LECTURES ON CURVATURE EQUATIONS

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Abstract. The Minkowski problem aims to reconstruct a convex body from its surface area measure. The solution to this problem is remarkable: a Borel measure μ on the unit sphere is the surface area measure of a convex body if and only if μ has its centroid at the origin and is not concentrated on a great subsphere. In this course, we focus on the smooth version of this problem and its close relatives. The aim is to provide the key components of the arguments for addressing the existence theory of an important class of curvature problems, the L_p-Minkowski and Christoffel-Minkowski problems.

Contents

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1. Background material

1.1. Geometry of hypersurfaces.

Definition 1.1. A subset M of $\mathbb{R}^{n+1}$ ($(n+1)$-dimensional Euclidean space) is said to be an n-dimensional submanifold (an embedded hypersurface), if for every point $p \in M$ there exists a neighbourhood O of p in $\mathbb{R}^{n+1}$, and an open set $\Omega \subset \mathbb{R}^n$, and a smooth map $X : \Omega \to \mathbb{R}^{n+1}$ such that X is a homeomorphism onto $M \cap O$, and $d_x X$ is injective for every $x \in \Omega$.

Remark 1.2. The homeomorphism on $M \cap O$ means that X is continuous, one-to-one, it maps Ω onto $M \cap O$, and its inverse is also continuous. Note that the continuity is interpreted with respect to the subspace topology on M induced from the inclusion into $\mathbb{R}^{n+1}$. Since open sets in the sub-topology are given by restriction of open sets in $\mathbb{R}^{n+1}$, this is equivalent to the statement that for every open set $\Omega' \subset \Omega$, there exists an open set $O' \subset \mathbb{R}^{n+1}$ such that $X(\Omega') = M \cap O'$.

Proposition 1.3. *Let M be a subset of $\mathbb{R}^{n+1}$. Then the following statements are equivalent.*

1. *M is an n-dimensional hypersurface;*
2. *M is an n-dimensional manifold, and can be given a differentiable structure in such a way that the inclusion map $i : M \to \mathbb{R}^{n+1}$ is an embedding;*
3. *M is locally the graph of a smooth function: For every $p \in M$, there exists an open set $O \subset \mathbb{R}^{n+1}$ containing p, an open set $\Omega \subset \mathbb{R}^n$, and a smooth map $\varphi : \Omega \to \mathbb{R}$ such that*

$$M \cap O = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; \quad x_{n+1} = \varphi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \Omega\}$$

In the remainder of this section, $\mathcal{M}^n$ denotes a (an arbitrary) hypersurface of $\mathbb{R}^{n+1}$. Later it will denote a "convex or strictly convex hypersurface"; see subsection 1.2.

Definition 1.4. Let $k \geq 1$. We say $\mathcal{M}^n$ is an embedded, C^k-smooth hypersurface of $\mathbb{R}^{n+1}$ if it is locally the (graphical) image of a C^k-smooth map:

$$X(x) = (X^1(x), \ldots, X^{n+1}(x)) = (x, \varphi(x)) : \Omega \to X(\Omega) \subset \mathbb{R}^{n+1}, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ and $\varphi : \Omega \to \mathbb{R}$ is a C^k-smooth function.

Smooth curves on $X(\Omega)$ are curves of the form $\gamma(t) = X(x(t))$, where the mapping $t \mapsto x(t)$ is a smooth curve lying in the domain Ω. Curves of the form

$$t \mapsto X(x_1, x_2, \ldots, x_{i-1}, t, x_{i+1}, x_n),$$

where x_i are fixed numbers, are called coordinate lines. The speed vectors of the coordinate lines are denoted by $\partial_i X$.

By definition, the vectors $\partial_i X$ are linearly independent for any $x \in \Omega$. The tangent plane of $\mathcal{M}^n$ at the point $p = X(x)$ is the (hyper)plane through p spanned by the vectors $\{\partial_i X\}_i$. We say v is a tangent vector to $X(\Omega)$ at p if $v \in \operatorname{span}\{\partial_1 X|_x, \ldots, \partial_n X|_x\}$.

A unit normal vector of the hypersurface at the point $p = X(x)$ is defined as a unit normal of the tangent plane. The cross product of vectors $\partial_i X$ is defined by

$$*(\partial_1 X \wedge \partial_2 X \wedge \cdots \wedge \partial_n X) = \det \begin{vmatrix} E_1 & E_2 & \cdots & E_{n+1} \\ \partial_1 X^1 & \partial_1 X^2 & \cdots & \partial_1 X^{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \partial_n X^1 & \partial_n X^2 & \cdots & \partial_n X^{n+1} \end{vmatrix}$$

Here $\{E_i\}_i$ is the standard coordinates basis of $\mathbb{R}^{n+1}$. Then, we may choose

$$N(x) := (-1)^n \frac{*(\partial_1 X \wedge \partial_2 X \wedge \cdots \wedge \partial_n X)}{\|*(\partial_1 X \wedge \partial_2 X \cdots \wedge \partial_n X)\|}.$$

A vector field <u>along</u> X is a mapping $v : \Omega \to T\mathbb{R}^{n+1}$ such that $v(x) \in T_{X(x)}\mathbb{R}^{n+1} (\sim \mathbb{R}^{n+1})$ for any $x \in \Omega$.

Let v be a vector field along X, $x_0 \in \Omega$ and u a tangent vector of $\mathcal{M}^n$ at $X(x_0)$. We define the (Euclidean covariant) derivative $D_u v$ as $D_u v = (v \circ \gamma)'(0)$, where $\gamma : [-1, 1] \to \Omega$ is a curve in Ω such that $\gamma(0) = x_0$ and $(X \circ \gamma)'(0) = u$. The derivative of N with respect to u at x_0 can be similarly defined and, in fact, is tangent to $\mathcal{M}^n$ at $X(x_0)$.

Let us denote by $T_p\mathcal{M}^n$ the tangent space at p. Moreover, the tangent bundle is defined as $T\mathcal{M}^n = \bigcup_{p \in \mathcal{M}^n} T_p\mathcal{M}^n$. The (induced) first fundamental form (namely the induced metric) of $\mathcal{M}^n$ at p is defined by

$$\mathrm{I}_p(u, v) = \langle u, v \rangle, \quad \forall u, v \in T_p\mathcal{M}^n.$$

This is a positive-definite symmetric bilinear map.

The linear map

$$A : T_p\mathcal{M}^n \to T_p\mathcal{M}^n, \quad v \mapsto D_v N$$

is called the <u>Weingarten map</u> or the shape operator at p. It can be shown that A_p is self-adjoint. The <u>second fundamental form</u> is defined as

$$\mathrm{II}_p(u, v) = \langle A_p(u), v \rangle.$$

The eigenvalues of A_p are called the <u>principal curvatures</u> of $\mathcal{M}^n$ at p. In particular, the determinant of A is the Gauss(ian) curvature, and its trace is the mean curvature.

We denote by g_{ij} and h_{ij} the components of I, II :

$$\begin{aligned} g_{ij} &= I(\partial_i X, \partial_j X) = \langle \partial_i X, \partial_j X \rangle \\ h_{ij} &= \langle A(\partial_i X), \partial_j X \rangle = \langle \partial_i N, \partial_j X \rangle = -\langle N, \partial^2_{ij} X \rangle, \end{aligned}$$

where we used $\langle N, \partial_i X \rangle = 0$. Note that

$$h_{ij} = \langle A(\partial_i X), \partial_j X \rangle = \sum_{k=1}^n \langle A_i^k \partial_k X, \partial_j X \rangle = \sum_k A_i^k g_{kj}.$$

Therefore, the eigenvalues of $[h_{ij}]$ with respect to the metric g (i.e. solutions of the equation $\det(h_{ij} - \lambda g_{ij}) = 0$, i.e. eigenvalues of the matrix $[h_{ik} g^{kj}]$) are the principal curvatures.

Here, $[g^{ij}]$ denotes the inverse matrix of $[g_{ij}]$. Note that the Gauss curvature is given by

$$\det A = \frac{\det \mathrm{II}}{\det g}.$$

Remark 1.5. Throughout this lecture note, we use the Einstein summation convention (repeated indices are implicitly summed over); e.g.

$$A_i^k g_{kj} = \sum_k A_i^k g_{kj}, \quad h_{ik} g^{kj} = \sum_k h_{ik} g^{kj}.$$

The Christoffel symbols are <u>defined</u> as the tangential components of $\partial^2 X$:

$$\partial^2_{ij} X = \Gamma^k_{ij} \partial_k X - h_{ij} N.$$

Christoffel symbols are related to the metric and its partial derivatives. Note that

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \quad \langle \partial^2_{ij} X, \partial_l X \rangle = \Gamma^k_{ij} g_{kl}.$$

For brevity write Γ_{ijl} for $\Gamma^k_{ij} g_{kl}$. We have

$$\sum_l \Gamma_{ijl} g^{lk} = \Gamma^k_{ij}.$$

Denote the (partial) derivative of g_{ij} with respect to k-th variable by $g_{ij,k}$. We have

$$\begin{aligned} g_{ij,k} =& \langle \partial^2_{ik} X, \partial_j X \rangle + \langle \partial_i X, \partial^2_{jk} X \rangle = \Gamma_{ikj} + \Gamma_{jki} \\ g_{jk,i} =& \langle \partial^2_{ji} X, \partial_k X \rangle + \langle \partial_j X, \partial^2_{ki} X \rangle = \Gamma_{jik} + \Gamma_{kij} \\ g_{ki,j} =& \langle \partial^2_{kj} X, \partial_i X \rangle + \langle \partial_k X, \partial^2_{ij} X \rangle = \Gamma_{kji} + \Gamma_{ijk}. \end{aligned}$$

