

## Lecture NO (4)

### Forces in Static Fluids

This section will study the forces acting on or generated by fluids at rest.

#### Objectives

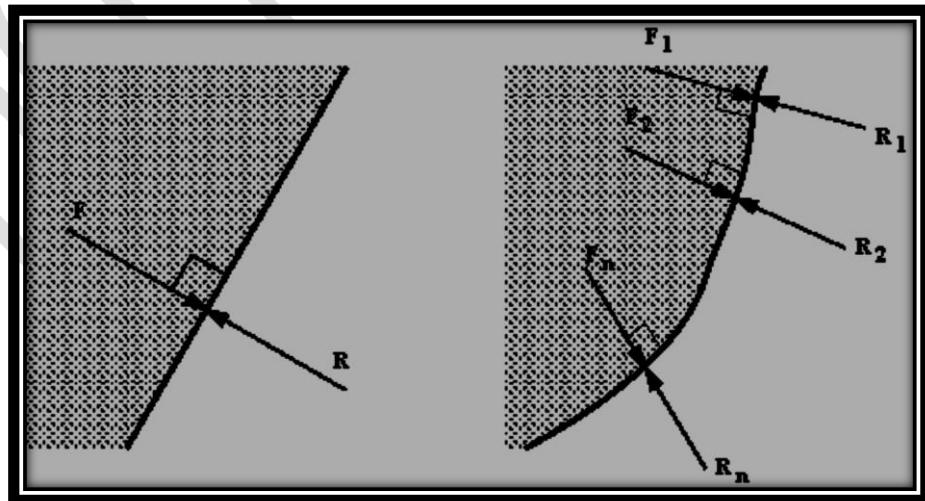
- Introduce the concept of pressure;
- Prove it has a unique value at any particular elevation;
- Show how it varies with depth according to the hydrostatic equation and
- Show how pressure can be expressed in terms of *head* of fluid.

This understanding of pressure will then be used to demonstrate methods of pressure measurement that will be useful later with fluid in motion and also to analyses the forces on submerge surface/structures.

#### (4.1) Fluids statics

The general rules of statics (as applied in solid mechanics) apply to fluids at rest. From earlier we know that:

- A static fluid can have no shearing force acting on it, and that
- Any force between the fluid and the boundary must be acting at right angles to the boundary.



Pressure force normal to the boundary

Note that this statement is also true for curved surfaces, in this case the force acting at any point is normal to the surface at that point. The statement is also true for any imaginary plane in a static fluid. We use this fact in our analysis by considering elements of fluid bounded by imaginary planes.

We also know that:

- For an element of fluid at rest, the element will be in equilibrium - the sum of the components of forces in any direction will be zero.
- The sum of the moments of forces on the element about any point must also be zero.

It is common to test equilibrium by resolving forces along three mutually perpendicular axes and also by taking moments in three mutually perpendicular planes and to equate these to zero.

#### (4.1.1) Pressure

As mentioned above a fluid will exert a normal force on any boundary it is in contact with. Since these boundaries may be large and the force may differ from place to place it is convenient to work in terms of pressure,  $p$ , which is the force per unit area.

If the force exerted on each unit area of a boundary is the same, the pressure is said to be *uniform*.

$$\text{Pressure} = \frac{\text{Force}}{\text{Area over which the force is applied}}$$

$$P = \frac{F}{A}$$

Units: Newton's per square metre,  $N m^{-2}$ ,  $kgm^{-1} s^{-2}$ .

(The same unit is also known as a Pascal,  $Pa$ , i.e.  $1Pa = 1 N m^{-2}$ )

