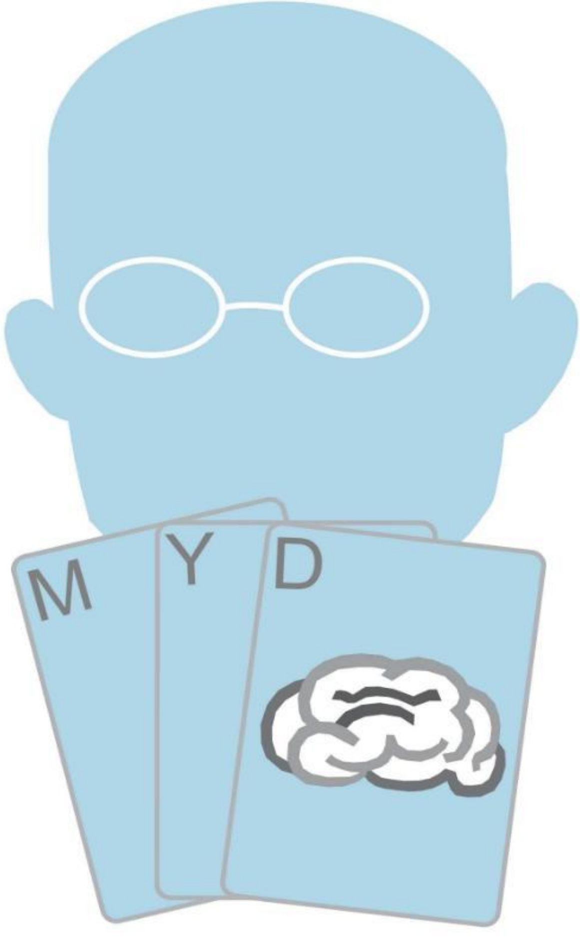
Math Puzzles Volume 3

Even More Riddles And Brain Teasers In Geometry, Logic, Number Theory, And Probability



Presh Talwalkar

About The Author

Presh Talwalkar studied Economics and Mathematics at Stanford University. His site *Mind Your Decisions* has blog posts and original videos about math that have been viewed millions of times.

Books By Presh Talwalkar

The Joy of Game Theory: An Introduction to Strategic Thinking. Game Theory is the study of interactive decision-making, situations where the choice of each person influences the outcome for the group. This book is an innovative approach to game theory that explains strategic games and shows how you can make better decisions by changing the game.

Math Puzzles Volume 1: Classic Riddles And Brain Teasers In Counting, Geometry, Probability, And Game Theory. This book contains 70 interesting brain-teasers.

Math Puzzles Volume 2: More Riddles And Brain Teasers In Counting, Geometry, Probability, And Game Theory. This is a follow-up puzzle book with more delightful problems.

Math Puzzles Volume 3: Even More Riddles And Brain Teasers In Geometry, Logic, Number Theory, And Probability. This is the third in the series with 70 more problems.

But I only got the soup! This fun book discusses the mathematics of splitting the bill fairly.

40 Paradoxes in Logic, Probability, and Game Theory. Is it ever logically correct to ask “May I disturb you?” How can a football team be ranked 6th or worse in several polls, but end up as 5th overall when the polls are averaged? These are a few of the thought-provoking paradoxes covered in the book.

Multiply By Lines. It is possible to multiply large numbers simply by drawing lines and counting intersections. Some people call it “how the Japanese multiply” or “Chinese stick multiplication.” This book is a reference guide for how to do the method and why it works.

The Best Mental Math Tricks. Can you multiply 97 by 96 in your head? Or can you figure out the day of the week when you are given a date? This book is a collection of methods that will help you solve math problems in your head and make you look like a genius.

Why You Should Study Math Puzzles

This is the third book I am publishing about mathematical brain teasers and riddles. What is the point of all of these math problems?

From a practical perspective, math puzzles can help you get a job. They have been asked during interviews at Google, Goldman Sachs, as well as other tech companies, investment banks, and consulting firms.

Math puzzles also serve a role in education. Because puzzles illustrate unexpected solutions and can be solved using different methods, they help students develop problem solving skills and demonstrate how geometry, probability, algebra, and other topics are intimately related. Math puzzles are also great for practice once you are out of school.

But mostly, math puzzles are worthwhile because they are just fun. I like to share these problems during parties and holidays. Even people who do not like math admit to enjoying them. So with that, I hope you will enjoy working through this collection of puzzles as much as I have enjoyed preparing the puzzles and their solutions.

This book is organized into easy, medium, and hard puzzles. It is never easy to organize puzzles by difficulty: some of the hard puzzles may be easy for you to solve and vice versa. But as a whole, the harder puzzles tend to involve higher-level mathematics, like knowledge of probability distributions or calculus.

Each puzzle is immediately accompanied with its solution, as I have done in the other two math puzzle ebooks. I have never been a fan of how print books put all the solutions at the end—it is too easy to peek at the solution for another problem's solution by mistake. In any case, while you are working on a problem, avoid reading the following section which contains the solution.

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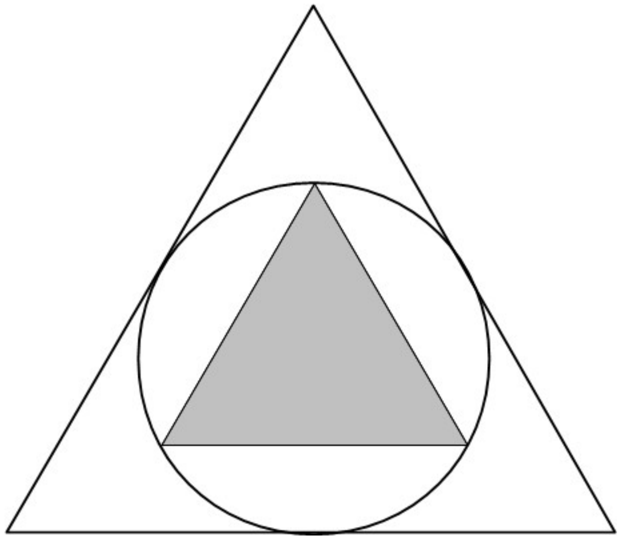
Part I: Easy Puzzles

Puzzle 1: Triangle Ratios

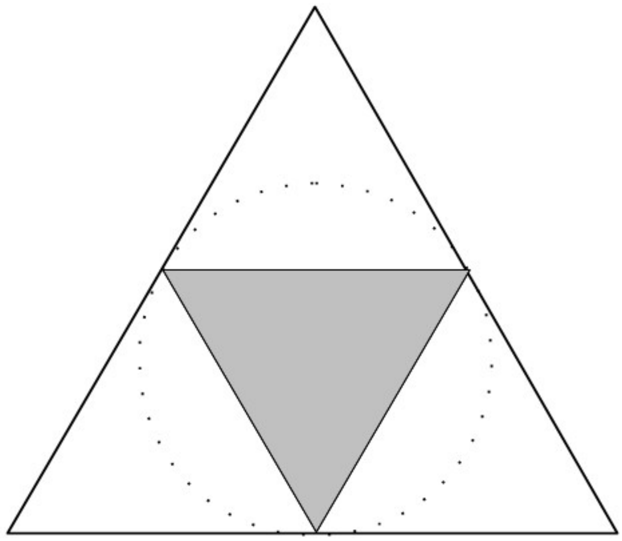
A circle is inscribed in an equilateral triangle. Then an equilateral triangle is drawn inside the circle, as pictured below.

What is the ratio of the area of the smaller equilateral triangle to the larger equilateral triangle?



Answer To Puzzle 1: Triangle Ratios

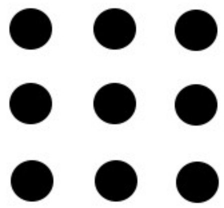
There’s no need to do complicated calculations. The easiest way is to rotate the triangle inside the circle to get the following figure.



Now the answer is obvious! The large equilateral triangle is divided into 4 small equilateral triangles: three of them are facing “up” while the shaded one is facing “down.” Since 4 small equilateral triangles make up the 1 large equilateral triangle, the shaded triangle is 1/4 the area of the larger equilateral triangle.

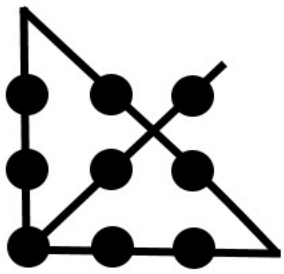
Puzzle 2: Nine Dots

Connect the all the dots without lifting your pen, using at most 4 straight lines, and without retracing any of the lines.



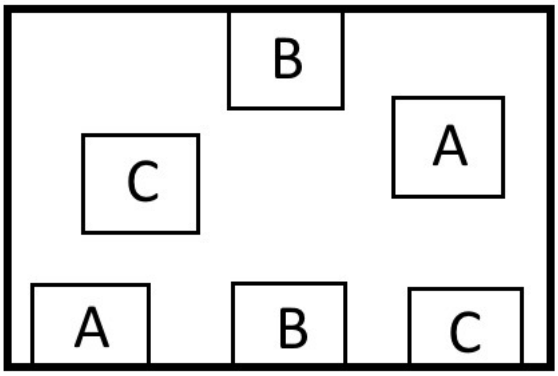
Answer To Puzzle 2: Nine Dots

Here is one way you can solve the puzzle. The trick is to think “outside the box.”



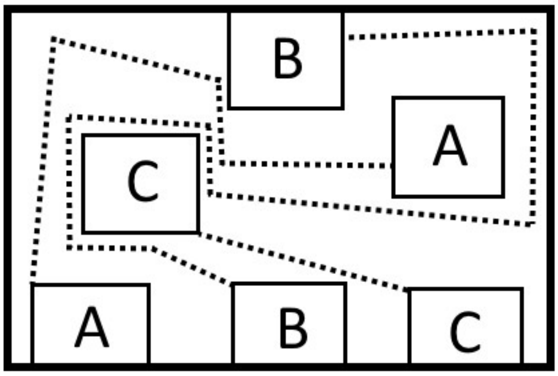
Puzzle 3: Six Boxes

Connect boxes of the same letter with lines. The lines cannot intersect and they have to stay inside of the boundary of the bordering box.



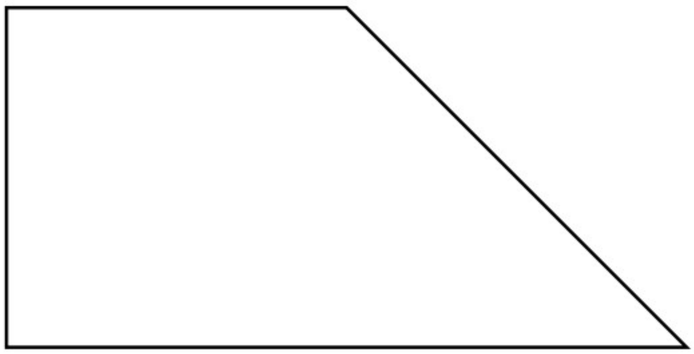
Answer To Puzzle 3: Six Boxes

Here is one way to solve the puzzle. To connect the two boxes with the letter B, wrap the line around the boxes for the letters A and C. Then there will be enough room to connect the remaining boxes without having to cross the path of any existing line.



Puzzle 4: Divide Into 4

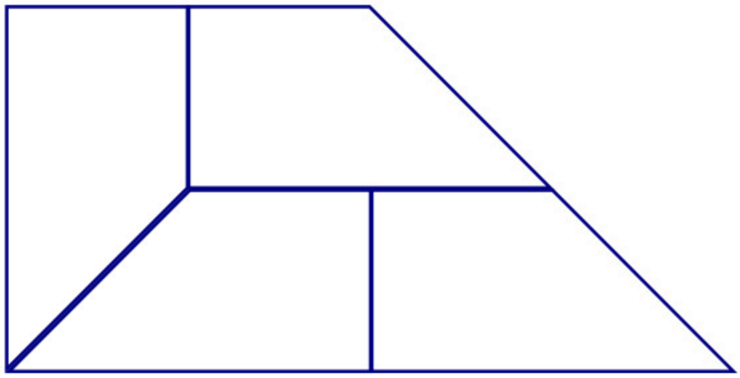
Divide the following shape into 4 equal parts, where each part is the same shape and size.



The dimensions are not listed because the figure is drawn to scale.

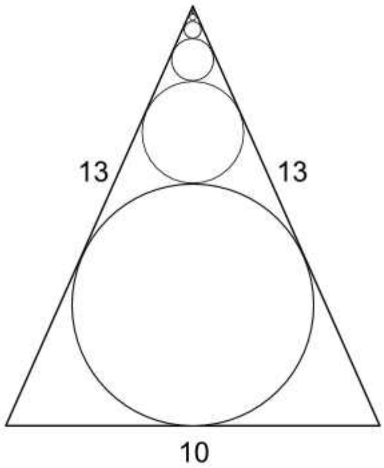
Answer To Puzzle 4: Divide Into 4

There’s a neat trick to solve the puzzle: you can break the shape into 4 smaller copies of itself.



Puzzle 5: Circles In Triangles

Infinitely many circles are inscribed into an isosceles triangle with sides of 10, 13, and 13, as shown in the following diagram. What is the sum of the circumferences of all the circles?

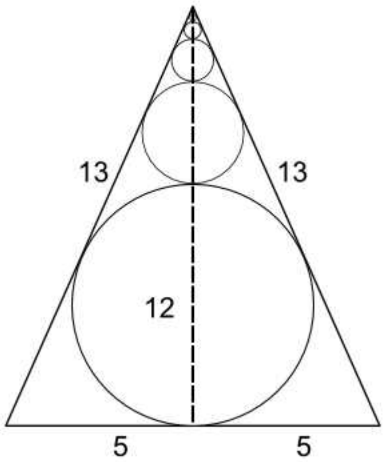


Answer To Puzzle 5: Circles In Triangles

You could solve for the circumference of each circle. An inscribed circle in a triangle has a radius equal to the area of the triangle divided by its semi-perimeter. In this problem, the largest circle has a radius of 60/18 = 10/3. By similar triangles, one could find out the ratio of each smaller circle. Skipping the details, each subsequent circle can be found to be 4/9 of the previous circle. The largest circumference is $2r\,\pi = (20/3)\,\pi$ and each subsequent circumference is 4/9 of that. So we could solve the problem by summing the infinite geometric series starting with $(20/3)\,\pi$ and having a common ratio of 4/9.

$(20/3)\,\pi(1 + 4/9 + (4/9)^2 + \ldots)$
$(20/3)\,\pi[1/(1-4/9)]$
$(20/3)\,\pi[9/5]$
$12\,\pi$

But there is a much easier way to solve this problem! Draw the altitude of the triangle, which divides the base of 10 into two equal segments of 5.



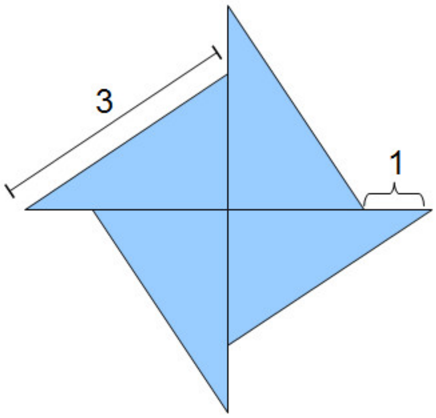
The isosceles triangle is now divided into two right-triangles with a hypotenuse of 13 and a leg of 5. Therefore this must be a 5-12-13 right triangle, meaning the altitude of the isosceles triangle is 12.

Furthermore, the length of the altitude is exactly equal to the sum of the diameters of all of the circles. The sum of the circumferences is π (sum of the diameters) = π (length of altitude) = $12\,\pi$.

Credit: I read about this problem in *Aha! Solutions* by Martin Erickson.

Puzzle 6: Area Of A Triangle

Pictured below are four right triangles, each with a hypotenuse of 3. The larger leg is 1 unit longer than the shorter leg. What's the area of a single triangle?



Answer To Puzzle 6: Area Of A Triangle

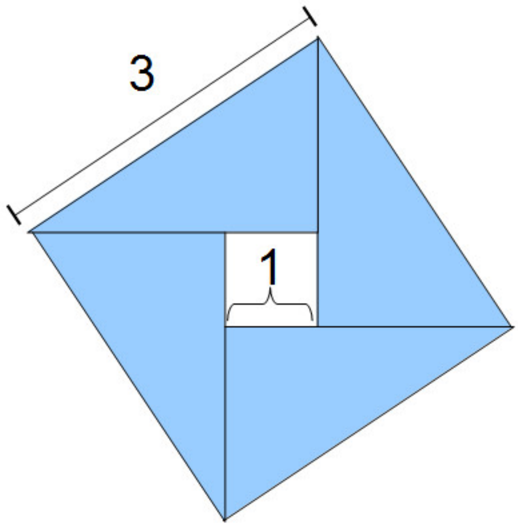
The answer is 2. Just like the last puzzle, there is a hard way and an easy way to arrive at the answer.

The hard way is to use algebra. If you label the longer leg as a and the shorter leg as b, then we are given $a = b + 1$.

We can use the Pythagorean Theorem to arrive at the equation $a^2 + b^2 = 3^2$. Substituting for a, we get $(b + 1)^2 + b^2 = 2b^2 + 2b + 1 = 9$. Now we simplify to get $2b^2 + 2b = 8$, and then we divide by 2 to get $b^2 + b = 4$.

The good part is we do not have to solve this equation. We ultimately want to find the area of the triangle, which is $ab/2 = (b + 1)b/2 = (b^2 +b)/2$. Since we derived that $b^2 + b = 4$, we can divide the equation by 2 to find $(b^2 +b)/2 = 2$. Thus, the area of a single triangle is 2.

There is a much more elegant way to solve this puzzle. Rearrange the four triangles to make a big square with a small square removed, as shown in the following diagram.



The area of the outer square with dimensions 3x3 is equal to the area of the 4 triangles plus the area of the internal square with dimensions 1x1.

Therefore, 3x3 = (area 4 triangles) + 1x1. In other words, 3x3 – 1x1 = (area 4 triangles).

This simplifies to 8 = (area 4 triangles). We divide this equation by 4 and conclude the area of a single triangle is 2!

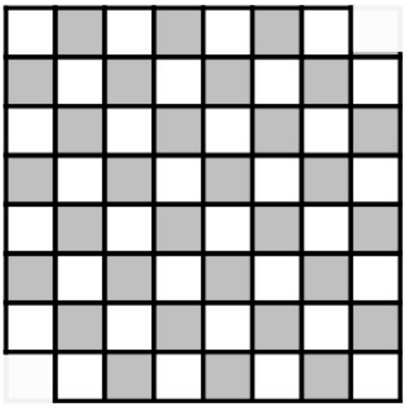
Incidentally, this rearrangement of four right triangles into the shape of a square with a smaller square inside is one method to prove the Pythagorean Theorem, credited to the Indian mathematician Bhaskara. Behold!

For a general right triangle with legs a and b, and a hypotenuse c, you can construct a square with side c composed of the four of the right triangles and a small square with side $a - b$. One way to compute the area of the square is by squaring its side, so the area is c^2. The other way to compute the area of the square is to add up the areas of the 4 triangles and the small inside square. The area of the 4 triangles is $2ab$ and the area of the small square is $(a - b)^2 = a^2 + b^2 - 2ab$. The sum of the areas of the triangles and the square is $a^2 + b^2$, and this equals the area of the entire square, c^2. Therefore we have proved the Pythagorean Theorem $a^2 + b^2 = c^2$.

I credit this puzzle to a reader of the Mind Your Decisions blog, Danny, who was inspired by the geometry of the UK flag.

Puzzle 7: Covering A Chessboard

A standard chessboard is a grid of 8 by 8 squares. You have dominoes that cover 2 squares at a time (that is, they measure 2 by 1). If two opposite corners of the chessboard are removed, can you cover the remaining 62 squares with 31 dominoes?



Answer To Puzzle 7: Covering A Chessboard

It is actually impossible to cover the board. The easiest proof involves coloring alternate squares of the board.

A chessboard alternates between black and white squares. Because each row has 8 squares and each column has 8 squares, there is a pattern in the coloring of the squares. In any given row, the first and last squares always have opposite colors. Similarly, in any given column, the first and last squares always have opposite colors. Putting these two facts together, we can reason that two diagonally opposite corners must have the same color.

When two diagonally opposite corners are removed, that means two squares of the same color are removed. Hence, the board contains 30 squares of one color and 32 squares of the other.

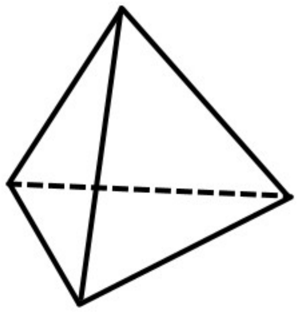
Each domino covers two adjacent squares, one white and one black. This means any arrangement of dominos must also cover an equal number of black and white squares. It is therefore impossible to cover a board with 30 squares of one color and 32 of the other using 2 by 1 dominos!

Puzzle 8: A Matchstick Problem

This puzzle is short but challenging. Using 6 matches, create 4 triangles of the same size.

Answer To Puzzle 8: A Matchstick Problem

The trick is to think in 3 dimensions. The 6 matches can be arranged into a tetrahedron whose 4 faces are triangles of equal size.



Puzzle 9: Medicine

To stay alive, you have to take 1 pill of medicine A and 1 pill of medicine B daily.

You take out 1 pill of medicine A and hold it. Then you lift the bottle for medicine B and tap out a pill into your hand. The only problem is 2 pills come out instead of just 1.

Then you realize something scary: all three pills look indistinguishable to you, as they were foolishly manufactured to be the same size, shape, and color.

Now you’re in a bit of a jam. You need to take 1 pill each of medicine A and medicine B, but you don’t know which pill is of what medicine. How can you do this, without throwing any of the pills away?

Answer To Puzzle 9: Medicine

Take out 1 more pill of medicine A and add it to the 3 pills. Carefully split each pill in half, placing one half into a Left pile and the other half into a Right pile. Now the Left and Right piles each contain 4 pill halves that came from 2 pills of medicine A and 2 pills of medicine B. Therefore, each pile contains the equivalent of 1 pill of medicine A and 1 pill of medicine B. Consume one pile today and the other pile the next day.

Of course, there are some medicines that cannot be split into half. But many medicines can, and devices like pill splitters exist. Technically the problem should specify the pills can be split in half, but emphasizing that detail essentially spoils the fun of the puzzle.

Puzzle 10: Rope Burning

You have a lighter and two ropes. Each rope burns up in 1 hour from end to end, but the ropes do not burn evenly.

How can you measure 45 minutes by burning the ropes?

Challenge: how would you measure 50 minutes?

Answer To Puzzle 10: Rope Burning

First arrange the two ropes so you are holding all four of the rope ends. At the start, light both ends of rope A and one end of rope B. Exactly when rope A burns up, light the other end of rope B. The time from the start to when rope B burns up is exactly 45 minutes.

To see why, first note that it takes rope A exactly 30 minutes to burn up. Intuitively this makes sense. If rope A burns in 60 minutes with one flame, then it would burn in 30 minutes = 60/2 with two flames. (*more precise proof below)

After 30 minutes, some amount of rope B is left. By similar logic, it will take 15 minutes to burn what remains of rope B with two flames.

Therefore, the total time to burn both ropes is 30 + 15 = 45 minutes.

How would you measure 50 minutes? We can measure 30 minutes by burning one rope at both ends, so we can solve this if we can measure 20 minutes using the other rope.

Since a rope burns in 30 minutes = 60/2 with two flames, the idea is a rope will burn 20 minutes = 60/3 if we can keep 3 flames going. How do we do that?

Light the rope at one end and somewhere in the middle. Now the rope will burn with 3 flames. At some point, the two flames headed towards each other might meet and cancel each other out. At exactly that instant, light the remaining rope somewhere in the middle to keep 3 flames going. Repeating this process as many times as needed (infinitely), the rope will always burn with 3 flames and so it will burn in 20 minutes.

*More precise proofs

Proof A: Think about rope A as the interval [0, 1]. We can burn the interval from either endpoint. In 30 minutes, we would burn either $[0, x]$ or $[y, 1]$. In the next 30 minutes, we would burn $[x, 1]$ or $[0, y]$ as the entire rope burns in 60 minutes.

Both the intervals $[0, x]$ or $[0, y]$ burn in 30 minutes time. This implies $x = y$. Hence, burning from both ends we burn $[0, x]$ and $[x, 1]$—that is, the entire rope—in a matter of 30 minutes.

Similarly, think about rope B as the interval [0, 1]. After 30 minutes some portion remains, call it $[0, z]$. We can burn this starting from either end, and we can show the intervals $[0, w]$ and $[w, z]$ would burn in 15 minutes each. So if we burn from both ends, the remaining rope would burn up in 15 minutes.

Proof B: For the 3 flames case, we will keep track of the distances burned by whichever flame is the single flame as a sequence $s_1, s_2, \ldots, s_n$ and the distances burned by the pair of flames going towards each other as $p_1, p_2, \ldots, p_n$, corresponding to the time periods $t_1, t_2, \ldots, t_n$.

From Proof A above, the pair of flames burns distances in half the time a single flame would take. So the distances $p_1, p_2, \ldots, p_n$ are burned in half the time the single flame would take. Since these distances are burned in time periods of $t_1, t_2, \ldots, t_n$, that means it would have taken the single flame $2t_1, 2t_2, \ldots, 2t_n$ to burn the same distances.

For the single flame to burn the entire rope, it needs to burn distances of $s_1, s_2, \ldots, s_n, p_1, p_2, \ldots, p_n$. The total time it would take is equal to $t_1 + t_2 + \ldots + t_n + 2t_1 + 2t_2 + \ldots 2t_n = 3t_1 + 3t_2 + \ldots 3t_n$.

We can equate this to 60 because a single flame burns the entire rope in 60 minutes. So we have $3t_1 + 3t_2 + \ldots 3t_n = 60$.

Dividing by 3, we get $t_1 + t_2 + \ldots t_n = 20$. In other words, a rope with 3 flames simultaneously burns in 20 minutes.

Puzzle 11: Cheryl's Birthday

A version of this problem appeared as a Math Olympiad question for high school students in Singapore. In April 2015, it was posted to Facebook, and the problem soon went viral generating debate about the correct answer. Here is the setup.

Albert and Bernard have just become friends with Cheryl, and they want to know when her birthday is. Cheryl gives them a list of 10 possible dates.

May 15, May 16, May 19
June 17, June 18
July 14, July 16
August 14, August 15, August 17

Cheryl tells Albert only the month and Bernard only the day.

Albert says, "I don't know when Cheryl's birthday is, but I know for sure that Bernard cannot know either."

Bernard then says, "At first I didn't know when Cheryl's birthday is, but now I do know."

Albert concludes, "Now I know when Cheryl's birthday is too."

So when is Cheryl's birthday? It seems impossible, but you can figure it out from the information given!

Answer To Puzzle 11: Cheryl's Birthday

The answer is July 16. What is more interesting is why this is the correct answer. Start by looking at the dates again.

May 15, May 16, May 19
June 17, June 18
July 14, July 16
August 14, August 15, August 17

Albert is told one of the months May, June, July, or August. And Bernard is told one of the days 14, 15, 16, 17, 18, or 19.

Albert first says, "I don't know when Cheryl's birthday is, but I know that Bernard cannot know either."

The first part of the sentence is obvious because of course Albert doesn't know the day. But the second part is a clue.

Albert is sure that Bernard cannot know the date. If Bernard was told 19, he would know the birthday was May 19 because that day is unique to one date. Similarly, if Bernard was told 18, he would know the birthday was June 18. How is Albert sure that Bernard didn't hear 18 or 19? It must be because Albert knows the birthday is not in May or June. In other words, Albert was told either July or August.

So the first sentence reduces our list to 5 dates.

July 14, July 16
August 14, August 15, August 17

Bernard then says, "At first I didn't know when Cheryl's birthday is, but now I do know."

Upon hearing Albert's statement, Bernard figured out the month must be July or August. He then also said now he could figure out the birthday. If he had been told 14, the month would still be ambiguous. So he must have been told 15, 16, or 17.

So Albert is left with 3 possible dates.

July 16
August 15, August 17

Albert finally explains, "Now I know when Cheryl's birthday is too."

If Albert was told August, he still would not be sure. So he must have been told July.

Therefore, Cheryl said July to Albert and 16 to Bernard. Cheryl's birthday is July 16.

Puzzle 12: Cheryl’s Hat Color

Albert, Bernard and Cheryl have to solve a logic puzzle for math class. The teacher lines them up with Cheryl in front, Bernard in the middle, and Albert in the back.

The teacher blindfolds them and puts either a white or black hat on each person. The hats, the teacher announces, are chosen from a group of 3 black and 2 white hats.

The teacher removes the blindfolds and asks, “Does anyone know the color of hat I put on them? If anyone can figure it out, you all pass this test.”

Albert, who can see both Bernard’s and Cheryl’s hat colors, goes first. He disappointingly says, “I don’t know.”

Bernard, who can see Cheryl’s hat color, goes next. He also says, “I don’t know.”

At this point the situation looks bad for the group. Cheryl is in the front of the line and *cannot see anyone’s hat color*.

But suddenly Cheryl exclaims, “I know!”

Cheryl explained her answer and it was correct. What was Cheryl’s hat color, and how did she know?

Answer To Puzzle 12: Cheryl’s Hat Color

Cheryl must have been wearing a black hat.

If Albert saw two white hats, he would know he has a black hat. So when Albert says “I don’t know,” that means he saw at most one white hat. If Bernard then saw a white hat he would know he must have a black hat. So when Bernard says “I don’t know,” that means he saw Cheryl had a black hat.

Upon hearing Albert and Bernard say “I don’t know,” Cheryl could figure out for sure that she had a black hat.

Here is a longer explanation. Imagine Cheryl was wearing a white hat. Then either Bernard has a white hat, Albert has a white hat, or both Albert and Bernard have black hats. We can write the cases where B means black, W means white, and we write the hat colors in order of Albert, Bernard and Cheryl.

BWW
WBW
BBW

Let’s consider BWW. This hat coloring means Albert saw two white hats. But if Albert saw two white hats, he would know he is wearing a black hat, as there are only two white hats in total. When Albert says, “I don’t know,” Cheryl can exclude BWW.

Similarly, let’s consider WBW and BBW. After Albert says “I don’t know,” Bernard can be sure Albert is not seeing two white hats. So if Bernard sees a white hat on Cheryl, he can be sure his own hat color must be black. When Bernard says, “I don’t know,” Cheryl can exclude WBW and BBW as possibilities.

In summary, if Cheryl was wearing a white hat then either Albert or Bernard would have deduced their hat color.

After Albert and Bernard say “I don’t know,” Cheryl can figure out she was wearing a black hat.

Surprisingly Cheryl, who cannot see anyone’s hat, can figure out her own hat color while the others cannot. This is a rare puzzle that shows you can sometimes learn from your ears more than you can see with your eyes.

Puzzle 13: Four Hats

A villain has taken 4 captives. He offers them a chance to go free if they can solve a puzzle.

He lines up three of the prisoners (A, B, and C) and a fourth prisoner D is placed in a separate room.

A B C | D

Each captive is wearing a hat. The villain explains to everyone there are 2 blue hats and 2 red hats. If anyone can guess the color of his hat, then they all are set free.

No communication is allowed. The only information is that captive A can see the colors of the hats on captives B and C; and captive B can see the color of the hat on captive C.

Here is the arrangement of colors:

A(blue) B(red) C(blue) | D(red)

After a couple of minutes, one of the captives calls out an answer. Who makes the guess and what color?

Answer To Puzzle 13: Four Hats

It is captive B who can figure out he is wearing a red colored hat. How can that be? Recall the distribution of hats.

A(blue) B(red) C(blue) | D(red)

Captive B makes his guess based on the following observation. All captive B sees is a blue hat on captive C. His hat could either be red or blue, but he is not sure.

However, he reasons as follows. What if his hat were colored blue? In that case, captive A would be looking at two blue hats on captives B and C. This immediately would indicate to captive A that his hat must be colored red, and captive A would have called out his answer promptly.

