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Lecture No (9)

Bernoulli's Equation

Situations frequently occur in which the following simplifying assumptions can reasonably be made:

1. The flow is steady.
2. There are no work effects; that is, the fluid neither performs work (as in a turbine) nor has work performed on it (as in a pump). Thus, $\omega = 0$ in Eqn. (2.18).
3. The flow is frictionless, so that $F = 0$ in Eqn. (2.18). Clearly, this assumption would not hold for long runs of pipe.
4. The fluid is incompressible; that is, the density is constant. This approximation is excellent for the majority of liquids, and may also be reasonable for some cases of gas flows provided that the pressure variations are moderately small.

Under these circumstances, the general energy balance reduces to:

$$\Delta\left(\frac{u^2}{2}\right) + g\Delta z + \frac{\Delta p}{\rho} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots.(9.1)$$

which is the famous Bernoulli's equation, one of the most important relations in fluid mechanics.

For flow between points 1 and 2 on the same streamline, or for any two points in a fluid under static equilibrium (in which case the velocities are zero), Eqn. (9.1) becomes:

$$\underbrace{\frac{u_1^2}{2} + gz_1 + \frac{p_1}{\rho}}_{Total\ energy\ at\ point\ 1} = \underbrace{\frac{u_2^2}{2} + gz_2 + \frac{p_2}{\rho}}_{Total\ energy\ at\ point\ 1} \quad \ldots\ldots.(9.2)$$

which states that although the kinetic, potential, and pressure energies may vary individually, their sum remains constant. Each term in (9.2) must have the same dimensions as the first one, namely, velocity squared or L^2/T^2. Further manipulations in the two principal systems of units yield the following:

1. *SI Units*

$$\frac{m^2}{s^2} = \frac{m}{kg}\underbrace{kg\frac{m}{s^2}}_{force\ (N)} = \frac{m.N}{kg} = \frac{J}{kg} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9.3)$$

which is readily seen to be energy per unit mass.

8.2.3.1. Head of fluid.

A quantity closely related to energy per unit mass may be obtained by dividing (9.3) through by the gravitational acceleration g:

$$\underbrace{\frac{u_1^2}{2} + gz_1 + \frac{p_1}{\rho}}_{Velocity+static+pressure\ head} = \underbrace{\frac{u_2^2}{2} + gz_2 + \frac{p_2}{\rho}}_{Velocity+static+pressure\ head} = H \qquad \ldots\ldots(9.4)$$

Each term in (9.3) has dimensions of length, and indeed the terms such as $\frac{u_1^2}{2g}$, z_1, $\frac{p_1}{\rho g}$, and H are called the velocity head, static head, pressure head, and total head, respectively.

A physical interpretation of fluid head is readily available by considering the steady flow of a liquid from a tank through the idealized frictionless pipe shown in Fig. 9.1(a). At point 1 (the free surface), the velocity u1 is virtually zero for a tank of reasonable size, and the pressure p1 there is also zero (gauge) because the free surface is exposed to the atmosphere. Thus, the velocity and pressure heads are both zero at point 1, so that the total head (H for example) is identical with the static head z_1, namely, the elevation of the free surface relative to some datum level. Hence, Eqn. (9.3) can be rewritten as:

$$z_1 = H = \underbrace{\frac{u_2^2}{2} + gz_2 + \frac{p_2}{\rho}}_{Each\ of\ these\ tearm\ is\ interpreted\ in\ fig\ (8.6)} \qquad \ldots\ldots(9.5)$$

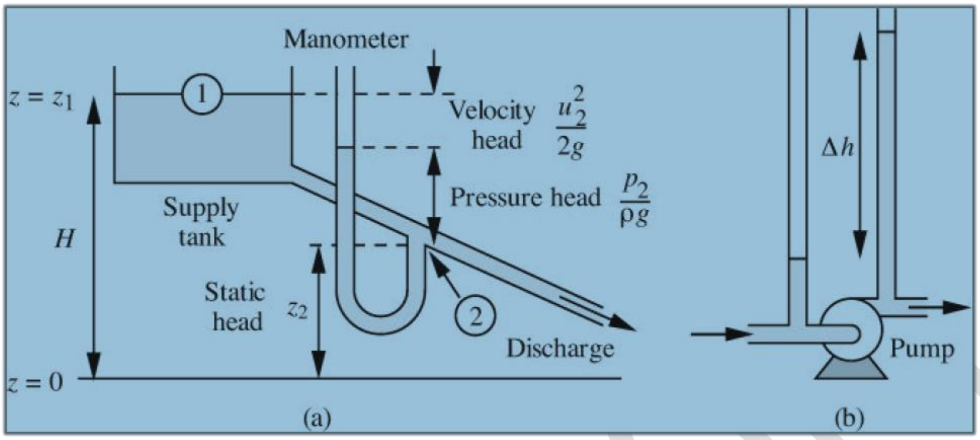

Fig. 9.1 (a) Physical interpretation of velocity, static, and pressure heads for pipe flow, and (b) pressure head increase across a pump.