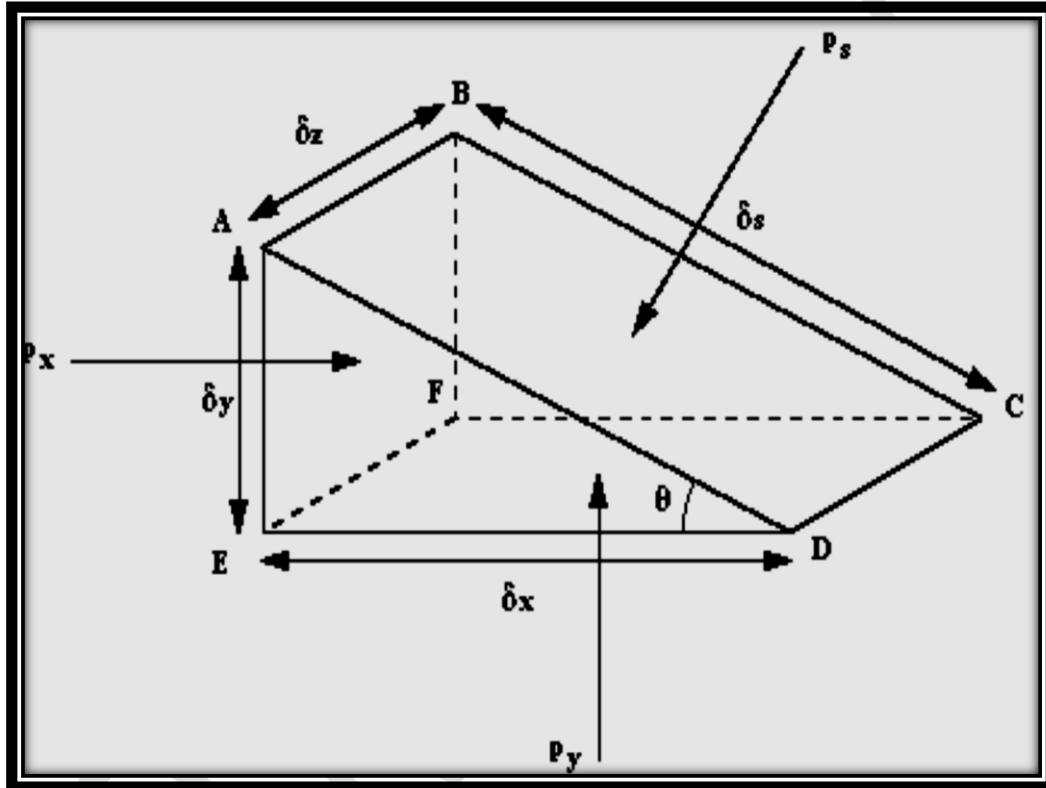
(Also frequently used is the alternative SI unit the *bar*, where  $1bar = 10^5 N m^{-2}$ )

Dimensions:  $ML^{-1}T^{-2}$ .

### (4.1.2) Pascal's Law for Pressure at A Point

*(Proof that pressure acts equally in all directions.)*

By considering a small element of fluid in the form of a triangular prism which contains a point P, we can establish a relationship between the three pressures  $p_x$  in the x direction,  $p_y$  in the y direction and  $p_s$  in the direction normal to the sloping face



Triangular prismatic element of fluid

The fluid is at rest, so we know there are no shearing forces, and we know that all forces are acting at right angles to the surfaces i.e.

$p_s$  acts perpendicular to surface ABCD,

$p_x$  acts perpendicular to surface ABFE and

$p_y$  acts perpendicular to surface FECD.

And, as the fluid is at rest, in equilibrium, the sum of the forces in any direction is zero.

Summing forces in the x-direction:

Force due to  $p_x$ ,

$$F_{xx} = P_x \times \text{Area}_{ABFE} = P_x \delta_z \delta_y$$

**Component of force in the x-direction due to  $p_s$ ,**

$$\begin{aligned} F_{xs} &= -P_s \times \text{Area}_{ABCD} \sin\theta \\ &= -P_s \delta_s \delta_z \frac{\delta_y}{\delta_s} \\ &= -P_s \delta_z \delta_y \\ \sin\theta &= \frac{\delta_y}{\delta_s} \end{aligned}$$

**Component of force in x-direction due to  $p_y$ ,**

$$F_{xy} = 0$$

**To be at rest (in equilibrium)**

$$\begin{aligned} F_{xy} + F_{xs} + F_{xx} &= 0 \\ P_x \delta_z \delta_y + (-P_s \delta_z \delta_y) &= 0 \\ P_x &= P_s \end{aligned}$$

**Similarly, summing forces in the y-direction. Force due to  $p_y$ ,**

$$\begin{aligned} F_{yy} &= P_y \times \text{Area}_{ABCD} = P_y \delta_x \delta_z \\ F_{ys} &= -P_s \times \text{Area}_{ABCD} \cos\theta \\ &= -P_s \delta_s \delta_z \frac{\delta_x}{\delta_s} \\ &= -P_s \delta_z \delta_x \\ \cos\theta &= \frac{\delta_x}{\delta_s} \end{aligned}$$

**Component of force due to  $p_x$ ,**

$$F_{yx} = 0$$

Force due to gravity,

$$\text{Weight} = - \text{Specific Weight} \times \text{Volume of Element}$$

$$= -\rho g \times \frac{1}{2} \delta_x \delta_y \delta_z$$

To be at rest (in equilibrium)

$$F_{yy} + F_{ys} + F_{xy} = 0$$

$$P_y \delta_x \delta_z + (-P_s \delta_z \delta_x) + \left( -\rho g \times \frac{1}{2} \delta_x \delta_y \delta_z \right) = 0$$

