$F(x, y) = \int e^x + ydx = xe^x + yx + h(y)$$

Then using the second equation to solve for $h(y)$,

$$\frac{\partial F}{\partial y} = x + h'(x)$$

Then, $N^* = x - e^{-y}$, which gives us $h'(y) = -e^{-y}$. So we get

$$h(y) = \int h'(y)dy = -\int e^{-y} = e^y + k$$

Then the general solution is

$$e^x + xy + e^{-y} = C$$

1.5 Linear First-Order ODEs

Definition 1.5.1. *A first order ODE that can be written under the form*

$$y' + f(x)y = r(x)$$

is called ***linear****.*

Example.

$$xy' + e^x y = \frac{\sin x}{1+x^2}$$

We can divide by x to get

$$y' + \frac{e^x}{x} y = \frac{\sin x}{x(1+x^2)}$$

This is linear with $f(x) = \frac{e^x}{x}$ and $r(x) = \frac{\sin x}{x(1+x^2)}$

1.5.1 Steps to Finding (Explicit) Solutions

Given a linear first-order ODE in the formm $y' + f(x)y = r(x)$, we can find the solution by following these steps. Start by writing the ODE in differential form by replacing y' with $\frac{dy}{dx}$,

$$\begin{aligned}
&\frac{dy}{dx} + f(x)y = r(x) \\
&\implies dy + f(x)ydx = r(x)dy \\
&\implies (f(x)y - r(x))dx + 1dy = 0
\end{aligned}$$

This ODE is not exact since

$$\frac{\partial M}{\partial y} = f(x)$$

$$\frac{\partial N}{\partial x} = 1$$

So we find the integrating factor,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

and we get our integrating factor,

$$\mu(x) = \exp\left(\int f(x)dx\right)$$

Note that

$$\mu'(x) = \exp\left(f(x)dx\right) \cdot \left(\int f(x)dx\right)' = \mu(x)f(x)$$

Now we can multply the ODE by $\mu(x)$,

$$(\mu(x)f(x)y - r(x)\mu(x))dx + \mu(x)dy = 0$$

Now we can check for exactness, of our new ODE,

$$\frac{\partial M^*}{\partial y} = \mu(x)f(x)$$

$$\frac{\partial n^*}{\partial x} = \mu(x)f(x)$$

Then we look for our function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M^* \text{ and } \frac{\partial F}{\partial y} = N^*$$

$$F(x, y) = \int \mu(x)dy = \mu(x)y + h(x)$$

Then we use the second equation to solve for $h(x)$,

$$\frac{\partial F}{\partial y} = \mu'(x)y + h'(x) = \mu(x)f(x) + h'(x)$$

Then, $M^* = \mu(x)f(x)y - \mu(x)r(x)$, which gives us that $h'(x) = -\mu(x)r(x)$. So we get

$$h(x) = \int \mu(x)r(x)dx + k$$

Then we get the function

$$F(x, y) = \mu(x)y - \int \mu(x)r(x)dx + k$$

and our (implicit) general solution is

$$\mu(x)y - \int \mu(x)r(x)dx = C$$

We can find an explicit solution by solving for y,

$$\begin{aligned}
&\mu(x)y - \int \mu(x)r(x)dx = C \\
\implies &\mu(x)y = \int \mu(x)r(x)dx + C \\
\implies &y = \frac{\int \mu(x)r(x)dx + C}{\mu(x)} \\
\implies &y = \frac{\int \mu(x)r(x)dx + C}{\exp\left(\int f(x)dx\right)} \\
\implies &y = \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right) \exp\left(\int f(x)\right)^{-1} \\
\implies &y = \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right) \exp\left(-\int f(x)\right)
\end{aligned}$$

Example. Solve the IVP

$$y' - 2xy = x \quad y(0) = \frac{1}{2}$$

Solution. Clearly this is a linear first order ODE witih $f(x) = -2x$, and $r(x) = x$. Then we can apply the formula to get the general solution

$$\begin{aligned} y &= \left(\int \exp \left(\int f(x) dx \right) r(x) dx + C \right) \exp \left(- \int f(x) dx \right) \\ &= \left(\int \exp \left(\int -x^2 dx \right) x dx + C \right) \exp \left(- \int -2x dx \right) \\ &= \left(\int x e^{-x^2} x dx + C \right) e^{x^2} \end{aligned}$$

We can solve the integral by substitution, set $u \coloneqq -x^2$, $du = -2x dx$, $dx = \frac{du}{-2x}$, then we get

$$\begin{aligned} \int x e^{-x^2} dx &= \int x e^u \frac{du}{-2x} \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^{-x^2} \end{aligned}$$

Then we get the explicit general solution

$$y = \left(-\frac{1}{2} e^{-x^2} + C \right) e^{x^2} = -\frac{1}{2} + C e^{x^2}$$

