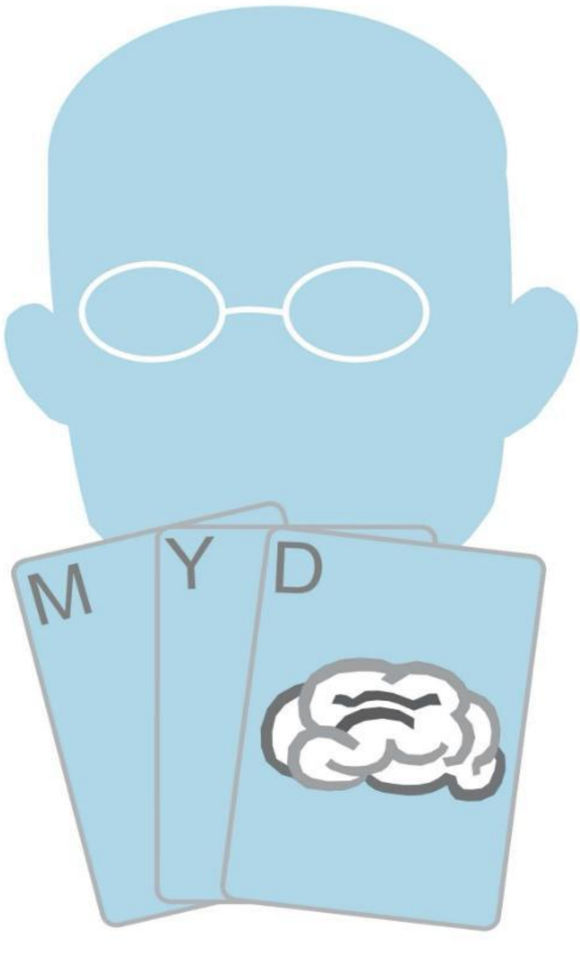
Math Puzzles Volume 2

More Riddles And Brain Teasers In Counting, Geometry, Probability, And Game Theory



Presh Talwalkar

Math Puzzles Volume 2: More Riddles And Brain Teasers In Counting, Geometry, Probability, And Game Theory

About The Author

Presh Talwalkar studied Economics and Mathematics at Stanford University. His site *Mind Your Decisions* has blog posts and original videos about math that have been viewed millions of times.

Books By Presh Talwalkar

The Joy of Game Theory: An Introduction to Strategic Thinking. Game Theory is the study of interactive decision-making, situations where the choice of each person influences the outcome for the group. This book is an innovative approach to game theory that explains strategic games and shows how you can make better decisions by changing the game.

Math Puzzles Volume 1: Classic Riddles And Brain Teasers In Counting, Geometry, Probability, And Game Theory. This book contains 70 interesting brain-teasers.

Math Puzzles Volume 2: More Riddles And Brain Teasers In Counting, Geometry, Probability, And Game Theory. This is a follow-up puzzle book with more delightful problems.

Math Puzzles Volume 3: Even More Riddles And Brain Teasers In Geometry, Logic, Number Theory, And Probability. This is the third in the series with 70 more problems.

But I only got the soup! This fun book discusses the mathematics of splitting the bill fairly.

40 Paradoxes in Logic, Probability, and Game Theory. Is it ever logically correct to ask "May I disturb you?" How can a football team be ranked 6th or worse in several polls, but end up as 5th overall when the polls are averaged? These are a few of the thought-provoking paradoxes covered in the book.

Multiply By Lines. It is possible to multiply large numbers simply by drawing lines and counting intersections. Some people call it "how the Japanese multiply" or "Chinese stick multiplication." This book is a reference guide for how to do the method and why it works.

The Best Mental Math Tricks. Can you multiply 97 by 96 in your head? Or can you figure out the day of the week when you are given a date? This book is a collection of methods that will help you solve math problems in your head and make you look like a genius.

Why Math Puzzles?

I am adding this introductory note in 2015 after completing volume 3 of the math puzzles series. What is the point of all of these math problems?

From a practical perspective, math puzzles can help you get a job. They have been asked during interviews at Google, Goldman Sachs, as well as other tech companies, investment banks, and consulting firms.

Math puzzles also serve a role in education. Because puzzles illustrate unexpected solutions and can be solved using different methods, they help students develop problem solving skills and demonstrate how geometry, probability, algebra, and other topics are intimately related. Math puzzles are also great for practice once you are out of school.

But mostly, math puzzles are worthwhile because they are just fun. I like to share these problems during parties and holidays. Even people who do not like math admit to enjoying them. So with that, I hope you will enjoy working through this collection of puzzles as much as I have enjoyed preparing the puzzles and their solutions.

Each puzzle is immediately accompanied with its solution. I have never been a fan of how print books put all the solutions at the end—it is too easy to peek at the solution for another problem's solution by mistake. In any case, while you are working on a problem, avoid reading the following section which contains the solution.

This book is organized into topics of counting, geometry, and probability and game theory. In each section, the puzzles are roughly organized with increasing difficulty. It is never easy to organize puzzles by difficulty: some of the hard puzzles may be easy for you to solve and vice versa. But as a whole, the harder puzzles tend to involve higher-level mathematics, like knowledge of probability distributions or calculus.

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Section I: Counting

These problems are generally about the number of ways to do things, but there are a few algebra and logic problems included as well.

Puzzle 1: 3 Lamps

In front of you are 3 switches, each of which operates a lamp in another room not visible to you. Your job is to identify which switch operates which lamp.

You are allowed to operate the switches, but you can only visit the other room once. How can you identify which switch operates which lamp?

My friend was asked this problem in an interview for a consulting job. It requires logical thinking and applying a bit of common sense.

Answer To Puzzle 1: 3 Lamps

Turn on one of the switches for a few minutes. Then turn it off and turn on another switch.

When you visit the other room, one lamp will be on. That corresponds to the last switch turned on.

Then touch the other two lamps and feel which bulb is warm. That corresponds to the switch you had turned on for a while and then turned off. The last lamp corresponds to the switch you did not touch at all.

Puzzle 2: Famous Logic Puzzle

This problem was asked in a famous psychological experiment.

Four cards are on a table and each has a number written on one side and a color on the other side. The faces of the cards showing are 3, 8, red, and brown. You are told that every card that shows an even number on one side has the color red on the other side.

Which cards do you need to flip over to test this rule?

Only about 10 percent of subjects got the answer correct. Can you figure it out?

Answer To Puzzle 2: Famous Logic Puzzle

The cards that need to be flipped over are “8” and “brown”. The rule is only broken if an even-card has another color, or a card with another color has an even number on the other side. So the only cards to check are “8” and “brown.”

Here is the logic in more detail when deciding whether each card should be flipped.

Flip over 3? The rule states the opposite side of even cards has red. The rule does not say what the opposite of an odd card has. We could have cards with an odd number on one side and red on the other. There is no need to flip 3.

Flip over 8? If the rule is true, this card better have red on the other side so “8” should be flipped.

Flip over red? The rule states the opposite side of even cards has red. It does not say what the opposite side of red cards has. So if we flip over the red card to find an odd number, that does not break the rule. No need to flip it.

Flip over brown? If we flip this card over and find an even number, then that would break the rule, because it would be an even card that has brown on the other side. Hence we must flip this card over.

The puzzle is not very hard, but the situation appears to make the logic hard to follow.

Experiments have found re-phrasing the problem in a familiar context is helpful. For instance, consider the rule that “every voter must be over 18 years old.” You are given profile cards with age on one side and whether a person voted on the other. The cards shown to you are, “16”, “voted”, “23”, and “not voted”. Which cards do you need to flip over to make sure everyone who voted was legally over 18 years old?

People generally solve this problem instantly (you should check “16” and “voted”).

This problem is known as the [Wason Selection Task].

Puzzle 3: Lost Money To A Buyer

I sold a concert ticket for $20 to someone responding to a Craigslist ad. The buyer paid with a $50 bill.

I didn’t have change on hand, but luckily my friend did. I gave him the $50 note for smaller bills, and I provided the buyer with $30 in change.

The next day my friend tried to use the bill, but it was counterfeit and not accepted.

The buyer did not respond to my emails and I realized I was probably scammed. I owned up to the mistake and repaid my friend $50.

If I originally paid $20 for the concert ticket, how much money did I lose in total?

Answer To Puzzle 3: Lost Money To A Buyer

The answer is $50. Here is why.

If the $50 bill had been legitimate, you would not have lost any money. So the only money you lost was for repaying your friend, which is $50.

The longer answer is to carefully account for every transaction.

You paid $20 to get the concert ticket. But you have not lost any money yet because you got the concert ticket, worth $20, in exchange.

Your friend then gives you $50 dollars in exchange for the counterfeit note. So you are actually up $50 at this point. You reduce this by $30 when you give change to the buyer, so you are up by $20. This balances out with the $20 concert ticket you give up, so you are now completely even.

The next day your friend tells you the note is fake. So you pay him $50, and so you are now down $50. Therefore, you have lost $50 in total.

Credit: This is an updated version of an accounting puzzle I read in the book *Decision Traps* by J. Edward Russo, Paul J. H. Schoemaker.

Puzzle 4: A Friend Puzzle

This puzzle is based on an actual situation I faced recently.

I wanted to have a get-together with coworkers from a previous job. But I was unsure if the guests all got along.

I decided to do a simple test on Facebook. I personally was friends with each of the 5 people I wanted to invite.

But was each person also mutual friends with everyone too?

I did not have to check each person's profile, because once I saw person A was friends with person B, it must be the case that person B is friends with person A.

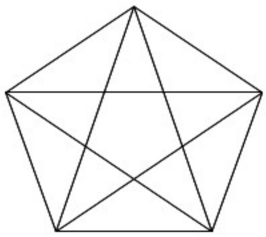
The question is this: for a group of 5 of friends that you suspect are mutual friends, how many profiles do you have to check, at minimum, to be sure they are all mutual friends? What about the same problem for *n* people?

Answer To Puzzle 4: A Friend Puzzle

In a group of 2 people, you only have to check 1 person's profile. If person A is friends with person B, then person B is necessarily friends with person A. (In mathematical parlance, the relation of mutual friends is "reflexive.")

Similarly, in a group of 3 people, you only need to check 2 people's profiles. If person A is friends with B and C, and person B is friends with C as well, then person C is necessarily friends with A and B too.

We can generalize: in a group of *n* people, one needs to check $n - 1$ profiles. It is necessary to check the first $n - 1$ people's friends' lists to make sure each person is friends with each other and the last person in the group. But once that is verified, the last person must necessarily be friends with everyone else.



You can think about the problem graphically. Represent the group as a set of *n* points, and draw a line between two points if the two people are friends. If the first $n - 1$ points are connected to every other point by a line, then the last point must also be connected to every other point (there are $n - 1$ degrees of freedom).

Thus, in a group of 5, one needs to check 4 people's profiles to make sure they are all mutual friends.

Puzzle 5: Family Crossing A River

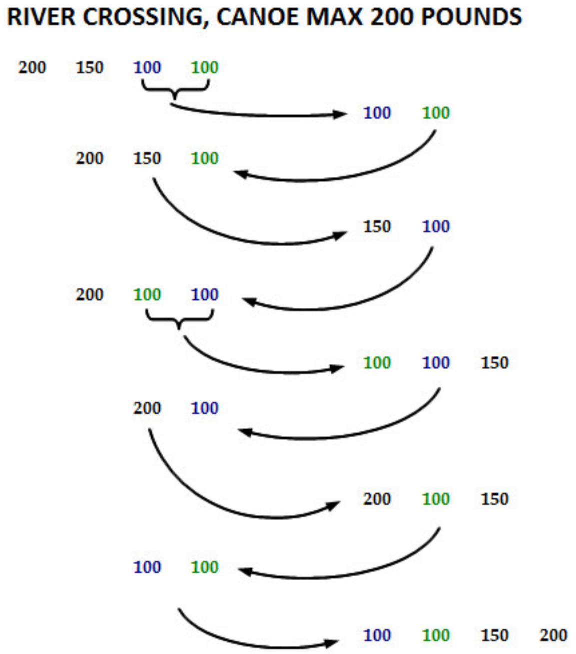
A family of four needs to cross a river. The father weighs 200 pounds, the mother 150, and each of two sons weighs 100.

If the canoe can carry a maximum of 200 pounds at a time, how can the family get across the river? How many trips are necessary?

Answer To Puzzle 5: Family Crossing A River

Obviously the father and mother have to make lone trips while the sons ferry the canoe back and forth.

In all, the family will need 9 trips as depicted below:



Credit: this is a variation of puzzle in the Math Mole newsletter.

Puzzle 6: Losing Weight

A 200 pound man starts a weight loss regimen that promises a loss of 1 percent of body weight per week.

How many weeks will it take to reach his goal of 150 pounds?

Answer To Puzzle 6: Losing Weight

The weight loss program has diminishing effectiveness each week.

In the first week, he will lose 200(0.01) = 2 pounds so he will weigh in at 198 pounds. In the second, he will lose a bit less weight: only 198(0.01) = 1.98 pounds. Each successive week he will weigh in lighter and therefore lose a bit less weight.

The easiest way to solve this problem is to track his weekly weight. If he loses 1 percent per week, that means each week he will weigh in at 99 percent of his previous week's weight. For a weight W, his week k weight will be 99 percent of his week k - 1 weight. In other words:

$W(k) = 0.99\ W(k\text{-}1)$

We know that his initial weight is 200, so the formula for weight after n weeks is:

$W(n) = 0.99\ W(n\text{-}1) = (0.99)^2\ W(n\text{-}2) = \ldots = (0.99)^n(200)$

We solve this equation for the target of 150 pounds to find:

$150 = (0.99)^n(200)$
$n = \log(150/200)/\log(0.99) = 28.62\ldots$

Therefore, he will drop below 150 pounds at the 29th week. To the extent this model is realistic, few people stay on a diet for over half a year, and that's perhaps one of the reasons weight loss is hard.

Puzzle 7: Writing Thank-You Notes

After a party, Alice and Bob prepare thank-you notes for the guests.

Alice by herself would take 10 hours, and Bob by himself would take 5 hours to write all the thank-you notes. How long will it take if they work together?

Answer To Puzzle 7: Writing Thank-You Notes

Logically the job should get done quicker if Alice and Bob work together. Let’s determine a mathematical expression for them working together.

We know that Alice can complete the job in 10 hours. So in t hours, she completes the fraction $t/10$ of the job. Similarly, in t hours, Bob can complete the fraction $t/5$ of the job.

We wish to solve for t so their joint work is 1 whole job: that is, their fractional amounts add up to 1. This gives the following equation:

$t/10 + t/5 = 1$

We can solve that $t = 10/3$, or 3 hours and 20 minutes.

Let’s derive a general equation. If Alice can complete the job in a hours and Bob in b hours, then we can add up their fractional efforts of Alice’s t/a and Bob’s t/b to be 1 complete job:

$t/a + t/b = 1$

If we multiply both sides by ab and simplify, we get the time is $t = ab/(a + b)$. As a bit of trivia, this fraction has a special name. It is equal to half of the *harmonic mean* of the numbers a and b.

Puzzle 8: Three Runners

Alice, Bob and Charlie complete a 100 yard race.

Alice finished the race 10 yards ahead of Bob. Subsequently Bob finished the race 10 yards ahead of Charlie.

How far ahead was Alice over Charlie when she finished the race?

Assume all three ran at constant speeds.

Answer To Puzzle 8: Three Runners

Since Alice finished 10 yards ahead of Bob, and Bob finished 10 yards ahead of Charlie, it is tempting to add up the distances to conclude Alice finished 20 yards ahead of Charlie. But that is not correct.

The reason is we have to consider running rates. Charlie is a slower runner than Bob. In the time it takes Bob to complete the final 10 yards, Charlie would have run less than 10 yards. So at the instant Alice finished, Charlie must have been closer than 10 + 10 yards.

How much closer would Charlie have been?

We can solve the problem in terms of relative speeds. When Alice completes the race of 100 yards, Bob is 10 yards behind, so he has run 90 yards in the same amount of time. This means Bob runs at 90 percent the speed of Alice (B = 0.9 A).

Similarly, when Bob completes the 100 yard race, Charlie is 10 yards behind, so he has run 90 yards in the same time. Hence Charlie runs at 90 percent the speed of Bob (C = 0.9 B).

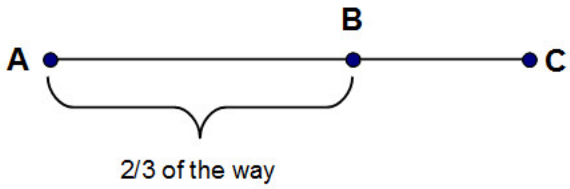
We can combine this equations to find that Charlie runs at 81 percent the speed of Alice (C = 0.9B = 0.9[0.9 A] = 0.81 A).

So when Alice runs 100 yards to complete the race, in that time Charlie would have run 81 yards. Therefore, Charlie is 19 yards behind Alice when she finishes.

Puzzle 9: Splitting A Shared Ride

A businessman hires a driver to take him from point A to point C and back. The round trip costs \$180.

During the ride, the driver gets a request from a pair of poor college students. They would like to tag along and join for the round trip from point B to C. Point B is 2/3 of the way between A and C.

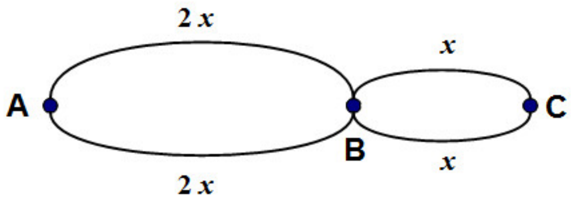


The businessman doesn't mind, and he in fact feels generous. He says to them: "If you can calculate the exact amount that each of you should fairly pay me, I'll let you ride for free."

What would be the fair price for each student? Assume the fare is calculated on distance alone, and the riders will split a shared distance equally.

Answer To Puzzle 9: Splitting A Shared Ride

Since point B is 2/3 of the way to C, we can imagine BC is a distance of x and then AB would be a distance of $2x$.



The round-trip from A to C is a distance of $6x$ and it costs \$180. So to cover a distance of x the fair price is \$30.

What is the fair price for each student?

The round-trip from B to C is a distance of $2x$, so the total cost is \$60. Since there are 3 riders for this portion of the trip, the fair price should split the cost 3 ways. Thus, the fair price for each student is 1/3 of \$60, or \$20.

Puzzle 10: Average Speed

For the first half of my trip, I went at 30 miles per hour (mph). How quickly do I need to drive the rest of the way so I average 60 mph for the complete trip?

Answer To Puzzle 10: Average Speed

Suppose the journey is 60 miles. In order to average 60 mph, you need to complete the entire trip in (60 miles)/(60 mph) = 1 hour.

If you travel half the way at 30 mph, then you have traveled 30 miles at 30 mph, which would have already taken you 1 hour. In order to average 60 mph for the entire journey, you would need to reach the destination instantly, or travel at an infinite speed! This is not possible because nothing travels faster than the finite speed of light.

The result holds for a trip of general distance. If you need to average $2X$ mph for a trip of D miles, then you need to complete the trip in (D miles)/($2X$ mph) = $D/(2X)$ hours. If you travel half the distance, $D/2$, at half the speed X, then you have already taken up ($D/2$ miles)/(X mph) = $D/(2X)$ hours.

In other words, you won't be able to average 60 mph if you have already gone half the distance at half the speed of 30 mph.

Puzzle 11: How Far Did I Jog?

How fast I jog depends on the path. I can jog 3 miles per hour uphill, 4 miles per hour on flat land, and 6 miles per hour downhill.

One day in San Francisco I jogged from my apartment to the Marina. It took me 50 minutes to get to the Marina, and 1 hour to return home on the reverse path.

How far is my apartment from the Marina?

Answer To Puzzle 11: How Far Did I Jog?

This seems like one of those impossible puzzles in which there is not enough information. But if you do the math diligently you can figure it out!

Let's say the path from my apartment to the Marina involves the following distances:

Apartment to Marina
a miles uphill
b miles of flat land
c miles downhill

We want to solve for the distance $a + b + c$.

The first trick is to realize the reverse path will have the same distances, but the uphill and downhill distances will be transposed. So we have:

Marina to Apartment
c miles uphill
b miles of flat land
a miles downhill

Now we can set up the equations to relate distance to jogging speed. If I jog (uphill distance) miles at 3 miles per hour, then that will take me (uphill distance)/3 hours to complete that distance. Similarly, it will take me (flat land distance)/4 hours to go on flat land and (downhill distance)/5 hours to go downhill.

Using the distances defined above, we can get two equations that relate the distances to the total time it takes to jog to and from the Marina.

Apartment to Marina
$a/3 + b/4 + c/6 = (5/6)$ hour

Marina to Apartment
$c/3 + b/4 + a/6 = 1$ hour

Now we want to solve for the distance $a + b + c$.

We proceed by getting rid of the fractions. We multiply both equations by 60 to get rid of the denominators and convert the time in hours into time in minutes.

We get the following equations:

Apartment to Marina
$20a + 15b + 10c = 50$

Marina to Apartment
$20c + 15b + 10a = 60$

Now we add these two equations together to get the following equation:

$30a + 30b + 30c = 110$

We divide this equation by 30 to solve for the sum of the distances.

$a + b + c = 3 + 2/3$ miles

That's the answer! The apartment is 3 and 2/3 miles from the Marina. And if you think about it, that makes for a round trip of 7 and 1/3 miles which is quite a good jog.

Puzzle 12: College Football Title

Not satisfied with the current system, suppose college football programs decide to crown their own champion.

Rather than organize a playoff bracket, they institute a free-for-all system. Each team is allowed to challenge any other team to a game. A team that loses a game is eliminated and ineligible to play further games.

The last team standing–the only team that never loses–is declared the national champion.

Suppose there are 120 teams. What’s the minimum number of games needed to crown a title? What’s the maximum number?

Answer To Puzzle 12: College Football Title

Number the teams as 1, 2, …, 120. Consider a simple albeit unrealistic example. Imagine team 120 beats every other team one by one. There will be 119 matches played.

You can experiment with other ways of scheduling the matches. But time and again, you'll find that 119 matches need to be played! This is both the minimum (necessary) and maximum (sufficient) number of matches.

Why is that?

The reason is each team plays until it loses. In the end, there are 119 teams that lose and 1 team that never loses. Therefore, the tournament always has 119 games, where each game produces 1 loser.

In general, for a similar single-elimination tournament with n teams, there will always be $n - 1$ games played.

Puzzle 13: Pairs Of Cards

Let's play a game of chance.

I have a standard 52 card deck that is shuffled well. I flip over the top two cards.

If they are both red, then you get the pair. If they are both black, then I get the pair. If they are different colors, then the pair is discarded.

I will continue flipping two cards at a time until the entire deck is dealt.

At the end, you and I will have some pairs of cards. If you end up with more pairs of cards than me, then lucky you, I will pay you $10 for every extra pair you have. Otherwise, you lose the game and walk away.

The game costs $1 to play. Are you willing to try your luck?

Answer To Puzzle 13: Pairs Of Cards

The game sounds too good to be true: if you end up with extra cards, you win a lot. If you don't, you only lose the $1 entrance fee. So what's the catch?

This is an example of a sucker bet. The somewhat surprising answer is we will both get exactly the same number of pairs every time, no matter how the deck is arranged or shuffled. I'll never have to pay you any money, but I'll profit from the entrance fee.

Why does this happen?

