MAT 2384 Summary Sheet: Ordinary Differential Equations

Order 1 ODE's

Standard Form: $y' = f(x,y)$
Differential Form: $M(x,y)dx + N(x,y)dy = 0$

Seperable First Order ODE's

Definition. A first order ODE is called seperable if it can be written in the form

$$F(x)dx = G(y)dy$$

Steps to Solving Seperable ODE's

1. Write $y' = \frac{dy}{dx}$
2. Seperate the ODE to write it in the form $F(x)dx = G(y)dy$
3. Integrate both sides
4. If an initial condition is given, solve for the integration constant C.

First Order ODE's with Homogeneous Coefficients

Definition. A function $F(x,y)$ is called homogeneous of degree k if

$$F(\lambda x, \lambda y) = \lambda^k \cdot F(x,y)$$

Definition. A first order ODE given in differential form is called homogeneous if both $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree.

Theorem. A first order ODE of homogeneous coefficients can be made seperable by changing the function by substituting $u = \frac{y}{x} \implies y = xu$ or $u\frac{x}{y} \implies y = \frac{x}{u}$.

Exact First Order ODEs

Definition. Given a function $F(x,y)$, the differential of F denoted by dF is defined by

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Remark. $dF = 0 \iff F(x,y) = C$.

Definition. A first order ODE is called exact if there exists $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial x} = N(x,y)$$

Theorem. A first order ODE is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact First Order ODE's

Steps To Solving

1. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Look for a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M, \ \frac{\partial F}{\partial y} = N$$

   Do this by integrating M with respect to x or N with respect to y then differentiate the equation with respect to the other variable respectively.
3. The general solution is $F(x,y) = C$
4. If an initial condition is given, solve for the integration constant C.

Example. Solve the following IVP

$$(6x - 2y^2 + 2xy^3)dx + (3x^2y^2 - 4xy)dy = 0$$

Solution. Check for exactness

$$\frac{\partial M}{\partial y} = -4y + 6xy^2 = \frac{\partial N}{\partial x}$$

$$F(x,y) = \int 3x^2y^2 - 4xydy = x^2y^3 - 2xy^2 + h(x)$$

$$\frac{\partial F}{\partial x} = 2xy^3 - 2y^2 + h'(x) = M \implies h'(x) = 6x$$

$$h(x) = \int 6xdx = 3x^2 + k$$

Our general solution is

$$F(x,y) = x^2y^3 - 2xy^2 + 3x^2 = C$$

ODEs With an Integrating Factor

Definition. We say that the function $\mu(x,y)$ is an integrating factor of a first order ODE in differential form if the following equation is exact

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

Theorem

If
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$$
for some function $g(y)$, then
$$\mu(y) = \exp\left(-\int g(y)dy\right)$$
If
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(x)$$
for some function $f(x)$, then
$$\mu(x) = \exp\left(\int f(x)dy\right)$$

Linear First-Order ODEs

Definition. A first order ODE is called linear if it can be written in the form

$$y' + f(x)y = r(x)$$

Steps to Finding General Solution

Given a linear first-order ODE in the form $y' + f(x)y = r(x)$, find y using

$$y = \frac{\int \mu(x)r(x)dx + C}{\exp\left(\int f(x)dx\right)}$$

$$y = \left(\int \exp\left(\int f(x)dx\right) r(x)dx + C\right) \exp\left(\int f(x)\right)^{-1}$$

Iterative Methods to Solving $f(x) = 0$

Theorem. (Intermediate Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $y \in \mathbb{R}$ between $f(a)$ and $f(b)$. Then there exists $z \in [a, b]$ such that $f(z) = y$.

Fixed Point Iteration

Definition. The value $x = r$ is a fixed point for $g(x)$ if $g(r) = r$.
Definition. The iteration sequence is given as

$$x_{n+1} = g(x_n)$$

with x_0 being given.

Theorem. Assume that the function $g(x)$ has a fixed-point s on an interal I, if

- $g(x)$ is continuous on I,
- $g'(x)$ is continuous on I, and
- $|g'(x)| < 1$ for all $x \in I$.

Then then the iteration sequence converges.