But since captive A kept quiet for a couple of minutes, he must not have been sure. That is, captive A must have been looking at one red and one blue hat. And since B knows C is wearing a blue hat, it must be the case that he is wearing a red hat.

Puzzle 14: Water Jug

This problem was featured in the film *Die Hard With a Vengeance*.

You have a 3-gallon and a 5-gallon jug that you can fill from a fountain of water.

How can you fill *exactly* 4 gallons of water?

Answer To Puzzle 14: Water Jug

Some people estimate 4 gallons by adding 3 gallons of water to 1/3 of the 3 gallon jug. But the riddle is asking for a precise measurement and so this solution does not work.

The trick is to realize 5 - 3 = 2, which means 3 - (5 - 3) = 1, and so 5 - (3 - (5 - 3)) = 4. In other words, it is possible to obtain 4 gallons from operations involving 3 and 5 gallons.

Here is one way to find the answer.

1. Fill up the 5-gallon jug.
2. Fill up the 3-gallon jug using the water from the 5-gallon jug (leaving 2 gallons in the 5-gallon jug).
3. Pour out the 3-gallon jug into the fountain.
4. Transfer the 2 gallons from the 5-gallon jug into the 3-gallon jug.
5. Fill up the 5-gallon jug.
6. Transfer water from the 5-gallon jug until the 3-gallon jug is full. Since the 3-gallon jug already had 2 gallons of water, there is room for just 1 gallon.
7. The amount of water in the 5-gallon jug is precisely 4 gallons.

If we denote the contents of the jugs as the pair (5-gallon jug amount, 3-gallon jug amount), the sequence of events is the following:

(5, 0)—>(2, 3)—>(2, 0)—>(0, 2)—>(5, 2)—>(4, 3)

That's not the only solution. Here is another way to obtain 4 gallons.

1. Fill up the 3-gallon jug.
2. Transfer to the 5-gallon jug.
3. Fill up the 3-gallon jug again.
4. Transfer water to fill up the 5-gallon jug, leaving 1 gallon in the 3-gallon jug.
5. Empty out the 5-gallon jug.
6. Transfer the 1 gallon to the 5-gallon jug.
7. Fill up the 3-gallon jug and transfer that to the 5-gallon jug.
8. The 5-gallon jug contains 4 gallons of water.

These steps can be written as the following sequence of pairs.

(0, 3)—>(3, 0)—>(3, 3)—>(5, 1)—>(0, 1)—>(1, 0)—>(1, 3)—>(4,0)

Incidentally, the reason we can find a solution is because the two numbers 5 and 3 are relatively prime—that is, they have no common divisors. We can actually generate any volume of water from 1 to 5 (in fact, we did get measurements of 1, 2, 3, 4, and 5 along the way in our solutions).

The water jug problem is an example of a Diophantine equation. The general problem is finding integer solutions for the polynomial equation $ax + by = c$. It can be proven that a solution (x, y) exists when the greatest common divisor of a and b is a factor of c.

So you can get 4 gallons from a 3 and 5 gallon jug, but it would be impossible to get 3 gallons from a 2 and 4 gallon jug (since 2 and 4 have a greatest common divisor of 2, which is not a factor of 3).

Puzzle 15: Dehydrated Food

Imagine you have 100 pounds of food that is 99 percent water by weight. In the hot sun, the food dehydrates until it is 98 percent water.

How much will the food weigh? Assume the only weight loss is from water evaporating away.

Answer To Puzzle 15: Dehydrated Food

The surprising answer is the food will be 50 pounds! Even though the water content drops by a mere 1 percentage point, the dehydrated food will lose 50 percent of its original weight.

A direct approach is to work out the algebra. The food starts at 100 pounds, and 1 percent, or 1 pound is non-water weight.

The dehydrated food is 98 percent water, so it is 2 percent non-water. The total weight of the dehydrated food is found by dividing the non-water weight by its percent weight. The answer is 50 pounds = (1 pound)/(2 percent). We can verify the water will make up 49 of the 50 pounds, which is 98 percent.

The dramatic weight loss might be easier to understand from the non-water component frame of reference. The non-water component weighs the same absolute amount in both cases, but its percent weight doubles in the dehydrated food. Since the non-water percent weight *doubles*, that means the dehydrated food must weigh *half* as much.

Now that you've solved this puzzle, you can easily solve the following related problems.

1. A gallon of milk weighs 8 pounds and is 90 percent water by weight. If the milk is converted into evaporated milk, which is 60 percent water, how much will the evaporated milk weigh? (answer: 2 pounds).

2. A soup broth weighing 10 pounds is 99 percent water by weight. I add salt until the broth becomes 98 percent water. How much will the soup broth weigh? (answer: about 10.1 pounds).

The dehydrated food puzzle is surprising because problems 1 and 2 sound similar but have drastically different answers. The weight change is large in problem 1 but quite small in problem 2.

Puzzle 16: Crossing Trains

There is a 100 mile train track that connects cities A and B. A train leaves from A to B and is scheduled to arrive 1 hour later. At the same time, a train leaves from B to A and will make the trip in 2 hours.

When the trains cross paths, how far will they be from city A?

Answer To Puzzle 16: Crossing Trains

There are a variety of methods to solve this problem.

For convenience, label the train going from A to B as train A and the one from B to A as train B.

Method 1: rates of speed

Train B requires double the time to make the trip; therefore, train B moves half as fast as train A. This means that in the same amount of time, train A moves twice the distance as train B.

By the time train B has traveled the distance x, train A has traveled $2x$. When they meet, the sum of the distances they traveled will be the entire journey of 100 miles.

Setting the sum of train distances equal to the distance between the towns gives the equation $2x + x = 100$. This has a solution $x = 33\ 1/3$. We want the distance from city A, which is the distance that train A has traveled. Since $2x = 66\ 2/3$, the answer is the trains meet 66 2/3 miles from city A.

Method 2: working together

The train problem is phrased in a somewhat competitive manner since the trains are moving in opposite directions. But we can re-frame the problem cooperatively as follows. Train A can complete 100 miles in 1 hour; train B can do it in 2 hours. *Working together*, how long will it take them to travel 100 miles?

At time t, train A has traveled the percentage $t / 1$ and train B has traveled $t / 2$. We need these percentages to add up to 1 whole job; therefore the equation is $t / 1 + t / 2 = 1$. We can solve this equation for $t = 2/3$.

In 2/3 of an hour, train A is 2/3 of the journey. Thus, the distance is (2/3)(100) = 66 2/3.

In general, the time it takes for two people working together is equal to half of the *harmonic mean* of the rates of work. The harmonic mean of two numbers x and y is $2/(1/x + 1/y)$. This might look complicated, but this formula often arises when averaging rates.

Train A moves at twice the speed of train B, so the rates can be thought of as 2 and 1. The time to meet is half of the harmonic mean, which is $1/(1/2 + 1/1) = 1/(3/2) = 2/3$. In 2/3 of an hour, train A has completed 2/3 of 100 miles, so the meeting point is (2/3)(100) = 66 2/3 miles from city A.

Method 3: algebra

This is my least favorite way of solving the problem, but it is assured to work, so it's worth knowing!

Train A moves at a speed of 100 miles per hour. Train B moves at a speed of 100/2 = 50 miles per hour. At a given time t, train A has moved a distance of $100t$ from city A.

Since train B starts 100 miles from city A, its distance at time t will be $100 - 50t$. When the two trains meet, the distances will be equal.

$100t = 100 - 50t$
$150t = 100$
$t = 2/3$

In 2/3 of an hour, train A and B are therefore $100(2/3) = 100 - 50(2/3) = 66\ 2/3$ miles away from city A.

Puzzle 17: Monks Crossing

There is a sacred hill where monks pray. At each hour, a monk leaves the base for the top of the hill. Also at each hour, a monk leaves from the top of the hill for the return journey. Traversing the hill takes 4 hours in either direction and there is only one path.

If a monk starts his journey from the bottom at 12 noon, how many monks will he pass along the way to the top?

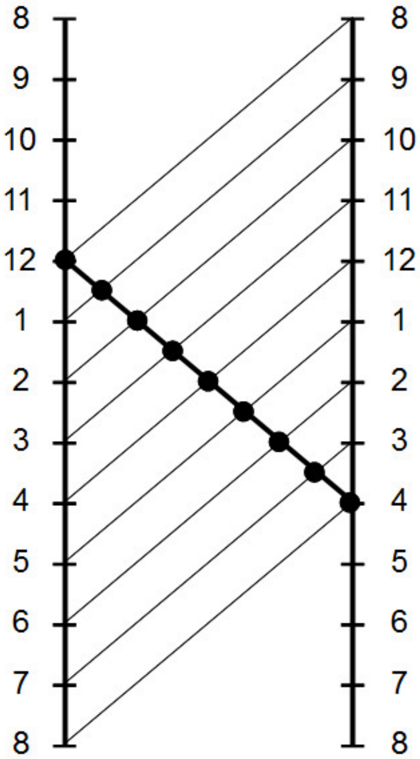
Answer To Puzzle 17: Monks Crossing

For each of the 4 hours the monk travels, there will be another monk that leaves the top to return. It would be tempting to think the answer is 4.

But this is incorrect. The reason is some monks are already on their way down. This factor contributes another 4 monks who get crossed. There is also one extra monk who gets crossed: the one who is leaving just as the monk reaches the top.

Thus, the answer is 9.

Here is a visual graph of the journey, where the right-hand numbers indicate the time at the top of the hill, and the lines going southwest are the paths of the monks returning to the base of the hill. The monks passed correspond to intersections in the graph.



I adapted this puzzle adapted from “Meeting of Ships” in the book *Famous Puzzles of Great Mathematicians* by Miodrag S. Petkovic.

Puzzle 18: Crossing Trucks

At exactly sunrise, a delivery truck starts driving from city A to B. At the same time, another delivery truck is going in the opposite direction from city B to A.

The two trucks pass each other at 12 noon. The truck driving to city B reaches its destination at 4pm. The truck going to city A takes much longer and reaches at 9pm.

What time was sunrise? Assume the two trucks moved at a constant rate, and they traveled along exactly the same road between the two cities.

Answer To Puzzle 18: Crossing Trucks

The puzzle is a bit more difficult than the previous “crossing” problems because neither the starting time, nor the distance, nor the rate of either truck is given. Remarkably, there is enough information to solve this problem.

There are a variety of methods to tackle the puzzle.

Method 1: standard approach

Let *C* denote the location of the crossing point. The trip can be thought of as two distances AC and CB.

Write *t* for the number of hours between sunrise and 12 noon.

The truck going from A to B was able to cover the distance CB in a manner of 4 hours. It took the other truck *t* hours to cover the same distance.

Similarly, the truck going from A to B covered the distance AC in *t* hours while truck B did the same in 9 hours.

Now we proceed in the following fashion. Each truck was said to be moving at a constant speed. So we will calculate the speed of each truck for each segment of the trip AC and CB.

The truck going from A to B had a constant speed given by AC/*t* and CB/4. Since the two expressions are for the same speed, their ratio is equal to 1. Therefore, (AC/*t*)/(CB/4) = (4 AC)/(*t* CB) = 1.

The truck going from B to A had a constant speed given by CB/*t* and AC/9. Again the ratio of these values must be equal to 1, which means (AC/9)/(CB/*t*) = (*t* AC)/(9 CB) = 1.

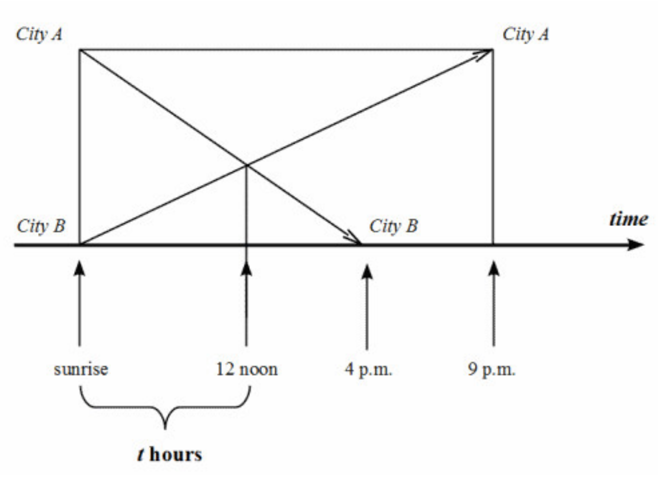
Now we have two equations that are both equal to 1, and therefore they must be equal to each other. A wonderful amount of cancellation happens when the two equations are set equal to each other.

(4 AC)/(*t* CB) = (*t* AC)/(9 CB)
(4 ~~AC~~)/(*t* ~~CB~~) = (*t* ~~AC~~)/(9 ~~CB~~)
4/*t* = *t*/9
$t^2 = 36$
t = 6 (because *t* was defined as positive, -6 doesn’t make sense)

Thus, the two trucks started 6 hours before noon, which means sunrise took place at 6 a.m.

Method 2: geometrical solution

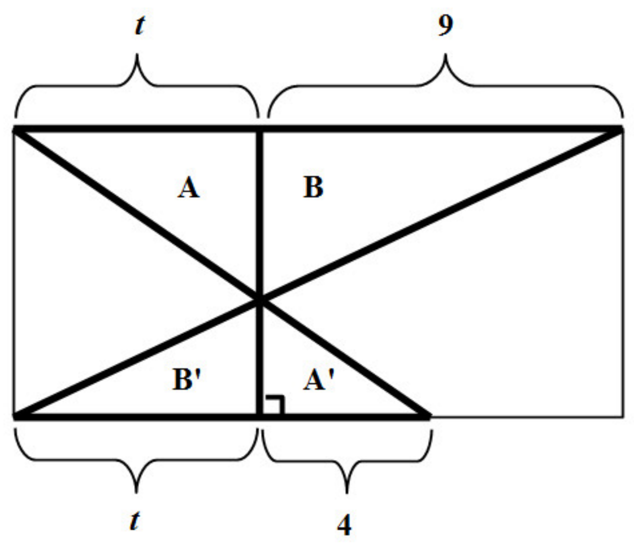
The path of the two trucks can be drawn out in a rectangular diagram as follows.



The horizontal axis indicates the time of day, the vertical axis indicates the distance of the truck from cities A and B, and the path of each truck is drawn as a directed line.

The two lines meet exactly at 12 noon, and the other times of sunrise, 4pm, and 9pm are all drawn as given in the setup.

To solve the problem, the relevant details can be re-drawn in this simplified diagram.



There are a multitude of similar triangles in the figure. Note that triangle A is similar to A', and B is similar to B'.

One can work out that this implies the following ratios are equal.

(leg of A')/(leg of A) = (leg of B')/(leg of B)
$4/t = t/9$

This is exactly the same equation as using the previous method. The answer is the same that $t = 6$ and the sunrise happened at 6 a.m.

Method 3: the long (and boring) algebra

This is the method that came to my mind first. While it is the least elegant, it does get the job done.

Write s_a for the speed of the truck going from A to B, and s_b for the truck B to A.

Write d for the distance of the trip, and write t for the number of hours between sunrise and 12 noon.

In a matter of t hours, the two trucks crossed. That means that they jointly traveled a distance of d (since C is the crossing point, AC + CB = AB).

This implies the following equation.

(I): $s_a t + s_b t = d$

We also know the following. One truck completed the journey in $t + 4$ hours, and the other truck did it in $t + 9$ hours. We can calculate the speeds by dividing the distance of the total trip by the time it took each to complete the trip. This yields the following equations.

$s_a = d/(t + 4)$
$s_b = d/(t + 9)$

Now we can substitute these back into equation **(I)**.

$[d/(t + 4)]t + [d/(t + 9)]t = d$

The term d cancels out, and we can solve the equation diligently.

$t/(t + 4) + t/(t + 9) = 1$
$t(t + 9) + t(t + 4) = (t + 4)(t + 9)$
$2t^2 + 13t = t^2 + 13t + 36$
$t^2 = 36$
$t = 6$ (as we reject the negative solution)

It took the trucks 6 hours from sunrise to meet at noon, so once again the answer is they started at 6 a.m.

Credit: this is a puzzle I saw on Reddit Math, where credit was given to an AMS interview with Vladimir Arnol'd (Yes the name Arnol'd has an apostrophe and that is not a typo!).

Puzzle 19: Slicing Vegetables

One of the vegetables I eat regularly is an ivy gourd, also known as tondli/tindora/gentlemen's toes. The vegetable is a type of cucumber and looks like a pickle.

The dish I prepare requires slicing the tondli into "coins" which are like mini pickle slices. Initially, I would chop up the tondli one by one. It would take me about 30 minutes to chop everything up.

I later realized I could increase efficiency by changing my cutting technique. I soon learned to chop 2 tondli pieces at a time, so I could slice up 2 pieces in the same amount of time I used to be able to chop 1. I can now chop 3 tondli pieces at a time.

The question is, how much time did I save when going from slicing 1 at a time to 2 at a time to 3 at a time?

Assume the number of tondli pieces is divisible by 3 so there are no "leftover" pieces I have to chop individually or two at a time.

Answer To Puzzle 19: Slicing Vegetables

I was able to save a lot of time when going from 1 piece to 2, but then sadly I did not save much time when going from 2 pieces to 3—even though in both cases I was cutting one more piece at a time.

The reason is there are diminishing returns with marginal increases to a rate of change. This is why driving faster and faster does not keep saving you much more time, or why cutting back on the frequency of coffee drinking will not improve your finances drastically. The same math applies to cutting vegetables more efficiently.

Let's work out a concrete example. Let's say there are 30 tondli pieces and it takes me 30 minutes to cut them 1 at a time. Then my rate of chopping is 1 tondli piece per minute.

If I increase that to 2 pieces per minute, then the time I would take is 15 minutes = (30 pieces)/(2 pieces/min). If I increased to 3 pieces per minute, then the time is 10 minutes = (30 pieces)/(3 pieces/min).

In other words, I would save 15 minutes by increasing to 2 pieces per minute, but only 20 minutes by increasing to 3 pieces per minute.

Although I save more time by cutting 3 at a time, I only end up saving 5 extra minutes over cutting 2 at a time. You can see there are diminishing returns to trying to increase my efficiency any more.

If I increase to 4 pieces at a time, then I would only end up saving 22.5 minutes. At 5 pieces at a time, I would save 24 minutes.

Even though I'm increasing the rate at which I slice the tondli by 1 extra piece each time, each time I end up saving less time in absolute minutes saved. On a practical note, it also becomes harder to chop more pieces at the same time.

So I figure that I'm pretty good chopping 3 pieces at a time and if I want to increase the efficiency in my life I'm better off working on some other skill besides chopping.

More Precise Proof

I made the above calculation using 30 pieces. That was an assumption made without loss of generality. The reason is the number of pieces cancels out in the final calculation of time savings.

For instance, suppose I start with N pieces of tondli. The time it takes me to chop 1 piece at a time is 30 minutes. Hence my rate of work is (N/ 30) pieces/min.

When I increase my rate to **2** pieces at a time, I will be chopping at the rate of ($\mathbf{2}N/30$) pieces/min. And for **3** pieces at a time, I will be chopping at a rate of ($\mathbf{3}N/30$) pieces/min.

If I chop $\boldsymbol{k}$ pieces at a time, then my rate is ($\boldsymbol{k}N/30$) pieces/min.

Then I need to calculate how much time it takes at each chopping rate. This is where the N cancels out. If I chop at 2 pieces per minute, then my time is 15 min = (N pieces)/($2N/30$). If I chop at 3 pieces per minute, then my time is 10 min = (N pieces)/($3N/30$) = 10 min.

If I chop at k pieces per minute, then my time is $30/k$ min = (N pieces)/($kN/30$).

Therefore, the calculation of time is independent of the number of pieces.

Puzzle 20: Moving Walkway

This is a neat mathematical problem, but moving walkways are an interesting topic on their own. This problem takes place at the famous moving walkway of Chicago's O'Hare airport.

If I walk while I'm on the moving walkway, it takes me 172 seconds to go from one end to the other. If I walk against the moving walkway—that is, I go the wrong way—it takes me five times as long to go from one end to the other, 860 seconds.

There are two questions. First, if I stood still on the moving walkway, how long would it take me to go from one end to the other? Second, if I just walked—without using the moving walkway—how long would that take me to move the same distance?

First, I've never actually done this experiment, but the numbers are relatively accurate to data on actual speeds, which is why the odd number of 172 seconds appears.

Second, you'll notice there is a lot of missing information. You don't know the moving walkway's speed, the distance traveled, or how fast I walk. That's the beauty of this puzzle: you can still solve the problem!

Answer To Puzzle 20: Moving Walkway

The problem is fairly easy to solve with the assistance of a formula from algebra or physics class. The formula is that distance (d) equals the product of the rate of speed (r) and the time taken (t).

That is, $d = rt$.

We have to modify the formula slightly for this problem. The speed at which I move is not a single rate. It is the combination of my walking speed (r_{me}) and the speed of the moving walkway ($r_{walkway}$). When I walk going in the same direction as the moving walkway, the rate will be the sum of my walking speed and the moving walkway's speed ($r_{me} + r_{walkway}$). When I go against the track, the rate will be the difference of my walking speed and the moving walkway's speed ($r_{me} - r_{walkway}$).

We are given two points of information: going the correct way took 172 seconds while going against the moving walkway took five times as long, 860 seconds. Note that in either case the distance traveled is the same.

Therefore, we can set up two equations, using the times given and letting distance and the rates of speed be unknown constants.

(I) Walking in the correct direction
$d = (r_{me} + r_{walkway})(172 \text{ seconds})$

(II) Walking in the wrong direction
$d = (r_{me} - r_{walkway})(860 \text{ seconds})$

Now we can solve the problem. The time it takes if I stand still on the walkway is given by the following equation.

$d = (r_{walkway})(\text{time if standing still})$

We can re-arrange the equation to solve for the time by dividing both sides by $r_{walkway}$. This gets us the next equation.

time if standing still = $d/r_{walkway}$.

Analogously, we can find out an equation for the time it takes to walk, not using the moving walkway.

time if not using the moving walkway = d/r_{me}

In short, we wish to solve for $d/r_{walkway}$ and d/r_{me}. We can actually do this from our equations (I) and (II). In summary, our problem is the following.

Solve for $d/r_{walkway}$ and d/r_{me}, given that:

(I) Walking in the correct direction
$d = (r_{me} + r_{walkway})(172 \text{ seconds})$

(II) Walking in the wrong direction
$d = (r_{me} - r_{walkway})(860 \text{ seconds})$

A little bit of algebra will get us to the answer. If we multiply the equation (I) by 5, and then subtract equation (II) from it, we eliminate the variable r_{me} to solve for the other variable. The result is the following.

5(I) - (II)
$4d = (r_{walkway})(1720 \text{ seconds})$
$d/r_{walkway} = 430$ seconds = the time it takes if I stood still

Similarly, we can we multiply the equation (I) by 5, and then add equation (II) to it to eliminate the variable $r_{walkway}$.

5(I) + (II)
$6d = (r_{me})(1720 \text{ seconds})$
$d/r_{me} = 1720/6 \approx 287$ seconds = the time it takes if I didn't use the moving walkway

The problem is solved, but let us take a moment to summarize the results, ranked from fastest to slowest options.

172 seconds = time if walking on moving walkway
287 seconds = time if walking, no moving walkway
430 seconds = time if standing still on moving walkway
860 seconds = time if walking against the moving walkway

The astute reader will notice something interesting about the results. It is faster to walk without any assistance than to stand still on the moving walkway. And yet, *this is what almost everyone does*.

Before taking the numbers too seriously, I should explain how I got the numbers for this problem. The O'Hare moving walkway is about 860 feet. I assumed it moved at 2 feet per second. The walking rate to construct this problem is 3 feet per second, which is just a tad over a very leisurely rate of 2 miles per hour (one mile every 30 minutes). So the numbers in this problem are fairly realistic.

Moving walkways actually slow us down?

I will end with a small tangent about scientific studies on the impact of moving walkways. As reported in The Telegraph, researchers found moving walkways at airports slow people down, as people often stand at busy times and their luggage can block movement. The time gained on the moving walkway is minimal, but the time wasted during congesting walkways is significant. That is, the moving walkway marginally increases speed when the airport is not busy, but it tends to slow everyone down otherwise (because of blockages and the fact that people like to stand on it).

New airports should take this lesson to heart. We as a world would probably move a little faster if we just walked, and the extra exercise is probably a good thing as well.

Puzzle 21: Wind Speed

Imagine you have just attained 1,000 miles per hour in a car on a one mile track, blowing away the official land speed record. You'd probably be pumped and ready to celebrate. But it's not quite time to break out the champagne.

You would actually be preparing the car and refueling. For you see, the official land speed record, as regulated by the Federation Internationale de l'Automobile, is based on the average speed for two runs *in opposite directions*.

The rule seems immediately fair, because if a factor like wind speed was beneficial in one direction, it would be detrimental in the other direction.

The situation raises an interesting mathematical question. If you were planning to break the land speed record, would you want the weather to have strong winds or no wind at all?

In other words, how does wind affect the average speed of the round trip?

Answer To Puzzle 21: Wind Speed

Initially it might seem that wind won't matter as its effect would "cancel out" between the two runs. Careful calculation shows that wind actually does matter. Specifically, wind will lower the average speed of the round trip.

Why is that the case? The reason is that speed is a rate, not an absolute quantity, and so equal gains and losses do not cancel out.

Let's do a simple example. Let's say your car runs at 55 mph, with wind at 5 mph. If wind simply added to your speed, that would mean you would drive 1 mile at 60 mph, and then you drive the 1 mile back at 50 mph. Doing the math, the average speed is not 55 mph. The time on the first run is 1/60 hour, and on the second it is 1/50 hour. For the round trip, this means it took (1/60 + 1/50) hours to drive 2 miles. The average speed for the round trip is therefore 54.5 mph.

We can generalize the equations to prove the round-trip with wind is slower.

With no wind, the average speed of the round trip is (2 miles)/(time 1 + time 2), where each time is (time) = 1 mile / (speed).

With wind, the two times will get changed. Let's say that wind adds a speed of *S* in the first run reduces it by the same amount in the other. Then (new time 1) = (1 mile)/(speed + S) and (new time 2) = (1 mile)/(speed - S).

The average speed for the round trip is:

(2 miles) / ((1 mile)/(speed + S) + (1 mile)/(speed - S)).

After a bit of algebra, this can be simplified to the following equation.

(no-wind average) * 1/(1 - $(S)^2$/$(\text{speed})^2$)

Now label X = $(S)^2$/$(\text{speed})^2$. We don't know the value of X, but we know that it will be non-negative since it is a squared term. Hence, the answer simplifies to the following equation.

average with wind = (no-wind average) * 1/(1 - X)

Since X is positive, this means the term 1/(1 - X) will be a fraction, and hence the average with wind will be a smaller number than the average without wind.

In short, the average speed of the round-trip is fastest with no wind.

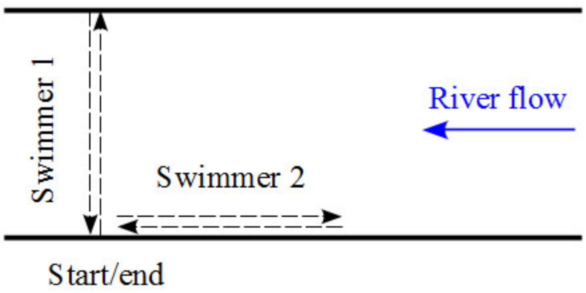
Puzzle 22: Michelson-Morley

In the 1800s scientists understood that light acted as a wave. Because sound waves travel through air, and tidal waves travel in water, they believed light waves traveled through a medium as well, what they called the *aether*. The theory implied that light from the sun had to “swim” in the aether to reach the earth. And just as a strong wind affects the propagation of sound, the presence of an aether wind would affect the flow of light. But how could one detect the aether and the speed of the aether wind?

In 1887, two scientists developed an apparatus to detect the aether wind. One of the scientists used a math puzzle to explain the principle behind the experiment. So consider the following problem.

Two equally strong swimmers race at a river. They start and end at the same point, but they take different paths. Swimmer 1 goes to the closest point across the river, 100 feet in width, and returns. Swimmer 2 goes directly upstream (against the current) for 100 feet and then returns by swimming downstream. Who wins the race?

Assume the river flows at 3 feet per second, and the swimmers have a speed of 5 feet per second.

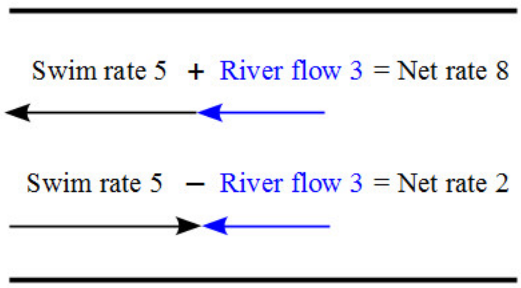


Note that both swimmers are influenced by the flow of the river. Swimmer 1 has to fight the flow of the river to make a straight line path across and back. Swimmer 2 is actively swimming against the river flow in one direction and then swimming with the river flow on the return trip.

Work out the puzzle. The solution will explain the connection of the problem to the aether wind.

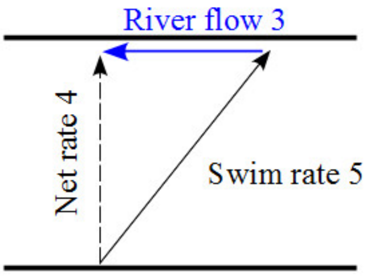
Answer To Puzzle 22: Michelson-Morley

The easier case is the swimmer 2, who swims upstream (against the current) in one direction and then downstream (with the current) in the other. The net speed is equal to the combined effect of the swimmer's speed and the river's flow.



Going downstream the swimmer's speed is the sum of his speed plus the river's flow (8 = 5 + 3), and going upstream his speed is his rate minus the river's flow (2 = 5 - 3). The time downstream is 100/8 = 12.5 seconds and the time upstream is 100/2 = 50 seconds. The total time is 62.5 seconds.

Swimmer 1 is trickier because he is swimming across the flow of the river. The path he swims will be the net result of his velocity and the river flow's velocity. We can think about this in terms of vector math: the swimmer's vector plus the river's vector must be equal a straight line vector.



In the diagram, the swimmer's vector is the hypotenuse and the river's flow is the short leg. The numbers have been chosen with a purpose: the swimmer's vector has a speed of 5 feet per second, the river's flow is 3 feet per second, and the vectors form a right triangle. So we must have a 3-4-5 right triangle! The swimmer's net speed is 4 feet per second. Note the size of the vectors will be the same for the trip across the river and the return trip, so the swimmer's speed is 4 feet per second on each trip.

Swimmer 1 takes 100/4 = 25 seconds for each of the two trips. In total, the cross-stream swimmer takes 50 seconds in all, which is shorter than the 62.5 seconds of the upstream swimmer.

The answer is swimmer 1 wins. And this turns out always to be true: swimmer 1, the cross-stream swimmer, always wins (so long as their swimming speed is faster than the flow of the river).

The Michelson-Morley experiment

In 1887 the scientists Albert A. Michelson and Edward W. Morley developed an apparatus to detect the aether wind. The device was called an interferometer, and it worked on the same principle as the swimming puzzle. (Michelson devised the puzzle to explain it to his children).

The interferometer used a set of mirrors to split up a beam of light into two beams. The two beams then "raced" just like the two swimmers. One beam of light (call it a red beam) was like the cross-stream swimmer that was sent to the closest point across a certain distance. The other beam (call it a blue beam) was like the upstream swimmer and it was sent the same distance against the aether wind.

If the aether wind existed, then the light beam that "swam" cross-stream would win every time. In other words, the red beam would arrive faster than the blue beam. This could be experimentally detected by the interaction of the light waves. The experimental results showed the two beams arrived at about the same time.

The experiment provided strong evidence that the aether wind did not exist and is celebrated as one of the most famous experiments in physics.

I read about this puzzle and its relationship to the experiment on the following website: http://galileo.phys.virginia.edu/classes/109N/lectures/michelson.html. The Michelson-Morley Experiment, Michael Fowler U. Va. Physics. 9/15/08.

