Lecture NO (13)
Compressible Gas Flow

13.1 Compressible Gas Flow in Pipelines

An example of this type of flow is closely approximated by the interstate underground pipeline transport of natural gas, in which the surrounding ground maintains essentially isothermal conditions. Therefore, consider the steady flow of an ideal gas of molecular weight M_w in a long-distance horizontal pipeline of length L and diameter D, as shown in

Fig. 13.1. The inlet and exit pressures and densities are p_1, ρ_1 and p_2, ρ_2, respectively. The pipeline is assumed to be sufficiently long in relation to its diameter that it comes into thermal equilibrium with its surroundings; thus, the flow is *isothermal*, at an absolute temperature T.

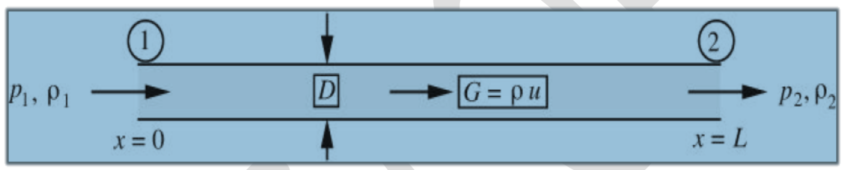

Fig. 13.1 Isothermal flow of gas in a pipeline.

$$d\left(\frac{u^2}{2}\right) + \frac{dp}{\rho} + df = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (13.1)$$

Note that the change in kinetic energy cannot necessarily be ignored, since fluctuations in density will cause the gas either to accelerate or decelerate. Expansion of the differential, substitution of an alternative expression for df, and division by u^2 produces:

$$\frac{du}{u} + \frac{dp}{\rho u^2} + 2f_F\frac{dx}{D} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (13.2)$$

Because of continuity, the mass velocity $G = \rho u$ is constant:

$$dG = 0 = \rho du + ud\rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \ldots\ldots (`13.3)$$

or, since the density is proportional to the absolute pressure:

$$\frac{du}{u} = -\frac{d\rho}{\rho} = -\frac{dp}{p} \ldots\ldots\ldots\ldots. \ldots\ldots (`13.4)$$

Also note that

$$\frac{1}{\rho u^2} = \frac{\rho}{G^2} = \frac{\rho_1}{p_1}\frac{p}{G^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (13.5)$$

From Eqns. (13.2), (13.4), and (13.5), there results:

$$= -\int_{p_1}^{p_2} \frac{dp}{p} = \frac{\rho_1}{p_1 G^2}\int_{p_1}^{p_2} p dp + \frac{2f_F}{D}\int_0^L dx = 0 \ldots\ldots (13.6)$$

Note that since$\dot{m}$ is constant, and the viscosity of a gas is virtually independent of pressure, the Reynolds number, Re = $\rho uD/\mu = GD/\mu$, is essentially constant. Hence, the friction factor f_F, which depends only on the Reynolds number and the roughness ratio, is justifiably taken outside the integral in Eqn. (13.6). Performing the integration:

$$\frac{2f_F L}{D} = (p_1^2 + p_2^2)\frac{\rho_1}{\rho_1 G^2} + ln\left(\frac{p_2}{p_1}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13.7)$$

The mass velocity in the pipeline is therefore:

$$G^2 = \frac{\rho_1}{p_1}\frac{(p_1^2 + p_2^2)}{\frac{4f_F L}{D} - ln\left(\frac{p_2}{p_1}\right)^2} = \frac{M_w}{RT}\frac{(p_1^2 + p_2^2)}{\frac{4f_F L}{D} - ln\left(\frac{p_2}{p_1}\right)^2} \ldots\ldots\ldots. (13.8)$$

The last term in the denominator, being derived from the kinetic-energy term of Eqn. (13.1), is typically relatively small; if indeed it can be ignored (such an assumption should be checked with numerical values), the *Weymouth* equation results:

$$G^2 = \frac{M_w}{RT}\frac{D}{4f_F L}(p_1^2 + p_2^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13.9)$$

However, if the ratio of the absolute pressures p_2/p_1 is significantly less than unity, the last term in the denominator of Eqn. (13.8) cannot be ignored. Consider the situation in which the exit pressure p_2 is progressively reduced below the inlet pressure, as shown in Fig. 3.18. As expected, equation (13.8) predicts an initial increase in the mass velocity **G** as p_2 is reduced below p_1. However, a maximum value of **G** is eventually reached when p_2 has fallen to a critical value; a further reduction in the exit pressure then apparently leads to a reduction in **G**.

