Lecture NO (6)

Forces on Submerged Surfaces in Static Fluids

We have seen the following features of statics fluids

- Hydrostatic vertical pressure distribution
- Pressures at any equal depths in a continuous fluid are equal
- Pressure at a point acts equally in all directions (Pascal's law).
- Forces from a fluid on a boundary acts at right angles to that boundary.

Objectives:

We will use these to analyses and obtain expressions for the forces on submerged surfaces. In doing this it should also be clear the difference between:

- Pressure which is a scalar quantity whose value is equal in all directions and,
- Force, which is a vector quantity having both magnitude and direction.

(6.1) Fluid pressure on a surface

Pressure is defined as force per unit area. If a pressure p acts on a small area ***dA*** then the force exerted on that area will be

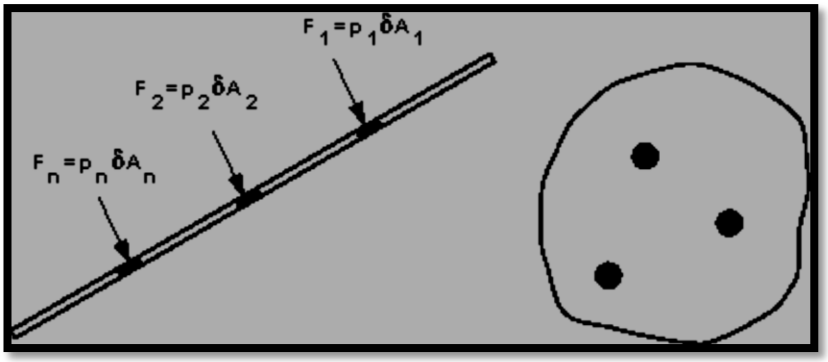
$$F = p\delta A$$

Since the fluid is at rest the force will act at right-angles to the surface.

General submerged plane

Consider the plane surface shown in the figure below. The total area is made up of many elemental areas.

The force on each elemental area is always normal to the surface but, in general, each force is of different magnitude as the pressure usually varies



We can find the total or *resultant* force, *R*, on the plane by summing up all of the forces on the small elements i.e.

$$R = p_1\delta A_1 + p_2\delta A_2 \dots\dots\dots\dots p_n\delta A_n = \sum p\delta A$$

This resultant force will act through the center of pressure, hence we can say

If the surface is a plane the force can be represented by one single resultant force, acting at right-angles to the plane through the center of pressure.

Horizontal Submerged Plane

For a horizontal plane submerged in a liquid (or a plane experiencing uniform pressure over its surface), the pressure, *p*, will be equal at all points of the surface. Thus the resultant force will be given by

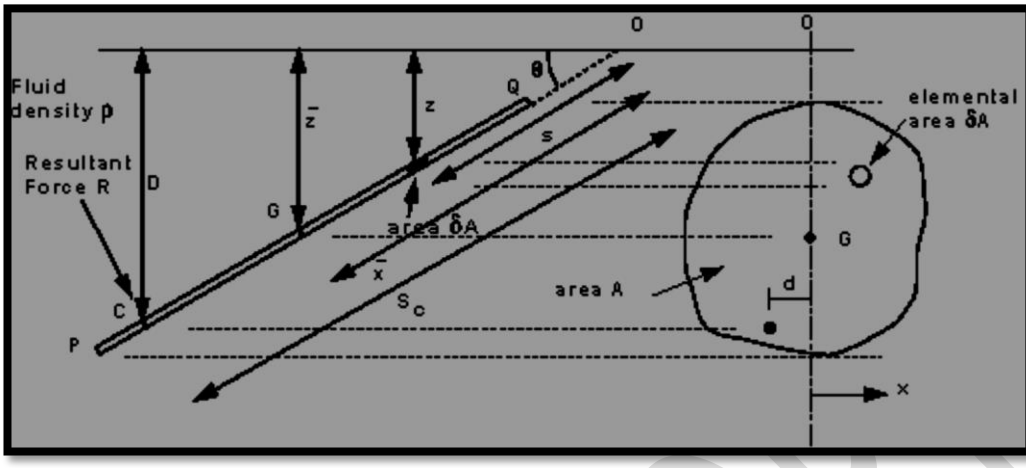
$$R = \text{pressure} \times \text{area of plane}$$

$$R = pA$$

Curved Submerged Surface

If the surface is curved, each elemental force will be a different magnitude and in different direction but still normal to the surface of that element. The resultant force can be found by resolving all forces into orthogonal co-ordinate directions to obtain its magnitude and direction. This will always be less than the sum of the individual forces, $\sum \delta pA$.

