Order 1 ODE's

Standard Form: $y' = f(x,y)$
Differential Form: $M(x,y)dx + N(x,y)dy = 0$

Seperable First Order ODE's

An ODE is called seperate if it can be written as

$$F(x)dx = G(y)dy$$

Steps to Solving.

1. Write $y' = \frac{dy}{dx}$, and seperate the ODE to write it in the form $F(x)dx = G(y)dy$
2. Integrate both sides
3. If an initial condition is given, solve for the integration constant C.

First Order ODE's with Homogeneous Coefficients

$F(x,y)$ is called homogeneous of degree k if it can be written

$$F(\lambda x, \lambda y) = \lambda^k \cdot F(x,y)$$

An ODE is differential form is homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree.
Steps to Solving.
A homogeneous ODE can be made seperable by substituting $u = \frac{y}{x}$, or $u = \frac{x}{y}$.

$$u = \frac{y}{x} \implies y = xu,\ dy = udx + xdu$$

$$u = \frac{x}{y} \implies y = \frac{x}{u}, \frac{dy}{dx} = \frac{x - u\frac{dx}{du}}{u^2}$$

Exact First Order ODEs

An ODE is called exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
Steps to Solving.

1. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Look for a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$ Integrate M with respect to x or N with respect to y then differentiate the equation with respect to the other variable respectively.
3. The general solution is $F(x,y) = C$

Integrating Factor

$\mu(x,y)$ is an integrating factor of a if the following equation is exact

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

Theorem. If

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$$

for some function $g(y)$, then

$$\mu(y) = \exp\left(-\int g(y)dy\right)$$

If

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(x)$$

for some function $f(x)$, then

$$\mu(x) = \exp\left(\int f(x)dy\right)$$

Linear First-Order ODEs

Definition. A first order ODE is called linear if it can be written in the form

$$y' + f(x)y = r(x)$$

Steps to Finding General Solution

Given a linear first-order ODE in the form $y' + f(x)y = r(x)$, find y using

$$y = \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right) \exp\left(-\int f(x)\right)$$

Bernoulli ODE's

A first order ODE is of Bernoulli type if it can be written as

$$y' + f(x)y = r(x)y^a$$

Steps to Solving

1. Let $u = y^{1-a}$, then compute $u' = (1-a)y^{-a}y'$.
2. Isolate y' from the original ODE and substitute into u'.
3. The resulting ODE is linear and solve for u.

Homogeneous ODEs

Constant Coefficients

General Form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Characteristic Equation

$$\lambda^n + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 = 0$$

If λ is a root with multiplicity k,

$$y_1 = e^{\lambda x},\ y_2 = xe^{\lambda x},\ y_3 = x^2 e^{\lambda x}, \ldots, y_k = x^{k-1}e^{\lambda x}$$

If $\alpha + i\beta$ is a pair of complex conjugate roots, then

$$y_1 = e^{\alpha x}\cos(\beta x),\ y_2 = e^{\alpha x}\sin(\beta x)$$

Euler-Cauchy

General Form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

Characteristic Equation. Differentiate $y = x^m$ and plug into ODE. If m is a root of the characteristic equation of multiplicity k, then it contributes the following equations to our basis of solutions

$$y_1 = x^m, y_2 = x^m \ln x, y_3 = x^m(\ln x)^2, y_k = x^m(\ln x)^{k-1}$$

If $\alpha \pm i\beta$ is a pair of complex conjugate roots of the characteristic equation, then the pair contributes the following 2 equations to our basis of solutions

$$y_1 = x^\alpha \cos(\beta \ln x), y_2 = x^\alpha \sin(\beta \ln x)$$

Undetermined Coefficients

General Form

$$a_n y^{(n)} + \cdots + a_1(x)y' + a_0 y = r(x)$$

All coefficients on the left $(a_n, \ldots, a_0)$ are constants and $r(x)$ is a polynomial, exponential, and/or sinusoidal.

Rules

$r(x)$	y_p
$Ke^{\lambda x}$	$Ae^{\lambda x}$
$p_n(x)$	$q_n(x)$
$K\sin(wx)$	$A\cos(wx) + B\sin(wx)$
$Ke^{\alpha x}\sin(wx)$	$Ae^{\alpha x}\cos(wx) + Be^{\alpha x}\sin(wx)$
$p_n(x)e^{\alpha x}$	$q_n(x)e^{\alpha x}$

Where $p_n(x)$ and $q_n(x)$ are polynomials of degree n.