We can solve for C using the initial condition $y(0) = \frac{1}{2}$,

$$\frac{1}{2} = -\frac{1}{2} + C \implies C = 1$$

So our explicit solution is

$$y = e^{x^2} - \frac{1}{2}$$

Example. Solve the IVP

$$y' - 4y = x; \ y(0) = \frac{15}{16}$$

Solution. This is a linear first-order ODE with $f(x) = -4$, and $r(x) = x$. Then we can apply the formula to get the general solution

$$\begin{aligned} y &= \left(\int \exp \left(\int f(x) dx \right) r(x) dx + C \right) \exp \left(- \int f(x) \right) \\ &= \left(\int \exp \left(\int -4 dx \right) x dx + C \right) \exp \left(- \int -4 dx \right) \\ &= \left(\int e^{-4x} x dx + C \right) e^{4x} \end{aligned}$$

Now we can solve the integral by parts, set $u = x$, $v' = e^{4x}$, $u' = 1$, $v = \int e^{-4x}dx = -\frac{1}{4}e^{-4x}$, then we get

$$\begin{aligned}\int e^{-4x}xdx &= uv - \int u'vdx \\ &= -\frac{1}{4}xe^{-4x} - \int -\frac{1}{4}e^{-4x}dx \\ &= -\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x}\end{aligned}$$

Then, we get the general solution

$$y = \left(-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C\right)e^{4x} = -\frac{1}{4}x - \frac{1}{16} + Ce^{4x}$$

Using our intial condition $y(0) = \frac{15}{16}$, we can solve for C,

$$\frac{15}{16} = -\frac{1}{16} + C \implies C = 1$$

Therefore, our explicit solution is

$$y = -\frac{1}{4}x - \frac{1}{16} + e^{4x}$$

Example. Solve the IVP

$$(1 + \cos x)y' - (\sin x)y = 2x;\ y(0) = \frac{1}{2}$$

Solution. We have to make the coefficient of y' to be 1, so we divide both sides by $1 + \cos x$ to get

$$y' - \frac{\sin x}{1 + \cos x}y = \frac{2x}{1 + \cos x}$$

Now we can use our formula for linear ODE's,

$$\begin{aligned}y &= \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right)\exp\left(-\int f(x)dx\right) \\ &= \left(\int \exp\left(\int -\frac{\sin x}{1 + \cos x}dx\right)\frac{2x}{1 + \cos x}dx + C\right)\exp\left(-\int -\frac{\sin x}{1 + \cos x}\right)\end{aligned}$$

Now computing the integral of $f(x)$, set $u = 1 + \cos x$, $du = -\sin x dx$

$$\begin{aligned}\int \frac{-\sin x}{1 + \cos x}dx &= \int \frac{-\sin x}{u}\frac{du}{-\sin x} \\ &= \int \frac{1}{u}du \\ &= \ln|u| \\ &= \ln(1 + \cos x)\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
y &= \left(\int \exp\left(\int -\frac{\sin x}{1+\cos x}dx\right)\frac{2x}{1+\cos x}dx + C\right)\exp\left(-\int -\frac{\sin x}{1+\cos x}\right) \\
&= \left(\int \exp(\ln(1+\cos x))\frac{2x}{1+\cos x}dx + C\right)\exp\left(-\ln(1+\cos x)\right) \\
&= \left(\int (1+\cos x)\frac{2x}{1+\cos x}dx + C\right)(1+\cos x)^{-1} \\
&= \frac{\left(\int 2x dx + C\right)}{1+\cos x} \\
&= \frac{x^2+C}{1+\cos x}
\end{aligned}
$$

Using our inital condition $y(0) = \frac{1}{2}$, we get

$$\frac{1}{2} = \frac{C}{2} \implies C = 1$$

Therefore, our explicit solution is

$$y = \frac{x^2+1}{1+\cos x}$$

1.6 First-Order Bernoulli ODE's

Definition 1.6.1. *A first-order ODE is called of* Bernouilli *type if it can be written in the form*

$$y' + f(x)y = r(x)y^a$$

for some $a \in \mathbb{R}$.

1.6.1 Steps to Solving Bernoulli type ODE's

1. Let $u = y^{1-a}$
2. Compute u':
$$u' = (1-a)y^{-1}y'$$
3. Isolate y' from the original ODE and substitute into u'
4. The resulting ODE is linear that we solve for u.

Example. Solve the IVP

$$y' + \frac{4}{x}y = -x^2y^2;\ x > 0,\ y(1) = \frac{1}{3}$$

Solution. This is a first order Bernouilli ODE, with $f(x) = \frac{4}{x}$, $r(x) = -x^2$, and $a = 2$. Let $u = y^{1-a} = y^{-1}$, then

$$\begin{aligned} u' &= -y^{-2}y' = -y^{-2}\left(-\frac{4}{x}y - x^3y^2\right) \\ &= \frac{4}{x}y^{-1} + x^3 \\ &= \frac{4}{x}u + x^3 \\ &= u' - \frac{4}{x}u = x^3 \end{aligned}$$

Now this is a linear first-order ODE in the function u with $f(x) = -\frac{4}{x}$ and $r(x) = x^3$. Then, the general solution is

$$\begin{aligned} u &= \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right) \exp\left(-\int f(x)\right) \\ &= \left(\int \exp\left(\int -\frac{4}{x}\right) x^3dx + C\right) \exp\left(-\int -\frac{4}{x}dx\right) \\ &= \left(\int \exp(-4\ln x)x^3dx + C\right) \exp(4\ln x) \\ &= \left(\int x^{-4}x^3dx + C\right) x^4 \\ &= \left(\int x^{-1}dx + C\right) x^4 \\ &= (\ln x + C)\, x^4 \end{aligned}$$

