Model Types

Typical structure of a time series is

$$X_t = m_t + Y_t + S_t$$

where m_t is a trend, S_t is a seasonal part, and Y_t is a stationary part.

- **White Noise:** Sequence of iid random variables $\{Z_t\}$ with mean 0 and variance σ_Z^2.
- **Random Walk:** $X_t = X_{t-1} + Z_t$, where $\{Z_t\}$ is white noise.
- **MA(q):** Let $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^2$,
$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} = \theta(B) Z_t$$
- **AR(p):** Let $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$,
$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t \implies Z_t = \phi(B) X_t$$
- **ARMA(p,q):** $\phi(B)X_t = \theta(B)Z_t$
$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

Autocovariance Functions

The autocovariance function of a time series $\{X_t\}$ is

$$\gamma_X(r,s) = \mathrm{Cov}(X_r, X_s) = E[(X_r - \mu_X(r))(X_s - \mu_X(s))]$$

ACF For Different Models

- **White Noise:** $\gamma_Z(t+h,t) = I(h=0)$
- **Random Walk:** $\gamma_S(t,t+h) = t\sigma_Z^2$
- **MA(1):** $\gamma_X(t+h,t) = \begin{cases} \sigma_Z^2(1+\theta^2) & h = 0 \\ \theta\sigma_Z^2 & h = \pm 1 \\ 0 & |h| > 1 \end{cases}$
- **AR(1):** $\gamma_X(h) = \sigma_Z^2 \dfrac{\phi^h}{1-\phi^2}$
- **ARMA(1,1):** $\gamma_X(0) = \sigma_Z^2\left(1 + \dfrac{(\phi+\theta)^2}{1-\phi^2}\right)$,
$$\gamma(1) = \sigma_Z^2\left[(\phi+\theta) + \frac{(\phi+\theta)^2\phi}{1-\phi^2}\right] \gamma(h) = \phi^{h-1}\gamma_X(1)$$
- **AR(2):** $\gamma_X(0) = \dfrac{\sigma_Z^2(1-\phi_2)}{(1+\phi^2)[(1-\phi_2)^2 - \phi_1^2]}$,
$$\gamma(1) = \frac{\gamma(0)\phi_1}{1-\phi_2},\ \gamma(h) = \phi_1\gamma(h-1) + \phi_2\gamma(h-2)$$
- **MA(q):** $\gamma_X(h) = \sigma_Z^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}$

ARMA Models

Solutions to ARMA Models

i) A stationary solution for ARMA(p,q) exists whenever the autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0$ for all $|z| = 1$. In other words, for ARMA to be stationary, if z solves $\phi(z) = 0$, then $|z| \neq 1$.

ii) A stationary and causal solution for ARMA(p,q) exists whenever the autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0$ for all $|z| \leq 1$. In other words, for ARMA to be stationary and causal, if z solves $\phi(z) = 0$, then $|z| > 1$.

Linear Processes

Definition Let $\{Z_t\}$ be white noise. Let $\psi_j,\ j \geq 0$, be a sequence of constants such that $\sum_{j=0}^\infty \psi_j^2 < \infty$. Then,
$$X_t = \sum_{j=0}^\infty \psi_j Z_{t-j}$$
is called a linear process.

Linear Representation of AR(1): AR(1) is obtained by setting $p = 1$ and $q = 0$, so $Z_t = \phi(B)X_t$ where $\phi(z) = 1 - \phi z$. Then, we define
$$\chi(z) = \frac{1}{\phi(z)}$$
This function has the power series expansion when $|\phi| < 1$,
$$\chi(z) = \frac{1}{\phi(z)} = \sum_{j=0}^\infty \phi^j z^j$$
Multiplying this with our equation for Z_t,
$$\chi(B)\phi(B)X_t = \chi(B)Z_t \iff X_t = \chi(B)Z_t$$
Thus, our linear representation of AR(1) is
$$X_t = \chi(B)Z_t = \sum_{j=0}^\infty \phi^j B^j Z_t = \sum_{j=0}^\infty \phi^j Z_{t-j}$$

Linear Representation of ARMA(1,1): Here we have $\phi(B)X_t = \theta(B)Z_t$ where $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$. Then define $\chi(z)$ the same way and multiply on both sides to get $X_t = \chi(B)\theta(B)Z_t$.
$$X_t = \sum_{j=0}^\infty \phi^j B^j (1+\theta B) Z_t = \sum_{j=0}^\infty \phi^j Z_{t-j} + \theta \sum_{j=0}^\infty \phi^j Z_{t-(j+1)}$$
We want to write this in the form:
$$\sum_{j=0}^\infty \psi_j Z_{t-j} = \sum_{j=0}^\infty \phi^j Z_{t-j} + \theta \sum_{j=0}^\infty \phi^j Z_{t-j-1}$$
$$\psi_0 Z_t + \sum_{j=1}^\infty \psi_j Z_{t-j} = \phi^0 Z_t + \sum_{j=1}^\infty (\phi^j + \theta\phi^{j-1}) Z_{j-1}$$
Hence $\psi_0 = 1,\ \psi_j = \phi^{j-1}(\theta+\phi),\ j \geq 1$.