Puzzle 23: Smartphone Battery

John's phone is a battery hog. It can last only 4 hours before the battery dies.

John has an identical spare battery. But there are days he needs even more time from his phone. So in desperation, John devises a battery swapping and charging scheme.

When the first battery dies, he immediately charges it while swapping in the spare battery. Then, when the spare battery dies, he immediately charges it while swapping in the other battery—which by that time has charged to 50%. He repeats this over and over until both batteries are dead.

How many hours can John get out of his two batteries with one phone charger?

Assume battery swaps are instantaneous, batteries discharge twice as fast as they charge, and both charging and discharging happen at a linear rate.

Answer To Puzzle 23: Smartphone Battery

At first you might be tempted to solve for the time each battery takes to die, and then sum up the infinite series of battery life times. I'll explain how to solve using that method. But there's a clever trick to solve it in a single linear equation! The trick takes some setup, but then the algebra is trivial.

The key is to remember the conservation of energy: the total amount of battery life discharged (energy used) is equal to the total amount of battery life charged (energy available, accounting for the initial charge on both batteries).

Energy used = Energy available

The energy used is the rate the batteries discharge times the discharge rate. The energy available is the battery charge time times the charging rate, plus the initial energy that both batteries had since they were charged at the start.

(discharge rate)(discharge time) = initial energy + (charge rate)(charge time)

Let's say the batteries have an energy of 1 when they are fully charged. Since the batteries discharge in 4 hours, they discharge at a rate of 0.25 energy units per hour. They charge at half that speed, or 0.125 energy units per hour. Both batteries are charged to start, so the initial energy is 2.

Let's say the total time until both batteries die is t. It takes 4 hours until the first battery dies, so the total time charging is t - 4. So we have the following equation.

$0.25t = 2 + (0.125)(t - 4)$

We can solve for t which is the total time until both batteries die.

$0.25t = 2 + 0.125t - 0.5$
$0.125t = 1.5$
$t = 12$

So John can get 12 hours from his two batteries, which is probably just enough to last for a day.

We can generalize this problem. If the charge rate is 50% of the discharge rate, John can always get 3 times whatever the battery life is. In other words, it's like John has a third spare battery!

Let's prove this. We again suppose a battery starts with 1 energy unit. If the battery life is B, then the discharge rate is $1/B$ energy units per hour, and the charge rate is half of that, or $0.5/B$ energy units per hour. The first charge starts after B hours, and the initial energy is still 2 since both batteries start charged.

So we solve the same problem for B.

$(1/B)t = 2 + (0.5/B)(t - B)$
$t/B = 2 + 0.5t/B - 0.5$
$0.5t/B = 1.5$
$t = 3B$

Thus, the total time is 3 times the battery life.

Further, we can generalize if the charging rate is a fraction $f < 1$ of the discharge rate. (If the charge time is as fast or faster than the discharge rate, then the battery charges until full each time, so the two batteries can last indefinitely). The rate of discharge is f/B energy units per hour.

$(1/B)t = 2 + (f/B)(t - B)$
$t/B = 2 + ft/B - f$
$(1 - f)t/B = 2 - f$
$t = B(2 - f)/(1 - f)$

So, for example, if the battery charges only 1/3 as fast as it discharges, then the total time is 2.5 the battery life.

Infinite Series

It is not too difficult to solve the problem explicitly using infinite series.

The first battery discharges in B hours. Let's call this time period t_1.

The spare battery then discharges in B hours. The next time period is t_2.

In the third period, the first battery has had B hours to charge, but only charges up a fraction f of the battery life. So the first battery will discharge in fB hours. This time period is t_3.

In the fourth period, the spare battery has had fB hours to charge (the time the first battery took to discharge in the last period), but only charges up a fraction f of the time. So the spare battery will discharge in f^2B hours. This time period is t_4.

Now we can see the pattern that the next period lasts f times as long as the previous one. That is, in each time period t_k, with $k > 1$, the battery discharges in $f^{k-2}B$ hours.

The total time until both batteries discharge completely is the sum of every time period.

$T = t_1 + t_2 + t_3 + \ldots$
$T = B + B + fB + f^2B + \ldots$
$T = B + B(1 + f + f^2 + \ldots)$

The sum of the infinite series in brackets is $1/(1 - f)$. Therefore, the total time is the following.

$T = B + B/(1 - f)$
$T = B(2 - f)/(1 - f)$

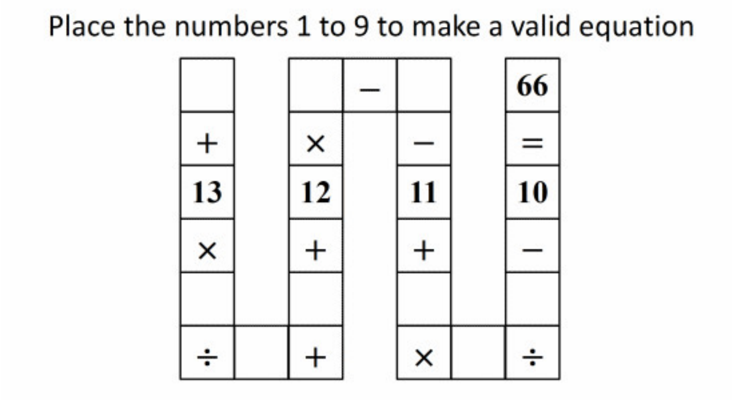
Note this is the same result as when we used the trick!

This problem is a variant of a puzzle from Cornell Engineering Magazine, brain teaser by Andy Ruina.

Puzzle 24: Vietnam Problem

This problem was posed to 3rd graders (8 year olds) in Vietnam. Many adults had trouble solving it and this problem was widely shared.

The problem is to place the numbers 1 to 9, using each number once, to make a valid equation. The expression is read from left to right wrapping around the corners like a snake.



Be sure to follow the order of operations: multiplication and division should be evaluated before addition and subtraction.

There are in fact many answers to this problem. See if you can figure out at least one before reading the solution.

Answer To Puzzle 24: Vietnam Problem

Here is one possible answer: fill the boxes, from the left, with the numbers 1, 2, 6, 4, 7, 8, 3, 5, 9. In all there are 136 possible answers found by computer. Before I present those, I will explain how I reasoned to find a single answer.

How many arrangements are there?

How many ways can you place the numbers 1 to 9 in the empty boxes?

There are 9 choices for the first box. Then there are 8 choices for the second box (as one number is already used up). Then there are 7 choices for the third box (as two numbers have been used), and so on, with each subsequent box having one fewer possibility.

In all there are 9x8x7x6x5x4x3x2x1 = 9! = 362,880 possible ways to arrange the numbers.

There are too many possibilities to check. So you'll have to think about the problem logically.

Write the expression in a line

A good first step is to re-write the expression in a single line. This will make it easier to understand the equation. I will use a slash mark (/) for the division symbol.

_ + 13 x _ / _ + _ + 12 x _ - _ - 11 + _ x _ / _ - 10 = 66

The next step is to consider the order of operations. Multiplication and division take precedence over addition and subtraction. So we can add parentheses around those steps so we know to evaluate them first.

_ + (13 x _ / _) + _ + (12 x _) - _ - 11 + (_ x _ / _) - 10 = 66

Now what do we do?

Guess and check!

I suspected the solution might be easy to find since it was given to 3rd graders. So I wondered: what would happen if I wrote the numbers 1 to 9 in ascending order?

1 + (13 x 2 / 3) + 4 + (12 x 5) - 6 - 11 + (7 x 8 / 9) - 10 = 52.889…

It was actually pretty close! What if I tried the numbers in descending order, from 9 to 1?

9 + (13 x 8 / 7) + 6 + (12 x 5) - 4 - 11 + (3 x 2 / 1) - 10 = 70.857…

Again this was pretty close! So I thought maybe I could adjust the numbers and get closer to 66.

Dividing by 7 was going to lead to a fraction. So I thought of swapping the 4 and 7. That didn't immediately work, so I thought of cycling the 8, 7, and 4 so I was dividing 4 by 8.

Since I would end up with 1/2, I thought of also shifting the 3, 2, and 1 around so I would have 3 divided by 2.

Here's what I got.

9 + (13 x 4 / 8) + 6 + (12 x 5) - 7 - 11 + (1 x 3 / 2) - 10 = 55

This was a whole number, which was good. And it was only 11 less than the desired result of 66. Then I saw an easy way to solve this: I could swap the 6 and the 5. This would increase the expression by -1 + 12 = 11 and get to 66.

9 + (13 x 4 / 8) + 5 + (12 x 6) - 7 - 11 + (1 x 3 / 2) - 10 = 66

So that is one answer.

Multitude of solutions

Then I thought about the expression.

9 + (13 x 4 / 8) + 5 + (12 x 6) - 7 - 11 + (1 x 3 / 2) - 10 = 66

The numbers 9 and 5 can obviously be swapped since addition is commutative, so that's another solution.

5 + (13 x 4 / 8) + 9 + (12 x 6) - 7 - 11 + (1 x 3 / 2) - 10 = 66

And since multiplication is commutative, the numbers 1 and 3 can be swapped.

5 + (13 x 4 / 8) + 9 + (12 x 6) - 7 - 11 + (3 x 1 / 2) - 10 = 66

We can get a fourth solution by swapping the 5 and the 9 again.

9 + (13 x 4 / 8) + 5 + (12 x 6) - 7 - 11 + (3 x 1 / 2) - 10 = 66

Besides those related solutions, there are other combinations of numbers that solve the puzzle too. Here are a few examples.

5 + (13 x 4 / 1) + 9 + (12 x 2) - 7 - 11 + (3 x 8 / 6) - 10 = 66

5 + (13 x 9 / 3) + 6 + (12 x 2) - 1 - 11 + (7 x 8 / 4) - 10 = 66

6 + (13 x 3 / 1) + 9 + (12 x 2) - 5 - 11 + (7 x 8 / 4) - 10 = 66

Each of these equations also has 3 related solutions when considering swapping numbers, as explained above.

How many solutions are there in all?

We know that any equation results in a set of 4 answers. Therefore, we know the total number of answers should be a multiple of 4. If someone told you there were 101 solutions, that wouldn't make sense because 101 is not a multiple of 4.

Brute Force calculation

I used a spreadsheet and tested out every single possible expression and found there are 136 solutions. This would be 34 distinct answers, each with 3 related equations from swapping numbers.

Here is my full list of answers, read from left to right. For example, the first set of numbers 1, 2, 6, 4, 7, 8, 3, 5, 9 corresponds to the expression 1+13x2/6+4+12x7-8-11+3x5/9-10 = 66.

So here are all 136 solutions to this problem.

1, 2, 6, 4, 7, 8, 3, 5, 9
1, 2, 6, 4, 7, 8, 5, 3, 9
1, 3, 2, 9, 5, 6, 4, 7, 8
1, 3, 2, 9, 5, 6, 7, 4, 8
1, 3, 2, 4, 5, 8, 7, 9, 6
1, 3, 2, 4, 5, 8, 9, 7, 6
1, 3, 4, 7, 6, 5, 9, 2, 8
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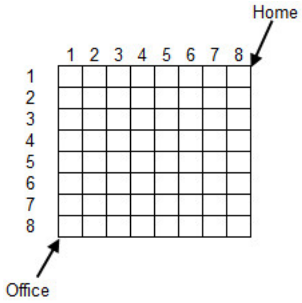
9, 6, 4, 3, 5, 8, 1, 7, 2
9, 6, 4, 3, 5, 8, 7, 1, 2
9, 8, 6, 2, 4, 1, 5, 7, 3
9, 8, 6, 2, 4, 1, 7, 5, 3

Source: the problem appeared in VN Express (Vietnamese) and I read about it from Alex Bellos via The Guardian.

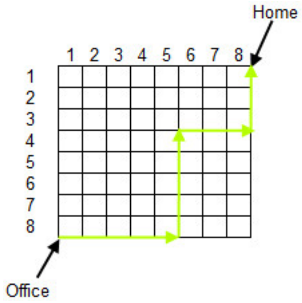
Part II: Medium Puzzles

Puzzle 1: Paths Home

Bob walks home from work. His home is 8 city blocks east and 8 city blocks north of his office. The walkways are exactly a rectangular grid.



One way Bob could walk home is by going 8 blocks east and then 8 blocks north. He could also walk 5 blocks east, 5 blocks north, then 3 blocks east and 3 blocks north, as depicted below.



Obviously there are many variations that Bob could take.

The question is, how many paths home are there, if Bob only walks east and north, one block at a time, and walks 16 blocks in total?

Answer To Puzzle 1: Paths Home

The trick is to approach this problem combinatorially. Bob's path can be represented as a list of 16 steps, each step being "east" or "north." For instance, walking 8 blocks east and then 8 blocks north is (east, east, east, east, east, east, east, east, north, north, north, north, north, north, north, north).

The question can be re-phrased in the following manner: how many ways are there to write a list of 16 items, where 8 of the items are east and the other 8 are north?

The problem translates into the number of ways to arrange 8 indistinguishable objects out of 16 (once you select 8 spots as "east" the remaining 8 spots will be "north"). The answer is 16 choose 8 = 12,870.

Puzzle 2: Two Sequences

Test your knowledge of patterns on the following two sequences.

Problem 1: What is the product of the following set of numbers?

(1 - 1/4)(1 - 1/9)(1 - 1/16)(1 - 1/25)(1 - 1/36)(1 - 1/49)

Can you generalize the sequence and the answer?

Problem 2: A sequence of numbers, following a logical rule, starts out as follows.

1, 11, 21, 1211, 111221, …

Can you figure out the next number?

Answer To Puzzle 2: Two Sequences

Here is the first sequence.

(1 - 1/4)(1 - 1/9)(1 - 1/16)(1 - 1/25)(1 - 1/36)(1 - 1/49)

Notice each term is 1 minus the reciprocal of a square. That is, each term is $(1 - 1/n^2) = (1 - 1/n)(1 + 1/n)$.

There is an interesting pattern if we write out each term in the expanded form. It turns out the middle terms are reciprocals of each other and will cancel out.

(1 - 1/4)(1 - 1/9)(1 - 1/16)(1 - 1/25)(1 - 1/36)(1 - 1/49)
= (1 - 1/2)(1 + 1/2)(1 - 1/3)(1 + 1/3)…(1 - 1/7)(1 + 1/7)
= (1/2) (3/2)(3/2) (4/3)(3/4) … (7/6)(6/7) (8/7)
= (1/2) (1) (1) (1) … (8/7)
= (1/2)(8/7)
= 8/14 = 4/7

We can generalize the sequence. When the product of consecutive integer terms $(1 - 1/n^2)$ ends on the square of k, we will have the following result.

$(1 - 1/4)(1 - 1/9)(1 - 1/16)\ldots(1 - 1/k^2)$
$=(1 - 1/2)(1 + 1/2)\ldots(1 - 1/k)(1 + 1/k)$
$= (1/2)\ (1)\ (1)\ (1) \ldots (1 + 1/k)$
$= (1/2)(1 + 1/k)$
$= (k + 1)/(2k)$

So the product is $(k + 1)/(2k)$.

The second problem is a different beast. There is a logical rule, but it is not as easy to see the pattern. Here is the sequence.

1, 11, 21, 1211, 111221, …

These numbers are the start of the look and say sequence.

The sequence starts out with 1. We read this number as “there is 1 term of 1.” Retaining only the numbers in this sentence, in order, gives 11, which is then the next term of the sequence.

Now we will read out 11 to get the next item in the sequence. We read this number as “there are 2 terms of 1.” We again retain only the numbers in this sentence to get the next term, 21.

You can see the pattern. Reading 21 out loud, we would say, “There is 1 term of 2 and 1 term of 1.” Retaining only the numbers in this sentence gives the next term 1211.

To solve the puzzle, we need to read 111221 as, “There are 3 terms of 1, 2 terms of 2, and 1 term of 1,” which gives 312211 as the next item in the sequence.

Puzzle 3: 100 Lamps

One hundred lamps are placed in a row on a long table. The lamps are labeled sequentially, with the first lamp numbered "1" and the last lamp "100."

The 1st person to enter the room hits the switch for each lamp, turning them all on. Then a 2nd person enters the room and hits the switch for every second lamp, which turns off the lamps numbered 2, 4, 6, etc.

A 3rd person hits the switch for every third lamp, numbered 3, 6, 9, etc. Note that some lamps get turned off—like lamp 3—while others get turned on—like lamp 6.

The pattern continues for the 4th, 5th, …, and 100th person entering the room, with the n^{th} person hitting the switch for the lamps numbered n, $2n$, etc.

After all 100 people are done, some lights will be on and some will be off. Will lamp 60 be on or off? Can you figure out exactly which lamps will be on and which will be off?

Answer To Puzzle 3: 100 Lamps

The solution could be found by simulating the experiment and seeing which lamps are on and off. But there is a more elegant method.

A simple example will illustrate. Consider lamp number 6. Which people will alter the state of this lamp? Clearly the 1st person will turn the lamp on, and then the 2nd person will turn it off. The 3rd person will then turn the lamp back on. Finally, the 6th person will turn the lamp off. No other person will alter the state of the lamp because every subsequent person starts at a lamp number higher than 6. Therefore, lamp number 6 ends up in the off position.

Which people altered the state of lamp 6? These were the 1st, 2nd, 3rd, and 6th persons. The connection should be made that these numbers are the factors of the number 6. Notice that the factors come in pairs because 6 = 1x6 = 2x3. The key observation is that each multiplicative pair corresponds to the lamp switch being hit two times in an (on position, off position) pairing.

Therefore, if a lamp number has an even number of factors, then it will end up in the off position. And if it has an odd number of factors, it will end up in the on position.

Returning to the question posed, the number 60 has 12 factors: the numbers 1, 2, 3, 4, 5, 6, 10, 12, 15, 30, and 60. This means the lamp gets turned on 6 times and turned off 6 times, so ultimately the lamp will be in the off position.

Will any lamp be on after all 100 people? The answer is yes. Certain lamp numbers only have an odd number of factors. For instance, consider the number 4 which has factors 1, 2, and 4. This is an odd number because the factor 2 pairs with itself: 4 = 1x4 = 2x2. The lamp number 4 ends up in the on position—it gets turned on by person 1, then off by person 2, and then on again by person 4.

Which other numbers have an odd number of factors? It is precisely the numbers where a factor pairs with itself, namely, the perfect squares. There are 10 perfect squares between 1 and 100 that will be in the on position at the end: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. All of the other lamps are composite numbers and will be off at the end.

Puzzle 4: Trailing Zeroes

The number 1,000 starts with the digit 1 and then finishes with 3 “trailing zeroes.” The number 10,000 has 4 trailing zeroes.

How many trailing zeroes are in the number 1000! = (1000)(999)…(1)?

Answer To Puzzle 4: Trailing Zeroes

What's the fastest way to solve this problem? You can get an answer from the website WolframAlpha. If you type the phrase "trailing zeroes in 1000!", you will get the answer of 249 immediately.

That is the correct computation, but the question is how did people do it before WolframAlpha?

Take a step back and consider why a number might have a trailing zero. The number 10 has 1 trailing zero because it is divisible by 10. The number 100 has 2 because it is divisible by 100 = 10^2, and the number 1,000,000 has 6 trailing zeroes because it is divisible by 10^6. A number like 1,010 has 1 trailing zero because it is divisible by 10 but not by 100 = 10^2.

The question of "how many trailing zeroes are there?" is really asking "what's the highest power of 10 that the number is divisible by?"

The number 1000! by its definition is clearly divisible by 10, 100, and 1,000, so to start, we know there are at least 3 trailing zeroes. In fact, since 10, 100 and 1000 are factors of 1000!, we know the number is also divisible by the product 10(100)(1000) = 10^6. Therefore, 1000! will have at least 6 trailing zeroes.

What's the highest power of 10 that 1000! is divisible by? Rather than continuing to guess and check, we can take a systematic approach. We wish to find the largest number n such that 1000! is divisible by 10^n. The idea, therefore, is to count the number of factors of 10 in 1000! For instance, the number 10 contributes one factor and the number 1,000 contributes 3 factors since 1,000 = 10^3.

Now comes the first trick. The number 10 is factored as 10 = 2 x 5. To find the largest number n such that 1000! is divisible by 10^n, we can equivalently find the largest n such that 1000! is divisible by $(2 \times 5)^n = (2)^n(5)^n$. In other words, we need to count how many factors of 2 and 5 are in the number 1000!

The second trick is to realize there will be an abundance of factors of 2 in comparison to 5. This is because every other number contains a factor of 2 whereas only every fifth number contains a factor of 5. So the problem reduces to counting the number of factors of 5 in the number 1000!

Now every 5th number contributes at least one factor of 5. That is, the numbers 5, 10, 15, …, 1000 all contribute one factor of 5. There are 1000/5 = 200 numbers that contribute a factor of 5.

Additionally, every 25th number contributes a second factor of 5, since 25 = 5^2. Therefore, the numbers 25, 50, …, 1000 all contribute a second factor of 5. There are 1000/25 = 40 such numbers.

The problem is solved by continuing this counting. Every 5^3 = 125th number contributes a third factor of 5, and there are 1000/125 = 8 such numbers. Finally the number 625 = 5^4 contributes a fourth factor of 5. (The next power of 5 is 3,125 = 5^5 which is larger than 1000 and so it is not a term in 1000!)

Tabulating these results, there are 200 + 40 + 8 + 1 = 249 factors of 5 in the number 1000! Each of these factors gets paired with a factor of 2, meaning there are 249 factors of 10 in the number 1000!

Therefore, 1000! is divisible by 10^{249} and it will have 249 trailing zeroes.

The same procedure works for finding the trailing zeroes in n! You can count the number of trailing zeroes by counting the factors of 5. This can be found from the expression $n/5 + n/25 + n/125 + \ldots$, which terminates at the largest power of 5 that is smaller than n.

Interesting fact: Peter Thiel, one of the co-founders of PayPal, told the New Yorker that he and co-founder Max Levchin would try to stump each other with math puzzles, citing this problem as an example.

Puzzle 5: Number Of Digits

How many digits are in the number 125^{100}?

Answer To Puzzle 5: Number Of Digits

Once again, this problem can be solved computationally. You can literally type in the phrase “digits in 125^100” on the website WolframAlpha to find the answer of 210 digits.

But that is no fun. The real puzzle is to solve the problem without using a calculator.

Function to find digits in a number

We will proceed with some examples to see if we can find a pattern. What we want is to find a function so that for any whole number x, we will have $f(x)$ = the number of digits in x.

If x is between 1 and 9, then the number only has 1 digit. Once we get to 10, the number has 2 digits. In fact, any number between 10 and 99 has 2 digits. It is at 100 that we move to 3 digits, and all the numbers from 100 to 999 have 3 digits.

We’re beginning to see a pattern. The critical point for a number having an extra digit is related to the powers of 10.

A number less than 10^1 has 1 digit
A number greater than 10^1 but less than 10^2 has 2 digits
A number greater than 10^2 but less than 10^3 has 3 digits
…
A number greater than 10^n but less than 10^{n+1} has n+1 digits

We’ve found a useful pattern. The question is now this: if we have a given number x, how can we determine which powers of 10 that is in between?

As luck would have it, there is a magical function that you already know that can do this: the logarithm! The logarithm (base 10) of a number x gives you the value y where $10^y = x$.

For instance, the logarithm of 150 is approximately 2.176, because $10^{2.176}$ is equal to 150. So we can conclude that 150 is between the powers 10^2 and 10^3, meaning that 150 has 3 digits. Another way of putting this: the answer 2.176 is between the numbers 2 and 3, and we need to round up to find the number of digits.

The mathematical function for “rounding up” is typically called the “ceiling” function. So that’s it: the function $f(x)$ = ceiling(log(x)) will provide us the number of digits in a whole, positive number x.

Actually there’s a problem with this function. For the number 10, we get ceiling(log(10)) = 1, and obviously 10 has 2 digits. So we can slightly modify the function. Instead of rounding up, we will round down and then add 1. The correct function is $f(x)$ = 1 + floor(log(x)) where “floor” means to round down.

Now we will apply this method to our original problem. We need to find log(125^{100}). Since 125 = 5^3, we can re-write this as log(5^{300}) = 300 log(5). If you know that log(5) is about 0.699, then you can estimate this is between 209 and 210.

Therefore, $f(125^{100})$ = 1 + floor(log(125^{100})) = 1 + 209 = 210. And so there are 210 digits in digits in 125^{100}.

Personally I’m satisfied with this answer. However, some people feel it is cheating to use a log-table or have any knowledge of the values of logarithms. So there is another way of solving this problem that is also quite clever.

Alternate method

I found this solution in a collection of difficult math problems. The idea is to think about the largest power of 10 less than 125^{100}, and that will involve a bunch of creative ways to re-write the expression.

The first step is to re-write the expression in terms of powers of 10 and 2, because 5 = 10/2.

$125^{100} = 5^{300} = 10^{300}/2^{300}$

It is commonly known that 2^{10} = 1024 (this is something computer scientists know, as everything is done in base 2. A kilo-byte is equal to 1024 bytes, not 1000 bytes).

We can also write 1024 = 1000 x 1.024 = 10^3 x 1.024 (aka “scientific notation”).

Therefore, we have $2^{300} = (10^3 \text{ x } 1.024)^{30} = 10^{90} \text{ x } 1.024^{30}$. Putting this all together, we find the following:

$125^{100} = 10^{300}/2^{300} = 10^{300}/(10^{90} \text{ x } 1.024^{30})$

We cancel the powers of 10 to get:

$125^{100} = 10^{210}/1.024^{30}$

Now we need to figure out how big 1.024^{30} is. This will require a bit of estimation from the binomial theorem. We have $(1 + x)^{30} = 1 + 30x + 435x^2 + \ldots$

The key is that $x = 0.024$ is a small number, so higher and higher powers of it will tend to zero. In this example, the term $30x = 0.72$, and $435x^2$ is about 0.25, and the next term will be something like 0.05. It is evident the terms are getting smaller quickly so we can ignore higher powers. Therefore, we can conclude that 1.024^{30} will be larger than 1, but it will be less than 10.

Therefore, we can bound $125^{100} = 10^{210}/1.024^{30}$ from below by dividing by 10, and we can bound it from above by dividing by 1.

This means that 125^{100} will be less than 10^{210} but it will be greater than 10^{209}. Therefore, there are 210 digits in the number 125^{100}.

Puzzle 6: Odd Even Sequence

From the numbers 1, 2, … n, start by writing an odd number, and keep writing larger numbers—alternating between even and odd—until you stop at n. Call this an *odd-even* sequence of n.

For example, if $n = 3$, then the only odd-even sequences are 123 and 3.

If $n = 8$, then the sequences 1278 and 58 are odd-even, but other sequences are not, like 278 (starts with even number), 138 (two odd numbers in a row), 1438 (going from 4 to 3 is a smaller number).

For an arbitrary n, can you figure out a pattern for how many odd-even sequences there are?

Answer To Puzzle 6: Odd Even Sequence

First, let's list out some sequences

n = 1, sequences = 1

n = 2, sequences = 12

n = 3, sequences = 123, 3

n = 4, sequences = 1234, 14, 34

n = 5, sequences = 5, 345, 145, 125, 12345

n = 6, sequences = 56, 36, 3456, 16, 1456, 1256, 123456, 1236

n = 7, sequences = 7, 567, 347, 367, 34567, 147, 14567, 127, 12567, 12367, 12347, 1234567, 167

Counting the number of sequences for each case, the sequence is 1, 1, 2, 3, 5, 8, 13, …

Remarkably, this is the Fibonacci sequence!

Write f(n) for the number of odd-even sequences of n. Now we need to understand why f(n) = f(n - 1) + f(n - 2). There is a correspondence between the number of sequences as follows. From a list of sequences of n - 1, we can append the number n to form an odd-even sequence of n. Therefore, there are at least f(n - 1) sequences for n. Similarly, for any sequence of n - 2, we can replace the last term n - 2 with the number n (this is legal because both n and n - 2 have the same parity). These sequences will all be different because none of them include the number n - 1. So we have f(n) is at least f(n - 1) + f(n - 2). Doing the logic in reverse will show that f(n) is at most f(n - 1) + f(n - 2): from a list of sequences of n, every sequence is either a concatenation of a sequence from n - 1 or it is a modification of one ending in n - 2.

Therefore f(n) = f(n - 1) + f(n - 2).

This puzzle is a variation of a problem on qbyte.

Puzzle 7: Divisible By 3

Let’s play a little game. Write down 5 positive whole numbers. Any 5 will do. If you hand me the list, there is an interesting thing I can do.

I can always select 3 of the numbers you wrote and their sum will be divisible by 3.

For instance, let’s say you wrote down 1, 2, 3, 4, 5. This is an easy one. When you hand me the list, I can quickly identify 1 + 2 + 3 = 6 which is divisible by 3.

If you instead wrote a different list like 11, 23, 7, 19, 54, the game will be a bit harder for me. But with a bit of work, I can figure out that 11 + 19 + 54 = 84, which is divisible by 3.

The challenge is: why is this always the case? Why is it that in a list of 5 positive whole numbers there is always a group of 3 numbers that have a sum divisible by 3?

Answer To Puzzle 7: Divisible By 3

It is instructive to work on a simpler problem before tackling the puzzle at hand. We will first prove that in any list of 3 positive whole numbers, there is a group of 2 numbers whose sum is divisible by 2.

Working out divisible by 2

This problem is much easier to work out. Suppose the first number in the list is even. If either the second or the third number is even, then that will be a pair of even numbers. Clearly their sum will be even (since even + even = even) and we are done. If neither the second nor third number is even, then that means both the second and third number must be odd. In that case, the second and third number will have an even sum (since odd + odd = even).

What if the first number in the list is odd? A similar argument works here. Either one of the second or third numbers is odd—creating a pair with an even sum (since odd + odd = even)—or both the second and third numbers are even—again creating a pair with an even sum (since even + even = even).

The proof can also be explained more succinctly using the terminology of number theory. Since all we care about is whether the numbers are divisible by 2, we can work modulo 2. Therefore, each of the three numbers is either 0 or 1. There are 3 numbers to be placed into two "slots" of 0 or 1, so by the pigeonhole principle at least two of the numbers are equal to 0 or two of the numbers are equal to 1. In either case the pair has an even sum.

(The pigeonhole principle states that if $n + 1$ pigeons are to be put into n pigeonholes, then at least one pigeonhole has more than one pigeon. An equivalent statement is the maximum is at least the average, or the minimum is at most the average.)

Divisible by 3

We will use the pigeonhole principle to solve the problem. Each of the 5 numbers can be written in modulo 3 since all we care about is divisibility by 3. Each of the 5 numbers will either be 0, 1, or 2.

The first case is if the list has one of each number. In that case, we are done since 0 + 1 + 2 = 3 which is divisible by 3.

So assume the list only contains two of the possible remainders (either 0, 1 or 0, 2 or 1, 2). In that case, we have 5 numbers that need to be placed into two "slots." By the pigeonhole principle, one of the slots must contain at least 3 numbers from the list. This triplet of numbers will be divisible by 3 (because 0 + 0 + 0 = 0, 1 + 1 + 1 = 3, and 2 + 2 + 2 = 6, and each sum is divisible by 3).

Therefore, in any list of 5 positive whole numbers, there is a group of 3 numbers that is divisible by 3.

As a concluding remark, the result can be generalized that in a group of $2n - 1$ numbers, there is always a subset of n numbers whose sum is divisible by n. This is known as the Erdos-Ginzburg-Ziv Theorem and is a bit more complicated to prove.

Credit: I first read about this puzzle on the Harvard University Physics Department Problem of the week. The link I have is broken, but this was problem 80 "divisible by 9" (possibly accessible from the Wayback Machine.)

Puzzle 8: Two Be Or Not Two Be

Out of the numbers 0 to 9,999,999, which is larger: the set of numbers where the digit 2 appears at least once or the set of numbers where none of the digits is equal to 2?

Answer To Puzzle 8: Two Be Or Not Two Be

Write each number as a 7-digit number, so that a number like 8 is written with six leading zeroes as 0000008.

A number that does not contain 2 is written using any of the other 9 digits: 0, 1, 3, 4, 5, 6, 7, 8, and 9.