Now, we know $u = y^{-1}$, so

$$y = \frac{1}{x^4(\ln x + C)}$$

Using our intial condition $y(1) = \frac{1}{3}$,

$$\frac{1}{3} = \frac{1}{1 \cdot (\ln 1 + C)} = \frac{1}{C} \implies C = 3$$

Therefore, the explicit solution to our IVP is

$$y = \frac{1}{x^4(\ln x + 3)}$$

Example. Solve the IVP

$$y' + \frac{2}{x}y = 2\sqrt{y};\ \ x > 0\ y(1) = 1$$

Solution. This is a first order Bernouilli ODE, with $f(x) = \frac{2}{x}$, $r(x) = 2$, and $a = \frac{1}{2}$. Let $u = y^{1-a} = y^{\frac{1}{2}} = \sqrt{y}$. Then,

$$\begin{aligned} u' &= \frac{1}{2}y^{-\frac{1}{2}}y' \\ &= \frac{1}{2}y^{-\frac{1}{2}}\left(-\frac{2}{x}y + 2y^{\frac{1}{2}}\right) \\ &= -\frac{1}{x}y^{\frac{1}{2}} + 1 \end{aligned}$$

Then we get the linear first order ODE in u

$$u' + \frac{1}{x}u = 1$$

Our general solution for u is

$$\begin{aligned} u &= \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right) \exp\left(-\int f(x)dx\right) \\ &= \left(\int \exp\left(\int \frac{1}{x}dx\right) 1dx + C\right) \exp\left(-\int \frac{1}{x}dx\right) \\ &= \left(\int \exp(\ln x)dx + C\right) \exp(-\ln x) \\ &= \left(\int xdx + C\right) \exp(-2\ln x) \\ &= \left(\frac{1}{2}x^2 + C\right) x^{-1} \\ &= \frac{x}{2} + \frac{C}{x} \end{aligned}$$

Then using your equation for u in terms of y,

$$u = \sqrt{y} \implies y = \left(\frac{x}{2} + \frac{C}{x}\right)^2$$

Using our initial condition $y(1) = 1$,

$$1 = \left(\frac{1}{2} + C\right)^2 \implies C = \frac{1}{2}$$

Therefore, the explicit solution to our IVP is

$$y = \left(\frac{x^2 + 1}{2x}\right)^2$$

Chapter 2

Second Order Linear Homogeneous ODEs

Definition 2.0.1 (Linear Independence)**.** *We say that the functions $y_1, y_2, \ldots, y_n$ are linearly indepedent on an interval I if*

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$$

Theorem 2.0.1. *Two functions y_1, y_2 are linearly indepedent if and only if $\frac{y_1}{y_2}$ does not equal a constant.*

2.1 Wronskian

Definition 2.1.1 (Wronskian)**.** *Let $y_1, y_2, \ldots, y_n$ be n functions such that the first $n-1$ derivatives of each function exists, and are continuous on I. The* Wronskian *of $y_1, y_2, \ldots y_n$ at a point $x \in I$ is the determinant*

$$W[y_1, y_2, \ldots, y_n](x) = \det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

Example. Take $y_1 = 1$, $y_2 = \sin x$, $y_3 = \cos x$. Then, the Wronskian is

$$\begin{aligned} W[y_1, y_2, y_3](x) &= \det \begin{bmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{bmatrix} \\ &= 1 \det \begin{bmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{bmatrix} - 0 + 0 \\ &= \cos^2 x - \sin^2 x \\ &= -(\cos^2 x + \sin^2 x) = -1 \end{aligned}$$

Theorem 2.1.1. *Let $y_1, y_2, \ldots, y_n$ be continuous functions with continuous first $n-1$ derivatives on an interval I. If $W[y_1, y_2, \ldots, y_n](x) \neq 0$ for some $x \in I$, then $y_1, y_2, \ldots, y_n$ are linearly independent on I.*

Example. $y_1 = x$, $y_2 = e^x$, $y_3 = e^{2x}$. Prove that $\{y_1, y_2, y_3\}$ are linearly independent on $\mathbb{R}$.

Solution. We'll use the Wronskian to show that these functions are linearly independent.

$$\begin{aligned} W[y_1, y_2, y_3](x) &= \det \begin{bmatrix} x & e^x & e^{2x} \\ 1 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{bmatrix} \\ &= \det \begin{bmatrix} 0 & -xe^x + e^x & -2xe^{2x} + e^{2x} \\ 1 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{bmatrix} \qquad (-xR_2 + R_1 \to R_1) \\ &= -\det \begin{bmatrix} -xe^x + e^x & -2e^{2x} + e^{2x} \\ e^x & 4e^{2x} \end{bmatrix} \\ &= -e^{3x}(3 - 2x) \end{aligned}$$

Definition 2.1.2. *An ODE of order n is called* linear *if it is of the form*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = r(x)$$

If $r(x) = 0$, then the ODE is called homogeneous. *If $r(x) \neq 0$, then the ODE is called* non-homogeneous.

