

MAT 2384 Midterm 2 Cheat Sheet

Homogeneous ODEs

Constant Coefficients

General Form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

Characteristic Equation

$$\lambda^n + a_{n-1} \lambda^{(n-1)} + \dots + a_1 \lambda + a_0 = 0$$

If λ is a root with multiplicity k ,

$$y_1 = e^{\lambda x}, y_2 = x e^{\lambda x}, y_3 = x^2 e^{\lambda x}, \dots, y_k = x^{k-1} e^{\lambda x}$$

If $\alpha + i\beta$ is a pair of complex conjugate roots, then it contributes the following 2 equations to our basis of solutions,

$$y_1 = e^{\alpha x} \cos(\beta x), y_2 = e^{\alpha x} \sin(\beta x)$$

Euler-Cauchy

General Form

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = 0$$

Characteristic Equation. Differentiate $y = x^m$ and plug into ODE. If m is a root of the characteristic equation of multiplicity k , then it contributes the following equations to our basis of solutions

$$y_1 = x^m, y_2 = x^m \ln x, y_3 = x^m (\ln x)^2, y_k = x^m (\ln x)^{k-1}$$

If $\alpha \pm i\beta$ is a pair of complex conjugate roots of the characteristic equation, then the pair contributes the following 2 equations to our basis of solutions

$$y_1 = x^\alpha \cos(\beta \ln x), y_2 = x^\alpha \sin(\beta \ln x)$$

Undetermined Coefficients

General Form

$$a_n y^{(n)} + \dots + a_1(x) y' + a_0 y = r(x)$$

All coefficients on the left (a_n, \dots, a_0) are constants and $r(x)$ is a polynomial, exponential, and/or sinusoidal.

Rules

- Rule 1: Basic Rule.** If $r(x) = k e^{\lambda x}$, then $y_p = A e^{\lambda x}$. If $r(x) = p_n(x)$, then $y_p = q_n(x)$. If $r(x) = k \cos(wx)$ or $k \sin(wx)$, then $y_p = A e^{\alpha x} \cos(wx) + B e^{\alpha x} \sin(wx)$. If $r(x) = p_n(x) e^{\lambda x}$, then $y_p = q_n(x) e^{\lambda x}$.
- Rule 2: The Modification Rule.** If any term of y_p is in the basis of solutions for y_H , then multiply by x until it's not.
- Rule 3: The Sum Rule.** If $r(x) = r_1(x) + \dots + r_n(x)$, then do each $r_i(x)$ separately and combine.

Variation of Parameters

General Form

$$a_n(x) y^{(n)} + \dots + a_1(x) y' + a_0 y = r(x)$$

Same solution with $y = y_H + y_p$ where y_H is the solution to the corresponding homogeneous ODE

$$a_n(x) y^{(n)} + \dots + a_1(x) y' + a_0 y = 0$$

$$y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$$

Where $\{y_1, y_2, \dots, y_n\}$ is a basis of solutions for the corresponding homogeneous ODE. u_1, u_2, \dots, u_n are functions that satisfy the following system of equations

$$\begin{cases} 0 = u'_1 y_1 + u'_2 y_2 + \dots + u'_n y_n \\ 0 = u'_1 y'_1 + u'_2 y'_2 + \dots + u'_n y'_n \\ \vdots \\ \frac{r(x)}{a_n(x)} = u'_1 y_1^{(n-1)} + u'_2 y_2^{(n-1)} + \dots + u'_n y_n^{(n-1)} \end{cases}$$

Systems of ODE's

General Form

$$\begin{cases} y'_1 = a_{11} y_1 + a_{12} y_2 + r_1(x) \\ y'_2 = a_{21} y_1 + a_{22} y_2 + r_2(x) \end{cases} \implies \vec{y}' = A \vec{y} + \vec{r}(x)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \vec{y}' = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}, \vec{r}(x) = \begin{bmatrix} r_1(x) \\ r_2(x) \end{bmatrix}$$

Homogenous Systems with Constant Coefficients

A system is homogeneous if $\vec{r}(x) = 0$, $\vec{y}' = A \vec{y}$.