Looking now at point 2, the static head is simply the elevation z_2 of that point above the datum level; the pressure head is the height above point 2 to which the liquid rises in the manometer—an amount that is just sufficient to balance the pressure p_2; and, by difference, the elevation difference between the top of the liquid in the tube and point 1 must be the velocity head Since the pipe diameter is constant, continuity also requires the velocity and hence the velocity head to be constant. Referring to Fig. 9.1(a), since the static head continuously decreases along the pipe, and the total head is constant, the pressure head must constantly increase. But since the pressure at the exit—or very shortly after it—is atmospheric, the pressure head must again be zero! The reader will doubtless ask: "Is there an anomaly?", and may wish to ponder whether or not the diagram is completely accurate as drawn. Fig. 9.1(b) shows that the pressure increase Δp across a pump is also equivalent to a head increase $\Delta h = \Delta p/\rho g$, being the increase in liquid levels in piezometric tubes placed at the pump inlet and exit. Note carefully that the above analysis is for an ideal liquid—one that exhibits no friction. In practice, there would be some loss in total head along the pipe.

Applications of Bernoulli's Equation

We now apply Eqn. (9.1) to several commonly occurring situations, in which useful relations involving pressures, velocities, and elevations may be obtained. The usual assumptions of steady flow, no external work, no friction, and constant density may reasonably be made in each case.

(a)Flow through a Small Orifice

We are to consider the flow from a tank through a hole in the side close to the base. The general arrangement and a close up of the hole and streamlines are shown in the figure below.

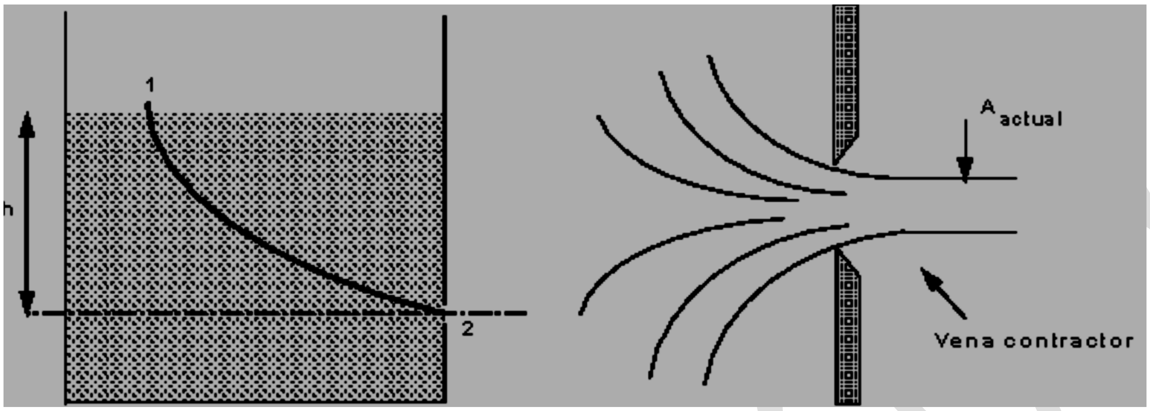

Fig. (9.2) Tank and streamlines of flow out of the sharp edged orifice

The shape of the hole's edges are as they are (sharp) to minimize frictional losses by minimizing the contact between the hole and the liquid - the only contact is the very edge.

Looking at the streamlines you can see how they contract after the orifice to a minimum value when they all become parallel; at this point, the velocity and pressure are uniform across the jet. This convergence is called the *vena contract*. (From the Latin 'contracted vein'). It is necessary to know the amount of contraction to allow us to calculate the flow. We can predict the velocity at the orifice using the Bernoulli equation. Apply it along the streamline joining point 1 on the surface to point 2 at the center of the orifice. At the surface velocity is negligible ($u_1 = 0$) and the pressure atmospheric ($p_1 = 0$). At the orifice the jet is open to the air so again the pressure is atmospheric ($p_2 = 0$). If we take the datum line through the orifice then $z_1 = h$ and $z_2 = 0$, leaving

$$h = \frac{u_2^2}{2g} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (9.6)$$

$$u_2 = \sqrt{2gh} \dots\dots\dots\dots\dots\dots\dots (9.7)$$

This is the theoretical value of velocity. Unfortunately, it will be an over estimate of the real velocity because friction losses have not been taken into account. To incorporate friction, we use the *coefficient of velocity* to correct the theoretical velocity,

$$u_{actual} = C_v u_{theortical} \dots \dots . \dots (9.8.)$$

Each orifice has its own coefficient of velocity; they usually lie in the range (0.97 - 0.99).