Solving this linear system of equations, we obtain

$$\Gamma_{ijk} = \frac{1}{2} \left(g_{ik,j} + g_{jk,l} - g_{ij,k} \right)$$

and

$$\Gamma^k_{ij} = \frac{1}{2} g^{lk} \left(g_{il,j} + g_{jl,i} - g_{ij,l} \right). \tag{1.1}$$

Next, we derive the Gauss and Codazzi equations. We calculate

$$\begin{aligned} \partial^3_{ijk} X =& \partial_k \left(\Gamma^l_{ij} \partial_l X - h_{ij} N \right) \\ =& \Gamma^l_{ij,k} \partial_l X + \Gamma^l_{ij} \partial^2_{lk} X - h_{ij,k} N - h_{ij} \partial_k N \\ =& \Gamma^l_{ij,k} \partial_l X + \Gamma^l_{ij} \left(\Gamma^s_{lk} \partial_s X - h_{lk} N \right) - h_{ij,k} N - h_{ij} h_{ks} g^{sl} \partial_l X \\ =& \left(\Gamma^l_{ij,k} + \Gamma^s_{ij} \Gamma^l_{sk} - h_{ij} h_{ks} g^{sl} \right) \partial_l X - (h_{ij,k} + \Gamma^l_{ij} h_{lk}) N. \end{aligned}$$

Comparing the coefficients of $\partial_l X$ in $\partial^3_{ijk} X$ and $\partial^3_{ikj} X$, we obtain the Gauss equation:

$$\Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^s_{ik} \Gamma^l_{sj} - \Gamma^s_{ij} \Gamma^l_{sk} = (h_{ik} h_{js} - h_{ij} h_{ks}) g^{sl}.$$

Comparing the normal components gives the Codazzi equation:

$$h_{ij,k} - h_{ik,j} = \Gamma^l_{ik} h_{lj} - \Gamma^l_{ij} h_{lk}. \tag{1.2}$$

Denote the right-hand side of Gauss' equation by Rm^l_{kji} :

$$\mathrm{Rm}^l_{kji} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^s_{ik}\Gamma^l_{sj} - \Gamma^s_{ij}\Gamma^l_{sk}. \tag{1.3}$$

Hence $\mathrm{R}_{kjil} := \mathrm{Rm}^m_{kji}\, g_{ml}$ satisfies the Gauss equation:

$$\mathrm{R}_{kjil} = h_{ik}h_{jl} - h_{ij}h_{lk}. \tag{1.4}$$

Recall that a (k,ℓ)-tensor over a linear space V is a multilinear function

$$T : V \times \cdots \times V \times V^* \times \cdots \times V^* \to \mathbb{R},$$

where we have k copies of V and ℓ copies of the dual space V^*. Here V^* is the dual space; that is, the vector space of linear functions $\omega : V \to \mathbb{R}$ (referred to as covectors).

Let $\{e_i\}$ be a basis of V, and $\{e^i\}$ be the corresponding dual basis of V^* (dual basis means $e^j(e_i) = \delta_{ij}$). The components of T (with respect to these bases) are

$$T^{j_i \cdots j_l}_{i_1 \cdots i_k} = T(e_{i_1}, \ldots, e_{i_k}; e^{j_1}, \ldots, e^{j_l}).$$

Given $p \in \mathcal{M}^n$, let $T^*_p\mathcal{M}^n$ be the dual vector space of $T_p\mathcal{M}^n$. The cotangent bundle is then defined as $T^*\mathcal{M}^n = \bigcup_{p \in \mathcal{M}^n} T^*_p\mathcal{M}^n$. A tensor field of type (k,ℓ) over $T\mathcal{M}^n$ is a mapping T that assigns to every point $p \in \mathcal{M}^n$ a tensor of type (k,ℓ) over $T_p\mathcal{M}^n$. We write $\Gamma^\ell_k(\mathcal{M}^n)$ for (k,ℓ)-tensor fields.

Let us denote the dual basis corresponding to $\{\partial_i := \partial_i X\}$ by $\{dx^i\}$. We say T is a smooth tensor field if its (local) components in these bases are smooth (and hence in any other bases). A $(0,1)$-tensor field is called a vector field. A $(1,0)$-tensor field is called a differential form (or a 1-form). For example, the first and second fundamental forms are $(2,0)$ tensors.

In local coordinates, we may write the metric as

$$g = g_{ij} dx^i \otimes dx^j,$$

where $g_{ij} := g(\partial_i, \partial_j)$. In general, a (k,ℓ)-tensor field α can be written as

$$\alpha = \alpha^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}},$$

where

$$\alpha^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} := \alpha(\partial_{i_1}, \partial_{i_2}, \ldots, \partial_{i_k}; dx^{j_1}, \ldots, dx^{j_\ell})$$

are the components of α.

Next, we will discuss the covariant derivative on $\mathcal{M}^n$. Recall that the differentiation of two vector fields v, w in $\mathbb{R}^{n+1}$ is defined as

$$D_v w|_p = (w \circ \gamma)'(0),$$

where $\gamma : [-1,1] \to \Omega \subset \mathbb{R}^{n+1}$ with $\gamma(0) = p$, $\gamma'(0) = v(0)$. Note that by the chain rule,

$$D_v w = (D_v w^i) E_i = v(w^i) E_i,$$

where w^i are the components of the vector field w in the basis $\{E_i\}_i$.

Let c be a constant and $f : \Omega \to \mathbb{R}$ be a smooth function. The vector field differentiation satisfies:

$$\begin{aligned} D_{v_1+v_2}w &= D_{v_1}w + D_{v_2}w \\ D_{cv}w &= c\partial_v w \\ D_v(w_1 + w_2) &= D_v w_1 + D_v w_2 \\ D_v(fw) &= v(f)w + fD_v w \\ D_{w_1}w_2 - D_{w_2}w_1 &= [w_1, w_2] \\ D_v\langle w_1, w_2\rangle &= \langle D_v w_1, w_2\rangle + \langle w_1, D_v w_2\rangle. \end{aligned}$$

Here, $[w_1, w_2]$ denotes the (Euclidean) Lie bracket of the vector fields w_1 and w_2.

Definition 1.6. A connection on $\mathcal{M}^n$ is a mapping that assigns to any two smooth (tangential) vector fields v, w a (tangential) vector field $\nabla_v w$ satisfying the following rules:

$$\begin{aligned} \nabla_{v_1+v_2}w &= \nabla_{v_1}w + \nabla_{v_2}w \\ \nabla_{cv}w &= c\nabla_v w \\ \nabla_v(w_1 + w_2) &= \nabla_v w_1 + \nabla_v w_2 \\ \nabla_v(fw) &= v(f)w + f\nabla_v w. \end{aligned}$$

We say a connection ∇ is symmetric or torsion-free if

$$\nabla_v w - \nabla_w v = (v(w^i) - w(v^i))\partial_i,$$

where $v = v^i\partial_i$ and $w = w^i\partial_i$. Define $[v, w] = \sum_{i=1}^n (v(w^i) - w(v^i))\partial_i$; the Lie bracket.

We say a connection ∇ is compatible with the (induced) metric on $\mathcal{M}^n$ if

$$v\langle w_1, w_2\rangle = \langle \nabla_v w_1, w_2\rangle + \langle w_1, \nabla_v w_2\rangle.$$

Using the Christoffel symbols defined in (1.1), we may define a connection on $\mathcal{M}^n$ as follows. For vector fields $v = v^i\partial_i$ and $w = w^i\partial_i$, <u>define</u>

$$\nabla_v w = \left(v(w^k) + v^i w^j \Gamma_{ij}^k\right)\partial_k,$$

where Γ_{ij}^k are the Christoffel symbols defined earlier. It is then easy to see that this connection is torsion-free and compatible with the metric. The fundamental theorem of Riemannian geometry states any other connection on $\mathcal{M}^n$ which is both symmetric and compatible with the induced metric g is the one defined above. This connection is called the <u>Levi-Civita connection</u> of g.