Variation of Parameters

General Form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0 y(x) = r(x)$$

Same solution with $y = y_H + y_p$ where y_H is the solution to the coresponing homogeneous ODE

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0 y(x) = 0$$

$$y_p = u_1 y_1 + u_2 y_2 + \cdots u_n y_n$$

Where $\{y_1, y_2, \ldots, y_n\}$ is a basis of solutions for the corresponding homogeneous ODE. $u_1, u_2, \ldots, u_n$ are functions that satisfy the following system of equations

$$\begin{cases} 0 = u_1' y_1 + u2' y_2 + \cdots + u_n' y_n \\ 0 = u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' \\ \vdots \\ \frac{r(x)}{a_n(x)} = u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \end{cases}$$

Solving IVP's With Laplace Transforms

Steps to Solving.

1. Set $Y = \mathcal{L}\{y(t)\}$.
2. Apply Laplace transform to both sides of the ODE.
3. Isolate Y after preforming the Laplace transform.
4. Then solve $y(t) = \mathcal{L}^{-1}\{Y\}$.

Laplace of a derivative

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$$

$$\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y(t)\} - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)$$

Systems of ODE's

General Form

$$\left.\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + r_1(x) \\ y_1' &= a_{21}y_1 + a_{22}y_2 + r_2(x) \end{aligned}\right\} \implies \vec{y}' = A\vec{y} + \vec{r}(x)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \vec{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \vec{r}(x) = \begin{bmatrix} r_1(x) \\ r_2(x) \end{bmatrix}$$

Homogenous Systems with Constant Coefficients

A system is homogeneous if $\vec{r}(x) = 0$, $\vec{y'} = A\vec{y}$.

Steps to Solving

1. Find the eigen values of A $|\lambda I - A| = 0$
2. If 2 distinct real eigenvalues, find eigenvectors V_1, V_2. $\vec{y} = c_1\vec{V_1}e^{\lambda_1 x} + c_2\vec{V_2}e^{\lambda_2 x}$
3. If λ with multiplicity 2, find generalized eigenvector ρ $Y = c_1 V e^{\lambda x} + c_2(xV + \rho)e^{\lambda x}$
4. If $\lambda = \alpha \pm i\beta$, then find eigenvector for $\lambda_1 = \alpha + i\beta$, compute gen. solution $y = c_1 V_1 + c_2 V_2$ with $\vec{V}e^{(\alpha+i\beta)x} = \vec{V}e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) = \vec{V_1} + i\vec{V_2}$

Non-Homogeneous Systems

Similar to non-homogeneous ODEs, use undetermined to solve $\vec{y} = \vec{y_H} + \vec{y_p}$. y_H is ODE with $\vec{r}(x) = 0$. Decompose $\vec{r}(x)$ as

$$r(\vec{x}) = \begin{bmatrix} 2x^3 + x^2 + x \\ 3e^x + x^2 + 2x + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} x^3 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} e^x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x^2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x$$

Solve each $r_1(x)$ using same rules as undetermined coefficients replacing constants with constant vectors

$$r_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} x^3 \implies \vec{y_p} = \vec{U}x^3 + \vec{V}x^2 + \vec{W}x + \vec{Z}$$

Linear Algebra and Trig Identities

Linear Algebra

Eigen Values

$$|A - \lambda I| = |\lambda I - A| = 0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigen Vectors. An eigenvector $\vec{V}$ is a vector that satisfies $[A - \lambda I|\vec{0}]$

Generalized Eigen Vector ρ The solution to, then pick specific value for the parameter t to get a specific vector, (i.e take $t = 0$, $t = 1$, etc).

$$[A - \lambda I|V]$$

Trig Identities

$$\cos^2\left(\frac{t}{2}\right) = \frac{1 + \cos(t)}{2}, \quad 2\sin(t)\cos(t) = \sin(2t)$$

Example of Non-Homogeneous System

Example.

