MAT 2384: Ordinary Differentials Lecture Notes

Last Updated:

May 10, 2023

Contents

Chapter 0

Introduction and Basic Terminology

Definition 0.0.1 (Differential Equations). *A differential equation is an equation involving an unknown function y (of one or many variables), derivatives of y, and other known functions of indepedent variables.*

Definition 0.0.2 (Order of Differential Equations). *The* order *of a differential equation is the highest order of a derivative appearing in the equation.*

If the unknown function y is a function of only one variable, $y = f(x)$, we saw that the differential equation is *ordinary*. If y is a function of two or more variables, we say the differential equation is a *partial* differential equation.

Example.

$$x^3y'' - 3e^x \sin xy' + 3y = \tan x$$

This is an ODE of order 2.

Example.

$$x_1x_2\frac{\partial^2 y}{\partial x_1 \partial x_2} - 3e^{x_1}\frac{\partial y}{\partial x_1} = 0$$

This is a PDE of order 2.

Note: In this course, we will only consider ODEs.

Definition 0.0.3. *We say that the function y is a* solution *to a differential equation on an interval I if y is well-defined on I and y satisfies the differential equation.*

Example. Consider the differential equation

$$y'' - 5y' + 4y = 0$$

Show that the function
$$y = Ae^x + Be^{4x}$$
is a solution for the differential equation on $\mathbb{R}$ for any constants A and B.

Solution: We have $y = Ae^x + Be^{4x}$ is well defined on $\mathbb{R}$.
$$y' = Ae^x - 4Be^{4x}$$
$$y'' = Ae^x + 16Be^{4x}$$
So,
$$y'' - 5y' + 4y = Ae^x + 16Be^{4x} - 5Ae^x - 20Be^{4x} + 4Ae^x + 4Be^{4x} = 0$$
Therefore, $y = Ae^x + Be^{4x}$ is a solution to the differential equation for any $A, B \in \mathbb{R}$. This is called the *general solution* to the differential equation.

Remark: The above example shows that a differential equation has infinitely many solutions.

Definition 0.0.4 (Initial Value Problem)**.** *An intial value problem (IVP) of order n consists of an ordinary differential equation of order n, and n initial coniditions of the form*
$$y(x_0) = y_0, \quad y'(x_0) = y_1 \dots$$
$$y^{(n-1)}(x_0) = y_{n-1}$$
Note: $y^{(i)}$ denotes the ith derivative of y.

Example. Consider the IVP of order 3
$$y''' - 3e^x y'' + 6xy' + 2y = x^2$$
$$y(0) = -1 \;\; y'(0) = 2 \;\; y''(0) = 1$$
Example. Solve the following IVP
$$y'' - 5y + 4y = 0$$
$$y(0) = 1 \;\; y'(0) = 2$$
Solution: We saw in the previous example that the general solution to this differential equation is
$$y = Ae^x + Be^{4x}$$
We can use the initial conditions to find the constants A and B.
$$y(0) = 1 \implies 1 = Ae^0 + Be^0 = A + B$$
$$y'(0) = 2 \implies 2 = Ae^0 - 4Be^0 = A + 4B$$
$$A + 4B - A - B = 2 - 1 \implies 3B = 1 \implies B = \frac{1}{3} \;\; A = \frac{2}{3}$$

Theorem 0.0.1 (Existence and Uniqueness Theorem for the First Order ODEs).
Consider the IVP:

$$y' = F(x, y), \quad y(x_0) = y_0$$

- ***Existence:*** *If $F(x, y)$ is continuous in an open rectangular region*

  $$R = \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$$

  of the xy-plane that contains the initial point (x_0, y_0), then there exists a solution $y(x)$ to the intial value problem that is defined in some open interval $I = (\alpha, \beta)$ containg x_0.

- ***Uniqueness:*** *If the partial derivative $\frac{\partial F}{\partial y}$ of the function $F(x, y)$ is continuous in the recnagular region R, then the solution $y(x)$ is unique.*

Note: We will always suppose this condition is satisfied in this course.

Chapter 1

Ordinary Differential Equations of First Order

The goal of this chapter is to solve ODE's of order 1.

