Lecture NO(3)
Dimensional Analysis

Learning Objectives

After completing this lecture, you should be able to:

■ apply the Buckingham pi theorem.

■ develop a set of dimensionless variables for a given flow situation.

■ discuss the use of dimensionless variables in data analysis.

■ apply the concepts of modeling and similitude to develop prediction equations.

3.1 Introduction

Any phenomenon is physical sciences and engineering can be described by the fundamentals dimensions mass, length, time, and temperature. Till the rapid development of science and technology the engineers and scientists depend upon the experimental data. But the rapid development of science and technology has created new mathematical methods of solving complicated problems, which could not have been solved completely by analytical methods and would have consumed enormous time. This mathematical method of obtaining the equations governing certain natural phenomenon by balancing the fundamental dimensions is called (Dimensional Analysis). Of course, the equation obtained by this method is known as (Empirical Equation).

3.2 Fundamentals Dimensions

The various physical quantities used by engineer and scientists can be expressed in terms of fundamentals dimensions are: Mass (M), Length (L), Time (T), and Temperature (ϴ). All other quantities such as area, volume, acceleration, force, energy, etc., are termed as " derived quantities".

3.3 Dimensional Homogeneity

An equation is called "dimensionally homogeneous" if the fundamentals dimensions have identical powers of [L T M] (i.e. length, time, and mass) on both sides. Such an equation be independent of the system of measurement (i.e. metric, English, or S.I.). Let consider the common equation of volumetric flow rate,

$$Q = Au \rightarrow L^3T^{-1} = L^2L\,T^{-1} = L^3T^{-1}$$

We see, from the above equation that both right and left hand sides of the equation have the same dimensions, and the equation is therefore dimensionally homogeneous.

Example (3.1)

a) Determine the dimensions of the following quantities in M-L-T system 1- force 2- pressure 3- work 4- power.

b) Check the dimensional homogeneity of the following equation

$$u_2 = \sqrt{2g(z_1 + z_2)}$$

Solutions

a)

1- F = m.g (kg.m/s^2) ≡ [MLT^{-2}]

2- P=F/A ≡ [(MLT^{-2}) (L^{-2})] (Pa) ≡ [ML^{-1}T^{-2}]

3- Work = F.L ≡ [(MLT^{-2}) (L)] (N.m) ≡ [ML^{2}T^{-2}]

4- Power = Work/time ≡ [(ML^{2}T^{-2}) (T-1)] (W) ≡ [ML^{-1}T^{-2}]

b)

L.H.S. u ≡ LT^{-1} $\quad R.H.S\ u \equiv (L\,T^{-2}L)^2 \equiv$ **LT^{-1}**

Since the dimensions on both sides of the equation are same, therefore the equation is dimensionally homogenous.

3.4 Methods of Dimensional Analysis

Dimensional analysis, which enables the variables in a problem to be grouped into form of dimensionless groups. Thus reducing the effective number of variables. The method of dimensional analysis by providing a smaller number of independent groups is most helpful to experimenter. Many methods of dimensional analysis are available; two of these methods are given here, which are:

1- Rayleigh's method (or Power series)

2- Buckingham's method (or π-Theorem)

3.4.1 Rayleigh's method (or Power series)

In this method, the functional relationship of some variable is expressed in the form of an exponential equation, which must be dimensionally homogeneous. If (y) is some function of independent variables$(x_1, x_2, x_3 \dots\dots\dots etc)$, then functional relationship may be written as;

$$\mathbf{y = f(x_1, x_2, x_3 \dots\dots\dots etc)}$$

The dependent variable (y) is one about which information is required; whereas the independent variables are those, which govern the variation of dependent variables.

The Rayleigh's method is based on the following steps:-

1- First of all, write the functional relationship with the given data.

2- Now write the equation in terms of a constant with exponents i.e. powers a, b, c,...

3- With the help of the principle of dimensional homogeneity, find out the values of a, b, c, … by obtaining simultaneous equation and simplify it.

4- Now substitute the values of these exponents in the main equation, and simplify it.

Example (3.2)

If the capillary rise (h) depends upon the specific weight (sp.wt) surface tension (σ) of the liquid and tube radius (r) show that:

$$\boldsymbol{h = r\emptyset\left(\frac{\sigma}{\text{sp. wt.}\, r^2}\right); where\ \emptyset\ is\ any\ function}$$

Solution:

Capillary rise (h)m $\equiv [\boldsymbol{L}]$

Specific weight $(\boldsymbol{sp.wt})\dfrac{\boldsymbol{N}}{\boldsymbol{m^3}}$ $(\boldsymbol{MLT^{-2}L^{-3}}) \equiv (\boldsymbol{ML^{-2}T^{-2}})$

Surface tension $\equiv \dfrac{\boldsymbol{N}}{\boldsymbol{M}}$, $(\boldsymbol{MLT^{-2}L^{-1}}) \equiv [\boldsymbol{MT^{-2}}]$