Now observe that since

$$D_{\partial_j}\partial_i X = \partial_{ij}^2 X = \Gamma_{ij}^k \partial_k X - h_{ij}N = dX(\nabla_{\partial_j}\partial_i) - h(\partial_i, \partial_j)N,$$

we obtain for any vector fields v, w:

$$D_v w = \nabla_v w - \mathrm{II}(v, w)N.$$

This provides a geometric interpretation of covariant derivatives on $\mathcal{M}^n$:

$$\nabla_v w = (D_v w)^\top.$$

The covariant differentiation can be extended so that it can act on (k,ℓ)-tensor fields. Let $\alpha \in \Gamma^k_\ell(\mathcal{M}^n)$ and $u \in T\mathcal{M}^n$. Then

$$\begin{aligned}(\nabla_u \alpha)(\omega^1, \dots, \omega^\ell, u_1, \dots, u_k) := {} & u(\alpha(\omega^1, \dots, \omega^\ell, u_1, \dots, u_k)) \\ & - \sum_i \alpha(\omega_1, \dots, \nabla_u \omega^i, \dots, \omega^\ell, u_1, \dots, u_k) \\ & - \sum_j \alpha(\omega_1, \dots, \omega^\ell, u_1, \dots, \nabla_u u_j, \dots, u_k).\end{aligned}$$

Here ω^i are 1-forms, and u_i are tangent vectors. For example,

$$\nabla_{\partial_j} \partial_i = \Gamma^k_{ij} \partial_k$$

and

$$\begin{aligned}\left(\nabla_{\partial_i} dx^j\right)(\partial_k) &= \partial_i(\delta^j_k) - dx^j(\nabla_{\partial_i} \partial_k) \\ &= -dx^j(\Gamma^\ell_{ik} \partial_\ell) \\ &= -\Gamma^j_{ik}.\end{aligned}$$

Remark 1.7. Very often, it is convenient to write $\nabla_i \alpha = \nabla_{\partial_i} \alpha$.

The second derivative of $\alpha \in \Gamma^k_\ell(\mathcal{M}^n)$, $\nabla^2_{u,v}\alpha \in \Gamma^k_\ell(\mathcal{M}^n)$ for $u, v \in T\mathcal{M}^n$, is defined by

$$\nabla^2_{u,v} \alpha = \nabla_u(\nabla_v \alpha) - \nabla_{\nabla_u v} \alpha. \tag{1.5}$$

We can rewrite the Codazzi equation (1.2) using the ∇ notation as follows:

$$\nabla_i h_{jk} = \nabla_j h_{ik}. \tag{1.6}$$

Note that here $\nabla_i h_{jk} = (\nabla_{\partial_i} h)(\partial_j, \partial_k)$ and $\nabla_j h_{ik} = \nabla_{\partial_j} h(\partial_i, \partial_k)$.
In terms of ∇, we can rewrite Rm as follows:

$$\begin{aligned}\mathrm{Rm}(u,v)w &= \nabla_v(\nabla_u w) - \nabla_u(\nabla_v w) + \nabla_{[u,v]} w \\ &= \nabla^2_{v,u} w - \nabla^2_{u,v} w.\end{aligned} \tag{1.7}$$

The corresponding $(4,0)$-tensor, the *Riemann* curvature, is also given by

$$\mathrm{R}(u,v,w,z) = g(\mathrm{Rm}(u,v)w, z).$$

The Riemannian curvature is a $(4,0)$-tensor. In the literature, some define the Riemann curvature as the opposite sign of ours.

Exercise 1.8. Verify (1.2) and (1.6) are in fact the same. Similarly, show that (1.3) and (1.7) are the same.

Let $u, v \in T\mathcal{M}^n$ and $\alpha \in \Gamma^k_\ell(\mathcal{M}^n)$ be a (k,ℓ)-tensor field. We define

$$\operatorname{Hess}\alpha(u,v) = \nabla^2_{u,v}\alpha = \nabla_u(\nabla_v\alpha) - \nabla_{\nabla_u v}\alpha.$$

We can extend the definition of Rm so that it can act on (k,ℓ)-tensor fields as well:

$$\operatorname{Rm}(u,v)\alpha = \nabla^2_{v,u}\alpha - \nabla^2_{u,v}\alpha = \operatorname{Hess}\alpha(v,u) - \operatorname{Hess}\alpha(u,v).$$

For any $(2,0)$-tensor field α and $u, v, w, z \in T\mathcal{M}^n$ we have

$$(\operatorname{Rm}(u,v)\alpha)(w,z) = -\alpha(\operatorname{Rm}(u,v)w, z) - \alpha(w, \operatorname{Rm}(u,v)z). \tag{1.8}$$

To verify this formula, we may extend u, v, w, z to vector fields so that all covariant derivatives vanish at a given point $p \in \mathcal{M}^n$. By definition of $\nabla^2_{u,v}$ (cf. (1.5)),

$$\begin{aligned}(\nabla^2_{v,u}\alpha)(w,z) &= (\nabla_v(\nabla_u\alpha))(w,z)\\ &= \nabla_v\left(u(\alpha(w,z)) - \alpha(\nabla_u w, z) - \alpha(w, \nabla_u z)\right)\\ &= v(u(\alpha(w,z))) - \alpha(\nabla^2_{v,u}w, z) - \alpha(w, \nabla^2_{v,u}z).\end{aligned}$$

Therefore,

$$\begin{aligned}(\nabla^2_{v,u}\alpha)(w,z) - (\nabla^2_{u,v}\alpha)(w,z) = {} & [v,u](\alpha(w,z))\\ & + \alpha(\nabla^2_{u,v}w, z) + \alpha(w, \nabla^2_{u,v}z)\\ & - \alpha(\nabla^2_{v,u}w, z) - \alpha(w, \nabla^2_{v,u}z).\end{aligned}$$

In view of $[u,v]f = \operatorname{Hess} f(u,v) - \operatorname{Hess} f(v,u) = 0$, the proof is complete.

Given a symmetric $(2,0)$-tensor field α, $\alpha^\sharp := g^*\alpha$ denotes a (1,1)-tensor field that is implicitly defined by

$$g(\alpha^\sharp(u), v) = \alpha(u,v).$$

In local coordinate, the components of $\alpha^\sharp$, α^j_i, are given by

$$\alpha^j_i = \alpha_{ik}g^{kj}.$$

To put it simply, $\sharp$ is the index-raising operator; e.g. $A = \mathrm{II}^\sharp$.

Exercise 1.9. Prove Simon's identity:

$$\nabla^2_{i,j}h_{kl} - \nabla^2_{k,l}h_{ij} = h_{ij}h^2_{kl} - h_{jk}h^2_{il} + h_{il}h^2_{jk} - h_{kl}h^2_{ij}.$$

Note that here, e.g. $\nabla^2_{i,j}h_{kl} = (\nabla^2_{\partial_i,\partial_j}h)(\partial_k, \partial_l)$.

Let u, v, w, z be vector fields so that all covariant derivatives vanish at a given point $p \in \mathcal{M}^n$. Using (1.8), we calculate

$$\begin{aligned}
(\nabla^2_{u,v}h)(w,z) &= \nabla_u(\nabla_v h)(w,z)\\
&= \nabla_u((\nabla_v h)(w,z))\\
&= \nabla_u((\nabla_w h)(v,z))\\
&= \nabla_u(\nabla_w h)(v,z)\\
&= (\nabla^2_{u,w}h)(v,z)\\
&= (\nabla^2_{w,u}h)(v,z) + (\mathrm{Rm}(w,u)h)(v,z)\\
&= (\nabla^2_{w,u}h)(v,z) - h(\mathrm{Rm}(w,u)v,z) - h(v,\mathrm{Rm}(w,u)z)\\
&= (\nabla^2_{w,z}h)(u,v) - h(\mathrm{Rm}(w,u)v,z) - h(v,\mathrm{Rm}(w,u)z).
\end{aligned}$$

Now, the claim follows from the Gauss equation (1.4).

Example 1.10. Let $\mathcal{M}^n$ be a smooth hypersurface $\mathcal{M}^n$, locally represented as the graph of a function $\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. That is, $\mathcal{M}^n$ is locally parameterized by $X(x) = (x, \varphi(x))$. Therefore, the tangent space is spanned by

$$\partial_i X = (E_i, \partial_i \varphi),$$

where $\{E_i\}$ is the standard basis of $\mathbb{R}^n$. The induced metric, unit *upward* normal vector as well as the second fundamental form is given by

$$g_{ij} = \delta_{ij} + \partial_i\varphi\partial_j\varphi, \quad N = \frac{(-D\varphi, 1)}{\sqrt{1+|D\varphi|^2}}.$$

Recall that by definition

$$h_{ij} = -\langle \partial^2_{ij}X, N\rangle = -\frac{\partial^2_{ij}\varphi}{\sqrt{1+|D\varphi|^2}}.$$

Moreover, we have

$$g^{ij} = \delta_{ij} - \frac{\partial_i\varphi\partial_j\varphi}{1+|D\varphi|^2}.$$

Exercise 1.11. Find a local parametrization of the unit sphere $\mathbb{S}^n$, and calculate its second fundamental form and Riemannian curvature.

1.2. **Convex geometry.**

The main reference of this section is [Sch14].

Definition 1.12. A non-empty, compact, convex set is called a convex body. (Note that with this definition a convex body need not have interior points). The boundary of a convex body with non-empty interior is called a convex hypersurface (which may not be smooth). Moreover, we say the hypersurface is strictly convex if *it does not contain any line segment.*

Definition 1.13. Let K be a convex body. The support function of K is defined by

$$s(x) = \max_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^{n+1}.$$

Remark 1.14. From the definition of the support function, it is clear that s is a sublinear function (i.e. one-homogeneous and convex). By [Sch14, Thm 1.7.1], a sublinear function uniquely determines a convex body whose support function is s. Let $H(K, x)$, $x \in \mathbb{S}^n$, denote the supporting hyperplane to K with the normal x. Then $s(x) = \langle y, x \rangle$, $y \in K \cap H(K, x)$.