Partial Autocorrelation Functions

Definition. The partial autocorrelation at lag h for a time series $\{X_t\}$, denoted $\alpha(h)$ is the autocorrelation between X_t and X_{t+h} after removing the effect of the intervening values $X_{t+1}, \ldots, X_{t+h-1}$. For lag 1 we have $\alpha(1) = \rho(1)$, and lag 2
$$\alpha(2) = \rho_{13.2} = \frac{\rho_X(2) - \rho_X^2(1)}{\sqrt{1-\rho_X^2(1)}\sqrt{1-\rho_X^2(1)}}$$

Interpreting PACF/ACF Plots

	AR(p)	MA(q)
PACF	Significant at each lag p Cuts off after lag p	Tails off (Geometric Decay)
ACF	Tails off (Geometric Decay)	Significant at lag q Cuts off after lag q

- Look at PACF plot first, if you have only p significant lags, then you have a pure AR(p) model. No need to look at ACF. If there is oscillating behavior for significant lags, indicates presence of MA part.
- If there was an MA(q) part, then look at ACF. If ACF becomes insignificant after lag q, then it is a pure MA(q) model. If it is significant past lag q, then it is a mixed ARMA model. No way to guess order of AR part, pick small $p \leq q$.

Yule Walker Equations

Denote $P_n X_{n+k}$ as the predicted value of X_{n+k} given that we have n observations.
$$P_n X_{n+k} = a_0 + a_1 X_n + a_2 X_{n-1} + \cdots a_n X_1$$
To find coefficients $a_0, \ldots, a_n$, solve the system
$$\Gamma_n \mathbf{a}_n = \gamma(n;k),\ \gamma(n;k) = (\gamma_X(k), \ldots, \gamma_X(k+n-1))$$
$$\Gamma_n = [\gamma_X(i-j)]_{i,j=1}^n,\ \mathbf{a}_n = (a_1, \ldots, a_n)^T$$
The mean squared prediction error is
$$\mathrm{MSPE}_n(k) = E\left[\left(X_{n+k} - \sum_{i=1}^n a_i X_{n+1-i}\right)^2\right]$$
$$= \gamma(0) - \mathbf{a}_n \gamma(n;k)$$

Durbin-Levinson Algorithm

$$P_n X_{n+1} = \phi_{n1} X_n + \cdots + \phi_{nn} X_1$$
$$\phi_{nn} = \left[\gamma_X(n) - \sum_{j=1}^{n-1} \phi_{n-1,j}\gamma_X(n-j)\right] v_{n-1}^{-1}$$
$$(\phi_{n,i})_{i=1}^{n-1} = (\phi_{n-1,i})_{i=1}^{n-1} - \phi_{nn}(\phi_{n-1,i})_{i=n-1}^1$$
$$v_n = v_{n-1}(1-\phi_{nn}^2), v_0 = \gamma_X(0), \phi_{11} = \rho_X(1)$$

Useful Formulas and Theorems

Autocovariance Using Linear Representation

$$\gamma_X(h) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

Evaluating Geometric Series

$$\sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r},\ |r| < 1$$

Yule-Walker and Durbin-Levinson for MA(1)

Yule-Walker procedure and Durbin-Levinson algorithm for MA(1) models $X_t = Z_t + \theta Z_{t-1}$, $\theta \in$, where Z_t are i.i.d. random variables with mean 0 and variance σ_Z^2.

a) Let $n = 2$. Using Yule-Walker equations to obtain coefficients a_1, a_2 in $P_2X_3 = a_1X_2 + a_2X_1$:

$$\Gamma_2 = \begin{pmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{pmatrix}$$

$$\Gamma_n \mathbf{a}_n = \gamma(n;1) \implies \Gamma_2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \gamma(2;1)$$

$$\implies \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \Gamma_n^{-1}\gamma(2;1)$$

$$\Gamma_2^{-1} = \frac{1}{\gamma_X^2(0) - \gamma_X^2(1)} \begin{pmatrix} \gamma_X(0) & -\gamma_X(1) \\ -\gamma_X(1) & \gamma_X(0) \end{pmatrix}$$