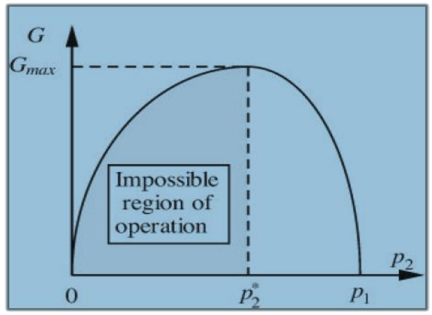

Fig. 13.2 Mass velocity as a function of the exit pressure.

The critical exit pressure is obtained by noting that at the maximum,

$$\frac{dG^2}{dp_2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13.10)$$

which, when applied to Eqn. (13.8), gives, after some algebra:

$$\left(\frac{p_1}{p_2^*}\right)^2 - ln\left(\frac{p_1}{p_2^*}\right)^2 = 1 + \frac{4f_F L}{D} \ldots\ldots\ldots\ldots\ldots (13.11)$$

which can be solved for p_2^*. The corresponding maximum mass velocity G_{max} can be shown to obey the equation

$$G_{max} = \frac{\rho_1}{p_1} p_2^* \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (A)$$

The derivation of Eqn. (13.12) is somewhat tricky. First eliminate ln between Eqns. (13.8) and (13.11), giving: Then substitute for (p_2^*) from Eqn. (A) in both the numerator and denominator of Eqn. (11.8). Rearrangement then yields Eqn.(13.12).

$$G_{max}^2 = \frac{p_1\rho_1}{1 + \frac{4f_F L}{D} + ln\frac{p_1\rho_1}{G_{max}^2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (13.12)$$

which can be solved for G_{max} by successive substitution or Newton's method. The curve in Fig. 13.2 in the region 0 < p2 < $p_2^*$is actually an illusion because it would involve a decrease of entropy, and a further reduction of p_2 below p_2^* does *not* decrease the flow rate. Instead, the exit pressure remains at $p_2^*$and there is a sudden irreversible expansion or *shock wave* at the pipe exit from p_2^* down to the pressure p_2 (< p_2^*) just outside the exit. The shock occurs over an extremely narrow region of a few molecules in thickness, in which there are abrupt changes in pressure, temperature, density, and velocity. It may also be shown that:

$$G_{max} = \sqrt{\rho_2^* p_2^*} = \rho_2^* u_2^* \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (13.13)$$

Also, the corresponding exit velocity

$$u_2^* = \sqrt{\frac{p_2^*}{\rho_2^*}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(13.14)$$

can be interpreted as the velocity of a hypothetical isothermal sound wave at the exit conditions, since we have the following relations for the velocity of sound and an ideal gas:

$$c = \sqrt{\frac{dp}{d\rho}} = \frac{p}{\rho} = \frac{RT}{M_w} \dots\dots\dots\dots\dots\dots\dots\dots.(13.15)$$

Thus, the velocity of a hypothetical isothermal sound wave is given by:

$$Isothermal : c = \sqrt{\frac{RT}{M_w}} \dots\dots\dots\dots\dots\dots.(13.16)$$

In practice, however, sound waves travel nearly isentropically, and the sonic velocity is then:

$$Isenttropically : c = \sqrt{\frac{\gamma RT}{M_w}} \dots\dots\dots\dots\dots\dots.(13.16)$$

in which $\gamma = c_p/c_v$ is the ratio of the specific heat at constant pressure to that at constant volume.

