Problem 4:

Goal

We want to calculate the entropy of some distributions, and if it is possible to find an analytic solution, we will do so. Otherwise, we will provide the numerical solution.

1. $p(n) \propto \exp(-\lambda n), \quad n = 0, 1, 2, \ldots$

First,we normalize p(n) to ensure that it sums to one, using the following normalization constant S, which ensures that the sum of probabilities becomes one:

$$p(n) = \frac{\exp(-\lambda n)}{S}$$

and the normalization constant S is given by:

$$S = \sum_{n=0}^{\infty} \exp(-\lambda n) = \frac{1}{1 - \exp(-\lambda)}, \quad \text{valid for} \quad \lambda > 0.$$

Then the normalized probability is this:

$$p(n) = \frac{\exp(-\lambda n)}{S} = \frac{\exp(-\lambda n)}{\frac{1}{1-\exp(-\lambda)}} = (1 - \exp(-\lambda)) \exp(-\lambda n)$$

Now we can calculate entropy from this formula:

$$H(X) = -\sum_{n=0}^{\infty} p(n) \log p(n)$$

Now we find an analytic solution:

$$H = -\sum_{n=0}^{\infty} ((1 - \exp(-\lambda)) \exp(-\lambda n)) \log_2 ((1 - \exp(-\lambda)) \exp(-\lambda n))$$

Using the property of logarithms $\log_b(x^a) = a \log_b(x)$.

$$\log_2 (\exp(-\lambda n)) = -\lambda n \log_2 e$$

Using the property of logarithms $\log_b(xy) = \log_b(x) + \log_b(y)$ The logarithm can be split:

$$\log_2\left((1-\exp(-\lambda))\exp(-\lambda n)\right) = \log_2(1-\exp(-\lambda))+\log_2(\exp(-\lambda n)) = \log_2(1-\exp(-\lambda))-\lambda n\log_2 e$$

We can break the summation into two parts:

$$H = -\log_2(1-\exp(-\lambda))\sum_{n=0}^{\infty}\left((1-\exp(-\lambda))\exp(-\lambda n)\right)+\lambda\log_2 e\sum_{n=0}^{\infty} n\left((1-\exp(-\lambda))\exp(-\lambda n)\right)$$

For the first part:

$$\sum_{n=0}^{\infty} p(n) = \sum_{n=0}^{\infty}\left((1-\exp(-\lambda))\exp(-\lambda n)\right) = 1 \quad \text{(Normalization)}$$

Thus:

$$-\log_2(1-\exp(-\lambda))\cdot 1 = -\log_2(1-\exp(-\lambda))$$

For the second sum:

$$\sum_{n=0}^{\infty} np(n) = \sum_{n=0}^{\infty} n(1-\exp(-\lambda))\exp(-\lambda n) = (1-\exp(-\lambda))\sum_{n=0}^{\infty} n\exp(-\lambda n)$$

Let $q = \exp(-\lambda)$ so the sum becomes:

$$\sum_{n=0}^{\infty} nq^n$$

This sum is a standard formula for the expected value of a geometric distribution with parameter q:

$$\sum_{n=0}^{\infty} nq^n = \frac{q}{(1-q)^2}$$

Proof:

Start with the geometric series:

$$S = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \quad \text{for } |q| < 1$$

Now, differentiate both sides with respect to q:

$$\frac{d}{dq}S = \frac{d}{dq}\left(\sum_{n=0}^{\infty} q^n\right) = \sum_{n=1}^{\infty} nq^{n-1}$$

The derivative of the geometric series is:

$$\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{1}{(1-q)^2}$$

Multiplying both sides of the derivative equation by q gives:

$$q\sum_{n=1}^{\infty} nq^{n-1} = q \cdot \frac{1}{(1-q)^2}$$

Thus, the sum becomes:

$$\sum_{n=0}^{\infty} nq^{n} = \frac{q}{(1-q)^2}$$

So we have this expression:

$$\sum_{n=0}^{\infty} np(n) = \frac{\exp(-\lambda)}{1-\exp(-\lambda)}$$

Now we can calculate Final Expression for Entropy:

$$H = -\log_2(1-\exp(-\lambda)) + \lambda \log_2 e \cdot \frac{\exp(-\lambda)}{1-\exp(-\lambda)}$$

To ensure that the series converges and the probability distribution is valid, we need to satisfy the following conditions:

Normalization Condition

The sum of the probabilities $p(n)$ must equal 1, i.e.,

$$\sum_{n=0}^{\infty} p(n) = 1.$$

For the distribution $p(n) \propto \exp(-\lambda n)$, the normalized form of $p(n)$ is given by:

$$p(n) = \frac{\exp(-\lambda n)}{S},$$

where S is the normalization constant.

$$S = \sum_{n=0}^{\infty} \exp(-\lambda n).$$

This is a geometric series, and its sum is:

$$S = \frac{1}{1-\exp(-\lambda)} \quad \text{for} \quad \lambda > 0.$$

Thus, the normalized probability distribution is:

$$p(n) = \frac{\exp(-\lambda n)}{1 - \exp(-\lambda)}.$$

Convergence of the Series:

The series $\sum_{n=0}^{\infty} \exp(-\lambda n)$ converges if $\lambda > 0$.