Tube radius (r)m $\equiv [L]$

$$h = f(\mathbf{sp.\,wt.}, \boldsymbol{\sigma}, \mathbf{r})$$
$$h = k\left(\mathbf{sp.\,wt}^{\boldsymbol{a}}., \boldsymbol{\sigma}^{\boldsymbol{b}}, \mathbf{r}^{c}\right)$$

$$[L] = \left([ML^{-2}T^{-2}]^{a}.,[MT^{-2}]^{b},[L]^{c}\right)$$

Now by the principle of dimensional homogeneity, equating the power of M, L, T on both sides of the equation

$$For \quad M \qquad 0 = a + b = a = -b$$

$$For \quad L \qquad 1 = -2a + c = 2a = c - 1 = -2b = c - 1 \therefore c = 1 - 2b$$

$$For \quad T \qquad 0 = -2a - 2b = 2a = -2b = a = -b$$

$$h = k\left(\mathbf{sp.wt}^{-b}.,\boldsymbol{\sigma}^{b},\mathbf{r}^{1-2b}\right)$$

$$h = k\,r\left(\frac{\sigma}{\mathrm{sp.wt}.\,r^{2}}\right)^{b} \therefore h = r\emptyset\left(\frac{\sigma}{\mathrm{sp.wt}.\,r^{2}}\right)$$

3.4.2 Buckingham's method (or π-Theorem)

It has been observed that the Rayleigh's method of dimensional analysis becomes cumbersome, when a large number of variables are involved. In order to overcome this difficulty, the Buckingham's method may be convenient used. It states that " If there are (n) variables in a dimensionally homogeneous equation, and if these variables contain (m) fundamental dimensions such as (MLT) they may be grouped into (n-m) non-dimensional independent π-terms".

Mathematically, if a dependent variable X1 depends upon independent variables (X_2, X_3, X_4, ……….X_n), the functional equation may be written as:

$$\mathbf{X_1} = \mathbf{k}(\mathbf{X_2},\mathbf{X_3},\mathbf{X_4},\ldots\ldots\ldots\ldots..\mathbf{X_n})$$

This equation may be written in its general form as;

$$f(\mathbf{X_2},\mathbf{X_3},\mathbf{X_4},\ldots\ldots\ldots\ldots..\mathbf{X_n}) = 0$$

In this equation, there are n variables. If there are m fundamental dimensions, then according to Buckingham's π -theorem;

$$f_1(\pi_2,\pi_3,\pi_4,\ldots\ldots\ldots\ldots..\pi_{n-m}) = 0$$

The Buckingham's π -theorem is based on the following steps:

Step 1 **List all the variables that are involved in the problem.**

Step 2 **Express each of the variables in terms of basic dimensions.**

Step 3 **Determine the required number of pi terms.**

Step 4 **Select a number of repeating variables, where the number required is equal to the number of reference dimensions (usually the same as the number of basic dimensions).**

Step 5 **Form a pi term by multiplying one of the nonrepeating variables by the product of repeating variables each raised to an exponent that will make the combination dimensionless.**

Step 6 **Repeat Step 5 for each of the remaining nonrepeating variables.**

Step 7 **Check all the resulting pi terms to make sure they are dimensionless and independent.**

Step 8 **Express the final form as a relationship among the pi terms and think about what it means**

.**Note:-**

Any π -term may be replaced by any power of it, because the power of a non-dimensional term is also non-dimensional e.g. π_1 may be replaced by $\pi_1^2, \pi_1^3, \pi_1^{0.5} \ldots\ldots\ldots.\ or\ by\ 2\pi_1, 3\pi_1, \frac{\pi_1}{2}\ , \ldots\ldots\ldots\ldots.etc$

2.4.2.1 Selection of repeating variables

In the previous section, we have mentioned that we should choose (m) repeating variables and write separate expressions for each π-term.

Though there is no hard or fast rule for the selection of repeating variables, yet the following points should be borne in mind while selecting the repeating variables:

1. The variables should be such that none of them is dimension les.

2. No two variables should have the same dimensions.

3. Independent variables should, as far as possible, be selected as repeating variables.

4. Each of the fundamental dimensions must appear in at least one of the m variables.

5. It must not possible to form a dimensionless group from some or all the variables within the repeating variables. If it were so possible, this dimensionless group would, of course, be one of the π-term.

6. In general the selected repeating variables should be expressed as the following: (1) representing the flow characteristics, (2), representing the geometry and (3) representing the physical properties of fluid.

7. In case of that the example is held up, then one of the repeating variables should be changed.

Example -2.5-

By dimensional analysis, obtain an expression for the drag force (F) on a partially submerged body moving with a relative velocity (u) in a fluid; the other variables being the linear dimension (L), surface roughness (e), fluid density (ρ), and gravitational acceleration (g).

