

# LECTURES ON CURVATURE EQUATIONS

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ABSTRACT. The Minkowski problem aims to reconstruct a convex body from its surface area measure. The solution to this problem is remarkable: a Borel measure  $\mu$  on the unit sphere is the surface area measure of a convex body if and only if  $\mu$  has its centroid at the origin and is not concentrated on a great subsphere. In this course, we focus on the smooth version of this problem and its close relatives. The aim is to provide the key components of the arguments for addressing the existence theory of an important class of curvature problems, the  $L_p$ -Minkowski and Christoffel-Minkowski problems.

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## 1. BACKGROUND MATERIAL

The main references for this section are [[ACGL20](#), [dCar92](#), [Lee09](#)].

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*Key words and phrases.* Continuity method, Degree theory, Minkowski problem, Christoffel-Minkowski problem, Full rank theorem,  $C^2$ -estimates.

### 1.1. Geometry of hypersurfaces.

**Definition 1.1.** Let  $k \geq 1$ . We say  $\mathcal{M} \subset \mathbb{R}^{n+1}$  is a  $C^k$  (embedded) hypersurface if for every point  $p \in \mathcal{M}$ , there exists a neighborhood  $O$  of  $p$  in  $\mathbb{R}^{n+1}$ , an open set  $\Omega \subset \mathbb{R}^n$ , and a  $C^k$  map  $X : \Omega \rightarrow \mathbb{R}^{n+1}$  such that  $X$  is a homeomorphism onto  $\mathcal{M} \cap O$ , and  $d_x X$  is injective for every  $x \in \Omega$ .

*Remark 1.2.* The homeomorphism on  $\mathcal{M} \cap O$  means that  $X$  is continuous, one-to-one, it maps  $\Omega$  onto  $\mathcal{M} \cap O$ , and its inverse is also continuous. Note that the continuity is interpreted with respect to the subspace topology on  $\mathcal{M}$  induced from the inclusion into  $\mathbb{R}^{n+1}$ . Since open sets in the subspace topology are given by restriction of open sets in  $\mathbb{R}^{n+1}$ , this is equivalent to the statement that for every open set  $\Omega' \subset \Omega$ , there exists an open set  $O' \subset \mathbb{R}^{n+1}$  such that  $X(\Omega') = \mathcal{M} \cap O'$ .

**Proposition 1.3.** *Let  $\mathcal{M}$  be a subset of  $\mathbb{R}^{n+1}$ . The following statements are equivalent:*

- (1)  $\mathcal{M}$  is a  $C^k$  (embedded) hypersurface;
- (2)  $\mathcal{M}$  is locally the graph of a  $C^k$  function: For every  $p \in \mathcal{M}$ , there exists an open set  $O \subset \mathbb{R}^{n+1}$  containing  $p$ , an open set  $\Omega \subset \mathbb{R}^n$ , and a  $C^k$  map  $\varphi : \Omega \rightarrow \mathbb{R}$  such that  $\mathcal{M} \cap O = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = \varphi(x_1, \dots, x_n), (x_1, \dots, x_n) \in \Omega\}$

Smooth curves on  $X(\Omega)$  are curves of the form  $\gamma(t) = X(x(t))$ , where the mapping  $t \mapsto x(t)$  is a smooth curve lying in the domain  $\Omega$ . Curves of the form

$$t \mapsto X(x_1, x_2, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n),$$

where  $x_i$  are fixed numbers, are called coordinate lines. The speed vectors of the coordinate lines are denoted by  $\partial_i X$ .

By definition, the vectors  $\partial_i X$  are linearly independent for any  $x \in \Omega$ . The tangent plane of  $\mathcal{M}$  at the point  $p = X(x)$  is the (hyper)plane through  $p$  spanned by the vectors  $\{\partial_i X\}_i$ . We say  $v$  is a tangent vector to  $X(\Omega)$  at  $p$  if  $v \in \text{span}\{\partial_1 X|_x, \dots, \partial_n X|_x\}$ .

A unit normal vector of the hypersurface at the point  $p = X(x)$  is defined as a unit normal of the tangent plane. The cross product of vectors  $\partial_i X$  is defined by

$$*(\partial_1 X \wedge \partial_2 X \wedge \dots \wedge \partial_n X) = \det \begin{vmatrix} E_1 & E_2 & \dots & E_{n+1} \\ \partial_1 X^1 & \partial_1 X^2 & \dots & \partial_1 X^{n+1} \\ \dots & \dots & \dots & \dots \\ \partial_n X^1 & \partial_n X^2 & \dots & \partial_n X^{n+1} \end{vmatrix}$$

Here  $\{E_i\}_i$  is the standard coordinate basis of  $\mathbb{R}^{n+1}$ . Then, we may choose

$$N(x) := (-1)^n \frac{*(\partial_1 X \wedge \partial_2 X \wedge \dots \wedge \partial_n X)}{\|*(\partial_1 X \wedge \partial_2 X \wedge \dots \wedge \partial_n X)\|}.$$

A vector field along  $X$  is a mapping  $v : \Omega \rightarrow T\mathbb{R}^{n+1}$  such that  $v(x) \in T_{X(x)}\mathbb{R}^{n+1} (\sim \mathbb{R}^{n+1})$  for any  $x \in \Omega$ .

Recall that the differentiation of two vector fields  $v, w$  in  $\mathbb{R}^{n+1}$  is defined as

$$D_v w|_p = (w \circ \gamma)'(0),$$

where  $\gamma : [-1, 1] \rightarrow \mathbb{R}^{n+1}$  is a curve with  $\gamma(0) = p$  and  $\gamma'(0) = v(0)$ . Note that by the chain rule,

$$D_v w = (D_v w^i) E_i = v(w^i) E_i,$$

where  $w^i$  are the components of the vector field  $w$  in the coordinate basis  $\{E_i\}_i$ .

We define the pull-back connection of  $D$  via  $X$  (denoted again by  $D$ ) as follows. Let  $v$  be a vector field along  $X$ ,  $x_0 \in \Omega$ , and  $u$  a tangent vector of  $\mathcal{M}$  at  $X(x_0)$ . Then  $D_u v$  is defined as  $D_u v = (v \circ \gamma)'(0)$ , where  $\gamma : [-1, 1] \rightarrow \Omega$  is a curve in  $\Omega$  such that  $\gamma(0) = x_0$  and  $(X \circ \gamma)'(0) = u$ . Note that  $D_u v = D_{dX^{-1}(u)} v$ . In particular,  $D_{\partial_i X} \partial_i X = D_{\partial_j} \partial_i X = \partial_{ij}^2 X$ . The derivative of  $N$  with respect to  $u$  at  $x_0$  can be similarly defined and, in fact, is tangent to  $\mathcal{M}$  at  $X(x_0)$ .

Let us denote by  $T_p \mathcal{M}$  the tangent space at  $p$ . The tangent bundle of  $\mathcal{M}$  is defined as  $T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p \mathcal{M}$ . The (induced) first fundamental form (namely the induced metric) of  $\mathcal{M}$  at  $p$  is defined by

$$I_p(u, v) = \langle u, v \rangle, \quad \forall u, v \in T_p \mathcal{M}.$$

This is a positive-definite symmetric bilinear map.

The linear map

$$A : T_p \mathcal{M} \rightarrow T_p \mathcal{M}, \quad v \mapsto D_v N = d_v N$$

is called the Weingarten map or the shape operator at  $p$ . It can be shown that  $A_p$  is self-adjoint. The second fundamental form is defined as

$$II_p(u, v) = \langle A_p(u), v \rangle.$$

The eigenvalues of  $A_p$  are called the principal curvatures of  $\mathcal{M}$  at  $p$ . In particular, the determinant of  $A$  is the Gauss(ian) curvature, and its trace is the mean curvature.

We denote by  $g_{ij}$  and  $h_{ij}$  the components of  $I, II$ :

$$\begin{aligned} g_{ij} &= I(\partial_i X, \partial_j X) = \langle \partial_i X, \partial_j X \rangle, \\ h_{ij} &= \langle A(\partial_i X), \partial_j X \rangle = \langle \partial_i N, \partial_j X \rangle = -\langle N, \partial_{ij}^2 X \rangle, \end{aligned}$$

where we used  $\langle N, \partial_i X \rangle = 0$ . Note that

$$h_{ij} = \langle A(\partial_i X), \partial_j X \rangle = \sum_{k=1}^n \langle A_i^k \partial_k X, \partial_j X \rangle = \sum_{k=1}^n A_i^k g_{kj}.$$

Therefore, the eigenvalues of  $[h_{ij}]$  with respect to the metric  $g$  (i.e., solutions of the equation  $\det(h_{ij} - \lambda g_{ij}) = 0$ , i.e., eigenvalues of the matrix  $[h_{ik} g^{kj}]$ ) are the principal curvatures. Here,  $[g^{ij}]$  denotes the inverse matrix of  $[g_{ij}]$ . Note that the Gauss curvature is given by

$$\det A = \frac{\det II}{\det g}.$$

*Remark 1.4.* Throughout this lecture note, we use the Einstein summation convention (repeated indices are implicitly summed over); e.g.,

$$A_i^k g_{kj} = \sum_{k=1}^n A_i^k g_{kj}, \quad h_{ik} g^{kj} = \sum_{k=1}^n h_{ik} g^{kj}.$$

The Christoffel symbols are defined as the tangential components of  $\partial^2 X$  :

$$D_{\partial_j X} \partial_i X = \partial_{ij}^2 X = \Gamma_{ij}^k \partial_k X - h_{ij} N.$$

Christoffel symbols are related to the metric and its partial derivatives. Note that

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \quad \langle \partial_{ij}^2 X, \partial_l X \rangle = \Gamma_{ij}^k g_{kl}.$$

For brevity, write  $\Gamma_{ijl}$  for  $\Gamma_{ij}^k g_{kl}$ . We have

$$\sum_{l=1}^n \Gamma_{ijl} g^{lk} = \Gamma_{ij}^k.$$

Denote the (partial) derivative of  $g_{ij}$  with respect to the  $k$ -th variable by  $g_{ij,k}$ . We have

$$\begin{aligned} g_{ij,k} &= \langle \partial_{ik}^2 X, \partial_j X \rangle + \langle \partial_i X, \partial_{jk}^2 X \rangle = \Gamma_{ikj} + \Gamma_{jki}, \\ g_{jk,i} &= \langle \partial_{ji}^2 X, \partial_k X \rangle + \langle \partial_j X, \partial_{ki}^2 X \rangle = \Gamma_{jik} + \Gamma_{kij}, \\ g_{ki,j} &= \langle \partial_{kj}^2 X, \partial_i X \rangle + \langle \partial_k X, \partial_{ij}^2 X \rangle = \Gamma_{kji} + \Gamma_{ijk}. \end{aligned}$$

Solving this linear system of equations, we obtain

$$\Gamma_{ijk} = \frac{1}{2} (g_{ik,j} + g_{jk,i} - g_{ij,k})$$

and

$$(1.1) \quad \Gamma_{ij}^k = \frac{1}{2} g^{lk} (g_{il,j} + g_{jl,i} - g_{ij,l}).$$

Next, we derive the Gauss and Codazzi equations. We calculate

$$\begin{aligned} \partial_{ijk}^3 X &= \partial_k (\Gamma_{ij}^l \partial_l X - h_{ij} N) \\ &= \Gamma_{ij,k}^l \partial_l X + \Gamma_{ij}^l \partial_{lk}^2 X - h_{ij,k} N - h_{ij} \partial_k N \\ &= \Gamma_{ij,k}^l \partial_l X + \Gamma_{ij}^l (\Gamma_{lk}^s \partial_s X - h_{lk} N) - h_{ij,k} N - h_{ij} h_{ks} g^{sl} \partial_l X \\ &= (\Gamma_{ij,k}^l + \Gamma_{ij}^s \Gamma_{sk}^l - h_{ij} h_{ks} g^{sl}) \partial_l X - (h_{ij,k} + \Gamma_{ij}^l h_{lk}) N. \end{aligned}$$

Comparing the coefficients of  $\partial_l X$  in  $\partial_{ijk}^3 X$  and  $\partial_{ikj}^3 X$ , we obtain the Gauss equation:

$$\Gamma_{ik,j}^l - \Gamma_{ij,k}^l + \Gamma_{ik}^s \Gamma_{sj}^l - \Gamma_{ij}^s \Gamma_{sk}^l = (h_{ik} h_{js} - h_{ij} h_{ks}) g^{sl}.$$

Comparing the normal components gives the Codazzi equation:

$$(1.2) \quad h_{ij,k} - h_{ik,j} = \Gamma_{ik}^l h_{lj} - \Gamma_{ij}^l h_{lk}.$$

Denote the right-hand side of Gauss' equation by  $\text{Rm}_{kji}^l$  :

$$(1.3) \quad \text{Rm}_{kji}^l = \Gamma_{ik,j}^l - \Gamma_{ij,k}^l + \Gamma_{ik}^s \Gamma_{sj}^l - \Gamma_{ij}^s \Gamma_{sk}^l.$$

Hence  $\text{R}_{kji} := \text{Rm}_{kji}^m g_{ml}$  satisfies the Gauss equation:

$$(1.4) \quad \text{R}_{kji} = h_{ik} h_{jl} - h_{ij} h_{lk}.$$

Recall that a  $(k, \ell)$ -tensor over a linear space  $V$  is a multilinear function

$$T : V \times \cdots \times V \times V^* \times \cdots \times V^* \rightarrow \mathbb{R},$$

where we have  $k$  copies of  $V$  and  $\ell$  copies of the dual space  $V^*$ . Here  $V^*$  is the dual space; that is, the vector space of linear functions  $\omega : V \rightarrow \mathbb{R}$  (referred to as covectors).

Let  $\{e_i\}$  be a basis of  $V$ , and  $\{e^i\}$  be the corresponding dual basis of  $V^*$  (dual basis means  $e^j(e_i) = \delta_{ij}$ ). The components of  $T$  (with respect to these bases) are

$$T_{i_1 \dots i_k}^{j_1 \dots j_\ell} = T(e_{i_1}, \dots, e_{i_k}; e^{j_1}, \dots, e^{j_\ell}).$$

Given  $p \in \mathcal{M}$ , let  $T_p^* \mathcal{M}$  be the dual vector space of  $T_p \mathcal{M}$ . The cotangent bundle is then defined as  $T^* \mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p^* \mathcal{M}$ . A tensor field of type  $(k, \ell)$  over  $T\mathcal{M}$  is a mapping  $T$  that assigns to every point  $p \in \mathcal{M}$  a tensor of type  $(k, \ell)$  over  $T_p \mathcal{M}$ . We write  $\Gamma_k^\ell(\mathcal{M})$  for  $(k, \ell)$ -tensor fields.

Let us denote the dual basis corresponding to  $\{\partial_i := \partial_i X\}$  by  $\{dx^i\}$ . We say  $T$  is a smooth tensor field if its (local) components in these bases are smooth (and hence in any other bases). A  $(0, 1)$ -tensor field is called a vector field. A  $(1, 0)$ -tensor field is called a differential form (or a 1-form). For example, the first and second fundamental forms are  $(2, 0)$  tensors.

In local coordinates, we may write the metric as

$$g = g_{ij} dx^i \otimes dx^j,$$

where  $g_{ij} := g(\partial_i, \partial_j)$ . In general, a  $(k, \ell)$ -tensor field  $\alpha$  can be written as

$$\alpha = \alpha_{i_1 \dots i_k}^{j_1 \dots j_\ell} dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_\ell}},$$

where

$$\alpha_{i_1 \dots i_k}^{j_1 \dots j_\ell} := \alpha(\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_k}; dx^{j_1}, \dots, dx^{j_\ell})$$

are the components of  $\alpha$ .

Let  $c$  be a constant and  $f : \Omega \rightarrow \mathbb{R}$  be a smooth function. Recall that the vector field differentiation in  $\mathbb{R}^n$  satisfies:

$$\begin{aligned} D_{v_1+v_2} w &= D_{v_1} w + D_{v_2} w, \\ D_{cw} &= cD_w, \\ D_v(w_1 + w_2) &= D_v w_1 + D_v w_2, \\ D_v(fw) &= v(f)w + fD_v w, \\ D_{w_1} w_2 - D_{w_2} w_1 &= [w_1, w_2], \\ D_v \langle w_1, w_2 \rangle &= \langle D_v w_1, w_2 \rangle + \langle w_1, D_v w_2 \rangle. \end{aligned}$$

Here,  $[w_1, w_2]$  denotes the (Euclidean) Lie bracket of the vector fields  $w_1$  and  $w_2$ :

$$[w_1, w_2] = \sum_{i=1}^n (w_1(w_2^i) - w_2(w_1^i)) E_i.$$

**Definition 1.5.** A connection on  $\mathcal{M}$  is a mapping that assigns to any two smooth tangential vector fields  $v, w$  a tangential vector field  $\nabla_v w$  satisfying the following rules:

$$\begin{aligned}\nabla_{v_1+v_2} w &= \nabla_{v_1} w + \nabla_{v_2} w, \\ \nabla_{cw} &= c\nabla_v w, \\ \nabla_v(w_1 + w_2) &= \nabla_v w_1 + \nabla_v w_2, \\ \nabla_v(fw) &= v(f)w + f\nabla_v w.\end{aligned}$$

We say a connection  $\nabla$  is symmetric or torsion-free if

$$\nabla_v w - \nabla_w v = (v(w^i) - w(v^i))\partial_i,$$

where  $v = v^i \partial_i$  and  $w = w^i \partial_i$ . Define  $[v, w] = \sum_{i=1}^n (v(w^i) - w(v^i))\partial_i$ ; the Lie bracket.

We say a connection  $\nabla$  is compatible with the (induced) metric on  $\mathcal{M}$  if

$$v\langle w_1, w_2 \rangle = \langle \nabla_v w_1, w_2 \rangle + \langle w_1, \nabla_v w_2 \rangle.$$

Using the Christoffel symbols defined in (1.1), we may define a connection on  $\mathcal{M}$  as follows. For vector fields  $v = v^i \partial_i$  and  $w = w^i \partial_i$ , define

$$\nabla_v w = (v(w^k) + v^i w^j \Gamma_{ij}^k) \partial_k,$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols defined earlier. It is then easy to see that this connection is torsion-free and compatible with the metric. The fundamental theorem of Riemannian geometry states any other connection on  $\mathcal{M}$  which is both symmetric and compatible with the induced metric  $g$  is the one defined above. This connection is called the Levi-Civita connection of  $g$ .

Now observe that since

$$D_{\partial_j X} \partial_i X = \partial_{ij}^2 X = \Gamma_{ij}^k \partial_k X - h_{ij} N,$$

we obtain for any vector fields  $v, w$  along  $X$ :

$$D_v w = \nabla_v w - \Pi(v, w)N,$$

where  $D$  denotes the pull-back connection via  $X$ . This provides a geometric interpretation of covariant derivatives on  $\mathcal{M}$ :

$$\nabla_v w = (D_v w)^\top.$$

The covariant differentiation can be extended so that it can act on  $(k, \ell)$ -tensor fields. Let  $\alpha \in \Gamma_\ell^k(\mathcal{M})$  and  $u \in T\mathcal{M}$ . Then

$$\begin{aligned}(\nabla_u \alpha)(\omega^1, \dots, \omega^\ell, u_1, \dots, u_k) &= u(\alpha(\omega^1, \dots, \omega^\ell, u_1, \dots, u_k)) \\ &\quad - \sum_i \alpha(\omega^1, \dots, \nabla_u \omega^i, \dots, \omega^\ell, u_1, \dots, u_k) \\ &\quad - \sum_j \alpha(\omega^1, \dots, \omega^\ell, u_1, \dots, \nabla_u u_j, \dots, u_k).\end{aligned}$$

Here,  $\omega^i$  are 1-forms, and  $u_i$  are tangent vectors. For example,

$$\nabla_{\partial_j} \partial_i = \Gamma_{ij}^k \partial_k,$$

and

$$\begin{aligned} (\nabla_{\partial_i} dx^j)(\partial_k) &= \partial_i(\delta_k^j) - dx^j(\nabla_{\partial_i} \partial_k) \\ &= -dx^j(\Gamma_{ik}^\ell \partial_\ell) \\ &= -\Gamma_{ik}^j. \end{aligned}$$

*Remark 1.6.* Very often, it is convenient to write  $\nabla_i \alpha = \nabla_{\partial_i} \alpha$ .

The second derivative of  $\alpha \in \Gamma_\ell^k(\mathcal{M})$ ,  $\nabla_{u,v}^2 \alpha \in \Gamma_\ell^k(\mathcal{M})$  for  $u, v \in T\mathcal{M}$ , is defined by

$$(1.5) \quad \nabla_{u,v}^2 \alpha = \nabla_u(\nabla_v \alpha) - \nabla_{\nabla_u v} \alpha.$$

We can rewrite the Codazzi equation (1.2) using the  $\nabla$  notation as follows:

$$(1.6) \quad \nabla_i h_{jk} = \nabla_j h_{ik}.$$

Note that here  $\nabla_i h_{jk} = (\nabla_{\partial_i} h)(\partial_j, \partial_k)$  and  $\nabla_j h_{ik} = \nabla_{\partial_j} h(\partial_i, \partial_k)$ .

In terms of  $\nabla$ , we can rewrite Rm as follows:

$$(1.7) \quad \begin{aligned} \text{Rm}(u, v)w &= \nabla_v(\nabla_u w) - \nabla_u(\nabla_v w) + \nabla_{[u,v]}w \\ &= \nabla_{v,u}^2 w - \nabla_{u,v}^2 w. \end{aligned}$$

The corresponding  $(4, 0)$ -tensor, the *Riemann* curvature, is also given by

$$\text{R}(u, v, w, z) = g(\text{Rm}(u, v)w, z).$$

The Riemannian curvature is a  $(4, 0)$ -tensor. In the literature, some define the Riemann curvature with the opposite sign.

*Exercise 1.7.* Verify (1.2) and (1.6) are in fact the same. Similarly, show that (1.3) and (1.7) are the same.

Let  $u, v \in T\mathcal{M}$  and  $\alpha \in \Gamma_\ell^k(\mathcal{M})$  be a  $(k, \ell)$ -tensor field. We define

$$\text{Hess } \alpha(u, v) = \nabla_{u,v}^2 \alpha = \nabla_u(\nabla_v \alpha) - \nabla_{\nabla_u v} \alpha.$$

We can extend the definition of Rm so that it can act on  $(k, \ell)$ -tensor fields as well:

$$(1.8) \quad \text{Rm}(u, v)\alpha = \nabla_{v,u}^2 \alpha - \nabla_{u,v}^2 \alpha = \text{Hess } \alpha(v, u) - \text{Hess } \alpha(u, v).$$

For any  $(2, 0)$ -tensor field  $\alpha$  and  $u, v, w, z \in T\mathcal{M}$ , we have

$$(1.9) \quad (\text{Rm}(u, v)\alpha)(w, z) = -\alpha(\text{Rm}(u, v)w, z) - \alpha(w, \text{Rm}(u, v)z).$$

To verify this formula, we may extend  $u, v, w, z$  to vector fields so that all covariant derivatives vanish at a given point  $p \in \mathcal{M}$ . By the definition of  $\nabla_{u,v}^2$  (cf. (1.5)),

$$\begin{aligned} (\nabla_{v,u}^2 \alpha)(w, z) &= (\nabla_v(\nabla_u \alpha))(w, z) \\ &= \nabla_v(u(\alpha(w, z))) - \alpha(\nabla_u w, z) - \alpha(w, \nabla_u z) \\ &= v(u(\alpha(w, z))) - \alpha(\nabla_{v,u}^2 w, z) - \alpha(w, \nabla_{v,u}^2 z). \end{aligned}$$

Therefore,

$$\begin{aligned} (\nabla_{v,u}^2 \alpha)(w, z) - (\nabla_{u,v}^2 \alpha)(w, z) &= [v, u](\alpha(w, z)) \\ &\quad + \alpha(\nabla_{u,v}^2 w, z) + \alpha(w, \nabla_{u,v}^2 z) \\ &\quad - \alpha(\nabla_{v,u}^2 w, z) - \alpha(w, \nabla_{v,u}^2 z). \end{aligned}$$

In view of  $[u, v]f = \text{Hess } f(u, v) - \text{Hess } f(v, u) = 0$ , the proof is complete.