Definition 1.0.1. *The* standard form *of an ODE of order 1 is an expression of the form*

$$y' = f(x, y)$$

We can rewrite y' as $\frac{dy}{dx}$ and we have the differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Example. Consider the differential equation

$$2xy' + 3y = 2y' + \sin x$$

The standard form is

$$2xy' - 2y' = \sin x - 3y \implies y' = \frac{\sin x - 3y}{2x - 2}$$

The differential form is

$$\begin{aligned} 2x\frac{dy}{dx} + 3y &= 2\frac{dy}{dx} + \sin x \\ \implies 2xdy + 3ydx &= 2dy + \sin xdx \\ \implies (3y - \sin x)dx(2x - 2)dy &= 0 \end{aligned}$$

1.1 Seperable First Order Ordinary Differential Equations

Definition 1.1.1. *A first order ODE is called* seperable *if it can be written in the form*

$$F(x)dx = G(y)dy$$

1.1.1 Solving Seperable ODE's

To solve a seperable ODE,

1. Write $y' = \frac{dy}{dx}$
2. Seperate the ODE to write it in the form
$$F(x)dx = G(y)dy$$
3. Take integrals of both sides
4. If an initial condition is given, solve for the constant of integration C.

Example. Solve the IVP

$$(y^2+1)y' = \frac{x}{y} \quad y(1) = 1$$

Solution: We can write $y' = \frac{dy}{dx}$ and we get

$$(y^2+1)\frac{dy}{dx} = \frac{x}{y} \implies (y^2+1)ydy = xdx$$

Taking integrals on both sides, we have

$$\int y^3 + ydy = \int xdx \implies \frac{y^4}{4} + \frac{y^2}{2} = \frac{x^2}{2} + C$$

Using our initial condition, we have $y = 1$ when $x = 1$, then

$$\frac{1}{4} + \frac{1}{2} = \frac{1}{2} + C$$

Therefore $C = \frac{1}{2}$ and the solution to the IVP is

$$\frac{y^4}{4} + \frac{y^2}{2} = \frac{x}{2} + \frac{1}{4}$$

This is called the *implicit solution* since we could not explcitly solve for y in terms of x.

Example. Solve the IVP

$$e^x y' = (x+1)y^2 \quad y(0) = -\frac{1}{2}$$

Solution:

$$e^x \frac{dy}{dx} = (x+1)y^2$$
$$\implies \frac{1}{y^2}dy = \frac{x+1}{e^x}dx$$
$$\implies \int \frac{1}{y^2}dy = \int (x+1)e^{-x}dx$$

We can use integration by parts to solve the right hand side integral. Let $u = x + 1$ and $dv = e^{-x}dx$, $u' = 1$, and $v = -e^{-x}$. Then

$$\begin{aligned}
\int (x+1)e^{-x}dx &= uv - \int u'vdx \\
&= -(x+1)e^{-x} - \int -e^{-x}dx \\
&= -(x+1)e^{-x} - e^{-x} + C
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{y^{-2+1}}{-2+1} &= -(x+1)e^{-x} - e^{-x} + C \\
-\frac{1}{y} &= -(x+1)e^{-x} - e^{-x} + C
\end{aligned}$$

Setting $y = -\frac{1}{2}$ and $x = 0$, we have

$$2 = -2 + C \implies C = 4$$

Therefore the implicit solution is

$$-\frac{1}{y} = -(x+1)e^{-x} - e^{-x} - 4$$

We can rewrite this as an explicit solution as

$$y = \frac{1}{(x+2)e^{-x} - 4}$$

1.2 First Order ODE's With Homogeneous Coefficients

Definition 1.2.1. *A function $F(x, y)$ of two variables is called* homogeneous *of degree k if*

$$F(\lambda x, \lambda y) = \lambda^k \cdot F(x, y)$$

This type of ODEs can be made seperable after a suitable change of variables of the unknown function.

Example.

$$F(x, y) = 3x^2y - 2xy^2 + y^3$$

We can check if its homogeneous by the definition,

$$\begin{aligned}
F(\lambda x, \lambda y) &= 3(\lambda x)^2(\lambda y) - 2(\lambda x)(\lambda y)^2 + (\lambda y^3) \\
&= 3\lambda^3x^2y - 2\lambda^3xy^2 + \lambda^3y^3 \\
&= \lambda^3(3x^2y - 2xy^2 + y^3) \\
&= \lambda^3F(x, y)
\end{aligned}$$

Therefore, $F(x,y)$ is homogeneous of degree 3. We can tell quickly if a polynomial is homogeneous is by looking at the exponents of each term. If the sum of the exponents of each term is the same, then the polynomial is homogeneous, with order being the sum of the exponents in each term (i.e x^2y has exponents 2,1, xy^2 has exponents 1,2, and y^3 has exponents 3, each sum to 3).