13.2 Compressible Flow in Nozzles

As shown in Fig. 13.3, the gas in the reservoir has an absolute pressure p_1 and a density ρ1; the final discharge is typically to the atmosphere, at an absolute pressure p_2. The transfer of gas is rapid and there is little chance of heat transfer to the wall of the nozzle, so the flow is adiabatic. Furthermore, since only short lengths are involved, friction may be neglected, so the expansion is isentropic, being governed by the equation:

$$\frac{p}{\rho^\gamma} = c = \frac{p_1}{{\rho^\gamma}_1} \ or = \rho = p_1 \left(\frac{p}{p_1}\right)^{1/\gamma} \dots\dots\dots\dots\dots\dots\dots..(13.17)$$

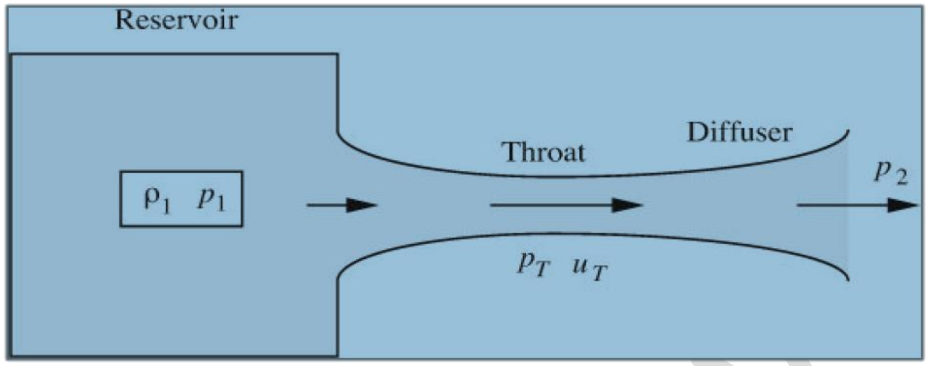

Fig. 13.3 Flow through a converging/diverging nozzle

in which *c* is a constant and $\gamma = c_p/c_v$, the ratio of specific heats. An ideal gas and constant specific heat are assumed.

For horizontal flow between the reservoir and some downstream position where the velocity is u and the pressure is p, Bernoulli's equation gives:

$$\frac{u^2}{2}-\frac{u_1^2}{2}+\int_{0}^{p}\frac{dp}{\rho}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13.18)$$

in which the integral is:

$$\int_{p_1}^{p}\frac{dp}{\rho}=\frac{p_1^{1/\gamma}}{\rho_1}\int_{p_1}^{p}\frac{dp}{p^{1/\gamma}}=\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left[\left(\frac{p}{p_1}\right)^{(\gamma-1)/\gamma}-1\right] \ldots\ldots\ldots\ldots\ldots\ldots (13.19)$$

Because the velocity u_1 in the reservoir is essentially zero, the last two equations yield a relation between the velocity and pressure at any point in the flowing gas:

$$u^2=\frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left[1-\left(\frac{p}{p_1}\right)^{(\gamma-1)/\gamma}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (13.20)$$

Since m = ρ uA at any location, where m is the mass flow rate and A is the cross-sectional area of the nozzle, Eqn. (13.20) may be rewritten as:

$$\left(\frac{m}{A}\right)^2=\frac{2\gamma}{\gamma-1}p_1\rho_1\left[1-\left(\frac{p}{p_1}\right)^{(\gamma-1)/\gamma}\right]\left(\frac{p}{p_1}\right)^{\frac{2}{\gamma}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (13.21)$$

The mass velocity *m/A* is clearly a maximum at the throat, where it has the value m/A_T. However, the pressure at the throat is still a variable, and a maximum of m/A_T occurs with respect to *p* when:

$$\frac{d\left({}^{m}/_{A_T}\right)}{Ad\left({}^{p}/_{p_1}\right)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (13.22)$$