To calculate the discharge through the orifice we multiply the area of the jet by the velocity. The actual area of the jet is the area of the vena contracts *not* the area of the orifice. We obtain this area by using a coefficient of contraction for the orifice

$$A_{actual} = C_c A_{orifice} \dots \dots \dots \dots \dots \dots . . \dots (9.9)$$

So the discharge through the orifice is given by

$$Q = uA \dots \dots \dots \dots \dots \dots . . \dots \dots \dots \dots \dots \dots . (9.10)$$

$$Q_{actual} = u_{actual} A_{actual} \dots \dots \dots \dots \dots \dots (9.11)$$

$$Q_{actual} = C_c C_v A_{orifice} A_{theortical} . . \dots . (9.12)$$

$$Q_{actual} = C_d A_{orifice} A_{theortical.} \dots \dots . . (9.13)$$

$$Q_{actual} = C_d A_{orifice} \sqrt{2gh} \dots \dots . . \dots . . (9.14)$$

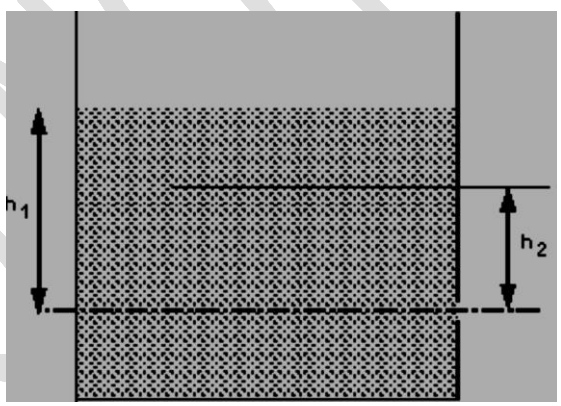

Fig (9.3) Tank emptying from level *h₁* to *h₂*.

The tank has a cross sectional area of *A*. In a time *dt* the level falls by *dh* or the flow out of the tank is

$$Q = Av$$

$$= A\frac{\delta h}{\delta t} \dots \dots . . \dots \dots . . \dots \dots . . \dots . . (9.15)$$

(-ve sign as δh is falling)

Rearranging and substituting the expression for Q through the orifice gives

$$\delta t = \frac{-A}{C_d A_{\circ}\sqrt{2g}} \frac{\delta h}{\sqrt{h}} \ldots \ldots \ldots \ldots (9.16)$$

This can be integrated between the initial level, h_1, and final level, h_2, to give an expression for the time it takes to fall this distance

$$t = \frac{-A}{C_d A_{\circ}\sqrt{2g}} \int_{h_1}^{h_2} \frac{\delta h}{\sqrt{h}} \ldots \ldots (9.17)$$

$$\frac{-A}{C_d A_{\circ}\sqrt{2g}} [2h]_{h_1}^{h_2}$$

$$\frac{-2A}{C_d A_{\circ}\sqrt{2g}} \left[\sqrt{h_2} - \sqrt{h_1}\right] \ldots \ldots (9.18)$$

(b)Submerged Orifice

We have two tanks next to each other (or one tank separated by a dividing wall) and fluid is to flow between them through a submerged orifice. Although difficult to see, careful measurement of the flow indicates that the submerged jet flow behaves in a similar way to the jet in air in that it forms a vena contract below the surface. To determine the velocity at the jet we first use the Bernoulli equation to give us the ideal velocity. Applying Bernoulli from point 1 on the surface of the deeper tank to point 2 at the center of the orifice gives

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 \ldots \ldots \ldots \ldots (9.19)$$

$$0 + 0 + z_1 = \frac{\rho g h_2}{\rho g} + \frac{u_2^2}{2g} + 0$$

$$u_2 = \sqrt{2g(h_1 - h_2)} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (9.20)$$

i.e. the ideal velocity of the jet through the submerged orifice depends on the *difference* in head across the orifice. And the discharge is given by

$$Q = C_d A_\circ u$$

$$Q = C_d A_\circ \sqrt{gh} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (9.21)$$