The vector $\gamma(2;1)$ is $\gamma(2;1) = (\gamma(1), \gamma(2))^T$. We have the system of equations

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\gamma_X^2(0) - \gamma_X^2(1)} \begin{pmatrix} \gamma_X(0) & -\gamma_X(1) \\ -\gamma_X(1) & \gamma_X(0) \end{pmatrix} \begin{pmatrix} \gamma_X(1) \\ \gamma_X(2) \end{pmatrix}$$

$$\implies \begin{cases} a_1 = \dfrac{\gamma_X(0)\gamma_X(1) - \gamma_X(1)\gamma_X(2)}{\gamma_X^2(0) - \gamma_X^2(1)} \\ a_2 = \dfrac{-\gamma_X(1)^2 + \gamma_X(0)\gamma_X(2)}{\gamma_X(0)^2 - \gamma_X(1)^2} \end{cases}$$

For the MA(1) model, the autocovariance at lag 2 is 0, so $\gamma_X(2) = 0$, and we are left with

$$a_1 = \frac{\gamma_X(0)\gamma_X(1)}{\gamma_X^2(0) - \gamma_X^2(1)},\ a_2 = -\frac{\gamma_X^2(1)}{\gamma_X^2(0) - \gamma_X^2(1)}$$

b) Applying Durbin-Levinson algo $P_2X_3 = \phi_{21}X_2 + \phi_{22}X_1$.

$$\begin{aligned} v_1 &= v_0[1 - \phi_{11}^2] = \gamma_X(0)[1 - \rho_X(1)] \\ &= \sigma_Z^2(1+\theta^2)\left(1 - \frac{\theta^2}{(1+\theta^2)^2}\right) \\ \phi_{22} &= [\gamma_X(2) - \phi_{11}\gamma_X(1)]v_1^{-1} = -v_1^{-1}\phi_{11}\gamma_X(1) \\ &= -\theta\sigma_Z^2 \frac{\theta}{1+\theta^2}\left(\frac{1+\theta^2}{\sigma_Z^2((1+\theta^2)^2 - \theta^2)}\right) \end{aligned}$$

$$\phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = \frac{\theta}{1+\theta^2} + \frac{\theta^2}{(1+\theta^2)^2 - \theta^2}\frac{\theta}{1+\theta^2}$$

Yule-Walker for AR(1)

Yule-Walker forecasting procedure for AR(p) models. Assume that Z_t are i.i.d random variables with mean 0 and variance σ_Z^2.

a) Apply the Yule-Walker procedure to obtain P_nX_{n+2} for AR(1) model. Compute $\text{MSPE}_n(2)$. Guess a general formula for P_nX_{n+k}. We guess the solution $\mathbf{a}_n = (\phi^2, \ldots, 0)^T$, check if its valid.

$$\begin{aligned} \gamma_X(n;2) &= (\gamma_X(2), \ldots, \gamma_X(n+1))^T \\ &= \frac{\sigma_Z^2}{1-\phi^2}(\phi^2, \ldots, \phi^{n+1})^T \end{aligned}$$

The variance covariance matrix Γ_n is

$$\begin{pmatrix} \gamma_X(0) & \gamma_X(1) & \cdots & \gamma_X(n-1) \\ \gamma_X(1) & \gamma_X(0) & \cdots & \gamma_X(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_X(n-1) & \gamma_X(n-2) & \cdots & \gamma_X(0) \end{pmatrix}$$

$$= \frac{\sigma_Z^2}{1-\phi^2} \begin{pmatrix} 1 & \phi & \cdots & \phi^{n-1} \\ \phi & 1 & \cdots & \phi^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \cdots & 1 \end{pmatrix}$$

Thus the Yule-Walker equations are

$$\frac{\sigma_Z^2}{1-\phi^2} \begin{pmatrix} 1 & \cdots & \phi^{n-1} & \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \cdots & 1 & \end{pmatrix} \begin{pmatrix} \phi^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{\sigma_Z^2}{1-\phi^2} \begin{pmatrix} \phi^2 \\ \vdots \\ \phi^{n+1} \end{pmatrix}$$