Given a symmetric  $(2, 0)$ -tensor field  $\alpha$ ,  $\alpha^\sharp := g^* \alpha$  denotes a  $(1, 1)$ -tensor field that is implicitly defined by

$$g(\alpha^\sharp(u), v) = \alpha(u, v).$$

In local coordinates, the components of  $\alpha^\sharp$ ,  $\alpha_i^j$ , are given by

$$\alpha_i^j = \alpha_{ik} g^{kj}.$$

To put it simply,  $\sharp$  is the index-raising operator; e.g.  $A = \Pi^\sharp$ .

Let us prove Simon's identity:

$$\nabla_{i,j}^2 h_{kl} - \nabla_{k,l}^2 h_{ij} = h_{ij} h_{kl}^2 - h_{jk} h_{il}^2 + h_{il} h_{jk}^2 - h_{kl} h_{ij}^2.$$

Note that here, for example,  $\nabla_{i,j}^2 h_{kl} = (\nabla_{\partial_i, \partial_j}^2 h)(\partial_k, \partial_l)$ .

Let  $u, v, w, z$  be vector fields so that all covariant derivatives vanish at a given point  $p \in \mathcal{M}$ . Using (1.8) and (1.9), we calculate

$$\begin{aligned} (\nabla_{u,v}^2 h)(w, z) &= \nabla_u(\nabla_v h)(w, z) \\ &= \nabla_u((\nabla_v h)(w, z)) \\ &= \nabla_u((\nabla_w h)(v, z)) \\ &= \nabla_u(\nabla_w h)(v, z) \\ &= (\nabla_{u,w}^2 h)(v, z) \\ &= (\nabla_{w,u}^2 h)(v, z) + (\text{Rm}(w, u)h)(v, z) \\ &= (\nabla_{w,u}^2 h)(v, z) - h(\text{Rm}(w, u)v, z) - h(v, \text{Rm}(w, u)z) \\ &= (\nabla_{w,z}^2 h)(u, v) - h(\text{Rm}(w, u)v, z) - h(v, \text{Rm}(w, u)z). \end{aligned}$$

Now the claim follows from the Gauss equation (1.4).

**Example 1.8.** Let  $\mathcal{M}$  be a smooth hypersurface, locally represented as the graph of a function  $\varphi : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ . That is,  $\mathcal{M}$  is locally parameterized by  $X(x) = (x, \varphi(x))$ . Therefore, the tangent space is spanned by

$$\partial_i X = (E_i, \partial_i \varphi),$$

where  $\{E_i\}$  is the standard basis of  $\mathbb{R}^n$ . The induced metric, unit *upward* normal vector, and the second fundamental form are given by

$$g_{ij} = \delta_{ij} + \partial_i \varphi \partial_j \varphi, \quad N = \frac{(-D\varphi, 1)}{\sqrt{1 + |D\varphi|^2}}.$$



Recall that by definition

$$h_{ij} = -\langle \partial_{ij}^2 X, N \rangle = -\frac{\partial_{ij}^2 \varphi}{\sqrt{1 + |D\varphi|^2}}.$$

Moreover, we have

$$g^{ij} = \delta_{ij} - \frac{\partial_i \varphi \partial_j \varphi}{1 + |D\varphi|^2}.$$

*Exercise 1.9.* Find a local parametrization of the unit sphere  $\mathbb{S}^n$ , and calculate its second fundamental form and Riemannian curvature.

**1.2. Convex geometry.** The main reference of this section is [Sch14].

**Definition 1.10.** A non-empty, compact, convex set is called a convex body. (Note that with this definition a convex body need not have interior points). The boundary of a convex body with non-empty interior is called a *convex* hypersurface (which may not be smooth but of course it is embedded). Moreover, we say the hypersurface is strictly convex if it does not contain any line segment.

**Definition 1.11.** Let  $K$  be a convex body. The support function of  $K$  is defined by

$$s_K(x) = \max_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^{n+1}.$$

*Remark 1.12.* Let  $K, L$  be two convex bodies such that  $L \subseteq K$ . Then  $s_L \leq s_K$ . In particular, if an origin-centred ball  $B_r$  is a subset of  $K$ , then  $s_K \geq r$ , where  $r$  is the radius of the ball.

*Remark 1.13.* We will often write  $s = s_K$  when the context is clear.

From the definition of the support function, it is clear that  $s_K$  is a sublinear function (i.e. one-homogeneous and convex). By [Sch14, Thm 1.7.1], a sublinear function  $s$  uniquely determines a convex body  $K$  whose support function is  $s_K = s$ . For  $x \in \mathbb{S}^n$ , define

$$\begin{aligned} H(K, x) &= \{y : \mathbb{R}^{n+1} : \langle x, y \rangle = s_K(x)\}, \\ H^-(K, x) &= \{y : \mathbb{R}^{n+1} : \langle x, y \rangle \leq s_K(x)\}. \end{aligned}$$

Then

$$K = \bigcap_{x \in \mathbb{S}^n} H^-(K, x).$$

$H(K, x)$  and  $H^-(K, x)$  are called, respectively, the supporting hyperplane and supporting half-space with the outer unit normal  $x$ .

**Definition 1.14.** Let  $k \geq 1$ . We say a convex body is  $C^k$  if it has *non-empty interior* and its boundary hypersurface  $\partial K$  is a  $C^k$  hypersurface (in the sense of Definition 1.1). For  $k \geq 2$ , we say a convex body is  $C_+^k$  if it is  $C^k$  and the outer unit normal map (the spherical image map)  $N : \partial K \rightarrow \mathcal{M}$  is a  $C^{k-1}$  diffeomorphism.

**Lemma 1.15.** [Sch14, Cor. 1.7.3]. *Let  $K$  be a convex body with support function  $s$ . Then  $s$  is differentiable at  $x \in \mathbb{S}^n$  if and only if  $H(K, x) \cap K$  contains only one point, say  $y$ . In this case, we have  $Ds(x) = y$ .*

**Lemma 1.16.** [Sch14, Lem. 2.2.12] *If  $K$  is  $C^1$  and strictly convex, then spherical image map  $N : \partial K \rightarrow \mathbb{S}^n$  is a homeomorphism.*

*Remark 1.17.* In view of **Lemma 1.15**,  $K$  is strictly convex if and only if  $s$  is  $C^1$ . Moreover, when  $s$  is  $C^1$ ,  $Ds(\mathbb{S}^n) = \partial K$ .

Since the support function is a one-homogenous function, we may consider it as a function on the unit sphere. If  $K$  is  $C^1$  and strictly convex, then for any  $x \in \mathbb{S}^n$ ,  $K \cap H(K, x)$  has only one element, say  $y$ . Moreover, since  $K$  is  $C^1$ , we must have  $N(y) = x$ . By **Lemma 1.16**,  $N^{-1}(x) = y$ . Due to **Lemma 1.15**, we have

$$Ds(x) = y = N^{-1}(x), \quad s(x) = \langle x, Ds(x) \rangle.$$

That is,  $s(x) = \langle x, N^{-1}(x) \rangle$ .

Denote the standard metric of  $\mathbb{S}^n$  and its Levi-Civita connection by  $g, \nabla$  (i.e. the induced structure from  $\mathbb{R}^{n+1}$ ). Consider a differentiable function  $f : \mathbb{S}^n \rightarrow \mathbb{R}$  and its one-homogeneous extension  $\bar{f} : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}$  defined by

$$\bar{f}(x) = |x|f\left(\frac{x}{|x|}\right).$$

Then for  $x \in \mathbb{S}^n$  we have  $D\bar{f}(x) = f(x)x + \nabla f(x)$  (to be more precise, perhaps we should use the notation  $\text{grad } f$  instead of  $\nabla f$ ). In particular, when  $K$  is  $C^1$  and strictly convex,

$$(1.10) \quad N^{-1}(x) = s(x)x + \nabla s(x)$$

*Exercise 1.18.* Show that  $D\bar{f}(x) = \nabla f(x) + f(x)x$  for all  $x \in \mathbb{S}^n$ , and

$$D^2\bar{f}|_{T\mathbb{S}^n} = \text{Hess } f + gf, \quad \Delta_{\mathbb{R}^{n+1}}\bar{f} = \Delta f + nf.$$

Hint:  $(D\bar{f})^\top$ , the tangential component, is  $\nabla f$  ( $= \text{grad } f$ ).

**Definition 1.19.** Let  $f : \mathbb{S}^n \rightarrow \mathbb{R}$  be a  $C^2$  function. We define the  $(1, 1)$ -tensor  $B[f]$  as

$$B[f] = (\nabla^2 f + gf)^\sharp.$$

In local coordinates,

$$B[f]_i^j := (\nabla_{i,k}^2 f + g_{ik}f)g^{kj}.$$

**Lemma 1.20.** *Let  $K$  be a  $C_+^2$  convex body with support function  $s$ . Then the eigenvalues of the  $(1, 1)$ -tensor  $B[s]$  are the principal radii of curvature of  $\mathcal{M} = \partial K$  (i.e. the eigenvalues of  $dN^{-1}$ ).*

*Proof.* Note that  $\mathcal{M}$  is  $C^2$  and  $N : \mathcal{M} \rightarrow \mathbb{S}^n$  is a  $C^1$  diffeomorphism. In particular, for any  $x \in \mathbb{S}^n$ ,  $H(K, x) \cap K$  contains exactly one point (otherwise, we would find two points on  $\mathcal{M}$  with outer unit normal  $x$ ). In view of (1.10),  $s$  is  $C^2$ .

Let us put  $Y = N^{-1} : \mathbb{S}^n \rightarrow \mathcal{M}$  for the Gauss map (the Gauss map parameterization). Let  $\varphi : U \subset \mathbb{R}^n \rightarrow \mathbb{S}^n$  be a local parametrization of  $\mathbb{S}^n$ . Then, the map

$$X = Y \circ \varphi : U \subset \mathbb{R}^n \rightarrow \mathcal{M}$$

is a local parametrization of  $\mathcal{M}$ .

Fix  $x_0 \in U$ . Suppose  $\{\partial_i \varphi\}_i$  is an orthonormal basis of  $T_{\varphi(x_0)} \mathbb{S}^n$ . In the parametrization  $X$  of  $\mathcal{M}$ , for every  $x \in U$ , we have  $s(x) = \langle X(x), \varphi(x) \rangle$ . At  $x_0$ , we calculate

$$\partial_i s = \langle \partial_i X, \varphi \rangle + \langle X, \partial_i \varphi \rangle = \langle X, \partial_i \varphi \rangle.$$

Differentiating this once more yields

$$\begin{aligned} \nabla_{i,j}^2 s &= \partial_{ij}^2 s - \Gamma_{ij}^k \partial_k s = \langle X, \partial_{ij}^2 \varphi - \Gamma_{ij}^k \partial_k \varphi \rangle + \langle \partial_j X, \partial_i \varphi \rangle \\ &= \langle X, \partial_{ij}^2 \varphi - \Gamma_{ij}^k \partial_k \varphi \rangle + h_{ij}, \end{aligned}$$

where  $h_{ij}$  is the second fundamental form of  $\mathcal{M}$  and  $\Gamma_{ij}^k = \langle \partial_{ij}^2 \varphi, \partial_k \varphi \rangle$  are the Christoffel symbols of  $(\mathbb{S}^n, \nabla, g)$ . Note that  $\partial_{ij}^2 \varphi - \Gamma_{ij}^k \partial_k \varphi$  is normal to  $\mathbb{S}^n$ . Now using  $\langle \varphi, \partial_i \varphi \rangle = 0$ , we compute

$$\langle \varphi, \partial_{ij}^2 \varphi - \Gamma_{ij}^k \partial_k \varphi \rangle = \langle \varphi, \partial_{ij}^2 \varphi \rangle = -\langle \partial_i \varphi, \partial_j \varphi \rangle = -\delta_{ij}.$$

Therefore,

$$\langle X, \partial_{ij}^2 \varphi - \Gamma_{ij}^k \partial_k \varphi \rangle = -\delta_{ij} s$$

and the second fundamental of  $\mathcal{M}$  at  $X(x_0)$  is given by

$$h_{ij} = \langle \partial_j X, \partial_i \varphi \rangle = \nabla_{i,j}^2 s + \delta_{ij} s.$$

Let us denote the induced metric on  $\mathcal{M}$  by  $\bar{g}$ . We have  $\varphi = N \circ X$  and

$$\partial_i \varphi = D_{\partial_i X} N = A(\partial_i X) = h_{ik} \bar{g}^{kl} \partial_l X.$$

On the other hand, at  $x_0$ :

$$\delta_{ij} = \langle \partial_i \varphi, \partial_j \varphi \rangle = h_{ik} \bar{g}^{kl} h_{jm} \bar{g}^{ms} \langle \partial_l X, \partial_s X \rangle = h_{ik} h_{jl} \bar{g}^{kl} = \sum_{k=1}^n B[s]_{ik} h_j^k.$$

Hence, the eigenvalues of  $B[s]$  are the principal radii of curvature.  $\square$

*Remark 1.21.* If  $K$  is  $C_+^k$ , then all principal curvatures are positive. To see this, note that  $dN^{-1}$  is of maximal rank and there is at least one point where all eigenvalues of  $B[s]$  are positive. Since  $\det dN^{-1} \neq 0$ , the principal radii of curvature are positive everywhere. We can also argue that there is at least one point that all principal curvatures are positive and since  $\det dN \neq 0$  ( $dN$  is of maximal rank), they are positive everywhere.

**Theorem 1.22.** *Suppose  $k \geq 2$ . Then  $K$  is  $C_+^k$  if and only if its support function  $s$  is  $C^k$  and the eigenvalues of  $h_{ij} = \nabla_{i,j}^2 s + g_{ij} s$  with respect to  $g$  (i.e. the eigenvalues of  $B[s]$ ) are all positive.*

*Proof.* Since  $K$  is  $C_+^k$ ,  $N$  is a  $C^{k-1}$  diffeomorphism and  $dN^{-1}$  is of maximal rank  $n$ . In particular, the support function  $s$  is  $C^k$ . In the proof of [Lemma 1.20](#), we have shown that the eigenvalues of  $dN^{-1}$  are the eigenvalues of  $B[s]$ . Since there is at least one point in  $\mathbb{S}^n$  at which the eigenvalues of  $B[s]$  are all positive, we conclude that  $B[s]$  is positive definite on the whole  $\mathbb{S}^n$ .

Now we prove the other direction of the claim. We need to show that if  $B[s]$  is positive-definite and  $s$  is  $C^k$ , then  $K$  is  $C_+^k$ . Let us translate  $K$  so that it encloses the origin in its interior. The polar body of  $K$ , denoted by  $K^\circ$ , is a convex body defined by

$$K^\circ = \{x : \langle x, y \rangle \leq 1, \forall y \in K\}.$$

The radial function of  $K^\circ$  is defined by

$$\rho^\circ(u) = \max\{\lambda \geq 0 : \lambda u \in K^\circ\}, \quad u \in \mathbb{S}^n.$$

By [\[Sch14, Lem. 1.7.14\]](#),  $\rho^\circ = s^{-1} : \mathbb{S}^n \rightarrow \mathbb{R}$ . Hence, the mapping

$$Z : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}, \quad u \mapsto \rho^\circ(u)u$$

is an injective  $C^k$  immersion with the image  $\partial K^\circ = Z(\mathbb{S}^n)$ . By [\[Lee13, Prop. 4.22-\(d\)\]](#), since  $\mathbb{S}^n$  is compact,  $Z$  is an embedding (with  $\partial K^\circ$  inheriting the subspace topology from  $\mathbb{R}^n$ ). Therefore,  $\partial K^\circ$  is a  $C^k$  hypersurface; cf. [\[Lee13, Prop. 5.2\]](#).

Note that the metric and the second fundamental form of  $K^\circ$  in this parameterization are given by

$$(1.11) \quad \bar{g}_{ij}^\circ = (\rho^\circ)^2 g_{ij} + \nabla_i \rho^\circ \nabla_j \rho^\circ,$$

$$(1.12) \quad \begin{aligned} h_{ij}^\circ &= \frac{1}{\sqrt{\rho^{\circ 2} + |\nabla \rho^\circ|^2}} (-\rho^\circ \nabla_{i,j}^2 \rho^\circ + 2 \nabla_i \rho^\circ \nabla_j \rho^\circ + \rho^{\circ 2} g_{ij}) \\ &= \frac{\rho^{\circ 3}}{\sqrt{\rho^{\circ 2} + |\nabla \rho^\circ|^2}} (\nabla_{i,j}^2 s + s g_{ij}) \\ &= \frac{\rho^{\circ 3}}{\sqrt{\rho^{\circ 2} + |\nabla \rho^\circ|^2}} h_{ij}. \end{aligned}$$

Since the eigenvalues of  $h$  with respect to  $g$  are all positive, by [Exercise 1.23](#), the eigenvalues of  $h_{ij}^\circ$  with respect to  $\bar{g}_{ij}^\circ$  are all positive as well. Hence,  $N^\circ : \partial K^\circ \rightarrow \mathbb{S}^n$  (the spherical image map of  $K^\circ$ ) is an immersion. Since  $\partial K^\circ$  is closed (i.e. compact and without boundary) and  $\mathbb{S}^n$  for  $n \geq 2$  is simply connected,  $N^\circ$  is a  $C^{k-1}$  diffeomorphism. That is,  $K^\circ$  is  $C_+^k$ , its support function  $s^\circ$  is  $C^k$  and the eigenvalues of  $\nabla^2 s^\circ + g s^\circ$  with respect to  $g$  are all positive.

We modify a part of the above argument so that it works for  $n \geq 1$  (we need more information on the map  $N^\circ$ ). Recall from [Remark 1.17](#) that  $Ds : \mathbb{S}^n \rightarrow \mathcal{M}$  is onto. By our assumption,  $Ds$  is  $C^{k-1}$  and it is an immersion. Therefore,  $Ds$  is a  $C^{k-1}$ -smooth local parametrization of  $\partial K$ . That is,  $K$  is (at least) of class  $C^{k-1}$ . Now that we know  $K$  is at least  $C^1$  and strictly convex, by [Lemma 1.16](#), we deduce  $Ds$  is a  $C^{k-1}$  diffeomorphism.

Moreover, by [Sch14, Rem. 1.7.14], we have

$$N^\circ : \partial K^\circ \rightarrow \mathbb{S}^n$$

$$N^\circ(\rho^\circ(u)u) = \frac{Ds(u)}{|Ds(u)|}.$$

Therefore,  $N^\circ$  is a composition of  $C^{k-1}$  diffeomorphisms. Hence,  $K^\circ$  is  $C_+^k$ , its support function  $s^\circ$  is  $C^k$  and the eigenvalues of  $\nabla^2 s^\circ + g s^\circ$  with respect to  $g$  are all positive. Repeating the above argument with  $K$  in place of  $K^\circ$  implies that  $K$  is  $C_+^k$ .  $\square$

*Exercise 1.23.* Show that the eigenvalues of  $h_{ij}^\circ$  with respect to  $\bar{g}_{ij}^\circ$  are all positive.

*Proof.* We have

$$[h_{ij}][\bar{g}^{oij}] = \frac{\sqrt{\rho^{\circ 2} + |\nabla \rho^\circ|^2}}{\rho^{\circ 3}} [h_{ik}^\circ \bar{g}^{\circ kj}],$$

where

$$\bar{g}^{oij} = \frac{1}{\rho^{\circ 2}} \left( g^{ij} - \frac{\nabla^i \rho^\circ \nabla^j \rho^\circ}{\rho^{\circ 2} + |\nabla \rho^\circ|^2} \right).$$

To prove the claim, we only need to consider points at which the gradient of the radial function  $\rho^\circ$  does not vanish. Around such a point, we introduce an orthonormal frame  $\{e_i\}$  of  $\mathbb{S}^n$  such that  $e_1 = \frac{\nabla \rho^\circ}{|\nabla \rho^\circ|}$ . Then  $\nabla \rho^\circ = (|\nabla \rho^\circ|, 0, \dots, 0)$ . Thus, in such a frame, we may express  $[h_{ij}][\bar{g}^{oij}]$  as follows:

$$\frac{\sqrt{\rho^{\circ 2} + |\nabla \rho^\circ|^2}}{\rho^{\circ 3}} [h_{ij}^\circ \bar{g}^{oij}] = [h_{ij}] \begin{pmatrix} \frac{1}{\rho^{\circ 2} + |\nabla \rho^\circ|^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\rho^{\circ 2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\rho^{\circ 2}} \end{pmatrix}.$$

The eigenvalues of  $[h_{ij}]$  (in the orthonormal frame  $\{e_i\}$ ) are the principal radii of curvature of  $\partial K$ , and the eigenvalues of the matrix on the left-hand side are

$$\left\{ \frac{\sqrt{\rho^{\circ 2} + |\nabla \rho^\circ|^2}}{\rho^{\circ 3}} \kappa_i^\circ \right\}_i.$$

Here,  $\kappa_i^\circ$  are the principal curvatures of  $\partial K^\circ$ . Since the determinant of the product matrix on the right-hand side is positive, we deduce that  $\kappa_i^\circ$  are all positive. Note that there is at least one point where all the principal curvatures of  $K^\circ$  are positive (e.g. a point where  $\rho^\circ$  attains its maximum; see (1.11) and (1.12)).  $\square$

*Exercise 1.24.* Suppose  $k \geq 2$ . Let  $f : \mathbb{S}^n \rightarrow \mathbb{R}$  be a  $C^k$  function. Show that if  $\det B[f] > 0$ , then  $f$  is the support function of a convex body of class  $C_+^k$ . (Hint: Show that the one-homogenous extension  $\bar{f}$  defined above is convex and hence sublinear.)

## 2. MINKOWSKI PROBLEM

The main reference for this section is Cheng-Yau [CY76] and Lutwak-Oliker [LO95]. The smooth Minkowski problem (the regular Minkowski problem) asks the following question:

**Question 2.1.** Given a positive smooth function  $f$  on the unit sphere, what is the necessary and sufficient condition on  $f$  that ensures the existence of a closed, strictly convex, smooth hypersurface whose Gauss curvature, as a function of the outer unit normal, is  $f^{-1}$ ? From our discussion in the previous section, the problem is equivalent to finding  $0 < s \in C^\infty(\mathbb{S}^n)$  such that

$$(2.1) \quad \det B[s] = f$$

We will see later the necessary and sufficient condition for the existence of a strictly convex solution (hypersurface) is

$$(2.2) \quad \int_{\mathbb{S}^n} \langle x, v \rangle f(x) dx = 0, \quad \forall v \in \mathbb{R}^{n+1}.$$

*Remark 2.2.* From now on  $\int = \int_{\mathbb{S}^n} dx$ ; i.e. all integrals are on the unit sphere.