When $\lambda > 0$, the terms $\exp(-\lambda n)$ decay exponentially as $n \to \infty$, ensuring that the series converges. If $\lambda \leq 0$, the terms $\exp(-\lambda n)$ either do not decay or grow (for $\lambda < 0$), causing the series to diverge.

Validity of the Probability Distribution:

For $p(n)$ to be a valid probability distribution, we need:

- $p(n) \geq 0$ for all $n \geq 0$. This condition is satisfied because $\exp(-\lambda n) \geq 0$ for all $n \geq 0$ when $\lambda > 0$.
- The normalization condition $\sum_{n=0}^{\infty} p(n) = 1$, which is ensured by the normalization constant S.

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2. $p(n) \propto \exp(-n^2), \quad n = 0, 1, 2, \ldots$

Calculate the entropy of the distribution given by

$$p(n) \propto \exp(-n^2), \quad n = 0, 1, 2, \ldots$$

First, we normalize $p(n)$ to ensure that it sums to one:

$$p(n) \propto \exp(-n^2)$$

To normalize it, we need to find the normalization constant S, which is the sum of the unnormalized distribution:

$$S = \sum_{n=0}^{\infty} \exp(-n^2)$$

Thus:

$$p(n) = \frac{\exp(-n^2)}{S}$$

Next, we calculate the entropy H of this distribution. The entropy is given by:

$$H = -\sum_{n=0}^{\infty} p(n) \log_2 p(n)$$

$$H = -\sum_{n=0}^{\infty} \frac{\exp(-n^2)}{S} \log_2 \left(\frac{\exp(-n^2)}{S} \right)$$

Using the property of logarithms $\log_b(xy) = \log_b(x) + \log_b(y)$ The logarithm can be split:

$$H = -\sum_{n=0}^{\infty} \frac{\exp(-n^2)}{S} \left(-n^2 \log_2(e) - \log_2(S) \right)$$

$$H = \log_2(e) \sum_{n=0}^{\infty} \frac{n^2 \exp(-n^2)}{S} + \log_2(S) \sum_{n=0}^{\infty} \frac{\exp(-n^2)}{S}$$

For the second sum:

$$\sum_{n=0}^{\infty} p(n) = \sum_{n=0}^{\infty} \frac{\exp(-n^2)}{S} = 1$$

Now we can calculate Final Expression for Entropy:

$$H = \log_2(e) \sum_{n=0}^{\infty} \frac{n^2 \exp(-n^2)}{S} + \log_2(S)$$

These sums can be computed numerically. For the entropy, we will need to compute the expected value n^2 and the normalization constant S numerically. Thus, the analytic solution is impossible in terms of elementary functions, but we can compute the entropy numerically.

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3. $p(n) \propto n^{-4}, \quad n = 1, 2, 3, \ldots$

First we normalize the distribution so that the sum of probabilities equals one. The normalization constant S is determined by the sum over all possible values of n:

$$p(n) = \frac{n^{-4}}{S}$$

And the normalization constant S is given by:

$$S = \sum_{n=1}^{\infty} n^{-4}$$

This series is Euler-Riemann zeta function with S=4:

$$\zeta(S) = \sum_{n=1}^{\infty} \frac{1}{n^S} = \frac{1}{1^S} + \frac{1}{2^S} + \frac{1}{3^S} + \dots$$

The normalized distribution is:

$$\zeta(4) = \frac{\pi^4}{90} \quad \Rightarrow \quad p(n) = \frac{n^{-4}}{\zeta(4)} = \frac{n^{-4}}{\frac{\pi^4}{90}}$$

Next, we calculate the entropy H of this distribution. The entropy is given by:

$$H = -\sum_{n=1}^{\infty} p(n) \log_2 p(n) = -\sum_{n=1}^{\infty} \frac{n^{-4}}{\zeta(4)} \log_2 \left(\frac{n^{-4}}{\zeta(4)} \right)$$

$$H = -\sum_{n=1}^{\infty} \frac{n^{-4}}{\zeta(4)} \left(\log_2 n^{-4} - \log_2 \zeta(4) \right)$$

After simplification entropy is:

$$H = \frac{4}{\zeta(4)} \sum_{n=1}^{\infty} n^{-4} \log_2 n + \log_2 \zeta(4) \sum_{n=1}^{\infty} \frac{n^{-4}}{\zeta(4)}$$

The second sum is 1:

$$H = \frac{4}{\zeta(4)} \sum_{n=1}^{\infty} n^{-4} \log_2 n + \log_2 \zeta(4)$$

The sum of a complicated series like $\sum_{n=1}^{\infty} n^{-4} \log_2 n$ does not simplify to a simple expression, so we need numerical evaluation to compute entropy.