(6.2)Resultant Force and Centre of Pressure on a submerged plane surface in a liquid.



This plane surface is totally submerged in a liquid of density ρ and inclined at an angle of θ to the horizontal. Taking pressure as zero at the surface and measuring down from the surface, the pressure on an element δA , submerged a distance z, is given by

$$p = \rho g z$$

And therefore the force on the element is

$$R = p\delta A = \rho g z \delta A$$

The resultant force can be found by summing all of these forces i.e.

$$R = \rho g \sum z\delta A$$

(Assuming ρ and g *as* constant).
The term $\sum z\delta A$ is known as the *1st Moment of Area* of the plane PQ about the free surface. It is equal to A z i.e.

$$\sum z\delta A = Az^{-}$$

= 1st moment of area about the line of the free surface

Where A is the area of the plane and z^{-} is the depth (distance from the free surface) to the centroid, G. This can also be written in terms of distance from point O (as $z^{-} = x^{-}$ sinθ)

$$\sum z\delta A = Az^{-} = A\, x^{-} \sin\theta$$

= 1st Moment of area about a line through O $\times \sin\theta$

Thus:

The resultant force on a plane

$$R = \rho g A z^{-}$$
$$= \rho g A x^{-} \sin\theta$$

This resultant force acts at right angles to the plane through the center of pressure, C, at a depth D. The moment of R about any point will be equal to the sum of the moments of the forces on all the elements δA of the plane about the same point. We use this to find the position of the center of pressure.

It is convenient to take moments about the point where a projection of the plane passes through the surface, point O in the figure.

Moment of R about O = Sum of moments of force

On all elements $\delta A$of about O

We can calculate the force on each elemental area:

$$Force\ on\ \delta A = \rho g z \delta A$$
$$= \rho g s \theta \delta A$$
$$\rho g s\ sin\theta \delta A \times s$$
$$= \rho g sin\theta \delta A s^2$$

ρ ,g andθ are the same for each element, so the total moment is

Sum of moments of forces on all elements of δA about O = $\rho g sin\theta \sum \delta A s^2$

We know the resultant force from above R= $\rho g A x^{-} sin\theta$, which acts through the center of pressure at C,

Moment of R about O = $\rho g A x^{-} sin\theta\ S_c$

Equating gives,

$$\rho g A x^{-} sin\theta \, S_c = \rho g x^{-} sin\theta \sum s^2 \, \delta A$$

Thus the position of the center of pressure along the plane measure from the point O is:

$$S_c = \frac{\sum s^2 \, \delta A}{A x^{-}}$$

It looks a rather difficult formula to calculate - particularly the summation term. Fortunately this term is known as the *2nd Moment of Area*, I_o , of the plane about the axis through O and it can be easily calculated for many common shapes. So, we know:

2nd moment of area about O = $I_o = \sum s^2 \, \delta A$

And as we have also seen that Ax^{-} = 1st Moment of area about a line through O,

Thus the position of the center of pressure along the plane measure from the point O is:

$$\boldsymbol{S_c = \frac{2^{nd}\ \text{Moment of area about a line through O}}{1^{nd}\ \text{Moment of area about a line through O}}}$$

And depth to the center of pressure is

$$D = S_c sin\theta$$

(6.2.1) How do you calculate the 2nd moment of area? To calculate the 2nd moment of area of a plane about an axis through O, we use the *parallel axis theorem* together with values of the 2nd moment of area about an axis though the centroid of the shape obtained from tables of geometric properties.

The parallel axis theorem can be written

$$I_o = I_{GG} + A(x^{-})^2$$

Where I_{GG} is the 2nd moment of area about an axis though the centroid G of the plane.

Using this we get the following expressions for the position of the center of pressure

$$S_c = \frac{I_{GG}}{Ax^-} + x^-$$

$$D = sin\theta\left(\frac{I_{GG}}{Ax^-} + x^-\right)$$

(In the examination the parallel axis theorem and the I_{GG} will be given)

Lateral position of Centre of Pressure

If the shape is symmetrical the centre of pressure lies on the line of symmetry. But if it is not symmetrical its position must be found by taking moments about the line OG in the same way as we took moments along the line through O, i.e.