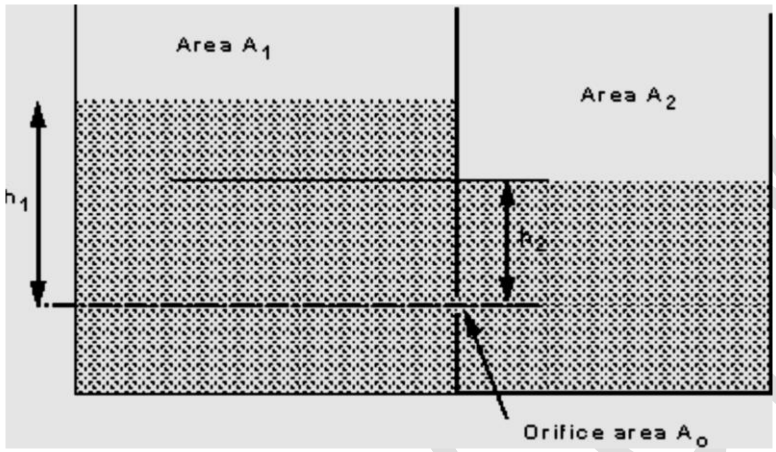

Fig (9.4.) Two tanks of initially different levels joined by an orifice

By a similar analysis used to find the time for a level drop in a tank we can derive an expression for the change in levels when there is flow between two connected tanks. Applying the continuity equation

$$Q = -A_1 \frac{\delta h_1}{\delta t} = A_2 \frac{\delta h_2}{\delta t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (9.22)$$

$$Q\delta t = -A_1 \delta h_1 = A_2 \delta h_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (9.23)$$

$$Q\delta t = -A_1 \delta h_1 = A_2 \delta h_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (9.24)$$

Also we can write $= -\delta h_1 + \delta h_2 = \delta h$

So

$$-A_1 \delta h_1 - A_2 \delta h_1 - A_2 \delta h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (9.25)$$

$$\delta h_1 = \frac{A_2 \delta h}{A_1 - A_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . \ldots (9.26)$$

Then we get

$$Q\delta t = -A_1 \delta h_1$$

$$C_d A_\circ \sqrt{2g(h_1 - h_2)}\ \delta t = \frac{A_2 A_1}{A_1 + A_2} \delta h \ \ldots\ldots\ldots . (9.27)$$

Re arranging and integrating between the two levels we get

$$\delta t \;=\; \frac{A_2 A_1}{A_1 + A_2 C_d A_\circ \sqrt{2g}} \frac{\delta h}{\sqrt{h}} \ldots\ldots\ldots\ldots\ldots\ldots (9.28)$$

$$t = \frac{A_2 A_1}{A_1 + A_2 C_d A_\circ \sqrt{2g}} \int_{h_{initial}}^{h_{final}} \frac{\delta h}{\sqrt{h}} \ldots\ldots\ldots\ldots\ldots (9.29)$$

$$t \;=\; \frac{2A_2 A_1}{A_1 + A_2 C_d A_\circ \sqrt{2g}} \left[\sqrt{h}\right]_{h_{initial}}^{h_{final}}$$

$$t = \frac{2A_2 A_1}{A_1 + A_2 C_d A_\circ \sqrt{2g}} \left[\sqrt{h_{initial}} - \sqrt{h_{final}}\right] \ldots\ldots (9.30)$$

(remember that *h* in this expression is the *difference* in height between the two levels (h_2 - h_1) to get the time for the levels to equal use $h_{initial} = h_1$ and $h_{final} = 0$). Thus we have an expression giving the time it will take for the two levels to equal.

(b)Orifice-plate "meter."

The Bernoulli principle—of a decrease in pressure in an accelerated stream—can be employed for the measurement of fluid flow rates in the device shown in Fig. 9.5. There, an *orifice plate* consisting of a circular disc with a central hole of area A_o is bolted between the flanges on two sections of pipe of cross-sectional area A_1.

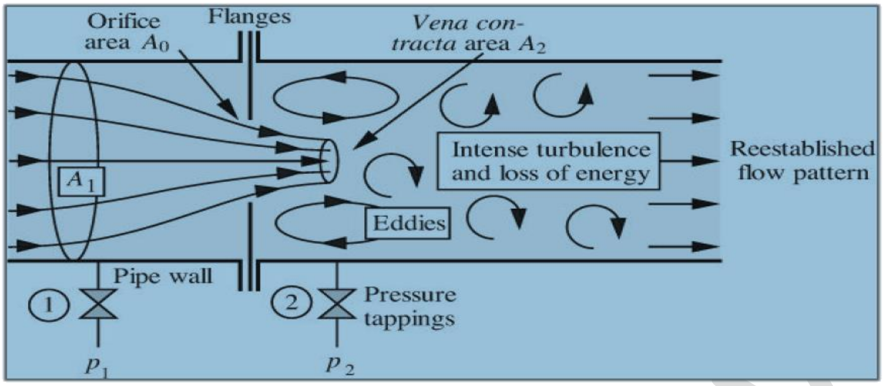

Fig.9.5 Flow through an orifice plate.