After some algebra, these last two equations give the critical pressure ratio at the throat, corresponding to the maximum possible mass flow rate:

$$\frac{p_{cT}}{p_1} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (13.23)$$

After several lines of algebra involving Eqns. (13.17),(1320), and (13.23), the corresponding velocity u_{cT} at the throat is found to be:

$$u_{cT}^2 = \left(\frac{2}{\gamma+1}\right)\frac{p_1}{\rho_1} = \frac{\gamma p_{cT}}{p_{cT}} = \frac{\gamma R T_{cT}}{M_w} \ldots\ldots\ldots\ldots. (13.24)$$

in which the subscript *c* denotes critical conditions. That is, the gas velocity at the throat equals the local sonic velocity, as in Eqn. (13.16). Under these conditions, known as choking at the throat, the critical mass flow rate mc is:

$$m_c = A_T\sqrt{\gamma p_1 \rho_1 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \ldots\ldots. (13.25)$$

Now consider what happens for various exit pressures p_2, decreasing progressively from the reservoir pressure p_1:

(A) If the exit pressure is slightly below the reservoir pressure, there will be a small flow rate, which can be computed from Eqn. (13.21) by substituting A = A_2 (the exit area) and p = p_2. Equation (13.21) then gives the variation of pressure in the nozzle. The flow is always subsonic.

(B) The same as A, except the mass flow rate is higher.

(C) If the exit pressure is reduced sufficiently, the velocity at the throat increases to the critical value given by Eqn. (13.24). In the diverging section, the pressure increases and the flow is subsonic.

(D) For an exit pressure lying between C and E, no continuous solution is possible. The flow, which is critical, is supersonic for a certain distance beyond the throat, but there is then a *very* sudden increase of pressure, known as a *shock*, and the flow is thereafter subsonic. The shock is an irreversible phenomenon, resulting in abrupt changes in velocity, pressure, and temperature over an extremely short distance of a few molecules in thickness.

(E) For the same critical mass flow rate as in C, Eqn. (13.21) has a second root, corresponding to an exit pressure at E in Fig. 13.4. In this case, however, there is a continuous *decrease* of pressure in the diffuser, where the flow is now *supersonic*.

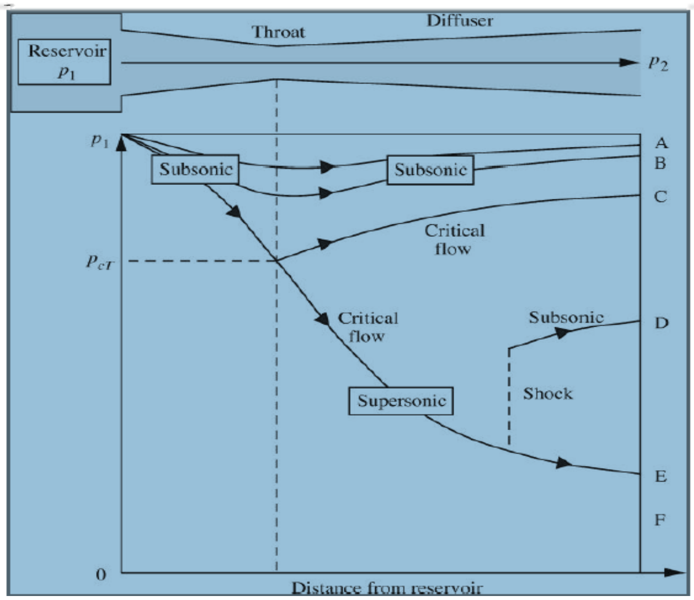

Fig. 13.4 Effect of varying exit pressure on nozzle flow.

(F) For an exit pressure lower than E, a further irreversible expansion occurs just outside the nozzle. If the diffuser section is absent, then the flow is essentially that through an orifice in a high-pressure reservoir. The flow will be subsonic if the exit pressure exceeds p_{cT}. If the exit pressure equals p_{cT}, then critical flow occurs with the sonic velocity through the orifice. And if it falls below this value, critical flow will still occur, but with a further irreversible expansion just outside the orifice.