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix}\vec{y} + \begin{bmatrix} 9x - 51 \\ 7 + e^{2x} \end{bmatrix}; \ \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution. Solve the corresponding homogeneous ODE

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix}\vec{y}$$

Find eigenvalues of A

$$\det\begin{bmatrix} 9 - \lambda & 18 \\ -2 & -3 - \lambda \end{bmatrix} = (9 - \lambda)(-3 - \lambda) + 36 = \lambda^2 - 6\lambda + 9$$

This gives us $\lambda = 3$ with multiplicity 2, find eigenvector $V = t\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ Find generalized eigenvector, set parameter $t = 0$,

$$[A - 3I|V] = \begin{bmatrix} 6 & 18 & -3 \\ -2 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{\rho} = \begin{bmatrix} -3t - 1/2 \\ t \end{bmatrix} \implies \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

The general solution to the homogeneous ODE is

$$\vec{y}_H = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} xe^{3x} + c_2\left(x\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}\right)e^{3x}$$

$$= \begin{bmatrix} -3c_1e^{3x} - 3c_2xe^{3x} - \frac{1}{2}c_2e^{3x} \\ c_1e^{3x} + c_2xe^{3x} \end{bmatrix}$$

For y_p, decompose $r(x)$ to get

$$r(x) = \begin{bmatrix} 9x - 51 \\ 7 + e^{2x} \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix} x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$$

Find y_p for each part of $r(x)$

$$y_p = Ux + V + We^{2x}$$

Rewrite the non-homogeneous system as

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix} y + \begin{bmatrix} 9 \\ 0 \end{bmatrix} x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$$

Compute $\vec{y'}_p$ and plug into the system to solve for constant vectors, $\vec{y'}_p = U + 2We^{2x}$, $\vec{y'}_p = A\vec{y_p} + \vec{r}(x)$

$$U + 2We^{2x} = A(Ux + V + We^{2x}) + \begin{bmatrix} 9 \\ 0 \end{bmatrix} x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$$

This gives us the three equations

$$AU + \begin{bmatrix} 9 \\ 0 \end{bmatrix} = 0, AV + \begin{bmatrix} -51 \\ 7 \end{bmatrix} = U, AW + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2W$$

Solve equations to find

$$U = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, V = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, W = \begin{bmatrix} 18 \\ -7 \end{bmatrix}$$

This gives us our particular solution

$$y_p = \begin{bmatrix} 3 \\ -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 18 \\ -7 \end{bmatrix} e^{2x} = \begin{bmatrix} 3x + 18e^{2x} \\ -2x + 3 - 7e^{2x} \end{bmatrix}$$

The general solution to the non-homogeneous ODE is

$$y = \begin{bmatrix} -3c_1e^{3x} - 3c_2xe^{3x} - \frac{1}{2}e^{3x} \\ c_1e^{3x} + c_2xe^{3x} \end{bmatrix} + \begin{bmatrix} 3x + 18e^{2x} \\ -2x + 3 - 7e^{2x} \end{bmatrix}$$

Fixed Point Iteration

For a given starting value x_0, The iteration sequence is given as $x_{n+1} = g(x_n)$

Theorem. Assume that the function $g(x)$ has a fixed-point s on an interal I, if

- $g(x)$ is continuous on I,
- $g'(x)$ is continuous on I, and
- $|g'(x)| < 1$ for all $x \in I$.

Then then the iteration sequence converges.

Steps to Solve Using Fixed-Point Iteration

Then the steps for solving are as follows,

1. Start with $f(x) = 0$
2. Rewrite $f(x) = 0$ under the form $x = g(x)$
3. Verify the iteration sequence $x_0, x_1 = g(x_0), \ldots, x_n = g(x_{n-1})$ converges using the above theorem
4. Compute the terms of the sequence and stop when 2 terms have the same required digits.

Newton's Method

Given an equation equation $f(x) = 0$ and a starting point x_0, the Newton's method is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Calculate values for x_n until you reach the accuracy.

Secant Method

Given 2 estimates for the roots x_0, x_1, compute the terms in the sequence until you reach the required accuracy.

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Lagrange Interpolation

Given points $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$, the Lagrange interpolation polynomial is

$$p_n(x) = L_0(x)f_0 + L_(1)(x)f_1 + \cdots + L_n(x)f_n$$

where

$$L_0 = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}$$

$$L_1 = \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}$$

$$L_2 = \frac{(x - x_0)(x - x_1)(x - x_3)\cdots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_n)}$$

The error formula for interpolation is

$$|f(x) - p_n(x)| = \left|(x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)(t)}}{(n+1)!}\right|$$