Solution:

Drag force (F) N ≡ $[MLT^{-2}]$

Relative velocity (u) m/s ≡ $[LT^{-1}]$

Linear dimension (L) m ≡ $[L]$

Surface roughness (e) m ≡ $[L]$

Density (ρ) kg/m³ ≡ $[ML^{-3}]$

Acceleration of gravity (g) m/s²≡ $[LT^{-2}]$

$$F = k(u, L, e, \rho, g) = f(u, L, e, \rho, g) = 0$$

n = 6, m = 3, □□□ = n – m = 6 – 3 = 3

$$n = 6\,, m = 3 \Rightarrow \pi = n - m = 6 - 3 = 3$$

No. of repeating variables = m = 3; The selected repeating variables is (u, L, ρ)

$$\pi_1 = u^{a1}L^{b1}\rho^{c1}F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\pi_2 = u^{a2}L^{b2}\rho^{c2}\, e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\pi_3 = u^{a3}L^{b3}\rho^{c3}g \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

For $\pi_1$equation (1)

$$[M^0L^0T^0] = [LT^{-1}]^{a1}[L]^{b1}[ML^{-3}]^{c1}[MLT^{-2}]$$

Now applied dimensional homogeneity

$$For\ M \qquad 0 = c_1 + 1 \Rightarrow c_1 = -1$$

$$For\ T \qquad 0 = -a_1 - 2 \Rightarrow a_1 = -2$$

$$For\ L \qquad 0 = a_1 + b_1 - 3c_1 + 1 \Rightarrow -2 + b_1 + 3 + 1 = 0 \therefore b_1 = -2$$

$$\pi_1 = u^{-2}L^{-2}\rho^{-1} \Rightarrow \pi_1 = \frac{F}{u^2\, L^2\, \rho}$$

For $\pi_2$equation (2)

$$[M^0L^0T^0] = [LT^{-1}]^{a2}[L]^{b2}[ML^{-3}]^{c2}[L]$$

Now applied dimensional homogeneity

$$For\ M \qquad 0 = c_2 + 0 \Rightarrow c_1 = 0$$

$$For\ T \qquad 0 = -a_2 + 0 \Rightarrow a_1 = 0$$

$$For\ L \qquad 0 = a_2 + b_2 - 3c_2 + 1 \Rightarrow 0 + b_2 + 0 + 1 = 0 \therefore b_2 = -1$$

$$\pi_2 = u^0L^{-1}\rho^0 \Rightarrow \pi_2 = \frac{e}{L}$$

For $\pi_3$equation (3)

$$[M^0L^0T^0] = [LT^{-1}]^{a3}[L]^{b3}[ML^{-3}]^{c3}[LT^{-2}]$$

$$For\ M \qquad 0 = c_3 + 0 \Rightarrow c_3 = 0$$

$$For\ T \qquad 0 = -a_3 - 2 \Rightarrow a_3 = -2$$

$$For\ L \qquad 0 = a_3 + b_3 - 3c_3 + 1 \Rightarrow -2 + b_3 + 0 + 1 = 0 \therefore b_3 = 1$$

$$\pi_3 = u^{-2}L\,\rho^0 \Rightarrow \pi_3 = \frac{gL}{u^2}$$

$$f(\pi_1, \pi_2, \pi_3) = 0 \Rightarrow f_1\left(\frac{F}{u^2\, L^2\, \rho}, \frac{e}{L}, \frac{gL}{u^2}\right) = 0$$

$$\therefore F = u^2\, L^2\, \rho \quad f\left(\frac{e}{L}, \frac{gL}{u^2}\right)$$

2.5 Dimensions of some important variables

Item	Property	Symbol	SI Units	M.L.T.
1-	Velocity	u	m/s	LT^{-1}
2-	Angular velocity	ω	Rad/s, Deg/s	T^{-1}
3-	Rotational velocity	N	Rev/s	T^{-1}
4-	Acceleration	a, g	m/s^2	LT^{-2}
5-	Angular acceleration	α	s^{-2}	T^{-2}
6-	Volumetric flow rate	Q	m^3/s	L^3T^{-1}
7-	Discharge	Q	m^3/s	L^3T^{-1}
8-	Mass flow rate	$\dot{m}$	kg/s	MT^{-1}
9-	Mass (flux) velocity	G	$kg/m^2.s$	$ML^{-2}T^{-1}$
10-	Density	ρ	kg/m^3	ML^{-3}
11-	Specific volume	υ	m^3/kg	L^3M
12-	Specific weight	sp.wt	N/m^3	$ML^{-2}T^{-2}$
13-	Specific gravity	sp.gr	[-]	[-]
14-	Dynamic viscosity	μ	kg/m.s, Pa.s	$ML^{-1}T^{-1}$
15-	Kinematic viscosity	ν	m^2/s	L^2T^{-1}