Example.

$$2xy''' + e^x y'' - y = \frac{1}{1+x^2}$$

This is a linear non-homogeneous ODE.

Theorem 2.1.2. *The set of all solutions to a homogeneous linear ODE of order n*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

is a vector space of dimension n.

2.1.1 Steps to finding a general solution of a homogeneous linear ODE

The theorem suggets the following steps to finding the general solution to a homogeneous linear ODE of order n:

1. Find n linearly indepdent solutions $y_1, y_2, \ldots, y_n$ to the ODE.
2. $\{y_1, y_2, \ldots, y_n\}$ form a basis of solutions to the ODE.

3. The general solution to the ODE is a linear combination of the basis functions
$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

In this chapter, we want to find the general solution to second order linear homogeneous ODEs
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

2.2 Second-Order Linear Homogeneous ODEs with Constant Coefficients

For this group of ODE's, we consider the case when $a_2(x), a_1(x), a_0(x)$ are constants. In this case, we can write the ODE as
$$ay'' + by' + cy = 0$$

We'll look at an example to illustrate the process of finding solutions for this type of ODE.

Example. Consider the ODE
$$y'' - y = 0$$

Solution. We know that we need 2 linearly independent solutions to the ODE. We have that $y'' - y = 0 \implies y'' = y$. Two functions that satisfy this equation are $y_1 = e^x$, and $y_2 = e^{-x}$. We can check that these are linearly indepedent by checking their ratio
$$\frac{e^x}{e^{-x}} = e^{2x}$$
Therefore, these solutions form a basis for the solution space for $y'' - y = 0$. Then the general solution of the ODE is a linear combination of the the basis functions
$$y = c_1 e^x + c_2 e^{-x}$$
The above example suggetss that we look for *exponential* solutions in the case of constant coefficients
$$y'' + ay' + by = 0$$
In general, we have
$$y = e^{\lambda x}$$
for some constant λ. Then we can differentiate y to get
$$y' = \lambda e^{\lambda x},\ y'' = \lambda^2 e^{\lambda x}$$
Plugging our equations into our ODE, we get
$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0$$
This is our *characteristic equation*. There are 3 possible cases for the roots of the (quadratic) characteristic equation,

- **Case 1:** We have two distinct real roots λ_1, λ_2, then $y_1 = e^{\lambda x}$, $y_2 = e^{\lambda_2 x}$. Then our general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

- **Case 2:** We have one real root λ with multiplicity 2. In this case, $y_1 = e^{\lambda x}$ is one solution, and our second solution is $y_2 = xe^{\lambda x}$. Then our general solution is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

- **Case 3:** We have two complex conjugate roots $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. In this case, we can show

$$y_1 = e^{\alpha x}\cos(\beta x),\ y_2 = e^{\alpha x}\sin(\beta x)$$

  is a bais of solutions, so

$$y = c_a e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

2.2.1 Steps to Solving Second Order Linear Homogeneous ODEs with Constant Coefficients

To summarize, the steps to solving the ODE

$$y'' + ay' + by = 0$$

are as follows.

1. Write the characteristic equation $\lambda^2 + a\lambda + b = 0$.
2. Find the roots of the characteristic equation.
3. If the characteristic equation has two distintc real roots λ_1, λ_2, then $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$ is a basis of solutions and the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

4. If the characteristic equation has a double real root $\lambda_1 = \lambda_2 = \lambda$. Then $\{y_1 = e^{\lambda x}, xe^{\lambda x}\}$ is a basis of solutions and the general solution is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

5. If the characteristic equation has 2 complex conjugate roots

$$\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$$

   Then $\{y_1 = e^{\alpha x}\cos(\beta x), y_2 e^{\lambda x}\sin(\beta x)\}$ is a basis of solutions. The general solution in this case is

$$y = c_1 e^{\lambda x}\cos(\beta x) + c_2 e^{\lambda x}\sin(\beta x)$$

Example. Solve the IVP

$$y'' - 5y' + 6y = 0,\ y(0) = -1,\ y'(0) = 2$$

Solution. This is a second order homogeneous ODE with constant coefficients. The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0$$

Now we can solve for the roots of the characteristic equation,

$$\lambda^2 - 5\lambda + 6 = 0 \implies (\lambda - 2)(\lambda - 3) = 0$$

We have 2 distinct real roots $\lambda_1 = 2$, $\lambda_2 = 3$. So our basis of solutions is

$$\{y_1 = e^{2x}, y_2 = e^{3x}\}$$

The general solution is

$$y = c_1 e^{2x} + c_2 e^{3x}$$

Now we can use our inital conditions

$$y' = 2c_1 e^{2x} + 3c_2 e^{3x}$$

$$y(0) = -1 \implies -1 = c_1 + c_2$$

$$y'(0) = 2 \implies 2 = 2c_1 + 3c_2$$

We can solve this system of equations

$$2c_1 + 3c_2 - 2(c_1 + c_2) = c_2 \implies c_2 = 2 - (-2) = 4$$

$$c_1 + 4 = -1 \implies c_1 = -5$$

Therefore our solution is

$$y = -5e^{2x} + 4e^{3x}$$

Example. Solve the IVP

$$y'' + 2y' + 2y = 0,\ y(0) = 1,\ y'(0) = 0$$

Solution. This is a second order homogeneous ODE with constant coefficients. The characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0$$