$\mathbf{a}_n$ is a solution since we end up with the proper covariance functions. The mean squared prediction error is

$$\text{MSPE}_n(2) = \gamma_X(0) - \mathbf{a}_n^T\gamma(n;k) = \gamma_X(0) - \phi^2\gamma_X(2)$$

b) Apply the Yule-Walker procedure to obtain P_nX_{n+1} for AR(2) model $X_t = \phi_1X_{t-1} + \phi_2X_{t-2} + Z_t$. Compute the corresponding $\text{MSPE}_n(1)$. **Solution.** Similarly, guess $\mathbf{a}_n = (\phi_1, \phi_2, \ldots, 0)^T$, verify it solves $\Gamma_n\mathbf{a}_n = \gamma(n;1)$. Then we find $P_nX_{n+1} = \phi_1X_n + \phi_2X_{n-1}$. The mean squared prediction error is

$$\text{MSPE}_n(1) = \gamma_X(0) - \phi_1\gamma_X(1) - \phi_2\gamma_X(2) = \sigma_Z^2$$

Recursive Method for ACF

We start with the AR(2) model. Then multiplying both sides by X_{t-h} and taking the expected value,

$$E(X_tX_{t-h}) = \phi_1E(X_{t-1}X_{t-h}) + \phi_2E(X_{t-2}X_{t-h})$$

This gives us the recursive formula for the covariance of AR(2) as

$$\gamma_X(h) = \phi_1\gamma_X(h-1) + \phi_2\gamma_X(h-2)$$

Then compute $\gamma_X(0), \gamma_X(1)$, Note $E(X_tZ_t) = \sigma_Z^2$

$$\begin{aligned} E(X_tX_{t-1}) &= \phi_1E(X_{t-1}X_{t-1}) + \phi_2E(X_{t-2}X_{t-1}) \\ &\iff \gamma_X(1) = \phi_1\gamma_X(0) + \phi_2\gamma_X(1) \\ E(X_tX_t) &= \phi_1E(X_tX_{t-1}) + \phi_2E(X_tX_{t-2}) + E(X_tZ_t) \\ \gamma_X(0) &= \phi_1\gamma_X(1) + \phi_2\gamma_X(2) + \sigma_Z^2 \end{aligned}$$

Rearranging and solving this system of equations for $\gamma_X(1)$ and $\gamma_X(0)$ gives

$$\gamma_X(h) = \begin{cases} \sigma_Z^2 \frac{1-\phi_2}{(1+\phi_2)((1-\phi_2)^2 - \phi_1^2)} & h = 0 \\ \sigma_Z^2 \frac{\phi_1}{(1+\phi_2)((1-\phi_2)^2 - \phi_1^2)} & h = 1 \\ \phi_1\gamma_X(h-1) + \phi_2\gamma_X(h-2) & h \geq 2 \end{cases}$$

Linear Representation for ARMA(1,2)

Here we have $\phi(B)X_t = \theta(B)Z_t$ with $\phi(z) = 1 - \phi z$, define

$$\chi(z) = \frac{1}{\phi(z)} = \frac{1}{1-\phi z} = \sum_{j=0}^{\infty} \phi^j z^j$$

Multiplying: $\chi(B)\ \chi(B)\underset{\theta}{\phi}(B)X_t = \chi(B)\theta(B)Z_t$ gives

$$\begin{aligned} X_t = \chi(z)\theta(B)Z_t &= \sum_{j=0} \phi^j B^j(1 + \theta_1B + \theta_2B^2)Z_t \\ &= \sum_{j=0}^{\infty} \phi^j Z_{t-j} + \sum_{j=0}^{\infty} \theta_1\phi^j Z_{t-j-1} + \sum_{j=0}^{\infty} \theta_2\phi^j Z_{t-j-2} \end{aligned}$$

We want to rewrite X_t in the form $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$, so

$$\sum_{j=0}^{\infty} \psi_j Z_{t-j} = \sum_{j=0}^{\infty} \phi^j Z_{t-j} + \sum_{j=0}^{\infty} \theta_1\phi^j Z_{t-j-1} + \sum_{j=0}^{\infty} \theta_2\phi^j Z_{t-j-2}$$

We can rewrite this as

$$\begin{aligned} \psi_0Z_t + \psi_iZ_{t-1} + \sum_{j=1}^{\infty} \psi_j Z_{t-j} &= \phi^0Z_t + (\theta + \phi_1)Z_{t-1} \\ &+ \sum_{j=2}^{\infty} (\phi^j + \theta_1\phi^{j-1} + \theta_2\phi^{j-2})Z_{t-j} \end{aligned}$$

Hence the linear representation of ARMA$(1,2)$ is

$$\psi_0 = 1,\ \psi_1 = \phi + \theta_1,\ \psi_j = \phi^{j-2}(\phi^2 + \theta_1\phi + \theta_2),\ j \geq 2$$