For a number that does not include the digit 2, there are 7 digits that can be any of the 9 digits not equal to 2. So there are 9 choices for the first digit, then 9 choices for the second digit, and so on for each of the nine digits. Therefore, there are 9(9)(9)…(9) = 9^7 = 4,782,969 numbers that do not contain the digit 2.

The remaining 10,000,000 - 4,782,969 = 5,217,031 are numbers that have at least one appearance of the digit 2.

Hence, the numbers with the digit 2 are more common.

An alternative method is to count the numbers that contain the digit 2. In a 7 digit number, a number with exactly 7 - x digits equal to number 2 can be chosen in [7 choose (7 - x)]9^x = (7 choose x)9^x ways. This is because there are (7 choose x) spots to choose the digits not equal to 2, and each of those can be any of 9 possible digits. We want at least one appearance of the digit 2, so we need to sum this formula from x = 0 to x = 6. Doing this on WolframAlpha we arrive at the same answer of 5,217,031.

Puzzle 9: Infinite Fraction

What is the value of x in the following equation?

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Answer To Puzzle 9: Infinite Fraction

The trick is noticing a repetition in the fraction. The quantity below the fraction is itself equal to x.

$$x = 1 + \cfrac{1}{\boxed{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = x}$$

This means an answer satisfies $x = 1 + 1/x$. Multiplying both sides by x and re-arranging terms, we get a quadratic equation $x^2 - x - 1 = 0$.

The solutions are $x = (1 + \sqrt{5})/2 \approx 1.618$ and $x = 0.5(1 - \sqrt{5}) \approx -0.618$.

Obviously x is the sum of positive terms, so we can rule out the negative solution as an extraneous solution.

Therefore, $x = (1 + \sqrt{5})/2$, and the golden ratio comes out of nowhere!

Incidentally, such infinitely repeated fractions are known as continued fractions and they have some interesting mathematical properties.

Puzzle 10: Nested Radical

What does the following expression equal?

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}}$$

Answer To Puzzle 10: Nested Radical

Here is a derivation of the answer.

$$x = \sqrt{2 + \sqrt{2 + \sqrt{\dots}}}$$

$$x^2 = 2 + \sqrt{2 + \sqrt{\dots}}$$

$$x^2 = 2 + x$$

$$(x+1)(x-2) = 0$$

$$x = 2$$

We can solve the expression for a general constant $a > 0$ using the quadratic formula:

$$x = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{\dots}}}}$$

$$x^2 = a + \sqrt{a + \sqrt{a + \sqrt{\dots}}}$$

$$x^2 = a + x$$

$$x = \frac{1 + \sqrt{1 + 4a}}{2}$$

Now consider the long-term behavior of this function compared to the simple square root function.

$$\lim_{a \to \infty} \frac{\frac{1+\sqrt{1+4a}}{2}}{\sqrt{a}} = 1$$

In other words, the nested radical converges to the more familiar square root function.

Puzzle 11: Infinite Exponents

Solve for *x* in the following equation:

$$x^{x^{x^{\cdots}}} = 2$$

Answer To Puzzle 11: Infinite Exponents

Once again, there is a trick that the entire exponent tower is the same as the whole, so we can solve as follows.

$$x^{x^{x^{\cdots}}} = 2$$

$$\implies x^{x^{x^{\cdots}}} \ln x = \ln 2$$

But since $x^{x^{x^{\cdots}}} = 2$

$$2 \ln x = \ln 2$$

$$x = e^{0.5 \ln 2} = \sqrt{2}$$

You can also solve the problem in another manner. Denote the exponent $u = x^{x \cdots}$. We know that $x^{x \cdots} = 2$, and so:

$x^{x \cdots} = x^2 = 2$

The answer is $x = \sqrt{2}$

Here an intuitive explanation of the solution (this is not a way you could have found the answer, but it somewhat explains why the answer is true).

Remember that when you raise a number to a power, you end up multiplying the exponents.

That is, $(x^a)^b = x^{ab}$

Since $\sqrt{2} = 2^{1/2}$, consider what happens when this number is raised to the power of 2.

We get $(2^{(1/2)})^2 = 2^{(1/2)2} = 2$

In the expression with infinite exponents, we have $2^{1/2}$ being raised to itself over and over again. Each time that $x = 2^{1/2}$ is raised to another power $x = 2^{1/2}$, the exponent in the base term (1/2) gets multiplied by the base term from the next exponent (2). These terms cancel out to 1, and the process repeats with each successive power. In the end, the expression turns out to be 2 to the power of 1, which is 2.

I should clarify everything done above is contingent on the infinite tower of exponents converging. If you do the same problem with the infinite exponents equaling 4, then you would get $x^4 = 4$, which is also an answer of $x = \sqrt{2}$. This is nonsense, since we just found out that $x = \sqrt{2}$ converges to a value of 2.

So to be more careful, we must consider when the tower converges in the first place. If the infinite tower converges to a value y, then our value x must satisfy the equation $x^y = y$. Thus we have $x = y^{1/y}$.

Now we can do a bit of work to show this converges when x is between the values of e^{-e} and $e^{1/e}$, which is roughly 0.066 to 1.44. So the infinite tower only has a solution when it is set to a value between $1/e$ and e, roughly between 0.368 and 2.718. That's why we got a nonsensical result when we set the equation equal to 4: that equation does not have a solution.

Puzzle 12: Guess My Polynomial

I'm thinking of a polynomial $f(x)$ where the coefficients can only take the values of -1, 0, and 1.

I will give you information. For any number a you submit, I will tell you the value of $f(a)$. But each time you ask me a number it will cost you a dollar.

What's the least cost method to figure out my polynomial?

Answer To Puzzle 12: Guess My Polynomial

Remarkably you can figure out my polynomial after asking for one number! The short answer is you need to ask for $a = 3$ and convert $f(3)$ to base 3. From there it will be easy to deduce the polynomial.

Let's do an example to see why. Suppose I tell you that $f(3) = 9$. The base 3 representation of the decimal number 9 is $(\mathbf{100})_3$ because $9 = \mathbf{1}(3)^2 + \mathbf{0}(3)^1 + \mathbf{0}(3)^0$. Since the base 3 representation of a number is unique, it must have been that $f(x) = \mathbf{1}(x)^2 + \mathbf{0}(x)^1 + \mathbf{0}(x)^0$.

In fact, whenever the base 3 representation of $f(3)$ contains only 0's and 1's, we can simply read off the base 3 representation as the coefficients of the polynomial.

Things are only slightly trickier when the base 3 number contains some 2's. For example, let's say $f(3) = 8$. We again convert into base 3 to get $f(3) = (22)_3$. If our polynomial could have coefficients of 0, 1, and 2, then we could just read off the values of the base 3 number as the coefficients. That is $f(\mathrm{x})$ would be $2x + 2$.

But our polynomial can only take values of 0, 1, and -1. So we need to do one more conversion: if the value for 3^k is 2, we convert that coefficient into a -1 by subtracting $3(3^k)$ and then we will add 3^{k+1}, which has the effect of increasing the placeholder for 3^{k+1}. This number base system is known as [balanced ternary](). By this process we can convert every unique representation in base 3 to a [unique representation]() in balanced ternary. Therefore, the value of $f(3)$ in balanced ternary will give us the coefficients of the polynomial.

In our example, $f(3) = 2(3) + 2$. Now we convert to balanced ternary. First we will change the 2 in the units value into a -1 by adding and subtracting 3. We will do this step in detail to illustrate the process.

That is, $f(3) = 2 + [(-3) + (3)] + 2(3) = (2 + -3) + (3 + 2(3)) = -1 + 3(3)$.

Now we write $3(3) = 3^2$ and include placeholders for 3^1 and 3^0. So we have $f(3) = (1)3^2 + (0)3^1 + (-1)3^0$, and therefore our balanced ternary representation is (1 0 -1). Our original polynomial must have been $f(x) = \mathbf{1}(x)^2 + \mathbf{0}(x)^1 + \mathbf{-1}(x)^0$.

To prove this procedure works generally, note the following.

For a polynomial $f(x) = c_n x^n + \ldots c_1 x + c_0$,

$f(3) = c_n 3^n + \ldots c_1(3) + c_0$

$f(3)$ in base 3 = $(c_n \ldots c_1\ c_0)$

And we can find the balanced ternary by converting from the unique representation in base 3.

Let's do a final example to illustrate once more. Let's say $f(3) = 99$. We have $99 = (10200)_3 = (1)3^4 + (2)3^2$. We need to get rid of the 2 so we add and subtract 3^3.

That is, $(1)3^4 + [3^3 - 3^3] + (2)3^2 = (1)3^4 + 3^3 + [-3^3 + (2)3^2] = (1)3^4 + (1)3^3 + (-1)3^2 + (0)3^1 + (0)3^0$.

Therefore $f(3) = (1\ 1\ {-1}\ 0\ 0)$ in balanced ternary, and $f(x) = x^4 + x^3 - x^2$.

And a fun fact of trivia, the Soviets developed a computer using [balanced ternary]() as an alternative to binary. But it seems whatever computational advantages it had did not stop binary computing from being adopted.

Puzzle 13: Find The Missing Number

Microsoft used to ask this problem in interviews to test knowledge of algorithms. I've re-characterized the puzzle in terms of a situation involving an Excel spreadsheet.

Your boss sends you a spreadsheet with a list of tasks. In one column your boss has written the task, and in the column next to it he has ranked each task in importance from 1 to 100. The list is unordered.

Your boss, however, is incompetent, and sends you a list with only 99 items—so you know one of the items is missing.

How can you quickly find out which task number is missing?

There are many ways to do this. The puzzle is to come up with an efficient algorithm.

Bonus: What if there are 2 tasks missing?

Answer To Puzzle 13: Find The Missing Number

A brute force method is to check if the list contains task 1, then task 2, and so on. This is obviously inefficient as you have to keep looping through the list.

A much better method is to sort the list from 1 to 100 and quickly scan to see which number is missing.

However, there are two problems with this method. First, it becomes impractical as the list gets bigger: you might be able to eyeball over 100 items, but you can't really scan 1,000 items very efficiently. Second, you are limited by how quickly Excel can sort the list**. For these reasons, sorting is not the best idea.

This is a technical point which I will elaborate. Computer scientists quantify the amount of time an algorithm will take to run, called a time complexity analysis. The time is related to the size of the data input n. For instance, let's say that I give you a list of n numbers and your job is to tell me the first number in the list. The time it takes only depends on how quickly you can access the first element—it doesn't matter if the list is one number or a million numbers. So this task can be done in a **constant amount of time, regardless of how big the input n is. On the other hand, let's say I ask you to find the largest item in the list. Now clearly this task will take longer, and it will also take longer as the list gets larger. In fact, the time it takes will be directly proportional to the list size—this is an example of **linear** time. If I ask you to sort the list in ascending order, how long will that take? That's actually a somewhat loaded question, as there are many ways you can sort the list. Regardless of the method, however, sorting is generally done in a time that is slightly worse than linear time. There is shorthand for various time classifications called big O notation. Constant time is denoted as O(1), linear time is O(n), and sorting is linearithmetic time is O(n log n).

So where does that leave us? How can we find the missing task number quickly?

There is a very neat algorithm that will find us the solution in linear time, and it can be performed using a single formula. In Big O notation, this algorithm works in linear O(n) time and uses a constant O(1) amount of memory.

The trick is to use a famous formula for the sum of the numbers from 1 to n, which is $n(n + 1)/2$.

So here's the algorithm: we will evaluate the formula:

= 100(100+1)/2 - SUM(task numbers)
= 5050 - SUM(task numbers)

Why does this formula work? Let's say the missing number is X. Then the formula SUM(task numbers) equals sum of the numbers from 1 to 100 excluding the missing task number. So we have SUM(task numbers) = 5,050 - X. Therefore, 5,050 - SUM(task numbers) = X so we exactly retrieve the missing number!

For a list of numbers from 1 to n, we would have the formula = $n(n+1)/2$ - SUM(task numbers).

This task will run as quickly as it takes to sum up n numbers, and that can be done in linear time. Furthermore, you only need a constant amount of memory to sum up n numbers.

Bonus: what if 2 numbers are missing?

A first approach is to find out the sum of the two missing numbers—using the above method—and then to find product of the two missing numbers, since we know the product of the first n numbers is n factorial. Then you'll have 2 equations in 2 variables.

Sum of missing: = $n(n + 1)/2$ - SUM(list)
Product of missing: = $n!$ - PRODUCT(list)

This is a very bad approach because the product $n!$ gets astronomical even for a modestly sized list of 100 items. Plus, you still have to solve for the 2 variables, and that's going to take more time and memory.

A better method is described in Data Streams: Algorithms and Applications by S. Muthukrishnan. The trick is to use another well-known formula: the sum of the squares of the first n numbers. This is $n(n + 1)(2n + 1)/6$.

Sum of missing: = $n(n + 1)/2$ - SUM(list)
Sum of squares of missing: = $n(n + 1)(2n + 1)/6$ - SUM(squares in list)

If the formulas return A and B, and the missing numbers are x and y, you end up needing to solve the two equations.

$$x + y = A$$
$$x^2 + y^2 = B$$

And these types of equations can be solved with relative efficiency (if interested, read more about it in the Stack Overflow discussion.)

The method can be generalized if k numbers are missing by adding up the sum of the cubes, fourth powers, and so on to the k^{th} power.

Puzzle 14: Poisoned Beer

Before a big celebration, a king receives a delivery of 1,000 kegs of beer. But there is a slight problem: one of the kegs has been poisoned.

The bad beer is indistinguishable in every way, except that it causes food poisoning about one hour after ingested.

The king is short on time and bodies. The party is in a couple of hours and he only has 10 guards to spare.

How can the king identify the bad keg?

Answer To Puzzle 14: Poisoned Beer

If the king had 1,000 guards, each guard could sample from a different keg. After one hour, only one guard would get sick, and that guard would uniquely identify the poisoned keg.

But the king has only 10 guards. So each guard needs to take a sample from multiple kegs. After one hour, some of the guards will get sick. The question is, can we then figure out which keg was poisoned? Can we give the beer samples so each combination of guards getting sick corresponds to a unique keg?

The answer is yes, and the trick is to use binary numbers.

Solving with 8 kegs and 3 guards

First let me explain the procedure in a simpler example, which can be generalized to solve the puzzle. Imagine there are 8 kegs and the king only has 3 guards.

Number the kegs from 0 to 7, and number the guards from 0 to 2. Here is how the guards should sample the kegs.

Guard 0 samples from 1, 3, 5, 7
Guard 1 samples from 2, 3, 6, 7
Guard 2 samples from 4, 5, 6, 7

Then see what happens after one hour. If none of the guards get sick, then the poisoned keg would be 0. Otherwise, some of the guards might get sick. Here's the neat part: every combination of guards that gets sick uniquely identifies one of the kegs!

For example, if guards 0 and 1 get sick, then that means keg 3 is poisoned. Why? The only kegs that both mutually drank were 3 and 7. But if keg 7 was poisoned, then guard 2 would have also gotten sick. So if only guards 0 and 1 get sick, then the poisoned keg must be keg 3.

You can verify that every combination of guards getting sick uniquely identifies a keg number. And similarly every keg number will uniquely sicken a subset of guards.

How did I come up with this sampling procedure? The method was to use binary numbers (base 2), which is the number system used in computers. The numbers 0 to 7 can be encoded in binary as follows.

0 = 000
1 = 001
2 = 010
3 = 011
4 = 100
5 = 101
6 = 110
7 = 111

Guard 0 then samples from a keg if the rightmost digit is a 1; guard 1 samples from a keg if the middle digit is 1; and guard 2 samples from a keg if the leftmost digit is 1.

In other words, here is the sampling procedure when the kegs are labeled using binary numbers.

Guard 0 samples from 001, 011, 101, 111
Guard 1 samples from 010, 011, 110, 111
Guard 2 samples from 100, 101, 110, 111

Notice we could do 8 kegs with 3 guards because $8 = 2^3$.

The puzzle asked us to do 1,000 kegs with 10 guards. This will be possible because $2^{10} = 1{,}024$, so we can actually test 1,024 kegs in this way. Let's explain the process in detail for completeness.

Solving the 1,000 keg case

First, number the beer kegs from 0 to 999. Then convert the beer keg numbers from decimal into binary. For example the beer keg 997, as a decimal number, would be written as 1111100101, as a binary number.

Number the 10 guards from 0 to 9. Then make guard k take a sample of a keg N if and only if the value of 2^k is 1 in the binary representation of N. In other words, guard k takes a sample from a keg if the corresponding digit of N in binary is equal to 1.

The keg 997, for example, has a binary representation 1111100101 = $1(2^9) + 1(2^8) + 1(2^7) + 1(2^6) + 1(2^5) + 0(2^4) + 0(2^3) + 1(2^2)+ 0(2^1) + 1(2^0)$.

Thus, only the guards 0, 2, 5, 6, 7, 8, and 9 drink a sample of beer from keg 997, and the other guards do not drink a sample from keg 997.

This process is done for each keg. The tasting should be done quickly, and each guard should only get a minimal tasting, since many guards will have to sample from many kegs of beer.

After one hour, if none of the guards are sick, then that means keg 0 is poisoned.

Otherwise, some of the guards will be sick, and this data can be used to uniquely figure out the poisoned keg. Create a binary number using the number of the guard for its position and write a 0 for each non-sick guard and a 1 for each sick guard. There is a unique binary number for each combination of guards getting sick, and this number uniquely corresponds to the poisoned keg.

So even with a few hours and 10 guards, the king can identify the 1 bad barrel from 1,000!

The solution is beautiful because the king can minimize the number of guards needed to find the bad keg. For N kegs, the king can find the bad keg using only $\log_2 N$ guards.

Credit: I heard about this puzzle as the Sultan’s Wine Bottle.

Puzzle 15: 100 Passengers

On a plane with 100 seats, 100 passengers board one by one. The first passenger loses his boarding pass and takes a random seat. The remaining passengers board the plane, and either

—take their assigned seat, if available
—take a random seat

This is a civil boarding process: no one puts up a fuss, knowing there are enough seats for everyone.

The math question is, what's the chance the last passenger to board ends up in the original assigned seat?

Answer To Puzzle 15: 100 Passengers

It would be very difficult to work out the conditional probabilities for each possible seating.

Notice if the first passenger gets in the correct seat, everyone will end up in the correct seat. What if the first passenger sits incorrectly? Then everyone sits in the correct seat until the person whose seat is already occupied. If that person sits in the first passenger's seat, then everyone following can sit in the correct seat. This pattern repeats for each person—basically either sit in the correct seat or fill a wrong seat. If ever someone fills the first passenger's seat, then everyone following can sit in the correct seat.

Therefore, the key observation is this: the last passenger will always end up sitting in either the correct, assigned seat OR will end up with the assigned seat of the first passenger. At every step of the seating process, these two seats can be filled with *equal probability*. Therefore, the probability of the last person getting the correct seat is 1/2.

I have seen this puzzle in a variety of sources, with some answers quite mathematically challenging. The answer I enjoy is credited to Peter Winkler's book *Mathematical Puzzles: A Connoisseur's Collection.*

Puzzle 16: 100 Floors

This is a problem that's been used as an interview puzzler at Google and some other companies.

You are in a 100 floor building and you have 2 identical marbles. Your job is to find out how strong the marbles are.

You are allowed to go to any floor and drop a marble to the ground. If the dropped marble does not break, you know the marble would not break from any lower floor. Also you can use the marble in another experiment. If the marble breaks into pieces, you cannot use it again, but you know it would break if dropped from any higher floor.

Your job is to find the highest floor where a marble can be dropped and does not break.

How can you do this efficiently? Your boss wants you to come up with a method where the worst case is minimized. What's the answer?

Answer To Puzzle 16: 100 Floors

The first trick to the puzzle is realizing that you have an advantage with 2 marbles. To see this, imagine that you only have 1 marble. Now you can only find the answer by brute force: you need to drop the marble from each floor, starting with the first floor, and proceeding higher and higher until you find out where the marble breaks (or doesn't break). In the worst case, you'll drop the marble on all 100 floors yielding 100 drops.

But with 2 marbles you can do much better by searching more aggressively. Let's say you drop a marble from the 50th floor. If it breaks, you know the answer is a lower floor than 50. So you can use the other marble and successively try floors 1, 2, .., 49 until you figure out where the marble breaks. On the other hand, let's say the marble does not break when dropped from the 50th floor. Now you know the answer must be higher than 50. In other words, you've eliminated all of the floors 1 to 50 using a single drop. So you only need to test the remaining 50 floors. Even if you test each floor one by one, that's 51 drops in all as a worst case. So we've already cut the worst case down from 100 with 1 marble to 51 using 2 marbles and a naive search strategy.

Can we do any better? It turns out we can.

Group searching

One search strategy is to jump up many floors with one marble, and then once that marble breaks do a linear search. For example, we can use one marble to test floors 25, 50, 75 and 100. If that marble survives floor 25 but breaks on floor 50, then we can test floors 26, 27, …, 49 with the other marble to find the answer.

The idea is to use one marble to do a rough search—so we can eliminate many floors at once—and then use the other marble to pinpoint the exact floor.

What's the best way to do the rough search? We can write out an expression that minimizes the expected number of drops. Our search method is to:

—Use one marble for floors k, $2k$, …, 100

—If the marble first breaks at floor X, we use the second marble to test floors $X - k + 1$, $X - k + 2$, …, $X - 1$

We assume the marble is equally likely to break at any of the 100 floors. The first marble is equally likely to break in any of the $100/k$ planned drops. So our average case will be halfway which is $(100/k)/2 = 50/k$ drops. Conditional on that, the second marble is expected to be tested in an additional $k - 1$ planned drops, which means an average of $(k - 1)/2$ planned drops.

The expected number of drops is the following expression.

E(drops) = (Average Marble 1) + (Average Marble 2)
$E(\text{drops}) = (50/k + k/2) + (k - 1)/2$

We can minimize this by taking the derivative with respect to k and setting that equal to zero.

$0 = -50/k^2 + 1/2$
$100 = k^2$
$k = 10$

Thus, $E(\text{drops}) = 50/10 + 9/2 = 9.5$

The group search means we should drop one marble from floors 10, 20, …, 100. Once the marble breaks, we test the previous 9 floors to find the answer.

The average case is 10 drops. However, the worst case is not that great: if the answer is 99, it would take 19 drops to figure this out (we'd use 10 drops for 10, 20, …, 100, then use another 9 with the second marble).

Is there any way to do better?

A modified group search

I admit I did not come up with this solution, so I don't think I would have passed the interview. However, it is quite an interesting method and worth understanding.

The problem with the group search is that it performs worse on higher floors than on lower floors. This is why it's good for the average case, but it is not good for the worst case scenario. And many times in life we want to minimize the worst case.

The general idea of the group search is good. But we need to modify it so the higher floors don't need as many drops. The way we can do that is to cheat the group search *by reducing the jumps after each drop*.

Let's say we make our first drop at floor X. If the marble breaks, we search the lower floors with a worst case of X drops in all. If it doesn't break, the group search would say to make the next drop at floor $2X$. But the modified group search says to make it at one floor less, $2X - 1 = X + (X - 1)$. The idea of reducing the jump is to account for the fact we already used 1 drop to test out floor X. This way we keep our worst case at X. Similarly, if the marble doesn't break at $2X - 1$, then our next drop should be at floor be $3X - 2 = X + (X - 1) + (X - 2)$, as we've used 2 drops to test out X and $X - 1$.

So what value of X will be best? There is a tradeoff here. On the one hand, we need to start high enough so that we will eventually test all 100 floors. (If you try $X = 10$, you'll see that you run out at 55 by testing floors 10, 19, 27, 34, 40, 45, 49, 52, 54, 55).

On the other hand, we wish to start as low as possible, since our worst case is X drops and we want to minimize the number of drops.

In order that we start high enough, we need to make sure we test all 100 floors. That means the final floor we test should at least be 100. Recall that the first floor we test is X, then the next floor is $X + (X - 1)$, then it is $X + (X - 1) + (X - 2)$, and so on. Continuing the pattern, the final floor that we test will be $X + (X - 1) + \ldots + 2 + 1$. We need this to at least be 100.

$X + (X - 1) + (X - 2) + \ldots + 2 + 1 \geq 100$

The series on the left is a familiar one: it is the sum of the numbers from 1 to X, and it has a known formula of $(X)(X + 1)/2$.

$(X)(X + 1)/2 \geq 100$

We now need to find the smallest value of X that solves this quadratic equation to *minimize* the worst case. Numerically, the smallest value is $X = 13.65$. We need to round this up so that our sum exceeds 100. Thus, our answer is $X = 14$.

So our algorithm is the following.

Marble 1, drop on floors: 14, 27, 39, 51, 61, 70, 78, 85, 91, 96, 100.

Marble 2: do a linear search between the highest floor the marble didn't break and the first time it did break.

The marvelous thing is this: the modified group search has a worst case of X, which is $X = 14$. This is a reduction from the regular group search of 19. (Try out a few examples. For instance, if the answer is 99, the modified group search uses 11 drops for marble 1, then marble 2 needs an extra 3 drops to test 97, 98, and 99. The worst case is 14 drops).

Furthermore, the modified group search has an expected value just a little bit above 10 (verified numerically). In other words, we've lowered the worst case by 5 drops without affecting the average case much at all!

In statistics class we often minimize the average case. It is important to remember to think about a worst case analysis, and a creative search might just do the trick.

Puzzle 17: 3 Locks

I have a safe with three keycard locks. That is, each lock is like a hotel door lock where you insert a card to operate the lock.

I have three cards that look identical, a correct card for each of the locks.

The safe operates as follows. If only one or two of the cards are inserted into the locks, nothing happens. If all three cards are inserted, then:

(a) a card in the wrong spot closes the lock;

(b) a card in the correct spot changes the status of the lock: if the lock was closed it becomes open, if it was open, it becomes closed.

The safe only opens when all three locks are open.

This morning I found out that my son played with the cards and the safe. Now the safe is closed, but I don’t know which locks are open or closed, and I can’t tell any of the cards from one another.

I’m in a bit of a tight spot. Can I eventually open the safe?

Answer To Puzzle 17: 3 Locks

I spent many hours trying to figure this one out. I kept trying to write out tables for the original position of the locks and trying to figure out a way to test the keycards.

The difficulty is that you can have the cards in the correct position but the safe might still not open. For example, suppose the locks are in the initial position open-closed-open. If by chance you put all three cards in the correct spot, this will change the position of the locks so they are closed-open-closed. The safe is closed in both cases and you really have no idea that you actually had the correct cards.

After fumbling around with some search strategies, I realized the trick to the puzzle was the following: you need to find a way to close all of the locks first. Then you can test out the cards in the different position, and if you get the right cards, the safe will open.

So how can you be sure that all of the locks are closed? The key fact is that when all of the cards are in the wrong position all of the locks will get closed. This kind of permutation in which all of the cards are in the wrong position is called a *derangement*.

With these insights in mind, let us solve the puzzle. Although we don't know which card belongs to which lock, we can test out different combinations to open the safe. Here is the answer.

First, take the three cards (in any order) and label them a, b, and c. Then test out the following card combinations in locks 1, 2, and 3 as follows, trying to open the safe each time:

abc
bca
cab
abc

(where abc means place card a in lock 1, card b in lock 2, and card c in lock 3)

Now, I claim the following: if any of these is the correct combination for the safe, then the safe will open. Why is that? Let's work out a case.

Suppose that abc is the correct combination. When you first try abc, the status of the three locks will change, and the safe may or may not open. But when you do bca and cab, both of these are derangements of abc where none of the cards are in the right spot. Therefore, all three locks will get closed. So when you do abc again, all three locks get opened and the safe will open.

Similarly, this procedure will open the safe if either bca or cab is the correct order for the cards.

If none of these four trials work, then you should do the following procedure, testing if the safe opens after each try.

bac
acb
cba
bac

By similar logic, under this procedure the safe will definitely open if any of bac, acb, or cba is the correct order for the keys.

Through these series of tests, we have created a procedure to test out all 6 possible ways the cards could correspond to their correct locks: abc, bca, cab, bac, acb, or cba. Therefore we have found a way to guarantee we can unlock the safe within 8 tries.

This is good news since it means the safe can get open. The bad side is that it means the safe can easily be cracked by anyone with mathematical sense. Sounds like it's time to get a new safe…

This delightful puzzle was suggested to me by a reader whose website is (written in Italian): Mau.

Puzzle 18: Which Foul?

The NBA has foul rules that are meant to encourage fair play and penalize excessive contact. In general, a team that commits a foul loses possession of the ball, the team that is fouled earns possession, and the fouled player might get to shoot free throws.

However, there are exceptions where teams get to designate the free throw shooter. When one team gets a technical foul, the other team gets to choose any of the players on the floor to take one free throw. Another circumstance is when the fouled player gets injured on the play and has to leave the game. If the injury happened out of bad luck, say landing on a twisted ankle, the defensive team gets to choose the replacement shooter. However, if the injury was the result of an excessive foul, like an elbow to the head, the offensive team can select the foul shooter.

While each foul rule has good intentions, the set of foul rules is a complicated mess that leads to some very unusual incentives. Additionally, in some plays it is not clear which fouls should be called, and the referees are left to judgment. In this puzzle, we imagine a very special circumstance and ask which foul you would prefer.

The last play in a close basketball game is often very exciting and full of action. It is also a time when many fouls happen at once and referees decide which fouls they choose to call.

So imagine the following situation. The score is tied and you're the coach. One of your players misses a shot as time expires, but there was contact on the play, and the player ends up injured. Players on the defensive team were also taunting. The refs had blown the whistle and they are reviewing which foul to give. Imagine they are deciding between the following two options.

1. Ignore the shooting contact and just call a technical foul. In this case, you get to choose your best player, who makes shots with probability p, to take one shot.

2. Call a shooting foul but not have it be flagrant. In this case, the other team would select your worst player, who makes shots with probability $p/2$.

If your player makes a shot, you win the game. Otherwise, the game goes into overtime where you are likely to lose (as one of your players got injured in the final play).

Which foul situation would you rather have?

I grant this situation is concocted. But if you've ever watched the NBA you will know that stranger things have happened!

Answer To Puzzle 18: Which Foul?

It's not intuitively clear whether you want one good shot or two bad shots. But mathematically there is a case to prefer the technical foul and having your best player take one shot.

In this problem, your team loses unless a shot is made. So we will calculate the probability of missing in both situations and compare.

The probability a good player misses the technical free throw is $1 - p$. The probability the bad player misses both free throws is $(1 - p/2)^2 = 1 - p + p^2/4$. Because p^2 is positive, this is larger than $1 - p$, so the chance the bad player misses both is larger.

One way to understand the result is that all we care about is having one shot made. In the technical foul shot, there is a p chance of making the shot and winning the game. On the other hand, we think about the bad player winning the game. The probability of making one shot is equal to 1 - Pr(miss both). This is a $1 - (1 - p/2)^2 = p - p^2/4$. So if the bad player shoots free throws, the chance your team wins is $p - p^2/4$, which is worse than the technical free throw chance of p. The difference of $p^2/4$ is the event the bad player makes both free throws. For this problem, making both shots is basically a "wasted" effort since only one made shot is needed to win the game.

It is not common you would prefer one free throw over two, but in this case it's better to have the one good look rather than two bad ones!

This puzzle is adapted from a Barry Nalebuff puzzler.

Puzzle 19: Choosing With A Coin

A group of 3 friends cannot agree on where to eat. Each person prefers a different restaurant and no one is willing to compromise.

Eventually they agree it would be fair to choose the restaurant at random. How can they use a coin to decide, making sure each restaurant is picked with equal chance?

What if the coin is biased, but they don't know if it produces more heads or tails?

Extension: how can you use a coin to choose between *n* items equally? What if the coin is biased?

Answer To Puzzle 19: Choosing With A Coin

If the coin is fair, there is a simple procedure for choosing between 3 items randomly. The group can flip the coin two times, and that will produce 4 events with equal probability: HH, HT, TH, and TT. Let the first three outcomes correspond to each of the three choices. If the fourth event happens (TT), then disregard it and repeat with two more flips.