We can solve for the roots of the characteristic equation,

$$\begin{aligned}
\lambda &= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{4 - 8}}{2} \\
&= \frac{-2 \pm \sqrt{-4}}{2} \\
&= \frac{-2 \pm 2\sqrt{-1}}{2} \\
&= -1 \pm i
\end{aligned}$$

We have 2 complex conjugate rootts, $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$. So our basis of solutions is

$$\{y_1 = e^{-x}\cos(x),\ y_2 = e^{-x}\sin(x)\}$$

The general solution is

$$y = c_1 e^{-x}\cos(x) + y_2 + c_2 e^{-x}\sin(x)$$

Using our initial conditions,

$$y' = -c_1 e^{-x}\cos(x) - c_1 e^{-x}\sin(x) - c_2 e^{-x}\sin(x) + c_2 e^{-x}\cos(x)$$

$$y(0) = 1 \implies 1 = c_1$$

$$y'(0) = 0 \implies 0 = -c_1 + c_2 \implies c_2 = c_1 = 1$$

Therefore our unique solution is

$$y = e^{-x}\cos(x) + y_2 + e^{-x}\sin(x)$$

Example. Solve the IVP

$$y'' + 4y' + 4y = 0,\ y(0) = y'(0) = 2$$

Solution. This is a second order homogeneous ODE with constant coefficients. The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

We can solve for the roots of the characteristic equation,

$$\lambda^2 + 4\lambda + 4 = 0 \implies (\lambda + 2)(\lambda + 2) = 0$$

Therefore our roots are $\lambda_1 = \lambda_2 = \lambda = -2$. So our basis of solutions is

$$\{y_1 = e^{-2x}, y_2 = xe^{-2x}\}$$

The general solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Now we can use our initial condition

$$y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$$

$$y(0) = 2 \implies 2 = c_1$$

$$y'(0) = 2 \implies 2 = -2c_1 + c2 \implies c_2 = 6$$

Therefore our unique solution is

$$y = 2e^{-2x} + 6xe^{-2x}$$

2.3 Second-order Euler-Cauchy Equations

Definition 2.3.1. *A second-order ODE is* Euler-Cauchy *if it has the following form*

$$x^2y'' + axy' + by = 0$$

With $x > 0$, $a, b \in \mathbb{R}$.

Unlike the case of constant coefficients where we looked for exponential functions, we look for solutions of the form

$$y = x^m,\ y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$$

Substituting these back into our ODE, we get

$$\begin{aligned} 0 &= x_2m(m-1)x^{m-2} + axmx^{m-1} + bx^m \\ 0 &= m(m-1)x^m + amx^m + bx^m \\ 0 &= m^2 - m + am + b \\ 0 &= m^2 + (a-1)m + b \end{aligned}$$

This is our characteristic equation for Euler-Cauchy equations. We again have 3 cases for the roots of our characteristic equation,

- **Case 1.** The characteristic equation has 2 distinct real roots m_1, m_2. In this case our basis of solutions is

$$y_1 = x^{m_1},\ y_2 = x^{m_2}$$

  The general solution is

$$y = c_1x^{m_1} + c_2x^{m_2}$$

- **Case 2.** The characteristic equation has a double real root $m_1 = m_2 = m$. In this case our basis of solutions is

$$y_1 = x^m,\ y_2 = x^m \ln x$$

  and the general solution is

$$y = c_1x^m +_2 x^m \ln x$$

- **Case 3.** The characteristic equation has complex conjugate roots $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$. In this case our basis of solutions is

$$y_1 = x^\alpha \cos(\beta \ln x),\ y_2 = x^\alpha \sin(\beta \ln x)$$

Example. Solve the IVP

$$x^2y'' - 3xy' + 4y = 0,\ y(1) = 2,\ y'(1) = 1$$

Solution. This is a second order Euler-Cauchy ODE. The characteristic equation is

$$m^2 - 4m + 4 = 0 \implies (m-2)^2 = 0$$

We have a double real root $m_1 = m_2 = 2$. So our basis of solutions is

$$\{y_1 = x^2, y_2 = x^2 \ln x\}$$

Our general solution is

$$y = c_1x^2 + c_2x^2 \ln x$$

Now we can use our initial conditions

$$y' = 2c_1x + 2c_2x \ln x + 2c_2x$$

$$y(1) = 2 \implies 2 = c_1$$

$$y'(1) = 1 \implies 1 = 2c_1 + 2c_2 \implies c_2 = 1 - 2c_1 = -3$$

The unique solution to our IVP is

$$y = 2x^2 - 3x^2 \ln x$$

Example. Find the general solution to the ODE

$$x_2y'' - 3xy' + 5y = 0$$

Solution. This is a second order Euler-Cauchy ODE. The characteristic equation is

$$m^2 - 4m + 5 = 0$$

The roots are

$$\begin{aligned} m &= \frac{4 \pm \sqrt{16 - 4(5)(1)}}{2(1)} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4 \pm 2\sqrt{-1}}{2} \\ &= 2 \pm i \end{aligned}$$