Bernoulli's equation applies to the fluid as it flows from left to right through the orifice of a reduced area because it is found experimentally that a contracting stream is relatively stable,

so that frictional dissipation can be ignored, especially over such a short distance. Hence, as the velocity increases, the pressure decreases. The following theory demonstrates that by measuring the pressure drop $p_1 - p_2$, it is possible to determine the upstream velocity u_1. Let u_2 be the velocity of the jet at the vena contracta. Bernoulli's equation applied between points 1 and 2, which have the same elevation ($z_1 = z_2$), gives:

$$\frac{u_1^2}{2} + gz_1 + \frac{p_1}{\rho} = \frac{u_2^2}{2} + gz_2 + \frac{p_2}{\rho} \dots\dots\dots\dots(9.31)$$

Conservation of mass between points 1 and 2 gives the continuity equation:

$$Q = A_1 u_1 = A_2 u_2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(9.32)$$

Elimination of u_2 between Eqns. (8.48) and (8.49) gives:

$$\frac{u_1^2}{2} + gz_1 + \frac{p_1}{\rho} = \frac{A_1^2 u_1^2}{2A_2^2} + gz_2 + \frac{p_2}{\rho} \dots\dots\dots\dots(9.33)$$

Solution for u_1 yields:

$$u_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\frac{A_1^2}{A_2^2} - 1\right)}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(9.34)$$

so that the volumetric flow rate Q is:

$$Q = A_1 u_1 = A_1 \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\frac{A_1^2}{A_2^2} - 1\right)}} = A_1 \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\frac{A_1^2}{C_c^2 A_o^2} - 1\right)}} \ldots .(9.35)$$

In Eqn. (9.35), the coefficient of contraction *Cc* is approximately 0.63 in most cases. However, the following version, which is somewhat less logical than Eqn. (9.35) and uses a dimensionless *discharge coefficient* C_D, is used in practice instead:

$$Q = C_D A_1 \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\frac{A_1^2}{A_o^2} - 1\right)}} \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(9.36)$$

Fig. 9.5 shows how C_D varies with two additional dimensionless groups, namely, the ratio of the orifice diameter to the pipe diameter, and the *Reynolds number* through the orifice:

$$\frac{D_o}{D_1}, Re_o = \frac{u_o D_O \rho}{\mu} \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(9.37)$$

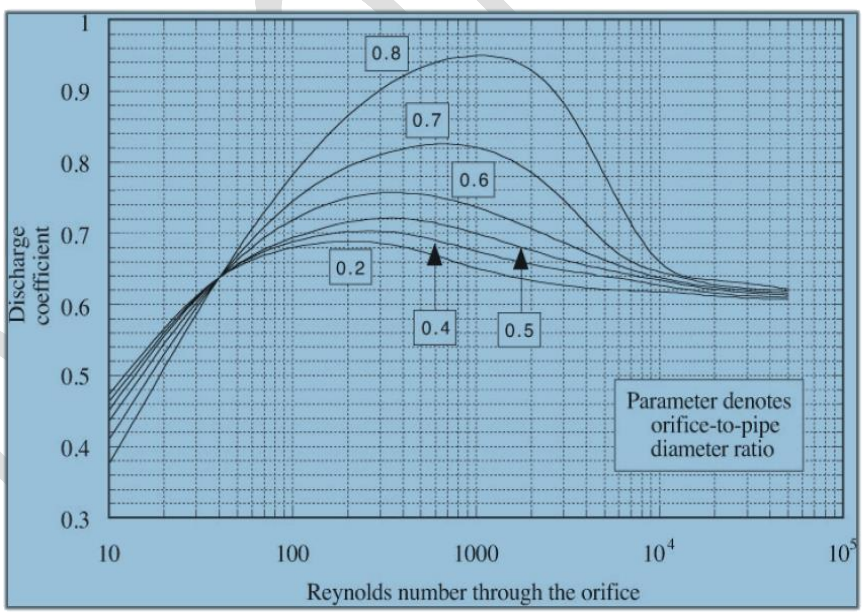

Fig. 9.6 Discharge coefficient for orifice plates. Based on values from G.G. Brown et al., Unit Operations, John Wiley & Sons, New York, 1950.