We say a convex body is C^k if it has *non-empty interior* and its boundary in the sense of Definition 1.4 is a C^k-smooth hypersurface of $\mathbb{R}^{n+1}$. We say a convex body is C^k_+ if it is C^k and its boundary is strictly convex.

Lemma 1.15. *Let K be a convex body with support function s. Then s is differentiable at $x \in \mathbb{S}^n$ if and only if there is a unique point $y \in \partial K$ whose outer unit normal is x. In this case, we have $Ds(x) = y$.*

Proof. See [Sch14, Cor. 1.7.3]. $\square$

Remark 1.16. In view of Lemma 1.15, K is strictly convex if and only if s is C^1.

Remark 1.17. If K is $C^{k,+}$, $k \geq 2$, then the second fundamental form of ∂K is positive-definite. Moreover, if K is of class C^k, $k \geq 2$, and its second fundamental form is positive-definite, then K it is strictly convex and hence of class $C^{k,+}$.

Since the support function is a one-homogenous function, we may consider it as a function on the unit sphere. By Lemma 1.15, if K is C^1_+, then

$$s(x) = \langle x, N^{-1}(x) \rangle, \quad Ds(x) = N^{-1}(x), \quad \forall x \in \mathbb{S}^n$$

where N is the *outer* unit normal of $\mathcal{M}^n := \partial K$.

Let us denote the standard metric of $\mathbb{S}^n$ and its Levi-Civita connection by g, ∇ (i.e. the induced structure from $\mathbb{R}^{n+1}$). Consider a differentiable function $f : \mathbb{S}^n \to \mathbb{R}$ and its one-homogeneous extension $\bar{f} : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ defined by

$$\bar{f}(x) = |x| f(\frac{x}{|x|}).$$

Then for $x \in \mathbb{S}^n$ we have $D\bar{f}(x) = f(x)x + \nabla f(x)$. In particular, when K is C^1_+ we have

$$N^{-1}(x) = s(x)x + \nabla s(x) \tag{1.9}$$

Exercise 1.18. Show that $D\bar{f}(x) = \nabla f(x) + f(x)x$ for all $x \in \mathbb{S}^n$, and

$$D^2 \bar{f}|_{T\mathbb{S}^n} = \mathrm{Hess}_{\mathbb{S}^n} f + gf, \quad \Delta_{\mathbb{R}^{n+1}} \bar{f} = \Delta_{\mathbb{S}^n} f + nf.$$

Hint: $(D\bar{f})^\top$, the tangential component, is ∇f.

Lemma 1.19. *Assume K is a C^2_+ convex body with support function s. Then the eigenvalues of $B[s]^i_j := g^{ik} \nabla^2_{j,k} s + \delta_{ij} s$ are the principal radii of curvature of $\mathcal{M}^n = \partial K$.*

Proof. Since ∂K is strictly convex and of class C^2, $N:\mathcal{M}^n\to\mathbb{S}^n$ (the outer unit normal) is a diffeomorphism of class C^1; see [Sch14, Lem. 2.2.12] and [Lee13, Prop. 4.22]. Hence by (1.9), s is C^2. Let us put $Y=N^{-1}:\mathbb{S}^n\to\mathcal{M}^n$ for the Gauss map (a.k.a. the Gauss map parameterization). Let $\varphi:U\subset\mathbb{R}^n\to\mathbb{S}^n$ be a local parametrization of $\mathbb{S}^n$. Hence $X=Y\circ\varphi:U\subset\mathbb{R}^n\to\mathcal{M}^n$ is a local parametrization of $\mathcal{M}^n$. Suppose $\{\partial_i\varphi\}_i$ is a an orthonormal frame on $\mathbb{S}^n$ (we might need to rotate the coordinates of $\mathbb{R}^n$). In this parametrization,

$$s(x)=\langle Y\circ\varphi(x),\varphi(x)\rangle.$$

We calculate

$$\partial_i s=\langle\partial_i X,\varphi\rangle+\langle X,\partial_i\varphi\rangle=\langle X,\partial_i\varphi\rangle$$

Differentiating this once more yields

$$\nabla^2_{i,j}s=\partial^2_{ij}s-\Gamma^k_{ij}\partial_k s=\langle X,\partial^2_{ij}\varphi-\Gamma^k_{ij}\partial_k\varphi\rangle+\langle\partial_j X,\partial_i\varphi\rangle=\langle X,\nabla^2_{i,j}\varphi\rangle+h_{ij},$$

where h_{ij} is the second fundamental form of $\mathcal{M}^n$ and

$$\Gamma^k_{ij}=\langle D_{\partial_j}\partial_i\varphi,\partial_k\varphi\rangle=\langle\partial^2_{ij}\varphi,\partial_k\varphi\rangle$$

are the Christoffel symbols of $(\mathbb{S}^n,\nabla,g)$.

Next using $\langle\varphi,\partial_i\varphi\rangle=0$, we compute

$$\langle\varphi,\nabla^2_{i,j}\varphi\rangle=\langle\varphi,\partial^2_{ij}\varphi\rangle=-\langle\partial_i\varphi,\partial_j\varphi\rangle=-\delta_{ij}.$$

Therefore, the second fundamental (in a <u>local orthonormal frame</u> of $\mathbb{S}^n$) is given by

$$h_{ij}=\nabla^2_{i,j}s+\delta_{ij}s.$$

Let us denote the induced metric on $\mathcal{M}^n$ by $\bar{g}$. We have

$$\partial_i\varphi=h_{ik}\bar{g}^{kl}\partial_l X$$

and thus

$$\delta_{ij}=\langle\partial_i\varphi,\partial_i\varphi\rangle=h_{ik}\bar{g}^{kl}h_{jm}\bar{g}^{ms}\langle\partial_l X,\partial_s X\rangle=h_{ik}h_{jl}\bar{g}^{kl}=\sum_k B[s]_{ik}h^k_j.$$

Hence, the eigenvalues of $B[s]$ are the principal radii of curvature. ☐

Theorem 1.20. *Suppose $k\geq 2$. Then K is C^k_+ if and only if its support function s is C^k and the eigenvalues of $h_{ij}=\nabla^2_{i,j}s+g_{ij}s$ with respect to g are all positive.*

Proof. From our discussion above, it is clear that if K is C^k_+, then the support function s is C^k, and the eigenvalues of $h_{ij}=\nabla^2_{i,j}s+g_{ij}s$ with respect to g are all positive. Now, we prove the other direction of the claim. Since s is differentiable, by Lemma 1.15, $\mathcal{M}$ is strictly convex. We need to show that $\mathcal{M}$ is of class C^k.

Let us translate K so that it encloses the origin in its interior. The polar body of K, denoted by K°, is a convex body defined by

$$K^\circ=\{x:\langle x,y\rangle\leq 1,\ \forall y\in K\}.$$

The radial function of K° is defined by

$$\rho^\circ(u) = \max\{\lambda \geq 0 : \lambda u \in K^\circ\}, \quad u \in \mathbb{S}^n.$$

By [Sch14, Lem. 1.7.14], $\rho^\circ = s^{-1} : \mathbb{S}^n \to \mathbb{R}$. Hence, the mapping

$$Z : \mathbb{S}^n \to \mathbb{R}^n,\ u \mapsto \rho^\circ(u)u$$

is a C^k-smooth, injective immersion. By [Lee13, Prop. 4.22], Z is, in fact, an embedding (with ∂K° inheriting the subspace topology from $\mathbb{R}^n$). Therefore, $\partial K^\circ = Z(\mathbb{S}^n)$ is a C^k-smooth hypersurface; cf. [Lee13, Prop. 5.2].

To see that K° is strictly convex, note that the metric and the second fundamental form of K° in this parameterization are given by

$$\bar{g}^\circ_{ij} = (\rho^\circ)^2 g_{ij} + \nabla_i\rho^\circ\nabla_j\rho^\circ, \tag{1.10}$$

$$\begin{aligned} h^\circ_{ij} &= \frac{1}{\sqrt{\rho^{\circ 2} + |\nabla\rho^\circ|^2}}\left(-\rho^\circ\nabla^2_{i,j}\rho^\circ + 2\nabla_i\rho^\circ\nabla_j\rho^\circ + \rho^{\circ 2}g_{ij}\right) \\ &= \frac{\rho^{\circ 3}}{\sqrt{\rho^{\circ 2} + |\nabla\rho^\circ|^2}}\left(\nabla^2_{i,j}s + sg_{ij}\right) \\ &= \frac{\rho^{\circ 3}}{\sqrt{\rho^{\circ 2} + |\nabla\rho^\circ|^2}}h_{ij}. \end{aligned} \tag{1.11}$$

Since the eigenvalues of h with respect to g are all positive, by Exercise 1.21, the eigenvalues of h°_{ij} with respect to $\bar{g}^\circ_{ij}$ are all positive as well. This implies that K° is strictly convex.

Therefore, K° is C^k_+, and its support function is C^k. Repeating this argument with K in place of K°, we see that $\mathcal{M}$ is the image of the C^k-smooth embedding $u \mapsto u/s^\circ(u)$, where s° is the support function of K°. Thus, it is of class C^k_+. □

Exercise 1.21. Show that the eigenvalues of h°_{ij} with respect to $\bar{g}^\circ_{ij}$ are all positive.

