ENSIMAG – GRENOBLE INP – UGA
Modélisation statistique et analyse de données
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ANNÉE 2023-2024

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TD: usual results for random vectors

Exercise 1

Let U and V be two independent random variables with uniform distribution over $[0, 1]$.

Let $X = U + V$ and $Y = U - V$.

1. Compute the expectation and covariance matrix of $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$.
2. Prove that X and Y are uncorrelated but not independent.

Exercise 2

Let X be a random vector in $\mathbb{R}^n$ and A be a deterministic $m \times n$ matrix.

1. Prove that
$$K_X = E\left[(X - \mathbb{E}[X])\ (X - \mathbb{E}[X])^T\right] = E\left[XX^T\right] - E[X]\ \ E[X]^T\ .$$
2. Prove that
$$K_{AX} = A\ K_X\ A^T\ .$$
3. Use the result obtained in (b) to derive again the results of Exercise 1.

Exercise 3

Let $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ be a Gaussian vector with mean $\mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and covariance $\Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

1. Compute the probability density function of Z.
2. Using $f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}$, compute the distribution of Y given $X = x$.
3. What is the best prediction of Y given $X = x$?

Exercise 4

Let $V = \begin{pmatrix} X \\ Y \end{pmatrix}$ be a Gaussian vector in $\mathbb{R}^2$. Let $Z = Y - E[Y] - \frac{Cov[X,Y]}{Var[X]}(X - E[X])$.

1. Compute $\mathbb{E}[Z]$ and $\mathrm{Var}[Z]$.
2. Prove that X and Z are independent.
3. Derive the distribution of Y given $X = x$.
4. Use (c) to derive again the result of Exercise 3.