(c)Pitot tube.

The device shown in Fig. 9.7 is also based on the Bernoulli principle, and is used for finding the velocity of a moving craft such as a boat or an airplane. Here, the *Pitot tube* is attached to a boat, for example, which is moving steadily with an unknown velocity $u1$ through otherwise stagnant water of density ρ. The submerged tip of the tube faces the direction of motion; the pressure at the tip can be found from the height h to which the water rises in the tube or—more practically—by a pressure transducer

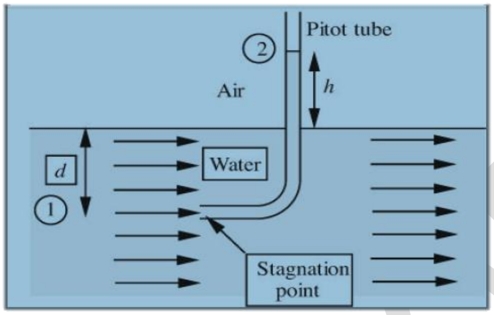

Fig. 9.7 Pitot tube.

For simplicity, consider the motion relative to an observer on the boat, in which case the Pitot tube is effectively stationary, with water approaching it with an upstream velocity $u1$. Opposite the Pitot tube, the oncoming water decelerates and comes to rest at the stagnation point at the tip of the tube. Application of Bernoulli′s equation between points 1 and 2 gives:

$$\frac{\quad}{2} + 0 + \frac{\quad}{\rho} = \frac{\quad}{2} + g(h + d) + \frac{\quad}{\rho} \ldots\ldots\ldots\ldots (9.38)$$

in which the first zero recognizes the datum level $z_1 = 0$ at point 1, and the second zero indicates that the water is stagnant with $u_2 = 0$ at point 2. But from hydrostatics, the pressure at point 1 is:

$$p_1 = p_2 + \rho g d \ldots\ldots\ldots\ldots (9.39)$$

Subtraction of Eqn. (8.55) from Eqn. (8.56) gives:

$$u_1 = \sqrt{2gh} \ldots\ldots\ldots\ldots\ldots (9.40)$$

so that the velocity u_1 of the boat is readily determined from the height of the water in the tube. In practice, a pressure transducer would probably be used for monitoring the excess pressure (corresponding to ρgh) instead of measuring the water level, but we have retained the latter because it is conceptually simpler.

A very similar device, called the Pitot-static tube, is shown in Fig. 8.7, and is employed for measuring the velocity at different radial locations in a pipe. Here, two tubes are involved. The left-hand tube simply measures the pressure and the movable right-hand one is essentially a Pitot tube as before. The velocity u1 at the particular transverse location where the Pitot tube is placed is given by:

$$u_1 = \sqrt{2gh} \ldots\ldots\ldots\ldots\ldots.(9.41)$$

(d) Pitot Static Tube

The necessity of two piezometers and thus two readings make this arrangement a little awkward. Connecting the piezometers to a manometer would simplify things but there are still two tubes. The *Pitot static* tube combines the tubes and they can then be easily connected to a manometer. A Pitot static tube is shown below. The holes on the side of the tube connect to one side of a manometer and register *the static head, (h_1),* while the central hole is connected to the other side of the manometer to register, as before, the *stagnation head (h_2).*