Steps to Solve Using Fixed-Point Iteration

Then the steps for solving are as follows,

1. Start with $f(x) = 0$
2. Rewrite $f(x) = 0$ under the form $x = g(x)$
3. Verify the iteration sequence $x_0, x_1 = g(x_0), \ldots, x_n = g(x_{n-1})$ converges using the above theorem
4. Compute the terms of the sequence and stop when the required accuracy is reached (i.e when 2 consecutive terms have the same k decimal places where k is the desired accuracy).

Newton's Method

Given an equation equation $f(x) = 0$ and a starting point x_0, the Newton's method is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Calculate values for x_n until you reach the accuracy.
Example. Approximate the root for the equation $x^3 + 12x - 3$ to 6 decimal places with $x_0 = 1.8$
Solution.
$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0.675138, \ldots, x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 0.248748$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} = 0.248718,\ x_5 = x_4 - \dfrac{f(x_4)}{f'(x_4)} = 0.248718$

Homogeneous ODEs

Constant Coefficients

General Form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Characteristic Equation

$$\lambda^n + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 = 0$$

If λ is a root with multiplicity k,

$$y_1 = e^{\lambda x},\ y_2 = xe^{\lambda x},\ y_3 = x^2 e^{\lambda x}, \ldots, y_k = x^{k-1}e^{\lambda x}$$

If $\alpha + i\beta$ is a pair of complex conjugate roots, then it contributes the following 2 equations to our basis of solutions,

$$y_1 = e^{\alpha x}\cos(\beta x),\ y_2 = e^{\alpha x}\sin(\beta x)$$

Euler-Cauchy

General Form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

Characteristic Equation. Differentiate $y = x^m$ and plug into ODE. If m is a root of the characteristic equation of multiplicity k, then it contributes the following equations to our basis of solutions

$$y_1 = x^m, y_2 = x^m \ln x, y_3 = x^m(\ln x)^2, y_k = x^m(\ln x)^{k-1}$$

If $\alpha \pm i\beta$ is a pair of complex conjugate roots of the characteristic equation, then the pair contributes the following 2 equations to our basis of solutions

$$y_1 = x^{\alpha}\cos(\beta \ln x), y_2 = x^{\alpha}\sin(\beta \ln x)$$

Undetermined Coefficients

General Form

$$a_n y^{(n)} + \cdots + a_1(x)y' + a_0 y = r(x)$$

All coefficients on the left $(a_n, \ldots, a_0)$ are constants and $r(x)$ is a polynomial, exponential, and/or sinusoidal.

Rules

- **Rule 1: Basic Rule.** If $r(x) = ke^{\lambda x}$, then $y_p = Ae^{\lambda x}$. If $r(x) = p_n(x)$, then $y_p = q_n(x)$. If $r(x) = k\cos(wx)$ or $k\sin(wx)$, then $y_p = Ae^{\alpha x}\cos(wx) + Be^{\alpha x}\sin(wx)$. If $r(x) = p_n(x)e^{\lambda x}$, then $y_p = q_n(x)e^{\lambda x}$.
- **Rule 2: The Modifcation Rule.** If any term of y_p is in the basis of solutions for y_H, then multiply by x until it's not.
- **Rule 3: The Sum Rule.** If $r(x) = r_1(x) + \cdots r_n(x)$, then do each $r_i(x)$ seperately and combine.

Variation of Parameters

General Form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0 y(x) = r(x)$$

Same solution with $y = y_H + y_p$ where y_H is the solution to the coresponing homogeneous ODE

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0 y(x) = 0$$

$$y_p = u_1 y_1 + u_2 y_2 + \cdots u_n y_n$$

Where $\{y_1, y_2, \ldots, y_n\}$ is a basis of solutions for the corresponding homogeneous ODE. $u_1, u_2, \ldots, u_n$ are functions that satisfy the following system of equations

$$\begin{cases} 0 = u_1' y_1 + u2' y_2 + \cdots + u_n' y_n \\ 0 = u_1' y_1' + u2' y_2' + \cdots + u_n' y_n' \\ \vdots \\ \frac{r(x)}{a_n(x)} = u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \end{cases}$$

Systems of ODE's

General Form

$$\left.\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + r_1(x) \\ y_1' &= a_{21}y_1 + a_{22}y_2 + r_2(x) \end{aligned}\right\} \implies \vec{y}' = A\vec{y} + \vec{r}(x)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \vec{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \vec{r}(x) = \begin{bmatrix} r_1(x) \\ r_2(x) \end{bmatrix}$$

Homogenous Systems with Constant Coefficients

A system is homogeneous if $\vec{r}(x) = 0$, $\vec{y}' = A\vec{y}$.