Let's prove it. To begin, note that a standard deck has 26 red cards and 26 black cards. All of the red cards end up in two piles once the game is over: either the pile of paired red cards or the pile of discarded red-black pairs. So we can write the equation:

26 = #cards in red-red pairs + #red cards in red-black pairs

Similarly, all the black cards end up in either the pile of paired black cards or the pile of discarded red-black pairs. So we have another equation:

26 = #cards in black-black pairs + #black cards in red-black pairs

Both equations are equal to 26, so we can equate the two expressions to get:

#cards in red-red pairs + #red cards in red-black pairs = #cards in black-black pairs + #black cards in red-black pairs

Now we can simplify this equation. The pile of red-black pairs has an equal number of red and black cards, so we have #red cards in red-black pile = #black cards in red-black pairs.

We can cancel those terms to conclude the following:

#cards in red-red pairs = #cards in black-black pairs

In other words, the number of red-red pairs must exactly equal the number of black-black pairs, no matter how the cards are ordered in the deck.

You will never have more red pairs than I have black pairs, so you always lose the game.

Puzzle 14: Ways To Eat A Chocolate Bar

I buy a chocolate bar that is divided into 10 bite-size squares in a line.

If I eat 1 or 2 squares every day, how many different ways are there for me to eat the entire bar?

Assume I start eating from the left side of the bar and continue eating to the right side without skipping any squares in between.

For example, I could eat 1 bar every day for 10 days. Or I could eat 2 bars first, then 1 bar each for 8 days.

Answer To Puzzle 14: Ways To Eat A Chocolate Bar

Let's work out smaller cases.

What if the bar only had 1 square? How many ways are there to eat just 1 square? There is clearly only one way: you eat a 1 square the first day.

What if the entire bar had 2 squares? You can see there are 2 different ways to eat the bar. You could either eat 1 square a day each day, or you could eat 2 squares on the first day.

What about 3 squares? Now comes the neat part. We don't have to count out the combinations. We can reason as follows. We can't eat 3 squares at once. So the only way we could end up eating 3 squares is if we eat 1 square or 2 squares on the first day. If we eat 1 square on the first day, that leaves 2 squares to be eaten (and we already counted the number of ways to do that). If we eat 2 squares on the first day, that leaves 1 square to be eaten (and we also already counted the number of ways to do that).

That means the following: we can count the number of ways to eat 3 squares by adding the number of ways for 2 squares left (2), plus the number of ways for 1 square left (1).

If f(n) is the number of ways to eat a bar with *n* squares by 1 or 2 squares at a time, then we have reasoned f(3) = f(2) + f(1) = 3.

Similarly, we can reason for *n* squares. In order to eat *n* squares in total, there are two possible ways to start. One way is to eat 1 square on the first day, leaving *n* – 1 squares not eaten (which can be counted as the number of ways to eat *n* - 1 squares). The other way is to eat 2 squares on the first day, leaving *n* – 2 squares not eaten (which can be counted as the number of ways to eat *n* - 2 squares).

So we have the following equation:

$f(n) = f(n-1) + f(n-2)$

This is the familiar formula for the Fibonacci sequence. (A bit of trivia: the Indian mathematician Pingala came upon this pattern 1,200 years before Fibonacci in the study of Sanskrit poetry.)

So we can use this recurrence relation to solve the puzzle:

Chocolate Bars Fibonacci

Squares	Number of ways
n	*f(n) = f(n-1) + f(n-2)*
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89

The answer is there are 89 different ways I can eat the chocolate bar.

Credit: this problem is adapted from the stair puzzle in William Poundstone's book *Are you Smart Enough to Work at Google?*

Puzzle 15: Flights Around The Country

Bob wants to visit 25 cities around the country. If he wants to fly to every city exactly once, and start and end the trip from his hometown, how many possible ways can he make the trip?

Assume there exists a flight between any two cities.

Bonus: Some of these itineraries will be impractical because one would want to fly to nearby cities for efficiency. So suppose Bob divides the trip into 5 geographic areas, each with 5 cities. When he visits a geographic area, he will visit all those cities together before going to another geographic area. In how many ways can Bob make this trip?

Answer To Puzzle 15: Flights Around The Country

There are 25 possible cities for the first stop, then 24 possible cities for the second, and so on.

Thus, there are (25)(24)… (1) = 25! possible ways to make the trip.

For the bonus, the correct answer is $(5!)^6$.

First we count the number of ways Bob can visit the different geographic areas. There are 5 choices for the first, then 4 for the second, then 3 and 2 and 1. So there are 5! = 120 ways to visit the different areas. In each area, Bob has to choose how to visit the 5 different cities. There are 5! ways to do this for each of the 5 areas, which gives $(5!)^5$ possibilities.

Thus, the total number of trips is: $5!(5!)^5 = (5!)^6$

That's still a lot of trips, but about $1/10^{13}$ smaller than the 25! number of trips from the original problem.

Puzzle 16: Order Of Eating Courses

I eat soup, salad, and a sandwich for lunch.

Some days I like to eat one course at a time; for instance, I'll finish my soup, then I'll have my salad, and only then I will eat my sandwich.

Other days I might like to eat all of the courses together. And finally, I might mix things up by pairing certain courses: I might eat my salad first, but then eat the soup and sandwich together.

How many possible ways are there for me to eat my lunch?

Assume my choices are to eat one course at a time, two courses at a time, or all three courses together, and the order of eating the courses matters.

Answer To Puzzle 16: Order Of Eating Courses

This problem can be solved by enumerating all the possibilities. Here's a listing of the possible ways I can eat my lunch. Write 1 to denote eating soup, 2 salad, and 3 the sandwich.

There are 6 ways I can eat my meal 1 course at a time. I will write (a, b, c) to mean I eat item a, then item b, and then item c. Here are the 6 ways: (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).

What if I pair 2 of the courses and eat them at once? There are 6 ways I can eat my meal in this case. I will write (ab, c) to mean I eat courses a and b together and then eat c after. I can also have (a, bc) to mean I eat a first and then b and c together. Here are the 6 ways: (1, 23), (2, 13), (3, 12), (12, 3), (13, 2), (23, 1).

If I eat all 3 courses at the same time, there is just 1 possibility: (123).

In all, there are 6 + 6 + 1 = 13 ways that I can eat my meal.

We can also solve the problem combinatorially. The number of ways to eat 1 course at a time is the number of permutations of 3 items. There are 3! = 6 ways to eat one course at a time. The number of ways to eat when some 2 courses are paired is found slightly differently. There are 3 choices for the course not to be paired, and that item can be eaten first or second, so there are 2 orderings. This is why there are 3(2) = 6 ways to eat in when one course has 2 items. Then there is just 1 way if all the items are eaten together. This means there are 6 + 6 + 1 = 13 ways in all.

Puzzle 17: Hot Sauce

My friend was cooking dinner, and he found himself in a predicament.

He made 6 quesadillas for dinner. On 2 of them he added hot sauce to the filling. He is the only person that enjoys the extra spice, and his wife and 2-year old daughter would be displeased if they got the wrong filling.

While cooking and flipping the quesadillas, he inadvertently forgot which quesadillas had the hot sauce filling. So he ended up with 6 identical quesadillas that he served at random. He went ahead and served himself 3 quesadillas, his wife 2, and his daughter 1.

My friend realized he was playing out a textbook probability question, and he was curious about the following questions.

Part A. What is the probability that he correctly serves the food and he ends up with both quesadillas that have hot sauce?

Part B. What is the probability that he ends up with 1 having hot sauce and his wife has 1 as well? What about him getting 1 hot sauce and his daughter getting the other?

Part C. What are the chances he gets none of the hot sauce quesadillas?

Can you help my friend out? What are the odds?

Answer To Puzzle 17: Hot Sauce

Write H for a quesadilla with hot sauce and N for a quesadilla without it. There are 6 identical-looking quesadillas: H, H, N, N, N, N.

Short explanation of ways to choose objects without order. This problem is about the number of ways to pick objects without regard to order. Suppose you want to pick k objects from N, and the order doesn't matter. Your first choice has N possibilities, your second choice has N - 1, your third choice has N - 2, and so on, until the last choice has N - k + 1 possibilities. So in all there are $N(N$ - $1)\ldots(N$ - k + 1) arrangements. However, the order of the items is not relevant. (As in, getting HN is the same as NH because in either case you have one hot sauce quesadilla.) So we divide by $k!$ to avoid double-counting the various arrangements. In summary, there are $N(N - 1)\ldots(N - k + 1)/k! = N!/[k!(N - k!)]$ ways to choose the items. The topic of choosing items from a set comes up frequently. In this book I'll abbreviate this as "N choose k" and proceed with the calculation.

Part A. My friend chooses 3 quesadillas for himself out of 6, so there are 6 choose 3 = 20 total ways that my friend can pick his 3 quesadillas. In how many ways will he get both of the ones with hot sauce? There is only 1 way he gets both of them (HH), and he can pick any of the four remaining non-hot sauce ones (N) for his third quesadilla. So there are 4 ways he could get both of the quesadillas with hot sauce for a probability of 4/20 = 20 percent. This is not too bad, but there is also an 80 percent chance he messes up, so it depends on how you look at it.

Part B. How many ways can my friend pick exactly 1 hot sauce for himself? There are 2 choices for the quesadilla with hot sauce. Then there are 4 choose 2 ways he can pick 2 out of the 4 quesadillas without hot sauce. So there are 2(4 choose 2) = 12 ways. Out of 20 ways in total, there are 12 where he gets exactly 1 with hot sauce, meaning there is a 60 percent chance of this case.

The other quesadilla with hot sauce filling either goes to the wife or the daughter. The wife eats 2 quesadillas and the daughter 1. In other words, there are 3 places to distribute the hot sauce quesadilla, of which the wife has 2 places and the daughter has 1. So in 2/3 of the 60 percent, which is 40 percent of the time, his wife gets the other hot sauce filling. And in 1/3 of 60 percent, which is 20 percent, his daughter gets the other hot sauce filling.

Part C. What's the chance my friend gets no hot sauce filling? That means he picks 3 of the 4 regular quesadillas, which can happen in 4 choose 3 = 4 ways. This is out of a total 20 ways, so he has a 20 percent chance of getting no hot sauce.

The most likely outcome is my friend will get exactly 1 quesadilla with hot sauce filling.

Puzzle 18: Wardrobe Choices

A finicky traveler is packing clothes for a 3-day trip. The traveler can never decide what to wear in advance and would like to have choices available for each day of the trip.

Each day the traveler wears a previously unworn pant and an unworn shirt. An outfit choice is a pant-shirt combination.

If the traveler would like to have at least 4 outfit choices each day, what is the minimum number of pants and shirts that the traveler must pack?

What if the trip lasts *n* days? What if the person wanted *k* choices each day?

Answer To Puzzle 18: Wardrobe Choices

This problem can be solved by thinking backwards from the end of the trip.

On the very last day, the traveler must have 4 outfit choices. So we need to have (pants x shirts) = 4 and we want (pants + shirts) to be minimal. The factors of 4 are either 4 and 1 (sum of 5) or 2 and 2 (sum of 4). The latter option has a smaller sum and corresponds to bringing fewer clothes.

Thus, the traveler would need 2 pants and 2 shirts for the last day of the trip.

For each of the days before, the traveler needs 1 extra pant and 1 extra shirt. Thus, for a 3-day trip, the traveler needs 2 + (1 + 1) = 4 pants and 2 + (1 + 1) = 4 shirts.

For an n day trip, the same logic applies. The traveler needs 2 pants and 2 shirts on the last day and will need 1 extra pant and 1 extra shirt for each previous day. So the traveler needs $2 + (n - 1) = n + 1$ pants and $2 + (n - 1) = n + 1$ shirts.

If the traveler wants k choices each day, then on the last day the traveler needs (shirts x pants) = k for a minimal number of (shirts + pants). This is minimized by considering the prime factorization of k to determine the pair of divisors with the least sum. Call those $x < k/x$. Then, for each previous day the traveler needs 1 extra shirt and 1 extra pant. The total number of pants and shirts needed will be:

pants: $x + (n - 1) = x + n - 1$
shirts: $k/x + (n - 1) = k/x + n - 1$

The traveler can also opt for k/x shirts and x pants on the last day giving similar solutions.

You can solve for x by considering the integers closest to the square root of k.

Why? Suppose we want factors a and b where $ab = k$ and the sum $a + b$ is minimal. Geometrically, this is like finding the least perimeter rectangle for a given area, which is a square. Alternately, we can use calculus to find the minimum happens in the continuous case when $a = b = \sqrt{k}$. In our problem we need whole number factors for the number of shirts/pants, so we consider the factors that are closest to $\sqrt{k}$.

Puzzle 19: Wedding Seating Arrangement

A group of 10 people sits down at a wedding table.

Only after sitting do they realize the table had name tags for each seat, and no one was sitting in the correct seat.

Prove that there is a way to rotate the table and its name tags so that at least 2 people will be in their correct seats.

Answer To Puzzle 19: Wedding Seating Arrangement

First note there are 10 ways to rotate the table: it can be rotated 1 to 10 seats from its current position. A rotation of 10 seats is the same as a rotation of 0 seats, which is the original position of the table.

Second, for each of the 10 people, there is some rotation that can bring that person to the properly assigned seat. Label the individuals 1 to 10 and call those rotations $r_1, r_2, \ldots, r_{10}$.

Each of these 10 rotations must be equal to one of the 10 possible rotations for the table. However, since no one was initially in the correct seat, we have to exclude the rotation where the table moves 10 seats (or is in the initial position). So each of the 10 rotations must be equal to one of the 9 possible rotations for the table.

If 10 rotations have to be equal to 9 possible values, then at least 2 of them have to be equal to the same value (by the pigeonhole principle). In other words, at least two of the rotations r_i, r_j must be equal. That is for some $r_i = r_j$ at least 2 people i and j will be in their correct seats.

Puzzle 20: A Fun Baseball Inequality

A player's *batting average* is defined as the number of hits (H) divided by the number of at-bats (AB). For example, someone who gets 2 hits in 10 tries has a batting average of 20 percent (often written to three decimal places as 0.200).

We can turn the numbers around to find another useful statistic. If we divide the number of at-bats by the number of hits, we get the *average number of at-bats before getting a hit*. For someone who gets 2 hits in 10 tries, we can say the person gets a hit in roughly every 5 tries at the plate.

Now here is the interesting question: what happens when you add the two numbers together?

Let's work through a couple of examples. If someone has 2 hits in 10 tries, then we have:

H/AB + AB/H = 2/10 + 10/2 = 5.4

If someone has 1 hit in 3 tries, then we have:

H/AB + AB/H = 1/3 + 3/1 = 3.333…

Finally, if someone gets 5 hits in 5 tries, then we have:

H/AB + AB/H = 5/5 + 5/5 = 2

Hmm, this looks like a pattern…it seems that the sum of the two statistics will always be greater than or equal to 2.

This is actually the case. But can you figure out why?

More generally, there is a stronger mathematical statement. If you pick any two numbers a and b, it will be the case that:

$|a/b + b/a| \geq 2$

Try it out with any two numbers–it will be true!

Can you prove why this must be the case?

Answer To Puzzle 20: A Fun Baseball Inequality

A reader of my blog sent me a direct proof. Since every squared number is non-negative, the squared difference of the numbers is non-negative. We can use this to find the result as follows:

$(a - b)^2 \geq 0$
$a^2 + b^2 \geq 2ab$
$|a/b + b/a| \geq 2$

The last step involves dividing both sides by *ab* and then taking the absolute value so that the final quantity on each side is positive.

I came upon a different proof which is longer but instructive.

The trick is to use the following fact: for any two non-negative numbers, the arithmetic mean (the simple average) is greater than or equal to the geometric mean (the square root of the product), with equality only when the two numbers are equal:

$(a + b)/2 \geq \sqrt{(ab)}$

For our purposes, we will rearrange the inequality as follows:

$a + b \geq 2\sqrt{(ab)}$

If we let a = H/AB and b = AB/H, the left hand side becomes the sum of the two ratios, and the right hand side becomes 2 because the product of reciprocals is 1.

So we have H/AB + AB/H ≥ 2.

This inequality can make for a nerdy party trick (ask someone for 2 numbers, then add up the ratio and reciprocal ratio) because it is not immediately obvious why it must be true.

Credit: I read about this problem in *Methods of Problem Solving, Book 2* by Tabov, JB, Taylor, PJ.

Puzzle 21: 12 Balls, 3 Weighings

You have 12 balls that look and feel identical. One of the balls has a slightly different weight than the others.

At your disposal is a weighing scale where you can place balls on each side.

How can you identify the different ball in at most 3 weighings?

Answer To Puzzle 21: 12 Balls, 3 Weighings

The task seems impossible at first because you don't know which balls are the same and which one is different, or whether the different one is lighter or heavier. The key observation is this: if two groups of balls weigh the same, then all the balls must be "good." This gives you a control group against which you can compare the remainder of the balls.

Number the 12 balls from 1 to 12, and group them into sets of 4 balls {1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12}.

Step 1: Weigh {1, 2, 3, 4} against {5, 6, 7, 8}.

Either the two groups weigh the same or one side is heavier. We'll first suppose the two groups are the same and then consider if one side is heavier a bit later.

If the two sides are equal, then we know all 8 of these are good balls. So we have to figure out the different ball from the group {9, 10, 11, 12}.

Step 2 (both sides equal): weigh {9, 10, 11} against the known good balls {1, 2, 3}

There are 3 things that can happen.

Case 1: The group {9, 10, 11} is lighter. Since {1, 2, 3} is a set of good balls, we know the lighter ball is in the group {9, 10, 11}. In the third weighing, weigh two of them against each other, say {9} against {10}. If one side is lighter, then that is the lighter ball. If the two sides are the same, then the other ball {11} is lighter.

Case 2: The group {9, 10, 11} is heavier. This case is similar. Since {1, 2, 3} is a set of good balls, we know the heavier ball is in the group {9, 10, 11}. In the third weighing, weigh two of them against each other, say {9} against {10}. If one side is heavier, then that is the heavier ball. If the two sides are the same, then the other ball {11} is heavier.

Case 3: If the two sides are the same, then the odd ball is the one not weighed yet, which is {12}. In the third weighing, weigh {12} against any of the other balls to find out if it is heavier or lighter.

This settles matters if the two sides were equal in the first weighing. Now suppose one side was heavier and call that side {1, 2, 3, 4}. Then we know the balls not weighed {9, 10, 11, 12} are all good balls, but we don't know which of the balls from 1 to 8 is different.

Now we need to do a clever weighing. We will remove 3 of the balls from the heavy side and transfer 3 from the light side. We'll also add 3 good balls to the lighter side.

Step 2 (one side heavier): weigh {1, 6, 7, 8} against {5, 9, 10, 11}

Again, there are 3 things that can happen.

Case 1: If the side with {1} is still heavier, then either {1} is the heavy ball or {5} is the light ball**. We can weigh either ball against a good ball {12} and figure out which one is true. (Say we weigh {1} against {12}. If they are the same, then {1} is good so {5} is lighter. Otherwise, {1} will be heavier.)

**Why must this be so? Notice we've moved 3 light balls to the side with {1}. So if the {1} side is heavy, that could not be a result of a light ball being on that side. So it must be that {1} is heavy. Otherwise, we know that {9, 10, 11} are good. So it could also be that {5} is light.

Case 2: If the side with {1} is now lighter, then one of the balls {6, 7, 8} is the light ball**. We can weigh two of them against each other and figure out the odd ball. (Say we weigh {6} against {7}. If they are the same, then {8} is light. Otherwise, they are not the same and the light one can be identified.)

**Why is this? There is only 1 odd ball. In two weighings the side with {6, 7, 8} was light, even against 3 known balls from the control group, so the light ball must be in that group.

Case 3: If two sides are now the same, then we know all these balls are good, so the balls {5, 6, 7, 8} must have been good. The second weighing also says {1} is good. Thus, one of the balls in the group {2, 3, 4} must be heavy. In a third weighing, weigh any two against each other to identify which one. (Say we weigh {2} against {3}. If they are the same, then {4} is heavy. Otherwise, they are not the same and the heavy one can be identified.)

Puzzle 22: Guessing A “Lost” Number Mathemagic

Today we will play a little game. It will require that you do a few calculations. Ready to play?

Okay, the first step is I want you to write down a 4-digit number. (Just don’t include too many zeros).

Next, I want you to add up the digits in the number. If you wrote 1234, then I want you to add 1 + 2 + 3 + 4 = 10. Call this number *X*.

Now comes the fun part. You get to cross out any digit of the original number so you end up with a 3-digit number. Think about which digit *from 1 to 9* that you want to “lose,” and cross it out.

Finally, take your three-digit number and subtract *X* (the sum you calculated earlier). Tell me the final number.

Now I didn’t know your original number, nor did I see the number you crossed out. But if you submit me that number then I can magically tell you which digit was “lost” that you crossed out!

An example of how this works

Let’s say you picked the 4-digit number 1234. The sum of the digits is 10 = *X*.

You cross out the 2 so you are left with 134. Subtracting *X* yields 124.

If all you did was tell me 124, I would know you crossed out 2, precisely the number you crossed out! This is even though I did not know your original number or see you cross anything out.

Can you figure out how I do the trick?

Answer To Puzzle 22: Guessing A “Lost” Number Mathemagic

Let’s say you start with a number *abcd* whose sum of digits is $a + b + c + d = X$.

Let’s say you cross out the digit b to be left with *acd*. The number you tell me is this number minus the sum of the original digits, which is $acd – X$.

How can I obtain that b was the missing digit from the value of $acd – X$?

Let’s step back for a moment and recall some basic number theory. The result seems mysterious when we view things as normal numbers, but it will be a lot easier if we work in modulo 9 (the remainder of a number after dividing by 9).

Part 1: Powers of 10 have a remainder of 1 (mod 9)

There’s a basic pattern that will help us out. A few examples will give the picture.

What is 10 modulo 9? That is easy: 10 gives a remainder of 1 when divided by 9.

What about 100? Well, 100 divided by 9 is 11 plus a remainder of 1. What about 1,000? Again, we will see that 1,000 modulo 9 is equal to 1 (as 1,000 is 1 more than 999).

We can see a pattern here:

$10^n \equiv 1 \pmod 9$

This should make intuitive sense: any power of 10 will be 1 more than a multiple of 9 (this is the same as pointing out any number 999…999 plus 1 will be a power of 10).

And let’s go one step further. Let’s multiply both sides of the equation by x. We then have:

$x(10^n) \equiv x \pmod 9$

So what does this mean? We actually have a powerful result here for our trick.

Part 2: the sum of digits

Let’s go back to the number *abcd*. This number really means $a(10^4) + b(10^3) + c(10^2) + d(10)$ because we are working in base 10.

Now what happens when we take this number modulo 9? We can use part 1’s result to get rid of the powers of 10, which are all congruent to 1 modulo 9.

So we have:

$a(10^4) + b(10^3) + c(10^2) + d(10) \equiv a + b + c + d \pmod 9$

So the original number and the sum of its digits are the same modulo 9. That’s the fact we will utilize for our trick.

Part 3: how the trick works

Okay, it’s been a lot of setup, but now we can answer why the trick works.

First, we start out with a number *abcd* and find the sum $a + b + c + d = X$.

Now we are told to delete some digit (say b), and subtract X, so we have $acd – X$.

The trick is to view this number modulo 9. We will have:

$acd – X \equiv a + c + d – (a + b + c + d) \equiv – b \pmod 9$

And voila! That is why the procedure will tell us the missing number. We end up with $-b$, and we can multiply by -1 and take that modulo 9 to obtain b. Equivalently, we can subtract the number from 9. If our result was 7, then we do 9 – 7 to find 2 was our missing number.

So the trick is simple; whatever number I am told, I calculate “answer = 9 – (number you told me) mod 9” and that is the missing number!

Important: don’t let them cross out 0

I originally did not realize an issue with this trick and ran into some trouble in a performance. You should make sure the other person picks a digit from 1 to 9 to “lose.”

The reason is that 0 and 9 are congruent modulo 9. This means if you ended up with the final number of 9, you won't be sure if the other person picked a 0 or a 9.

Doing this trick at parties

You can also perform this trick at parties using some mental math. Ask someone to pick a 4 digit number, and add up the digits. Then tell them to lose a digit *from 1 to 9* and subtract the sum of the original digits (this might sound complicated, but everyone I've asked can do this mental math with no problem).

Then ask them for the resulting number. Your job is then to:

1. Add up the digits until you get a single digit (this is known as "casting out the nines" and it's the same as taking a number modulo 9.

2. Subtract that number from 9 and to obtain the missing digit.

Credit: I read this trick first *Mathematical Magic* by William Simon. I have since read it in a few more books, so I'm not sure of the original source.

Puzzle 23: Piles Of Coins

This is a puzzle that was asked as a finance interview question. I want to give a fair warning this is a pretty hard problem, and it gave me a new level of appreciation for the intelligence of the quant guys on Wall Street.

A pile of 1,000 coins is split into two piles x and y. Multiply those together to get a number xy.

Subdivide each of the two piles further and get a number for each pile. (Say you divide x to get the new number x_1x_2, and y to get the new number y_1y_2).

Repeat the process until there are 1,000 piles, each with 1 coin.

Add all of the numbers together. What is the sum?

Does the answer change depending on how you split the pile?

Answer To Puzzle 23: Piles Of Coins

I would be stunned if anyone came up with an answer for the 1,000 case right off the bat. In interview questions, it is almost always a good idea to start with smaller cases and see if there is a pattern. This gives you some time to work through examples, and is shows the interviewer that you can break a problem down.

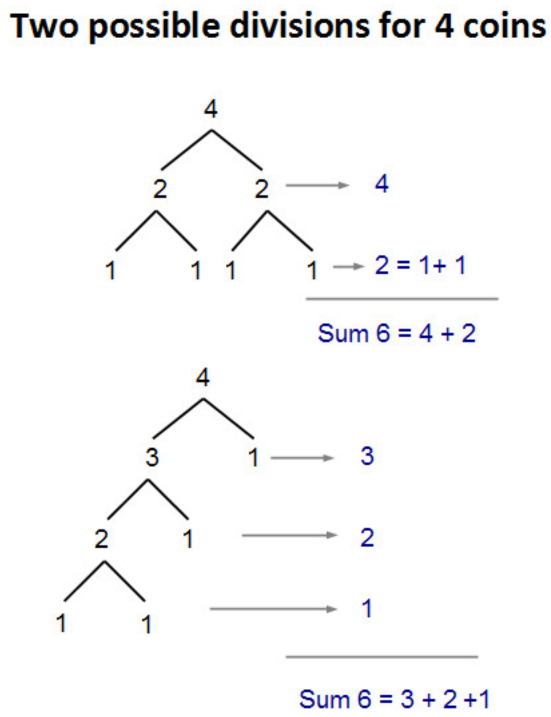
So let's try some smaller cases.

With 2 coins, there is only one possible division: 1-1. This gives a sum of 1.

With 3 coins, again there is only one possible division. You split the coins into 2-1 (gives the number 2) and then you split the 2 pile into 1-1 (gives the number 1). In all, the sum of the numbers is 3 = 2 + 1.

So far, it is hard to see any pattern.

What about 4 coins? This is where the problem gets interesting. There is more than one way to split the coins. We can do either of the following divisions:



Hmm, this is rather interesting that the sum is the same either way the coins are divided.

At this point, one might conjecture the final result is the same regardless of the way the coins are split. If that's the case, we can figure out a simple way to compute the sum and we can figure out the answer for 1,000 coins.

So what I'd do in the interview is work out a simple algorithm to compute the sum, and then I'd think of some way to prove this must be true.

A simple algorithm

I got this idea from the second possible way to divide 4 coins. I noticed that each time we were just removing 1 coin, and the numbers we summed were 3 + 2 + 1 = 6. Could this pattern be a hint to the sum?

Let's generalize this pattern. If we have n coins, a simple division is to just take away 1 coin from the pile each time. The piles will thus be (n – 1, 1), then (n – 2, 1), (n – 3, 1), …, then (2, 1), and finally (1, 1).

The sum we get will be:

sum = $(n - 1)(1) + (n - 2)(1) + \dots + (1)(1)$
sum = $(n - 1) + (n - 2) + \dots + 1$
sum = $n(n - 1)/2$

Basically, when we remove 1 coin from the pile at a time, the intermediate products end up being the integers from 1 to $n - 1$. So our final result is the sum of the numbers from 1 to $n - 1$. The formula for the sum of numbers 1 to k is $k(k + 1)/2$, and so the sum of the numbers 1 to $n - 1$ is $n(n - 1)/2$.

If we apply this formula to 1,000 coins, our final answer will be 1000(999)/2 = 499,500.

But we are still not done. It was a conjecture that the sum will be the same for an arbitrary division. Now we will have to prove that is actually the case.

Induction to the rescue!

To prove this formula, we will use induction.

We already worked out the cases of piles of 2, 3, and 4 coins, and one can verify the formula works on these base cases.

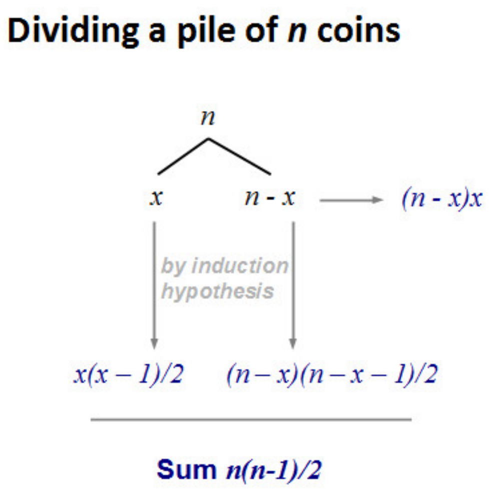
Now, we will prove the result generally holds. We assume that for piles of coins less than n the formula is true and holds regardless of the division method. Now we need to show the formula works for n.

In the very first step, we will divide the pile of n coins into piles of $n - x$ and x. Incidentally, the degenerate case of $x = 0$ means we left the pile as it was. But that is okay because 1000(0) = 0 so our ultimate sum is not affected.

Thus we can assume x is at least 1. The two piles will contribute the number $(n - x)(x)$ to our final sum. And now comes the neat part.

The two piles are both less than n coins. By the induction hypothesis, we can conclude the final result for each of these piles is the same regardless of the division process. Plus, we have a formula for what the final result will be for each pile.

So we add up the initial number $(n - x)(x)$ to the sums we get from the two applications of the induction hypothesis. The calculation we need to do is the following:



The pile of x has a sum $x(x - 1)/2$, the pile of $n - x$ has a sum $(n - x)(n - x - 1)/2$, and adding those to $(n - x)x$ gives a total sum of $n(n - 1)/2$.

This confirms the sum for a pile of n coins is $n(n - 1)/2$, which is what we sought out to prove.

Therefore, the formula holds and the answer for 1,000 coins is 1000(999)/2 = 499,500.

Puzzle 24: Piles Of Coins (Continuous Version)

This is an extension of the previous puzzle.

A line segment of length 1 is split into two lengths x and y. Multiply those together to get a number xy.

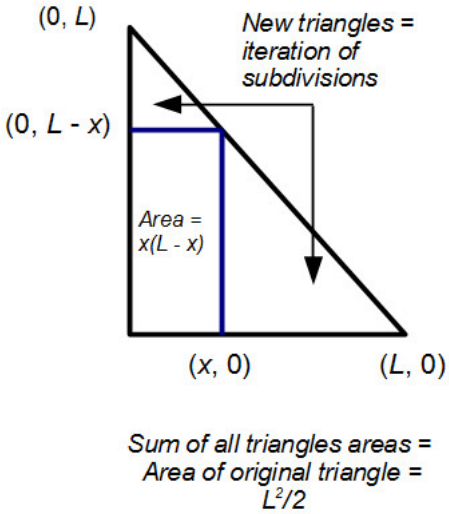
Subdivide each of the two line segments further and get a number for each of those new line segments. (Say you divide x to get the number x_1x_2, and y to get the number y_1y_2).

Repeat the process infinitely and add all of the numbers together. What is the sum? Generalize for a segment of length n.

Answer To Puzzle 24: Piles Of Coins (Continuous Version)

The answer for a length of 1 is 1/2, and for length of *n* it is $n^2/2$.

Here is a proof that was offered on a Reddit Math discussion to this problem.



Let's solve the problem for an interval of length *L*. Let's draw out a triangle in the Cartesian plane with corners (0, 0), (*L*, 0), and (0, *L*).

If the first division is *x* and *L* - *x*, then *x*(*L* - *x*) is the area of the rectangle with vertices (0, 0), (*x*, 0), (*x*, *L* - *x*), and (0, *L* - *x*). This rectangle is contained within the original triangle.

There are also two smaller right triangles, and these correspond to the new intervals for (0, *x*) and (*x*, *L*).

The process will be similar for each of those triangles: the division process will result in rectangles contained within those smaller triangles.

Adding up all the subdivision products is geometrically adding up the areas of all these small rectangles. In the limit, the sum of these rectangles will be equal to the area of the original isosceles right triangle with length *L*.

This triangle has area equal to (base)(height)/2 = (*L*)(*L*)/2 = $L^2/2$.

Note this will be the same result no matter which division method is used. So the answer is always $L^2/2$.

Puzzle 25: A Fun Math Sequence

This is a wonderful little math problem that I came across.

Consider the sequence S = {1, 1/2, 1/3, 1/4, …., 1/100}.

Pick any two numbers x and y and replace them with a new number equal to:

new term = $x + y + xy$

For instance, the numbers 1/4 and 1/8 would be replaced by 13/32.

Keep repeating the process until only one number remains. What number or numbers will result?

Answer To Puzzle 25: A Fun Math Sequence

The remarkable answer is the final number will be 100, and it will be the same regardless of the order in which you pick the numbers.

Why is that? Let's first prove the order does not matter, and then find a formula for the sum of the sequence.

Part 1: the order does not matter

Let's define the function f by

$f(x,y) = x + y + xy$

If we think for a minute, we can see why the order of picking the pairs does not matter. The reason is the function f only depends on addition and multiplication, both of which are associative and commutative so the order does not matter.

It suffices to prove the order does not matter for any three numbers we pick. Specifically, for any three numbers x, y, and z, we need to check that:

$f(f(x,y),z) = f(f(x,z),y) = f(f(y,z),x)$

We can verify all three formulations are equal to the following expression:

$x + y + z + xy + xz + yz + xyz$

The long and short is that the order in which we pick the pairs does not matter. You will always end up with the same final number.

Part 2: a formula for the final number

This is a problem where it helps to try a few smaller cases.

For instance, let's consider the sequence S(2) = {1, 1/2}. What final number results for this sequence? We can directly evaluate:

$f(1, 1/2) = 1 + 1/2 + (1)(1/2) = 2$

What happens when we add one more number for S(3) = {1, 1/2, 1/3}?

Now we can use a trick to evaluate the final number. We already know that picking the pairs 1 and 1/2 will be replaced by the number 2. So we can simply jump ahead and evaluate the pair of 2 and 1/3.

$f(2, 1/3) = 2 + 1/3 + (2)(1/3) = 3$

This is an interesting little pattern we have. We found the final result for S(2) was 2, and the final resulting term for S(3) was 3.

Could it be the case that S(n) will result in the final term of n?

In fact this is the answer. We will use induction to prove this.

We already checked the base cases above. So now we will assume the formula is true for all sequences less than k.

What will the result be for the sequence S(k)?

We know that S(k) is equal to S(k – 1) plus the additional term $1/k$. And we know the resulting term of S(k – 1) will be k – 1, by the induction hypothesis.

So we can evaluate the final term for S(k) as the result of the final pairs k – 1 and $1/k$:

$f(k - 1, 1/k) = k - 1 + 1/k + (k - 1)/k = k - 1 + 1 = k$

This proves that the final resulting term of S(k) will be k.

Extension: an interesting formula

While solving the problem, I came across a connection with set theory.

The concept I remembered was the *power set*, which is defined as the set of all subsets. For example, the power set for $\{x, y\}$ is the set that includes all of the subsets $\{\}$, $\{x\}$, $\{y\}$, and $\{x, y\}$.

Similarly, the power set for $\{x, y, z\}$ is the set that includes all 8 of the subsets $\{\}$, $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, and $\{x, y, z\}$.

Here's the connection with the original problem. Let's go back to the case when you have a set of three elements x, y, and z. The final number we arrive at is equal to

$x + y + z + xy + xz + yz + xyz$

Do you notice anything special about those terms in the sum? The terms can be found by writing out the power set for $\{x, y, z\}$, taking the product of terms from each subset, and then adding them up! (I'd nominate this sum to be called the "power set product sum" for lack of a better term).

The extension is we can solve for some crazy looking sums. For instance, in the power series $\{1, 1/2, 1/3, \ldots, 1/n\}$, we have proved the final term is equal to 100. Therefore, we have proven the following is equal to 100 as well:

$$\underbrace{(1+\frac{1}{2}+\cdots+\frac{1}{n})}_{\text{sum of elements of sets of size 1}}+\underbrace{(1\cdot\frac{1}{2}+\cdots+\frac{1}{n-1}\cdot\frac{1}{n})}_{\text{sum of product of pairs from sets of size 2}}+\underbrace{\cdots+(1\cdot\frac{1}{2}\cdots\frac{1}{n})}_{\text{continue summing the product of elements from sets of each size 3, 4, ..., n}}=n$$

In other words, the final term of problem 1 is equal to n, and that is also equal to an interesting sum so the two must be equal!

Or to write it out slightly differently:

$$(1+\frac{1}{2}+\cdots+\frac{1}{n})+(1\cdot\frac{1}{2}+\cdots+\frac{1}{n-1}\cdot\frac{1}{n})+\cdots+(1\cdot\frac{1}{2}\cdots\frac{1}{n})=n$$

$$\text{in other words} \sum_{\sigma\in P(S_n)}\prod_{x_i\in\sigma} x_i = n$$

$$\text{where } S_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right\}$$

$$\text{and } P() \text{ denotes its power set}$$

Combinatorial proofs can be very interesting indeed!

One more connection with algebra!

Consider the constants a, b, and c. Now define the function:

$F(x) = -x^3 + (x + a)(x + b)(x + c)$

If we expand the formula out, we get the following:

$F(x) = x^2(a + b + c) + x(ab + ac + bc) + abc$

Hmm, this is interesting. What if we evaluate this function at the value of 1? Then we end up with

$F(1) = a + b + c + ab + ac + bc + abc$

If we started with a set $\{a, b, c\}$, then this function evaluated at F(1) is exactly the final resulting term from our original problem!

The function can be extended for larger sets. If we have a set with n elements $a_1, a_2, \ldots, a_n$, then we can define the function

$F(x) = -x^n + (x + a_1)(x + a_2)\ldots(x + a_n)$

The function evaluated at 1, written as F(1), will precisely be equal to the final term of our original problem, as well as the complicated "power set product sum" that was described above.

A theorem based on the function

Let's add the term -1 to the harmonic set up to 100 so we have $\{-1, 1, 1/2, 1/3, \ldots, 1/100\}$. What will the final number be?

If you try to do this manually, it might not be apparent. But we can use the trick: let's think about the function!

We know the function can be written as:

$F(x) = -x^{101} + (x + -1)(x + 1)\ldots(x + 1/100)$

Now comes the neat part. We want to evaluate the function at $x = 1$. But notice something: one of the terms in parenthesis is $(x + -1)$. This term will be zero, and hence the entire product of the terms in parentheses will be 0 as well!

Thus, we can conclude that:

$F(1) = -1^{101} + 0 = -1$

So we have proven this: any set that contains the term -1 will necessarily result in the final number being -1.

Putting it all together

This simple puzzle results in some interesting math. Specifically, we have found the following three statements to be equivalent.

(1) Repeatedly applying the process $x + y + xy$ for a given set until the final term

(2) Evaluating the "power set product sum" for a given set

(3) Evaluating F(1) for the function described above

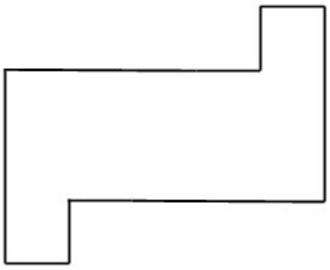
These are some neat results that came from a seemingly simple problem.

Section II: Geometry

Geometry is one of the oldest branches of mathematics. See if you can figure out these puzzles that test your knowledge of distances and angles.

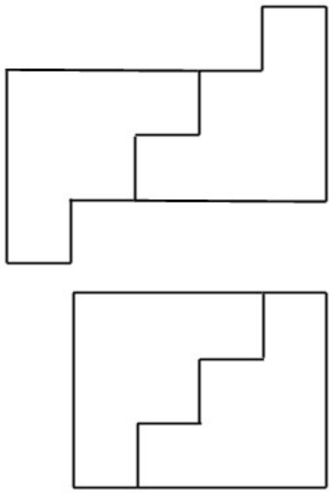
Puzzle 1: Make A Rectangle

In the following shape, make 1 continuous cut so that 2 pieces can be re-arranged to form a rectangle.



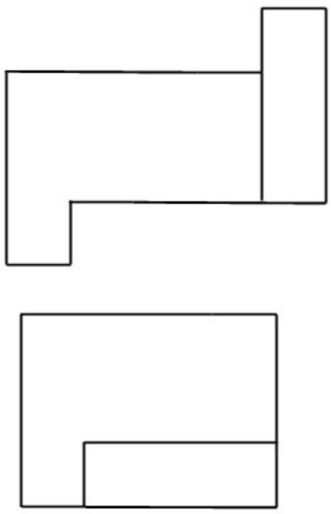
Answer To Puzzle 1: Make A Rectangle

The solution I had in mind was a staircase style cut.

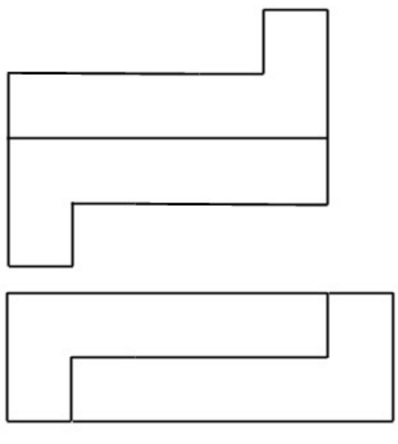


When I posted about this problem, it got picked up by the blog BoingBoing and I received a number of new solutions on Twitter.

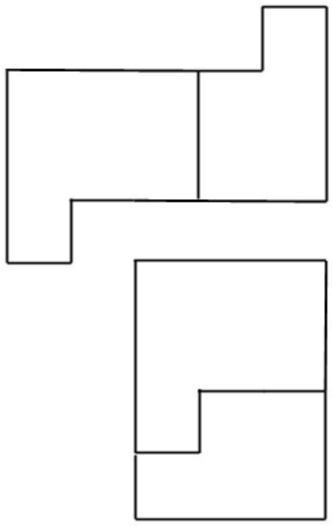
Erik Smith (@ErikSmith80) suggested a simple cut of one of the tabs.



DC Turner (@dcturner) thought about cutting horizontally across.



And Tim Heffernan (@Tim_Heffernan) came up with another solution that involves flipping one of the pieces over.



This problem was inspired by the many odd shapes I got while cutting wrapping paper. So it’s good there are many solutions as that means there are many ways to make unusual tabbed shapes back into a rectangle.

Puzzle 2: Clock Division

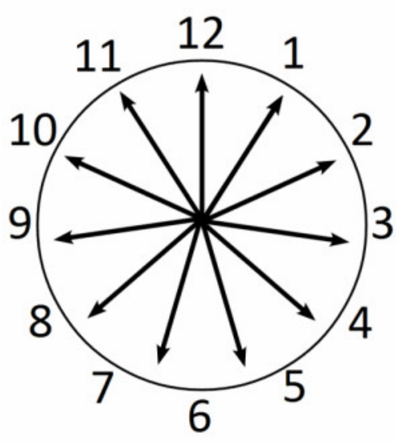
You need to divide a cake into equal pieces. The first problem is to divide it into 12 equal pieces. The second problem is to divide it into 11 pieces. You have an analog wall clock that you can use to determine angles.

Answer To Puzzle 2: Clock Division

Dividing into 12 pieces is easy: you can divide the cake at the 12 hours of the clock.

How do you divide into 11 pieces? Here's the trick: the hour and minute hands of an analog clock meet at exactly 11 times in a 12 hour period. Set the clock at 12:00 (which is one time the hands meet) to make the first slice. Then wind the minute hand until the hands meet next to get the next slice (this will be at about 1:05). Repeat the process to determine the rest of the angles!

Here is a rough sketch of the 11 divisions.



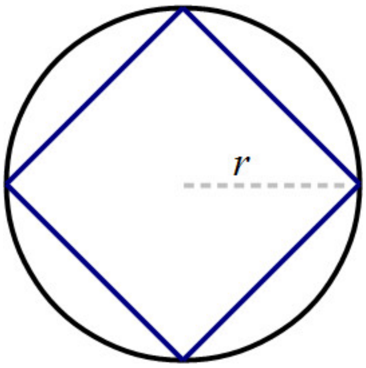
Credit: I came across this problem at Alok Goyal's puzzles.

Puzzle 3: Fitting A Square Peg In A Round Hole

What is a tighter fit: a square peg in a round hole, or a round peg in a square hole?

Answer To Puzzle 3: Fitting A Square Peg In A Round Hole

First, let us consider a square peg in a round hole, as in the following diagram.



If we let the circle have radius r, then that means the square's diagonal measures $2r$. The triangle formed by the square's diagonal and its two sides is an isosceles right triangle. By the Pythagorean Theorem, we can solve for the square's side, which is $r\sqrt{2}$.

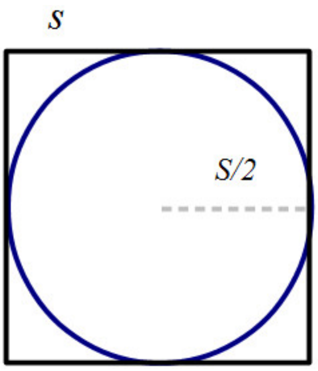
The square's area is the square of the side, so it is $2r^2$. This is to be divided by the circle's area, πr^2.

The ratio of the square's area to the circles is:

Square peg in round hole
$(2r^2)/(\pi r^2) = 2/\pi \approx 64$ percent

Thus, the square peg fills about 64 percent of the circle's area.

Now we calculate the other way around: a round peg in a square hole.



If we let the square have side s, then the circle's radius is $s/2$.

The circle has an area $\pi s^2/4$. This is to be divided by the square's area, s^2.

The ratio of the circle's area to the square is:

Round peg in square hole
$(\pi s^2/4)/(s^2) = \pi/4 \approx 79$ percent

The circle is about 79 percent of the square's area. This is larger than the 64 percent of the square filling the circle's area.

This means a round peg in a square hole is a better fit than the other way around.

Puzzle 4: Inscribed Rectangle

What is the largest area for a rectangle inscribed in a circle? That is, what percentage of the circle can the rectangle cover?

Answer To Puzzle 4: Inscribed Rectangle

One solution method is to use calculus. For a rectangle with height h and width w, the area is hw. The circle's diameter ($2r$) is the diagonal of the rectangle. The two sides of the rectangle and its diagonal form a right triangle. So by the Pythagorean Theorem, we have $h^2 + w^2 = 4r^2$. We can solve for the height: $h = \sqrt{(4r^2 - w^2)}$.

The area is $hw = w\sqrt{(4r^2 - w^2)}$. We can equivalently maximize the area squared, which is $w^2(4r^2 - w^2)$.

