Lecture No (8)
Equation of Motion

According to Newton's second law of motion, the net force F_x acting on a fluid element in the direction of x is equal to mass of the fluid element multiplied by the acceleration ax in the x-direction. Thus mathematically:

$$F_x = ma_x \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(8.1a)$$

In the fluid flow, the following forces are present:

1. F_g, gravity force.
2. F_p, the pressure force.
3. F_v, force due to viscosity.
4. F_t, force due to turbulence,
5. F_c, force due to compressibility.

The net force:

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x \dots\dots\dots\dots.(8.1b)$$

- If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x \dots\dots\dots\dots\dots..\dots\dots(8.1c)$$

and equation of motions are called **Reynolds's equations** of motion.

- For flow, where (F_t) is negligible, the resulting equations of motion are known as

Navier-Stokes Equation:

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x \dots\dots\dots\dots\dots..\dots\dots\dots\dots(8.1d)$$

- If the flow is assumed to be ideal, viscous force (Fv) is zero and equation of motions are known as **Euler's equation** of motion

$$F_x = (F_g)_x + (F_p)_x \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots...\dots\dots(8.1f)$$

8.1. General Conservation Laws

The study of fluid mechanics is based, to a large extent, on the conservation laws of three extensive quantities:

1. *Mass*—usually total, but sometimes of one or more individual chemical species.
2. Total *energy*—the sum of internal, kinetic, potential, and pressure energy.
3. *Momentum*, both linear and angular.

For a system viewed as a whole, *conservation* means that there is no net gain nor loss of any of these three quantities, even though there may be some redistribution of them within a system. A general *conservation law* can be phrased relative to the general system shown in Fig. 8.1, in which can be identified:

1. The *system* V.
2. The *surroundings* S.
3. The *boundary* B, also known as the *control surface*, across which the system interacts in some manner with its surroundings.

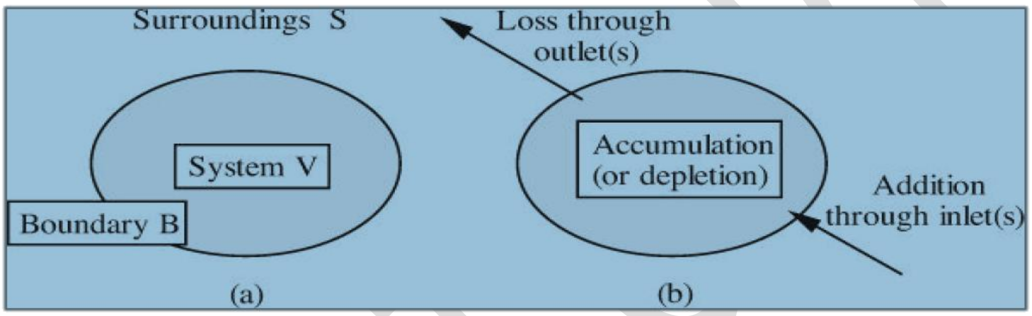

Fig. 8.1 (a) System and its surroundings; (b) transfers to and from a system. For a chemical reaction, creation and destruction terms would also be included inside the system.

The interaction between system and surroundings is typically by one or more of the following mechanisms:

1. A flowing stream, either entering or leaving the system.
2. A "contact" force on the boundary, usually normal or tangential to it, and commonly called a stress.
3. A "body" force, due to an external field that acts throughout the system, of which gravity is the prime example.
4. Useful work, such as electrical energy entering a motor or shaft work leaving a turbine.

Let X denote mass, energy, or momentum. Over a finite time period, the general conservation law for X is: Non-reacting system

$$X_{in} - X_{out} = \Delta X_{system} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.2)$$

For a mass balance on species i in a reacting system

$$X_{in}^{i} - X_{out}^{i} + X_{created}^{i} - X_{destroyed}^{i} = \Delta X_{system}^{i} \ldots\ldots (8.2a)$$

The symbols are defined in Table 8.1. The understanding is that the creation and destruction terms, together with the superscript *i*, are needed only for mass balances on species *i* in chemical reactions, which will not be pursued further in this text.

Table 8.1 Meanings of Symbols in Equation (8.2)

Symbol	*Meaning*
X_{in}	*Amount of X brought into the system*
X_{out}	*Amount of X taken out of the system*
$X_{created}$	*Amount of X created within the system*
$X_{destroyed}$	*Amount of X destroyed within the system*
ΔX_{system}	*Increase (accumulation)in the X content of the system*

It is very important to note that Eqn. (8.2) cannot be applied indiscriminately, and is only observed *in general* for the *three* properties of *mass*, *energy*, and *momentum*.