**Theorem 2.3** (Minkowski problem). *Let  $f \in C^\infty(\mathbb{S}^n)$  be a positive function. Suppose  $\int x_i f(x) = 0$  for all coordinate functions  $x_i$ ,  $i = 1, \dots, n+1$ . Then we can solve the equation (2.1) so that  $s$  is smooth. That is, we can find a closed strictly convex hypersurface in  $\mathbb{R}^{n+1}$  whose support function is  $s$  and Gauss curvature (as a function of the unit normal) is  $f^{-1}$ . Moreover, any two such hypersurface must coincide after a translation.*

Let  $k$  be a non-negative integer and let  $\alpha \in (0, 1]$ . The Banach space of real-valued functions on  $\mathbb{S}^n$  which are  $k$ -times continuously differentiable is denoted by  $C^k(\mathbb{S}^n)$  and it is equipped with the norm

$$\|f\|_{C^k} := \sum_{|\beta| \leq k} \sup_{\mathbb{S}^n} |\nabla^{(\beta)} f|.$$

Moreover,  $C^{k,\alpha}(\mathbb{S}^n)$  (or in short  $C^{k,\alpha}$ ) is the space of functions in  $C^k(\mathbb{S}^n)$  such that the norm

$$\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + \sup_{|\beta|=k} \sup_{x, y \in \mathbb{S}^n} \frac{|\nabla^{(\beta)} f(x) - \nabla^{(\beta)} f(y)|}{d_{\mathbb{S}^n}(x, y)^\alpha}$$

is finite. Here  $d_{\mathbb{S}^n}(x, y)$  denotes the (geodesic) distance between  $x$  and  $y$  in  $\mathbb{S}^n$ .

**2.1. Existence of solutions.** Suppose  $f \in C^{k,\alpha}(\mathbb{S}^n)$ . To solve (2.1), we use the continuity method. We define

$$f_t = tf + (1-t), \quad t \in [0, 1].$$

Note that for  $t = 0$ ,  $\det B[1] = 1$ . Hence, (2.1) has a smooth, strictly convex solution, the unit ball. Let  $S_\alpha = \{t \in [0, 1] : \text{the equation } \det B[s] = f_t \text{ has a } C^{k+2,\alpha} \text{ solution } s_t \text{ such that } B[s_t] > 0 \text{ and } \int s_t x_i = 0 \text{ for all } i = 1, \dots, n+1\}$ . For  $k \geq 3$ , we show that  $S_\alpha = [0, 1]$ . The method is divided into two steps. In the first step we prove  $S_\alpha$  is closed in  $[0, 1]$  and in the second step we prove  $S_\alpha$  is open in  $[0, 1]$ . Since  $0 \in S_\alpha$ , this clearly proves  $S_\alpha = [0, 1]$ .

2.1.1. *Closedness: A priori estimates.* Suppose  $\mathcal{M} = \partial K$  is a smooth, strictly convex hypersurface with support function  $s$ . Denote the eigenvalues of  $B[s]$  by  $\lambda_i$ . We define

$$\sigma_k = \sigma_k(B[s]) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq n.$$

Obviously, we have  $\det B[s] = \sigma_n$ . The extrinsic diameter of  $\mathcal{M}$  is defined as

$$\text{diam}(\mathcal{M}) = \max_{p, q \in \mathcal{M}} |p - q|,$$

where  $|p - q|$  denotes the Euclidean distance of  $p$  and  $q$ .

**Lemma 2.4.** *Let  $\tilde{K}$  be a convex body with non-empty interior (i.e. the volume of  $K$  is not zero). Then there is a unique point  $p$  (the Steiner point of  $\tilde{K}$ ) such that*

$$(2.3) \quad \int (s_{\tilde{K}}(x) - \langle x, p \rangle) x_i = 0 \quad \forall i = 1, \dots, n+1.$$

In fact,  $p = \frac{n+1}{|\mathbb{S}^n|} \int s_{\tilde{K}}(x) x$  and it lies in the interior of  $\tilde{K}$ .

*Proof.* Define

$$\ell(y) = \int (s_{\tilde{K}}(x) - \langle x, y \rangle)^2, \quad y \in \mathbb{R}^{n+1}.$$

Since  $\tilde{K}$  is compact,  $\ell$  attains a minimum in  $\tilde{K}$ , say at  $y_0$ . Suppose, for the sake of contradiction, that  $y_0$  is on the boundary of  $\tilde{K}$ . After changing coordinates, we may assume that  $y_0$  is at the origin and  $\tilde{K}$  lies below the hyperplane  $e_{n+1}^\perp$  so that  $-te_{n+1} \in \text{int } \tilde{K}$  for sufficiently small  $t > 0$ . Now we show that

$$\left. \frac{d}{dt} \right|_{t=0} \ell(-te_{n+1}) \text{ is negative.}$$

Let  $s = s_{\tilde{K}}$ ,  $x^+ = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n$  be such that  $x_{n+1} > 0$ , and define  $x^- = (x_1, \dots, -x_{n+1})$ . Moreover, let  $i(x^+)$  be a point on  $\partial \tilde{K}$  so that  $s(x^+) = \langle x^+, i(x^+) \rangle$ . Note that the  $x_{n+1}$ -component of  $i(x^+)$  is non-positive. Hence,

$$(2.4) \quad s(x^-) = \max_{y \in \tilde{K}} \langle x^-, y \rangle \geq \langle x^-, i(x^+) \rangle \geq \langle x^+, i(x^+) \rangle = s(x^+).$$

On the other hand,  $s((0, \dots, 0, 1)) = 0$ , while  $s((0, \dots, 0, -1)) > 0$ . Therefore, the set of points with  $s(x^-) > s(x^+)$  has positive measure. Moreover, due to (2.4),

$$\left. \frac{d}{dt} \right|_{t=0} \ell(-te_{n+1}) = 2 \int s(x) x_{n+1} = 2 \int_{\{x \in \mathbb{S}^n : x_{n+1} > 0\}} (s(x^+) - s(x^-)) x_{n+1} < 0.$$

That is,  $\ell(-te_{n+1}) < \ell(0)$  for some small  $t$ , a contradiction. Therefore,  $y_0 \in \text{int } \tilde{K}$  and  $D\ell|_{y_0} = 0$ . This implies that  $y_0 = p$  and  $p$  lies in the interior of  $\tilde{K}$ .  $\square$

**Lemma 2.5.** *Suppose  $\det B[s] = f$ . Then*

$$\text{diam}(\mathcal{M}) \leq c_n \left( \int f \right)^{\frac{n+1}{n}} \left( \inf_{y \in \mathbb{S}^n} \int |\langle y, x \rangle| f(x) \right)^{-1},$$

for some constant  $c_n$  depending only on  $n$ . In particular, if the support function  $s$  satisfies (2.2), then

$$0 < s \leq c_n \left( \int f \right)^{\frac{n+1}{n}} \left( \inf_{y \in \mathbb{S}^n} \int |\langle y, x \rangle| f(x) \right)^{-1}.$$

*Proof.* The argument is from [CY76]. Let  $p, q \in \mathcal{M}$  such that the line segment joining  $p$  and  $q$  has length  $L := \text{diam}(\mathcal{M})$ . We may assume 0 is in the middle of the line segment. Hence,  $\pm \frac{L}{2}u \in \mathcal{M}$  for some  $u \in \mathbb{S}^n$ , and for any  $x \in \mathbb{S}^n$  we have

$$s(x) = \max_{y \in K} \langle y, x \rangle \geq \frac{L}{2} |\langle u, x \rangle|.$$

Now we multiply both sides by  $f$  and integrate over  $\mathbb{S}^n$ :

$$\begin{aligned} L &\leq 2 \left( \int s f \right) \left( \int |\langle u, x \rangle| f(x) \right)^{-1} \\ &= 2 \left( \int s \sigma_n \right) \left( \int |\langle u, x \rangle| f(x) \right)^{-1} \\ &\leq 2 \left( \int s \sigma_n \right) \left( \inf_{y \in \mathbb{S}^n} \int |\langle y, x \rangle| f(x) \right)^{-1}. \end{aligned}$$

By (a special case of) Alexandrov-Fenchel's inequality [Sch14, Thm. 7.3.1] we have

$$\left( \int s \sigma_n \right)^{\frac{1}{n+1}} \leq c'_n \left( \int s \sigma_{n-1} \right)^{\frac{1}{n}}.$$

On the other hand, by Lemma 2.13, we have

$$\int s \sigma_{n-1} = n \int \sigma_n = n \int f.$$

Hence we obtain

$$L \leq c''_n \left( \int f \right)^{\frac{n+1}{n}} \left( \inf_{y \in \mathbb{S}^n} \int |\langle y, x \rangle| f(x) \right)^{-1}.$$

Since  $s$  satisfies (2.2) (i.e. the Steiner point of  $K$  is at the origin; cf. Lemma 2.4) we have

$$0 < s \leq L.$$

□

**Lemma 2.6.** Suppose  $\det B[s] = f$  and  $s$  satisfies (2.2). There exists  $C > 0$  such that

$$s \geq C.$$

Here  $C$  depends only on  $\min f$ .

*Proof.* The volume of  $K$  is given by

$$V(K) = \frac{1}{n+1} \int s \det B[s] = \frac{1}{n+1} \int s f.$$



Hence, we have

$$V(K) \geq c \int s$$

where  $c := \frac{1}{n+1} \min f$ . Moreover, by the Alexandrov-Fenchel inequality,

$$\int s \geq c_n V^{\frac{1}{n+1}},$$

we obtain  $V(K)^{\frac{n}{n+1}} \geq c'$ .

The following argument shows that  $s$  has a positive lower bound. Let  $P$  be the set of all convex bodies whose support functions satisfy (2.2), have diameters bounded by  $C_0$ , and volumes bounded below by  $C_1$ . We claim that there exists a constant  $C_2$  such that for any convex body  $\hat{K}$  in  $P$ ,  $s_{\hat{K}}$  is bounded below by  $C_2$ . If this is not the case, then we can find  $K_i$  such that  $s_{K_i}(u_i) \rightarrow 0$  for some  $u_i \in \mathbb{S}^n$ . By Blaschke's selection theorem, a subsequence of  $K_i$  converges to a convex body  $K_\infty$  (i.e.,  $\sup |s_{K_{i_k}} - s_{K_\infty}| \rightarrow 0$  as  $k \rightarrow \infty$  for some sequence  $i_k$ ) such that  $s_{K_\infty}(u) = 0$  for some  $u \in \mathbb{S}^n$ . That is,  $0 \in \partial K_\infty$ . However,

$$(2.5) \quad \int x_i s_{K_\infty} = 0, \quad \forall i \in \{1, \dots, n+1\}.$$

Since  $K_\infty$  has a non-empty interior, by Lemma 2.4 and (2.5) the Steiner point of  $K_\infty$  lies at the origin. Hence,  $s_{K_\infty}$  must be positive. This is a contradiction.  $\square$

**Lemma 2.7.** *We have  $|\nabla s| \leq L$ .*

*Proof.* Set  $r = s^2 + |\nabla s|^2$ . At any critical point  $x$  of  $r$ , we have

$$0 = \nabla_i r(x) = 2(\nabla_{i,j}^2 s + s g_{ij}) g^{jk} \nabla_k s.$$

Since  $\mathcal{M}$  is strictly convex,  $\nabla s(x) = 0$ . Therefore, we have

$$r \leq \max r = r(x) = s(x)^2 \leq L^2.$$

$\square$

Suppose  $t_k \in S_\alpha$  and  $t_k \rightarrow t_0$ . By these last three lemmas, we have  $C_1 \leq s_{t_k} \leq C_2$  and  $|\nabla s_{t_k}| \leq C_3$  where  $C_i$  are independent of  $k$ . Hence,  $s_{t_k}$  converges in  $C^{0,\alpha}$  to some function  $s_{t_0}$ . We will show that, in fact,  $s_{t_k}$  converges in  $C^{2,\alpha}$  (for any  $\alpha \in (0, 1)$ ) to  $s_{t_0}$  satisfying  $\det B[s_{t_0}] = f_{t_0}$ ,  $s_{t_0} \in C^{k+2,\alpha}(\mathbb{S}^n)$  and  $\int x_i s_{t_0} = 0$ . Our main tool is the interior  $C^2$ -estimate for Monge-Ampère type equations (cf. [Oli84, Thm. 4.2]) and the interior  $C^3$ -estimate of Calabi; see [Oli84, Thm. 4.5], we only prove the former.

We follow the exposition in [LO95]. To transform the standard interior  $C^2$  and  $C^3$  estimates for Monge-Ampère type equations in Euclidean domains to our equation (2.1), we need to extend  $s_k := s_{t_k}$  to a function  $\tilde{s}_k$  on  $(x_1, x_2, \dots, x_n, 1)$  and compute the equation it satisfies in a domain of  $\mathbb{R}^n$ ; see (2.9).

Let  $u_0 \in \mathbb{S}^n$  (for simplicity, take  $u_0 = e_{n+1}$ ), and let  $S^{u_0}$  denote the open hemisphere for which  $u_0$  is the pole. Let  $H$  denote the hyperplane tangent to  $S^{u_0}$  at  $u_0$  and  $H'$  denote the

hyperplane parallel to  $H$  that passes through the centre of  $\mathbb{S}^n$ . Choose cartesian coordinates so that  $x_1, \dots, x_n$  are the coordinates for  $H'$  and  $x^{n+1}$  is directed toward  $u_0 (= e_{n+1})$ . Hence,

$$\varphi(x) := (x, 1)/Q(x) \in S^{u_0}, \quad Q(x) := \sqrt{1 + |x|^2}.$$

is a local parameterization of  $\mathbb{S}^n$ .

With each function  $s_k$ , we associate the function

$$\tilde{s}_k(x) = Q(x)s_k(\varphi(x)), \quad x \in H'.$$

Then  $\tilde{s}_k$  converges uniformly to  $\tilde{s}_0$  on any compact subset of the hyperplane  $H'$ . In addition, we have  $\tilde{s}_k \geq C_1$ , and  $\tilde{s}_k \rightarrow \infty$  as  $|x| \rightarrow \infty$ .

Since  $Q(x)\varphi(x) = (x, 1)$ , we have

$$(2.6) \quad \varphi \partial_i Q + Q \partial_i \varphi = (0, \dots, 0, 1, 0, \dots, 0),$$

where 1 appears at  $i$ -th slot. Differentiating this one more time yields

$$(2.7) \quad \varphi \partial_{ij}^2 Q + \partial_i Q \partial_j \varphi + \partial_j Q \partial_i \varphi + Q \partial_{ij}^2 \varphi = (0, \dots, 0).$$

By the Gauss equation, we have

$$(2.8) \quad \partial_{ij}^2 \varphi = \Gamma_{ij}^k \partial_k \varphi - g_{ij} \varphi.$$

Since  $\langle \partial_r \varphi, \varphi \rangle = 0$  for any  $r$ , and  $\langle \partial_i \varphi, \partial_j \varphi \rangle = g_{ij}$ , taking the inner product of (2.7) with  $\partial_r \varphi$  we obtain

$$\partial_i Q g_{jr} + \partial_j Q g_{ir} + Q \Gamma_{ij}^m g_{mr} = 0.$$

On the other hand, we have<sup>1</sup>

$$\partial_i \tilde{s}_k = s_k \partial_i Q + Q \partial_i s_k$$

Moreover,

$$\begin{aligned} \partial_{ij}^2 \tilde{s}_k &= s_k \partial_{ij}^2 Q + \partial_i Q \partial_j s_k + \partial_j Q \partial_i s_k + Q \partial_{ij}^2 s_k \\ &= s_k \partial_{ij}^2 Q + (\partial_i Q g_{jr} + \partial_j Q g_{ir}) g^{rl} \partial_l s_k + Q \partial_{ij}^2 s_k \\ &= s_k \partial_{ij}^2 Q - Q \Gamma_{ij}^l \partial_l s_k + Q \partial_{ij}^2 s_k \\ &= s_k \partial_{ij}^2 Q + Q \nabla_{i,j}^2 s_k. \end{aligned}$$

Taking the inner product of (2.7) with  $\varphi$  and using (2.8) gives

$$\partial_{ij}^2 Q = Q g_{ij}.$$

Hence<sup>2</sup>

$$\partial_{ij}^2 \tilde{s}_k = (\nabla_{i,j}^2 s_k + s_k g_{ij}) Q,$$

and

$$f_{t_k} = \det B[s_k] = \frac{\det[\nabla_{i,j}^2 s_k + s_k g_{ij}]}{\det[g_{ij}]} = \frac{1}{Q^n} \frac{\det[\partial_{ij}^2 \tilde{s}_k]}{\det[g_{ij}]}.$$

<sup>1</sup>Note here that  $\partial_i s_k$  is the derivative of  $s_k : \mathbb{S}^n \rightarrow \mathbb{R}$  in the direction  $\partial_i \varphi$ .

<sup>2</sup>Compare with [Exercise 1.18](#).

Moreover, from (2.6) we have

$$\partial_i Q \partial_j Q + Q^2 \langle \partial_i \varphi, \partial_j \varphi \rangle = \delta_{ij}.$$

That is,

$$g_{ij} = \frac{1}{Q^2} (\delta_{ij} - \partial_i Q \partial_j Q).$$

Hence<sup>3</sup>,

$$\det[g_{ij}] = Q^{-2(n+1)}.$$

Therefore, we obtain

$$(2.9) \quad \det[\partial_{ij}^2 \tilde{s}_k] = f_{t_k} Q^{-n-2}.$$

Note that, in particular, this identity implies that the graphs  $\{(x, x^{n+1} = \tilde{s}_k(x)) : x \in H'\}$  are strictly convex. Consequently,  $\tilde{s}_0$  is a convex of function on  $H'$ .

Since  $\tilde{s}_0 \rightarrow \infty$  when  $|x| \rightarrow \infty$  (recall our uniform lower bound on the sequence  $s_k$ ), we can choose  $\lambda$ , sufficiently large, so that the set

$$P_0 := \{x \in H' : \tilde{s}_0(x) \leq \lambda\}$$

is a non-empty compact convex set in  $H'$ . Since the  $\tilde{s}_k$  converges to  $\tilde{s}_0$ , uniformly on each compact subset of  $H'$ , we may choose some  $\lambda$  so that a compact set  $\Theta$  is contained *strictly* inside all of the sets  $P_k := \{x \in H' : \tilde{s}_k(x) \leq \lambda\}$  as well as the set  $P_0$  (by strictly inside, we mean  $\Theta$  lies at a positive distance  $\delta$  from  $\partial P_k$  for all  $k$  large enough). Now consider the functions

$$\hat{s}_k := \tilde{s}_k - \lambda.$$

Obviously  $\hat{s}_k$  vanishes on the boundary of  $P_k$  and due (2.9), hence we have

$$(2.10) \quad \begin{cases} \det[\partial_{ij}^2 \hat{s}_k] = f_{t_k} Q^{-n-2} & \text{in } P_k, \\ \hat{s}_k = 0 & \text{on } \partial P_k. \end{cases}$$

Moreover, the r.h.s. of (2.10) is bounded away from 0 *uniformly* in  $k$ .

Let  $U$  be a convex domain in  $\mathbb{R}^n$  and  $\phi$  be function of  $2n+1$  variables in the domain

$$V = U \times [a, b] \times \mathbb{R}^n.$$

**Theorem 2.8** (Pogorelov; Oliker). [Oli84, Thm. 4.2], [GT01, Thm. 17.19]. *Suppose  $\phi \in C^2(\bar{V})$  and  $\phi > 0$  in  $V$ . Let  $z \in C^4(U) \cap C^1(\bar{U})$  and satisfies in  $U$  the equation*

$$(2.11) \quad \det[\partial_{ij}^2 z] = \phi(x, z, Dz).$$

---

<sup>3</sup>In fact, due to

$$\partial_i Q \partial_j Q = \frac{x_i x_j}{Q^2}.$$

we may write

$$g_{ij} = \frac{1}{Q^2} \left( \delta_{ij} - \frac{x_i x_j}{Q^2} \right), \quad [g_{ij}] = \frac{1}{Q^2} \left( I - \frac{1}{Q^2} x x^T \right),$$

Then we may use the fact that for a matrix of the form  $a(I - buu^T)$  (where  $u$  is a vector in  $\mathbb{R}^n$ ), the determinant is given by  $a^n(1 - b\|u\|^2)$ .

Assume further that the matrix  $[\partial_{ij}^2 z]$  is positive definite everywhere in  $U$ , and  $z|_{\partial U} = 0$ . Then the second derivatives of  $z$  at a point  $x \in U$  can be estimated from above and the estimates depends only on the  $C^1$ -norm of  $z$  in  $U$ , the  $C^2$ -norm of  $\phi$  in  $V$ , and the distance of  $x$  to the boundary of  $U$ .

*Proof.* The proof is from [Oli84] and [Gut16]. Since  $D^2 z$  is positive definite and  $z = 0$  on  $\partial U$ ,  $z < 0$  in  $U$ . Let  $\alpha$  be a unit vector in  $\mathbb{R}^n$  and  $z_\alpha = \langle Dz, \alpha \rangle$ ,  $z_{\alpha\alpha} = D^2 z(\alpha, \alpha)$ , and  $c$  be a positive constant. To prove the claim, we show that for a suitable  $c$  the function

$$w = -ze^{c\frac{z_\alpha^2}{2}}z_{\alpha\alpha},$$

is bounded above depending on the  $C^1$ -norm of  $z$  in  $U$ , the  $C^2$ -norm of  $\phi$  in  $V$ .

Note that the maximum of  $w$  is attained *inside*  $U$ , say at  $x_0$ . Moreover, there exists a unimodular matrix  $O$  (i.e.  $\det O = 1$ ), such that  $O^t[D^2 z(x_0)]O$  is diagonal and if  $\bar{z}(x) := z(Ox)$ , then  $\bar{z}_1(x) = z_\alpha(Ox)$  and  $\bar{z}_{11}(x) = z_{11}(Ox)$ . In particular,

$$[D^2 \bar{z}(O^{-1}x_0)] = O^t[D^2 z(x_0)]O$$

is diagonal; see [Remark 2.9](#) for the details. Hence, we may assume that  $\alpha = e_1$  and the matrix  $[z_{ij}] := [\partial_{ij}^2 z]$  is diagonal at  $x_0$ .