Definition 1.2.2. *A first order ODE given in differential form*

$$M(x,y)dx + N(x,y)dy = 0$$

is called of homogeneous coefficients *if both $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree.*

Example.

$$(3x^2 + 2y^2 + 2xy)dx - 4xydy = 0$$

Both terms are homogeneous of degree 2, therefore this is a differential equation of homogeneous coefficients.

Theorem 1.2.1. *A first order ODE of homogeneous coefficients can be made seperable by changing the function using one of the following substitutions:*

- *Set $u \coloneqq \frac{y}{x}$ or*
- $u \coloneqq \frac{x}{y}$

Example. Solve the following IVP

$$(x^2 - y^2)dx + 2xydy = 0 \quad y(1) = 2$$

Solution: This is a first order ODE with homogeneous coefficients. Let

$$u \coloneqq \frac{y}{x} \implies y = xu$$

$$\frac{dy}{dx} = 1 \cdot u + x \cdot \frac{du}{dx} \implies dy = udx + xdu$$

So, we have

$$(x^2 - y^2)dx + 2xydy = 0 \implies (x^2 - x^2u^2)dx + 2x(xu)(udx + xdu) = 0$$

Simplyfing, we get

$$\begin{aligned} x^2dx - x^2u^2dx + 2x^2u^2dx + 2x^3udu &= 0 \\ dx - u^2dx + 2u^2dx + 2xudu &= 0 \\ (1+u^2)dx + 2xudu &= 0 \\ (1+u^2)dx &= -2xudu \\ -\frac{1}{x}dx &= \frac{2u}{1+u^2}du \end{aligned}$$

Now that it's seperable, we can integrate both sides,

$$
\begin{aligned}
-\int \frac{1}{x} dx &= \int \frac{2u}{1+u^2} du \\
-\ln(x) &= \ln(1+u^2) + C
\end{aligned}
$$

Now using our initial condition, we have $y(1) = 2$. But, our differential equation is a function of u not y, so we must calculate u using our initial condition. So, $u(1) = \frac{y(1)}{1} = 2$. So,

$$-\ln 1 = \ln 5 + C \implies C = -\ln 5$$

Therefore, our solution is

$$
\begin{aligned}
\ln x &= \ln(1+u^2) - \ln 5 \\
\ln\left(\frac{5}{x}\right) &= \ln(1+u^2) \\
\frac{5}{x} &= 1 + u^2 \\
u^2 &= \frac{5}{x} - 1 \\
\frac{y^2}{x^2} &= \frac{5}{x} - 1 \\
y &= \sqrt{5x - x^2}
\end{aligned}
$$

We take the positive square root since if we took the negative square root, then $y(1) = -2$ which is not our initial condition.

Example. Solve the IVP

$$(2x + y)dx - xdy = 0 \quad y(1) = -2 \quad x > 0$$

Solution: This is a first order ODE with homogeneous coefficients. Let

$$u = \frac{y}{x} \implies y = xu$$

$$dy = udx + xdu$$

Substituting into our differential equation, we get

$$
\begin{aligned}
(2x + xu)dx - x(udx + xdu) &= 0 \\
(2 + u)dx - (udx + xdu) &= 0 \\
2dx + udx - udx - xdu &= 0 \\
2dx = xdu &\implies \frac{2}{x}dx = du
\end{aligned}
$$

This differential equation in u is sperable, so we can integrate

$$\int \frac{2}{x} dx = \int du$$
$$2 \ln x = u + C$$

Using your initial condition, $y(1) = -2$. so $u(1) = \frac{y(1)}{1} = -2$. Therefore,

$$2 \ln 1 = -2 + c \implies C = 2$$

Now solving for y,

$$u = 2 \ln x - 2$$
$$\frac{y}{x} = 2 \ln x - 2$$
$$y = x(2 \ln x - 2)$$

This is our explicit solution to the initial value problem.