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4. $p(n) \propto \alpha^n$, $n = 1, \dots, N$

First,we normalize p(n) to ensure that it sums to one, using the following normalization constant S, which ensures that the sum of probabilities becomes one:

$$\sum_{n=1}^{N} p(n) = 1 \quad \Rightarrow \quad \sum_{n=1}^{N} \frac{\alpha^n}{S} = 1$$

and the normalization constant S is given by:

$$\sum_{i=0}^{n} x^i = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad \text{for } x \neq 1$$

$$\sum_{i=1}^{n} x^i = x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} - 1 = x\frac{1 - x^n}{1 - x}, \quad \text{for } x \neq 1$$

Then base on above expression we have:

$$S = \sum_{n=1}^{N} \alpha^n = \alpha \frac{1 - \alpha^N}{1 - \alpha} \quad \text{for} \quad \alpha \neq 1$$

Then the normalized probability is this:

$$p(n) = \frac{\alpha^n}{\frac{\alpha(1-\alpha^N)}{1-\alpha}} = \frac{(1 - \alpha)\alpha^n}{\alpha(1 - \alpha^N)} = \frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N}$$

Now we can calculate entropy from this formula:

$$H(X) = -\sum_{n=0}^{\infty} p(n) \log p(n)$$

Now we find an analytic solution:

$$H = -\sum_{n=1}^{N} \frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N} \log_2 \left(\frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N} \right)$$

Using the logarithm property $\log_2(xy) = \log_2 x + \log_2 y$:

$$\log_2 \left(\frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N} \right) = \log_2(1 - \alpha) + \log_2 \alpha^{n-1} - \log_2(1 - \alpha^N)$$

After simplification entropy is:

$$H = -\sum_{n=1}^{N} \frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N} \left[\log_2(1 - \alpha) + (n - 1) \log_2 \alpha - \log_2(1 - \alpha^N) \right]$$

We can break the summation into three parts:

$$H = -\log_2(1-\alpha) \sum_{n=1}^{N} \frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N} - \log_2 \alpha \sum_{n=1}^{N} (n-1) \frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N} + \log_2(1-\alpha^N) \sum_{n=1}^{N} \frac{(1 - \alpha)\alpha^{n-1}}{1 - \alpha^N}$$

The first and third sums simplify using normalization $\sum_{n=1}^{n=N} p(n) = 1$.
The normalization ensures $\sum_{n=1}^{N} p(n) = 1$, so:

$$H = -\log_2(1-\alpha) - \log_2 \alpha \sum_{n=1}^{N} (n-1)p(n) + \log_2(1-\alpha^N).$$

Evaluate $\sum_{n=1}^{N} (n-1)p(n)$
Substitute $p(n) = \frac{(1-\alpha)\alpha^{n-1}}{1-\alpha^N}$:

$$\sum_{n=1}^{N} (n-1)p(n) = \frac{(1-\alpha)}{1-\alpha^N} \sum_{n=1}^{N} (n-1)\alpha^{n-1}.$$

1. The geometric series $\sum_{n=1}^{N} \alpha^{n-1}$:

$$\sum_{n=1}^{N} \alpha^{n-1} = \frac{1-\alpha^N}{1-\alpha}.$$

2. For $\sum_{n=1}^{N} n\alpha^{n-1}$, use the formula:

$$\sum_{n=1}^{N} n\alpha^{n-1} = \frac{1-\alpha^N}{(1-\alpha)^2} - \frac{N\alpha^N}{1-\alpha}.$$

$$\sum_{n=1}^{N} (n-1)\alpha^{n-1} = \frac{1-\alpha^N}{(1-\alpha)^2} - \frac{N\alpha^N}{1-\alpha} - \frac{1-\alpha^N}{1-\alpha}.$$

Final Expression for Entropy:

$$H = -\log_2(1-\alpha) - \log_2 \alpha \left[\frac{1-\alpha^N}{(1-\alpha)^2} - \frac{N\alpha^N}{1-\alpha} - \frac{1-\alpha^N}{1-\alpha} \right] + \log_2(1-\alpha^N).$$