13.3. Complex Piping Systems

The chemical engineer should be prepared to cope with pumping and piping systems that are far more complicated than the examples discussed thus far in this course. The only such type of a more complex system to be considered here is the simplest, which involves steady, incompressible flow. A modest complication of the systems already studied can be seen by glancing at the scheme in Fig. 13.5, in which water is pumped from a lowlying reservoir into a pipe that subsequently divides into two branches in order to feed two elevated tanks.

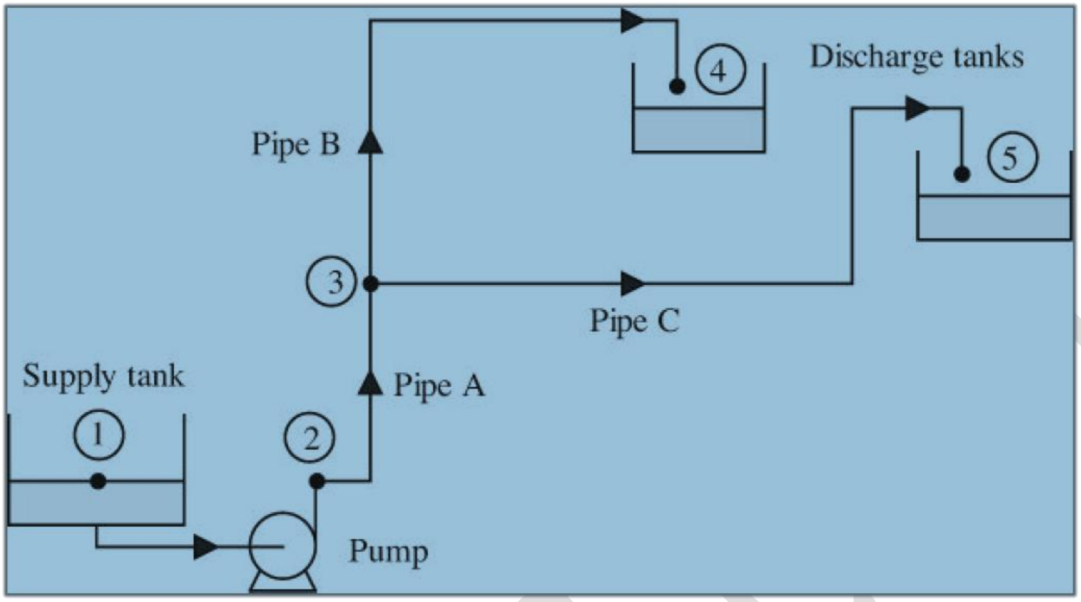

Fig. E3.7 Pumping and piping installation.

A system of simultaneous equations
based on the following principles is then developed:

(a) Continuity of mass (or volume, for an incompressible fluid): at any node, the sum of the incoming flow rates must equal the sum of the outgoing flow rates. For example, if pipe A leads into a node, and pipes B and C leave from it:

$$Q_A = Q_B + Q_C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (13.26)$$

(b) An energy balance for every segment of pipe that connects two nodes. It is customary to replace the mean velocity with the volumetric flow rate:

$$Q = \frac{\pi D^2}{4} m_c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (13.27)$$

so that a representative energy balance is:

$$g\Delta z + \frac{\Delta p}{\rho} = \frac{32 f_F Q^2 L}{\pi^2 D^5} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots. (13.28)$$

(c) An equation for each pump, such as:

$$\Delta p = a - bQ^2 \quad OR \ \frac{\Delta p}{\rho} + \omega = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13.29)$$

in which a and b are coefficients that depend on the particular pump. The second version in Eqn. (13.29) would only be used if any two of the following variables were specified: (a) the pump inlet pressure, (b) the pump discharge pressure, and (c) the work performed per unit mass flowing. The system of simultaneous equations will be nonlinear, because of the Q^2 terms appearing in the pipe and pump equations, and can be solved for the unknown pressures and flow rates by methods that are largely governed by the complexity of the system: