TD+

Consistance : $\left| \frac{\bar{y}(t+h) - \bar{y}(t)}{h} - \phi(t, \bar{y}(t), h) \right| \xrightarrow{h \to 0} 0$

Stabilité $\exists M > 0,\ \exists \bar{\varepsilon} > 0 \ /\ \forall h, \forall \varepsilon_i < \bar{\varepsilon},\ \max_k | y_k - z_k | \leq M . \max_{0 \leq i \leq N-1} |\varepsilon_i|$

Convergence : si $\forall y_0 \in \mathbb{R},\ \lim_{h \to 0} \max_k | y_k - \bar{y}(\quad | = 0$

1. $\left| \frac{\bar{y}(t+h) - \bar{y}(t)}{h} - \beta f(t_k, \bar{y}_k) - \gamma f(t_k + \alpha h, \bar{y}_k + \alpha h f(t_k, \bar{y}_k)) \right|$

$\xrightarrow[h \to 0]{} | \bar{y}'(t) - \beta f(t_k, \bar{y}_k) - \gamma f(t_k, \bar{y}_k) | = |(1 - \beta - \gamma) \bar{y}'(t)| = 0 \Leftrightarrow \underline{\beta + \gamma = 1}$

2. f est lipschitzienne, donc $\exists K \geq 0, \forall t \in [a,b], \forall y, z \in \mathbb{R}^n : \|f(t,y) - f(t,z)\| \leq K \|y - z\|$

Or, $\|\phi(t,y,h) - \phi(t,z,h)\| = \|\beta f(t,y) + \gamma f(t+\alpha h, y + \alpha h f(t,y))$
$- \beta f(t,z) - \gamma f(t+\alpha h, z + \alpha h f(t,z))\|$

$= \|\beta(f(t,y) - f(t,z)) + \gamma(f(\text{——}) - f(\text{——}))\|$

$\leq |\beta| \, \|f(t,y) - f(t,z)\| + |\gamma| \, \|f(\text{—}) - f(\text{—})\|$

$\leq |\beta| K \, \|y - z\| + |\gamma| K \, \|y + \alpha h f(t,y) - z - \alpha h f(t,z)\|$

$\leq |\beta| K \|y - z\| + |\gamma| K (\|y - z\| + |\alpha h| K \|f(t,y) - f(t,z)\|)$

$\leq \underbrace{K(|\beta| + |\gamma| + |\alpha h \gamma| K)}_{K'} \|y - z\|$

Donc ϕ est lipschitzienne.
De plus, ϕ est continue par op et compo

Donc la MPS est stable.

3. Stable & consistant donc convergent.

4. Erreur :

$\varepsilon(t) = \frac{y(t+h) - y(t)}{h} - \phi(t, y(t), h)$

$\frac{y(t+h) - y(t)}{h} = y'(t) + \frac{h}{2} f'(t) + o(h^2)$

$f(t + \alpha h, y(t) + \alpha h f(t)) = f(t + \alpha h, y(t + \alpha h) + o(h^2))$
$= f(t + \alpha h, y(t + \alpha h)) + o(h^2)$

D'où $\phi(t, y(t), h) = f(t) + \gamma \alpha h f'(t) + o(h^2)$

Donc $\varepsilon(t) = h f'(t) (\frac{1}{2} - \gamma \alpha) + o(h^2)$. Si $\gamma \alpha \neq \frac{1}{2}$ → ordre 1
Sinon → ordre 2.