So we have 2 complex conjugate roots $m_1 = 2 + i$, $m_2 = 2 - i$. So our basis of solutions is

$$y_1 = x^2 \cos(\ln x),\ y_2 = x^2 \sin(\ln x)$$

The general solution is

$$y = c_1x^2 \cos(\ln x) + c_2x^2 \sin(\ln x)$$

Example. Solve the IVP

$$x_2y'' + 5xy' + 4y = 0, \; y(1) = 0, \; y'(1) = 2$$

Solution. This is a second order Euler-Cauchy ODE. The characteristic equation is

$$m^2 + 4m + 4 = 0 \implies (x + 2)^2$$

We have a double real root $m_1 = m_2 = -2$. So our basis of solutions is

$$\{y_1 = x^{-2}, y_2 = x^{-2} \ln x\}$$

So our general solution is

$$y = c_1x^{-2} + c_2x^{-2} \ln x$$

We can use our initial conditions

$$y' = -2c_1x^{-3} + -2c_2x^{-3} \ln x + c_2x^{-3}$$

$$y(1) = 0 \implies 0 = c_1$$

$$y'(1) = 2 \implies 2 = -2c_1 + c_2 \implies c_2 = 2$$

The unique solution to the IVP is

$$y = 2x^{-2} \ln x$$

Example. Find the general solution of the ODE

$$x^2y'' - 2xy' + 2y = 0$$

Solution. This is a second order Euler-Cauchy ODE. The characteristic equation is

$$m^2 - 3y + 2 = 0 \implies (m - 2)(m - 1) = 0$$

The roots are $m_1 = 2$, $m_2 = 1$. So our basis of solutions is

$$\{y_1 = x^2, y_2 = x^1\}$$

The general solution is

$$y = c_1x^2 + c_2x$$

2.4 Higher-order Linear ODEs with Constant Coefficients.

We turn our attention to higher order linear homogeneous ODE's

$$a_m(x)y^{(m)} + \cdots + a_1(x)y' + a_0(x) = 0$$

Theorem 2.4.1. *If $a_m(x), \ldots, a_0(x)$ are continuous, then the set of solutions of the ODE is a vector space of dimension m.*

Like in the case of order 2, we consider two families of linear homogeneous ODE's, those with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = 0$$

For constants $a_{n-1}, \ldots, a_0$. Similarly to ODE's of order 2, we look for exponential solutions of the form $y = e^{\lambda x}$. Then we differentiate y

$$y' = \lambda e^{\lambda x},\ y'' = \lambda^2 e^{\lambda x}, y''' = \lambda^3 e^{\lambda x}, \ldots, y^{(n)} = \lambda^n e^{\lambda x}$$

Substituting these back into our ODE

$$\lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1}e^{\lambda x} + \cdots + a_1\lambda e^{\lambda x} + a_1 e^{\lambda x} = 0$$

Dividing by $e^{\lambda x}$, we get

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

This is our characteristic equation for any order linear homogeneous ODE with constant coefficients. The general solution depends on the roots of the characteristic equation,

- If λ is a root of multiplicity k of our characteristic equation, then it contributes the following equations to our basis of solutions,

  $$y_1 = e^{\lambda x},\ y_2 = xe^{\lambda x},\ y_3 = x^2e^{\lambda x}, \ldots, y_k = x^{k-1}e^{\lambda x}$$

  Each root will contribute k equations to our basis of solutions in this manner.

- If $\alpha + i\beta$ is a pair of complex conjugate roots, then it contributes the following 2 equations to our basis of solutions,

  $$y_1 = e^{\alpha x}\cos(\beta x),\ y_2 = e^{\alpha x}\sin(\beta x)$$

Example. Find the general solution of the following ODE

$$y^{(5)} - 2y''' + 2y'' - 3y' + 2y = 0$$

Solution. This is a linear homogeneous ODE with constant coefficients of order 5, our characteristic equation is

$$\lambda^5 - 2\lambda^3 + 2\lambda^2 - 3\lambda + 2 = 0$$

We start by finding 1 root, we'll guess and check the roots. We'll try $\lambda = 1$,

$$1 - 2 + 2 - 3 + 2 = 0$$

Therefore $\lambda - 1$ is a factor of our characteristic equation. We can use polynomial long division to find the other factors, and we get

$$(\lambda - 1)^2(\lambda + 2)(\lambda^2 + 1) = 0$$

So our roots are $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = i$, and $\lambda_4 = -i$. λ_1 is a real root with multiplicity 2, so it will contribute the following 2 equations to our basis of solutions

$$y_1 = e^x, \ y_2 = xe^x$$

$lambda_2 = -2$ is a real root with multiplicity 1, so it will contribute the following equation to our basis of solutions

$$y_3 = e^{-2x}$$

$\lambda_3 = i$ and $\lambda_4 = -i$ are to complex conjugate roots, they contribute 2 equations to our basis of solutions with $\alpha = 0$ and $\beta = 1$

$$y_4 = \cos(x), \ y_5 = \sin x$$

Therefore, our basis of solutions is

$$\{e^x, xe^x, e^{-2x}, \cos x, \sin x\}$$

The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-2x} + c_4 \cos x + c_5 \sin x$$