Steps to Solving

1. Find the eigen values of A $|\lambda I - A| = 0$
2. If 2 distinct real eigenvalues, find eigenvectors V_1, V_2. $\vec{y} = c_1\vec{V_1}e^{\lambda_1 x} + c_2\vec{V_2}e^{\lambda_2 x}$
3. If λ with multiplicity 2, find generalized eigenvector ρ $Y = c_1 V e^{\lambda x} + c_2(xV + \rho)e^{\lambda x}$
4. If $\lambda = \alpha \pm i\beta$, then find eigenvector for $\lambda_1 = \alpha + i\beta$, compute gen. solution $y = c_1 V_1 + c_2 V_2$ with
   $\vec{V}e^{(\alpha + i\beta)x} = \vec{V}e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) = \vec{V_1} + i\vec{V_2}$

Non-Homogeneous Systems

Similar to non-homogeneous ODEs, use undetermined to solve $\vec{y} = \vec{y_H} + \vec{y_p}$. y_H is ODE with $\vec{r}(x) = 0$. Decompose $\vec{r}(x)$ as

$$\vec{r(x)} = \begin{bmatrix} 2x^3 + x^2 + x \\ 3e^x + x^2 + 2x + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} x^3 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} e^x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x^2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x$$

Solve each $r_1(x)$ using same rules as undetermined coefficients replacing constants with constant vectors

$$r_1(x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} x^3 \implies \vec{y_p} = \vec{U}x^3 + \vec{V}x^2 + \vec{W}x + \vec{Z}$$

Newton's Divided-Difference Interpolation

Given a node x_i,

1. The *first divided difference* at x_i is defined as
$$f(x_i, x_{i+1}) = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$
2. The *second divided difference* at x_i is
$$f(x_i, x_{i+1}, x_{i+2}) = \frac{f(x_{i+1}, x_{i+2}) - f(x_i, x_{i+1})}{x_{i+2} - x_i}$$
3. In general, the kth divided difference at x_i is
$$f(x_i, x_{i+1}, \ldots x_{i+k}) = \frac{f(x_{i+1}, \ldots, x_{i+k}) - f(x_i, \ldots x_{i+k-1})}{x_{i+k} - x_i}$$

Newton's Interpolation Polynomial

$$\begin{aligned} p_n(x) = f_0 &+ f(x_0, x_1)(x - x_0) \\ &+ f(x_0, x_1, x_2)(x - x_0)(x - x_1) + \cdots \\ &+ f(x_0, \ldots, x_n)(x - x_0)(x - x_1)\cdots(x - x_n) \end{aligned}$$

Numerical Integration

Midpoint Rule

$$\int_a^b f(x)dx \approx h[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]$$

$$h = \frac{b-a}{n}, x_i^* = \frac{x_i + x_{i+1}}{2}$$

Error formula with M being $|f''(x)| \leq M$ for $x \in [a, b]$

$$E_M \leq \frac{M(b-a)^3}{24n^2}$$

Trapezoidal

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a) + 2f(x_1) + 2f(x_2) + \cdots + f(x_n)]$$

Error Formula

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}$$

Simpsons Rule

Divide $[a, b]$ into an $EVEN$ number of subintervals

$$\int_a^b f(x) \approx \frac{h}{3}[f(a) + 4(x_1) + 2f(x_2) + 4f(x_3) + \cdots + f(b)]$$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$

Error Formula Where M is the upperbound for the fourth derivative of $f(x)$

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$

Gaussian Quadrature

To convert $\int_a^b f(x)dx$ into the form $\int_{-1}^1 g(t)dt$, use the substitution