The element is small i.e.  $\delta_x$ ,  $\delta_y$  and  $\delta_z$  are small, and so  $\delta_x \delta_y \delta_z$  is very small and considered negligible, hence

$$P_y = P_s$$

Thus

$$P_y = P_s = P_x$$

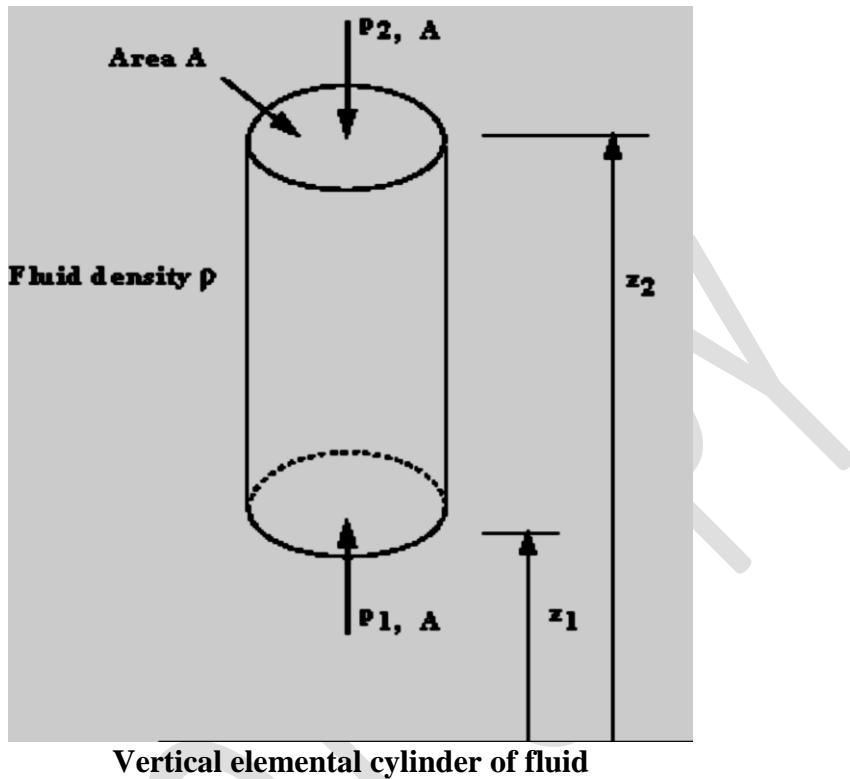
Considering the prismatic element again,  $p_s$  is the pressure on a plane at any angle  $\theta$ , the x, y and z directions could be any orientation. The element is so small that it can be considered a point so the derived expression  $P_y = P_s = P_x$ . Indicates that pressure at any point is the same in all directions.

(The proof may be extended to include the z axis).

*Pressure at any point is the same in all directions.*

*This is known as Pascal's Law and applies to fluids at rest.*

### (4.1.3) Variation of Pressure Vertically in a Fluid under Gravity



Vertical elemental cylinder of fluid

In the above figure we can see an element of fluid which is a vertical column of constant cross sectional area,  $A$ , surrounded by the same fluid of mass density  $\rho$ . The pressure at the bottom of the cylinder is  $p_1$  at level  $z_1$ , and at the top is  $p_2$  at level  $z_2$ . The fluid is at rest and in equilibrium so all the forces in the vertical direction sum to zero. i.e. we have