Taking the derivatives of

$$\log(w) = \log(-z) + c\frac{z_1^2}{2} + \log(z_{11})$$

in the direction  $x_i$ , at  $x_0$  we have

$$(2.12) \quad \frac{w_i}{w} = \frac{z_{11i}}{z_{11}} + cz_1 z_{1i} + \frac{z_i}{z} = 0.$$

Differentiating one more time with respect to  $x_i$ , we obtain

$$(2.13) \quad \frac{w_{ii}}{w} - \frac{w_i^2}{w^2} = \frac{z_{11ii}}{z_{11}} - \frac{z_{11i}^2}{z_{11}^2} + cz_1^2 + cz_1 z_{1ii} + \frac{z_{ii}}{z} - \frac{z_i^2}{z^2}.$$

Since  $z_{i1}|_{x_0} = 0$  when  $i \neq 1$  and  $w_i|_{x_0} = 0$ , after multiplying (2.13) by  $\frac{z_{11}}{z_{ii}}$  and summing over  $i$ , at  $x_0$  we get

$$(2.14) \quad \begin{aligned} \sum_i \frac{w_{ii} z_{11}}{w z_{ii}} &= \sum_i \frac{z_{11ii}}{z_{ii}} - \sum_i \frac{z_{11i}^2}{z_{11} z_{ii}} + cz_1^2 + cz_1 z_{11} \sum_i \frac{z_{1ii}}{z_{ii}} \\ &\quad + n \frac{z_{11}}{z} - \left(\frac{z_1}{z}\right)^2 - \frac{z_{11}}{z^2} \sum_{i \neq 1} \frac{z_i^2}{z_{ii}}. \end{aligned}$$

Here,  $-\sum_i \frac{z_i^2}{z^2} \frac{z_{11}}{z_{ii}}$  gives the last two terms. Moreover, (2.12) gives

$$\left(\frac{z_i}{z}\right)^2 = \frac{z_{11i}^2}{z_{11}^2} \quad \text{when } i \neq 1.$$

Substituting it in the last term of (2.14), we obtain

$$(2.15) \quad \begin{aligned} \sum_i \frac{w_{ii} z_{11}}{w z_{ii}} &= c z_{11}^2 + n \frac{z_{11}}{z} - \left( \frac{z_1}{z} \right)^2 + c z_1 z_{11} \sum_i \frac{z_{1ii}}{z_{ii}} \\ &+ \sum_i \frac{z_{11ii}}{z_{ii}} - \sum_i \frac{z_{11i}^2}{z_{11} z_{ii}} - \sum_{i \neq 1} \frac{z_{11i}^2}{z_{11} z_{ii}}. \end{aligned}$$

Now we differentiate equation (2.11) twice with respect to  $x_1$  (cf. Exercise 2.10):

$$(2.16) \quad (\log \phi)_1 = \sum_i \frac{z_{ii1}}{z_{ii}}$$

and

$$(2.17) \quad (\log \phi)_{11} = \sum_i \frac{z_{ii11}}{z_{ii}} - \sum_{i,j} \frac{z_{ij1}^2}{z_{ii} z_{jj}}.$$

Subtracting (2.17) from (2.15) term by term to cancel the  $\sum_i \frac{z_{11ii}}{z_{ii}}$ -term appearing in both identities, we obtain at  $x_0$ :

$$(2.18) \quad \begin{aligned} &\sum_i \frac{w_{ii} z_{11}}{w z_{ii}} - (\log \phi)_{11} \\ &= c z_{11}^2 + n \frac{z_{11}}{z} - \left( \frac{z_1}{z} \right)^2 + c z_1 z_{11} \sum_i \frac{z_{1ii}}{z_{ii}} - \sum_i \frac{z_{11i}^2}{z_{11} z_{ii}} - \sum_{i \neq 1} \frac{z_{11i}^2}{z_{11} z_{ii}} + \sum_{i,j} \frac{z_{ij1}^2}{z_{ii} z_{jj}} \end{aligned}$$

Note that

$$\sum_{i,j} \frac{z_{ij1}^2}{z_{ii} z_{jj}} - \sum_i \frac{z_{11i}^2}{z_{ii} z_{11}} - \sum_{i>1} \frac{z_{11i}^2}{z_{11} z_{ii}} = \sum_{i,j>1} \frac{z_{ij1}^2}{z_{ii} z_{jj}} \geq 0.$$

Using this last inequality and the inequality  $w_{ii}|_{x_0} \leq 0$ , from (2.18) we obtain

$$(2.19) \quad c z_{11}^2 + n \frac{z_{11}}{z} - \left( \frac{z_1}{z} \right)^2 + c z_1 z_{11} (\log \phi)_1 + (\log \phi)_{11} \leq 0 \quad \text{at } x_0.$$

We need to symbolically compute the derivatives of  $\log \phi$ . Set  $\phi(x, t, p) = \phi(x, z, Dz)$ . We calculate

$$(2.20) \quad (\log \phi)_1 = \frac{\phi_1 + \phi_t z_1 + \sum_i \phi_{p_i} z_{i1}}{\phi}$$

and

(2.21)

$$\begin{aligned}
(\log \phi)_{11} = & \left[ \phi_{11} + \phi_{1t}z_1 + \sum_i \phi_{1p_i}z_{i1} + \left( \phi_{t1} + \phi_{tt}z_1 + \sum_i \phi_{tp_i}z_{i1} \right) z_1 + \phi_t z_{11} \right. \\
& \left. + \sum_i \left( \phi_{p_i1} + \phi_{p_it}z_1 + \sum_j \phi_{p_ip_j}z_{j1} \right) z_{i1} + \sum_i \phi_{p_i}z_{i11} \right] / \phi \\
& - \left( \phi_1^2 + \phi_t^2 z_1^2 + \left( \sum_i \phi_{p_i}z_{i1} \right)^2 + 2\phi_1\phi_t z_1 + 2\phi_1 \sum_i \phi_{p_i}z_{i1} + 2\phi_t z_1 \sum_i \phi_{p_i}z_{i1} \right) / \phi^2.
\end{aligned}$$

Using (2.12), at  $x_0$  we find that

$$\sum_i \phi_{p_i}z_{i11} = -cz_1\phi_{p_1}z_{11}^2 - \frac{z_{11}}{z} \sum \phi_{p_i}z_i.$$

Therefore,

$$cz_1z_{11}(\log \phi)_1 + (\log \phi)_{11} = \left( \frac{\phi_{p_1p_1}}{\phi} - \frac{\phi_{p_1}^2}{\phi^2} \right) z_{11}^2 + Az_{11} + B,$$

where  $A$  and  $B$  denote quantities (depending on  $c$ ) admitting estimates in terms of the  $C^2$ -norm of  $\phi$  and  $C^1$ -norm of  $z$ . Substituting the last expression in (2.19) we get

$$\left( c + \frac{\phi_{p_1p_1}}{\phi} - \frac{\phi_{p_1}^2}{\phi^2} \right) z_{11}^2 + \left( A + \frac{n}{z} \right) z_{11} - \left( \frac{z_1}{z} \right)^2 + B \leq 0.$$

We may choose  $c$  so that the coefficient of  $z_{11}^2$  is positive. Multiplying the last inequality by  $z^2 e^{cz_1^2}$  and introducing  $w$  we get

$$\bar{C}w^2 + \bar{A}w + \bar{B} \leq 0,$$

where  $\bar{C} > 0$  and it admits an estimate from above and below in terms of the  $C^2$ -norm of  $\phi$ , and  $\bar{A}, \bar{B}$  admit uniform estimates in terms of the quantities indicated in the theorem. Thus everywhere  $w \leq \bar{w}$  in  $U$ , where  $\bar{w}$  depends on  $\bar{A}, \bar{B}$  and  $\bar{C}$ . Hence

$$z_{11} \leq \frac{\bar{w}}{|z|}.$$

Since  $z$  is concave upward and  $z|_{\partial U} = 0$ , we have (see Lemma 2.11):

$$\frac{2|z(x)|}{\text{dist}(x, \partial U)} \geq \frac{\max|z|}{d},$$

where  $d$  is the diameter of  $U$ . Hence

$$z_{11}(x) \leq \frac{2\bar{w}d}{\text{dist}(x, \partial U) \max|z|}.$$

Since  $\alpha$  was an arbitrary direction, the claim follows.  $\square$

*Remark 2.9.* ([Gut16, p. 63]) Let  $Q$  be an orthogonal matrix (i.e.  $Q^t = Q^{-1}$ ) such that  $Qe_1 = \alpha$ , and first let  $v(x) = z(Qx)$ . Then the first column of  $Q$  is the vector  $\alpha$  and we have  $v_1(x) = z_\alpha(Qx)$  and  $v_{11}(x) = z_{\alpha\alpha}(Qx)$ . Given an  $n \times n$  symmetric, positive-definite matrix  $A = [a_{ij}]$ , consider the matrix

$$B = \begin{bmatrix} 1 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & -\frac{a_{14}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

We have

$$B^t AB = \begin{bmatrix} a_{11} & 0 \\ 0 & B_1 \end{bmatrix},$$

where  $B_1$  is an  $(n-1) \times (n-1)$  matrix. Since  $A$  is symmetric and positive-definite, it follows that  $B_1$  is also positive-definite and symmetric. Hence, an orthogonal matrix  $O_1$  exists such that  $O_1^t B_1 O_1$  is diagonal. Let

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & O_1 \end{bmatrix}.$$

Now we choose  $A = [D^2 v(Q^{-1}x_0)]$  and set  $\bar{z}(x) = v(B\mathcal{O}x) = z(QB\mathcal{O}x)$ . Then

$$\begin{aligned} [D^2 \bar{z}((QB\mathcal{O})^{-1}x_0)] &= [D^2 \bar{z}((B\mathcal{O})^{-1}Q^{-1}x_0)] \\ &= (B\mathcal{O})^t [D^2 v(Q^{-1}x_0)] (B\mathcal{O}) \\ &= (B\mathcal{O})^t AB\mathcal{O} \end{aligned}$$

is diagonal. Combining the changes of coordinates, the matrix  $O = QB\mathcal{O}$  does the job.

*Exercise 2.10.* Verify (2.16) and (2.17). Let  $A(t)$  be a positive-definite symmetric matrix, and suppose  $A(t_0)$  is diagonal with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then

$$(2.22) \quad \frac{d}{dt} \log \det A(t) = \operatorname{tr} \left( A(t)^{-1} \frac{dA(t)}{dt} \right).$$

and

$$\frac{d^2}{dt^2} \log \det A(t) = \operatorname{tr} \left( -A(t)^{-1} \frac{dA(t)}{dt} A(t)^{-1} \frac{dA(t)}{dt} + A(t)^{-1} \frac{d^2 A(t)}{dt^2} \right).$$

Hence at  $t_0$ :

$$\begin{aligned} \left. \frac{d}{dt} \log \det A(t) \right|_{t=t_0} &= \sum_i \frac{\dot{a}_{ii}}{a_{ii}}, \\ \left. \frac{d^2}{dt^2} \log \det A(t) \right|_{t=t_0} &= \sum_{i,j} \left( -\frac{\dot{a}_{ij} \dot{a}_{ji}}{\lambda_i \lambda_j} \right) + \sum_i \frac{\ddot{a}_{ii}}{\lambda_i}. \end{aligned}$$

**Lemma 2.11.** *We have*

$$\frac{2|z(x)|}{\text{dist}(x, \partial U)} \geq \frac{\max|z|}{d}.$$

*Proof.* Let  $y \in U$ . Then by convexity of  $z$ , we have

$$z(y) \geq z(x) + \langle Dz(x), y - x \rangle.$$

Let  $0 < r < \text{dist}(x, \partial U)$ , then  $y_0 := x + r \frac{Dz(x)}{|Dz(x)|} \in U$  and

$$0 \geq z(y_0) \geq z(x) + r|Dz(x)|.$$

Hence

$$|Dz(x)| \leq \frac{-z(x)}{\text{dist}(x, \partial U)} = \frac{|z(x)|}{\text{dist}(x, \partial U)}.$$

Let  $y$  be a point where  $z$  attains its minimum. Then<sup>4</sup>

$$|z(x) - z(y)| \leq |y - x||Dz(x)| \leq d \frac{|z(x)|}{\text{dist}(x, \partial U)}.$$

Therefore,

$$\frac{\max|z|}{d} = \frac{|z(y)|}{d} \leq \frac{|z(x)|}{\text{dist}(x, \partial U)} + \frac{|z(x)|}{d} \leq \frac{2|z(x)|}{\text{dist}(x, \partial U)}.$$

□

Applying this theorem to  $\hat{s}_k$ , we find that

$$(2.23) \quad |\partial_{ij}^2 \hat{s}_k(x)| \leq C, \quad x \in P_k,$$

where  $C$  depends on the  $C^1$ -norm of  $\hat{s}_k$  in  $P_k$ , the  $C^2$ -norm of  $f_{t_k} Q^{-n-2}$  in  $P_k$ , and the distance of the point  $x$  to the boundary of  $P_k$ . Thus, we conclude that on the set  $\Theta$ , the estimate (2.23) is independent of  $k$ .

**Theorem 2.12** (Calabi; Oliker). [Oli84, Thm. 4.5] *Suppose  $\phi \in C^3(\bar{V})$  and that  $\phi > 0$  in  $\bar{V}$ . Let  $z \in C^5(U) \cap C^2(\bar{U})$  and satisfies in  $U$  the equation*

$$\det[\partial_{ij}^2 z] = \phi(x, z, Dz).$$

*Assume further that the matrix  $[\partial_{ij}^2 z]$  is positive definite everywhere in  $U$ , and  $z|_{\partial U} = 0$ . Then at every point  $x$  of  $U$  the  $C^3$ -norm of  $z$  admits a uniform estimate depending on the  $C^2$ -norm of  $z$  at  $x$ ,  $C^3$ -norm of  $\phi$  in  $\bar{V}$ , and the distance from  $x$  to the boundary of  $U$ .*

---

<sup>4</sup>To see this, define  $f(t) = z(y + t \frac{x-y}{|x-y|})$  for  $t \in [0, |x-y|]$ . Then  $f''(t) > 0$  and hence  $f'(t)$  is non-decreasing. Now for some  $t_* \in [0, |x-y|]$ , by the mean-value theorem

$$\frac{z(x) - z(y)}{|x-y|} = \frac{f(|x-y|) - f(0)}{|x-y|} = f'(t_*) \leq f'(|x-y|) = \langle Dz(x), \frac{x-y}{|x-y|} \rangle.$$



By the previous theorem, the third derivatives of  $\hat{s}_k$  in  $\Theta$  are bounded above, uniformly in  $k$ . Hence, by the Arzela-Ascoli theorem a subsequence of  $\hat{s}_k$  converges uniformly in  $\Theta$  to  $\hat{s}_0$  in  $C^{2,\alpha}$  for any  $\alpha \in (0, 1)$ . Thus  $\tilde{s}_0$  satisfies

$$\det(\partial_{ij}^2 \hat{s}_0) = f_{t_0} Q^{-n-2} \quad \text{in } \Theta.$$

Then, the theory of elliptic PDEs can be used to deduce that  $\hat{s}_0$  is, in fact,  $C^{k+2,\alpha}(\Theta)$ ; see [Fig17, Thm. A.42, Prop. A.43]. Therefore,  $t_0 \in S_\alpha$ .

2.1.2. *Openness.* Let  $S_n$  be the group of all permutations of  $\{1, \dots, n\}$  and

$$\text{sgn} : S_n \rightarrow \{-1, 1\}$$

be defined by  $\text{sgn}(\sigma) = 1$  ( $-1$ ) if  $\sigma$  is even (odd).

For smooth functions  $f_i \in C^2(\mathbb{S}^n)$  we define

$$Q[f_1, \dots, f_n] = \frac{1}{n!} \sum_{\sigma, \tau \in S_n} (-1)^{\text{sgn}(\tau) + \text{sgn}(\sigma)} B[f_1]_{\tau(1)}^{\sigma(1)} \cdots B[f_n]_{\tau(n)}^{\sigma(n)}.$$

Recall that

$$B[f]_i^j = g^{jk} (\nabla_{i,k}^2 f + f g_{ik}).$$

Note that  $Q[s, \dots, s] = \det B[s]$  and  $Q[s, \dots, s, 1, \dots, 1]$ , where  $s$  appears  $k$  times, is a multiple of  $\sigma_k$  of eigenvalues of  $B[s]$ .

**Lemma 2.13.**

(1)  $Q$  is independent of the order of its arguments:

$$Q[f_1, \dots, f_n] = Q[f_{\sigma(1)}, \dots, f_{\sigma(n)}]$$

for every permutation  $\sigma \in S_n$ .

(2) If  $B[f_i]$  for every  $i$  is positive-definite, then  $Q[f_1, \dots, f_n]$  is positive.

(3) If  $B[f_i]$  is positive definite for  $i \in \{2, \dots, n\}$ , then  $Q[f] := Q[f, f_2, \dots, f_n]$  is a non-degenerate second-order linear elliptic operator, given in local coordinates by an expression of the following form:

$$Q[f] = Q^{ij} (\nabla_{i,j}^2 f + g_{ij} f),$$

where  $Q^{ij}$  is a symmetric, positive-definite matrix.

(4) The following identity for any  $f_2, \dots, f_n$  as above holds

$$\sum_j \nabla_j Q^{ij} = 0.$$

*Proof.* Proofs of items (2)-(3) are given in [RK96, Ch. VI]. We prove (4). Define

$$(2.24) \quad Q_i^j = \frac{1}{n!} \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(1)=i, \tau(1)=j}} (-1)^{\text{sgn}(\tau) + \text{sgn}(\sigma)} B[f_2]_{\tau(2)}^{\sigma(2)} \cdots B[f_n]_{\tau(n)}^{\sigma(n)}.$$

Then we have

$$\begin{aligned} n! \sum_j \nabla_j Q_i^j &= \sum_j \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(1)=i, \tau(1)=j}} (-1)^{\text{sgn}(\tau) + \text{sgn}(\sigma)} \left( \nabla_j B[f_2]_{\tau(2)}^{\sigma(2)} \right) \cdots B[f_n]_{\tau(n)}^{\sigma(n)} \\ &\quad + \cdots + \sum_j \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(1)=i, \tau(1)=j}} (-1)^{\text{sgn}(\tau) + \text{sgn}(\sigma)} B[f_2]_{\tau(2)}^{\sigma(2)} \cdots \left( \nabla_j B[f_n]_{\tau(n)}^{\sigma(n)} \right). \end{aligned}$$

We may only consider the first term:

$$\begin{aligned} &\sum_j \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(1)=i, \tau(1)=j}} (-1)^{\text{sgn}(\tau) + \text{sgn}(\sigma)} \left( \nabla_j B[f_2]_{\tau(2)}^{\sigma(2)} \right) \cdots B[f_n]_{\tau(n)}^{\sigma(n)} \\ &= \sum_j \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(1)=i, \tau(1)=j}} (-1)^{\text{sgn}(\tau) + \text{sgn}(\sigma)} \left( \nabla_{\tau(1)} B[f_2]_{\tau(2)}^{\sigma(2)} \right) \cdots B[f_n]_{\tau(n)}^{\sigma(n)} \\ &= \frac{1}{2} \sum_j \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(1)=i, \tau(1)=j}} (-1)^{\text{sgn}(\tau) + \text{sgn}(\sigma)} \left( \nabla_{\tau(1)} B[f_2]_{\tau(2)}^{\sigma(2)} - \nabla_{\tau(2)} B[f_2]_{\tau(1)}^{\sigma(2)} \right) \cdots B[f_n]_{\tau(n)}^{\sigma(n)}. \end{aligned}$$

For any  $f \in C^2(\mathbb{S}^n)$ , we have  $\nabla_j B[f]_k^i = \nabla_k B[f]_j^i$  (cf. [Exercise 2.14](#)). Hence, this sum is zero. Moreover, we have

$$Q[f] = Q_k^i B[f]_i^k = Q_k^i g^{jk} (\nabla_{i,j}^2 f + f g_{ij}).$$

Now define  $Q^{ij} = Q_k^i g^{jk}$ . Then from the metric compatibility,  $\nabla g = 0$ , it follows that

$$\nabla_i Q^{ij} = 0.$$

□

*Exercise 2.14.* Show that for any  $f \in C^2(\mathbb{S}^n)$ ,  $\nabla_j B[f]_k^i = \nabla_k B[f]_j^i$ .

Suppose  $s \in C^2(\mathbb{S}^n)$  with  $B[s] > 0$ . Let  $f_2 = \cdots = f_n = s$  and denote the corresponding  $Q[\cdot] = Q[\cdot, s, \dots, s]$  by  $L_s[\cdot]$ . Note that we have

$$\begin{aligned} (2.25) \quad L_s[u] &= Q[u] = \frac{1}{n} \frac{d}{dt} \Big|_{t=0} Q[s + tu, s + tu, \dots, s + tu] \\ &= \frac{1}{n} \frac{d}{dt} \Big|_{t=0} \det B[s + tu]. \\ &= Q[u, s, \dots, s] \\ &= Q^{ij} (\nabla_{i,j}^2 u + g_{ij} u), \end{aligned}$$

where  $[Q^{ij}]$  depends on  $s$  (set  $f_2 = \cdots = f_n = s$  in [Lemma 2.13](#)-(2)). By [Lemma 2.13](#),

$$(2.26) \quad \int u L_s[v] = \int v L_s[u]$$

for all  $C^2$ -smooth functions  $u, v$ . That is,  $L_s$  is self-adjoint. Moreover, by approximation,  $L_s$  is self-adjoint when  $B[s]$  is non-negative definite.

*Remark 2.15.* In view of (2.22), we have

$$L_s[u] = \frac{1}{n} c^{ij} (\nabla_{i,j}^2 u + g_{ij} u),$$

where  $[c^{ij}]$  is the adjugate matrix (transpose of the cofactor matrix) of  $B[s]$ .

*Remark 2.16.* From the proof of Lemma 2.13, it is clear that we can define the operator  $Q^{ij}$  even when  $B[f_i]$ ,  $i = 2, \dots, n$ , are not positive definite and the identity  $\sum_j \nabla_j Q^{ij} = 0$  still holds.