Proof. We have

$$[h_{ij}][\bar{g}^{\circ ij}] = \frac{\sqrt{\rho^{\circ 2} + |\nabla\rho^\circ|^2}}{\rho^{\circ 3}}[h^\circ_{ik}\bar{g}^{\circ kj}],$$

where

$$\bar{g}^{\circ ij} = \frac{1}{\rho^{\circ 2}}\left(g^{ij} - \frac{\nabla^i\rho^\circ\nabla^j\rho^\circ}{\rho^{\circ 2} + |\nabla\rho^\circ|^2}\right).$$

To prove the claim, we only need to consider points for which the gradient of the radial function ρ° does not vanish. Around such a point, we introduce an orthonormal frame $\{e_i\}$ of $\mathbb{S}^n$ such that $e_1 = \frac{\nabla\rho^\circ}{|\nabla\rho^\circ|}$. Then $\nabla\rho^\circ = (|\nabla\rho^\circ|, 0, \ldots, 0)$. Thus, in such a frame, we may express $[h_{ij}][\bar{g}^{\circ ij}]$ as follows:

$$\frac{\sqrt{\rho^{\circ 2} + |\nabla\rho^\circ|^2}}{\rho^{\circ 3}}[h^\circ_{ij}\bar{g}^{\circ ij}] = [h_{ij}]\begin{pmatrix} \frac{1}{\rho^{\circ 2} + |\nabla\rho^\circ|^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\rho^{\circ 2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\rho^{\circ 2}} \end{pmatrix}.$$

The eigenvalues of $[h_{ij}]$ (in the orthonormal frame $\{e_i\}$) are the principal radii of curvature of ∂K, and the eigenvalues of the matrix on the left-hand side are

$$\left\{\frac{\sqrt{\rho^{\circ 2}+|\nabla\rho^\circ|^2}}{\rho^{\circ 3}}\kappa_i^\circ\right\}_i.$$

Here, κ_i° are the principal curvatures of ∂K°. Since the determinant of the product matrix on the right-hand side is positive, we deduce that κ_i° are all positive. Note that there is at least one point where all the principal curvatures of K° are positive (e.g., a point where ρ° attains its maximum; see (1.10) and (1.11)). □

Exercise 1.22. Suppose $k \geq 2$. Let $f : \mathbb{S}^n \to \mathbb{R}$ be a C^k function. Define

$$B[f]^i_j = g^{ik}\nabla^2_{j,k} f + \delta_{ij} f.$$

Show that if $\det B[f] > 0$, then f is the support function of a convex body of class C^k_+. (Hint: Show that $B[f] > 0$ and the one-homogenous extension $\bar{f}$ defined above is convex and hence sublinear.)

2. Minkowski problem

The main reference for this section is Cheng-Yau [CY76] and Lutwak-Oliker [LO95]. The smooth Minkowski problem (the regular Minkowski problem) asks the following question: Given a positive smooth function f on the unit sphere, what is the necessary and sufficient condition on f that ensures the existence of a closed, strictly convex, smooth hypersurface whose Gauss curvature, as a function of the outer unit normal, is f^{-1}? From our discussion in the previous section, the problem is equivalent to finding $0 < s \in C^\infty(\mathbb{S}^n)$ such that

$$\det B[s] = f \tag{2.1}$$

We will see later the necessary and sufficient condition for the existence of a strictly convex solution (hypersurface) is

$$\int_{\mathbb{S}^n} \langle x, v\rangle f(x) = 0, \quad \forall v \in \mathbb{R}^{n+1} \tag{2.2}$$

Note that for any f there exists a vector w such that $f(\cdot) + \langle \cdot, w\rangle$ satisfies (2.2).

Remark 2.1. From now on $\int = \int_{\mathbb{S}^n}$; i.e. all integrals are on the unit sphere.

Theorem 2.2 (Minkowski problem)**.** *Let $f \in C^\infty(\mathbb{S}^n)$ be a positive function. Suppose $\int x_i f = 0$ for all coordinate functions x_i. Then we can solve the equation (2.1), where s is smooth. That is, we can find a closed strictly convex hypersurface in $\mathbb{R}^{n+1}$ whose support function is given by s and whose Gauss curvature (as a function of the unit normal) is f^{-1}. Moreover, any two such hypersurface must coincide after a translation.*

Let k be a non-negative integer and let $\alpha \in (0, 1]$. The Banach space of real-valued functions on $\mathbb{S}^n$ which are k-times continuously differentiable is denoted by $C^k(\mathbb{S}^n)$ and it is equipped with the norm

$$\|f\|_{C^k} := \sum_{|\beta|\le k} \sup |\nabla^\beta f|.$$

Moreover, $C^{k,\alpha}$ is the space of functions in $C^k(\mathbb{S}^n)$ such that the norm

$$\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + \sup_{|\beta|=k} \sup_{x,\,y\in\mathbb{S}^n \,\&\, x\neq y} \frac{|\nabla^\beta f(x) - \nabla^\beta f(y)|}{d_{\mathbb{S}^n}(x,y)^\alpha}$$

is finite. Here $d_{\mathbb{S}^n}(x,y)$ denotes the (geodesic) distance between x and y in $\mathbb{S}^n$.

2.1. **Existence of solutions.** Suppose $f \in C^{k,\alpha}(\mathbb{S}^n)$. To solve (2.1), we use the continuity method. Let

$$f_t := tf + (1-t), \quad t \in [0,1].$$

Let $S_\alpha = \{t \in [0,1] :$ the equation $\det B[s] = f_t$ has a $C^{k+2,\alpha}$ solution s_t such that $B[s_t] > 0$ and $\int s_t x_i = 0$ for all $i\}$. For $k \ge 3$, using the method of continuity, we show that $S_\alpha = [0,1]$. The method is divided into two steps. In the first step we prove S_α is closed in $[0,1]$, and in the second we prove S_α is open in $[0,1]$. Since $0 \in S_\alpha$, this clearly proves $S_\alpha = [0,1]$.

2.1.1. *Closedness: A priori estimates.* Let $\mathcal{M}^n = \partial K$ be a smooth, strictly convex hypersurface with support function s. Let us denote the eigenvalues of $B[s]$ by λ_i. Define

$$\sigma_k = \sigma_k(B[s]) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}, \quad 1 \le k \le n.$$

Obviously, we have $\det B[s] = \sigma_n$. The extrinsic diameter of $\mathcal{M}^n$ is defined as

$$L = \max_{p,q\in\mathcal{M}^n} |p-q|,$$

where $|p-q|$ denotes the Euclidean distance of p and q.

Lemma 2.3. *Suppose* $\det B[s] = f$. *Then*

$$L \le c_n \left(\int f\right)^{\frac{n+1}{n}} \left(\inf_{y\in\mathbb{S}^n} \int |\langle y,x\rangle| f(x)\right)^{-1},$$

for some constant c_n depending only on n. In particular, if the support function s satisfies (2.2) *, then*

$$0 < s \le c_n \left(\int f\right)^{\frac{n+1}{n}} \left(\inf_{y\in\mathbb{S}^n} \int |\langle y,x\rangle| f(x)\right)^{-1}.$$

Proof. The argument is from [CY76]. Let $p, q \in \mathcal{M}^n$ such that the line segment joining p and q has length L. We may assume 0 is in the middle of the line segment. Hence $\pm\frac{L}{2}u \in \mathcal{M}^n$ for some $u \in \mathbb{S}^n$. For any $x \in \mathbb{S}^n$ we have

$$s(x) = \sup_{p\in K} \langle p,x\rangle \ge \frac{L}{2}|\langle u,x\rangle|.$$

Now, we multiply this by f and integrate over $\mathbb{S}^n$. We get

$$L \le 2\left(\int sf\right)\left(\int |\langle u, x\rangle| f(x)\right)^{-1}.$$

Then, by Alexandrov-Fenchel's inequality we have

$$\left(\int s\sigma_n\right)^{\frac{1}{n+1}} \le c_n'\left(\int s\sigma_{n-1}\right)^{\frac{1}{n}}.$$

On the other hand, by Lemma 2.12, we have

$$\int s\sigma_{n-1} = n\int \sigma_n = n\int f.$$

Hence we obtain

$$L \le c_n''\left(\int f\right)^{\frac{n+1}{n}}\left(\inf_{y\in\mathbb{S}^n}\int |\langle y, x\rangle| f(x)\right)^{-1}.$$

Since s satisfies (2.2) (i.e. the Steiner point of K is at the origin; see Remark 2.4) we have

$$0 < s \le L.$$

□

Remark 2.4. Consider a convex body $\tilde{K}$ with non-empty interior (i.e., the volume of $\tilde{K}$ is not zero) and

$$\ell(y) := \int (s_{\tilde{K}}(x) - \langle x, y\rangle)^2, \quad y \in \tilde{K}.$$

Due to the convexity of $\tilde{K}$, the minimum of ℓ is achieved at a unique point $y \in \tilde{K}$. Moreover, this point is characterized by

$$\int (s_{\tilde{K}}(x) - \langle x, y\rangle)x_i = 0 \quad \forall i = 1, \dots, n+1. \tag{2.3}$$

The point y is called the Steiner point of $\tilde{K}$. We show that y lies in the interior of $\tilde{K}$. Suppose, on the contrary, that y is on the boundary of $\tilde{K}$. After changing coordinates, we may assume that y is at the origin and $\tilde{K}$ lies below the hyperplane $e_{n+1}^{\perp}$ so that $-te_{n+1} \in \operatorname{int}\tilde{K}$ for small $t > 0$. Now we show that

$$\left.\frac{d}{dt}\right|_{t=0} \ell(-te_{n+1}) \quad \text{is negative.}$$

This implies that the minimum is not achieved at $y = 0$, which is a contradiction.

Let $x^+ = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n$ be such that $x_{n+1} > 0$, and define $x^- = (x_1, \dots, -x_{n+1})$. Moreover, let $i(x^+)$ be a point on $\partial\tilde{K}$ so that $s(x^+) = \langle x^+, i(x^+)\rangle$. Note that the x_{n+1}-component of $i(x^+)$ is non-positive. Hence,

$$s(x^-) \ge \langle x^-, i(x^+)\rangle \ge \langle x^+, i(x^+)\rangle = s(x^+). \tag{2.4}$$

On the other hand, $s((0,\dots,0,1)) = 0$, while $s((0,\dots,0,-1)) > 0$. Hence, the set of points with $s(x^-) > s(x^+)$ has positive measure. Moreover, due to (2.4),

$$\ell(-te_{n+1}) = \int s(x)x_{n+1} = \int_{\{x\in\mathbb{S}^n:\, x_{n+1}>0\}} (s(x^+) - s(x^-))x_{n+1} < 0.$$

Hence, $\ell(-te_{n+1}) < \ell(0)$ for some small t.

So far, we have shown that there is a point $y \in \operatorname{int}(\tilde{K})$ such that (2.3) holds. Suppose there is another point $y' \in \mathbb{R}^n$ (not necessarily in $\tilde{K}$) such that

$$\int (s_{\tilde{K}}(x) - \langle x, y'\rangle)x_i = 0 \quad \forall i = 1,\dots,n+1. \tag{2.5}$$

From the convexity of the functional ℓ, it follows that ℓ attains its global minimum both at y and y'. Hence, $\ell(ty + (1-t)y')$ is constant in $t \in [0,1]$. Taking the second derivative of $\ell(ty+(1-t)y')$ at $t = 0$, we see that $y = y'$. In particular, (2.3) holds for a *unique* point in $\mathbb{R}^n$, and that point is in the interior of $\tilde{K}$.

Lemma 2.5. *Suppose* $\det B[s] = f$ *and* s *satisfies* (2.2). *There exists* $C > 0$ *such that*

$$s \geq C.$$

Here C *depends only on* $\min f$.

Proof. The volume of K is given by

$$V(K) = \frac{1}{n+1}\int s \det B[s] = \frac{1}{n+1}\int sf.$$

Hence, we have

$$V(K) \geq c\int s$$

where $c := \frac{1}{n+1}\min f$. Moreover, by the Alexandrov-Fenchel inequality,

$$\int s \geq c_n V^{\frac{1}{n+1}},$$

we obtain $V(K)^{\frac{n}{n+1}} \geq c'$.

The following argument shows that s has a positive lower bound. Let P be the set of all convex bodies whose support functions satisfy (2.2), have diameters bounded by C^0, and volumes bounded below by C_1. We claim that there exists a constant C_2 such that for any convex body $\hat{K}$ in P, $s_{\hat{K}}$ is bounded below by C_2. If this is not the case, then we can find K_i such that $s_{K_i}(u_i) \to 0$ for some $u_i \in \mathbb{S}^n$. By Blaschke's selection theorem, a subsequence of K_i converges to a convex body K_∞ (i.e., $\sup|s_{K_{i_k}} - s_{K_\infty}| \to 0$ as $k \to \infty$ for some sequence i_k) such that $s_{K_\infty}(u) = 0$ for some $u \in \mathbb{S}^n$. That is, $0 \in \partial K_\infty$. However, we have

$$\int x_i s_{K_\infty} = 0, \quad \forall i \in \{1,\dots,n\}.$$

Since K_∞ has a non-empty interior, the Steiner point of K_∞, which is an *interior point*, lies at the origin. This is a contradiction in view of Remark 2.4. $\square$

Lemma 2.6. *We have* $|\nabla s| \leq L$.

Proof. Set $z = s^2 + |\nabla s|^2$. At any critical point x of z, we have

$$0 = \nabla_i z(x) = 2(\nabla^2_{i,j} s + s g_{ij}) g^{jk} \nabla_k s.$$

Since $\mathcal{M}^n$ is strictly convex, $\nabla s(x) = 0$. Therefore, we have

$$z \leq \max z = z(x) = s(x)^2 \leq L^2.$$

□

Suppose $t_k \in S_\alpha$ and $t_k \to t_0$. By these last three lemmas, we have $C_1 \leq s_{t_k} \leq C_2$ and $|\nabla s_{t_k}| \leq C_3$ where C_i are independent of k. Hence, s_{t_k} converges in $C^{0,\alpha}$ to some function s_{t_0}. We will show that, in fact, s_{t_k} converges in $C^{2,\alpha}$ (for any $\alpha \in (0,1)$) to s_{t_0} satisfying $\det B[s_{t_0}] = f_{t_0}$ and that s_{t_0} is of class $C^{k+2,\alpha}$ and $\int x_i s_{t_0} = 0$. Our main tool is the interior C^2-estimate for Monge-Ampère type equations (cf. [Oli84, Thm. 4.2]) and the interior C^3-estimate of Calabi; see [Oli84, Thm. 4.5].

We follow the exposition in [LO95]. To transform the standard interior C^2 and C^3 estimates for Monge-Ampère type equations in Euclidean domains to our equation (2.1), we need to extend $s_k := s_{t_k}$ to a function $\tilde{s}_k$ on $(x_1, x_2, \ldots, x_n, 1)$ and compute the equation it satisfies in a domain of $\mathbb{R}^n$; see (2.9).

Let $u_0 \in \mathbb{S}^n$, and let S^{u_0} denote the open hemisphere for which u_0 is the pole. Let H denote the hyperplane tangent to S^{u_0} at u_0 and H' denote the hyperplane parallel to H that passes through the centre of $\mathbb{S}^n$. Choose cartesian coordinates so that $x_1, \ldots, x_n$ are the coordinates for H' and x^{n+1} is directed toward $u_0 (= e_{n+1})$. Hence,

$$\varphi(x) := (x, 1)/Q(x) \in S^{u_0}, \quad Q(x) := \sqrt{1 + |x|^2}.$$

is a local parameterization of $\mathbb{S}^n$.

With each function s_k, we associate the function

$$\tilde{s}_k(x) = Q(x) s_k(\varphi(x)), \quad x \in H'.$$

Then $\tilde{s}_k$ converges uniformly to $\tilde{s}_0$ on any compact subset of the hyperplane H'. In addition, we have $\tilde{s}_k \geq C_1$, and $\tilde{s}_k \to \infty$ as $|x| \to \infty$.

Since $Q(x)\varphi(x) = (x, 1)$, we have

$$\varphi \partial_i Q + Q \partial_i \varphi = (0, \ldots, 0, 1, 0, \ldots, 0), \tag{2.6}$$

where 1 appears at i-th slot. Differentiating this one more time yields

$$\varphi \partial^2_{ij} Q + \partial_i Q \partial_j \varphi + \partial_j Q \partial_i \varphi + Q \partial^2_{ij} \varphi = 0. \tag{2.7}$$

By the Gauss equation, we have

$$\partial^2_{ij} \varphi = \Gamma^k_{ij} \partial_k \varphi - g_{ij} \varphi. \tag{2.8}$$

Since $\langle \partial_r \varphi, \varphi \rangle = 0$ for any r, and $\langle \partial_i \varphi, \partial_j \varphi \rangle = g_{ij}$, taking the inner product of (2.7) with $\partial_r \varphi$ we obtain

$$\partial_i Q g_{jr} + \partial_j Q g_{ir} + Q \Gamma^m_{ij} g_{mr} = 0.$$

On the other hand, we have

$$\partial_i \tilde{s}_k = s_k \partial_i Q + Q \partial_i s_k$$

which gives

$$\begin{aligned} \partial^2_{ij} \tilde{s}_k &= s_k \partial^2_{ij} Q + \partial_i Q \partial_j s_k + \partial_j Q \partial_i s_k + Q \partial^2_{ij} s_k \\ &= s_k \partial^2_{ij} Q + (\partial_i Q g_{jr} + \partial_j Q g_{ir}) g^{rl} \partial_l s_k + Q \partial^2_{ij} s_k \\ &= s_k \partial^2_{ij} Q - Q \Gamma^l_{ij} \partial_l s_k + Q \partial^2_{ij} s_k \\ &= s_k \partial^2_{ij} Q + Q \nabla^2_{i,j} s_k. \end{aligned}$$

Taking the inner product of (2.7) with φ and using (2.8) gives

$$\partial^2_{ij} Q = Q g_{ij}.$$

Hence[1]

$$\partial^2_{ij} \tilde{s}_k = (\nabla^2_{i,j} s_k + s_k g_{ij}) Q,$$

and

$$f_{t_k} = \det B[s_k] = \frac{\det[\nabla^2_{i,j} s_k + s_k g_{ij}]}{\det[g_{ij}]} = \frac{1}{Q^n} \frac{\det[\partial^2_{ij} \tilde{s}_k]}{\det[g_{ij}]}.$$

Moreover, from (2.6) we have

$$\partial_i Q \partial_j Q + Q^2 \langle \partial_i \varphi, \partial_j \varphi \rangle = \delta_{ij}.$$

That is,

$$g_{ij} = \frac{1}{Q^2} \left(\delta_{ij} - \partial_i Q \partial_j Q \right).$$

Hence,

$$\det[g_{ij}] = Q^{-2(n+1)}.$$

Therefore, we obtain

$$\det[\partial^2_{ij} \tilde{s}_k] = f_{t_k} Q^{-n-2}. \tag{2.9}$$

Note that, in particular, this identity implies that the graphs $x^{n+1} = \tilde{s}_k(x)$, $x \in H'$, are strictly convex. Consequently, $\tilde{s}_0$ is a convex of function on H'.

Since $\tilde{s}_0 \to \infty$ when $|x| \to \infty$, we can choose λ, sufficiently large, so that the set $P_0 := \{x \in H' : \tilde{s}_0(x) \leq \lambda\}$ is a non-empty compact convex set in H'. Since the $\tilde{s}_k$ converges to $\tilde{s}_0$, uniformly on each compact subset of H', we may choose some λ so that a compact set Θ is contained *strictly* inside all of the sets $P_k = \{x \in H' : \tilde{s}_k(x) \leq \lambda\}$ as well as the set P_0 (That is, at a positive distance δ from the boundaries ∂P_k for all k large enough). Now consider the functions

$$\hat{s}_k := \tilde{s}_k - \lambda.$$

Obviously $\hat{s}_k$ vanishes on the boundary of P_k and due (2.9), hence we have

$$\begin{cases} \det[\partial^2_{ij} \hat{s}_k] = f_{t_k} Q^{-n-2} & \text{in } P_k, \\ \hat{s}_k = 0 & \text{on } \partial P_k. \end{cases} \tag{2.10}$$

[1] Compare with Exercise 1.18.

Moreover, the r.h.s. of (2.10) is bounded away from 0 *uniformly* in k.

Let U be a convex domain in $\mathbb{R}^n$ and ϕ be function of $2n+1$ variables in the domain

$$V = U \times [a,b] \times \mathbb{R}^n.$$

Theorem 2.7 (Pogorelov; Oliker)**.** [Oli84, Thm. 4.2], [GT01, Thm. 17.19]. *Suppose $\phi \in C^2(\bar{V})$ and $\phi > 0$ in V. Let $z \in C^4(U) \cap C^1(\bar{U})$ and satisfies in U the equation*

$$\det[\partial^2_{ij} z] = \phi(x, z, Dz). \tag{2.11}$$

Assume further that the matrix $\partial^2_{ij} z$ positive definite everywhere in U, and $z|_{\partial U} = 0$. Then the second derivatives of z at a point $x \in U$ can be estimated and the estimates depends only on the C^1-norm of z in U, the C^2-norm of ϕ in V, and the distance of x to the boundary of U.

Proof. The proof is from [Oli84] and [Gut16]. Note that $z < 0$ in U. Consider the function

$$w = -z e^{c\frac{z_\alpha^2}{2}} z_{\alpha\alpha},$$

where α is a unit vector in $\mathbb{R}^n$ and $z_\alpha = \langle Dz, \alpha \rangle$ and $z_{\alpha\alpha} = D^2 z(\alpha, \alpha)$, and c is a positive constant to be determined.

Note that the maximum of w is attained *inside* U, say at x_0. Moreover, there exists a unimodular matrix O (i.e. $\det O = 1$), such that $O^t[\partial^2 z(x_0)]O$ is diagonal and if $\bar{z}(x) := z(Ox)$, then $\bar{z}_1(x) = z_\alpha(Ox)$ and $\bar{z}_{11}(x) = z_{11}(Ox)$. In particular,

$$[\partial^2 \bar{z}(O^{-1}x_0)] = O^t[\partial^2 z(x_0)]O$$

is diagonal; see Remark 2.8 for the details. Hence, we may assume that $\alpha = e_1$ and we have $z_{ij}(x_0) := \partial^2_{ij} z(x_0) = 0$ for $i \neq j$.

Now taking the logarithmic derivatives of w at x_0 in the variable x_i we get

$$\frac{w_i}{w} = \frac{z_{11i}}{z_{11}} + c z_1 z_{1i} + \frac{z_i}{z} = 0. \tag{2.12}$$

Differentiating one more time with respect to x_i, we obtain

$$\frac{w_{ii}}{w} - \frac{w_i^2}{w^2} = \frac{z_{11ii}}{z_{11}} - \frac{z_{11i}^2}{z_{11}^2} + c z_{1i}^2 + c z_1 z_{1ii} + \frac{z_{ii}}{z} - \frac{z_i^2}{z^2}. \tag{2.13}$$

Since $z_{i1} = 0$ when $i \neq 1$ and $w_i = 0$ (at x_0), we get after multiplying (2.13) by $\frac{z_{11}}{z_{ii}}$ and summing over i,

$$\begin{aligned}\sum_i \frac{w_{ii} z_{11}}{w z_{ii}} =& \sum_i \frac{z_{11ii}}{z_{ii}} - \sum_i \frac{z_{11i}^2}{z_{11} z_{ii}} + c z_{11}^2 + c z_1 z_{11} \sum_i \frac{z_{1ii}}{z_{ii}} \\ &+ n\frac{z_{11}}{z} - \left(\frac{z_1}{z}\right)^2 - \frac{z_{11}}{z^2} \sum_{i\neq 1} \frac{z_i^2}{z_{ii}}.\end{aligned} \tag{2.14}$$

When $i \neq 1$ (2.12) gives

$$\left(\frac{z_i}{z}\right)^2 = \frac{z_{11i}^2}{z_{11}^2}.$$

Substituting it in the last term of (2.14), we obtain

$$\begin{aligned}\sum_i \frac{w_{ii}z_{11}}{wz_{ii}} =& \sum_i \frac{z_{11ii}}{z_{ii}} - \sum_i \frac{z_{11i}^2}{z_{11}z_{ii}} + cz_{11}^2 + n\frac{z_{11}}{z} - \left(\frac{z_1}{z}\right)^2 \\ &+ cz_1z_{11}\sum_i \frac{z_{1ii}}{z_{ii}} - \sum_{i\neq 1} \frac{z_{11i}^2}{z_{11}z_{ii}}.\end{aligned} \tag{2.15}$$

Now we differentiate equation (2.11) twice with respect to x_1 :[2]

$$(\log\phi)_1 = \sum_i \frac{z_{ii1}}{z_{ii}} \tag{2.16}$$

and

$$(\log\phi)_{11} = \sum_i \frac{z_{ii11}}{z_{ii}} - \sum_{i,j} \frac{z_{ij1}^2}{z_{ii}z_{jj}}. \tag{2.17}$$

We rewrite (2.17) as follows

$$(\log\phi)_{11} = \sum_i \frac{z_{ii11}}{z_{ii}} + \sum_{i\neq j}\left(\frac{z_{ii1}z_{jj1}}{z_{ii}z_{jj}} - \frac{z_{ij1}^2}{z_{ii}z_{jj}}\right) - \sum_{i,j}\frac{z_{ii1}z_{jj1}}{z_{ii}z_{jj}}. \tag{2.18}$$

We may subtract (2.18) from (2.15) term by term to cancel the $\sum_i \frac{z_{11ii}}{z_{ii}}$-term appearing in both identities:

$$\begin{aligned}&cz_{11}^2 + n\frac{z_{11}}{z} - \left(\frac{z_1}{z}\right)^2 + cz_1z_{11}\sum_i \frac{z_{1ii}}{z_{ii}} - \sum_i \frac{z_{11i}^2}{z_{11}z_{ii}} - \sum_{i\neq 1}\frac{z_{11i}^2}{z_{11}z_{ii}} \\ &- \sum_{i\neq j}\frac{z_{ii1}z_{jj1}}{z_{ii}z_{jj}} + \sum_{i\neq j}\frac{z_{ij1}^2}{z_{ii}z_{jj}} + \sum_{i,j}\frac{z_{ii1}z_{jj1}}{z_{ii}z_{jj}} = \sum_i \frac{w_{ii}z_{11}}{wz_{ii}} - (\log\phi)_{11}.\end{aligned} \tag{2.19}$$

Note that

$$-\sum_{i\neq j}\frac{z_{ii1}z_{jj1}}{z_{ii}z_{jj}} + \sum_{i\neq j}\frac{z_{ij1}^2}{z_{ii}z_{jj}} + \sum_{i,j}\frac{z_{ii1}z_{jj1}}{z_{ii}z_{jj}} = \sum_{i,j}\frac{z_{ij1}^2}{z_{ii}z_{jj}}.$$

In addition,

$$\sum_{i,j}\frac{z_{ij1}^2}{z_{ii}z_{jj}} - \sum_i \frac{z_{11i}^2}{z_{ii}z_{11}} - \sum_{i>1}\frac{z_{11i}^2}{z_{11}z_{ii}} = \sum_{i,j>1}\frac{z_{ij1}^2}{z_{ii}z_{jj}} \geq 0.$$