Newton's Divided-Difference Interpolation

Given a node x_i,

1. The *first divided difference* at x_i is defined as
$$f(x_i, x_{i+1}) = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$
2. The *second divided difference* at x_i is
$$f(x_i, x_{i+1}, x_{i+2}) = \frac{f(x_{i+1}, x_{i+2}) - f(x_i, x_{i+1})}{x_{i+2} - x_i}$$
3. In general, the kth divided difference at x_i is
$$f(x_i, x_{i+1}, \dots x_{i+k}) = \frac{f(x_{i+1}, \dots, x_{i+k}) - f(x_i, \dots x_{i+k-1})}{x_{i+k} - x_i}$$

Newton's Interpolation Polynomial

$$\begin{aligned} p_n(x) = f_0 &+ f(x_0, x_1)(x - x_0) \\ &+ f(x_0, x_1, x_2)(x - x_0)(x - x_1) + \cdots \\ &+ f(x_0, \dots, x_n)(x - x_0)(x - x_1) \cdots (x - x_n) \end{aligned}$$

Numerical Integration

Midpoint Rule

$$\int_a^b f(x)dx \approx h[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]$$

$$h = \frac{b-a}{n}, x_i^* = \frac{x_i + x_{i+1}}{2}$$

Error formula with M being $|f''(x)| \leq M$ for $x \in [a, b]$

$$E_M \leq \frac{M(b-a)^3}{24n^2}$$

Trapezoidal

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a) + 2f(x_1) + 2f(x_2) + \cdots + f(x_n)]$$

Error Formula

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}$$

Simpsons Rule

Divide $[a, b]$ into an *EVEN* number of subintervals

$$\int_a^b f(x) \approx \frac{h}{3}[f(a) + 4(x_1) + 2f(x_2) + 4f(x_3) + \cdots + f(b)]$$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$

Error Formula Where M is the upperbound for the fourth derivative of $f(x)$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$

Gaussian Quadrature

To convert $\int_a^b f(x)dx$ into the form $\int_{-1}^1 g(t)dt$, use the substitution
$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$
Then the Gaussian Quadrature formula is
$$\int_{-1}^1 f(t)dt \approx w_1 f(t_1) + \cdots w_n f(t_n)$$

Table of Nodes and Coefficients

Order n	Nodes t_i	Coefficients w_i
1	0	2
2	−0.5773502692	1
	0.5773502692	1
3	−0.7745966692	0.555555556
	0	0.888888889
	0.7745966692	0.555555556
4	-0.8611363116	0.3478548451
	-0.3399810436	0.6521451549
	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	-0.9061798459	0.2369268850
	-0.5384693101	0.4786286705
	0.0	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850

Euler Method

Given a step size h between x values, we estimate values of y with
$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n, y_n)$$

Improved Euler Method

We use Euler's method to predict then correct each step with
$$x_{n+1} = x_n + h$$
$$y_{n+1}^c = y_n^c + \frac{h}{2}(f(x_n, y_n^c) + f(x_{n+1}, y_{n+1}^p))$$
$$y_{n+1}^p = y_n^c + hf(x_n, y_n)$$

Runge-Kutta Method of Order 4

Given $y(x_0) = y_0$, use the following formula to compute
$$x_{n+1} = x_n + h$$
$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$
$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$
$$k_4 = hf\left(x_n + h, y_n + k_3\right)$$
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k3 + k_4)$$

Examples for Numerical Integration

Midpoint Rule

Example. Estimate the following integral with a maximal absolute error of 0.001.
$$I = \int_0^{0.5} x \cos x dx$$
Solution. We have to first find n
$$f(x) = x \cos x, \; f'(x) = \cos x - x \sin x$$
$$f''(x) = -\sin x - \sin x - x \cos x = -2 \sin x - x \cos x$$
We can see that
$$|-2\sin x - x \cos x| \leq |-2 \sin x| + |-x \cos x| \leq 2.5$$
Thus $M = 2.5$, then by the error formula we have
$$|E_m| \leq \frac{2.5(0.5-0)}{24n^2} \leq 0.001 \implies n \geq \sqrt{\frac{2.5(0.5)^3}{24(0.001)}} = 2.79$$
We take $n = 3$. Then we can calculate h,
$$h = \frac{0.5-0}{3} = \frac{1}{6}, \; x_1^* = \frac{0 + 1/6}{2} = \frac{1}{12}$$
$$x_2^* = x_1^* + \frac{1}{6} = \frac{1}{4}, \; x_3^* = x_2^* + \frac{1}{6} = \frac{5}{12}$$
By the midpoint rule, we have
$$\int_0^{0.5} x \cos x dx \approx h[f(x_1^*) + f(x_2^*) + f(x_3^*)]$$
$$= \frac{1}{6}\left[\frac{1}{12}\cos\left(\frac{1}{12}\right) + \frac{1}{4}\cos\left(\frac{1}{4}\right) + \frac{5}{12}\cos\left(\frac{5}{12}\right)\right]$$

Trapezoidal Rule

Example. Estimate the value of the integral with a maximal absolute error of 0.01 to
$$\int_0^1 e^{-x^2} dx$$
Solution. First we compute n,
$$f(x) = e^{-x^2}, \; f'(x) = -2xe^{-x^2}, \; f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2}$$
$$f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2}, \; f'''(x) = e^{-x^2} 4x(3 - 2x^2)$$
$f''(x)$ is increasing, therefore
$$f''(0) \leq f''(x) \leq f''(1) \implies -2 \leq f''(x) \leq -2e^{-1} + 4e^{-1}$$
Thus we can take $M = 2$. Then by the error formula we have
$$\frac{2}{12n^2} \leq 0.01 \implies n \geq \sqrt{\frac{1}{6(0.01)}} \approx 4.08$$
Take $n = 5$. Length of each subinterval is $h = \frac{1-0}{5} = 0.2$.
$$\frac{0.2}{2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)]$$

Simpson's Rule

Example. Estimate the value of the integral with a maximal error of 0.001

$$\int_{0.5}^{1.5} x^2 \ln x dx$$

Solution. We start by computing n,

$$f(x) = x^2 \ln x,\ f'(x) = 2x\ln x + x,\ f''(x) = 2\ln x + 3$$

$$f'''(x) = \frac{2}{x},\ f^{(4)}(x) = -\frac{2}{x^2},\ f^{(5)}(x) = \frac{4}{x^3}$$

$f^{(4)}$ is increasing, therefore

$$f^{(4)}(0.5) \leq f^{(4)}(x) \leq f^{(4)}(1.5) \implies |f^{(4)}(x)| \leq 8$$

So we can take $M = 8$. Then by the error formula we have

$$\frac{8(1.5-0.5)}{180n^4} \implies n \geq \sqrt[4]{\frac{8}{180(0.001)}} \approx 2.58$$

We need an even n so we take $n = 4$. Then we can calculate h,

$$h = \frac{1}{4},\ x_0 = 0.5,\ x_1 = 0.5 + \frac{1}{4} = 0.75,\ x_2 = 0.75 + \frac{1}{4} = 1$$

$$x_3 = 1 + \frac{1}{4} = 1.25,\ x_4 = 1.25 + \frac{1}{4} = 1.5$$

Then we can approximate the integral,

$$\int_{0.5}^{1.5} x^2 \ln x \approx 0.123915$$

Guassian Quadrature

Use Gaussian Quadrature of order 4 to estimate the value of

$$\int_0^1 \sin(x^2)dx$$

Solution. First we substitute x with

$$x = \frac{b-a}{2}t + \frac{b+a}{2} = \frac{1}{2}t + \frac{1}{2}$$

$$\frac{dx}{dt} = \frac{1}{2} \implies dx = \frac{dt}{2}$$

$$\int_0^1 \sin(x^2) = \int_{-1}^1 \sin\left(\frac{(t+1)^2}{4}\right)\frac{1}{2}dt$$

From the table we have

$$w_1 = w_4 = 0.3479,\ w_2 = w_3 = 0.6521$$

$$f(t_1) = -f(t_4) = -0.8611,\ f(t_2) = -f(t_3) = -0.3399$$

Then using the formula

$$\int_{-1}^1 \sin\left(\frac{(t+1)^2}{4}\right)\frac{1}{2}dt \approx w_1 f(t_1) + w_2 f(t_2) + w_3(t_3) + w_4 f(t_4)$$

Estimating IVP's

Euler's Method

Example. Use Euler's method with $h = 0.2$ to estimate the IVP on $[0, 0.6]$.

$$y' = 2x + y,\ y(0) = -1$$

Solution. We have $f(x,y) = 2x + y$, $x_0 = 0$, $y_0 = -1$, and $h = 0.2$. Now we can caluclate each step with $x_1 = 0.2$, $x_2 = 0.04$, $x_3 = 0.6$.

$$y_1 = y_0 + hf(x_0, y_0) = -1 + 0.2(-1) = -1.2$$

$$y_2 = y_1 + hf(x_1, y_1) = -1.2 + 0.2(2(0.2) - 1.2) = -1.36$$

$$y_3 = y_2 + hf(x_2, y_2) = -1.36 + 0.2(2(0.4) - 1.36) = -1.472$$

Improved Euler's Method

Example. Using the same IVP as previous problem, We're given $f(x,y) = 2x+y$, $h = 0.2$, $y_0 = -1$, and $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.04$, $x_3 = 0.6$.

$$y_1^p = y_0 + hf(x_0, y_0) = -1 + 0.2(-1) = -1.2$$

$$y_1^c = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^p)\right]$$
$$= -1 + 0.1\left[-1 + 2(0.2) - 1.2\right] = -1.18$$

$$y_2^p = y_1^c + hf(x_1, y_1^c) = -1.18 + 0.2(2(0.2) - 1.18)$$
$$= -1.336$$

$$y_2^c = y_1^c + \frac{h}{2}\left[f(x_1, y_1^c) + f(x_2, y_2^p)\right]$$
$$= -1.18 + 0.1\left[2(0.2) - 1.18 + 2(0.4)\right] = -1.3116$$

$$y_3^p = y_2^c + hf(x_2, y_2^c) = -1.3116 + 0.2(2(0.4) - 1.3116)$$
$$= -1.41392$$

$$y_3^c = -1.3116 + 0.1\left[2(0.4) - 1.18 + 2(0.6) - 1.413192\right]$$
$$= -1.384152$$

Runge-Kutta Method of Order 4

Example. $y' = y - x^2 + 1;\ y(0) = \frac{1}{2}$

We can compute with a step size $h = 0.5$ on the interval $[0, 0.5]$, $x_1 = 0.5$.

$$k_1 = hf(x_0, y_0) = 0.5\left(\frac{1}{2} - 0 + 1\right) = 0.75$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$
$$= 0.5f(0.25, 0.875) = 0.90625$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right)$$
$$= 0.5f(0.25, 0.953125) = 0.9453125$$

$$k_3 = hf\left(x_0 + h, y_0 + k_3\right)$$
$$= 0.5f(0.5, 1.4453125) = 1.09765625$$

For $x_2 = 1$, the process repeats and we get 2 new points.

Laplace Transforms

Convolution

Example. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s-5}\right\}$

Solution. Rewrite the fraction and use the convolution,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s-5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\frac{1}{s+5}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$
$$= e^t * e^{-5t} = \int_0^t e^{t-x}e^{-5x}dx$$
$$= \int_0^t e^t e^{-x}e^{-5x}dx = e^t\int_0^t e^{-6x}dx$$
$$= e^t\left(-\frac{1}{6}e^{-6t} + 1\right) = \frac{1}{6}e^t - \frac{1}{6}e^{-5t}$$

Solving an IVP

Example. Solve the following IVP

$$y'' + 6y' + 9y = \begin{cases} 0 & 0 \leq t < 2 \\ e^{-3t} & t \geq 0 \end{cases},\ y(0) = 1,\ y'(0) = 0$$

Solution. We can rewrite the ODE as

$$y'' + 6y' + 9y = e^{-3t}u(t-2)$$

Applying the laplace transform to both sides,

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}u(t-2)\}$$
$$s^2Y - sy(0) - y'(0) + 6(sY - y(0)) + 9Y = \frac{e^{-2(s+3)}}{s+3}$$

This gives us

$$Y = \frac{s+6}{(s+3)^2} + \frac{e^{-2(s+3)}}{(s+3)^3}$$

Now we can compute the inverse laplace transform of Y

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s+6}{(s+3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2(s+3)}}{(s+3)^3}\right\}$$

The first part simply requires partial fractions,

$$\mathcal{L}^{-1}\left\{\frac{s+6}{(s+3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$
$$= e^{-3t} + 3te^{-3t}$$

The second part requires second shifting theorem with e^{-2s} and $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{e^{-2(s+3)}}{(s+3)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}e^{-6}}{(s+3)^3}\right\}$$
$$= e^{-6}\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s+3)^3}\right\}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\}$$
$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{(s+3)^3}\right\} = \frac{1}{2}t^2e^{-3t}$$

Thus,

$$y(t) = e^{-3t} + 3te^{-3t} + e^{-6}u(t-2)\frac{1}{2}(t-2)^2e^{-3(t-2)}$$