**Lemma 2.17.** *Suppose  $u, s \in C^2(\mathbb{S}^n)$  with  $B[s] > 0$ . Then*

$$\int u Q[s, u, s, \dots, s] = \int s Q[u, u, s, \dots, s].$$

*Proof.* By Lemma 2.13 and Remark 2.16, we may write

$$Q[s, u, s, \dots, s] = Q^{ij} (\nabla_{i,j}^2 s + g_{ij} s), \quad Q[u, u, s, \dots, s] = Q^{ij} (\nabla_{i,j}^2 u + g_{ij} u),$$

for some matrix  $[Q^{ij}]$  locally defined in (2.24). Since  $\nabla_j Q^{ij} = 0$ , by integration by parts

$$\int u Q^{ij} (\nabla_{i,j}^2 s + g_{ij} s) = \int s Q^{ij} (\nabla_{i,j}^2 u + g_{ij} u).$$

□

**Lemma 2.18.** *Suppose  $B[s]$  is non-negative definite. Then for any coordinate function  $x_i$ ,*

$$\int x_i \det B[s] = 0.$$

*Proof.* Suppose  $B[s]$  is positive definite. The volume of the convex body  $K$  with support function  $s$  is given by

$$V[s] = \frac{1}{n+1} \int s \det B[s] = \frac{1}{n+1} \int s Q[s, \dots, s].$$

Note that the support function of  $K + t e_i$ , where  $t \in \mathbb{R}$ , is given by  $s(\cdot) + t \langle \cdot, e_i \rangle$ . Hence,  $V[s + t \langle \cdot, e_i \rangle] = V[s]$ . For a smooth, strictly convex body, by Lemma 2.13, we have

$$0 = \frac{d}{dt} \Big|_{t=0} V[s + t \langle \cdot, e_i \rangle] = \int \langle \cdot, e_i \rangle \det B[s].$$

The general case follows by approximation. □

**Lemma 2.19.** *Let  $u \in C^2(\mathbb{S}^n)$  satisfy  $L_s[u] = 0$ , where  $s > 0$ ,  $B[s] > 0$ . Then for some vector  $(a_1, \dots, a_{n+1})$ , we have  $u(x) = \sum_{i=1}^{n+1} a_i x_i$ .*

*Proof.* By Lemma 2.17, we have

$$(2.27) \quad \int s Q[u, u, s, \dots, s] = \int u L_s[u] = 0.$$

Let  $x_0 \in \mathbb{S}^n$ . Take a local orthonormal frame on  $\mathbb{S}^n$  that diagonalizes  $B[s]$  at  $x_0$  and put

$$a_{ij} = \frac{B[u]_j^i}{\sqrt{B[s]_i^i B[s]_j^j}}.$$

By definition of  $Q$ , at  $x_0$ , we have

$$L_s[u] = \frac{1}{n!} \sum_i \sum_{\sigma \in S_n, \sigma(1)=i} B[s]_{\sigma(2)}^{\sigma(2)} \cdots B[s]_{\sigma(n)}^{\sigma(n)} B[u]_i^i = \det B[s] \sum_i a_{ii} = 0$$

and

$$\begin{aligned} Q[u, u, s, \dots, s] &= \frac{1}{n!} \sum_{i,j} \sum_{\sigma \in S_n, \sigma(1)=i, \sigma(2)=j} B[s]_{\sigma(3)}^{\sigma(3)} \cdots B[s]_{\sigma(n)}^{\sigma(n)} (B[u]_i^i B[u]_j^j - B[u]_j^i B[u]_i^j) \\ &= \det B[s] \left( \sum_{i,j} (a_{ii} a_{jj} - a_{ij}^2) \right). \end{aligned}$$

Now note that

$$\sum_{i,j} (a_{ii} a_{jj} - a_{ij}^2) = \frac{1}{2} \left( \left( \sum_i a_{ii} \right)^2 - \sum_i a_{ii}^2 \right) - \sum_{i,j} a_{ij}^2 \leq 0.$$

Thus, in view of (2.27), we get  $a_{ij} = 0$  for all  $i, j$ . Equivalently, we obtain

$$B[u]_j^i \equiv 0, \quad \forall i, j.$$

By [Exercise 1.18](#),  $D^2 \bar{u} \equiv 0$ , where  $\bar{u}(x) = |x|u(x/|x|)$ ,  $\forall x \in \mathbb{R}^{n+1} \setminus \{0\}$ . Hence, there exists some vector  $\vec{a}$  such that  $D\bar{u}(x) = \vec{a}$  for all  $x \in \mathbb{S}^n$ . That is,  $u \in \text{Span}\{x_1, \dots, x_{n+1}\}$ .  $\square$

We define the banach spaces  $B_1 = \{f \in C^{k+2,\alpha}(\mathbb{S}^n) : \int x_i f = 0 \text{ for } i = 1, \dots, n+1\}$  and  $B_2 = \{f \in C^{k,\alpha}(\mathbb{S}^n) : \int x_i f = 0 \text{ for } i = 1, \dots, n+1\}$ . Now consider the transformation

$$\begin{aligned} F : B_1 &\rightarrow B_2, \\ F(u) &= \det B[u]. \end{aligned}$$

We claim that  $F$  is locally invertible around any  $0 < s \in B_1$  with  $B[s] > 0$ . That is, if  $s \in B_1$  is the support function of some smooth, strictly convex hypersurface (hence  $s > 0$ ), then for any  $f$  in a small neighbourhood of  $\det B[s]$  (in the topology of  $B_2$ ), we can find  $u \in B_1$  such that  $F(u) = f$ .

To prove the above claim, in view of the inverse function theorem on Banach spaces, it is sufficient to show that the linearized operator of  $F$  at  $s$ ,  $L_s : B_1 \rightarrow B_2$ , is bounded, one-to-one and onto (i.e. an isomorphism). By the previous lemma  $L_s[u] = 0$  with  $u \in B_1$  implies that  $u = 0$ . Moreover, by the elliptic theory, for any  $\varphi \in B_2$ , we can find  $u \in B_1$  such that  $L_s[u] = \varphi$ . Therefore,  $L_s$  is an isomorphism.

Putting all our previous arguments together, we have shown  $S_\alpha = [0, 1]$  for  $k \geq 3$ . That is, for any  $f \in C^{k,\alpha}(\mathbb{S}^n)$  with  $k \geq 3$  and  $\int x_i f = 0$  for all  $i$ , there exists a solution to

$$\det B[s] = f$$

with  $s \in C^{k+2,\alpha}(\mathbb{S}^n)$ . In view of [Exercise 1.24](#),  $s$  is the support function of a convex body of class  $C_+^{k+2}$ .

*Remark 2.20.* Suppose  $0 < f \in C^{1,1}(\mathbb{S}^n)$  satisfies [\(2.2\)](#). In this case, we may approximate  $f$  in  $C^{1,1}$ -norm with positive functions  $f_l \in C^{k,\alpha}(\mathbb{S}^n)$  for  $k$  sufficiently large. To ensure the necessary sufficient condition is satisfied we replace  $f_l$  by  $\bar{f}_l$  given by

$$\bar{f}_l = f_l - \sum_{i=1}^n \left( \int x_i f_l \right) \left( \int x_i^2 \right)^{-1} x_i + 2\varepsilon_l,$$

where

$$-\varepsilon_l = \min \left\{ 0, \inf_{\mathbb{S}^n} \left( f_l - \sum_{i=1}^n \left( \int x_i f_l \right) \left( \int x_i^2 \right)^{-1} x_i \right) \right\}.$$

Now  $\det B[s_l] = \bar{f}_l$  have solutions in  $C^{k+2,\alpha}$ . By [Proposition 2.21](#), a subsequence of  $\{s_l\}_l$  converges in  $C^{2,\beta}$  ( $\forall \beta \in (0, 1)$ ) to a solution of

$$\det B[s] = f.$$

It can be shown that the solution  $s$  is, in fact, of class  $C^{3,\beta}$ , for all  $\beta \in (0, 1)$ ; cf. [\[Fig17, Thm. A.42, Prop. A.43\]](#).

*Proposition 2.21.* [\[GM03\]](#) For each integer  $\ell \geq 1$  and  $0 < \beta < 1$ , there exists a constant  $C$  depending only on  $\ell, \beta, \min f$  and  $\|f\|_{C^{\ell,1}}$  such that

$$\|s\|_{C^{\ell+1,\beta}} \leq C$$

for all strictly convex solutions of [\(2.1\)](#) satisfying the condition [\(2.2\)](#).

**2.2. Uniqueness.** Let  $s_i \in C^2(\mathbb{S}^n)$  with  $\det B[s_i] > 0$ . By the Minkowski inequality [\[Sch14, Thm. 7.2.1\]](#),

$$(2.28) \quad \left( \int s_1 \det B[s_2] \right)^{n+1} \geq \int s_1 \det B[s_1] \left( \int s_2 \det B[s_2] \right)^n,$$

and equality holds if and only if for some vector  $\vec{a}$  and constant  $c > 0$ , we have

$$s_2(x) - cs_1(x) = \langle x, \vec{a} \rangle, \quad \forall x \in \mathbb{S}^n.$$

Now suppose  $\det B[s_i] = f$ , for  $i = 1, 2$ . Then

$$\left( \int s_1 f \right)^{n+1} \geq \int s_1 f \left( \int s_2 f \right)^n \Rightarrow \int s_1 f \geq \int s_2 f.$$

Similarly, we have

$$\int s_2 f \geq \int s_1 f.$$

Hence equality holds in [\(2.28\)](#), for some vector  $\vec{a}$ , we have

$$s_2(x) - s_1(x) = \langle x, \vec{a} \rangle.$$

3.  $L_p$ -MINKOWSKI PROBLEM

We say  $f \in C^\infty(\mathbb{S}^n)$  is even if  $f(x) = f(-x)$  for all  $x \in \mathbb{S}^n$ . In this section, we prove the following theorem due to Lutwak-Oliker [LO95].

**Theorem 3.1** (even  $L_p$ -Minkowski problem). *Suppose  $1 < p \neq n + 1$ . Let  $f \in C^\infty(\mathbb{S}^n)$  be an even positive function. Then there exists an even solution  $0 < s \in C^\infty(\mathbb{S}^n)$  to*

$$(3.1) \quad \det B[s] = s^{p-1}f.$$

*That is, we can find an origin-symmetric, strictly convex hypersurface in  $\mathbb{R}^{n+1}$  with the support function  $s$  and Gauss curvature  $f^{-1}s^{1-p}$ .*

**3.1. Existence of solutions.** Let  $f_t := 1 - t + tf$  and  $S_\alpha = \{t \in [0, 1] : \text{the equation } \det B[s] = s^{p-1}f_t \text{ has a } C^{k+2,\alpha} \text{ solution } s_t > 0 \text{ such that } B[s_t] > 0 \text{ and } s_t \text{ is even}\}$ . For  $k \geq 3$ , using the method of continuity we show that  $S_\alpha = [0, 1]$ . Note that  $0 \in S_\alpha$ , hence  $S_\alpha$  is not empty. Recall from the previous section (case  $p = 1$ ) that the continuity method consists of two steps. To show closedness, we must prove for any sequence  $t_k \in S_\alpha$  such that  $t_k \rightarrow t_0$ , we have  $t_0 \in S_\alpha$ . To show openness, we need to show for any  $t \in S_\alpha \setminus \{1\}$  there is  $\varepsilon > 0$  such that  $(t - \varepsilon, t + \varepsilon) \in S_\alpha$ .

**3.1.1. Closedness.** Let  $K$  be a smooth, origin-symmetric, strictly convex body with support function  $s$ , and put  $\mathcal{M} = \partial K$ .

**Lemma 3.2.** *Let  $1 < p \neq n + 1$ . Suppose  $\det B[s] = s^{p-1}f$ . Then  $c \leq s \leq C$  with  $c, C$  depending only on  $f$ . Moreover, we have  $|\nabla s| \leq C$ .*

*Proof.* Case  $1 < p < n + 1$ : The volume of  $K$  is given by

$$V = V[s] = \frac{1}{n+1} \int s \det B[s] = \frac{1}{n+1} \int s^p f.$$

Hence we have

$$V \geq c_1 \int s^p$$

where  $c_1 := \frac{1}{n+1} \min f$ . Moreover, by the Alexandrov-Fenchel inequality,

$$\int s \geq c_2 V^{\frac{1}{n+1}} \Rightarrow \int s^p \geq c_3 V^{\frac{p}{n+1}},$$

where we used the Hölder inequality to get the r.h.s. Hence we obtain

$$V \geq c_4 V^{\frac{p}{n+1}} \Rightarrow V^{\frac{(n+1)-p}{n+1}} \geq c.$$

That is, the volume has a lower bound depending only on  $f$ .

Put  $Q[\cdot] = Q[\cdot, s, \dots, s]$ . By Lemma 2.13-(3),

$$\sigma_n = \det B[s] = Q^{ij}(\nabla_{i,j}^2 s + s g_{ij}) \Rightarrow Q^{ij} \nabla_{i,j}^2 s = \sigma_n - Q^{ij} g_{ij} s,$$

where  $Q^{ij}$  is the corresponding elliptic operator introduced in Lemma 2.13. Multiplying both sides by  $s^{1-p}$ , and using  $s^{1-p} \sigma_n = f$  we get

$$s^{1-p} Q^{ij} \nabla_{i,j}^2 s = f - Q^{ij} g_{ij} s^{2-p}.$$

Recall that by [Remark 2.15](#) in an orthonormal frame that diagonalizes  $B[s]$  (at some point),

$$Q^{ii} = \frac{1}{n} \frac{\det B[s]}{B[s]_i^i},$$

Hence  $Q^{ij}g_{ij} \geq \sigma_n^{\frac{n-1}{n}}$ . Now using  $\nabla_i Q^{ij} = 0$  (cf. [Lemma 2.13](#)) we obtain

$$\int s^{\frac{n+1-p}{n}} f^{\frac{n-1}{n}} = \int \sigma_n^{\frac{n-1}{n}} s^{2-p} \leq \int Q^{ij}g_{ij} s^{2-p} \leq \int f.$$

Let  $p, q \in \mathcal{M}$  such that the line segment joining  $p$  and  $q$  has length  $L$  (the extrinsic diameter of  $\mathcal{M}$ ). Since  $K$  is origin-symmetric, 0 is in the middle of the line segment. Hence  $\pm \frac{L}{2}u \in \mathcal{M}$  for some  $u \in \mathbb{S}^n$ . For any  $x \in \mathbb{S}^n$  we have

$$s(x) = \sup_{p \in K} \langle p, x \rangle \geq \frac{L}{2} |\langle u, x \rangle|.$$

Thus

$$\int f \geq \int s^{\frac{n+1-p}{n}} f^{\frac{n-1}{n}} \geq \left(\frac{L}{2}\right)^{\frac{n+1-p}{n}} \int f^{\frac{n-1}{n}}(x) |\langle u, x \rangle|^{\frac{n+1-p}{n}}.$$

Therefore,  $L$  has an upper bound depending only on  $f$ . Since  $K$  is origin-symmetric, this implies that  $s$  is positive and bounded. Moreover, since the volume is bounded below, the lower bound of  $s$  depends only on  $f$ .

Case  $p > n + 1$ . Let  $x_0$  be a point at which  $s$  attain its maximum. Then

$$s^n(x_0) \geq \det B[s](x_0) = s^{p-1}(x_0) f(x_0).$$

Therefore,  $(\max s)^{n+1-p} \geq \min f$ . Similarly  $(\min s)^{n+1-p} \leq \max f$ .

In either case the bound on the gradient of  $s$  follows as in [Lemma 2.7](#).  $\square$

By the previous lemma,  $|\nabla s_{t_k}|$  is uniformly bounded above in  $k$ . Thus a subsequence of  $\{s_{t_k}\}$  converges to a function  $s_{t_0}$  in  $C^\alpha(\mathbb{S}^n)$ . Then, as in the previous section, we may choose the sets  $P_k$  containing strictly a compact set  $\Theta$  such that<sup>5</sup>

$$\hat{s}_k(x) := Q(x)s_{t_k}(\varphi(x)) - \lambda,$$

satisfies

$$(3.2) \quad \begin{cases} \det[\partial_{ij}^2 \hat{s}_k] = f_{t_k}(\hat{s}_k + \lambda)^{p-1} Q^{-n-1-p} & \text{in } P_k, \\ \hat{s}_k = 0 & \text{on } \partial P_k. \end{cases}$$

Moreover, the r.h.s. of (3.2) is bounded away from 0 uniformly in  $k$ .

Applying [Theorem 2.8](#) and [Theorem 2.12](#), we deduce a uniform bound on  $C^3$ -norm of  $\{\hat{s}_k\}$  in  $\Theta$ . Thus  $\{\hat{s}_k\}$  converges in  $\Theta$  in  $C^{2,\alpha}$ -norm (for any  $\alpha \in (0, 1)$ ) to  $\hat{s}_0$ . Hence  $\hat{s}_0$  is a  $C^{2,\alpha}$  solution of

$$\det[\partial_{ij}^2 \hat{s}_0] = f_{t_0}(\hat{s}_0 + \lambda)^{p-1} Q^{-n-1-p} \quad \text{in } \Theta.$$

It can be shown that, in fact,  $\hat{s}_0 \in C^{k+2,\alpha}(\Theta)$ . Therefore,  $s_0$  is a  $C^{k+2,\alpha}$  solution of

$$\det B[s_0] = s_0^{p-1} f_{t_0} \quad \text{in } \mathbb{S}^n.$$

---

<sup>5</sup>Recall that  $\varphi(x) := (x, 1)/Q(x)$ ,  $Q(x) := \sqrt{1 + |x|^2}$ .

Hence  $t_0 \in S_\alpha$ .

3.1.2. *Openness.* Define

$$F(u) = u^{1-p} \det B[u].$$

To show that  $S_\alpha$  is open, we prove that  $F$  is locally invertible around any  $s \in B_1$  with  $B[s] > 0$ . Note that

$$\mathcal{L}_s[u] := \frac{d}{dt}\bigg|_{t=0} F(s + tu) = (1-p)s^{-p}u \det B[s] + ns^{1-p}L_s[u],$$

where  $L_s$  was defined in (2.25). Suppose now  $\mathcal{L}_s[u] = 0$  for some  $u \in B_1$ . Then  $s^p \mathcal{L}_s[u] = 0$ . That is,

$$(1-p)u \det B[s] + nsL_s[u] = 0.$$

Integrating this over  $\mathbb{S}^n$  and using that  $L_s$  is self-adjoint we obtain

$$\begin{aligned} \int (1-p)u \det B[s] + nuL_s[s] &= (n+1-p) \int u \det B[s] = 0 \\ (3.3) \quad &\Rightarrow \int u \det B[s] = 0. \end{aligned}$$

On the other hand, by a theorem of Hilbert-Alexandrov (cf. [RK96, p. 138]) or the local Aleksandrov-Fenchel inequality<sup>6</sup>, (3.3) implies that

$$\int uL_s[u] \leq 0,$$

and equality holds if  $u(x) = \langle \vec{a}, x \rangle$  for some vector  $\vec{a}$ . Since  $s^{p-1}u\mathcal{L}_s[u] = 0$ , we have

$$(p-1) \int s^{-1}u^2 \det B[s] = n \int uL_s[u].$$

Hence, we arrive at

$$\int u^2 s^{-1} \det B[s] \leq 0.$$

That is, the kernel of  $\mathcal{L}_s$  is trivial. Therefore by the inverse function theorem there is  $\delta > 0$  such that for all functions  $\|\tilde{f} - F(s)\|_{C^{k,\alpha}} < \delta$ , the equation  $F(\cdot) = \tilde{f}$  has a solution that is even and  $C^{k+2,\alpha}$ . In particular, for any  $t \in S_\alpha$ , the equation  $F(\cdot) = f_t$  for  $\tilde{t}$  sufficiently close to  $t$  has a solution such that  $s_{\tilde{t}}$  and  $\det B[s_{\tilde{t}}] > 0$ . Therefore,  $\tilde{t} \in S_\alpha$ .

---

<sup>6</sup>For  $s \in C^2(\mathbb{S}^n)$  with  $B[s] > 0$  and any  $u \in C^2(\mathbb{S}^n)$  we have :  $(\int sL_s[u])^2 \geq \int uL_s[u] \int sL_s[s]$ . See [Sch14, Thm. 7.6.8].



**3.2. Uniqueness.** Suppose  $0 < s_i \in C^2(\mathbb{S}^n)$  with  $\det B[s_i] > 0$ . Now, by the  $L_p$ -Minkowski inequality [Lut93] for  $p > 1$ ,

$$(3.4) \quad \left( \int s_1^p s_2^{1-p} \det B[s_2] \right)^{n+1} \geq \left( \int s_1 \det B[s_1] \right)^p \left( \int s_2 \det B[s_2] \right)^{n+1-p},$$

and equality holds if and only if for some constant  $c > 0$ , we have

$$s_2 = cs_1.$$

Suppose and  $1 < p < n + 1$  and  $\det B[s_i] = s_i^{p-1} f$ , for  $i = 1, 2$ . Then

$$\left( \int s_1^p f \right)^{n+1} \geq \left( \int s_1^p f \right)^p \left( \int s_2^p f \right)^{n+1-p} \Rightarrow \int s_1^p f \geq \int s_2^p f.$$

Similarly, we have

$$\int s_2^p f \geq \int s_1^p f.$$

Hence, equality holds in (3.4) and we have

$$s_1 = s_2.$$

The argument for the case  $p > n + 1$  is similar.

#### 4. CHRISTOFFEL-MINKOWSKI PROBLEM

The main reference for this section is Guan-Ma [GM03]. The regular Christoffel-Minkowski problem asks the following question: Given a positive smooth function  $f$  on the unit sphere, what are the necessary and sufficient conditions on  $f$  that ensure the existence of a smooth, closed, strictly convex hypersurface whose  $\sigma_k$ (radii of curvature), as a function of the outer unit normal, is  $f$ ? The problem is equivalent to finding a smooth function  $s : \mathbb{S}^n \rightarrow [0, \infty)$  with  $B[s] > 0$  such that

$$(4.1) \quad \sigma_k(B[s]) = f.$$

A necessary condition for the existence of a strictly convex solution (hypersurface) is

$$(4.2) \quad \int x_i f(x) = 0$$

for all coordinate functions  $x_i$ . However, this is no longer a sufficient condition when  $k < n$ , and we need to impose another condition on  $f$  to guarantee the existence of solutions.

**Theorem 4.1** (Christoffel-Minkowski problem). [GM03] *Let  $1 \leq k < n$  and  $f \in C^\infty(\mathbb{S}^n)$  be a positive function. Suppose  $\int x_i f(x) = 0$  for all coordinate functions  $x_i$  and  $B[f^{-\frac{1}{k}}]$  is non-negative definite. Then, we can find a closed, smooth, strictly convex hypersurface in  $\mathbb{R}^{n+1}$  whose support function  $s$  satisfies (4.1). Moreover, any two such hypersurface must coincide after a translation.*

**4.1. Continuity method.** Let  $f_t$ ,  $t \in [0, 1]$ , be a suitable path that  $f_0 = 1$  and  $f_1 = f$ . Let  $S_\alpha = \{t \in [0, 1] : \text{the equation } \sigma_k(B[s]) = f_t \text{ has a } C^{\ell+2, \alpha} \text{ solution } s_t \text{ such that } B[s_t] > 0 \text{ and } \int s_t x_i = 0 \text{ for all } i\}$ . We want to show that for large  $\ell$ , this set is both open and closed in  $[0, 1]$  and hence  $S_\alpha = [0, 1]$ . Our suitable path is defined so that

$$(4.3) \quad B[f_t^{-\frac{1}{k}}] = \nabla^2 f_t^{-\frac{1}{k}} + g f_t^{-\frac{1}{k}} > 0, \quad t \in [0, 1)$$

and  $\int f_t x_i = 0$  for all  $i$ . The existence of such a path is proved in [BIS23b].

This particular choice of path allows us to find a strictly convex solution via the full rank theorem [Theorem 4.22](#).

**Lemma 4.2.** [STW04] *Suppose  $1 \leq k < n$  and  $0 < f \in C^\infty(\mathbb{S}^n)$  satisfies*

$$\nabla^2 f^{-\frac{1}{k}} + f^{-\frac{1}{k}} g \geq 0, \quad \int x f(x) = 0.$$

*Then for each  $t \in [0, 1)$ , there exists  $z_t \in \mathbb{R}^{n+1}$ , such that*

$$f_t(x) = (1 - t + t f^{-\frac{1}{k}}(x) - \langle x, z_t \rangle)^{-k}$$

*satisfies*

$$\nabla^2 f_t^{-\frac{1}{k}} + f_t^{-\frac{1}{k}} \bar{g} > 0, \quad \int x f_t(x) = 0.$$

*Moreover, we have*

$$(4.4) \quad |z_t| \leq 1 + \max f^{-\frac{1}{k}}.$$

*Proof.* The proof is from [BIS23b]. Note that for  $0 \leq t \leq 1$ ,  $s_{L_t} := 1 - t + t f^{-\frac{1}{k}}$  is the support function of a convex body  $L_t$ , which is smooth and strictly convex for  $t < 1$ . Similar to the proof of [Lemma 2.4](#), we can show that there exists a point  $z_t$  in the interior of  $L_t$  such that

$$\begin{cases} \min_{v \in L_t} \int -\log(s_{L_t-v}), & k = 1 \\ \min_{v \in L_t} \int s_{L_t-v}^{-k+1}, & k > 1 \end{cases}$$

is attained. Hence, the support function of  $L_t - z_t$  given by

$$1 - t + t f^{-\frac{1}{k}}(x) - \langle x, z_t \rangle$$

is positive and satisfies the required integral condition. In fact, for  $k > 1$  and any  $w \in \mathbb{S}^n$ ,

$$\begin{aligned} 0 &= \frac{d}{dr} \Big|_{r=0} \int s_{L_t-z_t-rw}^{-k+1} = \frac{d}{dr} \Big|_{r=0} \int \frac{1}{(s_{L_t-z_t}(x) - r \langle x, w \rangle)^{k-1}} dx \\ &= (k-1) \int \frac{\langle x, w \rangle}{s_{L_t-z_t}^k(x)} dx \\ &= (k-1) \int \frac{\langle x, w \rangle}{(1 - t + t f^{-\frac{1}{k}}(x) - \langle x, z_t \rangle)^k} dx. \end{aligned}$$

That is,  $f_t(x) := (1 - t + tf^{-\frac{1}{k}}(x) - \langle x, z_t \rangle)^{-k}$  satisfies

$$\int x f_t(x) = 0.$$

Similarly, for  $k = 1$ ,  $f_t$  satisfies the above integral condition. Since  $z_t$  is in the interior of  $L_t$ , the upper bound on the norm of  $z_t$  follows.  $\square$

Before delving into the  $C^2$  estimate, let us explain the necessary condition (4.2) that  $f$  has to satisfy.