Example. Solve the following IVP,

$$y''' - 5y'' + 6y' = 0, \ y(0) = -1, \ y'(0) = 2, \ y''(0) = 0$$

Solution. This is a linear homogeneous ODE with constant coefficients of order 3. Our characteristic equation is

$$\lambda^3 - 5\lambda^2 + 6\lambda = 0 \implies \lambda(\lambda^2 - 5\lambda + 6) = 0 \implies \lambda(\lambda - 3)(\lambda - 2)$$

We have 3 roots, $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 2$. So we have the following equations

$$y_1 = 1, \ y_2 = e^{2x}, \ y_3 = e^{3x}$$

So our basis of solutions is

$$y = c_1 + c_2 e^{2x} + c_3 e^{3x}$$

Using our initial conditions,

$$y' = 3c_2 e^{3x} + 2c_3 e^{2x}$$

$$y'' = 9c_2 e^{3x} + 4c_3 e^{2x}$$

$$y(0) = -1 \implies -1 = c_1 + c_2 + c_3$$
$$y'(0) = 2 \implies 2 = 2c_2 + 3c_3$$
$$y''(0) = 0 \implies 0 = 4c_2 + 9c_3$$

We can use Guass Jordan elimination to solve this system of equations,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 3 & 2 \\ 0 & 4 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & \frac{3}{2} & 1 \\ 0 & 4 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & \frac{3}{2} & 1 \\ 0 & 0 & 3 & -4 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{8}{3} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -\frac{4}{3} \end{array}\right]$$

Therefore, we get $c_1 = -8/3$, $c_2 = 3$, and $c_3 = -4/3$. Thus our unique solution is
$$y = -\frac{8}{3} + 3e^{2x} - \frac{4}{3}e^{3x}$$

Example. Find the general solution for the following ODE
$$y^{(4)} - y = 0$$

Solution. This is a homogeneous linear ODE with constant coefficients of order 4. Our characteristic equation is
$$\lambda^4 - 1 = 0 \implies (\lambda^2 - 1)(\lambda^2 + 1) = 0 \implies (\lambda + 1)(\lambda - 1)(\lambda^2 + 1) = 0$$

Solving $\lambda^2 + 1 = 0$ we get $\lambda = \pm i$. So our roots are $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = i$, and $\lambda_4 = -i$. So our basis of solutions is
$$\{y_1 = e^{-x}, y_2 = e^{x}, y_3 = \cos x, y_4 = \sin x\}$$

Thus our general solution is
$$y = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x$$

Example. Find the general solution to the ODE
$$y^{(5)} - 6y^{(4)} + 13y''' - 14y'' + 12y' - 8y' = 0$$

Solution. This is a homogeneous linear ODE with constant coefficients of order 5. Our characteristic equation is
$$\lambda^5 - 6\lambda^4 + 13\lambda^3 - 14\lambda^2 + 12\lambda - 8 = 0$$

We'll guess and check for roots, we get that $\lambda = 2$ works so we can use polynomial long division to find the other factors. we get
$$(\lambda - 2)(\lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda + 4) = 0$$

If we take $\lambda = 1$, we get the second term is 0 so our second factor is $(\lambda - 2)$ and we can preform polynomial long division again to get

$$(\lambda - 2)^2(\lambda^3 - 2\lambda^2 + \lambda - 2)$$

We notice again that $\lambda = 2$ is a root, so we get

$$(\lambda - 2)^3(\lambda^2 + 1)$$

We cannot simplify further, so our basis of solutions is

$$\{y_1 = e^{2x}, y_2 = xe^{2x}, y_3 = x^2e^{2x}, y_4 = \cos(x), y_5 = \sin(x)\}$$

Therefore our general solution is

$$y = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x} + c_4\cos(x) + c_5\sin(x)$$

2.5 Higher-order Euler-Cauchy Equations.

Definition 2.5.1. *A linear homogeneous ODE of order n is called* Euler-Cauchy *if it can be written under the form*

$$x^my^{(m)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots a_2x^2y'' + a_1xy' + a_0y = 0$$

for $x > 0$

Similar to what we do in the case of second order Euler-Cauchy equations, we look for solutions of the form

$$y = x^m$$

We can differentiate y n times and subtitute back into our ODE to get the characteristic equation. The general form characteristic equation is long so this process is easier to be repeated for each order. Again, we have the roots of the characteristic equation fall into 2 cases,

- If m is a root of the characteristic equation of multiplicity k, then it contributes the following equations to our basis of solutions

$$y_1 = x^m,\ y_2 = x^m\ln x,\ y_3 = x^m(\ln x)^2 \ldots, y_k = x^m(\ln x)^{k-1}$$

- If $\alpha \pm i\beta$ is a pair of complex conjugate roots of the characteristic equation, then the pair contributes the following 2 equations to our basis of solutions

$$y_1 = x^\alpha\cos(\beta\ln x),\ y_2 = x^\alpha\sin(\beta\ln x)$$

Example. Assume that the fifth order Euler-Cauchy ODE has the following characteristic equation.