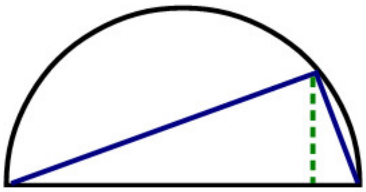
Taking the derivative with respect to w and solving, we get $w = r\sqrt{2}$. We substitute back to find the height is also the same, $h = r\sqrt{2}$. Since the rectangle's sides are equal, this rectangle is a square. The area is $hw = 2r^2$.

The circle has area πr^2, so the rectangle takes up $(2r^2)/(\pi r^2) = 2/\pi$ of the circle. This is approximately 64 percent.

There is another way to solve this problem that gives a different intuition as to why the rectangle of largest area is a square.

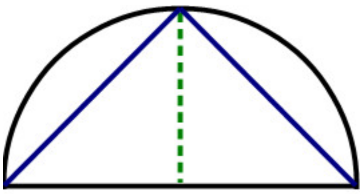
Note that the four interior angles in a rectangle are right angles. From geometry, an inscribed angle subtends an arc that measures twice as large. That is, the arc will measure 180 degrees.

This means two adjacent sides of an inscribed rectangle are contained in a semi-circle. Because the rectangle is symmetric about its diagonal, we can maximize the area of the entire rectangle by maximizing half the area of the rectangle, which is the area of the triangle between the two sides of the rectangle and the diameter of the circle.



The question is how: which triangle has the largest area?

The triangle's area is equal to the product of the circle's diameter times its height. The largest height will be when the triangle's height equals the circle's radius:



The largest triangle is an isosceles right triangle, which means the rectangle has adjacent sides that are equal. In other words, the rectangle is a square.

As calculated in the problem about a "square peg in a round hole," an inscribed square has area $2r^2$, which is approximately 64 percent of the circle's area.

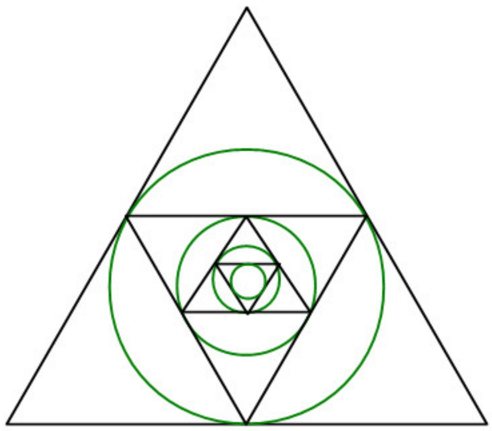
Puzzle 5: Infinitely Many Inscribed Circles

Inscribe a circle in an equilateral triangle with side length 1.

The inscribed circle intersects the equilateral triangle at the three midpoints of its sides. Connect these points to form another equilateral triangle.

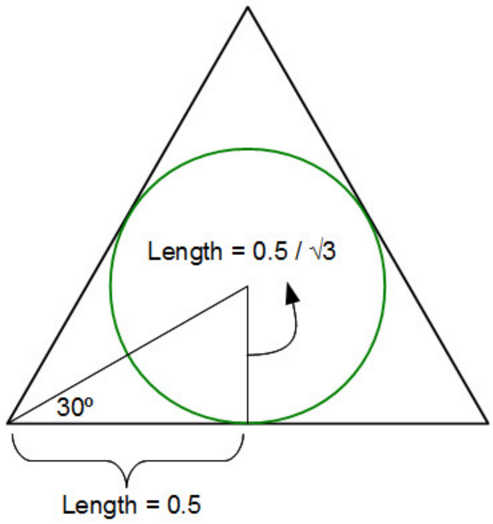
Now inscribe a circle into the smaller equilateral triangle.

Repeat the process indefinitely: draw nested equilateral triangles and inscribed circles.



What is the sum of the radii of the infinitely many inscribed circles?

Answer To Puzzle 5: Infinitely Many Inscribed Circles



We can find the radius of the first inscribed circle using textbook geometry. The radius is the leg of a 30-60-90 right triangle whose longer leg is 1/2. The radius of the first inscribed circle is $1/(2\sqrt{3})$.

This proportion is true generally since all equilateral triangles are similar. If an equilateral triangle had a different side length, then its inscribed circle would have a radius that is $1/(2\sqrt{3})$ of its side length.

In the problem, the next equilateral triangle is found by connecting the midpoints of the original triangle. Thus, it has a side length of 0.5. By the logic above, its inscribed circle has a radius that is $1/(2\sqrt{3})$ as big as its side length. That is, the radius is $1/(4\sqrt{3})$.

Each new equilateral triangle is also found by connecting the midpoints of the previous equilateral triangle, so each new triangle has a side length half as big as the previous one. This means each smaller inscribed circle is also half as big as the radius of the previous inscribed circle.

The sum of the radii of all the inscribed circles will form an infinite geometric series where the first term is the radius of the first inscribed circle, $1/(2\sqrt{3})$, and each new term is 1/2 the previous term.

We have the infinite series:

Sum of radii = $(1/\sqrt{3})\ (1/2 + 1/4 + 1/8 + \dots)$

The series inside the parentheses sums to 1, so we can solve the problem.

Sum of radii = $1/\sqrt{3} = \sqrt{3}/3$

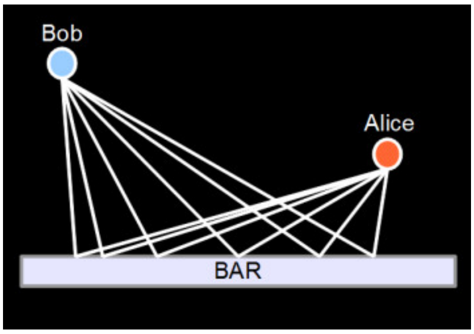
The sum of the radii is equal to twice the radius (is equal to the diameter) of the largest inscribed circle.

Puzzle 6: The Efficient Drink Order

Bob wants to buy a drink for his friend. To do so, Bob must walk to the bar, get the drink, and then walk to his friend.

There are a variety of paths Bob could walk, as illustrated below.

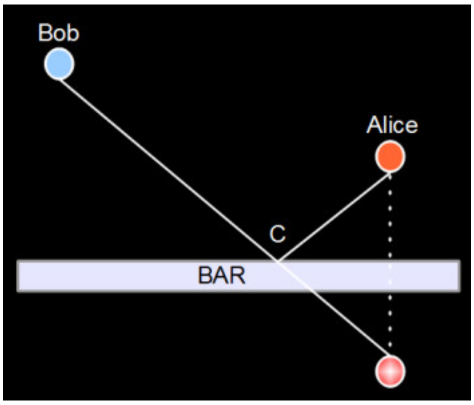
What's the shortest path he can take?



Answer To Puzzle 6: The Efficient Drink Order

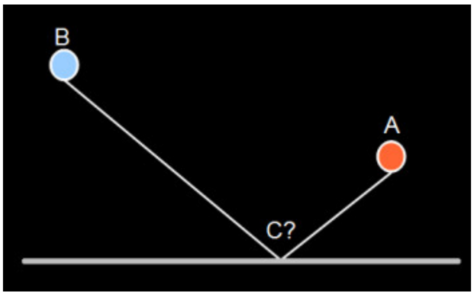
I had trouble solving this problem, until I remembered some high school geometry.

The trick is for Bob to visualize his friend on the other side of the bar. Specifically, he should reflect his friend's position across the bar. Then, Bob should connect a line from him to this imagined position. This line will intersect the bar at a point C. Bob should walk straight to this point C and then from C to his friend. This will be the shortest path.



The proposed solution is easy to follow. But why exactly is it the shortest path? Let's carefully prove why.

To do so, we take a step back and consider an idealized version. Represent Bob by point B and his friend by point A and consider the bar as a line segment.

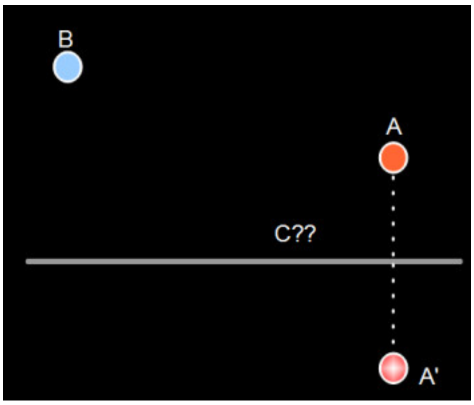


The problem is to find the point C which minimizes the distance:

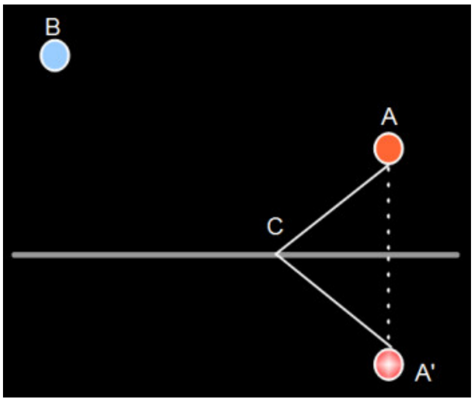
BC + CA

How can we determine point C?

The trick is to reflect point A across the line segment to its mirror image A'.



Now arbitrarily choose a point C along the line segment and draw the line segments CA and CA'.



The segments CA and CA’ are mirror images of each other, so they have equal length CA = CA’

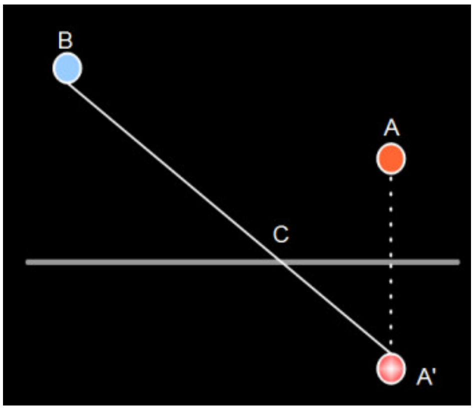
So here’s the interesting part. Our original problem was to minimize the distance:

BC + CA

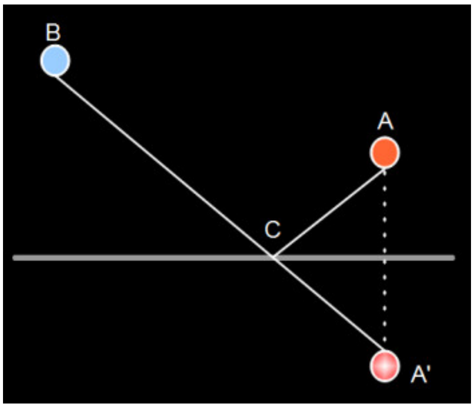
But we know CA = CA’. So we can rewrite our problem as minimizing the distance:

BC + CA’

Which point C minimizes the distance from B to A’? This problem is readily solved: between any two points in a plane, the shortest path is a straight line! So we will draw a line segment between B and A’, and its intersection with the line for the bar gives us the position of C.



We can reflect CA’ to get the line segment CA.



Since we have minimized the distance (BC + CA’), and we know CA’ = CA, we have also minimized the distance (BC + CA).

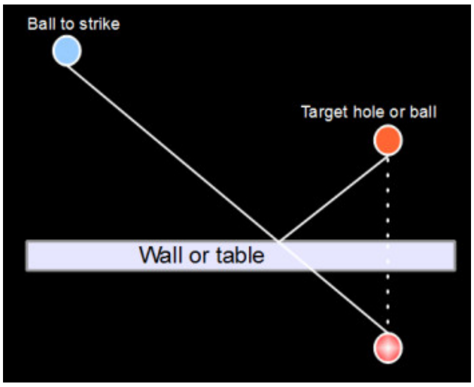
So the general procedure algorithm is:

1. Reflect point A, across the bar, to its mirror image A’.

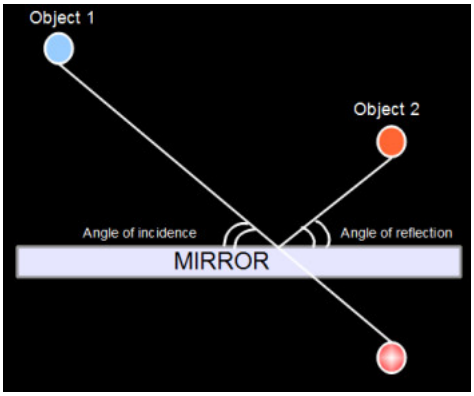
2. Draw a line segment from the point B to A’. Denote the intersection point with the bar as C.

3. Draw the line segments BC and CA. This is the shortest path.

The same geometric procedure is used to find the best shot in mini-golf or in billiards. If we wish to strike a ball so that it bounces off a wall and then goes to a desired target, we can imagine the same diagram.



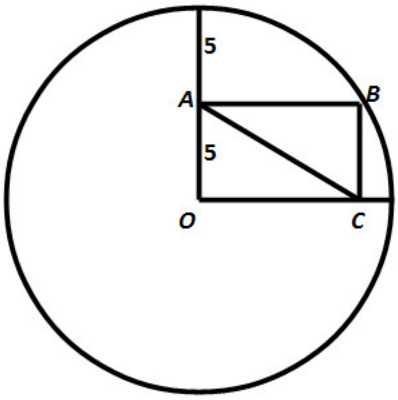
The problem also has a physics application. If a light is bounced off a mirror, then the path it will travel is the shortest distance, which is the path Bob takes to get the drink. When the light bounces off the mirror, the angle of incidence equals the angle of reflection.



This problem was described 2,000 years ago and is known as Heron's problem. It's quite an interesting geometry problem with many practical applications.

Puzzle 7: Circle Length

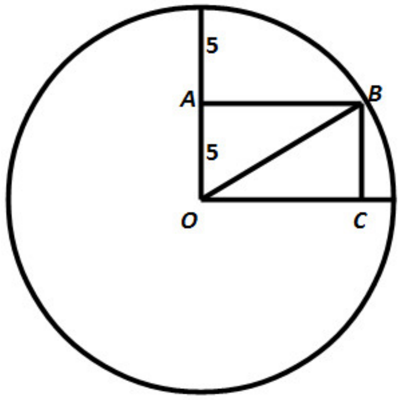
Rectangle ABCO is drawn in circle O as in the following diagram.



What is the length of segment AC?

Answer To Puzzle 7: Circle Length

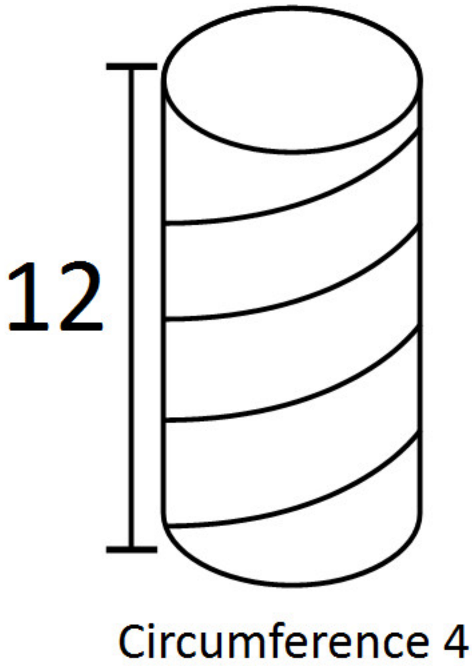
There are no special formulas needed to solve this puzzle. Note that diagonal OB is a radius of the circle, and the circle has a radius of 5 + 5 = 10.



In a rectangle, the two diagonals have the same length. So AC has the same length as OB, and its length is equal to 10.

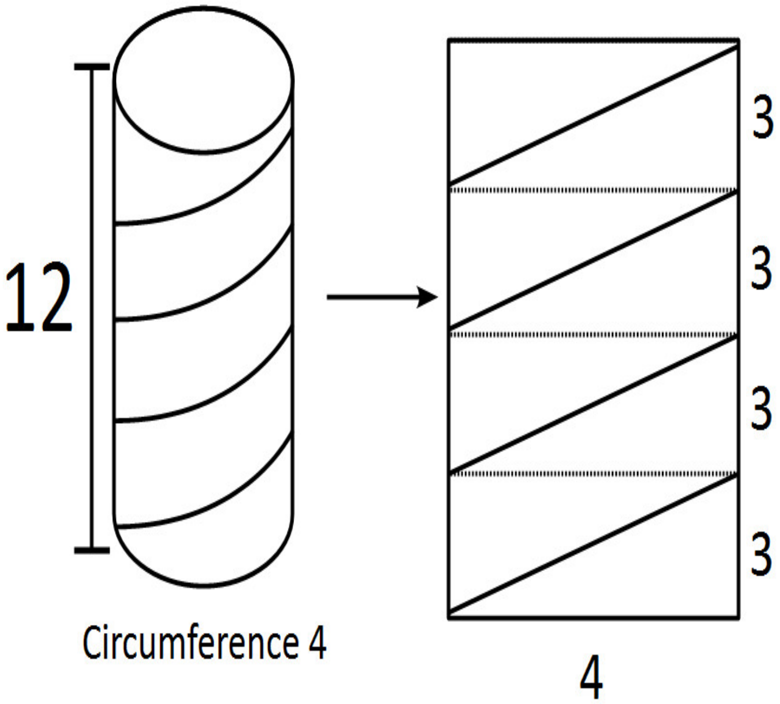
Puzzle 8: Length Of A Spiral

A tightly wrapped string makes 4 loops around a cylinder of height 12 and circumference 4. What is the length of the string?



Answer To Puzzle 8: Length Of A Spiral

The key is unwrapping the cylinder by flattening it out into a rectangle. Each of the spiral's loops becomes a right triangle.

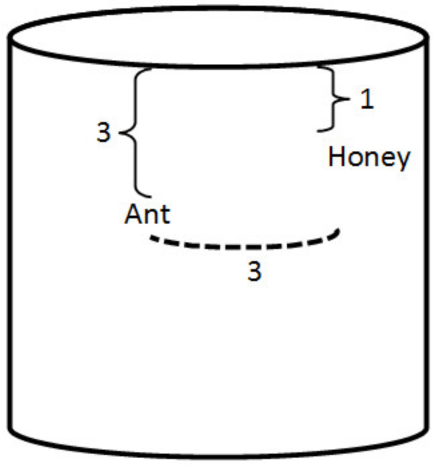


The height of each triangle is the diameter of the cylinder, which is 4, and the length of each triangle is the height divided by 4, which gives 3. So each triangle is a 3-4-5 right triangle and the hypotenuse of each is 5.

The total string length is four times as long as a single triangle's hypotenuse. So the answer is 20.

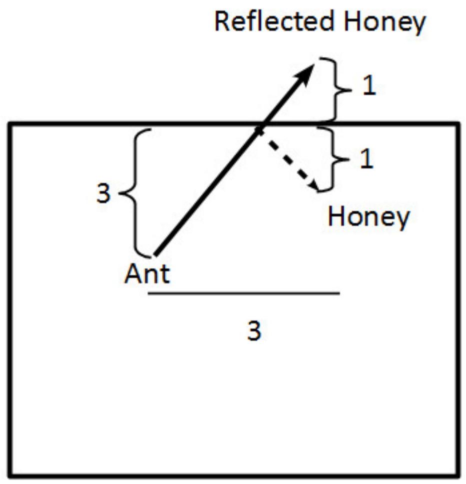
Puzzle 9: Ant Cylinder

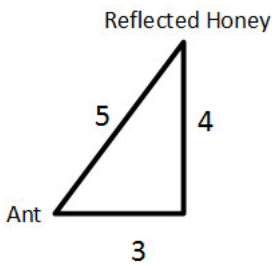
An ant wants to get to honey on a cylinder, as in the following diagram. What is the shortest path the ant can take, and what distance is it?



Answer To Puzzle 9: Ant Cylinder

There are two tricks to solve this puzzle. One is to unwrap the cylinder so it becomes a rectangle. The other trick is to reflect the position of the honey across the edge of the rectangle.



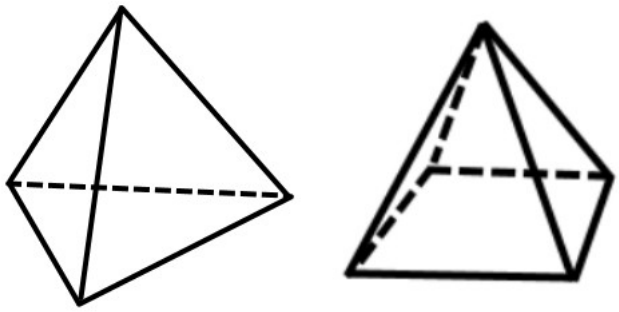


The shortest distance to the position of the reflected honey is a straight line. The mirror image of the path off the edge to the reflected honey also gives the second segment of the shortest path to the honey. This was explained in more detail in the puzzle "Efficient Drink Order."

The distance to the reflected honey is the hypotenuse of a right triangle with legs 3 and 4. So the distance is 5.

Puzzle 10: How Many Faces?

Consider a tetrahedron and a pyramid whose edges are the same length.



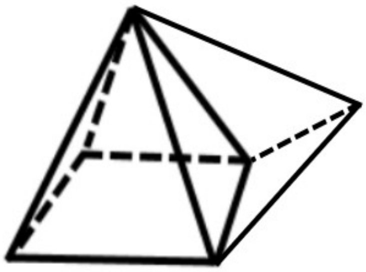
If the tetrahedron and the pyramid are attached on a triangular face, how many faces will the new shape have?

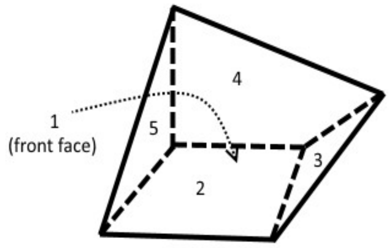
Answer To Puzzle 10: How Many Faces?

This question was asked on a PSAT Exam in 1980. American high school students take the test to qualify to become National Merit Scholars, a prestigious award that may lead to a $2,500 scholarship.

The test makers originally stated the correct answer was 7. The tetrahedron has 4 faces and the pyramid has 5 faces. When combined, each loses 1 face, so there are 4 + 5 - 1 - 1 = 7 faces in the new shape.

One student figured out this answer was wrong. When the two shapes are attached on a triangular face, the two adjacent faces are coplanar and line up perfectly to make one side. So the new shape loses 2 more faces. And so the resulting shape is like a ramp that only has 5 faces.



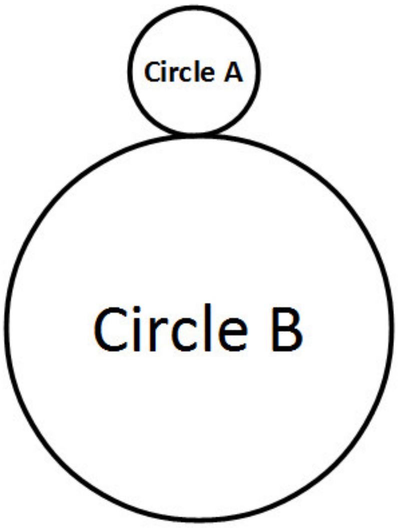


The test makers admitted the mistake and agreed to make 5 an acceptable answer as well.

The story is documented in the New York Times, “Youth Outwits Merit Exam, Raising 240,000 Scores.” Edward B. Fiske. March 17, 1981.

Puzzle 11: Circle Rotation

Circle A has 1/3 the radius of circle B. Circle A rolls around circle B until it returns to its starting position. How many revolutions of circle A are there in total?



This problem appeared in the 1982 SAT examination given to American high school students who were applying to college.

Here are the answer choices for the question.

(a) 3/2
(b) 3
(c) 6
(d) 9/2
(e) 9

What's the correct answer?

Answer To Puzzle 11: Circle Rotation

Amazingly, all five answer choices were wrong! The correct answer is 4.