**Lemma 4.3.** *Let  $s \in C^2(\mathbb{S}^n)$ . Suppose  $\sigma_k(B[s]) = f$ . Then*

$$\int x f(x) = 0.$$

*Proof.* For some constant  $c_k > 0$ , we have  $\sigma_k(B[s]) = c_k Q[s, \dots, s, 1, \dots, 1]$  where  $s$  appears  $k$  times. Let  $[Q^{ij}]$  denote the corresponding operator in Lemma 2.13. Due to Lemma 2.13,

$$\begin{aligned} c_k \int x_i f(x) &= \int x_i Q[s, \dots, s, 1, \dots, 1] \\ &= \int x_i Q^{ij} (\nabla_{i,j}^2 s + g_{ij} s) \\ &= \int s Q^{ij} (\nabla^2 x_i + g x_i). \end{aligned}$$

Since  $\nabla^2 x_i + g x_i = \text{Hess}_{\mathbb{R}^{n+1}} x_i = 0$ , the claim follows.  $\square$

**4.2. Closedness.** To show the closedness part of the continuity argument, we need some introduction to curvature functions. We follow the exposition in [ACGL20].

#### 4.2.1. Curvature functions.

**Definition 4.4.** A function  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $S_n$ -invariant (or symmetric) if

$$q(z_1, \dots, z_n) = q(z_{\sigma(1)}, \dots, z_{\sigma(n)}), \quad \forall \sigma \in S_n,$$

where  $S_n$  is the group of permutations of the set  $\{1, \dots, n\}$ .

Let  $S(n)$  denote the set of symmetric  $n \times n$  matrices. Then a function  $q : S(n) \rightarrow \mathbb{R}$  is said to be  $SO(n)$ -invariant (or symmetric) if  $q(P) = q(Q^{-1}PQ)$  for all  $Q \in SO(n)$ . Here  $SO(n)$  is the special orthogonal group of degree  $n$ .

*Remark 4.5.* Let  $\lambda$  denote the eigenvalue map. This is a multivalued map that assigns to a symmetric matrix  $P$  the set of  $n$ -tuples  $\lambda(P)$  with components given by its eigenvalues. Now, for any  $S_n$ -invariant function  $q$ , there is a corresponding  $SO(n)$ -invariant function  $\hat{q} : S(n) \rightarrow \mathbb{R}$  defined by  $\hat{q}(P) = q(z_1, \dots, z_n)$  for any  $(z_1, \dots, z_n) \in \lambda(P)$ . Since  $q$  is symmetric,  $\hat{q}$  is well-defined. Now suppose  $\hat{q}$  is  $SO(n)$ -invariant. Define  $q(z_1, \dots, z_n) = \hat{q}(P)$ , where  $P \in \lambda^{-1}([(z_1, \dots, z_n)])$  is the orbit of  $(z_1, \dots, z_n)$  under the  $S_n$  action. Since  $\hat{q}$  is symmetric, it takes the same value on any two symmetric matrices with equal eigenvalues. Hence,  $q$  is well-defined. In summary, every  $S_n$ -invariant function gives rise to a canonical

$SO(n)$ -invariant function and vice versa. Hence, from now on, we may use the letter  $q$  for both a function of matrix variables and as a function of eigenvalue variables.

**Example 4.6.**  $q(P) = \det P$  corresponds to  $q(z_1, \dots, z_n) = z_1 \cdots z_n$  and the map  $q$  defined by  $q(P) = \frac{1}{k!} \frac{d^k}{dt^k} \Big|_{t=0} \det(tP + I)$  corresponds to  $q(z) = \sum_{i_1 < \dots < i_k} z_{i_1} \cdots z_{i_k}$ .

Let  $q$  be a symmetric function. Define

$$\dot{q}_z^i v_i = \frac{d}{dt} \Big|_{t=0} q(z + tv), \quad \ddot{q}_z^{ij} v_i v_j = \frac{d^2}{dt^2} \Big|_{t=0} q(z + tv), \quad \forall z, v \in \mathbb{R}^n$$

and

$$\dot{q}_Z^{ij} V_{ij} = \frac{d}{dt} \Big|_{t=0} q(Z + tV), \quad \ddot{q}_Z^{kl,rs} V_{kl} V_{rs} = \frac{d^2}{dt^2} \Big|_{t=0} q(Z + tV), \quad \forall Z, V \in S(n).$$

**Theorem 4.7.** *Let  $q$  be a symmetric function. Then  $q$  is smooth with respect to the matrix variables if and only if it is smooth with respect to the eigenvalues variables. Moreover, the first and the second derivatives are related by the following formulae: For any diagonal matrix  $Z$  in the matrix domain of  $q$  with the ordered  $n$ -tuple of real eigenvalues  $z^7$ ,*

$$\dot{q}_Z^{kl} = \begin{cases} \dot{q}_z^k, & k = l, \\ 0, & k \neq l, \end{cases}$$

and if the eigenvalues are all distinct, then

$$(4.5) \quad \ddot{q}_Z^{kl,rs} V_{pq} V_{rs} = \sum_{k,l} \ddot{q}_z^{kl} V_{kk} V_{ll} + \sum_{k \neq l} \frac{\dot{q}_z^k - \dot{q}_z^l}{z_k - z_l} (V_{kl})^2,$$

for any  $V \in S(n)$ .

**Definition 4.8.** We say  $z \in \mathbb{R}^n$  is *simple* if its components are pairwise distinct.

**Lemma 4.9** (Concavity). *Let  $\Omega \subset \mathbb{R}^n$  be an open convex set. Suppose  $q : \Omega \rightarrow \mathbb{R}$  is smooth and symmetric. If  $q$  is concave, then for every simple  $z \in \Omega$ ,*

$$\frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} \leq 0$$

for each pair  $(i, j)$  with  $i \neq j$ .

*Proof.* Suppose  $q$  is concave and  $z \in \Omega$ . Take  $v \in \mathbb{R}^n$  and assume for some  $t_0 > 0$  and any  $t \in [0, t_0]$ ,  $z + tv \in \Omega$ . Then

$$\frac{d}{dt} \dot{q}^i(z + tv) v_i = \frac{d^2}{dt^2} q(z + tv) \leq 0 \quad \forall t \in [0, t_0].$$

Hence,

$$\dot{q}^k(z + tv) v_k \leq \dot{q}^k(z) v_k \quad \forall t \in [0, t_0].$$

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<sup>7</sup>To be more precise, let  $w_1, \dots, w_n$  be an orthonormal basis of eigenvectors for  $Z$  and  $z = (z_1, \dots, z_n)$  where  $z_i$  is the corresponding eigenvalue for the eigenvector  $e_i$

Set  $v = -(e_i - e_j)$ , where  $e_i$  is the basis vector in the direction of the  $i$ -th coordinate. For  $t_0$  sufficiently small we have

$$(4.6) \quad \dot{q}_z^i - \dot{q}_z^j \leq (\dot{q}^i - \dot{q}^j)|_{z-t(e_i-e_j)} \quad \forall t \in [0, t_0].$$

Suppose  $z_i > z_j$  and  $t_0 := \frac{z_i - z_j}{2} > 0$ . Then

$$(z - t_0(e_i - e_j))_i = (z - t_0(e_i - e_j))_j.$$

Let

$$\begin{aligned} w &:= (z_1, \dots, z_i - t_0, \dots, z_j + t_0, \dots, z_n) \\ &= (z_1, \dots, z_j + t_0, \dots, z_i - t_0, \dots, z_n) \\ &= z - t_0(e_i - e_j). \end{aligned}$$

Note that  $w$  is the midpoint of the line segment joining  $z$  and the point obtained from  $z$  by switching its  $i$ -th and  $j$ -th coordinates; hence,  $z - t(e_i - e_j) \in \Omega$  for all  $t \in [0, t_0]$ . Since  $q$  is symmetric,

$$(4.7) \quad \begin{aligned} q(z_1, \dots, z_i - (t_0 + r), \dots, z_j + t_0 + r, \dots, z_n) \\ = q(z_1, \dots, z_j + t_0 + r, \dots, z_i - (t_0 + r), \dots, z_n). \end{aligned}$$

for all  $r$  sufficiently small. Taking derivative of both sides of (4.7) with respect to  $r$  yields

$$-\dot{q}_w^i + \dot{q}_w^j = \dot{q}_w^i - \dot{q}_w^j.$$

That is,  $\dot{q}^i = \dot{q}^j$  at  $w$ . Hence, applying (4.6) at  $t_0$ , the claim follows.  $\square$

**Theorem 4.10.** *Let  $\Omega \subset \mathbb{R}^n$  be an open convex set. A smooth symmetric function  $q : \Omega \rightarrow \mathbb{R}$  is concave with respect to the eigenvalue variables if and only if it is concave with respect to the matrix variables.*

*Proof.* Due to identity (4.5), for any symmetric matrix  $V$  we have

$$\ddot{q}_Z^{ij,kl} V_{ij} V_{kl} = \ddot{q}_z^{ij} V_{ii} V_{jj} + 2 \sum_{i>j} \frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} (V_{ij})^2$$

at any diagonal matrix  $Z$  with simple eigenvalues  $z = (z_1, \dots, z_n)$ . Hence, the concavity of  $q$  at  $Z$  with respect to the matrix component implies the concavity of  $q$  at  $z$  with respect to the eigenvalues. The converse follows from the previous lemma.

To see if the claim holds at any diagonal matrix  $Z$ , observe that this is the limiting case along a sequence  $Z^{(k)}$  of diagonal matrices with distinct eigenvalues, which limits to  $Z$ . The general case follows from the invariance of  $q$  with respect to similarity transformations.  $\square$

*Remark 4.11.* Let  $s$  be the support function of a smooth, strictly convex body  $K$  and  $f = q(\lambda)$ , where as usual  $\lambda = (\lambda_1, \dots, \lambda_n)$  are the principal radii of curvature  $K$  (the eigenvalues of  $B[s]$ ), and  $q$  is a smooth symmetric function. By the previous remark, we may consider  $q$  as a function of matrix variables and write  $f = q(B[s])$ .

Choose an orthonormal frame of  $T\mathbb{S}^n$ , such that diagonalizes  $\alpha_{ij} = B[s]_i^j$  at a given point. Due to [Theorem 4.7](#), assuming the eigenvalues are distinct:

$$\begin{aligned}\nabla_i f &= \dot{q}_\alpha^{kl} \nabla_i \alpha_{kl}, \\ \nabla_{i,i}^2 f &= \dot{q}_\alpha^{kl} \nabla_{i,j}^2 \alpha_{kl} + \ddot{q}_\alpha^{kl,rs} \nabla_i \alpha_{kl} \nabla_i \alpha_{rs} \\ &= \dot{q}_\lambda^k \nabla_{i,i}^2 \alpha_{kk} + \sum_{p,q} \ddot{q}_\lambda^{kl} \nabla_i \alpha_{kk} \nabla_i \alpha_{ll} + \sum_{k \neq l} \frac{\dot{q}_\lambda^k - \dot{q}_\lambda^l}{\lambda_k - \lambda_l} (\nabla_i \alpha_{kl})^2.\end{aligned}$$

**4.2.2. Hyperbolic polynomials.** We write  $\mathbb{R}[x_1, \dots, x_m]$  for the set of polynomials in  $x_1, \dots, x_m$  with real coefficients. We say a homogeneous polynomial  $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $\xi \in \mathbb{R}^n$  if and only if

$$(4.8) \quad \begin{aligned} &\mathcal{P}(\xi) > 0, \quad \text{and} \\ &\forall x \in \mathbb{R}^n \quad \mathcal{P}(x + t\xi) \in \mathbb{R}[t] \quad \text{is real rooted.} \end{aligned}$$

Define the cone

$$(4.9) \quad \mathcal{C}(\mathcal{P}, \xi) = \{x \in \mathbb{R}^n : \mathcal{P}(x - t\xi) \in \mathbb{R}[t] \text{ has positive roots}\}.$$

Now we collect a few standard facts about the hyperbolic polynomials.

**Lemma 4.12.** *Assume  $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $\xi \in \mathbb{R}^n$ . Then  $D_\xi \mathcal{P}$  is hyperbolic in direction  $\xi$  as well, unless  $D_\xi \mathcal{P} = 0$ .*

**Lemma 4.13.** [\[Nui68\]](#) *Assume  $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $\xi \in \mathbb{R}^n$ . Then for any  $v \in \mathcal{C}(p, \xi)$ , the polynomial*

$$(4.10) \quad \mathcal{H}(x_1, \dots, x_n, x_{n+1}) = \mathcal{P}(x) - x_{n+1} D_v \mathcal{P}(x) \in \mathbb{R}[x_1, \dots, x_n, x_{n+1}]$$

*is hyperbolic in direction  $(v, 0)$ .*

**Lemma 4.14.** *If  $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $\xi \in \mathbb{R}^n$  and  $V \subset \mathbb{R}^n$  is a linear subspace containing  $\xi$ , then the restriction  $\mathcal{P} : V \rightarrow \mathbb{R}$  is also hyperbolic in direction  $\xi$ .*

**Theorem 4.15.** [\[Gül, Thm. 6.1\]](#), [\[BGLS01, Cor. 4.7\]](#) *Let  $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$  be hyperbolic in direction  $\xi \in \mathbb{R}^n$ . For  $v \in \mathcal{C}(p, \xi)$ , define the barrier in direction  $v$  as*

$$(4.11) \quad D_v \log \mathcal{P} = \frac{D_v \mathcal{P}}{\mathcal{P}}.$$

*Then  $\frac{\mathcal{P}}{D_v \mathcal{P}}$  is concave on  $\mathcal{C}(\mathcal{P}, \xi)$ .*

**Example 4.16.** Let  $\Gamma_+^n := \{z \in \mathbb{R}^n : z_i > 0 \text{ for each } i\}$ . Consider  $q = \sigma_k^{\frac{1}{k}} : \Gamma_+^n \rightarrow \mathbb{R}$ , defined by  $q(z) = \left(\sum_{i_1 < \dots < i_k} z_{i_1} \cdots z_{i_k}\right)^{\frac{1}{k}}$ . Then  $q$  is concave.

*Exercise 4.17.* Let  $\alpha = B[s] \geq 0$  and  $q = \sigma_k(B[s])$ . Show that  $\dot{q}_\alpha^{ij} \delta_{ij} \geq c_k q^{\frac{k-1}{k}}$  for some constant  $c_k > 0$ . *Hint:* In an orthonormal frame that diagonalizes  $B[s]$ :

$$\ddot{q}_\alpha^{ii} = \dot{q}_\lambda^i = \frac{\partial \sigma_k}{\partial \lambda_i}$$

and  $\dot{q}_\lambda^{ij} = 0$  for  $i \neq j$ .

*Remark 4.18.* Let  $\alpha = B[s]$  and  $q = \sigma_k(B[s])$ . Let  $u, s \in C(\mathbb{S}^n)$ . Consider  $Q[u, s, \dots, s, 1, \dots, 1]$  where  $s$  appears  $k-1$  times. By [Lemma 2.13](#), for some matrix  $[Q^{ij}]$  depending only on  $s$ :

$$Q[u, s, \dots, s, 1, \dots, 1] = Q_j^i (\nabla^j \nabla_i u + \delta_{ij} u).$$

From the definition of  $[Q_j^i]$  it is clear that in an orthonormal frame that diagonalizes  $B[s]$ :

$$\begin{aligned} Q_i^i &= \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=i}} B[s]_{\sigma(2)}^{\sigma(2)} \cdots B[s]_{\sigma(k-1)}^{\sigma(k-1)} B[1]_{\sigma(k)}^{\sigma(k)} \cdots B[1]_{\sigma(n)}^{\sigma(n)} \\ &= \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=i}} \lambda_{\sigma(2)} \cdots \lambda_{\sigma(k-1)} \\ &= c_{k,n} \frac{\partial \sigma_k}{\partial \lambda_i} \end{aligned}$$

for some constant  $c_{k,n}$  and  $Q_j^i = 0$  for  $i \neq j$ . That is, the two matrices  $[Q_j^i]$  and  $[q_\alpha^{ij}]$  are the same (up to a constant). In particular,  $\nabla_i q_\alpha^{ij} = 0$ .

**Lemma 4.19.** *Suppose  $s > 0$  is a smooth, strictly convex solution of  $\sigma_k(B[s]) = f$ . Then  $\max s$  is bounded, and the bound depends only on  $\min f, \max f$ .*

*Proof.* Let  $\alpha_{ij} = B[s]_i^j$  and  $q = \sigma_k(B[s])$ . Since  $q$  is one-homogeneous, we have  $\dot{q}_\alpha^{ij} \alpha_{ij} = kf$ . Due to  $\dot{q}_\alpha^{ij} \delta_{ij} \geq c_k q^{\frac{k-1}{k}}$  (see [Exercise 4.17](#)),  $\nabla_i q_\alpha^{ij} = 0$ , and using integration by parts,

$$\int s f^{\frac{k-1}{k}} \leq c_1 \int f.$$

We may assume  $s(1, 0, \dots, 0) = \max s$ . By convexity, we have  $s(x) \geq (\max s)x_1$ . Hence,

$$(4.12) \quad \max s \leq \frac{c_2 \int f}{\int_{\{x: x_1 > \frac{1}{2}\}} f^{\frac{k-1}{k}}}.$$

□

*Exercise 4.20.* Let  $\tau = \nabla^2 s + gs$ . Show that

$$g^{jk} \nabla_{i,k}^2 \sigma_1 = \Delta B[s]_i^j - n B[s]_i^j + \sigma_1 \delta_{ij}.$$

*Hint:* Prove

$$\nabla_{m,n}^2 B[s]_i^j = g^{pj} \nabla_{m,n}^2 \tau_{ip} = g^{pj} (\nabla_{i,p}^2 \tau_{mn} - \tau_{mn} g_{ip} + \tau_{ip} g_{mn}).$$

Now taking the trace of the left-hand side with respect to  $m, n$  gives

$$\begin{aligned} \Delta B[s]_i^j &= g^{mn} \nabla_{m,n}^2 B[s]_i^j = g^{mn} g^{pj} (\nabla_{i,p}^2 \tau_{mn} - \tau_{mn} g_{ip} + \tau_{ip} g_{mn}) \\ &= g^{pj} (\nabla_{i,p}^2 \sigma_1 - \sigma_1 g_{ip} + n \tau_{ip}) \\ &= g^{jk} \nabla_{i,k}^2 \sigma_1 - \sigma_1 \delta_{ij} + n B[s]_i^j. \end{aligned}$$

**Lemma 4.21.** *Suppose  $s > 0$  is a smooth, strictly convex solution of  $\sigma_k(B[s]) = f$ . Then for each  $\ell \geq 1$  and  $\gamma \in (0, 1)$ ,  $\|s\|_{C^{\ell+1, \gamma}} \leq C_{\ell, \gamma}$  for some constant depending only on  $f$ .*

*Proof.* We only need to consider the case  $k > 1$ . Define  $q = \sigma_k^{\frac{1}{k}}$ ,  $\alpha_{ij} = B[s]_i^j$  and  $\dot{q}^{ij} = \dot{q}_\alpha^{ij}$ . By [Exercise 4.20](#),

$$g^{jk} \nabla_{i,k}^2 \sigma_1 = \Delta \alpha_{ij} - n \alpha_{ij} + \sigma_1 \delta_{ij}.$$

Hence,

$$\begin{aligned} \sum_{i,j} \dot{q}^{ij} g^{jk} \nabla_{i,k}^2 \sigma_1 &= \dot{q}^{ij} \Delta \alpha_{ij} - n \dot{q}^{ij} \alpha_{ij} + \dot{q}^{ij} \delta_{ij} \sigma_1 \\ &= \dot{q}^{ij} \Delta \alpha_{ij} - n f^{\frac{1}{k}} + \dot{q}^{ij} \delta_{ij} \sigma_1 \\ &= \Delta q - g^{mn} \dot{q}^{ij,kl} \nabla_m \alpha_{ij} \nabla_n \alpha_{kl} - n f^{\frac{1}{k}} + \dot{q}^{ij} \delta_{ij} \sigma_1. \end{aligned}$$

Therefore, in an orthonormal frame that diagonalizes  $\alpha$ , using the concavity of  $q$  (cf. [Theorem 4.10](#)), we obtain

$$\begin{aligned} \dot{q}^{ij} \delta_{ij} \sigma_1 &= \dot{q}^{ij} \nabla_{i,j}^2 \sigma_1 + \ddot{q}^{ij,kl} \nabla_m \alpha_{ij} \nabla_m \alpha_{kl} + n f^{\frac{1}{k}} - \Delta f^{\frac{1}{k}} \\ (4.13) \quad &\leq \dot{q}^{ij} \nabla_{i,j}^2 \sigma_1 + n f^{\frac{1}{k}} - \Delta f^{\frac{1}{k}}. \end{aligned}$$

Note that  $\dot{q}^{ij} \delta_{ij} \geq c_k$  for some constant  $c_k > 0$ . Hence, at a point that  $\sigma_1$  attains its maximum, we have

$$(4.14) \quad c_k \sigma_1 \leq n f^{\frac{1}{k}} - \Delta f^{\frac{1}{k}}.$$

From this we obtain  $\|s\|_{C^2} \leq C$ , where  $C$  depends only on  $n$ ,  $\min f$  and  $\|f\|_{C^2}$ . The higher order estimates follow from Evans-Krylov and Schauder regularity theory.  $\square$

The following theorem completes the closedness of the set  $S_\alpha$ .

**Theorem 4.22** (Full rank theorem). *Let  $0 < f \in C^\infty(\mathbb{S}^n)$  and  $s : \mathbb{S}^n \rightarrow (0, \infty)$  be a smooth function, with  $B[s] \geq 0$  and  $\sigma_k(B[s]) = f$ . If  $B[f^{-\frac{1}{k}}] \geq 0$ , then  $B[s] > 0$ . That is,  $s$  is the support function of a closed, smooth, strictly convex hypersurface.*

**4.2.3. Full rank theorem.** In this section, we prove [Theorem 4.22](#). The original proof by Guan-Ma in [\[GM03\]](#) is very complicated. We outline a simple proof from [\[BIS23a\]](#) relying on a lemma from [\[BCD17\]](#).

**Definition 4.23** (Inverse concavity). Suppose  $q : \Gamma_+^n \rightarrow \mathbb{R}$  is positive,  $S_n$ -invariant. We say  $q$  is inverse-concave if the dual function  $q_* : \Gamma_+^n \rightarrow \mathbb{R}$  defined by

$$q_*(z_1^{-1}, \dots, z_n^{-1}) = q(z_1, \dots, z_n)^{-1}$$

is concave. Similarly, let  $q : S_+(n) \rightarrow \mathbb{R}$  be a positive  $SO(n)$ -invariant function. If the function

$$\begin{aligned} q_* : S_+(n) &\rightarrow \mathbb{R} \\ q_*(Z^{-1}) &= q(Z)^{-1} \end{aligned}$$

is concave, then we say  $q$  is inverse-concave.



