MAT 2384 Summary Sheet: Ordinary Differential Equations

Order 1 ODE's

Standard Form: $y' = f(x,y)$
Differential Form: $M(x,y)dx + N(x,y)dy = 0$

Seperable First Order ODE's

Definition. A first order ODE is called seperable if it can be written in the form

$$F(x)dx = G(y)dy$$

Steps to Solving Seperable ODE's

1. Write $y' = \frac{dy}{dx}$
2. Seperate the ODE to write it in the form $F(x)dx = G(y)dy$
3. Integrate both sides
4. If an initial condition is given, solve for the integration constant C.

First Order ODE's with Homogeneous Coefficients

Definition. A function $F(x,y)$ is called homogeneous of degree k if

$$F(\lambda x, \lambda y) = \lambda^k \cdot F(x,y)$$

Definition. A first order ODE given in differential form is called homogeneous if both $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree.

Theorem. A first order ODE of homogeneous coefficients can be made seperable by changing the function by substituting $u = \frac{y}{x} \implies y = xu$ or $u\frac{x}{y} \implies y = \frac{x}{u}$.

Exact First Order ODEs

Definition. Given a function $F(x,y)$, the differential of F denoted by dF is defined by

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Remark. $dF = 0 \iff F(x,y) = C$.

Definition. A first order ODE is called exact if there exists $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial x} = N(x,y)$$

Theorem. A first order ODE is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact First Order ODE's

Steps To Solving

1. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Look for a function $F(x,y)$ such that
$$\frac{\partial F}{\partial x} = M, \ \frac{\partial F}{\partial y} = N$$
Do this by integrating M with respect to x or N with respect to y then differentiate the equation with respect to the other variable respectively.
3. The general solution is $F(x,y) = C$
4. If an initial condition is given, solve for the integration constant C.

Example. Solve the following IVP

$$(6x - 2y^2 + 2xy^3)dx + (3x^2y^2 - 4xy)dy = 0$$

Solution. Check for exactness

$$\frac{\partial M}{\partial y} = -4y + 6xy^2 = \frac{\partial N}{\partial x}$$

$$F(x,y) = \int 3x^2y^2 - 4xydy = x^2y^3 - 2xy^2 + h(x)$$

$$\frac{\partial F}{\partial x} = 2xy^3 - 2y^2 + h'(x) = M \implies h'(x) = 6x$$

$$h(x) = \int 6xdx = 3x^2 + k$$

Our general solution is

$$F(x,y) = x^2y^3 - 2xy^2 + 3x^2 = C$$

ODEs With an Integrating Factor

Definition. We say that the function $\mu(x,y)$ is an integrating factor of a first order ODE in differential form if the following equation is exact

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

Theorem

If
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$$
for some function $g(y)$, then
$$\mu(y) = \exp\left(-\int g(y)dy\right)$$
If
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(x)$$
for some function $f(x)$, then
$$\mu(x) = \exp\left(\int f(x)dy\right)$$

Linear First-Order ODEs

Definition. A first order ODE is called linear if it can be written in the form

$$y' + f(x)y = r(x)$$

Steps to Finding General Solution

Given a linear first-order ODE in the form $y' + f(x)y = r(x)$, find y using

$$y = \frac{\int \mu(x)r(x)dx + C}{\exp\left(\int f(x)dx\right)}$$

$$y = \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right) \exp\left(\int f(x)\right)^{-1}$$

Numerical Methods

Iterative Methods to Solving $f(x) = 0$

Theorem. (Intermediate Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $y \in \mathbb{R}$ between $f(a)$ and $f(b)$. Then there exists $z \in [a, b]$ such that $f(z) = y$.

Fixed Point Iteration

Definition. The value $x = r$ is a fixed point for $g(x)$ if $g(r) = r$.
Definition. The iteration sequence is given as

$$x_{n+1} = g(x_n)$$

with x_0 being given.

Theorem. Assume that the function $g(x)$ has a fixed-point s on an interal I, if

- $g(x)$ is continuous on I,
- $g'(x)$ is continuous on I, and
- $|g'(x)| < 1$ for all $x \in I$.

Then then the iteration sequence converges.

Steps to Solve Using Fixed-Point Iteration

Then the steps for solving are as follows,

1. Start with $f(x) = 0$
2. Rewrite $f(x) = 0$ under the form $x = g(x)$
3. Verify the iteration sequence $x_0, x_1 = g(x_0), \ldots, x_n = g(x_{n-1})$ converges using the above theorem
4. Compute the terms of the sequence and stop when the required accuracy is reached (i.e when 2 consecutive terms have the same k decimal places where k is the desired accuracy).

Newton's Method

Given an equation equation $f(x) = 0$ and a starting point x_0, the Newton's method is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Calculate values for x_n until you reach the accuracy.
Example. Approximate the root for the equation $x^3 + 12x - 3$ to 6 decimal places with $x_0 = 1.8$
Solution.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.675138, \ldots, x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.248748$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.248718, \; x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.248718$$