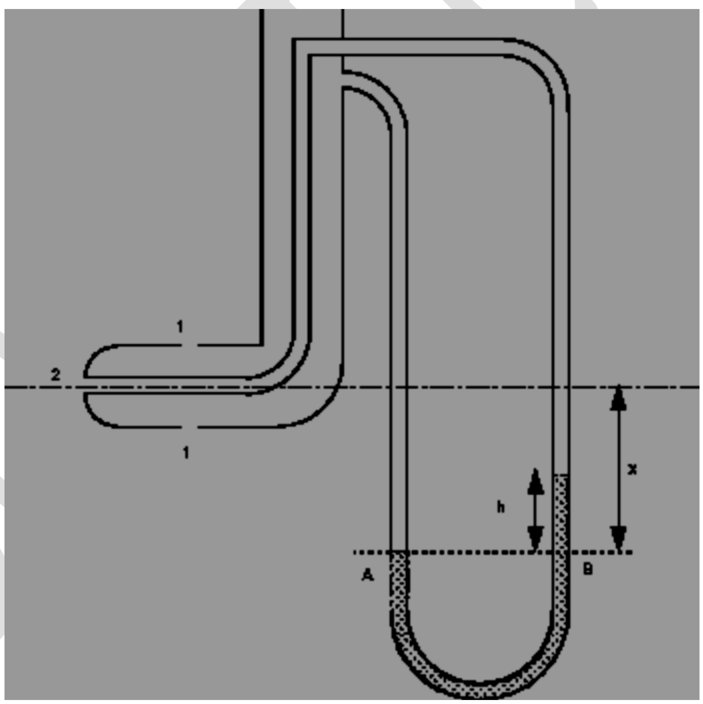

Fig. (9.8) A Pitot-static tube

Consider the pressures on the level of the center line of the Pitot tube and using the theory of the manometer

$$p_A = p_2 + \rho g X \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.42)$$

$$p_B = p_1 + \rho g(X - h) + \rho_{man} g h$$

$$p_A = p_B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(9.43)$$

$$p_2 + \rho g X = p_1 + \rho g(X - h) + \rho_{man} g h$$

We know that $p_2 = p_{static} = p_1 + \frac{1}{2}\rho u_1^2$, substituting this in to the above gives

$$p_1 + gh(\rho_{man} - \rho) = p_1 + \frac{1}{2}\rho u_1^2$$

$$u_1 = \sqrt{\frac{2gh(\rho_{man} - \rho)}{\rho}} \ldots\ldots\ldots\ldots\ldots..(9.44)$$

The Pitot/Pitot-static tubes give velocities at points in the flow. It does not give the overall discharge of the stream, which is often what is wanted. It also has the drawback that it is liable to block easily, particularly if there is significant debris in the flow.

3.5.8 Flow Over Notches and Weirs

A notch is an opening in the side of a tank or reservoir which extends above the surface of the liquid. It is usually a device for measuring discharge. A weir is a notch on a larger scale - usually found in rivers. It may be sharp crested but also may have a substantial width in the direction of flow - it is used as both a flow measuring device and a device to raise water levels.

3.5.8.1 Weir Assumptions

We will assume that the velocity of the fluid approaching the weir is small so that kinetic energy can be neglected. We will also assume that the velocity through any elemental strip depends only on the depth below the free surface. These are acceptable assumptions for tanks with notches or reservoirs with weirs, but for flows where the velocity approaching the weir is substantial the kinetic energy must be taken into account (e.g. a fast moving river).

3.5.8.2 A General Weir Equation

To determine an expression for the theoretical flow through a notch we will consider a horizontal strip of width b and depth h below the free surface, as shown in the figure below.

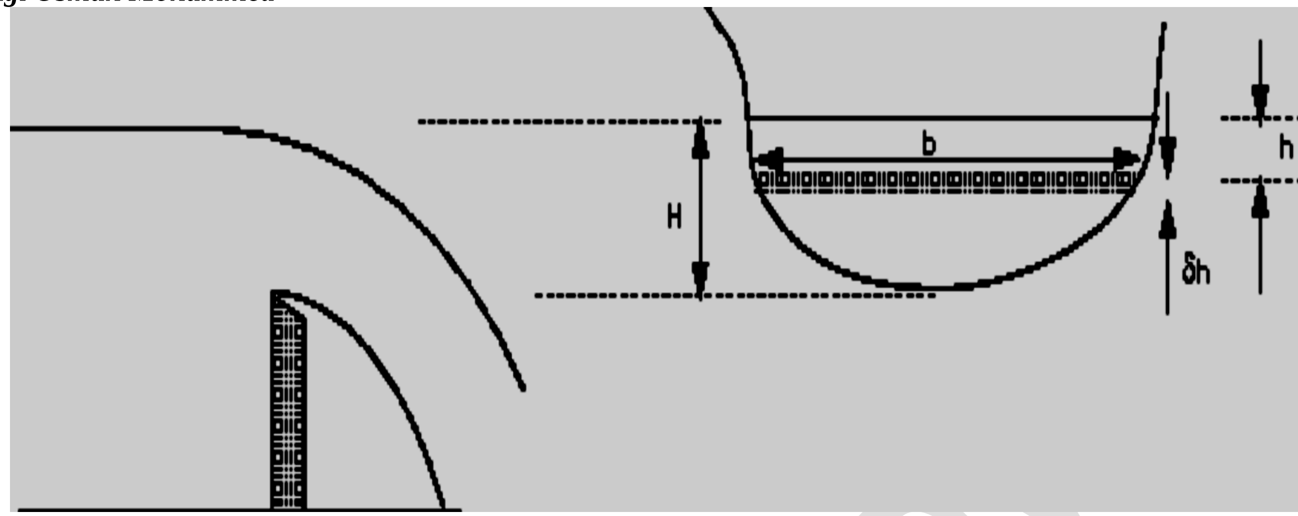

Fig (9.9) Elemental strip of flow through a notch

velocity through the strip, $u = \sqrt{2gh}$ ……………….……….…..(9.45)