$$R \times d = \text{Sum of the moments of the force on all elemnts of A about OG}$$

$$= \sum \rho g z \delta A x$$

But we have

$$R = \rho g \bar{z} A$$

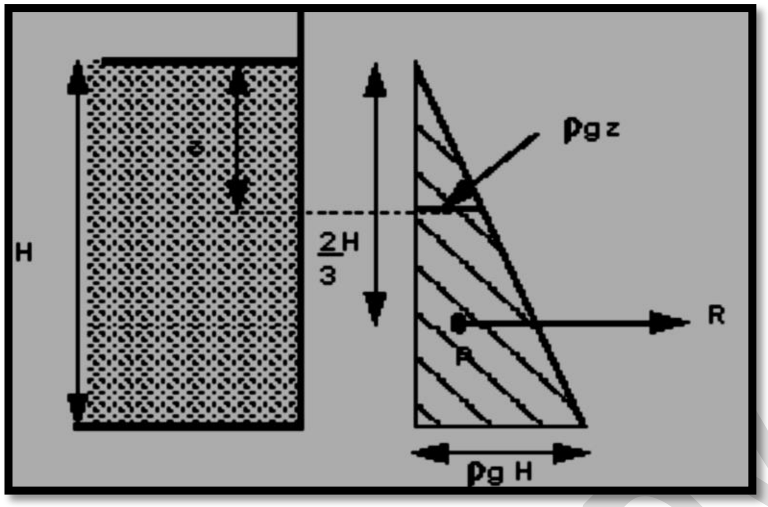
SO

$$R = \frac{\sum \delta A z x}{A\bar{z}}$$

(6.2.2)Submerged vertical surface - Pressure diagrams

For vertical walls of constant width it is usually much easier to find the resultant force and center of pressure. This is done graphically by means of a pressure diagram.

Consider the tank in the diagram below having vertical walls and holding a liquid of density ρ to a depth of *H*. To the right can be seen a graphical representation of the (gauge) pressure change with depth on one of the vertical walls. Pressure increases from zero at the surface linearly by $p = \rho g z$, to a maximum at the base of $p = \rho g H$.



Pressure diagram for vertical wall

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The area of this triangle represents the **resultant force per unit width** on the vertical wall, using SI units this would have units of Newtons permetre. So

$$Area = \frac{1}{2} \times AB \times BC$$

$$= \frac{1}{2} \times H\rho g H$$

$$= \frac{1}{2} \times H^2 \rho g$$

Resultant force per unit width

$$\boldsymbol{R = \frac{1}{2} \times H^2 \rho g}$$

The force acts through the centroid of the pressure diagram. For a triangle the centroid is at 2/3 its height, i.e. in the figure above the resultant force acts horizontally through the point $z = \frac{2}{3}H$

For a vertical plane the depth to the centre of pressure is given by

$$D = \frac{2}{3}H$$

This can be checked against the previous method:

The resultant force is given by:

$$R = \rho g A z^{-} = \rho g A x^{-} sin\theta$$

$$= \rho g (H \times 1) \frac{H}{2} sin\theta$$

$$= \frac{1}{2} \rho g H^2$$

and the depth to the centre of pressure by:

$$D = sin\theta \left(\frac{I_\circ}{A\bar{x}} \right)$$

and by the parallel axis theorem (with width of 1)

$$I_\circ = I_{GG} + A\bar{x}^2$$

$$= \frac{1 \times H^3}{12} + 1 \times H \left(\frac{H}{2} \right)^2$$

$$\frac{H^3}{12} + \frac{H^3}{4} = \frac{H^3}{3}$$

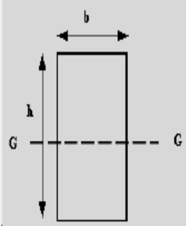
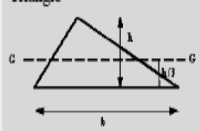
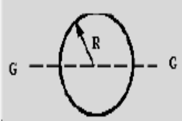
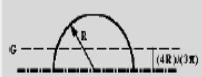
Giving depth to the center of pressure

$$D = \left(\frac{H^3/3}{H^2/2} \right) = \frac{2}{3} H$$

These two results are identical to the pressure diagram method.

The same pressure diagram technique can be used when combinations of liquids are held in tanks (e.g. oil floating on water) with position of action found by taking moments of the individual resultant forces for each fluid. Look at the examples to examine this area further.

More complex pressure diagrams can be draw for non-rectangular or non-vertical planes but it is usually far easier to use the moment's method.

Shape	Area A	2^{nd} moment of area, I_{GG}, about an axis through the centroid
Rectangle 	bd	$\frac{bd^3}{12}$
Triangle 	$\frac{bd}{2}$	$\frac{bd^3}{36}$
Circle 	πR^2	$\frac{\pi R^4}{4}$
Semicircle 	$\frac{\pi R^2}{2}$	$0.1102R^4$