$$\text{Force due to } p_1 \text{ on } A \text{ (upward)} = p_1 A$$

$$\text{Force due to } p_2 \text{ on } A \text{ (downward)} = p_2 A$$

$$\text{Force due to weight of element (downward)} = mg$$

$$\text{Mass density volume} = \rho g A(z_2 - z_1)$$

Taking upward as positive, in equilibrium we have

$$p_1 A - p_2 A - \rho g A(z_2 - z_1) = 0$$

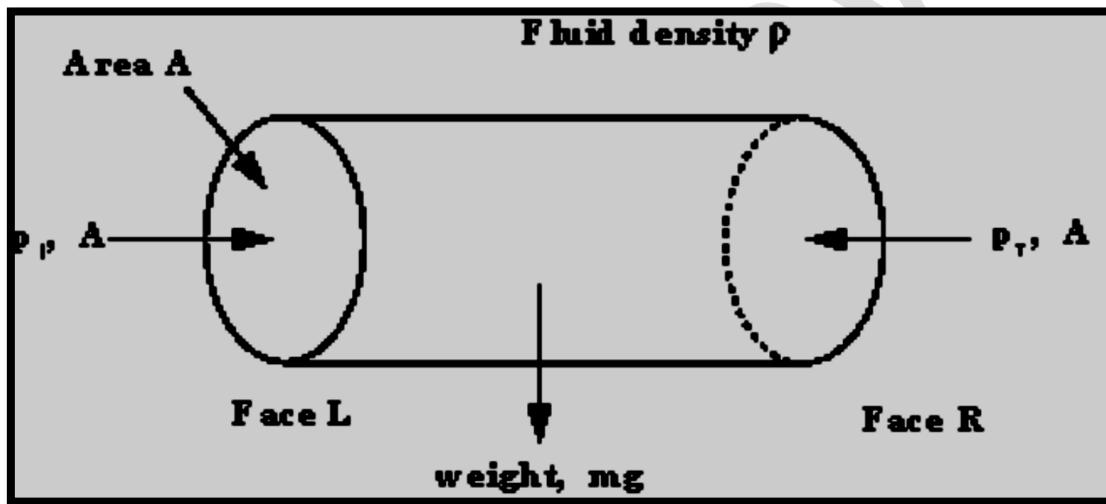
$$p_2 - p_1 - \rho g(z_2 - z_1) = 0$$

$$p_2 - p_1 = -\rho g(z_2 - z_1)$$

Thus in a fluid under gravity, pressure decreases with increase in height  $z = (z_2 - z_1)$ .

#### (4.1.4) Equality of Pressure at the Same Level in a Static Fluid

Consider the horizontal cylindrical element of fluid in the figure below, with cross-sectional area  $A$ , in a fluid of density  $\rho$ , pressure  $p_l$  at the left hand end and pressure  $p_r$  at the right hand end.



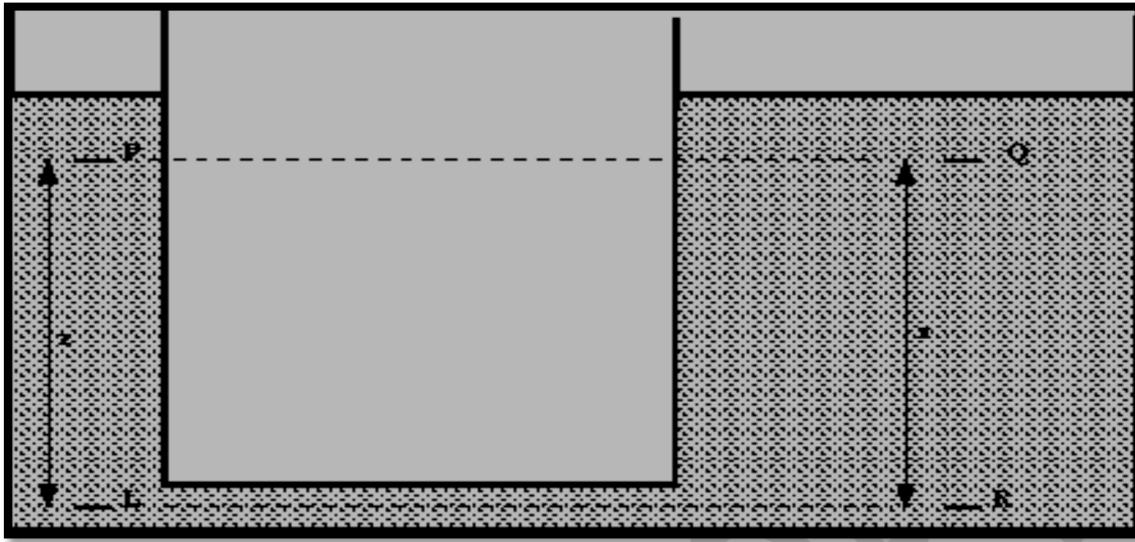
Horizontal elemental cylinder of fluid  
The fluid is at equilibrium so the sum of the forces acting in the x direction is zero.

$$p_l A = p_r A$$

$$p_l = p_r$$

*Pressure in the horizontal direction is constant*

This result is the same for any *continuous* fluid. It is still true for two connected tanks which appear not to have any direct connection, for example consider the tank in the figure below.