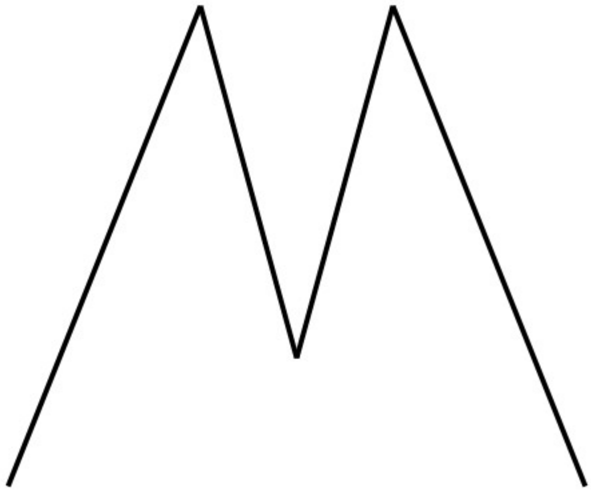
The test preparers thought that the small circle, which has 1/3 the radius, would revolve around the large circle 3 times. This would be true if the small circle were rolling in a straight line with length 3 times its circumference. However, the small circle in this problem was also rolling around a circle as well. That motion generates another revolution. Thus, the answer is 4.

You can get an idea of this with two coins of the same size: when you roll one coin around the other, it revolves around 2 times. One of the revolutions is because the coins have the same circumference. The other revolution is because the center of the rolling coin is also orbiting the center of the stationary coin. In American currency, you can try this with a quarter coin, but the coin might slip as you try to roll it. I have found it is easier with coins that have thicker edges, like a dollar coin or a half-dollar coin.

Credit: I had read about this debacle in a book, but I cannot remember the exact source. There is a nice discussion at Donald Sauter's website, which quotes an article from the Washington Post newspaper in May 25, 1982.

Puzzle 12: Non-Overlapping Triangles

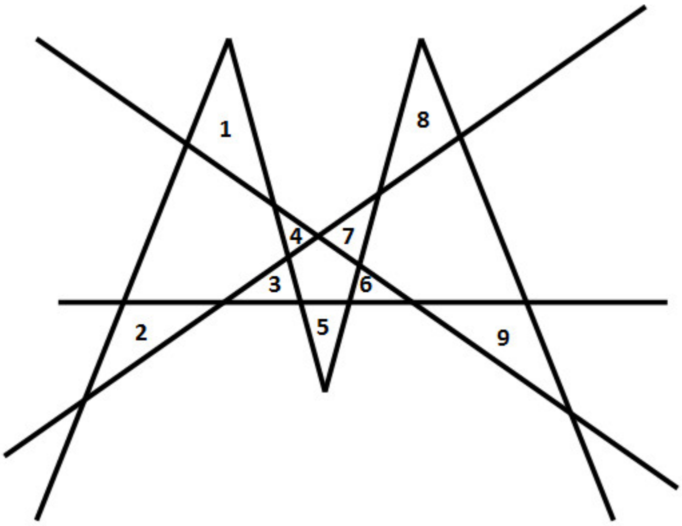
In the following figure, draw 3 straight lines to make 9 non-overlapping triangles.



Note: this problem appears in the Simpson's episode "Mathlete's Feat", the 22nd and final episode of season 26.

Answer To Puzzle 12: Non-Overlapping Triangles

You have to experiment with drawing the lines. Here's one solution.



Puzzle 13: How Many Partners?

The briefly-lived CBS show *Partners* advertised the teaser tagline, “4 friends, 3 couples.” The billboard depicted three men and one woman, leaving the audience in the dark about who is paired.

This inspired me to come up with the following math puzzle.

In how many ways is it possible to have 3 couples amongst 4 friends?

Assume that any two people in the group can form a couple and that people can have multiple simultaneous couplings.

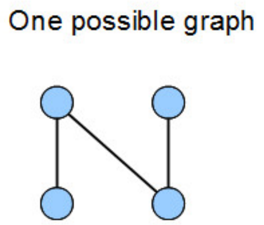
Examples of different ways to make 3 couples:

1. The woman could be coupled with each of the three men to make 3 couples in total.

2. Two of the men could form a couple, and one of them could also form a couple with the woman. The woman can also couple with the remaining male member of the group. There are 3 couples in total for this arrangement.

Answer To Puzzle 13: How Many Partners?

I found a solution using graph theory and combinatorics. We will depict each person as a node, and we can draw a line between two nodes if the two people are coupled.



The question is then rephrased: how many ways can we draw 3 edges on a graph with 4 nodes?

This is now a combinatorics question.

Part 1: How many possible edges can we draw?

An edge can connect any of the two nodes together. So we want to connect 2 of the four vertices, which can be done in:

(4 choose 2) = 4!/(2! 2!) = 6 possible edges

Part 2: How many ways for 3 edges?

From the 6 possible edges, we want to select 3 of these lines to make the 3 couples. This means there are:

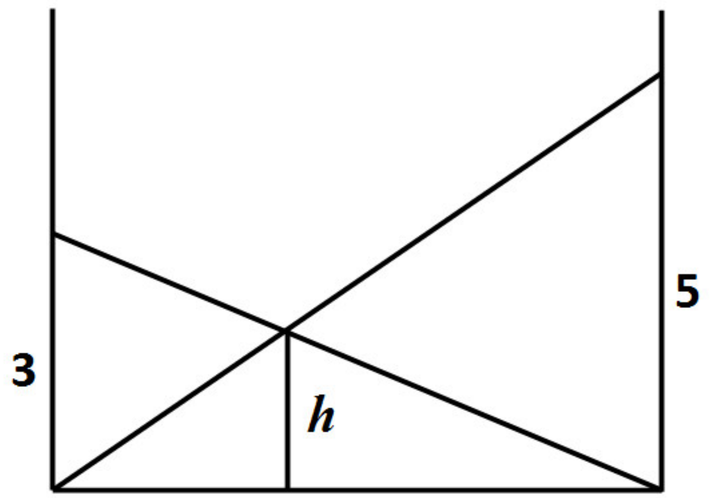
(6 choose 3) = 6!/(3! 3!) = 20 possible couples

This result surprised me: I did not realize there would be so many possible combinations.

Puzzle 14: Crossed Ladders

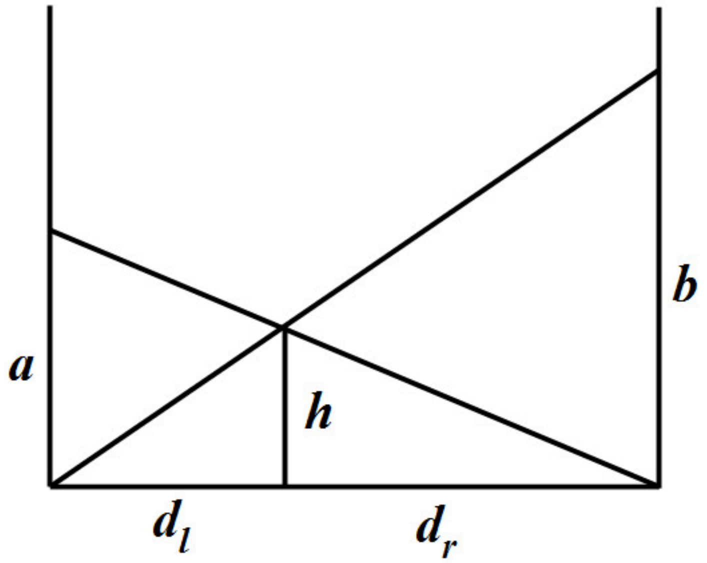
Two ladders are placed against walls to heights of 3 and 5. What is the height of the point where the ladders cross?

What's a general formula if the ladders are at heights *a* and *b* against the walls?



Answer To Puzzle 14: Crossed Ladders

Suppose the height is h and the ladders are at heights a and b. It will be useful to consider the horizontal distance from the crossing point to the left wall d_l and the same distance to the right wall d_r. These variables will eventually cancel out, but they will help in the calculation.



The triangle formed by the left wall, the distance between the walls, and the left ladder is similar to the triangle formed by the height to crossing, the right distance, and the left ladder to the crossing. This means the ratio of the legs of these triangles is equal. So we have the following.

$a/(d_l + d_r) = h/d_r$
$h = (ad_r)/(d_l + d_r)$

We can find another set of similar triangles when considering the other ladder to the right wall. So we find the following ratio.

$b/(d_l + d_r) = h/d_l$
$h = (bd_l)/(d_l + d_r)$

If we set the two equations for h equal to each other, it is evident that $ad_r = bd_l$ so that $d_r = (bd_l)/a$. We can substitute this into the first expression for h.

$h = (ad_r)/(d_l + d_r)$
$h = (bd_l)/(d_l + bd_l/a)$
$h = b/(1 + b/a)$
$h = (ab)/(a + b)$

Now we have a simple expression for h in terms of the wall heights a and b. When the ladders are against the walls at 3 and 5, then the height of crossing is $(3)(5)/(3 + 5) = 15/8 = 1.875$.

The equation $(ab)/(a + b)$ is half of the harmonic mean of a and b. The harmonic mean is used when adding up two rates of work. For example, if one person paints a house in a hours, and another person does it in b hours, how quickly will they do the job together? The answer is also half the harmonic mean. The crossed ladders problem seems completely unrelated to the problem of working together, and yet they both have the same solution. Quite interesting!

Puzzle 15: Fruit Label Stickers

Imagine a perfectly round grapefruit that is overly labeled with 5 fruit stickers.

Prove that the grapefruit can be cut in two equal halves in which one of the halves has 4 of the 5 stickers.

Note: A sticker along the boundary of the cut is counted as being on both halves.

Answer To Puzzle 15: Fruit Label Stickers

Pick any two stickers on the grapefruit. Now make your cut along the great circle that contains these two stickers (a great circle is a circle on the boundary of a sphere with the same radius as the sphere).

These 2 stickers can be counted as being on both of the halves. The remaining 3 stickers have to fit into either of the two halves. The possibilities for (one half, other half) are (3, 0), (2, 1), (1, 2), (0, 3). In each possibility, there is some half that contains at least 2 of these 3 stickers. This half ends up with at least 2 of these stickers, plus the 2 stickers from the boundary cut, for a total of at least 4 of the stickers.

Credit: this is adapted from a 2002 Putnam Exam question.

Puzzle 16: Castle Height

Your army is planning an invasion against a castle. You want to know the height of the castle to calibrate the catapult.

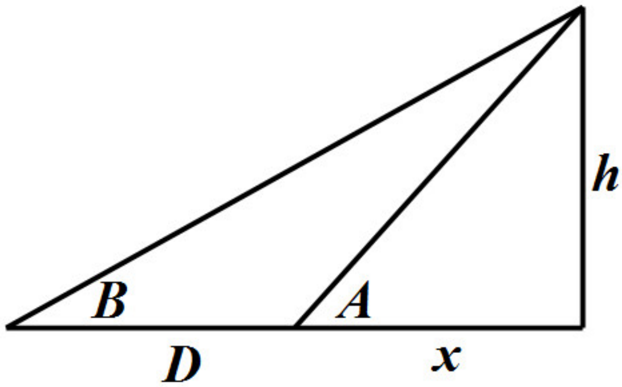
In the past, you calculated heights using right triangle trigonometry. Using basic survey tools, you measured the distance to the base of the object, and you also measured the viewing angle from that distance to the height. The height was equal to the distance times the tangent of the angle.

But this castle is surrounded by a moat, and you cannot accurately measure the distance to its base. How can you measure the height of the castle?

(Assume you take angle measurements from the ground-level.)

Answer To Puzzle 16: Castle Height

From one point measure the angle A to the height of the castle. Then go a distance D directly away from the castle. Measure the new angle B to the height, as in the following diagram.



We want to solve for h, but we will need to solve for x as an intermediate result.

From the diagram, we can deduce:

$x = h/\tan(A)$
$h = (D + x) \tan(B)$

We can substitute the value for x from the first equation into the second and then solve.

$h = (D + h/\tan(A)) \tan(B)$
$h(1 - \tan(B)/\tan(A)) = D \tan(B)$
$h = D \tan(B)/[1 - \tan(B)/\tan(A)]$
$h = D \tan(A) \tan(B)/[\tan(A) - \tan(B)]$

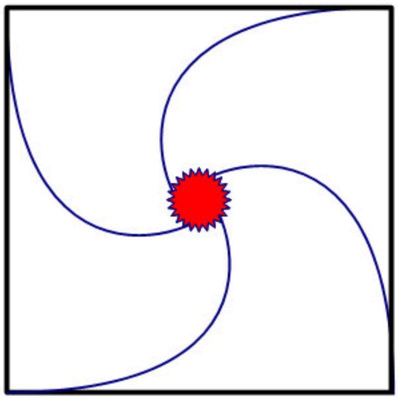
This provides the height of the castle in terms of the known distance D and the two known angles A and B.

For example, suppose the distance was 100 and the angles were A = 35 degrees and B = 30 degrees. Then the height would be approximately 329.

Puzzle 17: Bumper Cars On A Square

Four bumper cars start at different corners of a square arena. Each car is programmed to target the car that starts in the corner counter-clockwise to it. Each car moves directly toward its target and keeps adjusting its position so it is always headed straight towards its target. Once the cars collide they will come to a stop.

Here is a rough sketch of the path that the cars will take:

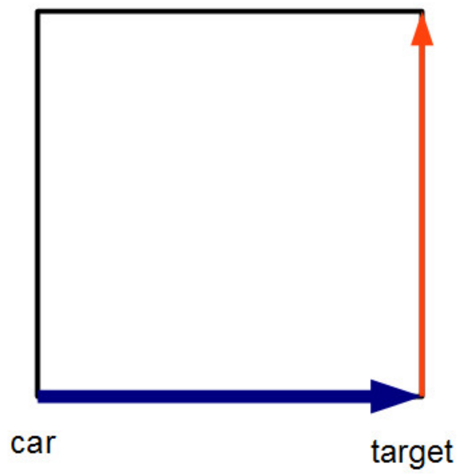


If one side of the square measures 25 meters, how much distance does each bumper car travel before collision?

Answer To Puzzle 17: Bumper Cars On A Square

This problem is more generally known as the mice problem. It can be solved using calculus. But we don't really need to do that.

The trick is to consider movement from the perspective of one car. The bumper car travels directly toward its target, which moves at a right angle to the line connecting the bumper car and its target, as pictured here:



Since the bumper car is always continuously adjusting its position towards its target, the perspective will be the same as if the bumper car were moving directly towards its target along the square.

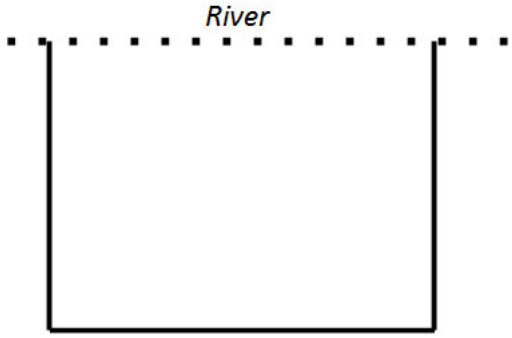
Therefore, the distance until they collide is the same as the distance of one side of the square, 25 meters.

In other words, the bumper cars would have moved the same distance if each was targeted to go to its adjacent corner.

Credit: I read about this in a column by Barry Nalebuff, Puzzles: Slot Machines, Zomepirac, Squash, and More. published by the American Economics Association Winter 1990.

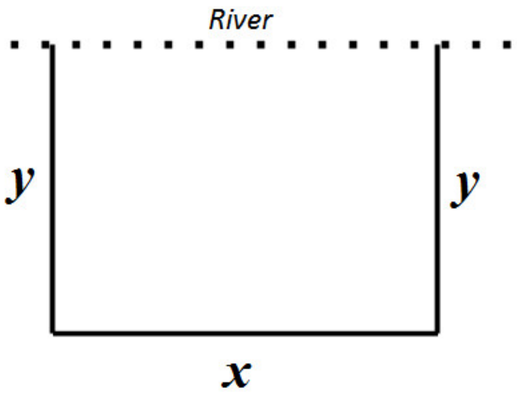
Puzzle 18: Optimize The Fence

You have 100 units of fencing. You want to enclose 3 sides of a rectangular area, with the 4th side of the rectangle being a river. What is the maximum area you can enclose?



Answer To Puzzle 18: Optimize The Fence

There are two ways to solve this problem. I'll explain the standard method first and then go over the creative solution.



Suppose the rectangle has side x parallel to the river and has two sides y perpendicular to the river. The perimeter of the rectangular shape is $x + 2y = 100$, which means $x = 100 - 2y$.

The area enclosed is xy. We can substitute for x to get an area in terms of the single variable y.

Area = $(100 - 2y)y$

We can solve for the maximum area either graphically or using calculus. If we take the derivative and set it equal to zero, we can solve for the maximum area as follows.

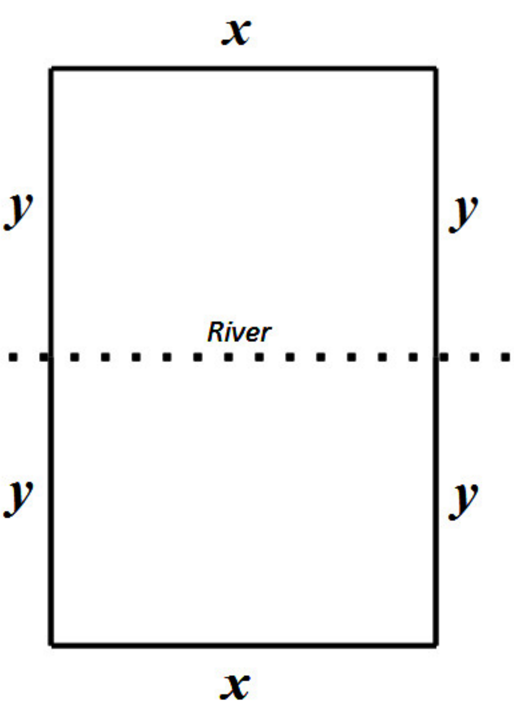
Derivative of Area = $100 - 4y = 0$
$y = 25$

We can then find $x = 100 - 2(25) = 50$, and so the area is $xy = (50)(25) = 1{,}250$.

This is a direct method, but there is a clever method to solving it faster!

Geometric Method

Reflect the 3-sided rectangle across the river to make a rectangle.



The big rectangle has a perimeter of $2x + 4y$, which is double the perimeter of the original shape. Also, the area is also twice the area of the original shape. Therefore, if we enclose the largest area for a given perimeter of this big rectangle, we will simultaneously solve for the largest area of the 3-sided shape.

For a given perimeter, what's the rectangle that encloses the most area? The answer is a square. Why?

Note that a rectangle with sides $a < b$ has the same perimeter as the square with sides $a + x = b - x$. That is, we remove a bit of the larger side until the two sides are equal. The square has an area of $(a + x)^2 = a^2 + 2ax + x^2$. This is larger by x^2 compared to the rectangle with unequal sides, which has an area $ab = a(a + 2x) = a^2 + 2ax$. So the rectangle that encloses the largest area for a given perimeter is a square.

Returning to the original problem, we can maximize the big rectangle by making it a square. This means its two sides of x and $2y$ must be equal.

This will also optimize the enclosed area for the small rectangular shape with 3 sides. In that figure, we had 100 units of fencing, which meant $x + 2y = 100$. Now we know $x = 2y$, so we can substitute to get $2y + 2y = 100$ and so $y = 25$. Then we have $x = 50$. And the area is $xy = (50)(25) = 1{,}250$.

It's the same answer and does not require calculus. Plus you can solve all related problems of maximizing a 3-sided rectangle by setting the single side parallel equal to 2 times a perpendicular side.

Puzzle 19: Cylinder Height

Canned food comes in many different sizes and shapes. You want to produce a cylindrical can that requires the minimum material (surface area) for a given volume. What ratio should you make the can's height to its radius?

Answer To Puzzle 19: Cylinder Height

Let the cylinder have a radius r and a height h. The volume of the cylinder is (base)(height) = (area of circle)(height) = $\pi r^2 h$.

The surface area of the cylinder is (2 area base) + (area tube) = $2\pi r^2 + 2\pi rh$.

For a given volume $V = \pi r^2 h$, we can solve for the height as $h = V/(\pi r^2)$. Now we substitute that into the equation for surface area.

Surface Area = $2\pi r^2 + 2\pi r(V/(\pi r^2))$
Surface Area = $2\pi r^2 + 2V/r$

Now the surface area is given as a function of one variable r, and we can take the derivative to find a condition for the minimum surface area.

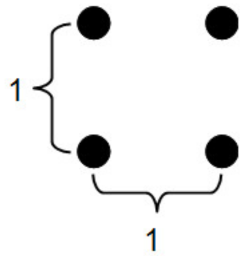
Derivative of Surface Area = $4\pi r - 2V/r^2 = 0$
Which implies $V = 2\pi r^3$

Now we have the minimum surface area is when $V = 2\pi r^3$. But for any cylinder, the volume is given by $V = \pi r^2 h$. Equating those expressions, we find the minimum surface area happens when $h = 2r$.

In other words, the height should be twice the radius.

Puzzle 20: Connect Four Towns

Four towns are located on the corners of a square with side length 1. A networking company wants to connect all four towns using straight lines and the least possible wiring.

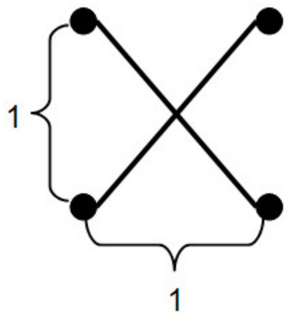


Can you think of an efficient method? You need not prove the answer is optimal, but if you find a good solution, calculate the amount of wiring necessary.

Answer To Puzzle 20: Connect Four Towns

The problem of finding the shortest interconnect between a set of points is known as the Steiner tree problem. The interesting thing is extra points, known as Steiner points, can serve as intermediate connections that reduce the total length of connections.

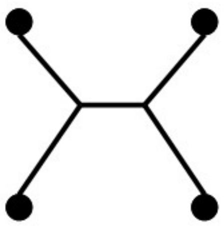
For example, the most obvious solution to the unit square is to connect the diagonal points.

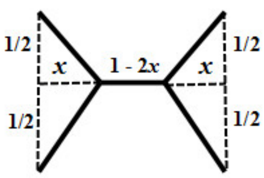


Each diagonal has a length of √2, so the total wiring is twice as long, 2√2 ≈ 2.83. Can we do any better? Surprisingly we can!

There are interesting patterns in soap bubbles or bubbles while boiling rice that can illustrate how one might have discovered the following unusual path of wiring.

Consider two points at a height of 1/2, each at a horizontal distance of x from a side of the square, and draw the connections as follows.





Let’s solve for the value of x that minimizes the total wiring length.

The one horizontal line has length 1 - 2x. The remaining four lines are congruent, and they are equal to the length of a hypotenuse of a right triangle with legs x and 1/2. The length of each hypotenuse is $\sqrt{(1/4 + x^2)}$, and we multiply that by 4 to get the total length. The total length is the following equation.

Length = $1 - 2x + 4\sqrt{(1/4 + x^2)}$

We can minimize this by taking the derivative and setting it equal to zero.

Derivative of Length = $-2 + (4x)/\sqrt{(1/4 + x^2)} = 0$
$\sqrt{(1/4 + x^2)} = 2x$
$1/4 + x^2 = 4x^2$
$3x^2 = 1/4$
$x = 1/(2\sqrt{3})$ [ignore negative solution]

When we evaluate the length at this value of x, the result is a total interconnection length of 1 + √3 ≈ 2.73. Note this is less than the result of 2.83 when we connected the diagonal points!

It turns out this unusual pattern is the shortest way to connect the four towns. You can read up on the Steiner tree problem to see how one might prove this interesting result.