1.3 Exact First Order ODEs

Definition 1.3.1. *Given a function $F(x, y)$ of two variables, the differential of $F(x, y)$ denoted by dF is defined by*

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

Example. Let

$$F(x, y) = 2x^2y^3 + \sin(x + 2y)$$

Then

$$dF = (4xy^3 + \cos(x + 2y))dx + (6x^2y^2 + 2\cos(x + 2y))dy$$

Remark:

$$dF = 0 \iff \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \iff \frac{\partial F}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial y} = 0$$

So, $F(x, y) = C$ is a constant function. Therefore,

$$dF = 0 \iff F(x, y) = C$$

Definition 1.3.2. *A first order ODE*

$$M(x, y)dx + N(x, y)dy = 0$$

is called exact *if there exists a continuous function $F(x, y)$ such that*

$$\frac{\partial F}{\partial x} = M(x, y) \text{ and } \frac{\partial F}{\partial x} = N(x, y)$$

So if $M(x, y)dx + N(x, y)dy = 0$ is exact, then

$$dF = 0 \implies F(x, y) = C$$

In summary, if $M(x,y)dx + N(x,y)dy = 0$ is exact, then find $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial x} = N(x,y)$$

Then, the (implicit) solution to the ODE is $F(x,y) = C$. Furthermore, since $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$, then

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial x \partial y}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial y \partial x}$$

So by the Clairaut-Schwarz Theorem, the ODE is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem 1.3.1 (Condition for Exactness)**.** *The first order ODE* $M(x,y)dx + N(x,y)dy = 0$ *(with* M, N *continuous) is exact if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

1.3.1 Steps to Solving Exact ODEs

1. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Look for a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \frac{\partial F}{\partial y} = N$$

3. The general solution to the ODE is $F(x,y) = C$.
4. If an intial condition is given, use it to find C.

Example. Solve the following IVP

$$(6x - 2y^2 + 2xy^3)dx + (3x^2y^2 - 4xy)dy = 0, \quad y(1) = -2$$

Solution. We first check exactness.

$$\frac{\partial M}{\partial y} = -4y + 6xy^2 = 6xy^2 - 4y$$

$$\frac{\partial N}{\partial x} = 6xy^2 - 4y$$

Therefore, this ODE is exact. Now we need to find a function $F(x,y)$ satisfying the partial derivatives. We can do this by integrating N with respect to y, so we have

$$\frac{\partial F}{\partial y} = 3x^2y^2 - 4xy$$

$$F(x,y) = \int 3x^2y^2 - 4xydy = 3x^2\int y^2dy - 4x\int ydy = x^2y^3 - 2xy^2 + h(x)$$

We add $h(x)$ since when integrating with respect to y, we are treating x as a constant so $h(x)$ is constant with respect to y. So we have

$$F(x,y) = x^2y^3 - 2xy^2 + h(x)$$

Now we can use the first equation to solve for $h(x)$,

$$\frac{\partial F}{\partial x} = 2xy^3 - 2y^2 + h'(x)$$

This equation is equal to M, so we can plug M in and get

$$M = 6x - 2y^2 + 2xy^3 = 2xy^3 - 2y^2 + h'(x) \implies h'(x) = 6x$$

Now we can solve for $h(x)$ by taking the integral,

$$h(x) = \int 6xdx = 3x^2 + C_1$$

Now, we get

$$F(x,y) = x^2y^3 - 2xy^2 + 3x^2 + C_1$$

So the general solution to the ODE is

$$x^2y^3 - 2xy^2 + 3x^2 + C_1 = C_2 \implies x^2y^3 - 2xy^2 + 3x^3 = C$$

Now using the intial condition $y(1) = -2$, then

$$1^2(-2)^3 - 2(1)(-2)^2 + 3(1)^2 = C \implies C = -13$$

Therefore, the solution to the IVP is

$$x^2y^3 - 2xy^2 + 3x^3 = -13$$

Example. Solve the IVP

$$(2x\cos(y) - 3x^2y + ye^{xy})dx + (-x^2\sin(y) + xe^{xy} - x^3)dy = 0, \quad y(0) = 1$$

Solution. We first check exactness,

$$\frac{\partial M}{\partial y} = -2x\sin(y) - 3x^2 + e^{xy} + ye^{xy}$$

$$\frac{\partial N}{\partial x} = -2x\sin(y) + ye^{xy} + e^{xy} - 3x^2$$

Therefore this ODE is exact, so we look for $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N$$

$$\begin{aligned}
F(x,y) &= \int 2x\cos(y) - 3x^2y + ye^{xy}dx \\
&= 2\cos(y)\int xdx - 3y\int x^2dx + y\int e^{xy}dx \\
&= x^2 - x^3y + y\frac{e^{xy}}{y} + h(y)
\end{aligned}$$