In other words, flip the coin two times and then let:

HH: choice 1
HT: choice 2
TH: choice 3
TT: do-over, flip the coin 2 more times and repeat

There will be some do-overs, but mostly this procedure will result in a choice in a matter of a few tries.

What if the coin is biased?

If heads occurs more frequently than tails, it will no longer be true that HH and HT occur with the same probability. So how can we choose between 3 items in this case?

It seems like we are stuck, but we can use a trick. Let's first see how we can choose between 2 items. In other words, let's make a "fair toss" from this unfair coin.

The procedure is this. Flip the coin 2 times. Let HT denote one choice and TH denote the other. If the flip is HH or TT, then disregard and repeat with two flips again.

We can prove this results in creating two events that happen with equal chance. To see why, let's say that H occurs with probability p and T with probability $1 - p$. When we flip the coin twice, we have:

HT occurs with $p(1 - \mathrm{p})$
TH occurs with $(1 - p)p$

We've created two events that happen with equal chance, even though the coin itself is biased. The trick was making sure to only consider outcomes where the number of H's equals the number of T's.

How can we generalize this for choosing between 3 items?

What we have to do is flip the coin 4 times. Now we disregard the outcome if the number of H's and T's is not equal. We are left with 6 choices in which there are 2 H's and 2 T's. We can label these as follows:

HHTT, HTHT: choice 1
HTTH, THHT: choice 2
THTH, TTHH: choice 3
any other result: disregard and repeat the 4 tosses again

This procedure will result in each of the three restaurants being chosen with equal chance.

(Obviously this is just one way to label the outcomes with choices. Any method that assigns 2 of the equally likely events to each of the 3 choices will be valid.)

Generalizing to *n*

Based on this logic we can make random choices between n choices using a coin. (This is not the most efficient way to do it, but it's easy to understand which is important so everyone can agree the procedure is fair!)

If the coin is fair: flip the coin k times with $2^k \geq n > 2^{k-1}$. Label n of the equally outcomes with each of the choices 1, 2, …, n. For any other outcome flip the coin again until it results in one of the labeled choices.

If the coin is not fair: We need to flip the coin so there are at least n outcomes where the number of heads is equal to the number of tails. That means we should flip the coin $2k$ times such that ($2k$ choose k) $\geq n$. Label n of the equally outcomes with each of the choices 1, 2, …, n. For any other outcome flip the coin again until it results in one of the labeled choices.

Puzzle 20: Sex Ratio

This problem comes from a seemingly paradoxical combination of statistics. As reported in The Economist, families in Nigeria have a preference for boys over girls. If they have a girl or two girls, they will try to have another child to get a boy. The surprising result is that this does not skew the sex ratio for the population: there are nearly as many boys as there are girls.

The puzzle is how is it possible that Nigeria has a nearly even sex ratio of girls to boys, when the average family has more girls than boys?

To be precise, specify the following setup. Suppose each family in Nigeria has a child until they get a boy and then stops. What will be the sex ratio in the population?

Answer To Puzzle 20: Sex Ratio

We will tabulate the proportion of families that have a boy on their first child, second child, and so on, and then count the number of boys versus girls. Since it is equally likely to have a boy or girl, we have the following proportions in the population.

1/2 of families have a boy
1/4 of families have a girl, then a boy
1/8 of families have 2 girls, then a boy
1/16 of families have 3 girls, then a boy
…. $1/2^n$ of families have n - 1 girls, then a boy

First we calculate the expected number of boys in the population for X families. This is easy: every family tries until they have a boy, so the answer is X.

We can also derive this answer by multiplying the proportion of families by the number of boys in the family. Note that 1/2 of families have 1 boy, 1/4 of families have 1 boy, and so on. Therefore, the expected number of boys is the following infinite series.

$(1/2)(X) + (1/4)(X) + (1/8)(X) + \dots$
$= X/2 + X/4 + X/8 + \dots$
$= X(1/2 + 1/4 + 1/8 + \dots)$
$= X$

What is the expected number of girls in the population per X families? We will need to use the infinite series method. Note that 1/2 of families have 0 girls, 1/4 of families have 1 girl, 1/8 of families have 2 girls, and so on. The expected number of girls is the following infinite series.

$(1/2)(0) + (1/4)(X) + (1/8)(2X) + (1/16)(3X) + \dots$
$= X(1/4 + 2/8 + 3/16 + 4/32 + \dots)$

We need to sum up the parenthetical series $S = n/2^{n-1}$ when n starts at 1. The trick is we will subtract $(1/2)S$.

$S = 1/4 + 2/8 + 3/16 + 4/32 + \dots$
$(1/2)S = 1/8 + 2/16 + 3/32 + \dots$

Subtract the second equation from the first, pairing terms with the same denominator:

$(1/2)S = 1/4 + 1/8 + 1/16 + \dots$
$(1/2)S = 1/2$
$S = 1$

Hence, across X families there will be X girls. Across the population of X families, there will be X boys and X girls.

The result is the sex ratio will be 50/50, even though every family tries for a boy, and most families will have at least one girl.

It’s a puzzling theoretical result that is borne out by actual data in Nigeria!

Puzzle 21: Unfound Errors

This is a delightful statistical problem I came across while I was publishing my last ebook.

I sent out a draft of my ebook to a couple of friends for review. Each independently proofread the text.

One person found 10 errors, another person found 6 errors, and it happened that 3 of those errors were commonly found by both people.

The question is, what is a reasonable estimate of the number of remaining, unfound errors?

In general, if two proofreaders independently find errors a and b, with a common number of errors c, then what is an estimate of the number of unfound errors?

Answer To Puzzle 21: Unfound Errors

Consider a much simpler problem. If you find 8 errors in a book, and you know you catch about 50 percent of the errors you see, what is your estimate of the number of unfound errors? The answer is 8 because you catch as many errors as you miss.

What if you know you catch about 75% of the errors you see? Now what is your estimate of the number of unfound errors? Clearly the answer will be less. You estimate there are 8/(0.75) = 10 total errors in the book, and since you caught 8, you estimate there are 2 errors you did not find.

These calculations were easy to make because the accuracy of proofreading was specified. But in practice we do not know the accuracy. We have to estimate it based on the observed data.

The problem of unfound errors is to estimate the number of unfound errors from two independent proofreaders. While we don't know the accuracy of each proofreader, we can use the fact that they caught a certain number of errors in common to develop our estimate of the total number of unfound errors.

Let's set up some notation to develop a general formula.

a = errors found by person 1 (observed)
p_a = accuracy of person 1 (unknown)
b = errors found by person 2 (observed)
p_b = accuracy of person 2 (unknown)
c = errors commonly found by 1 and 2 (observed)
e = total number of errors (unknown)
u = unfound errors (unknown)

The variables observed in practice are the errors found by each person and the number of common errors. So we know the variables a, b, and c.

From this we wish to estimate the number of unfound errors.

While we don't know the accuracy of each person, we do know how the accuracy relates to the total number of errors. Specifically, the errors found by the person equals the accuracy times the total number of errors.

errors found = (accuracy)(total number of errors)
$a = p_a e$
$b = p_b e$

What's the chance that an error is commonly found by both proofreaders? The chance that person 1 finds an error is p_a, and the chance that person 2 finds an error is p_b. So the chance they both find an error is the product of their accuracies $p_a p_b$.

(We make this claim because we are assuming the two proofreaders are independently searching for errors, and that each error is equally likely to be found. This is not a justifiable assumption if certain errors are more likely to be found or if the two proofreaders work jointly instead of independently).

Therefore, we can relate the number of common errors to the accuracy of the proofreaders.

errors both find = (accuracy a)(accuracy b)(total number of errors)
$c = p_a p_b e$

We can tweak this formula a bit as it will be useful in the derivation. Multiply both sides by e.

$ce = p_a p_b (e)(e)$

On the right side, there are two factors of e, each that can be paired with one of the accuracy factors. We will use the fact that the error times the accuracy rate is the number of errors found by a person. So we get the following.

$ce = (p_a e)(p_b e)$
$ce = ab$
$e = ab/c$

Now we are almost ready to solve the problem. The total number of errors is the errors that each person finds, less the common errors, plus the unfound errors. This means the following equation:

Total errors = errors a + errors b - common errors + unfound errors
Unfound errors = total errors - errors a - errors b + common errors
$u = e - a - b + c$

The only unknown in this equation is the total number of errors e. We use the formula $e = ab/c$ derived above for an estimate on the total errors.

Then we can derive the following formula.

$$
\begin{aligned}
u &= e - a - b + c \\
&= \frac{ab}{c} - a - b + c \\
&= \frac{ab - ac - bc + c^2}{c} \\
&= \frac{(a-c)(b-c)}{c}
\end{aligned}
$$

So to summarize, we have the formula for the number of unfound errors:

$u = (a - c)(b - c)/c$

where a: errors found by person 1
b: errors found by person 2
c: errors commonly found
u: unfound errors

This formula says the number of unseen errors is equal to the product of the unique errors each finds divided by the errors commonly found.

Returning to my ebook proofreading, where one person found 10 errors, another found 6, and 3 were common, the formula would estimate $u = (10 - 3)(6 - 3)/3 = 7$ unfound errors.

The formula isn't perfect: if both proofreaders find the same exact errors, then the estimate of unfound errors would be 0. Still, that rarely happens in practice so the formula is often useful.

Puzzle 22: Random Penalty Kicks

In the Mathland World Cup, the goal is represented by the interval [0, 1] (you can imagine 0 being one side of the goal and 1 being the other side).

In a penalty kick, the goalkeeper draws two random numbers—this region is "defended"—and the shooter simultaneously draws a single random number—this is the target of the "shot."

The shooter scores a goal only if the "shot" is outside of the region "defended." For instance, if the shot was 0.5, it would score a goal if the region was [0.6, 0.8], but it would be blocked if the region was [0.4, 0.55].

What is the probability the goalie blocks the shot?

Answer To Puzzle 22: Random Penalty Kicks

Trying to solve for the conditional expectations is very difficult. Luckily, there is a combinatorial approach that is much easier.

The goalkeeper draws two random numbers, A and B, and the shooter draws a single random number, C. The three draws can be ranked by magnitude as a permutation of three elements. For example, CAB means the third draw was the smallest number, then the first draw, and the second draw was the largest.

Because each draw was random, all rankings must be equally likely to occur by symmetry. Therefore, all 6 permutations of the three draws have equal chance. The problem is then: what is the chance that the draw C was in between draws A and B?

We can list out all six draws: ABC, BAC, CAB, CBA, ACB, BCA. The goalkeeper defends only in ACB and BCA. Therefore, the goalkeeper has a 2/6 = 1/3 chance of successfully blocking the shot.

We can solve the extension similarly. When the shooter takes N shots, the shooter takes N random draws and the goalie still has 2 random draws. Therefore the problem is: in a sequence of $N + 2$ random draws, what is the chance all of the draws are between the first two?

The problem can be solved combinatorially. All $(N + 2)!$ permutations of draws must be equally likely. The goalkeeper defends if and only if the first and second draws are the first and last (or last and first) elements in the permutation. The first draw can either be the first or last element, so there are 2 ways this can happen. All N shots must be between the first and last elements, and there are $N!$ ways to arrange these shots.

That is, the goal is defended if and only if

A***…***B or
B***…***A,
where ***…*** is a permutation of the shots

Hence there are $2N!$ ways to defend out of $(N + 2)!$ possibilities in all. Thus the goalkeeper defends in $2N!/(N + 2)! = 2/[(N + 1)(N + 2)]$ ways. As expected, the probability of defending decreases as the number of shots increases.

A similar problem was posted on the puzzle blog BayesianThink. The problem can be solved analytically conditioning on the first two draws, but even the case of 1 shot requires carefully setting up a double integral.

Puzzle 23: Highest Number Almost Always Wins

Two players secretly choose a whole number between 1, 2, …, N. The highest number wins, unless the lower number is exactly one less, in which case the lower number wins. Ties force a repeat.

The question is, what’s the best strategy in this game?

Answer To Puzzle 23: Highest Number Almost Always Wins

It turns out the game is rock-paper-scissors in disguise!

Let's work out some cases to see why. When $N = 3$, notice that 1 beats 2, 2 beats 3, but then 3 beats 1. This non-transitive nature of strategies turns out to be exactly the same as in the game rock-paper-scissors. In both games, the optimal strategy is to randomize evenly between the available strategies.

We can explicitly write the payoffs in a matrix and demonstrate the 1-2-3 game is the same game as rock-paper-scissors but with different labels.

		Your choice		
		1	2	3
My choice	1	-	win	lose
	2	lose	-	win
	3	win	lose	-

		Your choice		
		Rock	Scissors	Paper
My choice	Rock	-	win	lose
	Scissors	lose	-	win
	Paper	win	lose	-

This is an interesting result when there are three strategies. But what happens when $N= 4$? Now there are four possible options and that might change the game. However, there is a nice simplification that makes the game easy to solve.

It is pretty easy to see that the strategy of playing 1 is just a stupid choice. If you ever wanted to choose 1, you'd be better off playing 4 instead. Why? Here's a table of why playing 4 is better.

		Opponent's choice			
		1	2	3	4
Your choice	1	-	win	lose	lose
	4	win	win	lose	-

If you play 1, you will draw against 1, win against 2, and lose to 3 and 4. If you play 4 instead, you will win against 1 and 2, still lose to 3, and draw against 4. In other words, no matter what you think your opponent is doing, it's always better to play 4 instead of 1. (In game theory jargon, we say that 1 is a dominated strategy).

Therefore, there is no point to playing 1, and both players are only going to choose to play 2, 3, or 4.

Now comes the neat part. In this reduced game, note that 2 beats 3, 3 beats 4, and 4 beats 2. So we're again in a rock-paper-scissors game! (Another interpretation: this is the same game as 1, 2, 3, but each of the number labels has increased by 1).

This logic can be extended for any finite N. You will only want to play the highest three numbers N, N - 1, and N - 2, and the game will reduce to the classic game of rock-paper-scissors where you randomize between the three strategies.

Credit: I received this puzzle by email from Scott, a long-time reader of my blog.

Puzzle 24: Race To 15

Alice and Bob take turns selecting cards face-up numbered 1, 2, .., 9 without replacement. (Alice goes first, then Bob picks a remaining card, and so on).

The first player to make a set of 3 cards that adds up to 15 wins the game.

Is there a strategy to winning this game?

Answer To Puzzle 24: Race To 15

The short answer is that neither player can win! We can re-arrange the numbers 1 to 9 into a 3x3 magic square where every row, column, and diagonal sums to 15.

4 9 2
3 5 7
8 1 6

A player getting to 15 is equivalent to a player having 3 in a row on the magic square. In other words, this game is isomorphic to tic-tac-toe (aka noughts and crosses)! Since tic-tac-toe ends in a draw with proper play, so will the race to 15.

Part III: Hard Puzzles

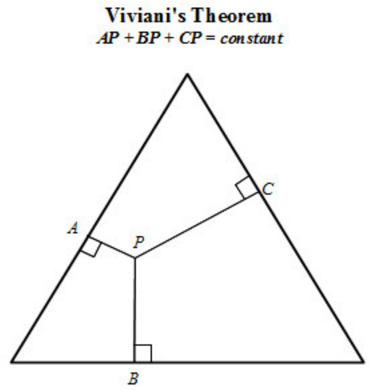
Puzzle 1: Probability Of Forming A Triangle

If you break a stick at 2 random points, what’s the probability the 3 lengths can make a triangle?

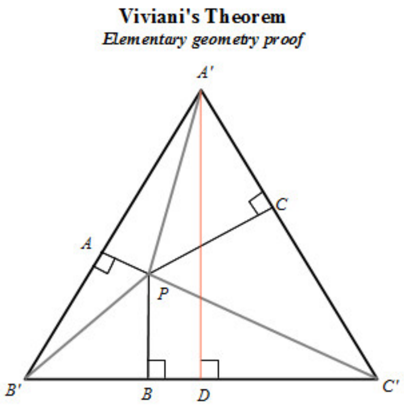
Answer To Puzzle 1: Probability Of Forming A Triangle

We take a slight detour by proving a preliminary result.

Consider an equilateral triangle. Viviani's Theorem states that for any point in its interior, the sum of the distances to its three sides is a constant.



Here is a simple proof. Consider a point P inside of the triangle. The sum of the distances to the three sides is AP + PB + CP. We will prove this is always equal to the altitude A'D.



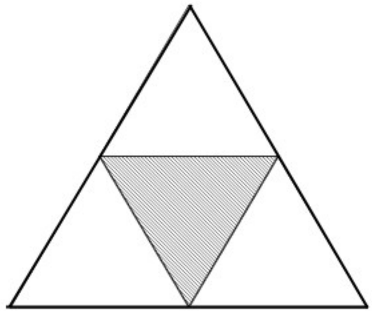
On the one hand, the area of the triangle is (1/2)(A'D)(B'C) = (1/2)(A'D)(side of triangle). On the other hand, the area is also the sum of the smaller triangles A'PB', A'PC', and B'PC'. These have a sum of (1/2)(AP)(A'B') + (1/2)(CP)(A'C') + (1/2)(BP)(B'C'). In an equilateral triangle all sides are congruent, so A'B' = A'C' = B'C'. So we can simplify the area of the small triangles as (1/2)(AP + CP + BP)(side of triangle). Setting this equal to our expression for the area involving the altitude, we have (1/2)(AP + CP + BP)(side of triangle) = (1/2)(A'D)(side of triangle). Cancelling terms, we find AP + CP + BP = A'D. That is, the sum of the distances to the three sides is equal to the constant length of an altitude.

Solving the original problem

Here's the insight. We can simulate breaking a stick of length 1 into three pieces x, y and z by selecting a random point inside an equilateral triangle with an altitude of 1, and letting the distances to the three sides be the three pieces x, y and z.

Why can we do this? Note that any point we select in the equilateral triangle leads to three distances x, y and z with the property that $x + y + z = 1$, so we can think about the distances as being the three broken pieces of a stick of length 1.

The three pieces can form a triangle so long as no piece is larger than 1/2. The set of points where no distance to a side is larger than 1/2 is an equilateral triangle formed by joining the midpoints of the sides.



The smaller equilateral triangle divides the larger triangle into 4 equally sized regions, and so its area is 1/4 of the larger equilateral triangle. Thus the answer is 1/4.

There are other methods to solve this problem. I learned about this proof from mathoverflow.

Puzzle 2: Three Napkins

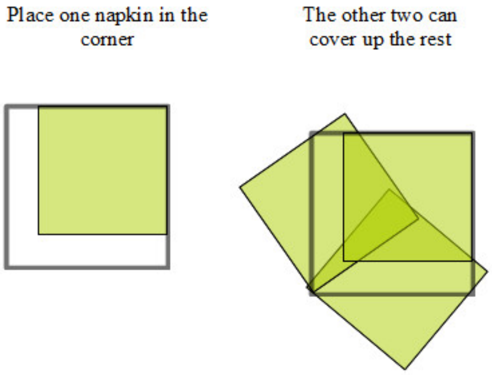
You are given three square napkins measuring 1 unit to a side. What’s the largest square table you can cover with the napkins? You cannot tear the napkins, but you can fold them or overlap them, and the napkins are allowed to drape over the side of the table.

Answer To Puzzle 2: Three Napkins

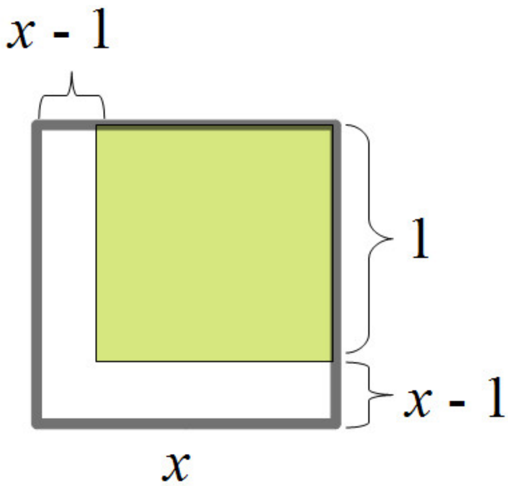
The three napkins have an area of 3 square units. If they could be cut, they could cover a table of exactly that size. But since they cannot be cut the table they can cover is significantly smaller, with an approximate area of 1.62 (and side length of approximately 1.27).

The trick is figuring out how to arrange the napkins and then solve for the maximum area they can enclose.

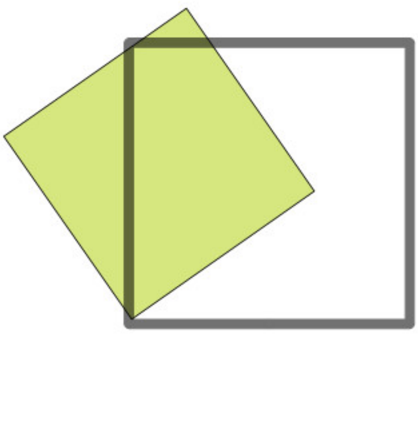
The arrangement is as follows: place one napkin in a corner of the square and then use the other remaining napkins to cover the remaining area.



Here is how we can solve for the maximum area. Imagine the square table has a side length of x. One of the napkins covers a 1 by 1 square, so the uncovered area is two strips with a height of x - 1.



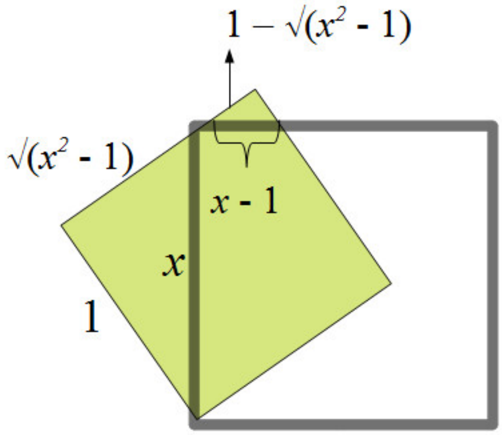
Now we wish to place the two remaining napkins so they cover the maximum size table. The way to do this is to place the napkin at an angle so that it just covers the strip. The configuration is as follows.



We justify this placement as optimal. If the napkin were rotated any more to the right, the upper left corner of the table would not get covered. If it were rotated any more to the left, then the strip it covers x - 1 would be less, and we're hoping to maximize x. Therefore this is the best way to place the napkin.

Now we can use some geometry to find the value of x. The parts of the napkin draping over the side of the table form two similar right triangles. The larger triangle has a hypotenuse x and its larger leg is 1. The smaller triangle has a hypotenuse of x - 1, and its larger leg can be found by subtraction: it is 1 minus the length of the leg of the other triangle, which is $\sqrt{(x^2 - 1)}$ by the Pythagorean Theorem.

It is a somewhat messy diagram.



The larger and smaller right triangles draping off the edges of the table are similar triangles, so the ratio of each triangle's hypotenuse to its longer leg must be equal. We get the following equation.

$x / 1 = (x - 1)/(1 - \sqrt{(x^2 - 1)})$

We simplify this equation by multiplying both sides by $(1 - \sqrt{(x^2 - 1)})$ and then squaring both sides.

$x(1 - \sqrt{(x^2 - 1)}) = x - 1$
$x(- \sqrt{(x^2 - 1)}) = -1$
$x^2(x^2 - 1) = 1$
$x^4 - x^2 - 1 = 0$

If we let $u = x^2$, then we have $u^2 - u - 1 = 0$, which is a famous equation with the positive solution being the golden ratio, $(1 + \sqrt{(5)})/2$. So the answer x is the square root of this, which is approximately 1.27, and the table has an area equal to the golden ratio, approximately 1.62.

This puzzle is credited to Sam Loyd. Square trisection has a long history and it was extensively studied by Muslims in the 8th to 15th centuries, as such problems arose when building mosaics in mosques. The solution presented is similar to problem of the week from Peter Malcolmson, Winter 1998, Week 9.

Puzzle 3: Hidden Treasure

An archaeologist finds the following set of directions written on a scroll.

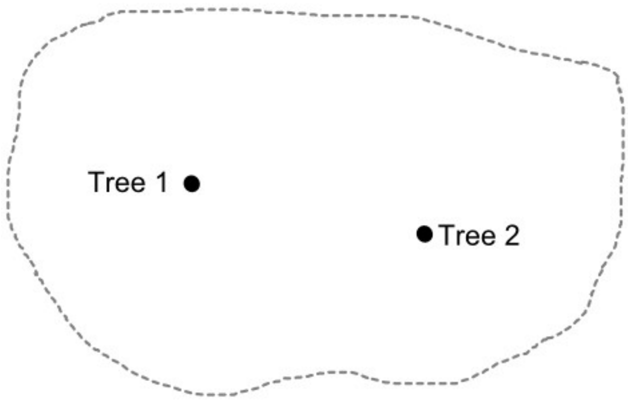
From the water well, walk to the tallest tree on the left and count your paces. Turn 90 degrees to your left, walk the same number of paces, and mark that spot.

Return to the water well. Walk to the tallest tree on your right and count your paces. Turn 90 degrees to your right, walk the same number of paces, and mark that spot.

The treasure is located at the exact middle of the two marked spots.

From studying historical documents, the archaeologist has confidently identified the two trees. However, the location of the water well is a mystery.

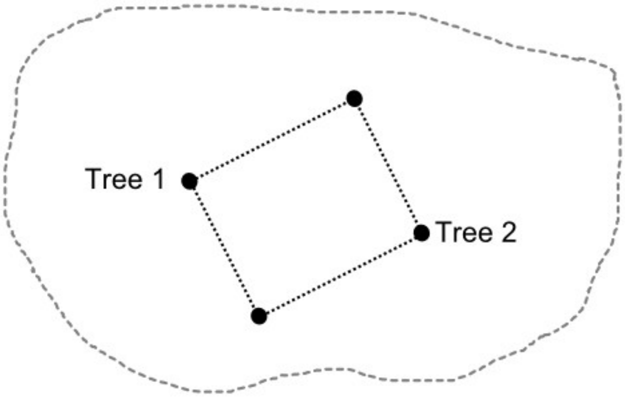
The archaeologist hands you a map (see picture below). Can you narrow the search for where the treasure could be buried?



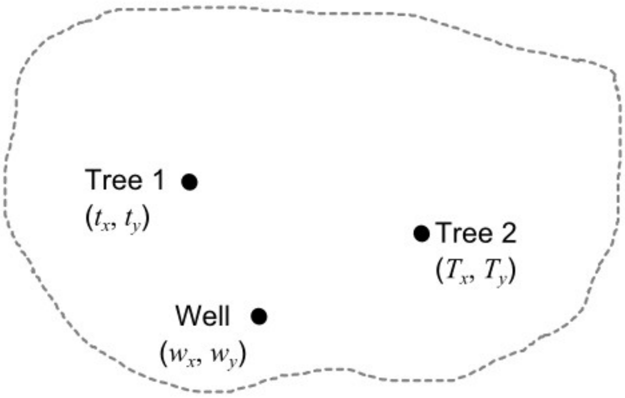
Answer To Puzzle 3: Hidden Treasure

Remarkably there are only two locations for the treasure, and you can find them without knowing the exact location of the water well!

Specifically, if you sketch a square with the two trees as opposite vertices, then the treasure could be buried at either of the two remaining vertices.



We can prove this using vector math. First, assume that Tree 1 is on the left, so the map looks like the following.



Imagine each point is determined by an x-coordinate and a y-coordinate, and write out the following coordinates.

Tree 1: (t_x, t_y)
Tree 2: (T_x, T_y)
Water Well: (w_x, w_y)

The vector from the water well to the first tree is given by $(t_x - w_x, t_y - w_y)$.

To turn left 90 degrees means to walk the vector given by $(-t_y + w_y, t_x - w_x)$.

The first marked spot is determined by adding the position of the first tree to the vector for walking.

Spot 1: $(t_x - t_y + w_y, t_y + t_x - w_x)$

Similarly we can solve for the vector for the second marked spot. The vector from the water well to tree 2 is given by $(T_x - w_x, T_y - w_y)$.

To turn right 90 degrees means to walk the vector given by $(T_y - w_y, -T_x + w_x)$.

The second marked spot is determined by adding the position of the second tree to the vector for walking.

Spot 2: $(T_x + T_y - w_y, T_y - T_x + w_x)$

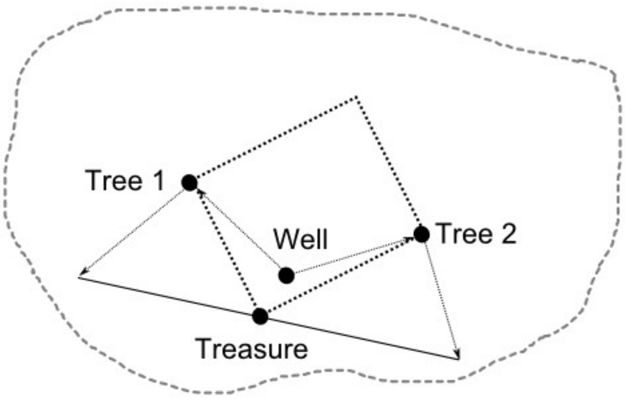
The midpoint between spots 1 and 2 is the average of the coordinates.

Treasure location: $0.5(t_x - t_y + T_x + T_y, t_y + t_x + T_y - T_x)$

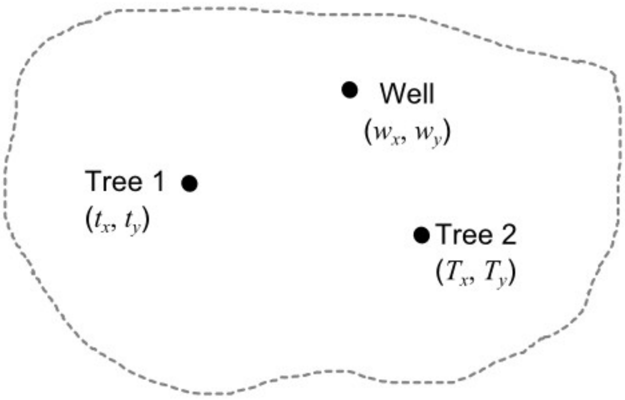
Treasure location: $0.5(\text{tree }1 + \text{tree }2) + 0.5(T_y - t_y, t_x - T_x)$

Notice how the coordinates for the water well cancel out, so the location of the treasure can be found from the location of the trees.

The second equation indicates the location is at the corner of a square defined by trees 1 and 2.



The above was given assuming tree 1 was on the left. However, from the water well it might have been that tree 2 was on the left.

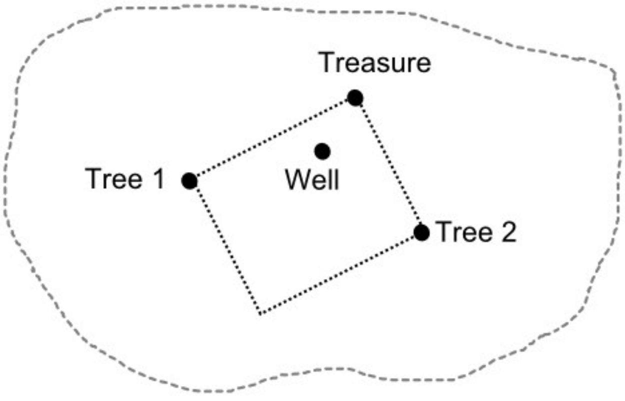


In that case, a similar analysis shows the treasure location would be the following.

Treasure location: $0.5(t_x + t_y + T_x - T_y, t_y - t_x + T_y + T_x)$

Treasure location: $0.5(\text{tree }1 + \text{tree }2) + 0.5(t_y - T_y, T_x - t_x)$

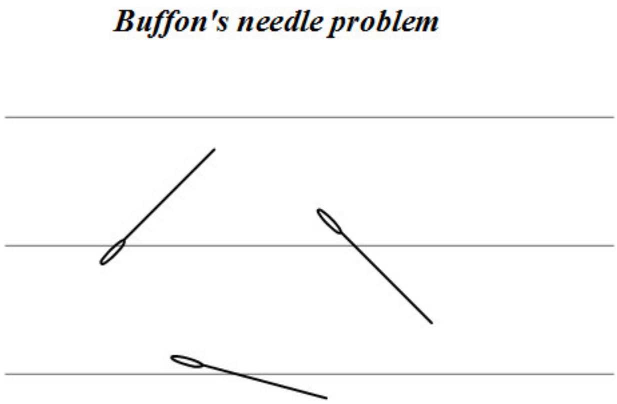
This corresponds to the other corner of the square.