Steps to Solving

- Find the eigen values of A $|\lambda I - A| = 0$
- If 2 distinct real eigenvalues, find eigenvectors V_1, V_2 . $\vec{y} = c_1 \vec{V}_1 e^{\lambda_1 x} + c_2 \vec{V}_2 e^{\lambda_2 x}$
- If λ with multiplicity 2, find generalized eigenvector ρ $Y = c_1 V e^{\lambda x} + c_2 (xV + \rho) e^{\lambda x}$
- If $\lambda = \alpha \pm i\beta$, then find eigenvector for $\lambda_1 = \alpha + i\beta$, compute gen. solution $y = c_1 V_1 + c_2 V_2$ with $\vec{V} e^{(\alpha + i\beta)x} = \vec{V} e^{\alpha x} (\cos(\beta x) + i \sin(\beta x)) = \vec{V}_1 + i \vec{V}_2$

Non-Homogeneous Systems

Similar to non-homogeneous ODEs, use undetermined to solve $\vec{y} = \vec{y}_H + \vec{y}_p$. y_H is ODE with $\vec{r}(x) = 0$. Decompose $\vec{r}(x)$ as

$$r(\vec{x}) = \begin{bmatrix} 2x^3 + x^2 + x \\ 3e^x + x^2 + 2x + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} x^3 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} e^x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x^2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x$$

Solve each $r_1(x)$ using same rules as undetermined coefficients replacing constants with constant vectors

$$r_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} x^3 \implies \vec{y}_p = \vec{U} x^3 + \vec{V} x^2 + \vec{W} x + \vec{Z}$$

Newton's Divided-Difference Interpolation

Given a node x_i ,

- The *first divided difference* at x_i is defined as

$$f(x_i, x_{i+1}) = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

- The *second divided difference* at x_i is

$$f(x_i, x_{i+1}, x_{i+2}) = \frac{f(x_{i+1}, x_{i+2}) - f(x_i, x_{i+1})}{x_{i+2} - x_i}$$

- In general, the k th divided difference at x_i is

$$f(x_i, x_{i+1}, \dots, x_{i+k}) = \frac{f(x_{i+1}, \dots, x_{i+k}) - f(x_i, \dots, x_{i+k-1})}{x_{i+k} - x_i}$$

Newton's Interpolation Polynomial

$$\begin{aligned} p_n(x) = & f_0 + f(x_0, x_1)(x - x_0) \\ & + f(x_0, x_1, x_2)(x - x_0)(x - x_1) + \dots \\ & + f(x_0, \dots, x_n)(x - x_0)(x - x_1) \dots (x - x_{n-1}) \end{aligned}$$

Numerical Integration

Midpoint Rule

$$\int_a^b f(x) dx \approx h[f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)]$$

$$h = \frac{b-a}{n}, x_i^* = \frac{x_i + x_{i+1}}{2}$$

Error formula with M being $|f''(x)| \leq M$ for $x \in [a, b]$

$$E_M \leq \frac{M(b-a)^3}{24n^2}$$

Trapezoidal

$$\int_a^b f(x) dx \approx \frac{h}{2} [f(a) + 2f(x_1) + 2f(x_2) + \dots + f(x_n)]$$

Error Formula

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}$$

Simpsons Rule

Divide $[a, b]$ into an *EVEN* number of subintervals

$$\int_a^b f(x) \approx \frac{h}{3} [f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + f(b)]$$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$

Error Formula Where M is the upperbound for the fourth derivative of $f(x)$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$

Gaussian Quadrature

To convert $\int_a^b f(x)dx$ into the form $\int_{-1}^1 g(t)dt$, use the substitution

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

Then the Gaussian Quadrature formula is

$$\int_{-1}^1 f(t)dt \approx w_1 f(t_1) + \dots w_n f(t_n)$$

Table of Nodes and Coefficients

Order n	Nodes t_i	Coefficients w_i
1	0	2
2	-0.5773502692 0.5773502692	1 1
3	-0.7745966692 0 0.7745966692	0.5555555556 0.8888888889 0.5555555556
4	-0.8611363116 -0.3399810436 0.3399810436 0.8611363116	0.3478548451 0.6521451549 0.6521451549 0.3478548451
5	-0.9061798459 -0.5384693101 0.0 0.5384693101 0.9061798459	0.2369268850 0.4786286705 0.5688888889 0.4786286705 0.2369268850