*Remark 4.24.* By [Theorem 4.10](#), a symmetric function is inverse-concave with respect to the eigenvalue variables if and only if it is inverse-concave with respect to the matrix variables.

**Theorem 4.25.** *Let  $q$  be a positive symmetric function. Then  $q$  is inverse concave if and only if the quadratic form  $Q_Z : S(n) \times S(n) \rightarrow \mathbb{R}$  defined by*

$$Q_Z(V, V) := \ddot{q}_Z(V, V) + 2\dot{q}_Z(VZ^{-1}V) - \frac{2\dot{q}_Z(V)\dot{q}_Z(V)}{q(Z)}$$

*is nonnegative definite for all  $Z \in S_+(n)$ , where juxtaposition of matrix variables denotes matrix multiplication. Equivalently,  $q$  is inverse-concave if and only if the quadratic form  $Q : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by*

$$Q_z(v, v) = \ddot{q}_z(v, v) - 2\frac{(\dot{q}_z(v))^2}{q(v)} + 2\dot{q}_z(vz^{-1}v)$$

*is nonnegative definite for all  $z \in \Gamma_+^n$  and, in addition,*

$$\frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} + \frac{\dot{q}_z^i}{z_j} + \frac{\dot{q}_z^j}{z_i} \geq 0$$

*for each  $i \neq j$  and each simple  $z$ , where  $z^{-1} := (z_1^{-1}, \dots, z_n^{-1})$  and the juxtaposition of eigenvalue variables denotes components-wise multiplication.*

*Proof.* Note that the derivative of  $Z^{-1}$  in the direction  $V$  is  $-Z^{-1}VZ^{-1}$ . Therefore, the first derivative of  $q_*(Z^{-1})$  with respect to  $Z \in S_+(n)$  in the direction  $V \in S(n)$  is given by

$$(4.15) \quad q(Z)^{-2}\dot{q}_Z(V) = \dot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}),$$

while the second derivative is given by

$$2\frac{(\dot{q}_Z(V))^2}{q(Z)^3} - \frac{\ddot{q}_Z(V, V)}{q(Z)^2} = \ddot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}, Z^{-1}VZ^{-1}) + 2\dot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}VZ^{-1}).$$

Moreover, replacing  $V$  in (4.15) by  $VZ^{-1}V$  gives

$$q(Z)^{-2}\dot{q}_Z(Z^{-1}VZ^{-1}) = \dot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}VZ^{-1}).$$

Hence,

$$\begin{aligned} q(Z)^{-2} \left( \ddot{q}_Z(V, V) - 2\frac{(\dot{q}_Z(V))^2}{q(Z)} + 2\dot{q}_Z(VZ^{-1}V) \right) \\ = -\ddot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}, Z^{-1}VZ^{-1}). \end{aligned}$$

To prove the second claim, we differentiate once  $q_*(z^{-1})$  with respect to  $z \in \Gamma_+$  in the direction  $v \in \mathbb{R}^n$  to obtain

$$(4.16) \quad \dot{q}_{*z^{-1}}(z^{-1}vz^{-1}) = q(z)^{-2}\dot{q}_z(v).$$

Differentiating once more, we obtain

$$-\ddot{q}_{*z^{-1}}(z^{-1}vz^{-1}, z^{-1}vz^{-1}) - 2\dot{q}_{*z^{-1}}(z^{-1}vz^{-1}vz^{-1}) = q(z)^{-2} \left( \ddot{q}_z(v, v) - 2\frac{(\dot{q}_z(v))^2}{q(v)} \right).$$

Now using (4.16), we get

$$-\ddot{q}_{*z^{-1}}(z^{-1}vz^{-1}, z^{-1}vz^{-1}) = q(z)^{-2} \left( \ddot{q}_z(v, v) - 2 \frac{(\dot{q}_z(v))^2}{q(v)} + 2\dot{q}_z(vz^{-1}v) \right).$$

Moreover, we have

$$\begin{aligned} \frac{\dot{q}_{*z^{-1}}^i - \dot{q}_{*z^{-1}}^j}{z_i - z_j} &= \frac{1}{q(z)^2(z_i - z_j)} (\dot{q}_z^i z_i^2 - \dot{q}_z^j z_j^2) \\ &= \frac{z_i z_j}{q(z)^2} \left( \frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} + \frac{\dot{q}_z^i}{z_j} + \frac{\dot{q}_z^j}{z_i} \right). \end{aligned}$$

□

*Remark 4.26.* Note that  $q : \Gamma_+^n \rightarrow \mathbb{R}$  defined by  $q(z) = \left( \sum_{i_1 < \dots < i_k} z_{i_1} \dots z_{i_k} \right)^{\frac{1}{k}}$  is inverse concave. Let  $s \in C^2(\mathbb{S}^n)$  and  $B[s] > 0$ . Now in [Theorem 4.25](#) take  $Z = \alpha = B[s]$  and  $V = \nabla_1 \alpha$ . Then at a point where  $g_{ij} = \delta_{ij}$  and  $B[s]$  is diagonal:

$$(4.17) \quad \sum_{i,j,k,l} \ddot{q}_\alpha^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} + 2 \sum_j \ddot{q}_\alpha^{ii} \frac{(\nabla_1 \alpha_{ij})^2}{\lambda_j} \geq 2 \frac{(\nabla_1 q(\alpha))^2}{q(\alpha)}.$$

**Corollary 4.27.** Let  $u \in C^2(\mathbb{S}^n)$  with  $B[u] \geq 0$  and  $\lambda_1 \leq \dots \leq \lambda_n$  denote the eigenvalues of  $\alpha_{ij} = B[u]_i^j$ . Let  $\{e_i\}_i$  be a local orthonormal frame of  $T\mathbb{S}^n$  with  $B[u]|_{x_0}(e_i, e_j) = \delta_{ij} \lambda_i$ . Suppose  $\mu$  is the multiplicity of the smallest eigenvalue  $\lambda_1 = 0$  at  $x_0$ . If  $Z \mapsto q(Z)$ ,  $Z \in S_+(n)$ , is inverse-concave and  $q(\alpha) = f > 0$ , then at  $x_0$ :

$$\sum_{i,j,k,l > \mu} \ddot{q}_\alpha^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} + 2 \sum_{j > \mu} \ddot{q}_\alpha^{ii} \frac{(\nabla_1 \alpha_{ij})^2}{\lambda_j} \geq 2 \frac{(\nabla_1 f)^2}{f}.$$

*Proof.* Define

$$V_{ij} = \begin{cases} \nabla_1 \alpha_{ij} & i, j > \mu \\ 0 & i \leq \mu \text{ or } j \leq \mu. \end{cases}$$

Moreover, define  $\alpha_\varepsilon = B[u] + \varepsilon \text{id}$ . By the previous lemma,

$$Q_{\alpha_\varepsilon}(V, V) \geq 0.$$

Hence, in view of (4.17),

$$\sum_{i,j,k,l} \ddot{q}_{\alpha_\varepsilon}^{ij,kl} V_{ij} V_{kl} + 2 \sum_j \ddot{q}_{\alpha_\varepsilon}^{ii} \frac{(V_{ij})^2}{\lambda_j + \varepsilon} \geq 2 \frac{(\nabla_1 q(\alpha_\varepsilon))^2}{q(\alpha_\varepsilon)^2}.$$

This implies that

$$\sum_{i,j,k,l > \mu} \ddot{q}_{\alpha_\varepsilon}^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} + 2 \sum_{j > \mu} \ddot{q}_{\alpha_\varepsilon}^{ii} \frac{(\nabla_1 \alpha_{ij})^2}{\lambda_j + \varepsilon} \geq 2 \frac{(\nabla_1 q(\alpha_\varepsilon))^2}{q(\alpha_\varepsilon)^2}.$$

Now let  $\varepsilon \rightarrow 0$ .

□

**Lemma 4.28.** [BCD17] Let  $u \in C^2(\mathbb{S}^n)$  and  $\lambda_1 \leq \dots \leq \lambda_n$  denote the eigenvalues of  $B[u]$  and  $\tau = \nabla^2 u + gu$ . Let  $\{e_i\}_i$  be a local orthonormal frame of  $T\mathbb{S}^n$  with  $\tau|_{x_0}(e_i, e_j) = \delta_{ij}\lambda_i$ . Suppose  $\mu$  is the multiplicity of the smallest eigenvalue at  $x_0$ . Let  $\psi$  be a smooth function such that  $\psi \leq \lambda_1$  everywhere and  $\psi(x_0) = \lambda_1(x_0)$ . Then, at  $x_0$ , we have

$$\nabla_{i,i}^2 \psi \leq \nabla_{i,i}^2 \tau_{11} - 2 \sum_{j>\mu} \frac{(\nabla_i \tau_{1j})^2}{\lambda_j - \lambda_1}.$$

Moreover, for any lower support  $\psi$  of  $\lambda_1$  at  $x_0$  we have  $\nabla_i \tau_{kl} = \nabla_i \psi \delta_{kl}$  for  $1 \leq k, l \leq \mu$ .

*Proof of the full rank theorem; Theorem 4.22.* Let and  $\alpha_{ij} = B[s]_i^j$ . Define  $q = \sigma_k^{\frac{1}{k}}$ . Then

$$\begin{aligned} \nabla_m f^{\frac{1}{k}} &= \dot{q}^{ij} \nabla_m \alpha_{ij}, \\ \nabla_{m,n}^2 f^{\frac{1}{k}} &= \dot{q}^{ij} \nabla_{m,n}^2 \alpha_{ij} + \ddot{q}^{ij,kl} \nabla_m \alpha_{ij} \nabla_n \alpha_{kl}. \end{aligned}$$

Let  $\tau = \nabla^2 s + gs$ . Recall from Exercise 4.20 that

$$\nabla_{m,n}^2 \alpha_{ij} = g^{pj} \nabla_{m,n}^2 \tau_{ip} = g^{pj} (\nabla_{i,p}^2 \tau_{mn} - \tau_{mn} g_{ip} + \tau_{ip} g_{mn}).$$

In particular, for  $m = n = 1$ :

$$\begin{aligned} \nabla_{1,1}^2 \alpha_{ij} &= g^{pj} \nabla_{i,p}^2 \tau_{11} - \tau_{11} \delta_{ij} + \alpha_{ij}, \\ \nabla_{1,1}^2 f^{\frac{1}{k}} &= \dot{q}^{ij} g^{pj} \nabla_{i,p}^2 \tau_{11} + \ddot{q}^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} - \tau_{11} \dot{q}^{ij} \delta_{ij} + f^{\frac{1}{k}}. \end{aligned}$$

Hence, at  $x_0$ :

$$\nabla_{1,1}^2 f^{\frac{1}{k}} = \dot{q}^{ii} \nabla_{i,i}^2 \tau_{11} + \ddot{q}^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} - \tau_{11} \dot{q}^{ij} \delta_{ij} + f^{\frac{1}{k}}.$$

In view of Lemma 4.28, at  $x_0$  we have

$$\dot{q}^{ii} \nabla_{i,i}^2 \psi \leq -\ddot{q}^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} - 2 \sum_{l>\mu} \frac{\dot{q}^{ii} (\nabla_i \tau_{1l})^2}{\lambda_l} + \nabla_{1,1}^2 f^{\frac{1}{k}} - f^{\frac{1}{k}} + \lambda_1 \dot{q}^{ij} \delta_{ij}.$$

Therefore,

$$\begin{aligned} \dot{q}^{ii} \nabla_{i,i}^2 \psi &\leq - \sum_{i,j,k,l>\mu} \ddot{q}^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} - 2 \sum_{l>\mu} \frac{\dot{q}^{ii} (\nabla_i \tau_{1l})^2}{\lambda_l} \\ &\quad - \sum_{(i,j,k,l) \in \Lambda} \ddot{q}^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} + \nabla_{1,1}^2 f^{\frac{1}{k}} - f^{\frac{1}{k}} + \lambda_1 \dot{q}^{ij} \delta_{ij}, \end{aligned}$$

where  $\Lambda$  is the complement of the set  $\{i, j, k, l > \mu\}$  in  $\{1, \dots, n\}^4$ . Due to Corollary 4.27, that  $\nabla \alpha$  is fully symmetric, Lemma 4.28, we have

$$\begin{aligned} (4.18) \quad \dot{q}^{ii} \nabla_{i,i}^2 \psi &\leq c |\nabla \psi| + \nabla_{1,1}^2 f^{\frac{1}{k}} - 2 f^{-\frac{1}{k}} |\nabla_1 f^{\frac{1}{k}}|^2 - f^{\frac{1}{k}} + \psi \dot{q}^{ij} \delta_{ij} \\ &= c |\nabla \psi| - f^{\frac{2}{k}} B[f^{-\frac{1}{k}}]_1^1 + \psi \dot{q}^{ij} \delta_{ij} \\ &\leq c |\nabla \psi| + \psi \dot{q}^{ij} \delta_{ij}. \end{aligned}$$

Here on the last step we used our assumption that  $B[f^{-\frac{1}{k}}] \geq 0$ . The strong maximum principle (cf. [BL99]) shows that if  $\lambda_1$  is zero at some point, then  $\lambda_1 \equiv 0$ . However, at a point where  $s$  attains its minimum, we have  $\lambda_1 > 0$ .  $\square$

Recall that  $S_\alpha = \{t \in [0, 1] : \text{the equation } \sigma_k(B[s]) = f_t \text{ has a } C^{\ell+2, \alpha} \text{ solution } s_t \text{ such that } B[s_t] > 0 \text{ and } \int x s_t(x) = 0\}$  and  $f_t$  is defined in Lemma 4.2. Let  $\ell$  be sufficiently large. Now consider a sequence  $t_i \in S_\alpha$  with  $t_i \rightarrow t_0$ . Then by the a priori estimates, after passing to a subsequence,  $s_{t_i}$  converges to some function  $s_{t_0} \in C^{\ell+2}(\mathbb{S}^n)$  so that  $B[s_{t_0}] \geq 0$ ,  $s_{t_0} \geq 0$ ,  $\int s_{t_0} x_i = 0$  for all  $i$ , and  $\sigma_k(B[s_{t_0}]) = f_{t_0}$ . If  $s_{t_0} > 0$ , then by the full rank theorem Theorem 4.22 we have  $B[s_{t_0}] > 0$  and so  $t_0 \in S_\alpha$  (recall that  $B[f_{t_0}^{-\frac{1}{k}}] \geq 0$ ). The following argument shows that in fact  $s_{t_0} > 0$ .

Let  $K$  be a convex body with non-empty interior. Define  $r_K$  to be the radius of the maximal ball contained in  $K$  and  $R_K$  be the radius of the minimal ball containing  $K$ . Consider the sequence  $s_{t_i}$  corresponding to convex bodies, say  $K_{t_i}$ . From [CW00, Lem. 2.2] and (4.14), it follows that  $R_{K_{t_i}}^2 \leq C r_{K_{t_i}}$  for some  $C$  depending on  $f_{t_i}$  (and hence only on  $f$ ). Note that  $R_{K_{t_i}}$  is uniformly bounded from below away from zero.<sup>8</sup> Hence,  $V(K_{t_i}) \geq c > 0$  for some constant depending on  $f$ . Now, letting  $t_i \rightarrow t_0$ , we deduce that  $s_{t_0}$  is the support function of a convex body  $K_{t_0}$  with *non-empty* interior. Since the Steiner point of  $K_{t_0}$  is at the origin, we have  $s_{t_0} > 0$ .

**4.3. Openness.** The argument to show openness is similar to the one we have seen for the Minkowski problem. We show that the kernel of the corresponding linearized operator only consists of linear functions.

Let  $s$  be the support function of a smooth, strictly convex body. Let  $q = \sigma_k$ . Define the linearized operator

$$L_{s,k}[u] = \frac{d}{dt}\bigg|_{t=0} \sigma_k(B[s + tu]), \quad u \in C^2(\mathbb{S}^n).$$

**Lemma 4.29.** *Suppose  $L_{s,k}[u] = 0$ , then  $u(x) = \langle \vec{a}, x \rangle$  for some vector  $\vec{a}$ .*

*Proof.* Let us define  $Q_k[u] = Q[u, s, \dots, s, 1, \dots, 1]$ , where  $s$  appears  $k-1$  times. Then, for any  $u \in C^2(\mathbb{S}^n)$ , by the (local) Aleksandrov-Fenchel inequality we have

$$(4.19) \quad \left( \int u Q_k[s] \right)^2 \geq \int u Q_k[u] \int s Q_k[s].$$

Equality holds if and only if  $u(x) = as(x) + \langle \vec{a}, x \rangle$  for some vector  $\vec{a}$  and constant  $a$ .

Now note that

$$\sigma_k(B[s + tu]) = c_k Q[s + tu, \dots, s + tu, 1, \dots, 1],$$

---

<sup>8</sup>From  $\sigma_k(B[s_{t_i}]) = f_{t_i}$ , we see that  $\max s_{t_i}^k \geq \min f_{t_i}$ . Therefore, from the construction of  $f_{t_i}$  we have  $\max s_{t_i} \geq \delta$  for some  $\delta > 0$  independent of  $t_i$ . Consider the family of convex bodies  $\mathcal{F} = \{K : \text{diam}(K) \leq C, s_K > 0, \max s_K \geq 1/C\}$ . Then from the Blaschke selection theorem it follows that  $R_K \geq c_0$  for all  $K \in \mathcal{F}$ .

where  $s + tu$  appears  $k$  times. Hence,

$$0 = L_{s,k}[u] = kc_k Q_k[u].$$

Since  $\int u Q_k[s] = \int s Q_k[u] = 0$  (cf. [Lemma 2.13](#)), we have equality in (4.19) and the claim follows (from  $Q_k[u] = 0$  we deduce that  $a = 0$ ).  $\square$

Having [Lemma 4.29](#) in hand, we can then proceed as in the case of the Minkowski problem to show the openness of  $S_\alpha$  in  $[0, 1]$ .

**4.4. Uniqueness.** Suppose we have

$$\sigma_k(B[s_1]) = \sigma_k(B[s_2]) = f.$$

By the Alexandrov-Fenchel inequality we have

$$(4.20) \quad \int s_2 \sigma_k(B[s_1]) \geq \left( \int s_1 \sigma_k(B[s_1]) \right)^{\frac{k}{k+1}} \left( \int s_2 \sigma_k(B[s_2]) \right)^{\frac{1}{k+1}}.$$

Moreover, equality holds if and only if  $s_2(x) = as_1(x) + \langle \vec{a}, x \rangle$  for some constant  $a$  and vector  $\vec{a}$ . Therefore,

$$\int s_2 f \geq \int s_1 f.$$

Similarly,  $\int s_1 f \geq \int s_2 f$ . That is, we have equality; hence, uniqueness holds up to a linear function (we must have  $a = 1$ ).

## 5. $L_p$ -CHRISTOFFEL-MINKOWSKI PROBLEM

In this section, we study the even  $L_p$ -Christoffel-Minkowski problem.

$$(5.1) \quad \sigma_k = f s^{p-1}.$$

For the case  $p \geq k + 1$  see [\[HMS04\]](#). Here, we focus on the (most interesting) case  $1 < p < k + 1$ . We prove the following theorem.

**Theorem 5.1.** [\[GX18\]](#) *Let  $1 \leq k < n$ ,  $1 < p < k + 1$  and  $0 < f \in C^\infty(\mathbb{S}^n)$  be an even function. Suppose  $B[f^{-\frac{1}{p+k-1}}]$  is non-negative definite. Then, there is a unique origin-symmetric, smooth, strictly convex hypersurface in  $\mathbb{R}^{n+1}$  satisfying (5.1).*

Again, our primary method of establishing this existence result is the continuity method in combination with a full rank theorem [Theorem 5.5](#). Regarding the continuity method, we only establish the closedness part, i.e., the  $C^0$  and  $C^2$  a priori estimates (verifying the openness part is similar to the case of the  $L_p$ -Minkowski problem). We follow the approach in [\[HI23\]](#).

**Lemma 5.2.** [\[CW00\]](#) *Let  $\mathcal{M}$  be a smooth, origin-symmetric, strictly convex hypersurface. Let  $R = \max s$  and  $r = \min s$ . We have either  $\frac{R}{r} \leq \sqrt{n+1}$  or  $\frac{R^2}{r} \leq C_n \max_{\mathbb{S}^n} \lambda_n$ , where  $C_n$  is a constant depending only on  $n$ .*

*Proof.* Suppose  $R > r\sqrt{n+1}$ . Due to convexity, we may find two perpendicular directions, say  $e_1, e_2$ , such that  $s(e_2) = r$  and

$$s(e_1) > \frac{R}{\sqrt{n+1}}.$$

Now project  $\mathcal{M}$  to the  $x_1x_2$ -plane and denote the corresponding convex body by  $P$ . Since  $P$  is origin-symmetric,  $(\pm R/2\sqrt{n+1}, 0)$  are in the interior of  $P$  and  $(0, \pm r) \in \partial P$ .

For simplicity, let  $P_r := \frac{1}{r}P$ , and write  $D$  for the disk of radius  $1/2$  centered at the origin. We have  $D \subset P_r$ . Let  $\tilde{D}$  denote the convex hull of  $D$  and  $(\pm R/(2r\sqrt{n+1}), 0)$ . Then  $\partial\tilde{D}$  is the union of four tangential segments to the circle  $\partial D$  and two closed arcs of  $\partial D$ . The four tangent lines are given by

$$y = \pm \frac{1}{\sqrt{\left(\frac{R}{r\sqrt{n+1}}\right)^2 - 1}} \left( x \pm \frac{R}{2r\sqrt{n+1}} \right).$$

Now note that the rectangle

$$-\frac{R}{4r\sqrt{n+1}} \leq x_1 \leq \frac{R}{4r\sqrt{n+1}}, \quad -\frac{1}{4} \leq x_2 \leq \frac{1}{4}$$

is contained in the interior of  $\tilde{D}$ .<sup>9</sup> Since the ellipse

$$E_0 = \left\{ (x_1, x_2) : 16r^2(n+1)\frac{x_1^2}{R^2} + 16x_2^2 \leq 1 \right\}$$

lies in this rectangle,  $rE_0$  is contained in the interior of  $P$ . Therefore, for some  $r/4 \leq h \leq r$ ,

$$E_1 := \left\{ (x_1, x_2) : 16(n+1)\frac{x_1^2}{R^2} + \frac{x_2^2}{h^2} \leq 1 \right\} \subset P,$$

while touching  $P$  at

$$\vec{a} := \left( \frac{R}{4\sqrt{n+1}} \cos \theta, h \sin \theta \right)$$

for some  $-\frac{\pi}{2} \leq \theta \neq 0 \leq \frac{\pi}{2}$ . We may assume  $-\frac{\pi}{2} \leq \theta < 0$ . Hence, by comparing the slope of the tangent line of  $E_1$  at  $\vec{a}$  with the one joining  $\vec{a}$  and  $(R/2\sqrt{n+1}, 0)$ , and in view of the convexity of  $P$ , we have

$$\frac{-h \sin \theta}{\frac{R}{2\sqrt{n+1}} - \frac{R}{4\sqrt{n+1}} \cos \theta} \geq -\frac{16(n+1)h^2 \frac{R}{4\sqrt{n+1}} \cos \theta}{R^2} \frac{1}{h \sin \theta}.$$

This gives

$$\sin^2 \theta \geq \frac{3}{4}.$$

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<sup>9</sup>The midpoint of the line segment joining the points  $(\frac{R}{2r\sqrt{n+1}}, 0)$  and  $(0, \frac{1}{2})$  must lie in the interior of  $\tilde{D}$ ; otherwise, the whole line segment has to be on the boundary of  $\tilde{D}$  and hence tangent to  $D$ .