discharge through the strip $\delta Q = Au = b\delta h\sqrt{2gh}$ ….……….……..(9.46)

integrating from the free surface, h = 0, to the weir crest, h = H gives the expression for the total theoretical discharge

$$Q_{theoretical} = \sqrt{2g} \int_0^H bh^{\frac{1}{2}}\, dh \text{ ……} (9.46)$$

This will be different for every differently shaped weir or notch. To make further use of this equation we need an expression relating the width of flow across the weir to the depth below the free surface.

3.5.8.3 Rectangular Weir

For a rectangular weir the width does not change with depth so there is no relationship between b and depth h. We have the equation,

b = constant = B

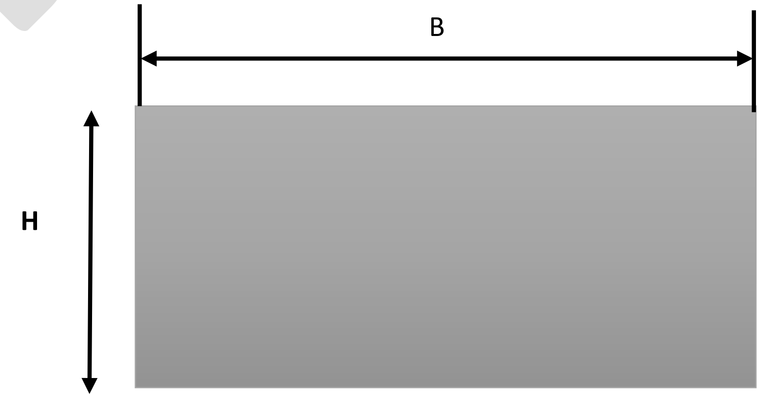

Fig .(9.10)A rectangular weir

Substituting this into the general weir equation gives

$$Q_{theoretical} = B\sqrt{2g} \int_0^H h^{\frac{1}{2}}\, dh \quad \ldots\ldots (9.47)$$

$$Q_{theoretical} = \frac{2}{3} B\sqrt{2g}\, h^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots (9.48)$$

To calculate the actual discharge we introduce a coefficient of discharge, C_d , which accounts for losses at the edges of the weir and contractions in the area of flow, giving

$$Q_{actual} = C_d \frac{2}{3} B\sqrt{2g}\, h^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots (9.49)$$

3.5.8.4 'V' Notch Weir

For the "V" notch weir the relationship between width and depth is dependent on the angle of the "V".

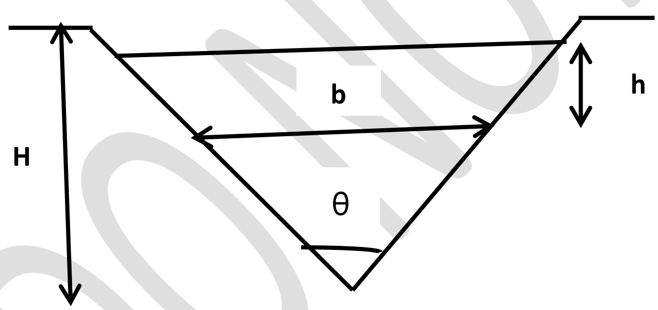

Fig (9.12)"V" notch, or triangular, weir geometry

If the angle of the "V" is q then the width, *b*, a depth *h* from the free surface is

$$b = 2(H - h)\tan\frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.50)$$

So the discharge is

$$Q_{theoretical} = 2\sqrt{2g}\, tan\frac{\theta}{2}\int_{0}^{H}(H-h)h^{\frac{1}{2}}\, dh \dots\dots\dots\dots (9.51)$$

$$2\sqrt{2g}\, tan\frac{\theta}{2}\left[\frac{2}{5}Hh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}}\right]_{0}^{H} = \frac{8}{15}\sqrt{2g}\, tan\frac{\theta}{2}H^{\frac{5}{2}}$$

And again, the actual discharge is obtained by introducing a coefficient of discharge

$$Q_{actual} = C_d\frac{8}{15}\sqrt{2g}\, tan\frac{\theta}{2}H^{\frac{5}{2}} \dots\dots\dots\dots\dots\dots\dots\dots (9.52)$$