So we have

$$F(x,y) = x^2\cos(y) - x^3y + e^{xy} + h(y)$$

Then,

$$\frac{\partial F}{\partial y} = -x^2\sin(y) - x^3 + xe^{xy} + h'(y) = N \implies h'(y) = 0$$

So $h(y)$ is a constant, say $h(y) = K$, then our general solution for $F(x,y)$ is

$$F(x,y) = x^2\cos(y) - x^3y + e^{xy} + k \implies x^2\cos(y) - x^3y + e^{xy} = C$$

Using the condition, $y(0) = 1$, we get

$$(0)^2\cos(1) - (0)^3(1) + e^{0\cdot 1} = C \implies C = 1$$

Therefore the (implicit) solution to the IVP is

$$x^2\cos(y) - x^3y + e^{xy} = 1$$

1.4 First Order ODEs With an Integrating Factor

Definition 1.4.1 (Integrating Factor)**.** *We say that the function $\mu(x,y)$ is an* integrating factor *of the first-order ODE*

$$M(x,y)dx + N(x,y)dy = 0$$

if the new ODE

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact.

In general, finding an integrating factor is not easy. However, there are some special cases where we can find an integrating factor easily.

Theorem 1.4.1. *For the ODE*

$$M(x,y)dx + N(x,y)dy = 0$$

1. *If*

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$$

for some function g of y only, then an integration factor exists given by

$$\mu(y) = \exp\left(-\int g(y)dy\right)$$

2. *If*

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(x)$$

for some function f of x only, then an integration factor exists given by

$$\mu(x) = \exp\left(\int f(x)dx\right)$$

Example. Solve the IVP

$$(y^4 + xy)dx + (xy^3 - x^2 + 2y^3e^y)dy = 0, \quad y(0) = 1$$

Solution. It's clear this ODE is not exact, so we need to find an integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4y^3 + x - y^3 + 2x = 3y^3 + 3x$$

If we dvidie by M, we get

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3(y^3+x)}{y^4+xy} = \frac{3(y^3+x)}{y(y^3+x)} = \frac{3}{y}$$

Therefore, we have our integrating factor

$$\mu(y) = \exp\left(-\int \frac{3}{y}dy\right) = \exp\left(-3\int \frac{1}{y}dy\right) = \exp\left(\ln(y^{-3})\right) = y^{-3}$$

We multiply the original ODE with $\mu(y) = y^{-3}$

$$y^{-3}(y^4 + xy)dx + y^{-3}(xy^3 - x^2 + 2y^3e^y)dy = (y + xy^2)dx + (x - x^2y^{-3}2e^y)$$

Now we can check the exactness of this ODE,

$$\frac{\partial M}{\partial y} = 1 - 2xy^{-3} \text{ and } \frac{\partial N}{\partial x} = 1 - 2xy^{-3}$$

Therefore, this ODE is exact, so we look for $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N$$

The first equation is simpler so we will start with that,

$$\begin{aligned} F(x,y) &= \int y + xy^{-2} dx \\ &= y \int dx + y^{-2} \int x dx \\ &= xy + \frac{x^2 y^{-2}}{2} \end{aligned}$$

Now we derive with respect to y and use the second equation,

$$\frac{\partial F}{\partial y} = x - x^2 y^{-3} + h'(y) = N = x - x^2 y^{-3} + 2e^y \implies h'(y) = 2e^y$$

Then

$$h(y) = \int h'(y) dy = \int 2e^y dy = 2e^y + k$$

So we get the function

$$F(x,y) = xy + \frac{x^2 y^{-2}}{2} + 2e^y + k$$

Then setting $F(x, y)$ equal to a constant to get our (implicit) general solution,

$$xy + \frac{x^2 y^{-2}}{2} + 2e^y = C$$

Using the initial condition, $y(0) = 1$, we get

$$(0)(1) + \frac{(0)^2 (1)^{-2}}{2} + 2e^1 = C \implies C = 2e$$

Therefore, the (implicit) solution to the IVP is

$$xy + \frac{x^2 y^{-2}}{2} + 2e^y = 2e$$