# Optimal Strategy for the Die Game

**Abstract**

We consider a probabilistic die game where players roll an $n$-sided die repeatedly in pursuit of a high target score, with the caveat that repeating any previously rolled number in a session(turn) causes the score to reset to zero. This creates a classic risk-reward trade-off. We analyze optimal stopping strategies using expected value analysis and derive asymptotic expressions for the maximum number of rolls a player should take. We show that this optimal roll count grows as $\sqrt[3]{n^2+n}$ or $\sqrt{n}$ depending on how the die is defined. These results offer insight into strategic decision-making in games of chance. While this game setup does not appear directly in prior literature, it draws parallels with optimal stopping theory, gambler's ruin, and memory-based risk games.

## 1. Game Description and Motivation

Consider a game played with a die with $n$ distinct sides. Each player can roll the die as many times as they like. The cumulative score in a turn, called the *current score*, is the sum of all rolls. If the player rolls a value they've already rolled in the same session, their score resets to zero. To avoid this, a player may stop at any point and let the opponent continue. The player who reaches a sufficiently large target score first wins.

We seek to answer: **What is the optimal number of rolls before one should stop?**

Assuming the target score is large, we apply the Law of Large Numbers to evaluate whether a player should continue rolling or stop, based on expected gains.

## 2. Situation 1: Die with Faces {1, 2, . . . , n}

Let:

- $n$: number of die faces
- $k$: number of rolls
- $S_k$: score after $k$ rolls with all distinct values
- $E_k$: expected score after rolling $k$ times

Since each face appears exactly once and values are increasing from 1 to $n$, the best-case distinct sequence of rolls yields:

$$S_k = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

If $k$ rolls have succeeded without repeating, the remaining possible outcomes are $n-k$, and all values are uniformly likely. The expected gain from one more roll is:

$$E_k = \frac{1}{n}\sum_{j=1}^{n-k}(S_k+k+j) = \frac{(n-k)(S_k+k)+\frac{(n-k)(n-k+1)}{2}}{n}$$
$$= \frac{(n-k)(n+k+2S_k+1)}{2n}$$

Substituting $S_k = \frac{k(k+1)}{2}$, we get:

$$E_k = \frac{(n-k)(k^2+2k+n+1)}{2n}$$

We now determine the optimal $k$ by comparing $S_k$ and $E_k$.

At optimal $k$:

$$S_k \geq E_k \Rightarrow \frac{k(k+1)}{2} \geq \frac{(n-k)(k^2+2k+n+1)}{2n} \Rightarrow k^3+2k^2+k \geq n^2+n$$

At $k-1$:

$$\frac{(k-1)k}{2} < \frac{(n-k+1)(k^2+n)}{2n} \Rightarrow k^3-k^2 < n^2+n$$

Dividing by $k^3$ gives:

$$1-\frac{1}{k} < \frac{n^2+n}{k^3} \leq 1+\frac{2}{k}+\frac{1}{k^2}$$

By the squeeze theorem:

$$\lim_{k\to\infty}\frac{n^2+n}{k^3} = 1 \Rightarrow k \sim \sqrt[3]{n^2+n}$$

**Example 1.** *Let $n = 12$. Then $k \approx \sqrt[3]{156} \approx 5.38$, so the player should not exceed 5 rolls.*

Table 1: Example with $n = 12$ and $k = 5$

| Roll Sequence | Score So Far | Probability of Next Being Safe |
|---|---|---|
| $\{1,2,3,4,5\}$ | 15 | 7/12 |
| Expected value if continuing | $\approx 14$ | - |

*So, a smart player should stop after 5 rolls.*

**Example 2.** *Let $n = 6$ (standard die). Then $k \approx \sqrt[3]{36+6} \approx 3.48$, so the player should stop after 3 rolls.*

Table 2: Example with $n = 6$ and $k = 3$

| Roll Sequence | Score So Far | Probability of Next Being Safe |
|---|---|---|
| $\{1,2,3\}$ | 6 | 3/6 |
| Expected value if continuing | $\approx 5.25$ | - |

**Example 3.** *Let $n = 20$. Then $k \approx \sqrt[3]{400+20} \approx 7.37$, so the player should stop after 7 rolls.*

Table 3: Example with $n = 20$ and $k = 7$

| Roll Sequence | Score So Far | Probability of Next Being Safe |
|---|---|---|
| $\{1,2,3,4,5,6,7\}$ | 28 | 13/20 |
| Expected value if continuing | $\approx 27.3$ | - |

# 3. Situation 2: Die with All Faces Equal to 1

Let each die face be labeled distinctly but valued as 1. Then:

$$S_k = k, \quad E_k = \frac{(k+1)(n-k)}{n}$$

Optimal $k$ satisfies, $S_k \geq E_k$:

$$k \geq \frac{(k+1)(n-k)}{n} \Rightarrow k^2 + k \geq n$$

Also at $k-1$, $S_k < E_k$ :

$$k - 1 < \frac{k(n-k+1)}{n} \Rightarrow k^2 - k < n$$

Then:

$$1 - \frac{1}{k} < \frac{n}{k^2} < 1 + \frac{1}{k} \Rightarrow \lim_{k\to\infty} \frac{n}{k^2} = 1 \Rightarrow k \sim \sqrt{n}$$

**Example 4.** *Let $n = 100$. Then $k \approx \sqrt{100} = 10$, so the player should stop after 10 rolls.*

Table 4: Example with $n = 100$ and $k = 10$

| Roll Sequence | Score So Far | Probability of Next Being Safe |
|---|---|---|
| $\{1,1,\ldots,1\}$ (10 times) | 10 | 90/100 |
| Expected value if continuing | $\approx 10.89$ | - |

**Example 5.** *Let $n = 50$. Then $k \approx \sqrt{50} \approx 7.07$, so the player should stop after 7 rolls.*

Table 5: Example with $n = 50$ and $k = 7$

| Roll Sequence | Score So Far | Probability of Next Being Safe |
|---|---|---|
| $\{1,1,\ldots,1\}$ (7 times) | 7 | 43/50 |
| Expected value if continuing | $\approx 7.52$ | - |

## 4. Conclusion

We derived the optimal number of die rolls a player should attempt before stopping, balancing risk and expected gain. For a die with distinct integer faces from 1 to $n$, the optimal roll count scales like $\sqrt[3]{n^2+n}$. If each face has equal value, it scales like $\sqrt{n}$. This strategy minimizes the risk of losing all progress due to repeat rolls.

Future directions may include considering non-uniform dice and dynamic targets.