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

Then the Gaussian Quadrature formula is

$$\int_{-1}^{1} f(t)dt \approx w_1 f(t_1) + \cdots w_n f(t_n)$$

Table of Nodes and Coefficients

Order n	Nodes t_i	Coefficients w_i
1	0	2
2	−0.5773502692	1
	0.5773502692	1
3	−0.7745966692	0.555555556
	0	0.888888889
	0.7745966692	0.555555556
4	-0.8611363116	0.3478548451
	-0.3399810436	0.6521451549
	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	-0.9061798459	0.2369268850
	-0.5384693101	0.4786286705
	0.0	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850

Linear Algebra and Integrals

Linear Algebra

Eigen Value

$$|A - \lambda I| = |\lambda I - A| = 0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigen Vectors. An eigenvector $\vec{V}$ is a vector that satisfies

$$[A - \lambda I|\vec{0}]$$

Generalized Eigen Vector ρ The solution to, then pick specific value for the parameter t to get a specific vector, (i.e take $t = 0$, $t = 1$, etc).

$$[A - \lambda I|V]$$

Integrals

$$\int xe^{ax} = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}, \int \ln x = x\ln x - x$$

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax), \int \cos(ax)dx = \frac{1}{a}\sin(ax)$$

Example of Non-Homogeneous System

Example.

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix}\vec{y} + \begin{bmatrix} 9x - 51 \\ 7 + e^{2x} \end{bmatrix};\ \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution. Solve the corresponding homogeneous ODE

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix}\vec{y}$$

Find eigenvalues of A

$$\det\begin{bmatrix} 9-\lambda & 18 \\ -2 & -3-\lambda \end{bmatrix} = (9-\lambda)(-3-\lambda) + 36 = \lambda^2 - 6\lambda + 9$$

This gives us $\lambda = 3$ with multiplicity 2, find eigenvector $V = t\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ Find generalized eigenvector, set parameter $t = 0$,

$$[A - 3I|V] = \begin{bmatrix} 6 & 18 & -3 \\ -2 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{\rho} = \begin{bmatrix} -3t - 1/2 \\ t \end{bmatrix} \implies \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

The general solution to the homogeneous ODE is

$$\vec{y}_H = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}xe^{3x} + c_2\left(x\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}\right)e^{3x}$$
$$= \begin{bmatrix} -3c_1e^{3x} - 3c_2xe^{3x} - \frac{1}{2}c_2e^{3x} \\ c_1e^{3x} + c_2xe^{3x} \end{bmatrix}$$

For y_p, decompose $r(x)$ to get

$$r(x) = \begin{bmatrix} 9x - 51 \\ 7 + e^{2x} \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{2x}$$

Find y_p for each part of $r(x)$

$$y_p = Ux + V + We^{2x}$$

Rewrite the non-homogeneous system as

$$\vec{y}' = \begin{bmatrix} 9 & 18 \\ -2 & -3 \end{bmatrix}y + \begin{bmatrix} 9 \\ 0 \end{bmatrix}x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{2x}$$

Compute $\vec{y}'_p$ and plug into the system to solve for constant vectors, $\vec{y}'_p = U + 2We^{2x}$, $\vec{y}'_p = A\vec{y}_p + \vec{r}(x)$

$$U + 2We^{2x} = A(Ux + V + We^{2x}) + \begin{bmatrix} 9 \\ 0 \end{bmatrix}x + \begin{bmatrix} -51 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{2x}$$

This gives us the three equations

$$AU + \begin{bmatrix} 9 \\ 0 \end{bmatrix} = 0, AV + \begin{bmatrix} -51 \\ 7 \end{bmatrix} = U, AW + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2W$$

Solve equations to find

$$U = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, V = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, W = \begin{bmatrix} 18 \\ -7 \end{bmatrix}$$

This gives us our particular solution

$$y_p = \begin{bmatrix} 3 \\ -2 \end{bmatrix}x + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 18 \\ -7 \end{bmatrix}e^{2x} = \begin{bmatrix} 3x + 18e^{2x} \\ -2x + 3 - 7e^{2x} \end{bmatrix}$$

The general solution to the non-homogeneous ODE is

$$y = \begin{bmatrix} -3c_1e^{3x} - 3c_2xe^{3x} - \frac{1}{2}e^{3x} \\ c_1e^{3x} + c_2xe^{3x} \end{bmatrix} + \begin{bmatrix} 3x + 18e^{2x} \\ -2x + 3 - 7e^{2x} \end{bmatrix}$$

Extra Notes