Now the claim follows from estimating the radius of curvature of  $E_1$  at  $\vec{a}$  from below:

$$\frac{4\sqrt{n+1} \left( \frac{R^2}{16(n+1)} \sin^2 \theta + h^2 \cos^2 \theta \right)^{\frac{3}{2}}}{hR} \geq \frac{|\sin \theta|^3 R^2}{16(n+1)h} \geq \frac{1}{C_n} \frac{R^2}{r},$$

where  $C_n := \frac{128(n+1)}{\sqrt{27}}$ . □

From (5.1), we have the following basic estimates:

$$(5.2) \quad R^{p-k-1} \leq \frac{C_{n,k}}{\min f}, \quad r^{p-k-1} \geq \frac{C_{n,k}}{\max f}.$$

Therefore, there is a lower bound on  $R$  and an upper bound on  $r$ . To obtain the  $C^0$  and  $C^2$  estimates, the following gradient estimate plays a crucial role in our argument.

**Lemma 5.3.** [HI23] *For any  $0 < \gamma < \frac{2(p-1)}{k}$ , there exists a constant  $\beta \geq 2$ , depending on  $\gamma, k, p, \min f$ , and  $\|f\|_{C^1}$ , such that*

$$(5.3) \quad \frac{s^2 + |\nabla s|^2}{s^\gamma} \leq \beta R^{2-\gamma}.$$

*Proof.* Let  $\rho^2 = s^2 + |\nabla s|^2$  and  $\zeta = \frac{\rho^2}{s^\gamma}$ , where  $0 < \gamma < \frac{2(p-1)}{k}$  and set  $\tau = \nabla^2 s + g s$ . Assume  $\max \zeta > R^{2-\gamma}$  (that is,  $\beta > 1$ ). Note that

$$\begin{aligned} \nabla_i \zeta &= 2 \frac{(s \nabla_i s + g^{k\ell} \nabla_{i,k}^2 s \nabla_\ell s)}{s^\gamma} - \gamma \frac{\rho^2}{s^{\gamma+1}} \nabla_i s \\ &= 2 \frac{g^{k\ell} (s g_{ik} + \nabla_{i,k}^2 s) \nabla_\ell s}{s^\gamma} - \gamma \frac{\rho^2}{s^{\gamma+1}} \nabla_i s \\ &= 2 \frac{g^{k\ell} \tau_{ik} \nabla_\ell s}{s^\gamma} - \gamma \frac{\rho^2}{s^{\gamma+1}} \nabla_i s \end{aligned}$$

Therefore, at a point  $x_0$  where  $\zeta$  attains its maximum, we have

$$\tau^{\sharp g} \nabla s = \frac{\gamma \rho^2}{2 s} \nabla s.$$

At  $x_0$ ,  $\nabla s \neq 0$  is an eigenvector of  $B[s] = \tau^{\sharp g}$ . Hence, we can find an orthonormal basis  $\{e_i\}$  for  $T_{x_0} \mathbb{S}^n$  such that  $e_1 = \frac{\nabla s}{|\nabla s|}$  and  $\tau|_{x_0}$  is diagonal. In particular,  $\tau_{1i} = 0$  for  $i = 2, \dots, n$ , while

$$(5.4) \quad \tau_{11} = \frac{\gamma \rho^2}{2 s}.$$

Also note that at  $x_0$ :

$$(5.5) \quad \nabla_1 s = |\nabla s| \neq 0, \quad \nabla_i s = 0 \quad i = 2, \dots, n.$$

Let  $\zeta_{;ii} = \nabla_{i,i}^2 \zeta$ ,  $s_i = \nabla_i s$  and  $\tau_{\ell ii} = \nabla_\ell \tau_{ii}$ . Then, at  $x_0$  we have

$$\begin{aligned} \nabla_i g^{k\ell} (\tau_{ik} \nabla_\ell s) &= g^{k\ell} \nabla_i \tau_{ik} \nabla_\ell s + g^{k\ell} \tau_{ik} \nabla_{i,\ell} s \\ &= g^{k\ell} \nabla_k \tau_{ii} \nabla_\ell s + g^{k\ell} \tau_{ik} (\tau_{i\ell} - g_{i\ell} s) \\ &= \tau_{\ell ii} s_\ell + \tau_{ii}^2 - s \tau_{ii}. \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta_{;ii} &= \frac{2}{s^\gamma} (\tau_{\ell ii} s_\ell + \tau_{ii}^2 - s \tau_{ii}) - 4\gamma \frac{\tau_{ii} s_\ell s_i}{s^{\gamma+1}} \\ &\quad - \frac{\gamma \rho^2 (\tau_{ii} - s)}{s^{\gamma+1}} + \gamma(\gamma+1) \frac{\rho^2 s_i^2}{s^{\gamma+2}}, \end{aligned}$$

and

$$\begin{aligned} \frac{\zeta_{;ii}}{\zeta} &= \frac{2}{\rho^2} (\tau_{\ell ii} s_\ell + \tau_{ii}^2 - s \tau_{ii}) - 4\gamma \frac{\tau_{ii} s_i^2}{s \rho^2} \\ &\quad - \frac{\gamma(\tau_{ii} - s)}{s} + \gamma(\gamma+1) \frac{s_i^2}{s^2}. \end{aligned}$$

Let  $q = \sigma_k$ . Note that  $\nabla \tau$  is fully symmetric, and  $q$  is  $k$ -homogeneous. Using (5.4), (5.5), and that  $[\zeta_{;ii}] \leq 0$  and  $\zeta_i = 0$  at  $x_0$  for  $i = 1, \dots, n$ , we have

$$\begin{aligned} 0 &\geq \frac{2}{\rho^2} \dot{q}^{ii} (\tau_{\ell ii} s_\ell + \tau_{ii}^2 - s \tau_{ii}) - 4\gamma \dot{q}^{11} \frac{\tau_{11} s_1^2}{s \rho^2} \\ &\quad - \gamma \dot{q}^{ii} \frac{\tau_{ii}}{s} + \gamma \dot{q}^{ii} + \gamma(\gamma+1) \dot{q}^{11} \frac{s_1^2}{s^2} \\ &= \frac{2}{\rho^2} ((s^{p-1} f)_1 s_1 + \dot{q}^{ii} \tau_{ii}^2 - k s^p f) - 2\gamma^2 \dot{q}^{11} \frac{s_1^2}{s^2} \\ &\quad - \gamma k s^{p-2} f + \gamma \dot{q}^{ii} + \gamma(\gamma+1) \dot{q}^{11} \frac{s_1^2}{s^2}. \end{aligned}$$

Therefore,<sup>10</sup>

$$\begin{aligned} 0 &\geq 2(p-1) \frac{s^{p-2} s_1^2 f}{\rho^2} + 2 \frac{s^{p-1} s_1 f_1}{\rho^2} - 2k \frac{s^p f}{\rho^2} \\ &\quad + \left( \frac{\gamma^2 \rho^2}{2 s_1^2} - \gamma(\gamma-1) \right) \dot{q}^{11} \frac{s_1^2}{s^2} - \gamma k s^{p-2} f. \end{aligned}$$

For  $0 < \gamma \leq 2$ ,

$$\frac{\gamma^2 \rho^2}{2 s_1^2} - \gamma(\gamma-1) \geq \frac{\gamma^2}{2} - \gamma(\gamma-1) = \gamma \left( 1 - \frac{\gamma}{2} \right) \geq 0.$$

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<sup>10</sup>Here, from the term  $\dot{q}^{ii} \tau_{ii}^2 = \sum_i \dot{q}^{ii} \tau_{ii}^2$ , we only kept the  $\dot{q}^{11} \tau_{11}^2$  part and then used (5.4). Moreover, we dropped the term  $\gamma \dot{q}^{ii}$ .



Now if for some  $\beta \geq 2$  we had

$$\left. \frac{\rho^2}{s^\gamma} \right|_{x_0} \geq \beta R^{2-\gamma},$$

then

$$(5.6) \quad s_1^2 \geq \beta R^{2-\gamma} s^\gamma - s^2 \geq \frac{\beta}{2} R^{2-\gamma} s^\gamma.$$

Hence, using (5.6), we obtain

$$\begin{aligned} 0 &\geq \frac{2s^{p-2}f}{\rho^2} \left( (p-1)s_1^2 + ss_1(\log f)_1 - ks^2 - \frac{k\gamma}{2}(s^2 + s_1^2) \right) \\ &= \frac{2s^{p-2}f}{\rho^2} \left( \left( p-1 - \frac{k\gamma}{2} \right) s_1^2 + ss_1(\log f)_1 - \left( k + \frac{k\gamma}{2} \right) s^2 \right) \\ &\geq \frac{2s^{p-2}f}{\rho^2} \left( \left( p-1 - \frac{k\gamma}{2} \right) \frac{\beta}{2} R^{2-\gamma} s^\gamma - c_1 \beta^{\frac{1}{2}} R^{1-\frac{\gamma}{2}} s^{1+\frac{\gamma}{2}} - c_2 s^2 \right) \\ &= \frac{2s^{p+\gamma-2}f}{\rho^2} \left( \left( p-1 - \frac{k\gamma}{2} \right) \frac{\beta}{2} R^{2-\gamma} - c_1 \beta^{\frac{1}{2}} R^{1-\frac{\gamma}{2}} s^{1-\frac{\gamma}{2}} - c_2 s^{2-\gamma} \right) \\ &\geq \frac{2s^{p+\gamma-2}R^{2-\gamma}f}{\rho^2} \left( \left( p-1 - \frac{k\gamma}{2} \right) \frac{\beta}{2} - c_1 \beta^{\frac{1}{2}} - c_2 \right). \end{aligned}$$

Here, we used  $0 < \gamma < \frac{2(p-1)}{k}$  on the last three lines. Moreover, the constant  $c_1$  depends on  $\min f$  and  $\|f\|_{C^1}$ , and the constant  $c_2$  depends on  $k$ . However, we would obtain a contradiction for  $\beta$  large enough.  $\square$

**Proposition 5.4.** *Let  $1 < p < k+1$ . Suppose  $s > 0$  is an even, smooth, strictly convex solution of  $\sigma_k(B[s]) = fs^{p-1}$ . Then for each  $\ell \geq 1$  and  $\gamma \in (0, 1)$ , we have*

$$1/C \leq s \leq C, \quad \|s\|_{C^{\ell+1, \gamma}} \leq C_{\ell, \gamma}$$

where  $C > 0$  is a constant depending only on  $n, \gamma, k, p, \min f$  and  $\|f\|_{C^2}$ .

*Proof.* Let  $q = \sigma_k^{\frac{1}{k}}$ . Note that  $q^k = fs^{p-1}$ . In view of the identity (see Exercise 4.20)

$$\sigma_1 \delta_{ij} = g^{pj} \nabla_{i,p}^2 \sigma_1 + nB[s]_i^j - \Delta B[s]_i^j.$$

By concavity of  $q$  (similar calculation was performed leading to (4.13)), we obtain

$$(5.7) \quad 0 \leq \dot{q}^{ij} \delta_{ij} \sigma_1 \leq \dot{q}^{ij} g^{pj} \nabla_{i,p}^2 \sigma_1 + n(s^{p-1}f)^{\frac{1}{k}} - \Delta(s^{p-1}f)^{\frac{1}{k}}.$$

We calculate

$$\begin{aligned} -k\Delta(s^{p-1}f)^{\frac{1}{k}} &= (1-p)s^{\frac{p-1}{k}-1}f^{\frac{1}{k}}\sigma_1 - n(1-p)(s^{p-1}f)^{\frac{1}{k}} \\ &\quad + \frac{1}{k}(1-p)(p-k-1)s^{\frac{p-1}{k}-2}|\nabla s|^2 f^{\frac{1}{k}} \\ &\quad + 2(1-p)s^{\frac{p-1}{k}-1}g(\nabla s, \nabla f^{\frac{1}{k}}) - ks^{\frac{p-1}{k}}\Delta f^{\frac{1}{k}}. \end{aligned} \quad (5.8)$$

For  $p > 1$ , at a point where  $\sigma_1$  attains its maximum we have

$$\sigma_1 \leq c_1 \left( \frac{|\nabla s|^2}{s} + R \right).$$

Due to [Lemma 5.3](#),

$$\sigma_1 \leq c_1 \left( \frac{|\nabla s|^2}{s} + R \right) \leq c_2 (s^{\gamma-1} R^{2-\gamma} + R) \leq c_3 R \left( \frac{R}{r} \right)^{1-\gamma}.$$

By [Lemma 5.2](#), for some constant  $C$  depending on  $n, \gamma, k, p, \min f$  and  $\|f\|_{C^2}$  we have

$$\left( \frac{R}{r} \right)^\gamma \leq C.$$

Now the uniform lower and upper bounds on the support function and the uniform upper bound on the principal radii of curvature follow from [\(5.2\)](#). The higher-order estimates follow from the Evans-Krylov and the Schauder regularity theory.  $\square$

**Theorem 5.5** (Full rank theorem). *Let  $p > 1$ ,  $0 < f \in C^\infty(\mathbb{S}^n)$  and  $s : \mathbb{S}^n \rightarrow (0, \infty)$  be a smooth function, with  $B[s] \geq 0$  and  $\sigma_k(B[s]) = fs^{p-1}$ . If  $B[f^{-\frac{1}{p+k-1}}] \geq 0$ , then  $B[s] > 0$ . That is,  $s$  is the support function of a closed, smooth, strictly convex hypersurface.*

*Proof.* Arguing exactly as in the proof of [Theorem 4.22](#) (see [\(4.18\)](#)) we obtain

$$\dot{q}^{ii} \nabla_{i,i}^2 \psi \leq c |\nabla \psi| + \psi \dot{q}^{ij} \delta_{ij} + \nabla_{1,1}^2 (fs^{p-1})^{\frac{1}{k}} - 2(fs^{p-1})^{-\frac{1}{k}} |\nabla_1 (fs^{p-1})^{\frac{1}{k}}|^2 - (fs^{p-1})^{\frac{1}{k}}.$$

We need to manipulate the last three terms so that  $B[f^{-\frac{1}{p+k-1}}] \geq 0$  can be used. Let  $\tilde{f} = f^{\frac{1}{k}}$  and  $\alpha = \frac{p-1}{k}$ . In the following the semicolon ; notation denotes the covariant derivatives (e.g.  $(s^\alpha \tilde{f})_{;11} = \nabla_{1,1}^2 (s^\alpha \tilde{f})$  and  $\nabla_{1,1}^2 \tilde{f} = \tilde{f}_{;11}$ ). We compute

$$\begin{aligned} & (s^\alpha \tilde{f})_{;11} - s^\alpha \tilde{f} - 2 \frac{((s^\alpha \tilde{f})_1)^2}{s^\alpha \tilde{f}} \\ &= (\alpha s^{\alpha-1} s_{11} + \alpha(\alpha-1) s^{\alpha-2} s_1^2) \tilde{f} + 2\alpha s^{\alpha-1} s_1 \tilde{f}_1 \\ & \quad + s^\alpha \tilde{f}_{;11} - s^\alpha \tilde{f} - 2 \frac{(\alpha s^{\alpha-1} s_1 \tilde{f} + s^\alpha \tilde{f}_1)^2}{s^\alpha \tilde{f}} \\ &= \alpha s^{\alpha-1} (\lambda_1 - s) \tilde{f} - \alpha(\alpha+1) s^{\alpha-2} s_1^2 \tilde{f} - 2\alpha s^{\alpha-1} s_1 \tilde{f}_1 \\ & \quad + s^\alpha \left( \tilde{f}_{;11} - \tilde{f} - 2 \frac{(\tilde{f}_1)^2}{\tilde{f}} \right) \\ &\leq \alpha s^{\alpha-1} \tilde{f} \lambda_1 + s^\alpha \left( \tilde{f}_{;11} - (\alpha+1) \tilde{f} - \frac{\alpha+2}{\alpha+1} \frac{(\tilde{f}_1)^2}{\tilde{f}} \right), \end{aligned}$$

where we used the Cauchy-Schwarz inequality and that  $\alpha \geq 0$ :

$$-2\alpha s^{\alpha-1} s_1 \tilde{f}_1 - \alpha(\alpha+1) s^{\alpha-2} s_1^2 \tilde{f} \leq \frac{\alpha}{\alpha+1} s^\alpha \frac{(\tilde{f}_1)^2}{\tilde{f}}.$$

Moreover, we have

$$\begin{aligned}
& \tilde{f}_{;11} - (\alpha + 1)\tilde{f} - \frac{\alpha + 2}{\alpha + 1} \frac{(\tilde{f}_1)^2}{\tilde{f}} \\
&= -(\alpha + 1)\tilde{f}^{\frac{\alpha+2}{\alpha+1}} \left( (\tilde{f}^{-\frac{1}{1+\alpha}})_{;11} + \tilde{f}^{-\frac{1}{1+\alpha}} \right) \\
&= -\frac{p-1+k}{k} \tilde{f}^{\frac{\alpha+2}{\alpha+1}} \left( (f^{-\frac{1}{k+p-1}})_{;11} + f^{-\frac{1}{k+p-1}} \right).
\end{aligned}$$

Putting everything together, we have shown

$$\begin{aligned}
\dot{q}^{ij} \nabla_{i,i}^2 \psi &\leq c|\nabla \psi| + \psi \dot{q}^{ij} \delta_{ij} + \alpha s^{\alpha-1} \tilde{f} \psi - \frac{p-1+k}{k} s^\alpha \tilde{f}^{\frac{\alpha+2}{\alpha+1}} \left( (f^{-\frac{1}{k+p-1}})_{;11} + f^{-\frac{1}{k+p-1}} \right) \\
&\leq c|\nabla \psi| + \left( \dot{q}^{ij} \delta_{ij} + \alpha s^{\alpha-1} \tilde{f} \right) \psi.
\end{aligned}$$

Therefore, by the strong maximum principle for viscosity solutions, if  $\lambda_1$  is zero somewhere, then it is identically zero. However, since  $s > 0$ , at the point where minimum of  $s$  is attained we have  $\lambda_1 > 0$ .  $\square$

For  $t \in [0, 1]$ , define  $f_t := (1-t+t f^{-\frac{1}{p+k-1}})^{-(p+k-1)}$ . Since  $f_t$  is even, we have  $\int x f_t(x) = 0$ . Moreover, it is easy to see that  $B[f_t^{-\frac{1}{p+k-1}}] \geq 0$ . Now define the set  $S_\alpha = \{t \in [0, 1] : \text{the equation } \sigma_k(B[s]) = s^{p-1} f_t \text{ has an even } C^{\ell+2, \alpha} \text{ solution } s_t \text{ such that } B[s_t] > 0\}$ . Let  $\ell$  be sufficiently large. Let  $t_i \in S_\alpha$  be a sequence with  $t_i \rightarrow t_0$ . By [Proposition 5.4](#), after passing to a subsequence,  $s_{t_i}$  converges to some function  $s_{t_0} \in C^{\ell+2}(\mathbb{S}^n)$  so that  $B[s_{t_0}] \geq 0$ ,  $s_{t_0} > 0$ , and  $\sigma_k(B[s_{t_0}]) = f_{t_0} s_{t_0}^{p-1}$ . By [Theorem 5.5](#), we have  $B[s_{t_0}] > 0$  and so  $t_0 \in S_\alpha$ .

**5.1. Uniqueness.** Let  $s_1, s_2$  be solutions of  $s^{1-p} \sigma_k(B[s]) = f$ . Recall from [\(4.20\)](#) that

$$\int s_2 \sigma_k(B[s_1]) \geq \left( \int s_1 \sigma_k(B[s_1]) \right)^{\frac{k}{k+1}} \left( \int s_2 \sigma_k(B[s_2]) \right)^{\frac{1}{k+1}}.$$

Moreover, equality holds if and only if  $s_2(x) = a s_1(x) + \langle \vec{a}, x \rangle$  for some constant  $a$  and vector  $\vec{a}$ . Using this inequality we obtain

$$\int s_2 s_1^{p-1} f \geq \left( \int s_1^p f \right)^{\frac{k}{k+1}} \left( \int s_2^p f \right)^{\frac{1}{k+1}}.$$

Since  $p > 1$ , by the Hölder inequality we have

$$\int s_2 s_1^{p-1} f \leq \left( \int s_1^p f \right)^{\frac{p-1}{p}} \left( \int s_2^p f \right)^{\frac{1}{p}}.$$

Combining the two inequalities yields

$$\left( \int s_1^p f \right)^{\frac{1}{k+1} - \frac{1}{p}} \geq \left( \int s_2^p f \right)^{\frac{1}{k+1} - \frac{1}{p}}.$$

Similarly, we obtain

$$\left( \int s_2^p f \right)^{\frac{1}{k+1} - \frac{1}{p}} \geq \left( \int s_1^p f \right)^{\frac{1}{k+1} - \frac{1}{p}}.$$

Hence, the equality holds and  $s_1 = s_2$ .

## 6. PRESCRIBED $L_p$ CURVATURE PROBLEM

The prescribed curvature problem asks the following question.

**Question 6.1.** Given a positive, smooth function  $f : \mathbb{S}^n \rightarrow \mathbb{R}$ , is there a closed, smooth, strictly convex hypersurface whose  $k$ -th elementary symmetric function of the principal curvatures,  $S_k$ , as a function of the unit normal vector, is  $f$ ?

The  $L_p$  version of this problem is stated as follows.

**Question 6.2.** Given a smooth function  $f : \mathbb{S}^n \rightarrow (0, \infty)$ , is there a closed, smooth, strictly convex hypersurface with the support function  $s$  such that, for some constant  $c$ ,

$$(6.1) \quad f s^{p-1} S_k = c \quad ?$$

In this section, we give the solutions to these two problems for the case  $1 \leq p < k + 1$  and provided  $f$  is even:

**Theorem 6.3.** [HI23, GG02] *Let  $1 \leq p < k + 1$ ,  $1 \leq k < n$ , and  $\ell \geq 2$ . Let  $f \in C^\ell(\mathbb{S}^n)$  be a positive, even function. Then there exists an origin-symmetric,  $C^{\ell+1, \alpha}$ -smooth (for all  $0 < \alpha < 1$ ), strictly convex hypersurface with the support function  $s$  such that*

$$f s^{p-1} S_k = 1.$$

To do so, we need to establish the  $C^0$  and  $C^2$  a priori estimates (our method from the previous section also works here) and apply a degree-theoretic argument; the continuity method does not work.

## 7. OPEN QUESTIONS

**Question 7.1.** Is there a version of [Theorem 5.5](#) for  $p < 1$  without any additional or very restrictive assumptions on  $f$ ? An application, for example, would be an extension of [Theorem 5.1](#) to a range  $p < 1$ .

**Question 7.2.** The proof of [Lemma 5.3](#) does not use any particular structure of  $q = \sigma_k$ , except for the homogeneity (i.e. it applies to any curvature function, whether it is concave or not). Is there a version of this gradient estimate (analytic and geometric) that works for some range of  $1 - k < p < 1$ ? An application, for example, would be an extension of [Theorem 6.3](#) to a range  $1 - k < p < 1$ .

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