1 Feuille1

Exercice 1

1) Rappel : Formule de Taylor avec reste de Lagrange
Si $f \in \ C^{n+1}([a,b] \in \mathbb{R})\ et\ x,\ x+h \in [a,b]\ alors\ \exists\theta \in [0,1]$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f^{(2)}(x) + ... + \frac{h^n}{n!}f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x+\theta h)$$

$$E_i = \frac{1}{h^2}(A\tilde{U}_i) + F_i = -\frac{1}{h^2}[\tilde{U}_{i+1} - 2\tilde{U}_i + \tilde{U}_{i-1}] + q_i\tilde{U}_i - f_i$$

$$E_i = -\frac{1}{h^2}[u(x_i+h) - 2u(x_i) + u(x_i-h)] + q(x_i)u(x_i) - f(x_i)$$

$$u(x_i+h) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u^{(2)}(x_i) + \frac{h^3}{6}u^{(3)}(x_i) + \frac{h^4}{24}u^{(4)}(x_i + \theta_i^+ h)$$

$$u(x_i-h) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u^{(2)}(x_i) - \frac{h^3}{6}u^{(3)}(x_i) + \frac{h^4}{24}u^{(4)}(x_i + \theta_i^- h)$$

$$\theta_i^+ h, \theta_i^- h \in]0,1[$$

$$u(x_i+h) - 2u(x_i) + u(x_i-h) = h^2u^{(2)}(x_i) + \frac{h^4}{24}[u^{(4)}(x_i+\theta_i^- h) + u^{(4)}(x_i+\theta_i^+ h)]$$

On obtient alors pour E_i :

$$E_i = -u^{(2)}(x_i) + q(x_i)u(x_i) - f(x_i) + \frac{h^2}{24}[u^{(4)}(x_i+\theta_i^- h) + u^{(4)}(x_i+\theta_i^+ h)]$$

$$Or\ \ -u^{(2)}(x_i) + q(x_i)u(x_i) - f(x_i) = 0\ Cf\ (1)$$

$$Donc\ \forall i\ |E_i| = \frac{h^2}{24}|u^{(4)}(x_i+\theta_i^- h) + u^{(4)}(x_i+\theta_i^+ h)|$$

$$|E_i| \leq \frac{h^2}{24}x2\underset{[0,1]}{Sup}|u^{(4)}|$$

$$||E||_\infty \leq \frac{h^2}{12}||u^{(4)}||_\infty$$

2) Suggestion : Montrer que $(Ax = y) \implies ||x||_\infty \leq \frac{||y||_\infty}{\delta}$ On note (P) cette propriété. Cela implique M inversible.
$(Mx = 0 \implies ||x||_\infty \leq 0)\ donc\ ||x||_\infty = 0 \implies x = 0$
De plus cela implique

$$|||M|||_\infty \leq \frac{1}{\delta} \implies |||M^{-1}|||_\infty = \underset{y\neq 0}{Sup}\frac{||M^{-1}y||_\infty}{||y||_\infty} \leq \frac{1}{\delta}$$

Pour montrer (P) :

$$(Mx = y) \iff \forall i y_i = \sum_{j=1}^{N} m_{ij}x_j$$

$$\forall i|y_i| = |m_{ii}x_i - \sum_{j\neq i} -m_{ij}x_j|$$

$$y_i \geq |m_{ii}x_i| - |\sum_{j\neq i} -m_{ij}x_j|$$

$$y_i \geq |m_{ii}x_i| - \sum_{j\neq i} |m_{ij}||x_j|$$

$$y_i \geq (|m_{ii}x_i| - \sum_{j\neq i} |m_{ij}|)||x||_\infty$$

$$\exists i_0 \in 1,...,N \ : ||x||_\infty = |x_{i_0}$$

$$|y_{i_0}| \geq (|m_{i_0i_0}x_i| - \sum_{j\neq i_0} |m_{i_0j}|)||x||_\infty$$

$$||y||_\infty \geq \underset{i}{Min}((|m_{ii}x_i| - \sum_{j\neq i} |m_{ij}|)||x||_\infty)$$

$$||x||_\infty \leq \frac{1}{\delta}||y||_\infty$$

$$avec\ \delta = \underset{i}{Min}((|m_{ii}x_i| - \sum_{j\neq i} |m_{ij}|)||x||_\infty) > 0$$

Remarque :

$$|||M|||_\infty = \underset{1\leq i\leq n}{max}\sum_{j=1}^{n} |m_{ij}| = \underset{x\neq 0}{Sup}\frac{||Mx||_\infty}{||x||_\infty}$$

3) On suppose $q > 0$ sur $[0, 1]$
Notons $q_- = \underset{[0,1]}{Inf} q, q_- > 0$

$$\delta = Min(h^2q_2, ..., h^2q_{N-1}, 1 + h^2q_1, .., 1 + h^2q_N) \geq h^2q_-$$

$$|||A^{-1}|||_\infty \leq \frac{1}{\delta} \leq \frac{1}{h^2q_-}$$

4)On a une solution exacte aux noeuds x_i

$$\frac{1}{h^2}A\tilde{U} = F + E$$

$$\frac{1}{h^2}AU = F$$

U est la solution numérique.

$$\frac{1}{h^2}A(\tilde{U} - U) = E \iff \tilde{U} - U = h^2A^{-1}E$$

$$||\tilde{U} - U||_\infty \leq h^2|||A^{-1}|||_\infty||E||_\infty \leq h^2\frac{1}{h^2q_-}\frac{h^2}{12}||u^{(4)}||_\infty$$

$$\underset{1\leq i\leq N}{Max}|U(x_i) - \tilde{U}(x_i)| \leq h^2\frac{||u^{(4)}||_\infty}{12q_-}$$

On a un schéma de convergence d'ordre 2.