Two tanks of different cross-section connected by a pipe

We have shown above that  $p_l = p_r$  and from the equation for a vertical pressure change we have

$$p_l = p_p + \rho zg$$

$$p_r = p_q + \rho zg$$

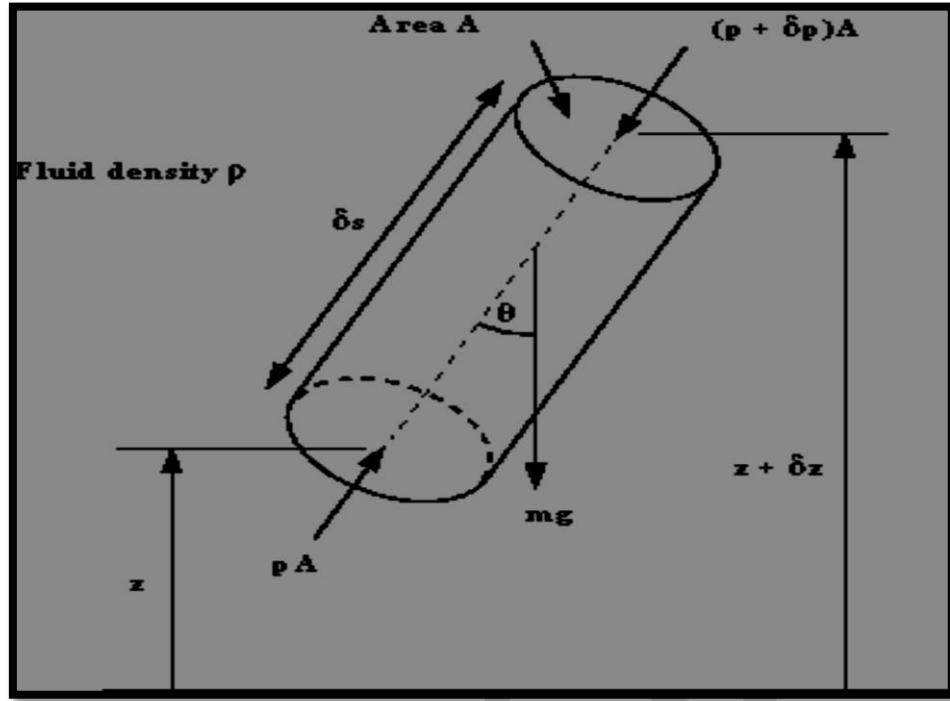
$$p_p + \rho zg = p_q + \rho zg$$

$$p_p = p_q$$

This shows that the pressures at the two equal levels, P and Q are the same

### General Equation for Variation of Pressure in a Static Fluid

Here we show how the above observations for vertical and horizontal elements of fluids can be generalized for an element of any orientation.



Cylindrical element of fluid at an arbitrary orientation.

Consider the cylindrical element of fluid in the figure above, inclined at an angle  $\theta$  to the vertical, length  $\delta_s$ , cross-sectional area  $A$  in a static fluid of mass density  $\rho$ . The pressure at the end with height  $z$  is  $p$  and at the end of height  $z + \delta z$  is  $p + \delta p$ .

The forces acting on the element are

$p_A$  acting at right - angles to the end of the face at  $z$

$(p + \delta p)A$  acting at right - angles to the end of the face at  $z + \delta z$

$$mg = \rho A \delta_s g$$

There are also forces from the surrounding fluid acting normal to these sides of the element. For equilibrium of the element the resultant of forces in any direction is zero. Resolving the forces in the direction along the central axis gives

$$p_A - (p_A + \delta p)A - \rho A \delta_s g \cos\theta = 0$$

$$\delta p = -\rho A \delta_s g \cos\theta$$

$$\frac{\delta_p}{\delta_s} = -\rho A g \cos\theta$$

Or in the differential form

$$\frac{dp}{ds} = -\rho A g \cos\theta$$

If  $\theta = 90^\circ$  then  $s$  is in the  $x$  or  $y$  directions, (i.e. horizontal), so the weight of the element acting vertically down

$$\left(\frac{dp}{ds}\right)_{\theta=90^\circ} = \frac{dp}{dx} = \frac{dp}{dy} = 0$$

Confirming that pressure on any horizontal plane is zero.

If  $\theta = 0$  then  $s$  is in the  $z$  direction (vertical) so

$$\left(\frac{dp}{ds}\right)_{\theta=0} = \frac{dp}{dz} = -\rho g$$

Confirming the result

$$\frac{p_2 - p_1}{z_2 - z_1} = \rho g$$

$$p_2 - p_1 = \rho g(z_2 - z_1)$$