I read about this puzzle on the site Gurmeet delightful puzzles. There is an interesting brute force hacking of the puzzle on github that has a nice visualization with d3.js.

Puzzle 4: Buffon’s Needle

If a needle of length $L < 1$ is dropped randomly onto a floor with horizontal lines spaced 1 unit apart, what is the chance that the needle intersects one of the lines?



Answer To Puzzle 4: Buffon's Needle

I believe the real trick is setting up the problem. Once that is done, the actual computation is relatively easy.

What does it mean for a needle to fall randomly? And what condition is necessary for the needle to intersect the horizontal lines?

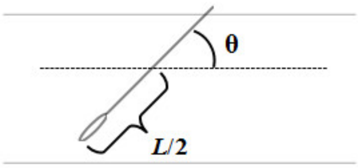
There are a variety of equivalent approaches. The basic idea is to fix some reference point on the needle and assume that the reference point is equally likely to land at any spot on the floor. From there, one considers the angle the needle makes with the horizontal lines.

The approach I prefer takes the following two conditions.

(1) The *midpoint* of the needle is equally likely to land anywhere on the floor.

(2) Relative to the horizontal line, the needle makes two complementary angles. Label the smaller angle as θ and note that θ can take any value from 0 to 90 degrees (mathematically it makes more sense to work in radians, so from now on the range of θ will be between 0 and π/2 radians).

Geometry of needle

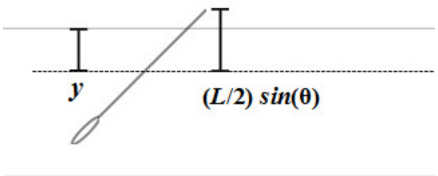


Now comes the time for calculation. Under what condition will the needle intersect one of the horizontal lines?

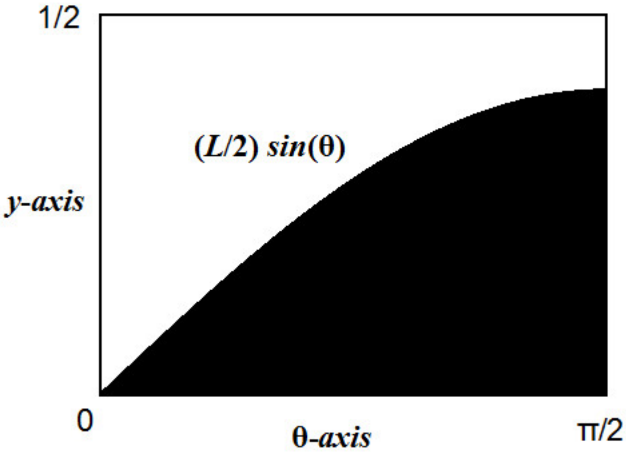
What's needed is to calculate the "vertical" distance of the needle to the nearest horizontal line based on the position of the midpoint and the angle θ. Using basic trigonometry, the distance to the nearest horizontal line will be equal to the length of half the needle times the sine of the angle, that is, "vertical" distance of needle = (L/2) sin θ.

When will the needle intersect the horizontal line? If y is the distance from the horizontal line to the midpoint of the needle, then the intersection happens if y < (L/2) sin (θ), that is, the needle's "vertical" length exceeds the distance to the horizontal. Note that y takes values between 0 and 1/2, as the horizontal lines are spaced 1 unit apart, and the midpoint cannot be farther than half the distance (1/2) to some line.

Trigonometry of intersection



Now we are set to calculate the answer. The range of possible needle drops can be represented graphically. One axis is the distance y which ranges from 0 to 1/2. The other axis is the angle θ, which ranges from 0 to π/2. Intersection happens only when y < (L/2) sin (θ), which is the shaded region in the graph.



What is the probability the needle intersects? The entire sample space has an area of the rectangle, $(1/2)(\pi/2) = \pi/4$. The event of an intersection happens in the shaded region, which can be calculated by the following integral:

$$\int_0^{\pi/2} \frac{L}{2} \sin\theta d\theta = \frac{L}{2}$$

The probability is the ratio of the shaded area to the entire area, which is $(L/2)(\pi/4) = 2L/\pi$

And that is how Buffon's needle problem can be solved and why the constant π appears in the answer.

The proof using calculus actually turns out to be the "easy" proof. Another method, Barbier's proof, derives the answer by a clever geometric argument. It is presented in Chapter 21 of *Proofs from the Book*, which is a collection of the most elegant proofs.

Puzzle 5: 52 Cards From 52 Decks

Each of 52 people has a standard deck of cards (which has 52 cards). Each person draws a single card at random from his respective deck of cards.

What is the probability that *no one* draws the Ace of spaces?

Answer To Puzzle 5: 52 Cards From 52 Decks

The stated problem is relatively easy to solve. In any given deck, there is a 51/52 chance that the ace of spaces is not drawn. Since each person draws the card independently, there is a $(51/52)^{52} \approx 36.4$ percent chance that no one draws the ace of spades.

But there is an interesting significance to this answer. It is very close to $1/e = 0.3679\ldots$

What is the connection?

In a deck of n cards, the chance that a particular card is not drawn is $(1 - 1/n)$. The chance that none of n people draws the card is the product:

$(1 - 1/n)(1 - 1/n)\ldots(1 - 1/n) = (1 - 1/n)^n$

The limit of this quantity as n goes to infinity is precisely $1/e$. So Euler's constant pops up in a probability problem.

Puzzle 6: Random Holidays

A mathematically inclined boss tells his employees the number of workdays for the next calendar year will be determined by chance. The boss will roll a 365-sided die, numbered 1 to 365.

The die will be rolled 365 times. The number of workdays will be the number of rolls with distinct outcomes (and the number of rolls with repeated outcomes will be the number of days off from work).

How many days off from work can the employees expect?

Challenge: Solve the problem for a calendar year of size N. In other words, the year has N days, and the boss rolls an N-sided die a total of N times.

Answer To Puzzle 6: Random Holidays

We need to count the expected number of distinct rolls. Define a random variable X_i to be 1 if the roll i appears any time in the 365 rolls and 0 otherwise. This means $E(X_i)$ is the probability that roll i appears as one of the 365 rolls.

How many distinct numbers appear in 365 rolls? We need to sum up the chance that the roll 1 appears, the roll 2 appears, and so on. In other words, the number of different rolls is the expected sum that each of the 365 rolls appears. This is the sum of $E(X_i)$, where i goes from 1 to 365, written as the following.

E(distinct rolls) = $E(X_1) + \ldots + E(X_{365})$

By symmetry and independence of each roll, the chance that roll 1 appears is the same that roll 2 appears, or 3 appears, or any roll i appears. Thus all of the $E(X_i)$ must be equal. So we can re-write the sum as 365 times $E(X_1)$.

E(different rolls) = 365 $E(X_1)$

What is the probability that the roll 1 appears? We can more easily calculate the chance it does not appear. There is a 364/365 chance a given roll is not a 1. Over 365 rolls, that leads to a $(364/365)^{365}$ chance the roll 1 does not appear at all. The chance that the roll 1 appears is the complement event of $1 - (364/365)^{365} = E(X_1)$.

Hence, we have:

E(different rolls) = $365(1 - (364/365)^{365})$
E(different rolls) ≈ 230.9

In other words, the boss ends up with about 230 workdays, with the remaining 135 days are off from work.

In America, it is customary to work on 260 weekdays less 10 federal holidays and 10 days of paid time off. So the average worker might have about 240 days of work.

That's not much different from our probabilistic answer! Perhaps someone was rolling the dice when they decided which days we should work and which days we get off as holidays.

Bonus: Calendar Years of Any Size

In fact, the procedure will result in about 63 percent of workdays, no matter how large the calendar year is!

Let's see why.

On an N-sided die, there is a $(1 - 1/N)$ chance a particular roll i does not appear. In N rolls, the chance the roll i does not appear at all is $(1 - 1/N)^N$.

If we sum this over the N days, and then divide by the N total workdays, the terms of N will cancel. That means the proportion of workdays in a year is $(1 - 1/N)^N$.

As N goes to infinity, the limit of this term is $1 - 1/e \approx 0.63$. Interestingly, in a calendar of any size, the method results in working about 63 percent of days.

Puzzle 7: Matching Cards

Alice and Bob play a card game. Each begins with a well-shuffled deck of 52 cards.

Each flips over the top card of their decks face up. Then each flips over the next card face up, and so on, for the entire deck.

If ever the two cards flipped over are an exact match, then Alice wins $1. Otherwise, Bob wins $1.

Is the game fair? If not, does it favor Alice or Bob?

Answer To Puzzle 7: Matching Cards

The first step is to simplify the problem. Without loss of generality, let us label the cards in Alice's deck as 1 = first card, 2 = second card, 3 = third card, …, and so on so 52 = last card.

Bob's deck contains the same cards but they might be in a different order. Mathematically, the order of cards in Bob's deck can be represented as permutation of the numbers 1 to 52.

For instance, if every card in Alice's deck is shifted over by 1 position in Bob's deck, then we can write the two decks as:

Alice's deck: 1, 2, 3, 4, 5, …52
Bob's deck: 2, 3, 4, 5, …52, 1

Now Alice wins the game if some card is in the same position in Bob's shuffled deck.

So Alice winning the game is analogous to the following question: what is the probability that some card ends up in the same position in the deck after being shuffled?

Work out small cases

Let's consider a simple example with a 2-card deck. There are two possible orders for Bob's deck:

case A: 1, 2
case B: 2, 1

Alice wins the game in case A and loses in case B. So Alice has a 50 percent chance of winning.

Now let's consider a 3-card deck. Note there can be 3 possible cards in the first position, 2 possible cards in the second position, and 1 possible card for the third position. In all, there are 3! = 6 total number of possible orderings (permutations) for Bob's deck. These are:

case A: **1**, 2, 3
case B: **1**, 3, 2
case C: 2, 1, **3**
case D: 2, 3, 1
case E: 3, 1, 2
case F: 3, **2**, 1

Alice wins the game in cases A, B, C, and F, hence she has a 66 2/3 percent chance of winning the game.

It appears the game is in Alice's favor. Let's work out the general solution to see why.

Derangements

The probability that Alice wins is the same as the chance that a card ends up in the same position in the deck after being shuffled. In other words, we are looking for the number of permutations in which *some* number does not change position.

Bob's chances of winning are the complement event. We want to find the chance that after shuffling a deck that *none* of the cards is in the same position.

It turns out that it's easier to solve for Bob's chance of winning. In fact, there is a special term for a permutation in which every number is moved around in order. Such a permutation is known as a *derangement* (which we briefly encountered in the 3 locks puzzle).

The concept comes up so frequently that there is special notation. The total number of permutations is written as $n!$ (n factorial) and is equal to the product of smaller positive integers: n (n-1) (n-2) … 1.

The number of derangements can be written as $!n$, which is pronounced "n subfactorial."

What is the total number of derangements? The formula can be found if we carefully count out the number of permutations.

Here is the procedure. Let's imagine we have written out all $n!$ permutations. We can classify the permutations in different categories. Some of the permutations will have at least 1 number in the same position, some will have at least 2 numbers in the same position, and so on.

We want to count the number of permutations in which *none* of the numbers are in the same position. The trick is we can subtract out the permutations where some of the numbers are in the same position, being careful not to double count.

The inclusion-exclusion formula will allow us to do exactly this. The basic formulation for two events is:

Pr(A or B) = Pr(A) + Pr(B) - Pr(A and B)

The idea is to add up the probability of event A and event B, but then subtract out the probability of both happening together to avoid double-counting.

For more than 2 events, there is a similar formula of the inclusion-exclusion principle. We will use this to count the number of derangements.

the number of derangements !n = Total permutations – permutations at least 1 number same position + permutations at least 2 numbers same position … +/- permutations n numbers same position

The formula works for the following reason. We start with the total number of permutations, and then we subtract out all the ones in which at least one number ends up in the same position. After we do that, we need to add back in the permutations where at least 2 numbers are in the same position to avoid double-counting, and so on, subtracting "at least" events for odd numbers and adding back in the "at least" events for even numbers.

Let's demonstrate the formula for the case of n = 3. Recall there are 6 total permutations:

case A: 1, 2, 3
case B: 1, 3, 2
case C: 2, 1, 3
case D: 2, 3, 1
case E: 3, 1, 2
case F: 3, 2, 1

Let us classify them according to how many numbers end up in the same position.

4 permutations at least 1 number same position: A, B, C, F
1 permutations at least 2 numbers same position: A
1 permutations 3 numbers same position: A

There are 6 total permutations, and so the inclusion-exclusion formula gives us:

!3 the number of derangements = 6 - 4 + 1 - 1 = 2

The case is a bit trivial, but it does provide us the correct answer that there are 2 derangements for the case n = 3.

And again, the constant e appears!

Now we go ahead and solve the problem.

The total number of derangements is:

the number of derangements !n = Total permutations – permutations at least 1 number same position + permutations at least 2 numbers same position …. +/- permutations n numbers same position

Now we work out a formula for the intermediate terms. How many ways are there to keep at least k numbers in the same positions and then arrange the remaining (n - k) numbers? The answer is there are "n choose k" [abbreviated as $C(n,k)$] ways to choose the k numbers and then there are $(n - k)!$ ways to arrange the remaining numbers:

$C(n,k)(n - k)!$

Substituting into the inclusion-exclusion formula yields:

$!n = n! - C(n,1)(n - 1)! + C(n,2)(n - 2)! + \ldots +/- C(n,n)(n - n)!$

Now we substitute that $C(n,k) = n!/[(n-k)!\ k!]$ and do a bit of algebra to find the formula for derangements is:

$!n = n!(1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n/n!)$

The series inside the parentheses is an interesting one: the series approaches the term $1/e$, which is about 37 percent.

So we now solve our original problem. We have that:

Pr(Bob wins) = Pr(derangement) = (number of derangements)/(number of permutations)

Hence, we have:

Pr(Bob wins) = !52 / 52! = (1 - 1/1! + 1/2! + 1/3! + … + 1/52!) = about 37 percent

Thus we will also have

Pr(Alice wins) = 1 - Pr(Bob wins) = about 63 percent

The long and short is that chances are favorable that some card will match between the two decks and Alice is the clear favorite to win the game.

Puzzle 8: Avoid A Red Card

Let's play a game. I'm holding a deck of two cards: one red and one blue. I shuffle the deck and you select a card.

If you picked the red card, the game ends. If you picked the blue card, then:

—I place the blue card back in the deck
—I add another red card to the deck
—You select a card from the new, shuffled deck

The game continues in this manner until you select a red card. When the game ends, you get paid a dollar amount equal to the number of red cards that were in the deck. For example,

—If you select a red card right away, you get paid $1.
—If you select a blue card three times, then the deck will have 4 red cards and 1 blue card. If you select a red card, you get paid $4.

Obviously you wish to avoid a red card as long as you can. The problem is, it gets more and more likely to select a red card as the game continues.

What is the expected value of the game?

Answer To Puzzle 8: Avoid A Red Card

I work out problems like this by trying to write out the expectation and seeing if there is a pattern in the expression.

At first, there is 1 red card and 1 blue card. Therefore, there is a 1/2 chance of selecting a red card and getting paid $1.

There is correspondingly a 1/2 chance you select a blue card and will make it to the second stage. At this point, there would be 2 red cards and 1 blue card, meaning a 2/3 chance of selecting a red card and getting paid $2.

There is a (1/2)(1/3) chance of selecting a blue card two times in a row to make it to the third stage. The deck would then have 3 red cards and 1 blue card, for a 3/4 chance of selecting a red card and getting paid $3.

We can see a pattern emerging. If the first time you select a red card is at the n^{th} stage, then the probability of selecting only blue cards up until that point is (1/2)(1/3)…(1/(n+1)), the chance of selecting the red card was (n/(n+1)), and the payout will be n.

The expected payout will be the sum of the probability of reaching a stage times the payout of the stage. The expected value of the game is given by the following series:

$$\text{Expected value} = \left(\frac{1}{2}\right)(\$1) + \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(\$2) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)(\$3) + \cdots$$

We can simplify this expression a bit to get the following:

$$\text{Expected value} = \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(2) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)(3) + \ldots$$

$$= \frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \ldots$$

So we're dealing with the infinite sum of $n^2/(n + 1)!$ from 1 to infinity.

This is the part of the problem where I take a slightly different (and arguably cheap) approach from a pure mathematician. I know I should try to work out the formulas and realize some pattern. But it's tempting to hack the answer in a spreadsheet.

So I'll evaluate this series for several terms.

n	$n^2/(n+1)!$	Sum
1	0.500000	0.50000
2	0.666667	1.16667
3	0.375000	1.54167
4	0.133333	1.67500
5	0.034722	1.70972
6	0.007143	1.71687
7	0.001215	1.71808
8	0.000176	1.71826
9	0.000022	1.71828
10	0.000003	1.71828

The terms get vanishingly small even by about n = 10, so this should be a pretty good approximation. Now I think about if the result is anything I recognize. It turns out the number 1.72828 is similar to something I already know: it is 1 less than e = 2.71828…

So again, e comes out of nowhere!

Now comes the issue of how to prove the series converges to e - 1. I looked up a few formulas about e and learned about some neat tricks.

First, we will decompose $n^2/(n + 1)!$ into two different terms.

$$\frac{n^2}{(n+1)!} = \frac{n(n+1)-n}{(n+1)!}$$

$$= \frac{n(n+1)}{(n+1)!} - \frac{n}{(n+1)!}$$

$$= \frac{1}{(n-1)!} - \frac{n}{(n+1)!}$$

Now we can work on summing up each of those infinite series. The series with the terms $1/(n - 1)!$ will be easy: let's focus on the series with the terms $n/(n + 1)!$ This again will require some manipulation.

$$\frac{n}{(n+1)!} = \frac{n+1-1}{(n+1)!}$$

$$= \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}$$

$$= \frac{1}{n!} - \frac{1}{(n+1)!}$$

Now we will sum this series.

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n!} - \frac{1}{(n+1)!}$$

$$= (\frac{1}{1!} - \frac{1}{2!}) + (\frac{1}{2!} - \frac{1}{3!}) + (\frac{1}{3!} - \frac{1}{4!}) + ...$$

$$= 1$$

The series in the second step is known as a *telescoping series* because the second term cancels out with the third, the fourth term cancels out with the fifth, and so on. The only term that survives is the first term of 1.

Now we put everything together to calculate the expected payout of the game.

$$\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - \frac{n}{(n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - 1$$

$$= (1 + \frac{1}{1!} + \frac{1}{2!} + ...) - 1$$

$$= e - 1$$

So the value of the game is e - 1, which is about 1.72828…

Puzzle 9: Broken Coffee Machine

Your boss tells you to bring him a cup of coffee from the company vending machine.

The problem is the machine is broken. When you press the button for a drink, it will randomly fill a percentage of the cup between 0 and 100 percent.

You know you need to bring a full cup back to your boss.

What's the expected number of times you will have to fill the cup?

Example

The machine fills the cup 10 percent, then 30 percent, then 80 percent: the cup is full plus 20 percent that you throw away or drink yourself. It took 3 fills of the cup.

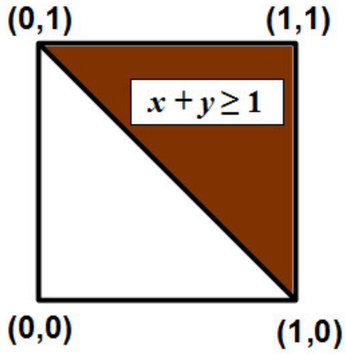
Answer To Puzzle 9: Broken Coffee Machine

We tackle this problem by working in steps.

You won't fill the cup on the first try (or precisely there is a 0 percent chance of that happening). What are the odds that you will fill it in 2 attempts?

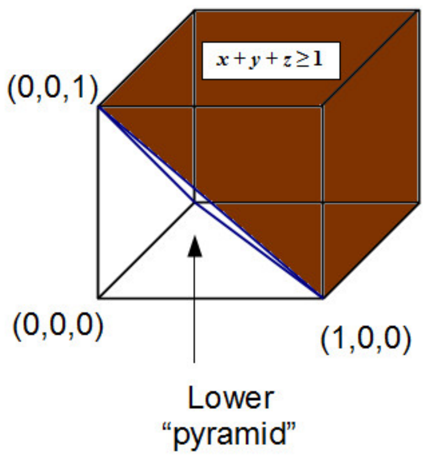
The problem is one of probability. We want to find the chance that $x + y \geq 1$ when both x and y are chosen randomly between 0 and 1.

We can figure this out using a nice visualization. We can consider x and y as coordinates of a square, and then consider the area for which $x + y \geq 1$. Here is the diagram.



The condition $x + y = 1$ splits the square into two equal regions, both with area 1/2. Hence, the probability is 1/2.

Now, what is the probability you will fill the cup in 3 attempts? We similarly want to calculate the probability that $x + y + z \geq 1$ when the three are chosen randomly from 0 to 1. We can again use the help of a diagram:



The entire cube has a volume of 1. With careful calculation, we can find the lower pyramid—where the sum is less than 1—has a volume of 1/6. Hence, the event we are interested in, corresponding to the upper region, has a probability of 5/6.

Now we generalize this logic. What is the probability that you fill the cup in 4 tries? What about n tries?

What we want to do is the following. We can consider the n draws of the random variables and represent them as a point inside an n-dimensional shape of volume 1 with boundary coordinates (1,0,0…,0), (0,1,0…,0), … (0,0,0…,1).

Just as in the 2-dimensional and 3-dimensional cases, we want to find the volume of the lower "pyramid" of the shape that corresponds to the event that the sum of the random variables does not exceed 1.

This "pyramid" in which each edge has length 1 is a shape that comes up frequently in mathematics. It is known as a simplex, and it is a generalization of a triangle in n-dimensional space.

Because the shape comes up so frequently, there is a formula for its volume. Specifically, an *n* dimensional simplex has volume 1/(n!).

Note how this fits in with the results we have already derived. For the 2-dimensional case, the lower triangle had area 1/2 = 1/2!, and in the 3-dimensional case we said the pyramid had a volume of 1/6 = 1/3!

The volume of the simplex is the probability that the sum does not exceed 1. We want the complement event: the chance the sum does exceed 1, which is 1 - 1/n!

This is the probability that the cup will get filled with *n* draws.

Solving the problem

But that's not what we want to figure out! We wish to find the expected number of attempts. If we fill the cup on the 2nd attempt, then we do not try any more: we stop and take the cup back to our boss.

We wish to find the attempt *n* that first fills up the cup. That is, we want to know the cup gets filled on attempt *n*, but that it was not full on the previous attempt *n* - 1.

The probability we want is:

P(*n*) = Pr(full on *n*) - Pr(full *n* - 1)

Using the result above, we find this is:

(1 - 1/*n*!) - (1 - 1/(*n*-1)!) = (*n* - 1)/*n*!

The expected value is the probability of filling up at the *n* attempt (the function P(*n*)) weighted by the number of attempts *n*, summed across all possible values of *n*. This is:

$$\sum_{n=2}^{\infty} nP(n) = \sum_{n=2}^{\infty} n\frac{(n-1)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

And amazingly, the number *e* = 2.71828… is the answer!

There are several sources for this puzzle. One is a book by John Derbyshire, *Prime Obsession*. Another is a blog post "And e appears from nowhere!" from mostlymaths.net.

Puzzle 10: The Straw That Broke The Camel's Back

This problem is a mathematical treatment of an idiom. *The straw that broke the camel's back* refers to how the trivial weight a straw can finally break an overburdened camel's back. The proverb refers to situations where a seemingly small final act precipitates a large reaction or failure. The moral is that the final straw was really no different from any others, but it just happened to have extra significance due to the accumulated weight.

The puzzle investigates this claim: is it really true that the last straw is no different from any others?

Let's make the problem precise. One by one straws are placed on a mathematical camel's back. The straws have random weight, being drawn independently from the uniform distribution 0 to 1. The camel's back breaks when the total weight of the straws exceeds 1.

Is the final straw really the same as any other straw? What's the expected weight of the straw that breaks the camel's back?

Answer To Puzzle 10: The Straw That Broke The Camel's Back

It turns out the idiom is wrong: the straw that breaks the camel's back will tend to be *heavier* than average.

The intuition is that a typical straw has an average weight of 0.50. The last straw, however, is something that broke the camel's back. Since heavier straws are more likely to break the camel's back, this would imply the last straw will be heavier than average.

Let's do a concrete example. Suppose you keep drawing randomly between 1 cent and 5 cent coins until you exceed 1 dollar. The first coin is equally likely to be 1 or 5 cents—for an average of 3 cents. What about the last coin? Note that if you're at 99 cents, you will exceed a dollar with either coin. But if you're at 98, 97, 96, or 95 cents, you will only exceed a dollar if you draw the 5 cent coin. Therefore, the distribution of the last coin shifts towards the coin of larger denomination.

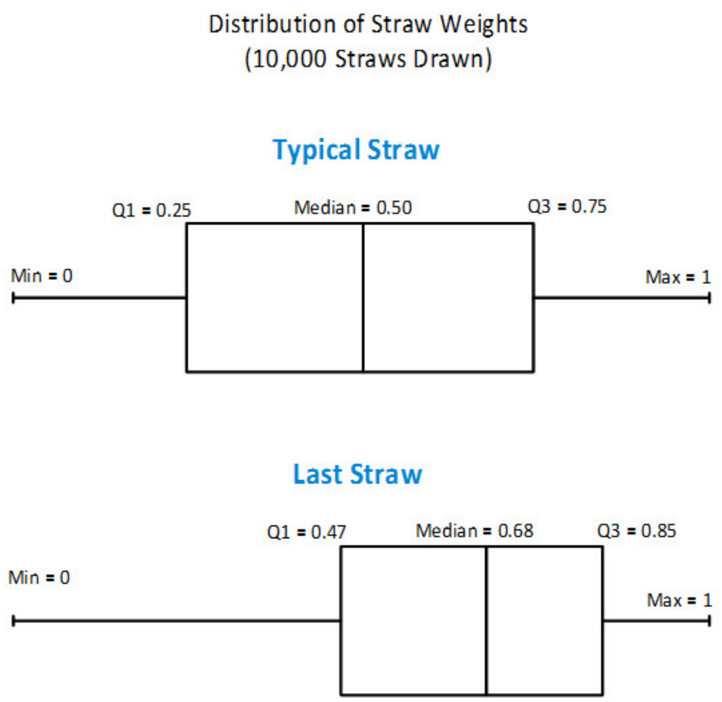
Or think about it in terms of the casino game blackjack. You receive cards valued from 1 to 11 and you bust if your total exceeds 21. Considering only hands where you bust, what's the average value of the last draw you received? Note that the typical card drawn is between 1 and 11 (several face cards are 10), with an average between 6 and 7. But you won't bust on certain cards. For instance, the only way you can bust with a 1 is if you were at 21 and you hit. That's not going to happen, so you'll never bust on a 1. Similarly, the only way you can bust with a 2 is if you were at 20 and took a hit. This is almost never going to happen, so you don't bust on a 2. It's also less likely you will bust on a 3 or a 4. Since low valued cards are less likely to cause you to bust, this means that bust cards will have a higher average value.

Now that we have the intuition of why the final straw is heavier, let us solve the problem with the uniform distribution.

Numerical Analysis

A problem like can be simulated in a spreadsheet. Use the function RAND() to generate a random draw from 0 to 1. In the next column have a running total, and whenever the sum exceeds 1, zero the running total and place that draw into the third column.

From 10,000 draws, there are approximately 3,000 draws that make the sum exceed 1. Here is a box and whisker plot comparing the weight of a typical straw to the weight of a straw that broke the camel's back (made the sum exceed 1).



The typical straw is exactly what we'd expect: it's evenly spread out from 0 to 1. The last straw is definitely skewed towards larger weights, with a median of 0.68.

A typical straw has a mean weight of 0.50 while the last straw has a mean weight of 0.64. The exact value, which we will derive below, is equal to 2 - $e/2$. And suddenly the constant e appears out of nowhere once again.

Analytic solution

First some notation. Let X_1, X_2, …, X_n be n independent draws from the uniform distribution. Denote the sum as $S_n = X_1 + X_2 + \ldots + X_n$.

What we wish to do is find the expected value of X_k, where k is the first value where the sum exceeds 1. In other words, we would have $S_{k-1} < 1$ and $S_k \geq 1$.

We will take a step back and derive another useful function. For x between 0 and 1, define $g(x)$ as follows

$$g(x) = \sum_{n \geq 1} \Pr(S_n \leq x)$$

The function $g(x)$ is the sum of the probabilities that the uniform sum does not exceed x. We can explicitly calculate this from the probability distribution of the uniform sum, which is known as the Irwin-Hall distribution.

The function $g(x)$ is the sum of the probabilities that the uniform sum does not exceed x. In other words, this is the probability that $X_1 + X_2 + \ldots + X_n \leq x$ for $0 < x \leq 1$ and each X_i is between 0 and 1. Geometrically this shape is a simplex whose volume equals $x/(n!)$.

Alternately, we can explicitly calculate the probability from the distribution function of the uniform sum, which is known as the Irwin-Hall distribution. The probability distribution function for the uniform sum of the n^{th} draw, where $0 < y < 1$, is given by is $y^{n-1}/(n-1)!$ We need to take the integral of this between 0 and x to find the cumulative probability that $S_n \leq x$. The result is $x/n!$ and the answer agrees with the derivation of the volume of the simplex method.

Now we can derive $g(x)$ since we know the n^{th} term of the summation is $x/(n!)$.

$$g(x) = \sum_{n \geq 1} \Pr(S_n \leq x)$$

$$= \sum_{n \geq 1} \frac{x^n}{n!}$$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= e^x - 1$$

What a happy coincidence! We started out with a probability sum and that led us into an infinite series which involved the constant e.

We're halfway there. We now need to compute the expectation of the final draw. We need to compute the probability density function $f(x)$, which will be the first draw where the uniform sum exceeds 1. So our expression is the following.

$$f(x) = \sum_{n \geq 1} \Pr(S_n < 1 \leq S_n + x)$$

Now we do a neat trick. We can write an equivalent expression for the inequality. First we will subtract S_n from each term, then we will multiply through by -1 (which switches the direction of the signs), and then finally we add 1 to each term. The steps are the following.

$S_n < 1 \leq S_n + x$
$0 < 1 - S_n \leq x$
$0 > -1 + S_n \geq -x$
$1 > S_n \geq 1 - x$

Now we can re-write $f(x)$ in terms of $g(x)$, which we already have solved for.

$$f(x) = \sum_{n \geq 1} \Pr(S_n < 1 \leq S_n + x)$$

$$= \sum_{n \geq 1} \Pr(1 - x \leq S_n < 1)$$

$$= \sum_{n \geq 1} \Pr(S_n < 1) - \Pr(S_n < 1 - x)$$

$$= g(1) - g(1 - x)$$

$$= e - e^{1-x}$$

Now that we have an expression for $f(x)$, we can compute the expectation of the draw that makes the sum exceed 1.

$$E(x) = \int_0^1 x f(x) dx$$

$$= \int_0^1 x(e - e^{1-x}) dx$$

$$= 2 - e/2 \approx 0.641$$

So the final draw, the one that makes the sum exceed 1, is about 28 percent larger than the average draw of 0.50.

The moral

While the math got a bit involved, the intuition nudged us to the qualitative lesson. The straw that breaks the camel's back is a wrong idiom. In our mathematical example, it was true that the final straw was identical and independently drawn from 0 to 1—so in one sense the last straw really is the same as any other. However, the fact that it broke the camel's back means something—it means the final straw is heavier than average.

