Exercice 1

jeudi 9 février 2017 09:53

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-m)^2}{2\sigma^2}}$$

$$(g \circ f)' = f' \times g' \circ f$$

1) et 2) $f_X(x) = \frac{d}{dx} P(X < x)$

$$\frac{d}{dx} P(aX + b < x) = \frac{d}{dx} P\left(X < \frac{x-b}{a}\right) = \frac{1}{a} f_X\left(\frac{x-b}{a}\right) = \frac{1}{\sigma a\sqrt{2\pi}} e^{\frac{-\left(\frac{x-b}{a}-m\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma a\sqrt{2\pi}} e^{\frac{-(x-b-am)^2}{2(\sigma a)^2}} \Rightarrow \text{densité de loi } \mathcal{N}(am+b, a^2\sigma^2)$$

Appliquons pour $a = \frac{1}{\sigma}, b = \frac{-m}{\sigma}$:

Donc, $\frac{X-m}{\sigma}$ suit $\mathcal{N}\left(m\frac{1}{\sigma} - \frac{m}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0,1)$

3) $\phi(x) = \int_{-\infty}^{x} f_U(t)dt$

Par parité de f_U:

$$\phi(-x) = \int_{-\infty}^{-x} f_U(t)dt \underset{u=-t}{=} \int_{\infty}^{x} -f_U(-u)du = \int_{x}^{\infty} f_U(u)du$$

$$= \int_{-\infty}^{\infty} f_U(t)dt - \int_{-\infty}^{x} f_U(t)dt = 1 - \phi(x)$$

Autre solution :

$\phi(x) = P(X \leq x)$

$= P(-X \leq x) = P(X \geq -x)$

$= 1 - P(X \leq -x) = 1 - \varphi(-x)$

4) $P(X \leq 1) = P(\sigma U + m \leq 1) = P(2U + 3 \leq 1) = P(U \leq -1) = 1 - P(U \leq 1)$

$= 1 - 0.8413 = 0.1587$ (Voir tables en annexe, page 114)

5) $u_\alpha = \phi^{-1}\left(1 - \frac{\alpha}{2}\right)$ (Question 3)

$\phi(u_\alpha) = 1 - \frac{\alpha}{2} = 1 - \phi(-u_\alpha)$

$\phi(-u_\alpha) = \frac{\alpha}{2}$

$-u_\alpha = \phi^{-1}\left(\frac{\alpha}{2}\right)$

$P(U \in [-u_\alpha, u_\alpha]) = 1 - 2P(U \leq -u_\alpha) = 1 - \alpha$

6) $f_{\chi_n^2}(x) = \frac{2^{\frac{-n}{2}}}{\Gamma\left(\frac{n}{2}\right)} e^{\frac{-x}{2}} x^{\frac{n}{2}-1} \mathbb{1}_{\mathbb{R}^+}(x)$

$n = 1 \Rightarrow \frac{2^{\frac{-1}{2}}}{\sqrt{\pi}} e^{\frac{-x}{2}} x^{\frac{-1}{2}} \mathbb{1}_{\mathbb{R}^+}(x)$

$Y = U^2 \geq 0$
$\forall t \geq 0, P(Y \leq t) = P(U^2 \leq t) = P\left(-\sqrt{t} \leq U \leq \sqrt{t}\right)$
$= 1 - 2P\left(U \leq \sqrt{t}\right) = 1 - 2\phi\left(-\sqrt{t}\right)$
$\Rightarrow f_Y(t) = \frac{1}{2\sqrt{t}} \times 2f\left(\sqrt{t}\right)$
$= \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}}\ e^{\frac{-1}{2}\left(\sqrt{t}\right)^2} = f_{\chi_1^2}(t)$

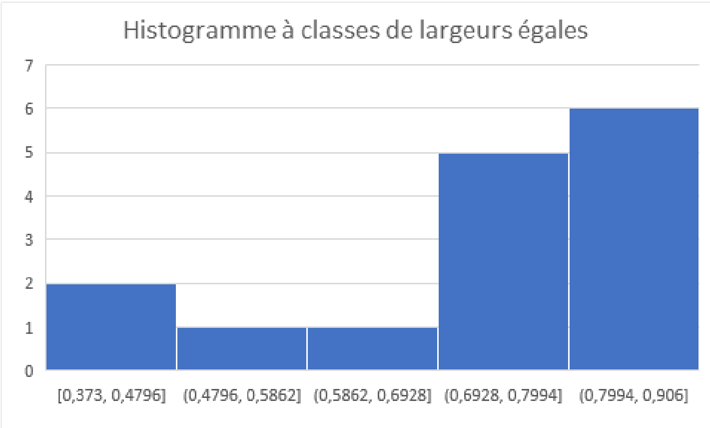
Exercice 2

jeudi 16 février 2017 10:29

1) On applique Sturges :
$k = 1 + \log_2 15 \approx 5$
$a_0 = 0.373, a_5 = 0.906$

Classes de même largeur :

classes	$]0.373, 0.480]$	$]0.480, 0.586]$	$]0.586, 0.693]$	$]0.693, 0.799]$	$]0.799, 0.906]$
n_i	2	1	1	5	6
$\frac{n_i}{n}$	13.3%	6.7%	6.7%	33.3%	40%
h_i	1.25	0.63	0.63	3.13	3.75



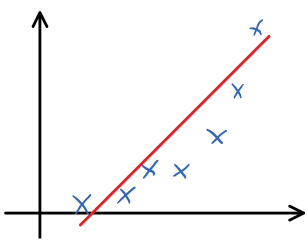
Classes de même effectif :

classes	$]0.373, 0.565]$	$]0.565, 0.721]$	$]0.721, 0.809]$	$]0.809, 0.861]$	$]0.861, 0.906]$
n_i	3	3	3	3	3
$\frac{n_i}{n}$	20%	20%	20%	20%	20%
h_i	1.04	1.28	2.27	3.85	4.44

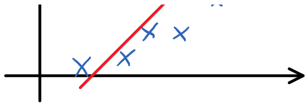
2) Les histogrammes ne sont pas compatibles avec une hypothèse $X_i \sim \mathcal{U}(0, \theta)$

$$F_{\mathcal{U}(0,\theta)}(x) = \frac{x}{\theta} \to \left(x_i^*, \frac{i}{n}\right)_{i=1,\dots,n}$$

$$\mathbb{F}_n(x_i^*) = \frac{i}{n} \approx l$$



$$\mathbb{F}_n(x_i^*) = \frac{i}{n} \approx l$$



$$\text{Pour } 0 \leq x \leq \theta, F(x) = \left(\frac{x}{\theta}\right)^c$$
$$\Rightarrow \ln\big(F(x)\big) = c \ln x - c \ln \theta$$
$$\underset{x=x_i^*}{\Longrightarrow} \ln\big(F(x_i^*)\big) \approx c \ln x_i^* - c \ln \theta$$
$$\ln\big(\mathbb{F}_n(x_i^*)\big) = \ln\left(\frac{i}{n}\right)$$

On trace :

$*$ Abscisse : $\ln(x_i^*)$

$*$ Ordonnée : $\ln\left(\frac{i}{n}\right)$