Equation (8.2) can also be considered on a basis of unit time, in which case all quantities become rates; for example, ΔX_{system} becomes the rate, dX_{system}/dt, at which the X-content of the system is increasing, x_{in} (note the lower-case ″x″) would be the rate of transfer of X into the system, and so on, as in Eqn. (8. 2):

$$x_{in} - x_{out} = \frac{dX_{system}}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.3)$$

8.2.1. Mass Balances

The general conservation law is typically most useful when *rates* are considered. In that case, if *X* denotes mass *M* and *x* denotes a mass "rate" *m* (the symbol $\dot{m}$ can also be used) the *transient* mass balance (for a *non-reacting system*) is:

$$m_{in} - m_{out} = \frac{dM_{system}}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.4)$$

In which the symbols have the meanings given in Table8.2.

Table 8.2 Meanings of Symbols in Equation (8.4)

Symbol	*Meaning*
m_{in}	*Rate of addition of mass into the system*
m_{out}	*Rate of removal of mass from the system*
$\frac{dM_{system}}{dt}$	*Rate of accumulation of mass in the system (will be negative of the for depletion mass*

8.2.1.1.Steady-state mass balance for fluid flow.

A particularly useful and simple mass balance—also known as the *continuity equation*—can be derived for the situation shown in Fig.8.2, where the system resembles a wind sock at an airport. At station 1, fluid flows steadily with density ρ_1 and auniform velocity u_1 normally across that part of the surface of the system represented by the area A_1.

In steady flow, each fluid particle traces a path called a *streamline*. By considering a large number of particles crossing the closed curve C, we have an equally large number of streamlines that then form a surface known as a *stream tube*, across which there is clearly no flow. The fluid then leaves the system with uniform velocity u_2 and density ρ_2 at station 2, where the area normal to the direction of flow is A_2.

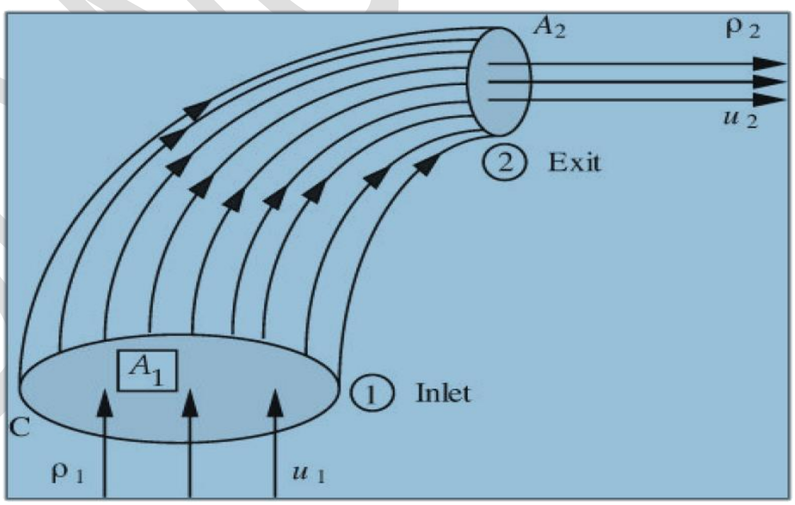

Figure (8.2) ***Flow through a stream tube*** **mass**

Referring to Eqn. (8.4), there is no accumulation of mass because the system is at steady state. Therefore, the only nonzero terms are m1 (the rate of addition of mass) and m_2 (the rate of removal of mass), which are equal to $\rho_1 A_1 u_1$ and $\rho_2 A_2 u_2$, respectively, so that Eqn. (8.4) becomes:

$$\underbrace{A_1\rho_1 u_1}_{in} - \underbrace{A_2\rho_2 u_2}_{out} = 0 \ (at\ steady\ state\ ) \dots\dots\dots\dots\dots (8.5a)$$

which can be rewritten as:

$$\underbrace{A_1\rho_1 u_1}_{in} - \underbrace{A_2\rho_2 u_2}_{out} = m \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\dots\dots\dots (8.5b)$$

where m (= m_1 = m_2) is the mass flow rate entering and leaving the system. For the special but common case of an incompressible fluid, $\rho_1 = \rho_2$, so that the steady-state mass balance becomes:

$$A_1 u_1 = A_2 u_2 = \frac{m}{\rho} = Q \dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots\dots.. (8.6)$$

in which Q is the volumetric flow rate. Equations (2.5a/b) would also apply for non-uniform inlet and exit velocities, if the appropriate

mean velocities u_{m1} and u_{m2} were substituted for u_1 and u_2. However, we shall postpone the concept of non-uniform velocity distributions to a more appropriate time, particularly to those chapters that deal with microscopic fluid mechanics.