Section III: Probability And Game Theory

These problems are about games of chance and randomness.

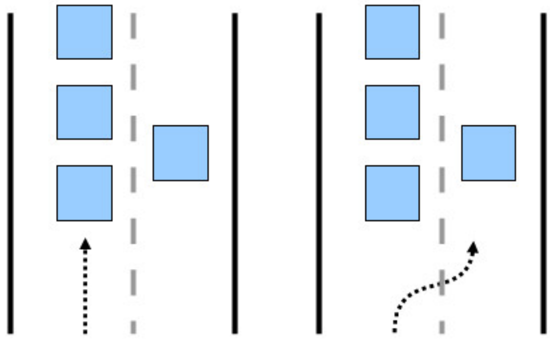
Puzzle 1: Which Lane Is Better?

You're on a two-lane highway and there are cars ahead of you in both lanes. You want to go straight ahead, but some of the cars in front of you might slow down to turn and delay you.

You are in the left lane. In that lane, there are 3 cars, each with an independent 20 percent chance of stopping to turn. If any of the cars stop, you have to stop too.

Over in the right lane there is only one car. But your experience indicates a 50 percent chance the car will stop to turn.

Which lane are you better off in?



Answer To Puzzle 1: Which Lane Is Better?

In the left lane, there are 8 possible ways the 3 cars might decide to stop (S) or not (N): SSS, SNS, SNN, SSN, NSS, NNS, NSN, NNN. You have to stop if even one car stops. So you want to know the event in which no car stops. The probability a single car does not stop is 0.8, calculated as 1 minus the probability it does stop. The probability of NNN is $(0.8)^3$ = 51.2 percent.

In the right lane, the car stops with a 50 percent chance, so it will not stop with a 50 percent chance.

You have a slightly better chance of not stopping in the left lane.

Puzzle 2: Medical Conspiracies

A 2014 study from the University of Chicago found that many American adults do not completely trust the medical profession. It found that about 1/2 of adults believed in at least 1 conspiracy out of the 6 conspiracies posed to them. Should this be surprising?

Summary: "You're Not Alone: Medical Conspiracies Believed By Many". Andrew M. Seaman. Reuters. Accessed at Reuters.com

Study: "Medical Conspiracy Theories and Health Behaviors in the United States." J. Eric Oliver, Thomas Wood. JAMA Intern Med. 2014;174(5):817-818. doi:10.1001/jamainternmed.2014.190. Accessed at Jama Network.com.

Answer To Puzzle 2: Medical Conspiracies

The result is less shocking than the title indicates. Suppose an adult had 90 percent confidence in published medical results. When presented with 6 conspiracies, the chance of disagreeing with all 6 is the chance of siding with 6 published medical results: $(0.90)^6$ = 53 percent. In other words, the chance of believing at least 1 conspiracy is about 47 percent.

On the other hand, if medical results were held to a 95 percent confidence level (as they meet the 5 percent significance level), then the chance of disagreeing with all 6 conspiracies should be $(0.95)^6$ = 74 percent. But even then, there would be 26 percent–more than 1 in 4 adults–that would still agree with at least 1 medical conspiracy.

Puzzle 3: Cards In Three Piles

Here's a game you can bust out at parties on your less mathematically inclined friends.

Offer the other person a standard deck of cards and tell them to shuffle it in any way they like.

Now tell them to divide the deck into 3 equal piles of 14 cards each, with the cards facing down (no peeking!).

Separate the top card from each pile.

Here's how the payouts work. Tell them your "lucky" cards are aces, fours, and jacks. You will win \$1 from your friend if any of the top cards is revealed to be a lucky card.

Otherwise, you pay \$1 to your friend.

It seems like the odds are against you as you only win on aces, fours and jacks. If you play for a while, you will win a bit more on average, and someone might challenge that you have rigged the deck in some way.

But there's no sleight of hand required in this game. Can you figure out why the game is to your advantage?

Answer To Puzzle 3: Cards In Three Piles

It’s true you only win on the 12 cards that are aces, fours, and jacks. But you also get 3 tries to win, and that’s how the game works out in your favor.

The way to proceed is to calculate the probability the other person will win–that none of the cards shown is an ace, a four, or a jack.

The probability the first card is not an ace, a four, or a jack is 40/52: there are 40 such cards out of the entire deck.

For the second card, there are now remaining 39 safe cards out of 51. So that chance is 39/51.

For the final card, there are 38 out of 50 safe cards.

The chance the other person wins is the product of these probabilities:

Pr(other person wins) = (40/52)(39/51)(38/50) = 0.447…

Remarkably the other person has less than a 50 percent chance of winning.

The probability you win is the complementary probability, which is a tad over 55 percent. The game does not appear to be in your favor, but the odds are and that’s what makes it a good sucker bet.

Credit: I read about this in *Mathematical Magic* by William Simon.

Puzzle 4: Random Music

People were so angry at Apple's iPod shuffle function replaying the same songs that a conspiracy theory developed that Apple was replaying particular songs to boost their popularity. Many people have felt angry that music devices do not properly randomize their music. But maybe it's all just a coincidence? Let's explore the mathematics of randomness in the following problem.

Bob likes to listen to soft music in the morning, so he makes a playlist of 3 songs that he plays on shuffle/random.

If Bob listens to 3 songs, what is the probability he will hear all 3 different songs?

Assume the shuffle function is purely random. That is, for any song that is playing, each of the songs A, B, and C has an equal chance of being played next.

What if there are n songs in a playlist put on shuffle? Then what are the odds the first n songs are all n different songs?

Answer To Puzzle 4: Random Music

If Bob hears song A first, there are 9 equally likely possibilities for the next 2 songs played.

AA, AB, AC, BA, BB, BC, CA, CB, CC

Of those 9, only 2 possibilities correspond to hearing all three songs: BC and CB. The probability Bob will listen to all 3 songs is 2/9, a modest 22 percent. In other words, there is a 78 percent chance one of the songs will be repeated.

In general, if there are n songs in a playlist, there are n^n equally likely possibilities for the first n songs to be played.

(n possible songs)(n possible songs)….(n possible songs) = n^n

If we want to play n songs without repeat, there are $n!$ arrangements of the n songs, because there is 1 less choice for playing each subsequent song:

(n possible songs)(n - 1 possible songs)….(1 possible songs) = $n!$

The probability of playing n songs without any repeats is $n!/n^n$.

As n increases, this ratio diminishes to 0. In other words, it becomes increasingly unlikely to play n songs at random without having any repeats. For example, in a 10 song playlist there is a mere 0.00036288 probability of not repeating a song in 10 plays. When you consider that people put hundreds of songs in their music playlists, it is nearly certain that a song will repeat.

This problem shows why a pure random distribution is not always suitable for consumer products. Most music listeners would want the shuffle algorithm to mix up the music so you do not keep hearing the same songs over and over again. And it's particularly annoying when the same song is played back to back.

Does Apple's iPod play music randomly? No one has proven the claim. And even if iTunes did play songs completely at random, people may not perceive it as random.

Puzzle 5: Buying Fresh Fruit

Alice is ill and sends her husband Bob to the fruit market. The only problem is Bob is inexperienced at picking fruit, and he can't distinguish between ripe and rotten fruit.

The apple stand has 200 apples, of which 20 are rotten.

If Bob selects 10 apples at random (he doesn't know what he's doing), what are the odds that:

1. He buys no rotten apples?
2. He buys all 10 rotten apples?
3. He buys x rotten apples?
4. He buys 2 or fewer rotten apples?

Bonus: Bob is told he better come home with 10 good apples. He decides that he could just purchase more apples to be safe. What's the minimum number of apples he must buy so he is sure to bring home 10 good apples?

Answer To Puzzle 5: Buying Fresh Fruit

The distribution in this problem is known as the *hypergeometric* distribution: it's the odds of selecting successes from a discrete number of draws out of a population with a fixed number of successes.

Bob is selecting 10 apples out of 200. The number of ways he can pick these is "200 choose 10":

$$\binom{200}{10}$$

This will be the denominator for the rest of the parts. Now we work out the number of ways he can pick the rotten apples. (The factorials in this problem involve very large numbers. I used the website WolframAlpha to compute these numbers).

Part 1: he buys no rotten apples

There are 180 good apples. If Bob is to avoid the rotten apples, he must select his 10 apples from this set. Therefore, there are 180 choose 10 ways that Bob can select all good apples. The odds of this happening are:

$$\frac{\binom{180}{10}}{\binom{200}{10}} = .3398$$

This is not bad; he has a better than 1/3 chance of getting the job done correctly.

Part 2: he buys all 10 rotten apples

There are 20 bad apples, from which Bob needs to select all 10 of his apples. The odds of that are:

$$\frac{\binom{20}{10}}{\binom{200}{10}} = 8.23 \times 10^{-12}$$

At that minuscule level, he's unlikely to select only rotten apples.

Part 3: he buys *x* rotten apples?

From the 20 bad apples, Bob is selecting x rotten apples and the remaining $10 - x$ apples as good (we assume that x is between 0 and 10).

The odds of that are:

$$\frac{\binom{20}{x}\binom{180}{10-x}}{\binom{200}{10}}$$

Part 4: he buys 2 or fewer rotten apples?

This is as an extension of part 3. We need to add up the cases in which Bob selects 0, 1, and 2 rotten apples:

$$\frac{\binom{20}{0}\binom{180}{10} + \binom{20}{1}\binom{180}{9} + \binom{20}{2}\binom{180}{8}}{\binom{200}{10}} = .935$$

All in all, Bob has a 93.5 percent chance which is quite good.

Bonus: all good apples

This is actually the easiest question to solve–and perhaps the most practical of the problems!

At most Bob can choose all 20 of the rotten apples. So if he chooses 30 apples, he is guaranteed that at most 20 are bad and the remaining 10 will be good.

So Bob gets a few more apples, but he plays the game like he's in a MasterCard credit card commercial: Cost of 30 apples: $15. Making the wife happy: priceless.

Puzzle 6: The Pawn Chase

Alice and Bob play a game on a board with 100 squares in each row and column (a 100x100 board). Alice has a pawn in one corner and Bob has the pawn in the opposite corner of the same row. In this game, a pawn can only move horizontally or vertically, not diagonally.

On each turn, a player moves the pawn one square. Alice and Bob move in turn. Alice's object is to capture Bob's pawn by going to the same square, and Bob's goal is to evade Alice.

If Bob goes first, who has the winning strategy?

Answer To Puzzle 6: The Pawn Chase

Remarkably Bob can always evade Alice. Consider coloring the squares on the board white and black in alternate colors, as on a standard chessboard.

In a given row, the 1st, 3rd, 5th, and odd squares will all have the same color (suppose it's white). The even squares will all have opposite color of black.

Alice and Bob start out in opposite corners of the same row. In a 100 square row, that means Alice starts on 1 (white) and Bob starts on 100 (black).

On each move, Alice and Bob move to an adjacent square, which will have the opposite color.

On the first move, Bob goes to a white square. On Alice's first move, she moves to a black square. On his second turn, Bob moves to a black square while Alice will move to a white square. In every odd round, Bob is on a white square while Alice is on a black. And on every even round, Bob is on a black square while Alice is on a white square.

Bob will always move to a square that Alice cannot reach, and therefore he can evade Alice indefinitely.

Puzzle 7: Who Will Toss More Heads?

Let's play a game with coins.

You have 100 dimes, and I have 99 pennies. At the same time, we will toss our coins in the air and let them fall on the floor. Then we meticulously count the outcomes of our tosses.

You win if you show more heads than I do. What's the probability that you will win?

Answer To Puzzle 7: Who Will Toss More Heads?

If I toss 99 coins, and you toss 100 coins, there are 2^{99+100} ways the game can play out. That's way too many outcomes to list out, so let's try to figure out the solution another way.

Let's think about the game using a smaller number of coins. Let's say you have 2 dimes, and I have 1 penny.

We can enumerate the 8 possible outcomes and list out the times you win (where you toss more heads than I do).

(my coin, your coins): outcome
(H, HH): you win
(H, HT): you lose
(H, TH): you lose
(H, TT): you lose
(T, HH): you win
(T, HT): you win
(T, TH): you win
(T, TT): you lose

You will show more heads in 4 of the 8 possible outcomes, so you have a 50 percent chance of winning.

Hmmm, that's actually interesting. Even though you toss one more coin than I do, the game is fair and there is a 50/50 chance either of us will win!

Why is that?

We can use a bit of logic to solve the game generally. The key is you have one extra coin than me.

As such, it is impossible that you toss both the same number of heads AND the same number of tails as I do. You will either toss more heads or more tails than I do. There are two equally likely possibilities:

1. You show more heads than I do, and equal or fewer tails.
2. You show more tails than I do, and equal or fewer heads.

By symmetry, these two events occur with the same probability. Therefore, the chance of each is 50 percent. The first event corresponds to you winning, and its probability is 50 percent.

Here's another way to solve the problem. If we were both tossing the same number of coins, then our expected number of heads would be the same and so you expect 0 more heads than me. When you toss the extra coin, half the time it's a tails and you lose, and the other half the time it's a heads and you win. So you win in half of the games.

Puzzle 8: Wrong Diagnosis?

In a 1945 study, doctors were asked to determine if boys needed tonsillectomies.

From a group of 389 boys, a panel of doctors judged 45 percent needed a tonsillectomy. Here is the interesting part.

Another panel of doctors then examined *only* the 214 boys already judged as healthy. They determined that 46 percent of them still needed a tonsillectomy! (To be consistent with the first panel, they should have said none needed it.)

Finally a third panel examined the 116 who were judged healthy by the first two panels of doctors. They deemed that 44 percent needed the procedure!

The study illustrated that doctors diagnosed about half of any group as needing a tonsillectomy.

This is the basis for the puzzle.

Suppose there is a condition with a 50 percent prevalence in the population. There is a procedure to cure the condition, but doctors cannot accurately judge who has the condition. Their accuracy rate of diagnosis is 50 percent.

To remedy the poor accuracy rate, it is decided that 3 panels of doctors will judge patients. The first panel will judge the group. Anyone who is diagnosed as sick will get treatment.

Then, a second panel will only judge the group of people who didn't already get treatment. Anyone diagnosed as sick in this group will also have the procedure done.

Finally, the third panel will repeat the process: they will look at people judged healthy from the first two panels, and then give treatment to anyone they deem as sick.

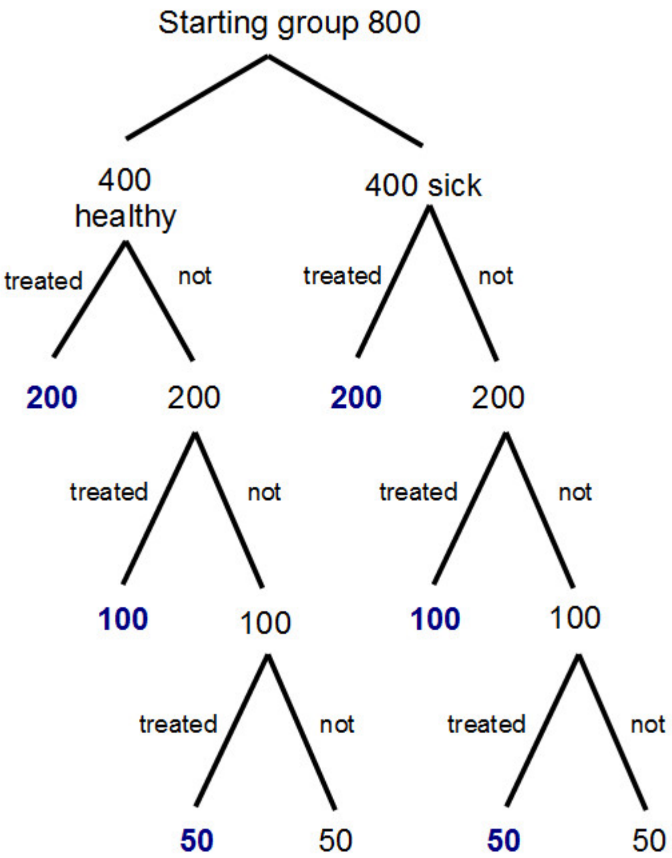
In a group of 800 people, here are the questions:

1. How many people will appropriately get the procedure?
2. How many will need the procedure, but not get it?
3. How many will incorrectly get the procedure?
4. How many will be healthy and not get the procedure?

Assume that exactly 50 percent of healthy people are correctly diagnosed/incorrectly diagnosed, and that exactly 50 percent of unhealthy people are treated correctly/untreated incorrectly.

Answer To Puzzle 8: Wrong Diagnosis?

Here is the tree depicting the possible diagnoses.



In the end, we have 700 people who get treated for the procedure, and we can deduce the answers by adding up the relevant numbers in the tree.

1. How many will appropriately get the procedure? There are 350 people who get treated out of 400 who need it. This is an 87.5 percent success rate.

2. How many will need the procedure, but not get it? There are 50 sick people who never get treated out of 400 total. This is a 12.5 percent rate of undetection.

3. How many will incorrectly get the procedure?. There are 350 people who get the procedure that didn't need it. For a healthy person, there is an 87.5 percent chance of wrongly having the procedure done.

4. How many will be healthy and not get it? There are 50 people who are healthy and do not get the procedure.

Notice how the treatment can be painted as good or bad depending on the statistic quoted.

A panel of doctors might point out they correctly treated 87.5 percent of patients who needed it. A consumer group might warn that 87.5 percent of healthy people will wrongly be treated.

Both are correct, and that is all the more reason people can be confused about medical statistics!

Credit: this example is mentioned in the book *Decision Traps* by J. Edward Russo, Paul J. H. Schoemaker. The 1945 study was "Pseudocia pediatrica" published in the *New England Journal of Medicine* by H. Bakwin.

Puzzle 9: A Dice And Coin Game

Let’s play a game. I will first roll a standard die. I will then toss a pair of fair coins as many times as the die indicates.

If the pair of coins shows two heads in any of the tosses, then I win the game. Otherwise, you win the game.

Should you play this game?

Examples of game play

Game 1: Die shows 1. I toss the pair of coins once and get HT. You win.

Game 2: Die shows 3. I toss the pair of coins 3 times. I get HT, TT, and HH. I win the game.

Answer To Puzzle 9: A Dice And Coin Game

A naive approach is to enumerate the possible tosses. But you will quickly find out there are too many cases to list.

The reason is the number of outcomes increases exponentially with the outcome of the die roll. To see this, consider the following.

When a pair of coins is tossed, there are 4 possible outcomes: HH, HT, TH, and TT. So if the die shows 1, there are 4 possible results. But if the die shows 2, the pair of coins is tossed two times, each with 4 possible results. Thus, there are $4^2 = 16$ possible results. Extending this logic, when the die shows 3, there are $4^3 = 64$ results, and when the die shows 4, 5, or 6, there are 256, 1024, and 4,096 possible results, respectively.

In all, there are 5,460 possible outcomes which is quite a bit to list out.

The trick to solving the problem is to find a pattern. We can readily solve for the case when the die shows 1: there is a 25 percent chance that HH shows, and a 75 percent chance of HH not showing (HT, TH, TT).

What about when the die shows 2? Now we want to calculate the chance that HH shows at least once or does not show at all.

We will use a standard technique that avoids the double counting trap. The probability that HH appears in *at least one* trial is 1 minus the probability that HH shows up in *neither* trial. That is:

Pr(HH when die shows 2) = 1 – Pr(not HH)Pr(not HH)
Pr(HH when die shows 2) = 1 – (3/4)(3/4) = 7/16

We can use a similar trick to solve for the probability that HH shows up in at least one result for the other rolls.

Pr(HH shows ≥1 on a roll n) = $1 - (3/4)^n$

Once we know the chance of winning for each roll, we need to remember that each roll of the die is equally likely. So we can average out these cases to solve for the chance of winning. The following table summarizes the calculations:

Die	Pr(HH) in some flip
1	25.00%
2	43.75%
3	57.81%
4	68.36%
5	76.27%
6	82.20%
Average	**58.9%**

In the end, there is nearly a 59 percent chance that HH will appear. So I win in 59 percent of the cases and you win in only 41 percent.

So even though HH only occurs with a 1 in 4 chance when the pair is tossed once, the possibilities of repeating the tosses more than compensate for the low chance of winning on a single toss.

Puzzle 10: A Dice Game

Warren Buffett once challenged Bill Gates to a game of dice similar to this. Each would pick a die, and the person who rolled the higher number more often would win. The dice all had 6 faces, but their faces had the following numbers.

3, 3, 3, 3, 3, 3
0, 0, 4, 4, 4, 4
1, 1, 1, 5, 5, 5
2, 2, 2, 2, 6, 6

Buffett graciously offers that Gates choose first. After inspecting the dice, Gates said he would prefer Buffett to choose first. Why did neither want to go first?

Answer To Puzzle 10: A Dice Game

Whoever goes first has a 2/3 chance of losing! It's like a game of rock-paper-scissors in which rock beats paper, paper beats scissors, and scissors beats rock. For any given choice, there is another choice that beats it. And since rock < paper < scissors < rock, the strategies are not transitively ordered so the game has a non-transitive property.

Dice with this property are known as non-transitive dice.

Let's label the dice:
A: 3, 3, 3, 3, 3, 3
B: 0, 0, 4, 4, 4, 4
C: 1, 1, 1, 5, 5, 5
D: 2, 2, 2, 2, 6, 6

It turns out dice A loses to B on average, B loses to C, C loses to D, and D loses to A. So whoever goes second can always pick a dice that tends to roll higher numbers.

Let's verify the dice hold this property.

Dice A (3, 3, 3, 3, 3, 3) loses to B (0, 0, 4, 4, 4, 4)

Dice A always rolls a 3. Dice B will produce a 4, which wins for sure, in 4/6 = 2/3 of its rolls, and it will produce a 0, which loses for sure, in 2/6 = 1/3 of its rolls. So dice B wins in 2/3 of its rolls and beats dice A.

Dice B (0, 0, 4, 4, 4, 4) loses to C (1, 1, 1, 5, 5, 5)

In 1/3 of its rolls dice B produces a 0 and it will lose for sure to dice C, which has 1s and 5s. In the other 2/3, dice B produces a 4. Of these times, 3/6 of the time dice C will produce a 5 and win. So dice C wins (1/3) + (2/3)(1/2) = 2/3 of the time.

Dice C (1, 1, 1, 5, 5, 5) loses to D (2, 2, 2, 2, 6, 6)

In 1/2 of its rolls dice C produces a 1 and it will lose for sure to dice D, which has 2s and 6s. In the other 1/2, dice B produces a 5. Of these times, 2/6 of the time dice D will produce a 6 and win. So dice D wins (1/2) + (1/2)(2/6) = 2/3 of the time.

Dice D (2, 2, 2, 2, 6, 6) loses to A (3, 3, 3, 3, 3, 3)

In 4/6 of its rolls dice D produces a 2 which loses for sure to dice A. So dice A wins 2/3 of the time.

We've shown that each dice loses 2/3 of the time to some other dice. It's pretty interesting this is possible. And if you find yourself playing a dice game with unusual numbers, you might want to consider letting the other person choose first and be on the look-out for non-transitive dice.

Source: The dice numbers and the story are described at Microsoft Research's website for Non-Transitive Dice.

Puzzle 11: Which Card Will Be Revealed?

Let’s play a card game.

I have a standard 52 deck of cards that is shuffled well.

I will flip up each card one by one until the *first ace* appears. Then I will place the next card in the deck face down on the table.

The outcome of the game depends on what this card is. I win if the card is revealed to be the ace of spades, and you win if the card is the king of diamonds. It’s a draw otherwise.

What is the probability you will win this game?

Hint: work out a smaller case.

Answer To Puzzle 11: Which Card Will Be Revealed?