I think this provides a lesson for interpersonal conflicts. When a relationship turns sour, the offending party often replies that they were acting exactly the same way as before, and they blame the other person for over-reacting. In fact, this might very well be true: if one's behavior is drawn from a uniform distribution, then the final act was a realization of exactly the same stochastic process. However, it is not true that the person is over-reacting: the final straw tends to be worse behavior than average.

In fact, the situation for personal relationships is likely even more skewed. Imagine you allow each friend to offend you up to a limiting point. Each time they betray you or harm you adds to the weight of the relationship. Now a small offense is quickly forgiven and forgotten. However it is the larger acts that stay in your memory. So small acts are unlikely to break up a relationship—the final straw will be something more egregious than average.

In conclusion, the straw that breaks the camel's back is a wrong idiom. It is not simply the same inconsequential act that overloads a system; the final straw is worse than average, precisely because it is the final straw.

I thank users Jason and Did on Math Stack Exchange who explained how to derive a solution when I was stuck.

Puzzle 11: Random Numbers Ratio

This problem was asked on the 1993 Putnam Exam, question B3.

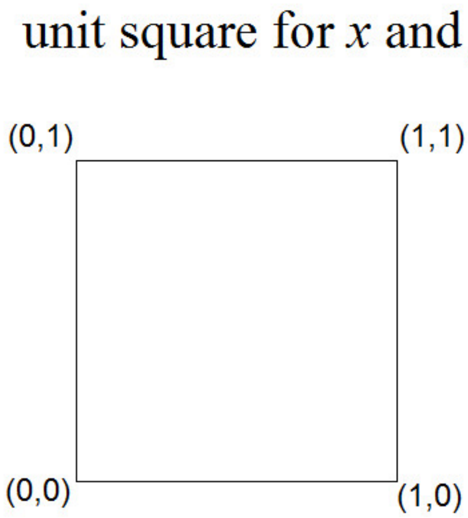
Let x and y be two random numbers between 0 and 1. What is the probability that x/y rounds to an even number?

It is tempting to think, “Since x/y ranges from 0 to infinity, and a number is equally likely to be even or odd, the probability of rounding to even or odd must be the same. Therefore the answer is 50 percent.”

But (luckily) this answer is wrong! What is the answer?

Answer To Puzzle 11: Random Numbers Ratio

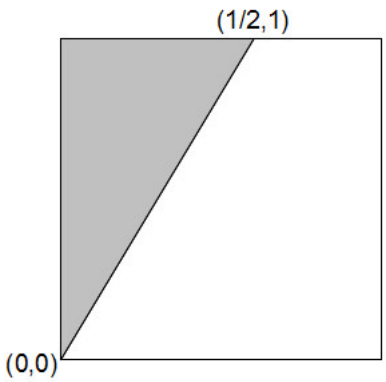
This problem is an example of geometric probability, and it will help to draw a diagram. The two random numbers x and y can be written as an ordered pair (x, y) and be plotted on a graph. Since both x and y vary between 0 and 1, the possible draws correspond to the points on the unit square.



The event that x/y rounds to an even number is a subset of this unit square. To find the probability of the event, we need to calculate what percentage of the unit square corresponds to the event. Since the unit square has an area of 1, this means the area of the event is the probability that it occurs.

In order for x/y to round to an even number, it must round to one of the numbers 0, 2, 4, …, and so on.

When does x/y round to 0? This happens only when x/y is between 0 and 1/2. Written another way, it happens when $0 \leq x/y < 1/2$. Multiplying by y, this means $0 \leq x < y/2$. We can sketch this area by shading the area between the lines $x = 0$ and $x = y/2$, as follows.

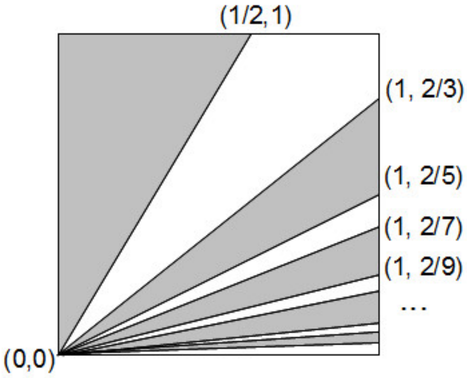


The region is a triangle with a height of 1 and a base of 1/2, so its area is 1/4.

We can similarly find the conditions for x/y to round to 2, 4, 6, or any even number. Note that:

x/y rounds to 2 if and only if x/y is between 3/2 and 5/2, which means $(2/5)y \leq x < (2/3)y$
x/y rounds to 4 if and only if x/y is between 7/2 and 9/2, which means $(2/7)y \leq x < (2/9)y$
….Pattern continues….

Here is a sketch of some of these conditions (not to scale).



Notice that each region is a triangle with a height of 1 and a base equal to the difference of two bounding values (e.g. 2/3 - 2/5). The region for x/y rounding to $2k$, for positive integers k, will be a triangle with a height of 1 and a base of $[2/(4k - 1) - 2/(4k + 1)]$. Therefore the area is $[1/(4k - 1) - 1/(4k + 1)]$.

We can sum up the areas of the triangles to find the probability.

Pr(x/y rounds to even) = Pr(x/y rounds to 0) + Pr(x/y rounds to 2) + …
Pr(x/y rounds to even) = 1/4 + (1/3 - 1/5) + (1/7 - 1/9) + …

At this point you could evaluate the partial sum and estimate the series will converge to about 46 percent.

However, there is an infinite series named after the German mathematician Leibniz that can help us find an exact answer. (Tangential trivia: the formula was first discovered by the Indian mathematician Madhava three hundred years before).

The remarkable formula is $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots$

The proof is that $\pi/4 = \arctan(1)$, which has the Taylor series expansion of $1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots$

Let's use this formula to re-write the answer to the puzzle.

Pr(x/y rounds to even) = 1/4 + (1/3 - 1/5) + (1/7 - 1/9) + …
Pr(x/y rounds to even) = 1/4 + (1 - 1) + (1/3 - 1/5) + (1/7 - 1/9) + …
Pr(x/y rounds to even) = (1/4 + 1) - 1 + 1/3 - 1/5 + 1/7 - 1/9 + …
Pr(x/y rounds to even) = 5/4 - (1 - 1/3 + 1/5 - 1/7 + 1/9) + …
Pr(x/y rounds to even) = 5/4 - π/4

And remarkably this probability problem has an answer that involves π!

The probability of rounding to an even number is about 46 percent, so that means the probability of rounding to an odd number is about 54 percent. The two are not equal!

Puzzle 12: π^e or e^π

Which quantity is greater: e^π or π^e? The challenge is to solve this without a calculator.

Answer To Puzzle 12: π^e or e^π

The answer is e^π is larger. There are several ways you can solve this problem. I've listed a few proofs below. I feel the first few proofs are the approaches that might come to mind, but they are relatively involved proofs.

Method 5 is my favorite, but it had me wondering "how the heck did they think of that?"

Method 1: comparing e^x to x^e

In one expression, the value of e is in the base and in the other it is in the exponent. One way to approach the problem is to compare the two quantities.

Define a function $f(x) = e^x - x^e$.

We can evaluate this function at the values of interest.

$f(e) = e^e - e^e = 0$
$f(\pi) = e^\pi - \pi^e$

We are basically interested in finding out if $f(\pi)$ is positive or negative. We know that $f(e)$ is equal to zero, so we might get an idea of whether the function is increasing or decreasing by considering its derivative. So we calculate.

$f(x) = e^x - x^e$
$f'(x) = e^x - ex^{e-1}$

When $y > e$, we have:

$f'(y) = e^y - ey^{e-1} > e^y - ee^{e-1} = e^y - e^e > 0$

The last step is because $y > e$, and the exponential function is increasing. Since $f'(x)$ is positive for $x > e$, this implies that $f(x)$ is increasing for values $x > e$.

Since $\pi > e$, we have $f(\pi) > f(e) = 0$. This will lead to the result.

$f(\pi) = e^\pi - \pi^e > 0$

Hence: $e^\pi > \pi^e$

Method 2: π to e(ln(π))

The natural logarithm is a monotonically increasing function. This means we can take the natural log of two expressions and compare the size of the resulting values. Thus we can compare:

$\ln(e^\pi) = \pi$
$\ln(\pi^e) = e \ln(\pi)$

Our attack will be similar to method 1. Notice that in the above expressions, the value of π appears once as its own value and once as an argument to the natural log function. We can create the following function to compare the quantities.

$f(x) = x - e \ln(x)$

Note that:

$f(e) = e - e \ln(e) = 0$
$f(\pi) = \pi - e \ln(\pi) = ??$

If we can figure out whether $f(\pi)$ is positive or negative, we can solve our problem. We proceed by looking at the derivative.

$f'(x) = 1 - e/x$

When $y > e$, we will have $e/y < 1$, so:

$f'(y) > 0$

We have just reasoned that $f'(x)$ is positive for values larger than e, therefore $f(x)$ will be increasing for values larger than e, such as π .

We can finish the proof

$f(\pi) = \pi - e \ln(\pi) > 0$

This implies $\pi > e \ln(\pi)$

And using each side as an exponent to the base e, we get $e^{\pi} > \pi^{e}$.

Method 3: consider $\pi/\ln(\pi)$

When we take the natural logarithms of both sides we get the following.

$\ln(e^{\pi}) = \pi$
$\ln(\pi^{e}) = e \ln(\pi)$

Let's say that we divide the first equation by the second. Then we end up with

$\pi/(e \ln \pi) = \pi/(\ln \pi)\ (1/e)$

We now have an equation that involves the ratio of π to the natural log of π . As in the first two methods, we will define a function that will yield light on the size of the quantities involved.

Let $f(x) = x/(\ln x)$, then we find the derivative (skipping a few steps)

$f'(x) = (-1+ \ln x)/(\ln x)^2$
$f'(x) = 0$ when $x = e$
and $f'(x) > 0$ when $x > e$
and $f'(x) < 0$ when $x > e$

Therefore we have found that $f(x)$ has a local minimum at $x = e$. So we can conclude:

$f(e) < f(\pi)$
$e/\ln(e) < \pi /(\ln \pi)$
$e < \pi /(\ln \pi)$
$e \ln(\pi) < \pi$

And using each side as an exponent to the base e, we get $\pi^{e} < e^{\pi}$.

Method 4: exponentiating to the reciprocal

Let $f(x) = x^{1/x}$.

Differentiating (by taking the natural log of both sides, using implicit differentiation, and then doing a lot of algebra), this simplifies to the expression:

$f'(x) = x^{1/x}(1/x^2)(1 - \ln x)$

Clearly $f'(e) = 0$. We can also do a bit of reasoning to conclude this is a global maximum.

(When $x < e$, all terms in the derivative are positive so the function is increasing. But when $x > e$ the first two quantities are positive but $(1 - \ln x)$ will be negative so the function is decreasing. In other words, the function increases up to $x = e$ and then decreases, so $x = e$ is a global maximum).

We use this to derive the result.

$f(e) = e^{1/e} > f(\pi) = \pi^{1/\pi}$

Now we raise both sides to the power of e to get $e^1 > \pi^{e/\pi}$.

Now we raise both sides to the power of π to get $e^{\pi} > \pi^{e}$.

Method 5: Taylor series

We start with the Taylor series expansion of e^x.

$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

For positive values of x, we know the partial sum has to be less than the infinite sum, so we can conclude the partial sum of just the first two terms will be less than the entire sum:

$e^x > 1 + x$

Believe it or not, this is all you really need to know in this elegant proof! We will pick a clever choice of value for x, namely $x = \pi /e - 1$. When we substitute we find:

$e^{\pi/e - 1} > 1 + (\pi/e - 1) = \pi/e$

We can re-write this as the following since $e^{-1} = (1/e)$

$(1/e)\ e^{\pi/e} > \pi/e$

Now we can cancel out the (1/e) on both sides.

$e^{\pi/e} > \pi$

Finally, we raise both sides to the power of e.

$e^{\pi} > \pi^{e}$

And magically we are done!

Just for the record

$e^{\pi} = 23.14069\ldots$ versus $\pi^{e} = 22.45915\ldots$

So you could do this on a calculator, but it's oh so much more fun to solve it with calculus ;)

Credit: I learned about some of the solution methods on Math Stack Exchange.

Puzzle 13: Guess The 1,000th Number

I have written a long, long sequence of positive whole numbers. Can you guess the 1,000 item on my list?

I will give you 3 clues about my sequence of numbers $\{a_n\}$.

1. The list of numbers is strictly increasing ($a_{k+1} > a_k$).

2. The second number in my list is 2 ($a_2 = 2$).

3. My list always has $a_{mn} = a_m a_n$ when the two numbers m and n are relatively prime. (Call this the multiplicative property). For example, $a_{60} = a_4 a_{15}$ because 4 and 15 are relatively prime.

What possible number(s) could I have written for a_{1000}?

Answer To Puzzle 13: Guess The 1,000th Number

The method of attack is to work in small steps. We try to figure out values of early entries in the list and see if there is a pattern.

We start by noting the value for a_1 is easy. We know $a_1 < a_2 = 2$ by the increasing property. The only positive whole number less than 2 is 1, so we must have $a_1 = 1$.

The next step is to figure out the value for a_3. This requires a bit of guesswork and hacking to play around with the properties given.

The multiplicative property leads us to consider multiples of 3, and the increasing property allows us to compare the size of larger values to another.

Here is the sequence of steps to show that $a_3 = 3$. We start out by considering a_{15} and then showing how repeatedly applying the multiplicative property and the increasing list order leads to the conclusion.

Here we go!

$$
\begin{aligned}
a_{15} &= a_3 a_5 && \text{(multiplicative property)}\\
&< a_{18} && \text{(increasing list)}\\
&= a_2 a_9 && \text{(multiplicative property)}\\
&< a_2 a_{10} && \text{(increasing list)}\\
&= a_2 a_2 a_5 && \text{(multiplicative property)}\\
&= (a_2)^2 a_5 &&\\
&= 4a_5 && (a_2 = 2)\\
\implies a_3 a_5 &< 4a_5 && \text{putting it all together}\\
\implies a_3 &< 4 &&\\
\implies a_3 &= 3 && \text{3 is the only integer between 2 and 4}
\end{aligned}
$$

This proof struck me as requiring a bit of MacGyver-like intuition, but for that very reason it was amazing! It took a bit of work, but we were able to conclude that $a_3 = 3$.

Now comes the next leap to guess that the pattern holds in general. That is, we will guess that $a_n = n$ and see if we can prove that by induction.

Here is the proof. Let us assume that $a_n = n$ for all values less than or equal to k. We have already proven the base cases of $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$. We wish to prove $a_{k+1} = k + 1$.

Case 1 (*k* is odd): If k is an odd number, then $2k$ will be an even number. We use the fact that 2 and k are relatively prime to derive the following.

$a_{2k} = a_2 a_k$

Now we use the induction hypothesis on $a_k = k$ and the fact $a_2 = 2$.

$a_{2k} = 2k$

We know that $a_k = k$ and $a_{2k} = 2k$. What could be the values of the intermediate entries in the list $a_{k+1}, a_{k+2}, \ldots, a_{2k-1}$?

Note there are exactly $k - 1$ entries in the list. There are also exactly $k - 1$ whole numbers greater than k and less than $2k$. There is only one way to order the numbers in a strictly increasing order, namely, we must have that $a_{k+1} = k + 1$, $a_{k+2} = k + 2$, …, and $a_{2k-1} = 2k - 1$.

Therefore we have proven $a_{k+1} = k + 1$ and the induction is complete.

Case 2 (*k* is even): This proof is almost entirely the same. By the induction hypothesis we have $a_{k-1} = k - 1$. Now we know that $k - 1$ is odd, so we follow the steps of Case 1. This will result in $a_k = k$.

Therefore, $a_{1000} = 1{,}000$.

The remarkable conclusion is the following: the only possible list of positive numbers that are increasing, have the second entry equal to 2, and have the multiplicative property is the natural numbers! This is an interesting number theory result illustrating how a few properties force a particular ordering.

Source of puzzle: this is puzzle 42 from the website qbyte.org.

Puzzle 14: How Many Primes?

Consider the set of natural numbers: 1, 2, 3, 4, …

It is well known this list has an infinite number of primes (a proof is presented in the solution as well).

But what about other lists of numbers? How many primes are in these lists? For example, consider the set of even numbers: {2, 4, 6, …}. Now obviously the only prime number is 2, because every subsequent term is divisible by 2 and therefore cannot be prime. Similarly, the only prime number in the list {3, 6, 9, …} is the first number 3, because every subsequent term is divisible by 3. And to generalize, the only possible prime number in a list of multiples of a number a: {a, $2a$, $3a$, …} could be the first number a.

Now for the challenge. Consider a list that starts with 2 and increases by 3 each time: 2, 5, 8, 11, 14, 17, …

How many prime numbers are in this list? Will it be a finite number—like the multiples of a—or will there be an infinite number of primes?

Answer To Puzzle 14: How Many Primes?

Before tackling the main problem, it will be instructive to recall Euclid's proof that there are infinitely many prime numbers in the natural numbers {1, 2, 3, …}.

Consider any finite list of primes in ascending order: p_1, p_2, …, p_k. Euclid's brilliant idea was to demonstrate one could always find a larger prime number.

Create a new number N that is the product of all the primes minus one.

$N = p_1p_2\ldots p_k - 1$

Obviously this number N is larger than p_k. If this number N is itself prime, then we have created a prime number larger than p_k.

Otherwise, it must be the case that N is not prime and has some prime factor. But what could be a factor of N? Clearly p_1 cannot be a factor of N because dividing N by p_1 will have a remainder of 1. Similarly, none of the primes p_2, …, p_k can be factors of N, as each will leave a remainder of 1. Therefore, N must have at least one prime factor p_{k+1} that is larger than p_k.

In either case, that means for any finite list of primes, one can find a larger prime, so there are infinitely many primes.

A preliminary result for the list 2, 5, 8, …

Does the sequence of numbers starting with 2 and increasing by 3 contain a finite or infinite number of primes? The proof will be similar to Euclid's proof.

One preliminary fact is needed. Notice that every item in this list is one less than a multiple of 3. In other words, every term can be written as in the form $3k - 1$. What can be said about the factors of such numbers?

The preliminary fact is this: every number of the form $3k - 1$ must have at least one prime factor that is also one less than a multiple of 3, with a similar form $3j - 1$.

Here is the reason why. Consider what happens when an integer is divided by 3. There are 3 possible cases that can occur: it can have no remainder ($3a$), it can have a remainder of 1 ($3a + 1$), or it can have a remainder of 2, which is equivalent to saying it is one less than a multiple of 3 ($3a - 1$).

We wish to prove that every number of the form $3k - 1$ has at least one prime factor of the form $3j - 1$. This is relatively easy to prove.

If the number has any prime factors of the form $3j$, then the resulting number would have to be a multiple of 3. So this will not produce a number of the form $3k - 1$.

If the number only has prime factors of the form $3j + 1$, then the resulting number would end up as 1 more than a multiple of 3. (To see why, multiply two numbers: $(3a + 1)(3b + 1) = 9ab + 3a + 3b + 1$. The first three terms are divisible by 3 so the +1 means it is one more than a multiple of 3.)

So the number cannot have prime factors of the form $3j$, nor can have only prime factors of the form $3j + 1$. It must clearly have at least one factor of the form $3j - 1$. (And we can demonstrate how this works: $(3a - 1)(3b + 1) = 9ab + 3a - 3b - 1$. The first three terms are divisible by 3, so the -1 means this number is one less than a multiple of 3).

Proving there are infinite primes in 2, 5, 8, …

Now to tackle the main problem. Consider any finite list of primes with the largest prime p. We will show we can always find a larger prime than p in the list.

Create a new number N that is 3 times the product of all the numbers in the list less than p, and then subtract 1 from it.

$N = 3(2 \cdot 5 \cdot 8 \cdot 11 \cdots p) - 1$

The first thing to notice is this number N is of the form $3j - 1$ and hence would be part of the list {2, 5, 8, …}. Secondly, this number N is obviously larger than p.

If the number N is prime itself, then that means it will be a larger prime than p and we are done.

Otherwise, the number N is not prime and has a prime factor. From the preliminary proof, one of the prime factors must be of the form $3j - 1$. But by construction, this prime factor cannot be one of the terms 2, 5, 8, …, p, since each of those leaves a remainder of 1 when dividing into N. Hence, this prime factor is a larger number than p and of the form $3j - 1$.

In conclusion, we have proved the list {2, 5, 8, …} cannot have a largest prime and therefore contains infinitely many prime numbers.

You might be wondering if the result can be generalized. In fact, it can. It turns out that any arithmetic sequence $\{a, a + b, a + 2b, \ldots\}$ has an infinite number of primes when a and b are coprime. The result is known as Dirichlet's theorem on arithmetic progressions. The proof is beyond the scope of this book, but interested readers can search for the theorem and find proofs online.

I read about this fun problem in the book *The Art of the Infinite* by Robert and Ellen Kaplan.

Puzzle 15: Powers Of 2

If *n* is a positive number, what is the probability that 2^n has the leading digit of 1?

For example, $2^{10} = 1024$ has a leading digit of 1.

Answer To Puzzle 15: Powers Of 2

You can get pretty close to the correct answer by calculating a few powers of 2. For positive *n*, the first 10 powers of 2^n are 2, 4, 8, 16, 32, 64, 128, 256, 512, and 1024. The three numbers 16, 128, and 1024 have a leading digit of 1, suggesting a probability of 3/10 = 0.30.

The exact probability has a limit of $\log_{10}(2) \approx 0.30103$. Where does the logarithm come from, and how can we derive this answer?

We need to be careful whenever we calculate probabilities over an infinite sample. Let's define f(*n*) to count the number of times the leading digit is 1 for powers of 2 not exceeding 2^n. For example, f(10) counts the number of times the leading digit is 1 for powers of 2 not exceeding $2^{10} = 1024$. We counted the numbers 16, 128, and 1024, which means f(10) = 3.

The problem is to calculate f(*n*)/*n* as the limit of *n* goes to infinity. We can do this by calculating an explicit formula for f(*n*), based on two observations.

Claim 1: For a given number of digits *k*, there is always some power of 2^n that has *k* digits and a leading digit of 1.

From our example of calculating the powers up to 2^{10}, we know there is always some power of 2^n that has 1, 2, 3, and 4 digits. We can get a 5 digit number by doubling the largest power with 4 digits. Similarly, we can get a 6 digit number by doubling the largest power with 5 digits. Proceeding by induction, we can get always get a power of 2 for any number of digits.

On the other hand, the smallest power of 2^n for any given number of digits has a leading digit of 1. This can be proved by contradiction. If the smallest power had the leading digit of 2 (or larger), then we could definitely divide that number in half to get another number with the same number of digits. This contradicts that the number with the leading digit of 2 (or larger) was the smallest power.

Combining these facts, we know 2^n achieves a leading digit of 1 exactly once for a given number of digits *k*.

Claim 2: f(*n*) = floor(*n* log 2).

We'll explain the formula with an example. Suppose 2^N has 20 digits. What is the value of f(*N*)? We can use claim 1 to find the answer. From claim 1, there must be some power of 2^n with 20 digits with a leading digit of 1. There must also be a success with 19 digit numbers, 18 digit numbers, 17 digit numbers, …, and so on to the 2 digit number example of 16. So we have f(*N*) = 19 = 20 - 1. Generalizing the logic, f(*N*) is equal to the number of digits in 2^N minus 1.

How many digits are there in 2^n? We have solved this type of problem in the puzzle about how many digits there are in 125^{100}.

The answer is related to powers of 10. Numbers less than 10 have 1 digit, numbers less than 100 have 2 digits, and numbers less than 10^x have *x* digits. For an arbitrary number *y*, the number of digits is the value *x* such that $y < 10^x$. If we take logarithms base 10 on both sides, we have log(*y*) < *x*. So we want the largest whole number less than *x*, so we need to round down *y* and then add 1. This is written as floor(log(y)) + 1.

Using $y = 2^n$, we conclude the number of digits in 2^n is floor(2 log *n*) + 1. We need to subtract 1 to arrive at f(*n*) = floor(*n* log 2).

Solving the problem

Finally we take the limit of the ratio.

f(*n*)/*n* = floor(*n* log 2) / *n*

The floor is equal to *n* log 2 less some fractional amount ε between 0 and 1. Thus, we find the following:

$(n \log 2 - \varepsilon) / n$
$= \log 2 + \varepsilon/n$
$= \log 2$ as $n \to \infty$

The probability 2^n has a leading digit of 1 is just above 30 percent at $\log 2 \approx 0.301$.

Credit: This is problem 90 of *Challenging Mathematical Problems with Elementary Solutions, Volume 1* by A.M. Yaglom and I.M. Yaglom.

Puzzle 16: Smaller Envelope

In one sealed envelope I have written a positive number x and in another I have written a different positive number y.

I randomly give you one of the envelopes and you can look at the number.

Can you tell me whether the number is the smaller one? Can you do it with better than 50/50 odds?

Answer To Puzzle 16: Smaller Envelope

At first it seems like the best you can do is take a random guess. After all, the only information you are given is the value of one of the envelopes. You don't know if that value is the smaller value or the larger value.

It seems you are doomed to a 50 percent chance of winning. But remarkably you can do better! While the strategy does not perform much better than 50 percent, it is provable that it is slightly better than random guessing.

Here is the strategy. Open the envelope and look at the value x. Then take a fair coin and count the number of times until the first head appears. Define the number z as the number of tosses plus 1/2 (we add 1/2 so that z is always in between two positive numbers). If $z > x$, declare that x is the smaller value; otherwise say that x is the larger value.

How often will this strategy win? There are three cases to consider. First, if z is in between x and y, then we say x is smaller and we will be correct. In the other two cases, z is either greater than both x and y, or z is less than both x and y. For these cases we have a 50/50 chance of being right, as the envelope was given at random.

Therefore, this strategy wins slightly more than 50 percent of the time!

The trick is that we are conditioning on the value of x. When x is a low value like 1, we are sure to say x is the smaller number and we will be correct, and analogously, when x is a large value, we are very unlikely to toss so many tails in a row, so we are almost surely going to say x is the larger value.

Many people cannot believe this result and I got many comments that this puzzle is somehow misleading or the analysis was wrong.

I pointed out this problem appears in section 4.3 of Grinstead and Snell's Introduction to Probability and I jokingly said I would never write about a counter-intuitive probability result again.

Puzzle 17: Russian Roulette Option

In Russian roulette, a certain number of bullets are placed in a revolver. A turn involves spinning the cylinder to randomize the location of the bullets, at which point a player puts the gun to his head and pulls the trigger. If the player is lucky to survive, the game continues with the next player.

Consider two different versions of the game with identical six-shooter guns.

Situation 1: You are playing the game with one bullet.

Situation 2: You are playing the game with four bullets.

In each game, you are given the option to pay money to remove a single bullet. If your preferences are to be logically consistent, should you pay more money to remove the bullet in situation 1, situation 2, or would you pay the same amount?

Answer To Puzzle 17: Russian Roulette Option

Almost everyone (myself included) would pay more money in situation 1 to remove the single bullet. The gut feeling is that it's worth more money to survive with certainty than to reduce the odds of death in situation 2.

But this is wrong according to rational choice theory! It is logically sensible to pay more to remove a bullet in situation 2 if you prefer being alive and having more money. The problem is known as Zeckhauser's Paradox.

Let's see why. Consider the events D = Dead and A = Alive. Also consider L_x as being alive after paying x dollars and L_y as being alive after paying y dollars.

If you pay x dollars to remove one bullet from six, then you are saying the event of being alive after paying x dollars is equal to the utility for the lottery of playing the game, in which there is a 1/6 chance of death and a 5/6 chance of being alive. (In Von Neumann and Morgenstern utility theory a rational agent is indifferent between two lotteries with the same expected outcome. So the value x you are willing to pay is the one that makes you indifferent—any more and you are overpaying, any less and you'd prefer to remove the risk.)

Therefore, with a utility function u, we have:

$$\mathrm{u}(L_x) = (1/6)\ \mathrm{u}(D) + (5/6)\ \mathrm{u}(A)$$

Similarly, when you are willing to pay y dollars to remove a bullet from four to three, that means you are indifferent in the lotteries where you play the game with four bullets or pay to play the game with three bullets. This means the following equation:

$$(3/6)\ \mathrm{u}(L_y) + (3/6)\ \mathrm{u}(D) = (4/6)\ \mathrm{u}(D) + (2/6)\ \mathrm{u}(A)$$

We can simplify the above equation to get:

$$\mathrm{u}(L_y) = (2/6)\ \mathrm{u}(D) + (4/6)\ \mathrm{u}(A)$$

If we take $\mathrm{u}(D) = 0$, then since we prefer to be alive that means $\mathrm{u}(A) > 0$. So we have derived the following equations:

$$\mathrm{u}(L_x) = (5/6)\ \mathrm{u}(A)$$
$$\mathrm{u}(L_y) = (4/6)\ \mathrm{u}(A)$$

Let's subtract the second equation from the first, and notice the result is positive:

$$\mathrm{u}(L_x) - \mathrm{u}(L_y) = (1/6)\ \mathrm{u}(A) > 0$$

In other words, you prefer to be alive after paying x dollars to being alive after paying y dollars. But since you prefer to pay less—since having more money is better—that must mean x is a smaller amount of money than y!

Therefore, under the Von Neumann and Morgenstern utility theory, you should be willing to pay more for situation 2 where you remove one bullet from four.

This goes against intuition, so let us offer a few justifications for the logic.

The main mental block is that people prefer certain outcomes versus risk reductions that could have greater impact. One of my friends was eating a bacon cheeseburger with fries and having a beer. I asked if he wanted ketchup and he told me he didn't eat high fructose corn syrup as it was unhealthy. It was evidently easier for him to remove one source of risk entirely than to reduce his calories in other areas of habit.

The tendency to favor certainty is related to what is called the *zero risk bias*. In one study, people were asked how much they would pay to remove a pesticide that would cause 15 adverse reactions in 10,000 containers. People were willing to pay $1.04 to reduce the risk from 15 reactions to 10, but they would pay more than twice that—$2.41—to reduce the risk from 5 reactions to 0. The absolute number of cases is the same in both, but the idea of "zero risk" led people to pay more.

I came across this paradox in Ken Binmore's book *Playing for Real: A Text on Game Theory*.

Puzzle 18: Group Testing

This problem is interesting because it has historical significance in World War II. In fact, the sampling technique led to a cost savings of 80 percent!

Consider the following setup. You have 1,000 soldiers that might have a disease with a known prevalence $p = 0.05$. Which soldiers have the disease? One method is to use a brute force technique and test every single soldier. The disadvantage is you'll need to run 1,000 tests and each test could be costly.

An alternative is to use group testing. What you do is you pool the blood samples of k soldiers together and test that pooled sample. If the test result is negative, then you know that *no one in the group has the disease*. So that single test suffices for all k soldiers. If the test is positive, however, you are out of luck: you'll have to perform individual tests for each of the k soldiers (meaning $k + 1$ tests in all).

The problem is, what value of k will be best to reduce the number of expected tests? (Assume the test works for arbitrary pooled samples).

Solve for $N = 1{,}000$ soldiers and $p = 0.05$.

Challenge: find an expression for general N and p.

Answer To Puzzle 18: Group Testing

In a group of 1,000 soldiers and $p = 0.05$, the optimal group size is $k = 5$, which results in an expected number of about 427 tests. This is a big reduction from 1,000 tests.

We solve the problem generally to find a neat approximation. The probability that an individual has the disease is p, which means the probability someone does not have the disease is $(1 - p)$. In a group of k, the chance that no one has the disease is the product of k terms:

$\Pr(\text{all negative}) = (1 - p)(1 - p)\ldots(1 - p) = (1 - p)^k$

The chance that a group test is positive is the chance that at least one person has the disease. This is the complement event of no one having the disease.

$\Pr(\text{positive}) = 1 - \Pr(\text{all negative}) = 1 - (1 - p)^k$

Now we set up the expression for the number of tests. If we divide the soldiers into groups of size k, there will be N/k pooled groups of blood samples to test. If the pooled sample is negative, we only need to do 1 test. If the pooled sample is positive, we need to run k additional tests for each person, resulting in $k + 1$ tests in all. Thus, we can write an expression for the number of tests.