8.2.2. Energy Balances

Equation (8.1) is next applied to the general system shown in Fig. 8.3, it being understood that property *X* is now *energy*. Observe that there is both flow into and from the system. Also note the quantities defined in Table 8.3.

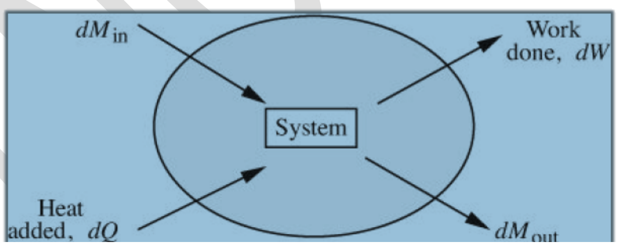

Fig. 8.3 Energy balance on a system with flow in and out.

Table 2.3 Definitions of Symbols for Energy Balance

Symbol	*Definition*
dM_{in}	Differential amount of mass entering the system
dM_{out}	Differential amount of mass leaving the system
dQ	Differential amount of heat added *to* the system
dW	Differential amount of useful work done *by* the system
e	Internal energy per unit mass
g	Gravitational acceleration
M	Mass of the system
u	Velocity
ρ	Density

A differential energy balance results by applying Eqn. (8.2) over a short time period. Observe that there are two transfers into the system (incoming mass and heat) and two transfers out of the system (outgoing mass and work). Since the mass transfers also carry energy with them, there results:

$$dM_{in}\left(e+\frac{p}{\rho}+gz+\frac{u^2}{2}\right)_{in}=dM_{out}\left(e+\frac{p}{\rho}+gz+\frac{u^2}{2}\right)_{out}+dQ-dW$$

$$=d\left[M\left(e+\frac{p}{\rho}+gz+\frac{u^2}{2}\right)\right]_{system} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(8.7)$$

in which each term has units of energy or work. In the above, the system is assumed for simplicity to be homogeneous, so that all parts of it have the same internal, potential, and kinetic energy per unit mass; if such were not the case, integration would be needed throughout the system. Also, multiple inlets and exits could be accommodated by means of additional terms. Since the density ρ is the reciprocal of υ, the volume per unit mass, e + p/ρ = e + pv, which is recognized as the enthalpy per unit mass. The flow energy term p/ρ in Eqn. (8.7), also known as injection work or flow work, is readily explained by examining Fig.8.4. Consider unit mass of fluid entering the stream tube under a pressure p_1. The volume of the unit mass is:

$$\frac{1}{\rho_1}=A_1\frac{1}{A_1\rho_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots(8.8)$$

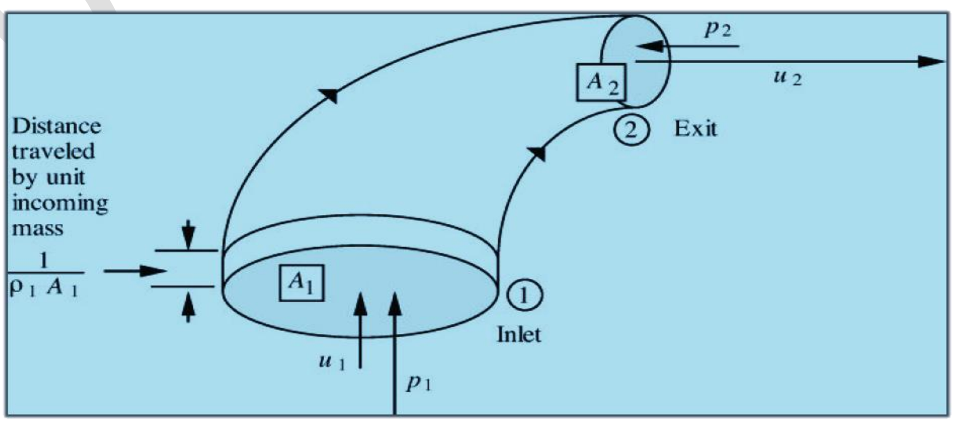

Fig. 8.4 Flow of unit mass to and from stream tube.

which is the product of the area A_1 and the distance $1/\rho_1 A_1$ through which the mass moves. (Here, the "1" has units of mass.) Hence, the work done on the system by p_1 in pushing the unit mass into the stream tube is the force $p_1 A_1$ exerted by the pressure multiplied by the distance through which it travels:

$$\rho_1 A_1 \times \frac{1}{A_1 \rho_1} = \frac{\rho_1}{\rho_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.9)$$

Likewise, the work done by the system on the surroundings at the exit is:

$$\rho_2 A_2 \times \frac{1}{A_2 \rho_2} = \frac{\rho_2}{\rho_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.10)$$