At first it can appear the odds are in your favor. You might reason: “The dealer cannot win if the first ace is revealed to be the ace of spades, so that gives me an edge.”

The problem is you have not carefully counted the possibilities, so you are actually falling into a trap of reasoning.

Let us consider the same game that is played with three cards: the ace of spades (AS), ace of hearts (AH), and king of diamonds (KD).

Because the deck is well shuffled, that means all 3! = 6 arrangements of the deck are equally likely. Here is what happens in each of the 6 possible orderings of the deck.:

AS, AH, KD: card revealed = AH
AS, KD, AH: card revealed = KD
AH, KD, AS: card revealed = KD
AH, AS, KD: card revealed = AS
KD, AH, AS: card revealed = AS
KD, AS, AH: card revealed = AH

Do you see something interesting? The ace of spades wins in exactly 2 of the 6 possible orderings of the deck, as does the king of diamonds!

So we have this: each of the cards has an equal chance (1/3) of being revealed after the first ace and the probability of winning is the same to both players.

This is definitely a bit counter-intuitive, but it makes sense when you think about it. From a randomized deck, each card has an equal chance of being at any position in the deck. Thus, both the ace of spades and king of diamonds are equally likely to be revealed after the first ace and be the winning cards.

General solution

We can generalize this logic to a 52 card deck. Note there are 52! ways to order the deck. How many possible ways are there so that the ace of spades is the winning card?

We can count as follows. Imagine removing the ace of spades from the deck. Of the remaining 51 cards, there are 51! ways to arrange these cards. For each arrangement, we can locate the first ace and then place the ace of spades to be the immediate next card. There is one placement for the winning card. So in all, there are 51! ways to have the ace of spades immediately follow the first ace.

The chance the ace of spades is the winning card is 51!/52! = 1/52.

The same logic demonstrates that the king of diamonds also wins with a probability of 1/52.

Hence, the probability either player wins the game is 1/52.

Credit: Prof. W. Kahan’s practice problems 19 Oct. 2007

Puzzle 12: The Three Coin Puzzle

Let's play a game. I have three coins. One is a regular coin with a heads and a tails (HT), one is two-sided with heads (HH), and the other is two sided with tails (TT).

I will draw a coin at random and I will place it on the table so one side is randomly face up.

If the face shows tails, we'll discard the draw and start over because I am superstitious against the tails side of coins.

If the side showing is heads, then the other side could be heads or tails. We will bet on the result. I will give you $10 if the other side is tails, and you will give me $10 if the other side is heads.

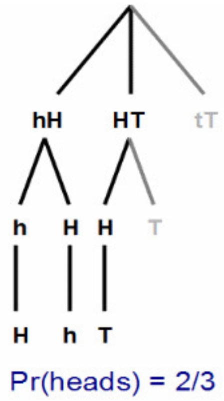
Is this a fair game? If not, what stakes should be set?

Answer To Puzzle 12: The Three Coin Puzzle

At first glance, the game appears to be fair. As a heads is showing, you know you are dealing with the HH coin or the HT coin, and not the TT coin. There is an equal chance I picked the HH or the HT coin, so it appears there is a 50/50 chance the other side is heads or tails.

This is not the case, and the game is in fact stacked in the favor of showing a heads. There is a 2/3 chance that a tails will show versus a 1/3 chance a heads will show.

How can that be? Here is a probability tree to illustrate.



Here's the logical explanation. I begin by picking a coin at random from the coins hH, HT, and tT which is the first part of the tree. I have intentionally written lower-cased letters for one side of the two sided coins to emphasize that each side is distinct.

When the face of heads is shown, that means you can eliminate that drawing the coin tT, and the non-relevant branches of the tree are colored in grey.

So there are 3 equally likely possibilities:

1. The heads is h from the hH coin, and the other side is H.
2. The heads is H from the hH coin, and the other side is h.
3. The heads is H from the HT coin, and the other side is T.

In 2 of the 3 cases, the other side is a heads. The key reason is that when you see a heads, it could be either side of the two sided hH coin.

The game is not fair. As there is a 2:1 chance that a heads shows, you should ask for 2:1 odds if you win, so you get paid $20 if you win and pay $10 if you lose.

Credit: I adapted this problem from a [Car Talk puzzler](#).

Puzzle 13: Cake Cutting

Devise a procedure so that 2 people who do not trust each other can split a cake evenly. What if there are 3 people?

Answer To Puzzle 13: Cake Cutting

The 2 person case is summarized as, “I cut, you choose.” The person who cuts is the second to choose. This gives the cutter an incentive to make the pieces as equal as possible so as not to be left with the smaller piece.

Mathematically, if the cutter divides the cake into fractional pieces (x, $1 - x$), the second person will choose the larger piece, which leaves the cutter with the smaller piece, $\min(x, 1 - x)$. The maximum value for this happens when $x = 1 - x$ so $x = 50$ percent of the cake.

The 3 person case is a bit harder. The “last diminisher” or “inspection” method is one way so that each person feels he is getting at least 1/3 of the cake.

Say the people are A, B, and C. First A cuts the cake into a slice that A deems to be 1/3 of the cake. Then B can trim this slice smaller or pass it along (the trimming is put back onto the cake). Finally C can trim this slice or pass (again the trimming would go back to the cake). The last person who cut the slice is the person that gets the slice.

The procedure guarantees the person who gets the slice feels it is at least 1/3 of the cake. Why? The first move has A cutting the cake. If A cuts it smaller than 1/3, then B and C will pass and A gets the slice. So A has every incentive to cut a slice that is at least 1/3.

When B examines the slice, B will leave it alone if it is 1/3 or less. But if it is more than 1/3, then B will trim a very small amount to have a chance at getting a piece that is more than 1/3. When C gets it, C will leave it alone if it is 1/3 or less or trim a negligible portion if it is 1/3 or more.

In summary, the last person who trims the slice felt it was at least 1/3 of the cake, and that person gets the slice.

Now there are two people left to divide the remaining cake. They can do this by the “I cut, you choose” method for 2 people.

Puzzle 14: Spreadsheet Random Numbers

The spreadsheet function rand() generates a random number between 0 and 1.

Let’s say I generate two numbers using the rand() function.

What is the probability the two numbers have a sum less than 1 AND a product less than 3/16?

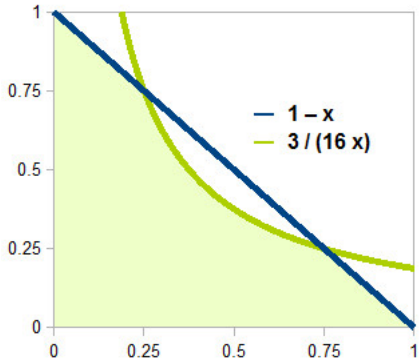
Answer To Puzzle 14: Spreadsheet Random Numbers

This puzzle is an example of geometric probability. The answer is most readily found by creating a graph.

Consider the two random numbers as an ordered pair (x, y). Since each value ranges from 0 to 1, the sample space is the square [0,1] x [0,1]. The sample space has an area of 1.

The chance the sum is less than 1 corresponds to the region $x + y < 1$ and the chance the product is less than 3/16 corresponds to the region $xy < 3/16$. We can plot those curves and shade the area that corresponds to those conditions. The area will then be the probability of the event.

The condition $x + y < 1$ is a line and the condition $xy < 3/16$ is a hyperbola. The event we want is the area below the curved line in the following graph:



The points of intersection between the curves are 1/4 and 3/4. When x is less than 1/4 or greater than 3/4, we want the area under the curve 1 - x. For the region between 1/4 and 3/4, we want the area under the curve 3/(16x). The area of the event can be found by the integrals:

$$\int_0^{0.25} (1-x)dx + \int_{.25}^{.75} \frac{3}{16x}dx + \int_{0.75}^{1} (1-x)dx = 0.25 + \frac{3}{16}\ln 3$$

The probability is 1/4 + (3/16)ln(3), or about 0.4559.

A numerical estimate

You can also estimate the answer. In the spreadsheet program, use the rand() function for two cells like a1 and b1, and make a cell c2 as the formula:

=if(and(a1+b1 < 1, a1*b1 < 3/16), 1, 0)

Copy the formulas down for at least a few thousand cells (there’s a lot of variance), and take the average of the values in column C. You’ll end up with about 0.4559.

Puzzle 15: Unloading Deliveries

You are awaiting two deliveries A and B.

Delivery A takes 30 minutes to unload and delivery B takes 1 hour to unload.

You only have enough staff to unload one delivery at a time. If one truck arrives while the other is being unloaded, it will have to wait until the first job is done.

If the two trucks arrive randomly during an 8-hour period, what is the chance that neither truck will have to wait?

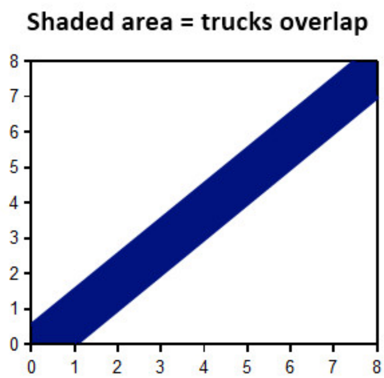
Answer To Puzzle 15: Unloading Deliveries

We will solve the problem geometrically. Consider (a, b) as coordinates for the delivery times of truck A and truck B. The values for a and b range from 0 hours to 8 hours.

The sample space is a square with side 8, so its area is 8x8 = 64. We will now find the proportion of the area for the event that the trucks do not meet to find the probability.

We will consider the complement event that some truck has to wait. There are two ways that a truck might have to wait. Either truck A arrives within an hour after truck B (corresponding to $b \leq a \leq b + 1$), or truck B arrives within 0.5 hours after truck A (corresponding to $a \leq b \leq a + 0.5$).

We can graph the lines for the boundary conditions and shade in the area corresponding to the inequalities as follows:



The unshaded area is two isosceles right triangles with legs 7 and 7.5. The total area of the unshaded part is the sum of the two triangle areas, which is $0.5(7^2 + 7.5^2) = 52.625$.

The probability that neither truck will need to wait is this area divided by the total area of 64. This is 52.625/64, which is about 82 percent.

Puzzle 16: Statistical Independence

Two events are statistically independent if the outcome of one does not affect the probability of the other. For instance, if I flip two coins, the fact that one coin shows heads in no way affects the outcome of the other coin.

Recall events A and B are independent if Pr(A | B) = Pr(A). That is, the knowledge of event B tells us nothing about the chances of event A.

Got it? Now let's take on the following problem.

A family has two or three children. Consider the events:

(A) The family has children of both sexes.
(B) There is at most one girl.

Are the two events independent or not? That is, does knowing the family has children of both sexes give you knowledge about whether there the family has at least one girl? Consider for family sizes of 2 and 3 children.

Answer To Puzzle 16: Statistical Independence

This is one of those really strange probability questions. The two events are independent if the family has 3 children, but they are not independent if the family has only 2 children!

When a family has 2 children, there are four possibilities for boys (b) and girls (g).

bb, bg, gb, gg

Let's consider the events in question:

(A) The family has children of both sexes.
(B) There is at most one girl.

Here are the ways each event can occur.

Event A: bg, gb (2/4)
Event B: bb, bg, gb (3/4)

If event B happens, out of the 3 ways, there are 2 ways in which event A can also happen (bg, gb). Thus, $Pr(A|B) = 2/3 \neq Pr(A) = 2/4$. Thus, the two events are not independent.

In the case of 3 children, there are 8 ways to have boys and girls.

bbb, bbg, bgb, gbb, bgg, gbg, ggb, ggg

Let's list out the ways the family can have children of both sexes (event A) and compare that to the ways they can have at most one girl (event B).

Event A: bbg, bgb, gbb, bgg, gbg, ggb (6/8)
Event B: bbg, bgb, gbb, bbb (4/8)

If event B happens, out of the 4 ways, there are 3 ways in which event A can also happen (bbg, bgb, gbb). Thus, $Pr(A|B) = 3/4 = Pr(A) = 6/8$.

Similarly, if event A happens, out of the 6 ways, there are 3 ways that event B can happen (bbg, bgb, gbb). Thus $Pr(B|A) = 3/6 = 1/2 = Pr(B) = 4/8$.

Hence, events A and B are independent when the family has 3 children.

So statistical independence can be a tricky thing as it depends on the sample space that you are dealing with. When a family goes from 2 to 3 children, suddenly the two events A and B become independent.

Credit: I came across this problem in William Feller's classic text *An Introduction to Probability Theory and Its Applications, Volume 1*.

Puzzle 17: Lost Child

A child gets lost during a school trip to the museum. The child is most likely in the west wing of the museum (a 70 percent chance), but he may have wandered to the east wing (30 percent chance).

There are 6 chaperons who are trying to find the child. Suppose each independently has a 20 percent chance of finding the child when searching in the correct wing (and a 0 percent chance if searching in the incorrect wing).

How should the team split up between west and east wings to maximize the odds of finding the child?

(If their search is unsuccessful, they will enlist the museum to make sure they have a 100 percent chance of finding the child. But they prefer to search privately to remedy the situation).

Answer To Puzzle 17: Lost Child

Like many probability problems, this one is easier to frame in terms of the complementary probability of not finding the child.

If each has an independent 20 percent chance of finding the child, then each has an 80 percent chance of not finding the child. So if N people search for a child, the chance of failure will be $(0.80)^N$.

Suppose the team is divided into x people in the west wing, and $6 - x$ in the east wing. The odds of not finding the child can be split up into the cases of finding the child in either the west wing or the east wing. So the probability is described by the function $f(x)$ as follows:

$f(x) = \text{Pr(west)}(0.80)^x + \text{Pr(east)}(0.80)^{6-x}$

$f(x) = (0.70)(0.80)^x + (0.30)(0.80)^{6-x}$

We wish to minimize the chance they do not find the child, so we minimize this function. We can take the derivative and set it equal to zero, or we can directly compute the cases for the seven possible search divisions:

$f(0) = 0.779\ldots$
$f(1) = 0.658\ldots$
$f(2) = 0.571\ldots$
$f(3) = 0.512\ldots$
$f(4) = 0.479\ldots$
$\boldsymbol{f(5) = 0.469\ldots}$
$f(6) = 0.484\ldots$

The minimum occurs when $x = 5$ which corresponds to a roughly 47 percent chance they do not find the child. The complement is a 53 percent chance they find the child.

So if the team sends 5 chaperons to the west wing and 1 to the east wing, they will have an expected 53 percent chance of finding the child.

Note that if the child was just in one location, the team would have roughly a 74 percent = $(1 - 0.8^6)$ of finding the child. The complication of an additional location makes the search quite a bit harder.

Puzzle 18: Snowball Puzzle

Alice and Bob take turns throwing a snowball at a target. Alice hits with accuracy 1/3 and Bob with 1/4.

If Alice goes first, how many throws will it take, on average, until someone hits the target?

What if Bob goes first?

Answer To Puzzle 18: Snowball Puzzle

There is a neat trick to solving this puzzle.

Let us write n for the number of tosses until the target is hit.

There is a 1/3 chance Alice hits on the first toss. If Alice misses with probability 2/3, then there is a (2/3)(1/4) chance that Bob will hit the target on the second toss.

If neither hits on the first toss (chance of that is (2/3)(3/4)), then we are basically in the same position again, except the total number of tosses must be increased by 2.

Hence we have the equation:

average tosses = Pr(first toss hit)(1) + Pr(second toss hit)(2) + Pr(no hit in first or second)(average + 2)

$n = (1/3)(1) + (2/3)(1/4)(2) + (2/3)(3/4)(n + 2)$

We can solve this equation and find $n = 10/3 = 3.33333\ldots$

So it will take on average 3 and 1/3 tosses to hit the target.

Note this fits in line with our intuition. The average number of tosses a person will take is 1 divided by their accuracy. Thus, Alice alone would take 3 tosses, and Bob alone would take 4. By alternating turns, they end up somewhere in the middle, but closer to Alice because she goes first.

If instead Bob went first, the expectation would be:

average tosses = Pr(first toss)(1) + Pr(second toss)(2) + Pr(no hit in first or second)(average + 2)

$n = (1/4)(1) + (3/4)(1/3)(2) + (3/4)(2/3)(n + 2)$

Which yields $n = 3.5$.

Since Bob goes first, and he is less accurate, the expected number of tosses is a tad bit higher.

Puzzle 19: Election Rule Change

A bill is proposed so that any change would require a 3/5 majority to pass instead of a simple 1/2 majority. If the bill passes, how much harder will it be to make law changes?

Assume the group has 100 members modeled as random voters: each has a 50 percent chance of voting "yes" or "no."

(Use the normal approximation to estimate the answer. In other words, solve for the probability p of a law change under a simple majority and then the probability q under a 3/5 majority. The new rule means a law change would pass only q/p times as frequently as under a simple majority.)

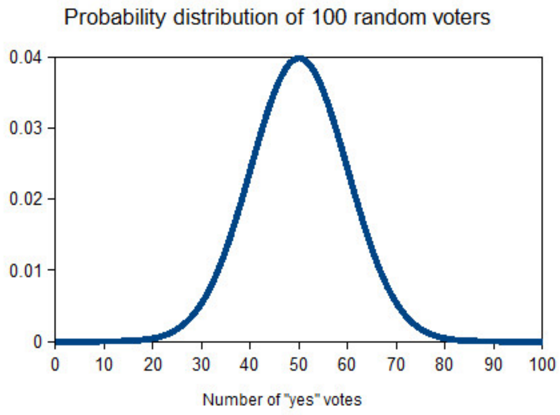
Answer To Puzzle 19: Election Rule Change

Since each voter has a 50 percent chance of voting either way, each voter can be treated as coin toss, with a “head” meaning “yes” for a rule change.

The question is now: how many times does a coin toss return at least 51 heads in 100 tosses? Intuitively this happens about 50 percent of the time.

We can use some statistics to verify this. Since each voter is treated as an independent draw, we can approximate the distribution of votes using the Central Limit Theorem. This states that the distribution of n voters saying “yes” will converge to the normal distribution with a mean of 50 and a standard deviation of 5 (this is calculated as $\sqrt{(100)(0.5)(0.5)}$ because a binomial variable with success p in n trials has a standard deviation equal to $\sqrt{np(1 - p)}$).

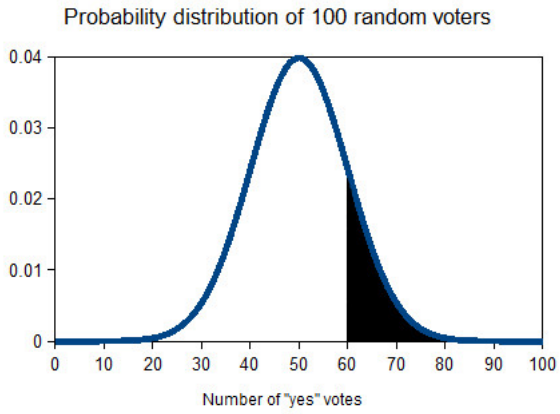
Here is what the probability distribution of votes looks like:



If the rule requires a simple majority, then it needs at least 50 votes to pass. We can calculate this as 1 minus the probability of getting 50 or fewer votes. This probability is 50 percent, because the normal distribution is symmetric around its mean of 50.

To solve the 3/5 majority rule case, we will solve the following question: how often will we get more than 60 votes, if each voter has a 50/50 chance of saying yes, and there are 100 voters?

We wish to find the chance of getting at least 60 votes, which is the shaded area in this graph:



We can calculate this area using a spreadsheet program (function NORMDIST) or a table of normal values.

The precise syntax is: “=1 – NORMDIST(50;60;5;1)” ≈ 2.3 percent.

This is astonishingly low! To emphasize and summarize:

Probability of a law change under
Simple 1/2 majority: about 50 percent
A 3/5 majority rule: about 2.3 percent

How much harder is it to pass laws under the rule 3/5 majority? Only 4.6 percent of bills passing under simple majority would also pass under a 3/5 majority!

The point is this: the rule change seems to understate the magnitude of the new requirement. On the face, the new rule requires 10 percent more voters, but that translates into meaning just 4.6 percent of the law changes that would pass under a simple majority would also pass under the new 3/5 majority requirement.

The lesson is one needs to be very careful about percentages. In this example, asking for 10 percent more votes makes it very hard to change the law.

Puzzle 20: Rolls Before Repeat

What’s the expected number of rolls until a standard die duplicates a number?

Answer To Puzzle 20: Rolls Before Repeat

The 1st roll is always a unique number, and the 7th roll is always a repeated number. So the answer should be somewhere in the middle, like 3.5. The exact answer is 1223/324, which is about 3.77.

Let E(x) denote the expected number of rolls to see a duplicate if x numbers are already seen. If we have seen all 6 numbers already, then the next roll is guaranteed to be a repeat. So E(6) = 1.

What is E(5)? We can figure this out in terms of E(6). There is a 5/6 chance the roll gives a duplicate–in which case we are done in 1 roll. And there is a 1/6 chance we will see a new number–in which case the average time is E(6) plus the 1 roll we just used. So we have the following equation for E(5):

$E(5) = (5/6)(1) + (1/6)(1 + E(6))$
$E(5) = 1 + (1/6)E(6)$
$E(5) = 1 + (1/6)1 = 7/6$

What is E(4)? We can calculate this similarly in terms of E(5). There is a 4/6 chance we will see a duplicate–in which case we are done in that 1 roll. And there is a 2/6 chance we will see a new number–in which case we get to the case of having seen 5 numbers, plus we add the roll that got us there.

$E(4) = (4/6)(1) + (2/6)(1 + E(5))$
$E(4) = 1 + (2/6)E(5)$
$E(4) = 1 + (1/6)(7/6) = 25/18$

There is a pattern to the calculations. If we have seen x numbers, then there is an x/6 chance we see a repeat–and we are done in that 1 roll. Or there is a (1-x)/6 chance we see a new number–in which case we get to the case of having seen $x + 1$ numbers, plus we add the roll that got us there. So we have the formula:

$E(x) = (x/6)(1) + ((1 - x)/6)(1 + E(x + 1))$
$E(x) = 1 + ((1 - x)/6)E(x + 1)$

We can use this formula to calculate having seen 3, 2, 1, and then 0 numbers. We find E(3) = 61/36, E(2) = 115/54, E(1) = 899/324, and finally E(0) = 1223/324.

So when we start out–and no numbers have yet been seen–the expected number of rolls until we see a duplicate is E(0) = 1223/324, which is about 3.77.

I give credit to Math StackExchange for suggesting the recursive approach.

Puzzle 21: Hat Guessing

Three prisoners are given one chance for freedom. The prison warden lines them up in a circle and randomly selects a black or white hat to place on each person's head. Each prisoner can see the other hats but not his own hat color, and no one is allowed to communicate. Each prisoner writes down a guess of his own hat color. If all 3 of them are correct, then they go free.

The 3 prisoners have 5 minutes to prepare a strategy before the game begins. What strategy can they employ to improve their odds of freedom?

What if the game involved n prisoners with the same rules? Can you think of a way for them to escape better than random chance?

Answer To Puzzle 21: Hat Guessing

The 3 prisoners can guarantee a surprising 50 percent win rate. What's more surprising is this strategy generalizes to *n* prisoners as well! How can so many prisoners guess accurately?

Here's the insight. The number of black hats (or white hats) will either be an even number or an odd number. Since the hat colors are chosen at random, it is equally likely the total number of black hats will be even or odd. So the prisoner's beforehand all agree to guess the total number will be either odd or even, and each person bases his guess on the hats he sees to keep the total number odd or even. If the total number of hats actually matches their planned strategy, then everyone will guess correctly.

Let's see how this works with 3 people. There are 8 possible ways the hats will be placed (B = black, W = white, and BBW means prisoners 1 and 2 got black and 3 got white).