Linear Algebra and Integrals

Linear Algebra

Eigen Value

$$|A - \lambda I| = |\lambda I - A| = 0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigen Vectors. An eigenvector \vec{V} is a vector that satisfies

$$[A - \lambda I] \vec{0}$$

Generalized Eigen Vector ρ The solution to, then pick specific value for the parameter t to get a specific vector, (i.e take $t = 0, t = 1$, etc).

$$[A - \lambda I]V$$

Integrals

$$\int x e^{ax} = \frac{1}{a} x e^{ax} - \frac{1}{a^2} e^{ax}, \int \ln x = x \ln x - x$$

$$\int \sin(ax) dx = -\frac{1}{a} \cos(ax), \int \cos(ax) dx = \frac{1}{a} \sin(ax)$$

Example of Non-Homogeneous System

Example.

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix} \vec{y} + \begin{bmatrix} 9x - 51 \\ 7 + e^{2x} \end{bmatrix}; \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution. Solve the corresponding homogeneous ODE

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix} \vec{y}$$

Find eigenvalues of A

$$\det \begin{bmatrix} 9 - \lambda & 18 \\ -2 & -3 - \lambda \end{bmatrix} = (9 - \lambda)(-3 - \lambda) + 36 = \lambda^2 - 6\lambda + 9$$

This gives us $\lambda = 3$ with multiplicity 2, find eigenvector

$V = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ Find generalized eigenvector, set parameter $t = 0$,

$$[A - 3I]V = \begin{bmatrix} 6 & 18 & -3 \\ -2 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{\rho} = \begin{bmatrix} -3t - 1/2 \\ t \end{bmatrix} \implies \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

The general solution to the homogeneous ODE is

$$\begin{aligned} \vec{y}_H &= c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} x e^{3x} + c_2 \left(x \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right) e^{3x} \\ &= \begin{bmatrix} -3c_1 e^{3x} - 3c_2 x e^{3x} - \frac{1}{2} c_2 e^{3x} \\ c_1 e^{3x} + c_2 x e^{3x} \end{bmatrix} \end{aligned}$$

For y_p , decompose $r(x)$ to get

$$r(x) = \begin{bmatrix} 9x - 51 \\ 7 + e^{2x} \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix} x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$$

Find y_p for each part of $r(x)$

$$y_p = Ux + V + W e^{2x}$$

Rewrite the non-homogeneous system as

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix} \vec{y} + \begin{bmatrix} 9 \\ 0 \end{bmatrix} x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$$

Compute \vec{y}'_p and plug into the system to solve for constant vectors, $\vec{y}'_p = U + 2W e^{2x}$, $\vec{y}'_p = A \vec{y}_p + \vec{r}(x)$

$$U + 2W e^{2x} = A(Ux + V + W e^{2x}) + \begin{bmatrix} 9 \\ 0 \end{bmatrix} x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$$

This gives us the three equations

$$AU + \begin{bmatrix} 9 \\ 0 \end{bmatrix} = 0, AV + \begin{bmatrix} -51 \\ 7 \end{bmatrix} = U, AW + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2W$$

Solve equations to find

$$U = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, V = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, W = \begin{bmatrix} 18 \\ -7 \end{bmatrix}$$

This gives us our particular solution

$$y_p = \begin{bmatrix} 3 \\ -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 18 \\ -7 \end{bmatrix} e^{2x} = \begin{bmatrix} 3x + 18e^{2x} \\ -2x + 3 - 7e^{2x} \end{bmatrix}$$

The general solution to the non-homogeneous ODE is

$$y = \begin{bmatrix} -3c_1 e^{3x} - 3c_2 x e^{3x} - \frac{1}{2} c_2 e^{3x} \\ c_1 e^{3x} + c_2 x e^{3x} \end{bmatrix} + \begin{bmatrix} 3x + 18e^{2x} \\ -2x + 3 - 7e^{2x} \end{bmatrix}$$

Extra Notes