$E(\text{tests}) = (\text{Number groups})[\Pr(\text{test negative})(1) + \Pr(\text{test positive})(k + 1)]$
$E(\text{tests}) = (N/k)[(1 - p)^k + (1 - (1 - p)^k)(k + 1)]$
$E(\text{tests}) = N[1 + 1/k - (1 - p)^k]$

We now have an equation that relates the number of tests to the group size k. We find the minimum by taking the partial derivative with respect to k and setting it equal to zero.

$0 = 1/k^2 + (1 - p)^k \ln(1 - p)$

The logarithm term appears because the derivative of p^k is equal to $p^k \ln(p)$.

Now we will use some tricks. We use the Taylor expansion of $\ln(1 - p) \approx -p - p^2/2$ and we use the binomial theorem $(1 - p)^k \approx 1 - kp + 0.5(k - 1)kp^2$. We multiply these together and approximate at the first power of p to get the following.

$(1 - p)^k \ln(1 - p) \approx -p$

Substituting this back into our derivative, we have:

$0 \approx 1/k^2 - p$
$k \approx 1/\sqrt{p}$

We substitute this back into our original expression for the number of tests, and we again use some approximations to find a compact answer.

$E(\text{tests}) = N[1 + 1/k - (1 - p)^k]$
$E(\text{tests}) \approx N[1 + \sqrt{p} - (1 - p)^{\sqrt{p}}]$
$E(\text{tests}) \approx N[1 + \sqrt{p} - (1 - \sqrt{p})]$
$E(\text{tests}) \approx 2N\sqrt{p}$

In the case of $N = 1000$ and $p = 0.05$, the exact answer (found numerically) was to have groups of 5 leading to about 427 tests.

The approximation, which is much easier to compute, is fairly accurate. It yields an answer of 4.47 groups (rounds to 5) with an expected number of tests at 448.

Sources and further reading. This problem appears in the statistics textbook by William Feller, *Introduction to Probability Theory and Its Applications* (1950). I also did some research on the history of this problem. According to one book, group testing was originally created by Robert Dorfman to screen soldiers for syphilis, but its use was limited because the test was not accurate in groups larger than 8 or 9 soldiers. However, combinatorial group testing has found other applications in "medical, chemical and electrical testing, coding, drug screening, pollution control, multiaccess channel management, and recently in data verification, clone library screening and AIDS testing." Pretty neat stuff!

Puzzle 19: Basketball Knockout

You can I take turns shooting free throws. We keep playing until one person—declared the winner—makes a shot following an opponent's missed shot.

If we each have probability p of making a shot, and I go first, what is the chance that I will win?

Bonus: What value of p gives the first player the best chance to win?

Exclude the cases $p = 0$ and $p = 1$ in which case no one ever wins.

Answer To Puzzle 19: Basketball Knockout

The key to solving this game is to understand the game "resets" itself after certain circumstances. For example, if the first two shots are made, the chance that the first player wins is exactly the same as when the game started. Precisely such a game is an example of a *stationary process*, because the probability distribution is "stationary" and does not change over time.

We will find it easier to solve for the probability that the second shooter wins the game. Since there are no draws, the probability the first player wins is one minus the probability the second player wins.

When player 2 takes a shot, there are two possible histories: player 1 has either made the shot or missed it. Let x denote an indicator variable for the event of winning the game after a missed shot and y the same after a made shot. Finally let the event of player 2 winning be denoted by the indicator variable z. (An indicator variable is 1 for a success and 0 for a failure. The expected value of z is the probability of player 2 winning the game).

We can express the expected value of z in terms of the conditional events that player 1 missed and made the previous shot, remembering the probability of making a shot is p.

$E(z) = E(z \mid \text{player 1 missed}) + E(z \mid \text{player 1 made})$
$E(z) = \Pr(\text{player 1 missed})\, E(x) + \Pr(\text{player 1 made})\, E(y)$
$E(z) = (1 - p)E(x) + pE(y)$

Now we can use the stationarity of the game to create expressions for $E(x)$ and $E(y)$.

First we solve for $E(x)$. Following a miss, player 2 can win in that round by hitting a shot. If, however, player 2 also misses the shot, then player 2 can only win if in the next move player 1 also misses a shot—in which case player 2 is once again trying to win after a missed shot, which is the probability $E(x)$. Putting these facts together, we can solve for $E(x)$.

$E(x) = E(x \mid \text{player 2 makes}) + E(x \mid \text{player 2 and 1 miss})$
$E(x) = \Pr(\text{player 2 makes})(1) + \Pr(\text{players 2 and 1 miss})\, E(x)$
$E(x) = p + (1 - p)^2E(x)$
$E(x)(1 - (1 - p)^2) = p$
$E(x) = 1/(2 - p)$

Now we similarly solve for $E(y)$. Following a made shot, player 2 cannot win in that round. There are three ways that player 2 can win.

(1) Both players make their next shots, in which case player 2 is trying to win following a made shot and the chance is $E(y)$.

(2) Both players miss their next shots, in which case player 2 is trying to win following a missed shot, and the chance is $E(x)$.

(3) Player 2 makes a shot and player 1 misses the next one, in which case player 2 is again trying to win following a missed shot, and the chance is $E(x)$.

The conditional expectation is therefore the following.

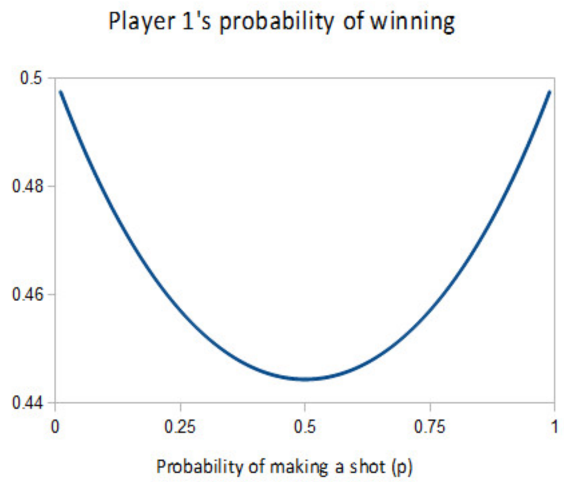
$E(y) = E(y \mid \text{event 1}) + E(y \mid \text{event 2}) + E(y \mid \text{event 3})$
$E(y) = \Pr(\text{event 1})E(y) + \Pr(\text{event 2})E(x) + \Pr(\text{event 3})E(x)$
$E(y) = p^2E(y) + (1 - p)^2E(x) + p(1 - p)E(x)$
$E(y)(1 - p^2) = (1 - p)^2E(x) + p(1 - p)E(x)$
$E(y)(1 - p)(1 + p) = (1 - p)^2E(x) + p(1 - p)E(x)$
$E(y)(1 + p) = (1 - p)E(x) + pE(x)$
$E(y)(1 + p) = E(x)$
$E(y) = E(x)/(1 + p)$
$E(y) = 1/[(1 + p)(2 - p)]$

Now we can solve for the overall chances of player 2 winning.

$E(z) = pE(x) + (1 - p)E(y)$
$E(z) = (1 + p - p^2)/[(1 + p)(2 - p)]$

The probability that player 1 wins is $1 - E(z) = 1/[(1 + p)(2 - p)]$. (It actually should come as no surprise that player 1's probability of winning is the same as when player 2 is trying to win after a made shot: when the game starts, the situation is identical to one in which player 1 is shooting after player 2 has already made a shot.)

Here is a graph of winning probabilities as p varies between 0 and 1 (remember p = 0 means no one ever scores and p = 1 means both always score—so those games never end.)



The probability that player 1 wins is always less than 50 percent, and this makes sense: player 1 is always at a disadvantage by going first. The worst case is when p = 0.5, in which the first player has a 4/9 chance of winning. The game is more fair if the players are both very poor shooters (low value of p) or they are very good shooters (high value of p), as the most likely case is they will both miss or make their shots in the first round, negating player 2's advantage of shooting second.

Puzzle 20: Cards On A Square

On a square-shaped Lazy Susan turntable, a card is placed on each corner. You are in a dark room and cannot see if the cards are face-up or face-down. Your job is to get all four of the cards to be face-up or face-down.

On a given turn, you can flip over any or all of the cards. Then you press a button to verify your guess. If you are correct, you win and the game ends. Otherwise, the square Lazy Susan rotates a random number of times (each rotation being 90 degrees). Then you start another turn.

Is there a way you can win this game for sure?

Answer To Puzzle 20: Cards On A Square

Here are the 8 moves you can take to win. You should be pressing the button after each move.

1. Do nothing
2. Flip two opposite corners
3. Flip two consecutive cards
4. Flip two opposite corners
5. Flip a single card
6. Flip two opposite corners
7. Flip two consecutive cards
8. Flip two opposite corners

Step 1 wins if all the cards were face-up or face-down to begin with.

Step 2 wins if alternate cards are face-up and face-down, such as up-down-up-down. Notice that rotating the cards will not change the fact that alternate cards are face-up and face-down. Therefore, by flipping opposite corners you will always end up making all of the cards the same orientation.

Steps 3 and 4 win when there are two cards of each orientation that are adjacent, such as down-down-up-up. Flipping two consecutive cards will either win, or will change the cards to be face-up and face-down as sequentially alternate cards. Then when you flip two opposite corners you will win.

If none of the above works, then it must be the case that exactly one card is a different orientation from the rest. Either it is exactly one face-up card or it is exactly one face-down card.

The first four moves involve changing an even number of cards, and so the result will still have exactly one card of opposite orientation.

In step 5 you flip over one card. If you pick the correct card, then you win the game. Otherwise, you will end up with two cards of one orientation and then two cards of the other. The cards of opposite orientation will be positioned in opposite corners or they will be adjacent.

But now we’ve reduced the situation to one that we just solved. We can win by repeating the same moves from steps 2 to 4 (flip opposite corners, flip adjacent cards, flip opposite corners).

So you can always win this game, even though you cannot see the cards and the cards can be rotated after each move.

Puzzle 21: Even Betting

Your friend wants to make an even-payoff bet on the outcome of the entire World Series. That is, he wants to make a \$100 bet so that if his team is the champion he will win \$100, and if his team loses he will lose all his money.

The problem is he uses a bookie that takes bets only on individual games, and not the entire outcome. The bookie is, however, offering even-payout bets for each game and for any dollar amount.

How much should your friend bet on each game so that he can simulate an even-payout \$100 bet on the outcome of the entire series?

Here's an example of what your friend should not do. Let's say he bets \$100 on the first game. If his team loses, he's out of all his money. So even if his team wins the next four games, he ends up with nothing. Your friend wanted to bet so that he would win \$100 if his team won the series. So clearly his bet of \$100 on the first game was a bad idea.

Can you help your friend create an even-payout bet on the entire series, taking into account the many ways the series can end in 4, 5, 6, or 7 games?

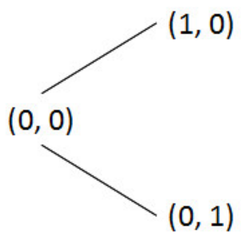
Answer To Puzzle 21: Even Betting

The difficulty in the problem is the series can end in a variable amount of games but your friend still wants an even-payout bet regardless of the how many games were played. That is, your friend wants to win \$100 if his team wins in a 4 game sweep, wins in 5 games and loses any of the previous four games, wins in 6 games and loses two of the previous five games, or wins in 7 and loses three of the previous six games. And similarly, your friend wants to end up with \$0 if his team loses (if the other team wins in 4, 5, 6, or 7 games).

We can break the problem down into three steps. First we'll draw a tree that shows the many ways the series can end. Second, we'll calculate how much money your friend should have when making the bets. Finally, we'll deduce the amount of the bets from the money he needs to make or lose for each outcome.

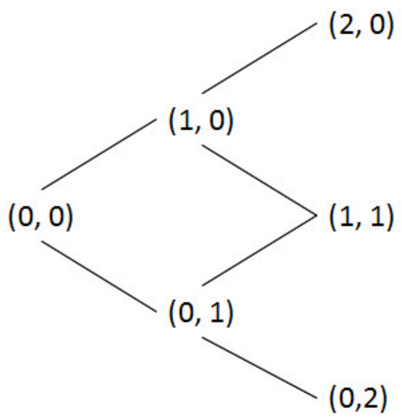
Part 1: Visualizing the Series

Let's draw a tree to visualize the possible ways the series could take place. The series always starts with both teams having no wins, which is written (0, 0) = (friend's team wins, other team wins). After the first game, one of the teams gets a win, so the state (0, 0) can either become (1, 0) or (0, 1).

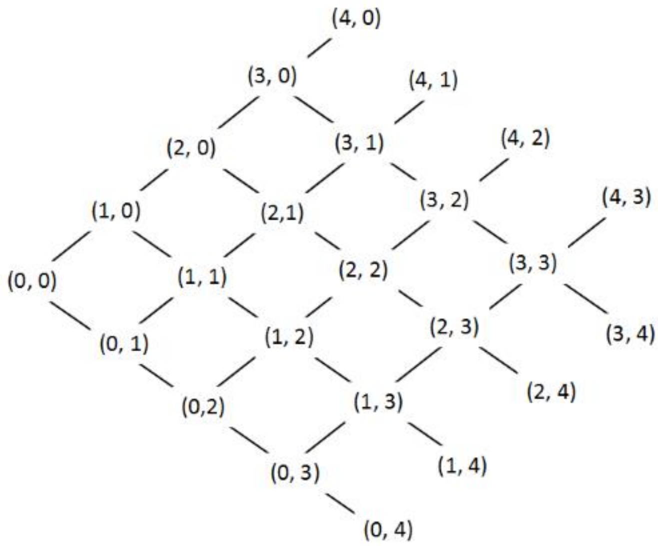


We can continue this thought process. From the state (1, 0), your friend's team could win and lead to the state (2, 0). Or your friend's team could lose leading to the state (1, 1).

Similarly, the state (0, 1) either results in the state (0, 2) or (1, 1).



For the third game, and every game after, we can draw out branches to complete the tree.

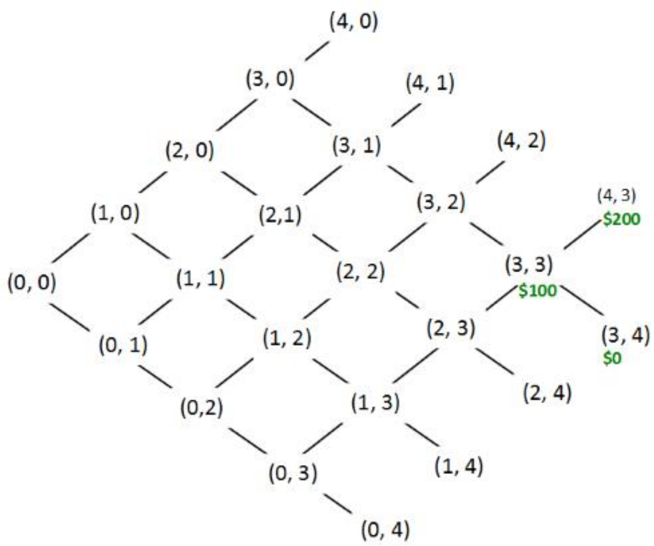


Part 2: Calculate the Money Using Backwards Induction

Let's imagine the entire seven games are necessary to decide the series. That is, both teams are tied at 3 games apiece after six games.

For your friend to make an even-payout \$100 bet on the entire series, he needs to make an even-payout \$100 bet on this particular seventh game. So after the seventh game, he should end up with \$0 if his team loses, and he should end up with \$200 if his team wins (he wins \$100 on a \$100 bet).

This means your friend should have \$0 at the state (3, 4) and he should have \$200 at the state (4, 3). And he clearly needs \$100 at the state (3, 3) so he can make the \$100 even-payout bet.

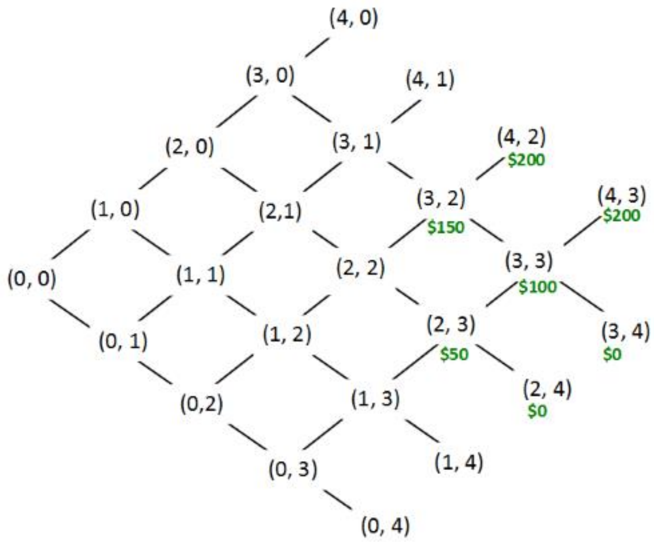


Now let's continue this backwards reasoning one step more. Perhaps the series does not get to seven games. How much money would your friend have needed for the other possible outcomes of 6 games of play?

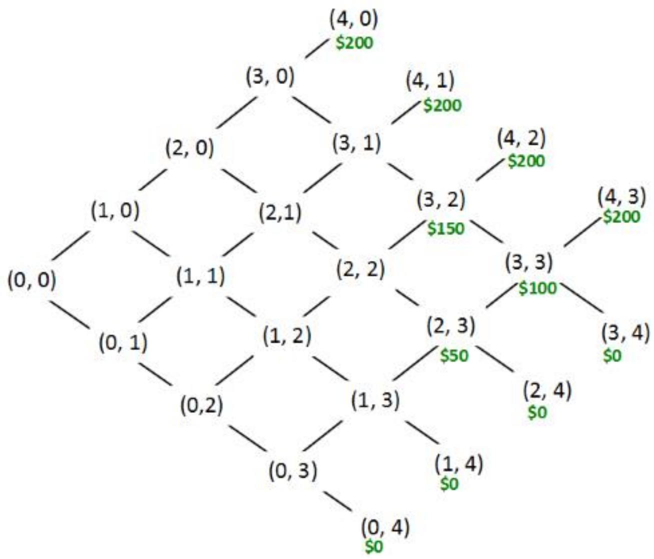
If the series ended in 6 games, your friend should end up with \$200 if his team won the series (4, 2), and \$0 if his team lost (2, 4).

Now comes the trick. We can deduce the money he should have at (3, 2) has to be the *average* of the money for the resulting two outcomes of (4, 2) and (3, 3). The reason is that each bet has to be an even-payout bet. Therefore, the money needed at any previous state is the average of the two states that follow it.

Similarly, the money your friend needs at (2, 3) is the average of the money in states (3, 3) and (2, 4).

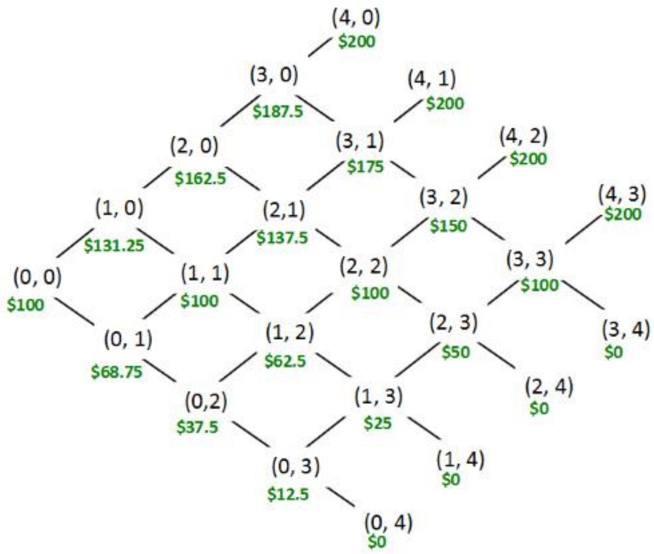


Now let's fill in the rest of the tree. The states (4, 1) and (4, 0) should be \$200 as they correspond to when your friend's team wins, and the states (1, 4) and (0, 4) should be \$0 as they are for when your friend's team loses.



Now we can find the money needed in the rest of the states by averaging two states that are known. We can first solve for the column of (3, 1), (2, 2), and (1, 3). Then we can use those results to solve for the column of (3, 0), (2, 1), (1, 2), and (0, 3).

We continue this process for the remaining columns and we can easily fill out the rest of the tree.



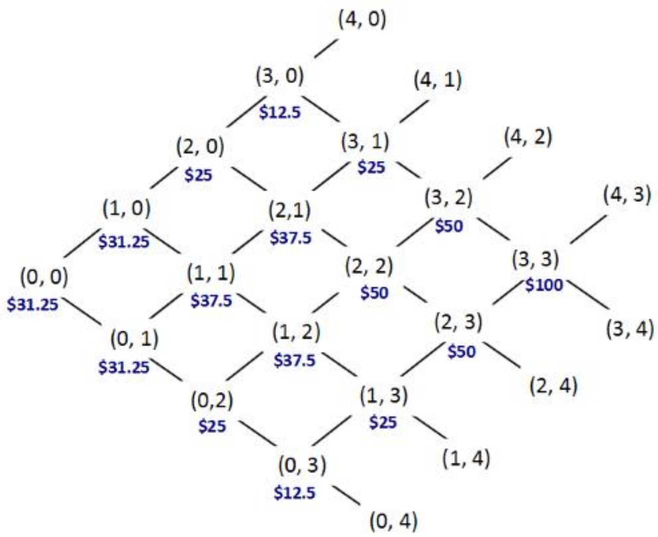
Now we know how much money your friend needs to have at each possible state.

Part 3: Calculate the Bets

How much should your friend bet? We can deduce this from the tree we have just created.

Your friend should bet on a game so that he ends up with the dollar amounts given. In the first game, he has $100 and he should either end up with $131.25 or $68.75. So he needs to make an even payout bet worth ($131.25 - $100) = ($100 - $68.75) = $31.25 dollars.

We can similarly calculate the bets necessary at each state by subtracting the amount if he wins (or loses) from the current money he has. Here is the schedule of bets for the entire series.



For a given state, your friend should bet the dollar amount that is written under that state for the next game.

For example, if the series is tied at 2 games each, then we are at (2, 2) and your friend should make an even-payout bet of $50 on the fifth game.

To summarize the strategy: Games 1, 2: Bet $31.25. Game 3: Bet $25, unless the series is tied where you should bet $37.50. Game 4: Bet $12.50 if the one team is ahead 3-0; otherwise, bet $37.50. Game 5 (if necessary): Bet $25, unless the series is tied 2-2 where you bet $50. Game 6 (if necessary): Bet $50. Game 7 (if necessary): Bet $100.

It is quite interesting that you can mathematically simulate an even-payout bet on a series from the combination of even-payout bets on individual games. And this type of math is not just for amusement: this is a flavor of the math economists use to price stocks or other investments in financial markets.

Puzzle 22: Rolling A 100-Sided Die

You roll a 100-sided die. You then have two choices: (1) you can cash out and get paid the dollar amount of the roll, or (2) you can pay $1 to roll the die again.

You get the same two choices on every roll, and there is no limit to the number of rolls you can make.

What is the expected value of the game?

Answer To Puzzle 22: Rolling A 100-Sided Die

This puzzle is great because it is deceptively difficult. Not only do you have to calculate a somewhat difficult expectation, but you also have to figure out the implicit question: what is the optimal value for cashing out?

Let's break the problem down into two steps. First, we will come up with a method to solve the game, and then we will figure out the best way to play the game.

Step 1: expected value for 50

When you roll the 100-sided die, you will get a number between 1 and 100. If you get a high roll like 90 you will simply cash out. And similarly, if you get a low value like 10, you would rather pay $1 to try your chances again.

Clearly, in order to play this game you need some sort of rule on when you cash out versus rolling again. What is a reasonable value? A sensible point is to consider the average roll, which is about 50. If you roll below the average, you could consider yourself unlucky and give yourself another chance. If you rolled higher than average, you could feel lucky and just cash out.

So consider the strategy: "If the roll is 50 or higher, cash out; otherwise, pay $1 and roll again." What is the expected value of this strategy?

Intuitively, you can get an idea of the answer. If you only cash out between 50 and 100, then your expected payout will be something like $75. But you only roll this high about half the time. Some of the time you will have to pay a dollar, or a few dollars, until you get such a high roll. This back of the envelope calculation suggests the answer will be $75 minus a few dollars—so maybe the game is worth $73.

This estimation is a crucial step in the process. It gives you a ballpark answer, and it also allows you to double-check your answer when you go through the more rigorous calculation. What is the exact answer?

Write r_k denote the value of the k^{th} roll. Also write $p = \Pr(r_1 \geq 50)$.

We need to compute an expectation that sums up the probability of "getting a roll of 50 or higher on the roll k" times "expected value of a roll of 50 or higher on roll k."

I will spare you the details; this method results in the following infinite series.

$\text{E(strategy 50)} = pr_1 + (1 - p)p(r_2 - 1) + (1 - p)^2p(r_3 - 2) + \ldots$

You *could* work out this series and figure out the answer. But there is a very useful trick in games like this, and it simplifies the calculation tremendously.

Here is the crucial observation that capitalizes on the "memoryless" feature of the game (more precisely, the game is a stationary Markov process).

If you win on the first roll, you cash out and the game is over. But if you don't win on the first roll, what happens? Well, you pay $1 in order to roll again. At this point, the *game is exactly the same as when you started, except you have paid $1.* Therefore, the expected value of the game, conditional on it being the second roll, is the expected value of the game minus $1.

The expectation to the game is described by the following equation.

E(game) = Pr(win 1st roll)E(win 1st roll) + Pr(not win 1st roll)[E(game) - 1]

$\text{E(strategy 50)} = \Pr(r_1 \geq 50)\ \text{E}(r_1 | r_1 \geq 50) + \Pr(r_1 < 50)\ [\text{E(strategy 50)} - 1]$

Now the expectation is a lot easier to solve. We can calculate some of the unknowns. The probability of rolling 50 or higher is $\Pr(r_1 \geq 50) = 51/100$, not rolling 50 or higher is $\Pr(r_1 < 50) = 49/100$, and the expected value of a roll between 50 and 100 is the average value, so $\text{E}(r_1 | r_1 \geq 50) = 75$.

We can solve the equation for the value of the game using the strategy of cashing out on rolls of 50 or higher.

E(strategy 50) = (51/100)(75) + (49/100)[E(strategy 50) - 1]
E(strategy 50) = [38.25 - 0.49]/(0.51)
E(strategy 50) = 74.04…

Note this answer is reasonable given the earlier ballpark estimate that the game was worth about \$73.

It is tempting at this point to think the game is worth about \$74. But in fact we have not solved the problem yet. We only know the game is worth *at least* \$74.

We calculated the game using the strategy of accepting rolls that were 50 or higher. What would happen if we increased the threshold to 51, or 60, or even 90?

There are two forces at play. On the one hand, if you increase the threshold, you increase your average payout. On the other, higher rolls are rarer, and you will have to pay \$1 for each roll as you wait to meet the threshold.

What is the optimal strategy?

Step 2: calculating the optimal expected value

The question did not explicitly state it, but the expected value of the game should be interpreted as finding the maximum value assuming optimal play. (Optimizing money is a type of assumption to make when interviewing for investment banking jobs, naturally.)

The task is now to consider the strategy: "If the roll is n or higher, cash out; otherwise, pay \$1 and roll again." What is the expected value of this strategy?

We proceed as before by exploiting the stationarity of the game. That is, we can write out the expected value in terms of winning on the first roll plus the probability of not winning on the first roll. If we don't win on the first roll, the value of the game is the same as before, but it is decreased by the \$1 we paid to keep rolling. So the general expectation is the following.

$E(\text{strategy } n) = \Pr(r_1 \geq n)\, E(r_1 | r_1 \geq n) + \Pr(r_1 < n)\, [E(\text{strategy } n) - 1]$

This equation is relatively easy to handle when we solve for some of the unknowns. The probability of rolling n or higher is $\Pr(r_1 \geq n) = 0.01(100 - n + 1)$, not rolling n or higher is $\Pr(r_1 < n) = 0.01(n - 1)$, and the expected value of a roll between n and 100 is the average value, so $E(r_1 | r_1 \geq n) = 0.5(100 + n)$.

So we get the following equation.

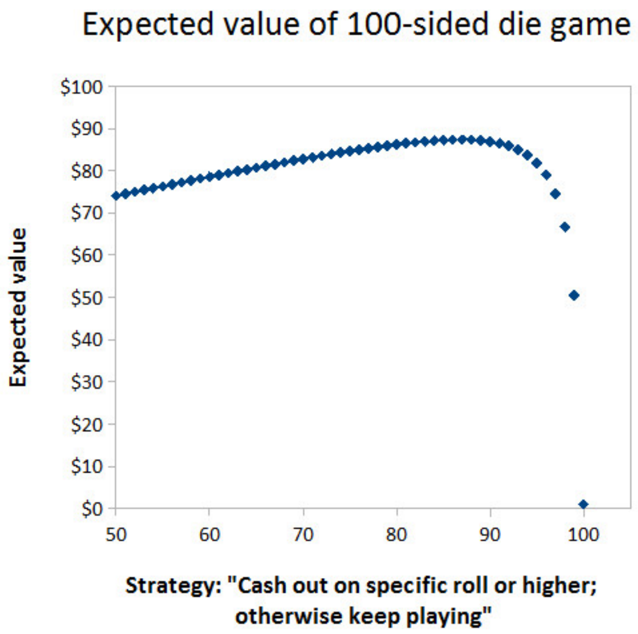
$E(\text{strategy } n) = 0.01(100 - n + 1)0.5(100 + n) + 0.01(n - 1)[E(\text{strategy } n) - 1]$

After a bit of algebra, this equation simplifies to the following.

$E(\text{strategy } n) = 51 + 0.5n - 100/(101 - n)$

We can solve the problem analytically by taking the derivative and finding the maximum value.

But since the problem is discrete, we can readily calculate and graph the payouts.



The maximum payout is $87.357… which happens for the strategy *n* = 87.

The graph is quite interesting. Note the expected payout increases gradually from 50 to the maximum of 87. It then decreases slowly until the last 3 values (for rolls 98, 99, and 100). At this point, the expected value falls precipitously! The reason is these high rolls are so rare that the game turns on you: you will wait so long until you get the roll that the gain from waiting is offset by the loss from paying $1 each time.

At 97, the payout is still a decent $74.5. But it drops to $66.67 for 98, and then $50.5 for 99. If you only accept the roll of 100, then your expected payout drops all the way to $1! This result makes sense. On average, you wait 100 rolls to get a payout of 100. Since your first roll was free, that means you expect to pay $99 for a chance to win $100, which is a net payout of $1.

Another interesting part is the expected payouts from 80 to 90 are within $1 of each other. So this is a game where being close to the optimal strategy is sufficient to get near the maximum value.

I think this part of the game offers a lesson. You don't have to be the smartest person to play the game right. You just need to have enough sense to wait for sufficiently high rolls. The analogy is that business people are rarely the smartest people in the room. But that's fine: they can still make a lot of money by being close to the right answer. If you insist on perfection, like a roll of 100, you might get there. But it might come at a high personal cost if the rolls are unlucky (many singers, artists, and movie stars are recognized late in life or even after their deaths).

In summary, this is game where there are a few ways to play the game really badly, like accepting low rolls, and many good ways to play the game reasonably. Then there are the tremendously stupid strategies of waiting for the highest of rolls. In this game and in life, you have to know when to cash out.

Credit: I first read this problem on Pratik Poddar's blog.

More From Presh Talwalkar

I hope you enjoyed this book. If you have a comment or suggestion, please email me presh@mindyourdecisions.com

The Joy of Game Theory: An Introduction to Strategic Thinking. Game Theory is the study of interactive decision-making, situations where the choice of each person influences the outcome for the group. This book is an innovative approach to game theory that explains strategic games and shows how you can make better decisions by changing the game.

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About The Author

Presh Talwalkar studied Economics and Mathematics at Stanford University. His site *Mind Your Decisions* has blog posts and original videos about math that have been viewed millions of times.

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