1. BBB
2. BBW
3. BWB
4. BWW
5. WWW
6. WWB
7. WBW
8. WBB

Suppose they agree beforehand to bet the number of black hats will be odd. This actually happens in cases 1, 4, 6, and 7, and they will all guess correctly in these cases.

For example, in BBB, each prisoner sees 2 black hats. So each person guesses his own hat is black in order for the total number of black hats to be 3 (odd). So each person guesses correctly and they win.

Similarly, consider BWW. Prisoner 1 sees 2 white hats, and will guess his own hat is black so the total number of black hats is odd (1). Prisoner 2 sees 1 black hat and so will guess his hat is white (a guess of black would mean there would be 2 total black hats, which is wrong, since they are betting the total number is odd). Prisoner 3 also sees 1 black hat and will guess his hat is white. So in BWW, all 3 guess correctly.

You can verify all 3 prisoners guess correctly in WBW and WWB as well.

So the 3 prisoners all guess correctly in 4 out or 8 cases. They win in 4/8 = 50 percent of the combinations of hats.

The strategy generalizes to *n* prisoners. Beforehand they all agree to bet the total number of black hats is odd (or even), and they all coordinate their own guess based on the hats they see. When their guess matches the realized hat colors, which happens 50 percent of the time, they will all guess correctly and escape.

Puzzle 22: Playing With A Loaded Die

A casino offers a game called “Roll me twice.”

A single die is rolled 2 times. You win if the same number shows in the 2 consecutive rolls.

It costs $1 to play, and you get $6 if you win.

The hitch is the die might be rigged: the casino does not guarantee the die is fair, and in fact, loading the die is part of the game.

Should you be willing to play this game? Why or why not?

Answer To Puzzle 22: Playing With A Loaded Die

First, consider the game if the die is fair and there is a 1/6 chance each number appears.

In that case, there are 36 possible rolls, and you win in the 6 outcomes of (first roll, second roll) = (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6). You win in 6/36 = 1/6 games, and the game pays out \$6 for a \$1 cost to play.

The game is fair and the gamble has an expected value of 0 to you.

Pr(win)(profit) + Pr(lose)(loss) = (1/6)5 + (5/6)(-1) = 0

What happens if the die is rigged? Is there a way they can make the game unfair to you?

The neat answer is that rigging the die will only increase your odds of winning! Take an extreme case that the die is rigged to show the number 1 every time. In that game, you will always roll (1, 1) and you will always win the game.

In fact, the lowest probability of winning happens when the die is fair. Let's prove this.

Let's model the die as a vector:

$v = (p_1, p_2, p_3, p_4, p_5, p_6)$

We will consider p_i as the probability of rolling outcome i.

The probability of winning the game is the dot product of v with itself. It is a result from linear algebra that this quantity is minimized when every variable is set equal to each other, so the probability of rolling any number is 1/6.

Alternately, we can prove this directly. Since the probabilities sum to 1, we can re-write the last probability in terms of the previous ones:

$p_6 = 1 - p_1 - p_2 - p_3 - p_4 - p_5$

Now the probability of winning is the following expression:

$(p_1)^2 + (p_2)^2 + (p_3)^2 + (p_4)^2 + (p_5)^2 + (1 - p_1 - p_2 - p_3 - p_4 - p_5)^2$

We want to minimize this over the five variables. If you take the partial derivative with respect to each variable and set them equal to zero, there is a system of equations whose solution is that each variable should be equal to the others. Thus, the smallest winning probability is 1/6.

It's amazing that even though we don't know how the die is configured, we can still conclude whether the game is favorable.

Puzzle 23: Auctioning Off A Gift Card

Alice and Bob get third place in pub trivia one night and win a $6 gift card.

They decide to play a game of skill to determine who gets the card.

Each person secretly bids a whole dollar amount (including $0) for the gift card. The person who bids more gets the card and pays that amount to the other person. If they both bid the same amount, then they will use a coin flip to determine the winner and neither has to pay the other.

Each agrees to bid cash they have on hand. Alice has $3 on her, and Bob has $5. How should each person bid? Is Bob favored because he has more cash?

(Assume each values the gift card as $6, same as cash)

Examples of play

Alice bids $2 and Bob bids $1. Alice gets the gift card and pays Bob $2. So Alice nets a payoff of $4 (=6 – 2) and Bob nets $2.

Alice bids $1 and Bob bids $1. They flip a coin for the card. The expected payoff to each is half the card value, $3.

Answer To Puzzle 23: Auctioning Off A Gift Card

Alice has four strategies: 0, 1, 2, and 3, and Bob has 6 strategies: 0, 1, 2, 3, 4, 5.

There are 24 entries we have to fill out in this matrix. We will view the game from Alice's perspective (the values are expected net dollars). Note that Bob's payoffs are $6 minus the entries in this matrix.

If Alice and Bob bid the same, then Alice expects to net $3, which is half of the gift card value. If Alice bids less than Bob, then she loses and gets payment from Bob of his bid. If Alice bids more than Bob, she pays her bid and gets the gift card–so her payout is 6 minus her bid.

Using that logic, here is the payoff matrix for the game:

Payoff to Alice from auction

		Bob					
		0	1	2	3	4	5
Alice	0	3	1	2	3	4	5
	1	5	3	2	3	4	5
	2	4	4	3	3	4	5
	3	3	3	3	3	4	5

Notes
1. Payoffs are expected values
2. Coin flip is used for tie-breaking rule
3. Bob's payoffs are 6 minus entries listed

Solving the game

Alice and Bob each want to get the most of the game. How can we figure out the best bids for each person?

Alice will imagine that Bob bids a particular value, say 0. Then she'll look at her highest payout (to maximize her gain) in that column, which is to bid 1. But if Alice bids 1, what will Bob bid? He will consider the row with 1, and he'll look for the smallest value (he is minimizing his loss). This is when Bob bids 2.

We then repeat the process: what will Alice bid if Bob is picking 2, and so on.

Formally, we want to look for best-responses to each strategy for each player. The algorithm is this: for a given column, we highlight the row strategy that gives the LARGEST payoffs. Then, for a given row, we highlight the column strategy that gives the SMALLEST payoff (this is Bob's minimization).

Another way is we can see how Bob would react to Alice's moves and then keep going until we find equilibrium strategies. The game is in its solution when each player is happy responding to the other person's strategy.

Alice plays		Bob plays		Alice plays
0	→	1	→	2
1	→	2	→	2, 3
2	→	**2, 3**	→	**2, 3**
3	→	**2, 3**	→	**2, 3**

The game turns out to be pretty easy to solve. Alice and Bob can each play either the numbers 2 or 3. This yields four equilibrium points: (2, 2), (2, 3), (3, 2), and (3, 3).

Each of these outcomes has an expected payoff of $3.

It's actually slightly better to play 2 because if the other person plays incorrectly then you win even more (if Alice plays 2 and Bob plays 1, then she gets $4. If Alice played 3, on the other hand, she would only get $3).

The surprise: the game is fair!

Even though Alice has less money, the auction ends up being fair. The reason is that has enough money to bid the optimal strategies of 2 and 3.

Of course, at the end of the day, Alice and Bob notice something. Each has an expected payoff of \$3: which is exactly the same as if they had just flipped a coin to award the gift card.

It just goes to show that complicated auctions are not always better: you might end up with the same outcome after going through a lot more work.

Puzzle 24: Higher Or Lower

I have randomly selected a whole number from 1 to 100. After you guess, I will tell you if you are correct or if my number is higher or lower than your guess. But each time you guess it costs you $1. How can you minimize your cost?

Now consider a game theory variation. If I know you're using that strategy, how might I pick numbers that are difficult for you to guess? Note: this game is very difficult to solve. So the puzzle is to consider how you might play rather than explicitly solving for the equilibrium.

For a challenge, try to solve a simplified version where I choose only from the numbers 1, 2, and 3. Solve for my optimal choosing strategy and your optimal guessing strategy and calculate the expected cost of the game (the expected number of guesses).

Answer To Puzzle 24: Higher Or Lower

Let's solve the regular guessing game where the number is picked randomly. What's the best strategy?

First guess 50, and then guess the average of the remaining possible numbers based on whether I say "higher" or "lower." For example, if I say "higher," then guess 75 which is the average of the possible numbers 51 to 100. If I say "lower," then guess 25. Repeat this strategy until you find the correct number.

In each stage you eliminate about half of the numbers, which is the best you can do since you don't know the secret number. This method is called a binary search and you can find the secret number in at most 7 guesses. If you are guessing from 1 to N, then at each step you remove half the numbers, so the strategy has a worst case of $\log_2 N$ steps (rounded up).

Many people intuitively come to the binary search method. It's the kind of strategy we used in the "old days" to look up words in a dictionary or tune a transistor to the correct radio station, swaying back and forth until we got the frequency just right.

But this only works if the secret number is randomly chosen. What if the chooser is playing strategically too? Suddenly the hide and seek game becomes much harder!

Game theory intuition

Suppose the secret number is not picked randomly. Knowing the guesser is using a binary search, the chooser will select 1, 49, 51, or 99, all of which take a full 7 guesses in a binary search. (To get 49, for example, the guesser will do 50, 25, 37, 43, 46, 48, and then 49.)

So if the chooser is consciously picking hard numbers, the guesser will have to randomly deviate from the binary strategy or else end up in the worst case all the time.

This version of the game is known as Dresher's Guessing Game, and it's extraordinarily difficult to solve because the number of strategies grows exponentially with N. In general, the guesser has to increase the probability of choosing the difficult numbers like 1, 49, 51, or 99.

The chooser can make the game very difficult on the guesser. For $N = 100$, the average number of guesses, when both are playing optimally, is 296/51, which is about 5.8. In other words, the chooser forces the guesser to make nearly 6 guesses when the worst case is 7 guesses in a binary search. In this game the chooser has the advantage of forcing nearly the worst case even when the guesser is using an optimal strategy.

For more about solving this game in general, see "An Asymptotic Solution of Dresher's Guessing Game" by Robbert Fokkink and Misha Stassen. The American Mathematical Monthly. 04/2012; DOI: 10.4169/amer.math.monthly.119.04.337. http://www.researchgate.net/publication/234168299_Tossing_coins_to_guess_a_secret_number

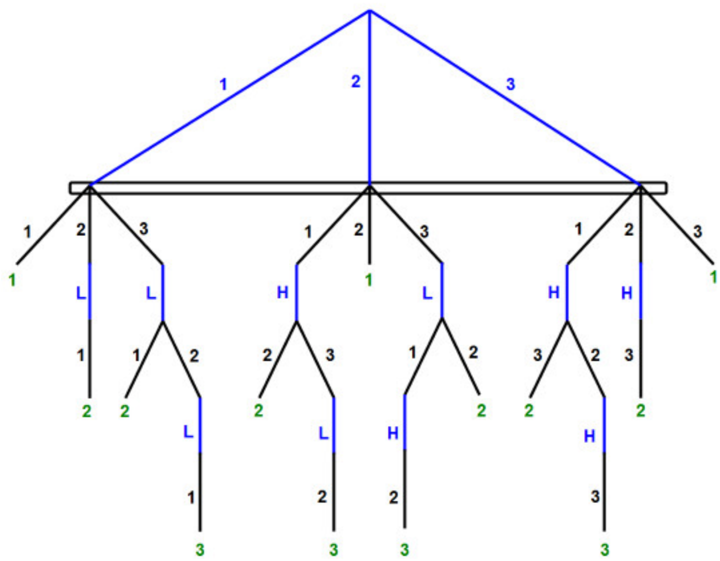
Solving the 1-2-3 guessing game

Melvin Dresher presented a simplified version of the game for the numbers 1, 2, and 3. This problem is presented in *Games of Strategy: Theory and Applications* published in 1961. (At the time of this writing, the book was available online for free at http://www.rand.org/pubs/commercial_books/CB149-1.html)

The chooser has three strategies: pick the number 1, pick the number 2, or pick the number 3.

The guesser's strategy is more complicated. In the first round, the guesser can pick 1, 2, or 3. If correct, then the game ends. If not, then the guesser will hear "lower" or "higher" and guess appropriately. So we need to specify what to do after hearing "lower" or "higher."

Here's the game in extensive form drawn as a tree. Player 1 (blue) secretly picks a number 1, 2, or 3. Player 2 (black) guesses a number, after which player 1 indicates the number is correct or says Lower (L) or Higher (H) accordingly. Player 2 then guesses again until selecting the correct number. The payoff to the game is the number of guesses, written in green.



As you can see, even this simple game has a complicated game tree!

To solve the game, we'll convert the game into normal form. We can then write the game as a matrix and solve for the Nash equilibrium.

The chooser has 3 choices: to pick 1, 2, or 3.

The guesser (player 2) has a strategy that includes what to do on the first guess, and what to do after hearing "lower" or "higher." So we can write a strategy as a 3-tuple of numbers (first guess, guess if hear "lower", guess if hear "higher").

There is one more consideration. If the first guess is 1 and wrong, then the next guess will necessarily have to be higher. The component guess if hear "lower" is not applicable in this case, so we put "-" to denote this case.

The guesser has 5 strategies in total.

(1, -, 2)
(1, -, 3)
(2, 1, 3)
(3, 1, -)
(3, 2, -)

The payoff to the game depends on the combination of the guesser's strategy and the chooser's strategy.

For example, consider the guesser's strategy (2, 1, 3). This says the guesser picks 2 at first, and then guesses 1 or 3 depending on hearing "lower" or "higher." So the game ends in 1 guess if the chooser selected 2, it ends in 2 guesses if the chooser selected 1 or 3.

Or consider the guesser's strategy (1, -, 2). This says the guesser picks 1 at first, and then guesses 2 for the next guess, and will guess 3 if needed. The game ends in 1 guess if the chooser selected 1, it ends in 2 guesses if the chooser selected 2, and it ends in 3 guesses if the chooser had selected 3.

With this notation, we can write the game in normal form by expressing the number of guesses for each guesser/chooser strategy combination.

		Guesser				
		(1, -, 2)	(1, -, 3)	(2, 1, 3)	(3, 1, -)	(3, 2, -)
	1	1	1	2	2	3
Chooser	2	2	3	1	3	2
	3	3	2	2	1	1

This game does not have a solution in pure strategies. When Dresher wrote the book in 1961, it was fairly difficult to solve this game for mixed strategies. The book uses this game as an example of how to use matrix manipulation to arrive at a solution. It takes several pages of text to derive the solution.

Arbitrary games could be solved using a method of linear programming, which was suitable for computers. However, at the time, personal computers were not common. Today grade school kids probably have more power in their cell phones than those academics had in their computer labs. So let's take a moment to appreciate that mathematicians spent considerable effort coming up with algorithms to solve these games.

So let's solve the game the easy way. I can simply input this payoff matrix into an online zero sum game solver (http://www.zweigmedia.com/RealWorld/gametheory/games.html) to find the mixed strategies.

Here is the solution to the game.

The guesser should never play (1, -, 2) or (3, 2, -), and the guesser should play (1, -, 3) for 20 percent of the time, (2, 1, 3) for 60 percent of the time, and (3, 1, -) for 20 percent of the time.

The strategy of (2, 1, 3) is the binary search strategy. It is still played 60 percent of the time—more than any other strategy—but it is not played all the time.

The chooser should optimally play 1 for 40 percent of the time, 2 for 20 percent of the time, and 3 for 40 percent of the time. This reveals the strategic nature of the game: the chooser is intentionally picking the extreme numbers 1 and 3 as a counter-measure to the efficiency of the binary search.

In response, the guesser should occasionally pick 1 and 3 on the first guess, followed by the other extreme number on the second guess—since the extreme numbers 1 and 3 are picked more often than 2. The guesser should never pick 1 or 3 and then follow it by a guess of 2.

The game ends up being quite favorable to the chooser. The expected number of guesses is 1.8. This is pretty crazy if you think about it!

The reason is a binary search in the 1-2-3 game has a worst case of 2 guesses. By playing strategically, the chooser forces the game very close towards the worst case, even when the guesser knows the chooser is hiding strategically.

So this verifies the guessing game is generally in favor of the chooser who can push the guesser close to the worst case of a binary search.

How would you solve this game more generally for numbers 1 to 100, for example? The problem becomes extremely difficult to solve as the strategies grow exponentially.

As mentioned above, someone has solved this using computer assistance. For more on that, see "An Asymptotic Solution of Dresher's Guessing Game" by Robbert Fokkink and Misha Stassen. The American Mathematical Monthly. 04/2012; DOI: 10.4169/amer.math.monthly.119.04.337. http://www.researchgate.net/publication/234168299_Tossing_coins_to_guess_a_secret_number

The general result is the chooser can force the guesser close to the worst case of a binary search.

So at first the game seems favorable to the guesser who can efficiently find the number. But strategically the game tilts towards the chooser who has the edge in this game of hide and seek.

Puzzle 25: What Do A Classic Physics Problem, An Infinite Tower, And Coin Flipping Have In Common?

This puzzle is 3 problems. You might see a hidden connection between them as you solve each problem.

Puzzle 1: a classic physics puzzler

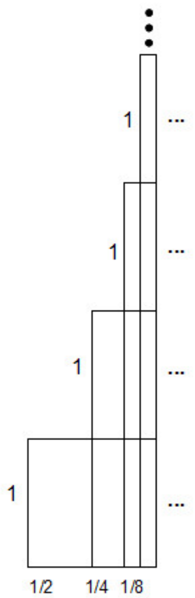
A particle moves at speed 1 for half a second, then at a speed of 2 for a quarter of a second, then at a speed of 3 for an eighth of a second, and so on.

How much distance does the particle travel?

Puzzle 2: infinite tower

In flatland, an ambitious architect designs an infinite tower. The structure begins with a 1/2 x 1 block, then attached to it are two blocks of size 1/4 x 1–stacked on top of each other, then there are three blocks of 1/8 x 1–again stacked one on top of each other, and so on.

The tower looks like this:



As each level adds to the height of the tower, the tower will have an infinite height.

But a mathematician notes the tower will actually have a finite area. What is the area of the tower?

Puzzle 3: coin flipping until heads

Let’s play a game. I will flip a fair coin until I get a result of heads. I will pay you a dollar for each flip I make. What’s your expected value to the game?

Answer To Puzzle 25: What Do A Classic Physics Problem, An Infinite Tower, And Coin Flipping Have In Common?

The savvy reader will note these are all really the same problem. Each puzzle boils down to the infinite sum:

$$S = \frac{1}{2} + \frac{1}{4}(2) + \frac{1}{8}(3) + ... + \frac{1}{2^n}(n) + ...$$

And the answer is 2 (to be derived below).

But here's the interesting part: each problem can be approached using a different solution method. And that's the remarkable thing about mathematical thinking: we can connect seemingly disparate areas of our life using the same method of logical thinking. Or, if you're having trouble solving a problem in one frame, you can consider using the tools of another area to solve it.

As someone who enjoys game theory, I prefer the solution method outlined in the third puzzler. Here are the solution methods for each part.

Physics problem: an analytic approach

Each second n, the particle moves for $(1/2)^n$ seconds. Hence we want to evaluate the infinite series:

$$S = \frac{1}{2} + \frac{1}{4}(2) + \frac{1}{8}(3) + ... + \frac{1}{2^n}(n) + ...$$

We can use a little trick to solve this problem. We will subtract the sum from half of itself:

$$S = \frac{1}{2} + \quad \frac{2}{4} + \frac{3}{8} + ... +$$
$$\frac{1}{2}S = \qquad \frac{1}{4} + \frac{2}{8} + ...$$

$$\implies \frac{1}{2}S = \frac{1}{2} + \quad \frac{1}{4} + ... = 1$$

$$\implies S = 2$$

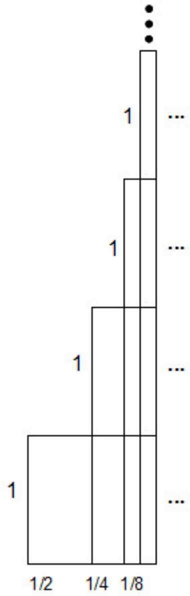
Thus, we arrive at the answer that the sum is 2.

Infinite tower: a geometric approach

In the above figure, we notice there is one block of area 1/2, then two blocks of area 1/4, and so on. We get the same infinite series as before:

$$S = \frac{1}{2} + \frac{1}{4}(2) + \frac{1}{8}(3) + ... + \frac{1}{2^n}(n) + ...$$

The amazing thing is we can solve the problem geometrically by re-arranging the blocks.



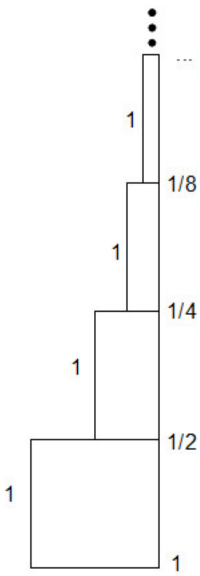
We first need to find the length of the base of the tower. Note that the first block adds the length of 1/2, the next 1/4, the next of 1/8, and so on. We end up with the infinite series:

$$L = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 1$$

This is a familiar series that sums to 1.

The base of the tower is 1. What is the base of the second layer of the tower, excluding the very bottom block? This will be 1/2 since it is the length of the bottom base minus the 1/2 block. Similarly, the base of the third layer of the tower is 1/4 since it is the bottom base minus the 1/2 block and the 1/4 block. And in general, the base of the next layers can also be calculated.

Thus we can cleverly re-draw the figure as follows:



What is the area of this tower?

Now the first block has an area of 1, the next 1/2, the next an area of 1/4, and so on. We end up with the infinite series:

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} ... = 2$$

Again, we find the area is 2, the same answer as before.

I learned about the equivalence of the physics problem and the infinite tower in the book *Famous Puzzles of Great Mathematicians* by Miodrag S. Petkovic. The author explains how the geometric approach was a 14th century solution to the physics problem.

Coin flipping: a game approach

This is my favorite approach to the problem. If you flip a coin until it shows heads, and you get paid for each flip, then you get paid n dollars with probability $(1/2)^n$.

The expected payout results in same infinite series as the physics problem:

$$S = \frac{1}{2} + \frac{1}{4}(2) + \frac{1}{8}(3) + ... + \frac{1}{2^n}(n) + ...$$

We can solve the problem using the two methods above. But there is another interesting approach using probability theory.

Here's how we can think about the game. Let's say that our average payout is x. We can decompose our expected value into two cases. In one case, we win on the first toss, in which case we get paid \$1 with probability 1/2.

Now comes the interesting part. Half of the time, we don't win on the first toss. What is our conditional expected payout? Think about it. After the first toss, the game is exactly the same as when we started, except that we have already earned \$1 for sure by surviving the first toss. Hence, our conditional expected value from this point onward is $x + 1$.

That is, we have the simple equation:

$x = \Pr(\text{heads})(1) + \Pr(\text{tails})(1 + x)$
$x = 0.5(1) + 0.5(x + 1)$

We can also solve this equation to find $x = 2$.

So this is a fitting example to end the book since it shows how 3 problems from physics, geometry, and probability/game theory all relate to the same infinite series. Furthermore, each solution method offers an interesting way to approach the problem, and I hope that will inspire you as you come across challenging mathematical problems.

More From Presh Talwalkar

I hope you enjoyed this ebook. If you have a comment or suggestion, please email me presh@mindyourdecisions.com

About The Author

Presh Talwalkar studied Economics and Mathematics at Stanford University. His site *Mind Your Decisions* has blog posts and original videos about math that have been viewed millions of times.

Books By Presh Talwalkar

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Math Puzzles Volume 1: Classic Riddles And Brain Teasers In Counting, Geometry, Probability, And Game Theory. This book contains 70 interesting brain-teasers.

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Math Puzzles Volume 3: Even More Riddles And Brain Teasers In Geometry, Logic, Number Theory, And Probability. This is the third in the series with 70 more problems.

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