8.2.2.1. Steady-state energy balance.

In the following, all quantities are per unit mass flowing. Referring to the general system shown in Fig. 8.5, the energy entering with the inlet stream plus the heat supplied to the system must equal the energy leaving with the exit stream plus the work done by the system on its surroundings. Therefore, the right-hand side of Eqn. (8.7) is zero under steady-state conditions, and division by $d_{Min} = dM_{\text{out}}$ gives:

$$e_1 + \frac{p_1}{\rho_1} + gz_1 + \frac{u_1^2}{2} + q = e_2 + \frac{p_2}{\rho_2} + gz_2 + \frac{u_2^2}{2} + w \ldots\ldots \quad (8.11)$$

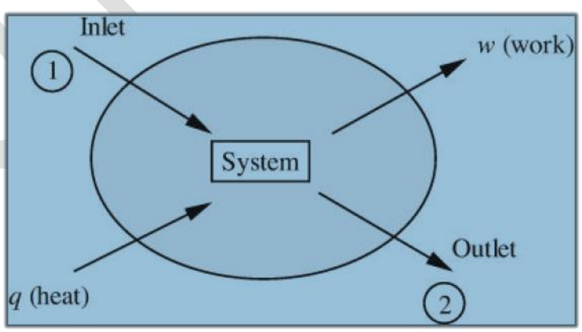

Fig. 8.5 Steady-state energy balance.

in which each term represents an energy per unit mass flowing.

For an infinitesimally small system in which *differential* changes are occurring, Eqn. (2.11) may be rewritten as:

$$de + d\left(\frac{p}{\rho}\right) + d(gz) + d\left(\frac{u^2}{2}\right) + d(pv) = dq - dw \ldots\ldots\ldots\ldots \quad (8.12)$$

in which, for example, *de* is now a *differential* change, and $\upsilon = 1/\rho$ is the volume per unit mass. Now examine the increase in internal energy *de*, which arises from frictional work *dF* dissipated into heat, heat addition *dq* from the surroundings, less work *pdv* done by the fluid. That is: *de* = *dF* + *dq* – *pdv*. Thus, eliminate the change *de* in the internal energy from Eqn. (2.12), and expand the term d(pv), which

$$\underbrace{d\mathcal{F} + dq + pdv}_{de} + d\left(\frac{u^2}{2}\right) + d(gz) + \ d(pv) + \underbrace{pdv + vdp +}_{d(pv)}$$

$$= dq - dw \ldots\ldots\ldots\ldots\ldots (8.13)$$

which simplifies to the differential form of the *mechanical* energy balance, in which heat terms are absent:

$$d\left(\frac{u^2}{2}\right) + d(gz) + \frac{dp}{\rho} + d\mathcal{F} + dw = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.14)$$

For a finite system, for flow from point 1 to point 2, Eqn. (2.14) integrates to:

$$\Delta\left(\frac{u^2}{2}\right) + \Delta(gz) + \int_1^2 \frac{dp}{\rho} + \mathcal{F} + w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.15)$$

in which a finite change is consistently the final minus the initial value, for example:

$$\Delta\left(\frac{u^2}{2}\right) = \frac{u_2^2}{2} - \frac{u_1^2}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.16)$$

An energy balance for an incompressible fluid of constant density permits the integral to be evaluated easily, giving:

$$\Delta\left(\frac{u^2}{2}\right) + \Delta(gz) + \frac{\Delta p}{\rho} + \mathcal{F} + w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.17)$$

In the majority of cases, g will be virtually constant, in which case there is a further simplification to:

$$\Delta\left(\frac{u^2}{2}\right) + g\Delta z + \frac{\Delta p}{\rho} + \mathcal{F} + w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.18)$$

which is a *generalized Bernoulli* equation, *augmented* by two extra terms—the frictional dissipation, *F*, and the work *w* done by the system. Note that *F* can *never* be negative—it is impossible to convert heat entirely into useful work. The work term *w* will be positive if the fluid flows through a turbine and performs work on the environment; conversely, it will be negative if the fluid flows through a pump and has work done on it.

Power. The *rate* of expending energy in order to perform work is known as *power*, with dimensions of ML^2/T^3, typical units being W (J/s) and ft lb_f/s. The relations in Table 8.4 are available, depending on the particular context.

Table 8.4 Expressions for Power in Different Systems

Flowing stream:	mw (m = mass flow rate, w = work per unit mass)
Force displacement:	Fv (F = force, v = displacement velocity)
Rotating shaft:	$T\omega$ (T = torque, ω = angular velocity of rotation)
Pump:	$Q\Delta p$ (Q = volume flow rate, Δp = pressure increase)