$$(m-1)^3(m^2+4) = 0$$

Then, the root $m = 1$ has multiplicity 3 and contributes the following 3 functions to our basis of solutions

$$y_1 = x^1, y_2 = x^1 \ln x, y_3 = x^1 (\ln x)^2$$

Our second factor $m^2 + 4$ has roots $m = \pm 2i$. So it contributes to our basis of solutions

$$y_4 = x^0 \cos(2 \ln x), y_5 = x^0 \sin(2 \ln x)$$

Therefore, our general solution is

$$y = c_1 x + c_2 x \ln x + c_3 x (\ln x)^2 + c_4 \cos(2 \ln x) + c_5 \sin(2 \ln x)$$

Example. Solve the IVP

$$x^3 y''' - 2x^2 y'' + 4xy' - 4y = 0,\ y(1) = 0,\ y'(1) = -3,\ y''(1) = 3$$

Solution. This is a linear homogeneous ODE of Euler-Cauchy type. Assume $y = x^m$, then we can differentiate y 3 times to get

$$y' = mx^{m-1},\ y'' = m(m-1)x^{m-2},\ y''' = m(m-1)(m-2)x^{m-3}$$

Substituting these back into our ODE, we get

$$\begin{aligned}
0 &= x^3 m(m-1)(m-2)x^{m-3} - 2x^2 m(m-1)x^{m-2} + 4xmx^{m-1} - 4x^m \\
&= m(m-1)(m-2)x^m - 2m(m-1)x^m + 4mx^m - 4x^m \\
&= m(m-1)(m-2) - 2m(m-1) + 4m - 4 \\
&= m(m-1)(m-2) - 2m(m-1) + 4(m-1) \\
&= (m-1)(m(m-2) - 2m + 4) \\
&= (m-1)(m^2 - 4m + 4) \\
&= (m-1)(m-2)^2
\end{aligned}$$

So we have our roots $m_1 = 1$, $m_2 = m_3 = 2$. Thus we have the basis of solutions

$$\{y_1 = x^1, y_2 = x^2, y_3 = x^2 \ln x\}$$

Our general solution is

$$y = c_1 x + c_2 x^2 + c_3 x^2 \ln x$$

Using our intial conditions,

$$y' = c_1 + 2c_2 x + 2c_3 x \ln x + c_3 x$$

$$y'' = 2c_2 + 2c_3 \ln x + 2c_3 + c_3 = 2c_2 + 2c_3 \ln x + 3c_3$$

$$y(1) = 0 \implies 0 = c_1 + c_2$$

$$y'(1) = -3 \implies -3 = c_1 + 2c_2 + c_3$$

$$y''(1) = 3 \implies 3 = 2c_2 + 3c_3$$

We can solve this system of equations using Guass Jordan elimination, and we find $c_1 = 12$, $c_2 = -12$, $c_3 = 9$. Therefore our unique solution is

$$y = 12x - 12x^2 + 9x^2 \ln x$$

Example. Solve the IVP

$$x^3y''' + x^2y'' - 2xy' + 2y = 0,\ y(1) = 1,\ y'(1) = -2,\ y''(1) = 0$$

Solution. This is a linear homogeneous ODE of Euler-Cauchy type. Assume $y = x^m$, then we can differentiate y 3 times to get

$$y' = mx^{m-1},\ y'' = m(m-1)x^{m-2},\ y''' = m(m-1)(m-2)x^{m-3}$$

Then substituting these back into our ODE, we get

$$\begin{aligned}
0 &= x^3m(m-1)(m-2)x^{m-3} + x^2m(m-1)x^{m-2} - 2xmx^{m-1} + 2x^m \\
&= m(m-1)(m-2)x^m + m(m-1)x^m - 2mx^m + 2x^m \\
&= m(m-1)(m-2) + m(m-1) - 2m - 2 \\
&= m(m-1)(m-2) + m(m-1) - 2(m-1) \\
&= (m-1)(m(m-2) + m - 2) \\
&= (m-1)(m^2 - m - 2) \\
&= (m-1)(m-2)(m+1)
\end{aligned}$$

We have three roots $m_1 = 1$, $m_2 = 2$, and $m_3 = -1$. So we have the basis of solutions

$$\{x^1, x^2, x^{-1}\}$$

Our general solution is

$$y = c_1x + c_2x^2 + c_3x^{-1}$$

Using our initial conditions,

$$y' = c_1 + 2c_2x - c_3x^{-2}$$

$$y'' = 2c_2 + 2c_3x^{-3}$$

$$y(1) = 1 \implies 1 = c_1 + c_2 + c_3$$

$$y'(1) = -2 \implies -2 = c_1 + 2c_2 - c_3$$

$$y''(1) = 0 \implies 0 = 2c_2 + 2c_3 \implies 0 = c_2 + c_1$$

We can solve this system of equations using Guass Jordan elimination, and we find $c_1 = c_2 = 1$ and $c_3 = -1$. Therefore our unique solution is

$$y = x + x^2 - 1x^{-1}$$