Ex1

1 - $f(x) \neq f(y)$ donc $T(x,y) \neq 0$. On définit alors $\varphi : (x,y) \mapsto x - \dfrac{f(x)}{T(x,y)}$.
Ainsi :
$x_{k+1} = \varphi(x_k, x_{k-1}) = \varphi(x_k, y_k)$.

D'où : $z_{k+1} = \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \varphi(x_k, y_k) \\ x_k \end{pmatrix} = \phi(z_k)$

Donc $\boxed{\phi : (x,y) \mapsto (\varphi(x,y), x)^T}$.

2 - $\lim\limits_{y \to x} T(x,y) = f'(x)$ donc T est C^1 au voisinage de (a,a) (f étant C^2).
φ est C^1 par opérations de telles fonctions et car $f'(a) \neq 0$.
Donc ϕ est ~~continue~~ C^1 au voisinage de (a,a).

3 -
$D\phi(a,a) = \begin{pmatrix} \dfrac{\partial\varphi}{\partial x}(a,a) & \dfrac{\partial\varphi}{\partial y}(a,a) \\ \dfrac{\partial x}{\partial x}(a,a) & \dfrac{\partial x}{\partial y}(a,a) \end{pmatrix}$

Or $\dfrac{\partial\varphi}{\partial x} = 1 - \dfrac{f'T - \frac{\partial T}{\partial x} f}{T^2}$ et $\dfrac{\partial\varphi}{\partial y} = \dfrac{f \frac{\partial T}{\partial y}}{T^2}$

$\dfrac{\partial\varphi}{\partial x}(a,a) = 0$. $\dfrac{\partial\varphi}{\partial y}(a,a) = 0$

Donc : $\boxed{D\phi(a,a) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$.

$\rho(D\phi(a,a)) = 0 < 1$ et : $\|(x_1, y_1) - (a,a)\| \leq \|x_1 - a\| + \|x_0 - a\| \longrightarrow 0$

Donc $\|(x_1, y_1) - (a,a)\| \to 0$ donc $\phi(x_k, y_k) = \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} \xrightarrow[k \to +\infty]{} \begin{pmatrix} a \\ a \end{pmatrix}$
$\lim\limits_{k \to +\infty} x_k = a$. ↙? ↗?

Ex2 :

1 - $x_{k+1} = x_k - \dfrac{f(x)}{f'(x)}$ $\left(f'(x) = -\dfrac{1}{x^2} \neq 0 \right)$

$\dfrac{f(x)}{f'(x)} = -x^2 \left(\dfrac{1}{x} - a \right) = -(x - ax^2)$

On a alors $x_{k+1} = x_k + x_k - a x_k^2 = x_k(2 - a x_k) (= \phi(x_k))$

2 - Cas : $x_0 \in]0, \frac{1}{a}]$, $]\frac{1}{a}, \frac{2}{a}[$, $x_0 = 0$, $x_0 < 0$, $x_0 > \frac{2}{a}$.