Using this last inequality and the inequality $w_{ii} \leq 0$ at x_0, from (2.19) we obtain

$$cz_{11}^2 + n\frac{z_{11}}{z} - \left(\frac{z_1}{z}\right)^2 + cz_1z_{11}(\log\phi)_1 + (\log\phi)_{11} \leq 0. \tag{2.20}$$

Now, we compute explicitly the derivatives of $\log\phi$. Set $\phi(x,t,p) = \phi(x,z,Dz)$. We have

$$(\log\phi)_1 = \frac{\phi_1 + \phi_t z_1 + \sum_i \phi_{p_i} z_{i1}}{\phi} \tag{2.21}$$

[2] We have $\frac{d}{dt}\log\det A(t) = \mathrm{tr}(A(t)^{-1}A(t)')$ and $\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)A(t)'A^{-1}(t)$.

and

(2.22)

$$(\log\phi)_{11} = \Bigg[\phi_{11} + \phi_{1t}z_1 + \sum_i \phi_{1p_i} z_{i1} + \Big(\phi_{t1} + \phi_{tt}z_1 + \sum_i \phi_{tp_i} z_{i1}\Big) z_1 + \phi_t z_{11} + \sum_i \Big(\phi_{p_i 1} + \phi_{p_i t} z_1 + \sum_j \phi_{p_i p_j} z_{j1}\Big) z_{i1} + \sum_i \phi_{p_i} z_{i11}\Bigg] /\phi - \left(\phi_1^2 + \phi_t^2 z_1^2 + \Big(\sum_i \phi_{p_i} z_{i1}\Big)^2 + 2\phi_1\phi_t z_1 + 2\phi_1 \sum_i \phi_{p_i} z_{i1} + 2\phi_t z_1 \sum_i \phi_{p_i} z_{i1}\right) /\phi^2.$$

Using (2.12), at x_0 we find that

$$\sum_i \phi_{p_i} z_{i11} = -cz_1\phi_{p_1} z_{11}^2 - \frac{z_{11}}{z} \sum \phi_{p_i} z_i.$$

Therefore,

$$cz_1 z_{11}(\log\phi)_1 + (\log\phi)_{11} = \left(\frac{\phi_{p_1p_1}}{\phi} - \frac{\phi_{p_1}^2}{\phi^2}\right) z_{11}^2 + Az_{11} + B,$$

where A and B denote quantities admitting estimates in terms of the C^2-norm of ϕ and C^1-norm of z. Substituting the last expression in (2.20) we get

$$\left(c + \frac{\phi_{p_1p_1}}{\phi} - \frac{\phi_{p_1}^2}{\phi^2}\right) z_{11}^2 + \left(A + \frac{n}{z}\right) z_{11} - \left(\frac{z_1}{z}\right)^2 + B \le 0.$$

We may choose c so that the coefficient of z_{11}^2 is positive. Multiplying the last inequality by $z^2 e^{cz_1^2}$ and introducing w we get

$$\bar{C}w^2 + \bar{A}w + \bar{B} \le 0,$$

where $\bar{C} > 0$ and admits an estimate from above and from below in terms of the C^2-norm of ϕ, and $\bar{A}, \bar{B}$ admit uniform estimates in terms of the quantities indicated in the theorem. Thus everywhere $w \le \bar{w}$ in U, where $\bar{w}$ depends on $\bar{A}$, $\bar{B}$ and $\bar{C}$. Hence

$$z_{11} \le \frac{\bar{w}}{|z|}.$$

Since z is concave upward and $z|_{\partial U} = 0$, we have (see Exercise 2.10):

$$\frac{2|z(x)|}{\operatorname{dist}(x,\partial U)} \ge \frac{\max|z|}{d},$$

where d is the diameter of U. Hence

$$z_{11}(x) \le \frac{2\bar{w}d}{\operatorname{dist}(x,\partial U)\max|z|}.$$

Since α was an arbitrary direction, the claim follows. $\square$

Remark 2.8. Let Q be an orthogonal matrix (i.e. $Q^t = Q^{-1}$) such that $Qe_1 = \alpha$, and first let $v(x) = z(Qx)$. Then the first column of Q is the vector α and we have $v_1(x) = z_\alpha(Qx)$ and $v_{11}(x) = z_{\alpha\alpha}(Qx)$. Given an $n \times n$ symmetric, positive-definite matrix $A = [a_{ij}]$, consider the matrix

$$B := \begin{bmatrix} 1 - \frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & -\frac{a_{14}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

We have

$$B^t AB = \begin{bmatrix} a_{11} & 1 \\ 0 & B_1 \end{bmatrix},$$

where B_1 is an $(n-1) \times (n-1)$ matrix. Since A is symmetric and positive-definite, it follows that B_1 is also positive-definite and symmetric. Hence, an orthogonal matrix O_1 exists such that $O_1^t B_1 O$ is diagonal. Let

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & O_1 \end{bmatrix}.$$

Now we choose $A = [\partial^2 v(Q^{-1}x_0)]$ and set $\bar{z}(x) = v(B\mathcal{O}x) = z(QB\mathcal{O}x)$. Then

$$\begin{aligned} [\partial^2 \bar{z}((QB\mathcal{O})^{-1}x_0)] &= [\partial^2 \bar{z}((B\mathcal{O})^{-1}Q^{-1}x_0)] \\ &= (B\mathcal{O})^t [\partial^2 v(Q^{-1}x_0)](B\mathcal{O}) \\ &= (B\mathcal{O})^t A(B\mathcal{O}) \end{aligned}$$

is diagonal. Combining the changes of coordinates, the matrix $O = QB\mathcal{O}$ does the job.

Exercise 2.9. Verify (2.16) and (2.17).
Hint: Let $A = [a_{ij}]$, $B = [b_{ij}]$ be two square matrices and assume that A is invertible. Then

$$\frac{d}{dt}_{|t=0} \det(A + tB) = (\det A)\operatorname{tr}(A^{-1}B). \tag{2.23}$$

In particular, if A is diagonal, then

$$\frac{d}{dt}_{|t=0} \log\det(A + tB) = \sum_i \frac{b_{ii}}{a_{ii}}.$$

Now note that $D^2 z(x + te_1) = D^2 z(x) + tD^2 z_1 + o(t)$. Hence

$$\partial_1 \log\det[\partial^2 z_{ij}(x)] = \frac{d}{dt}_{|t=0} \log\det(D^2 z(x + te_1)) = \sum_i \frac{z_{ii1}(x)}{z_{ii}(x)}.$$

Exercise 2.10. Show that

$$\frac{2|z(x)|}{\operatorname{dist}(x, \partial U)} \geq \frac{\max|z|}{d}.$$

Proof. Let $y \in U$. Then by convexity of z, we have

$$z(y) \geq z(x) + \langle Dz(x), y - x \rangle.$$

Let $0 < r < \operatorname{dist}(x, \partial U)$, then $y_0 := x + r\frac{Dz(x)}{|Dz(x)|} \in U$ and

$$0 \geq z(y_0) \geq z(x) + r|Dz(x)|.$$

Hence

$$|Dz(x)| \leq \frac{-z(x)}{\operatorname{dist}(x, \partial U)} = \frac{|z(x)|}{\operatorname{dist}(x, \partial U)}.$$

Now let y be a point where z attains its minimum. Then

$$|z(x) - z(y)| \leq |y - x||Dz(x)| \leq d\frac{|z(x)|}{\operatorname{dist}(x, \partial U)}.$$

Therefore,

$$\frac{\max|z|}{d} = \frac{|z(y)|}{d} \leq \frac{|z(x)|}{\operatorname{dist}(x, \partial U)} + \frac{|z(x)|}{d} \leq \frac{2|z(x)|}{\operatorname{dist}(x, \partial U)}.$$

□

Applying this theorem to $\hat{s}_k$, we find that

$$|\partial^2_{ij}\hat{s}_k(x)| \leq C, \tag{2.24}$$

where C depends on the C^1-norm of $\hat{s}_k$ in P_k, the C^2-norm of $f_{t_k}Q^{-n-2}$ in P_k, and the distance of the point x to the boundary of P_k. Thus, we conclude that on the set Θ, the estimate (2.24) is independent of k.

Theorem 2.11 (Calabi; Oliker)**.** [Oli84, Thm. 4.5] *Suppose $\phi \in C^3(\bar{V})$ and that $\phi > 0$ in $\bar{V}$. Let $z \in C^5(U) \cap C^2(\bar{U})$ and satisfies in U the equation*

$$\det[\partial^2_{ij} z] = \phi(x, z, Dz).$$

Assume further that the matrix $\partial^2_{ij} z$ positive definite everywhere in U, and $z|_{\partial U} = 0$. Then at every point x of U the C^3-norm of z admits a uniform estimate depending on the C^2-norm of z at x, C^3-norm of ϕ in $\bar{V}$, and the distance from x to the boundary of U.

By the previous theorem, the third derivatives of $\hat{s}_k$ in Θ are bounded above, uniformly in k. Hence, by the Arzela-Ascoli theorem a subsequence of $\hat{s}_k$ converges uniformly in Θ to $\hat{s}_0$ in $C^{2,\alpha}$ for any $\alpha \in (0, 1)$. Thus $\tilde{s}_0$ satisfies

$$\det(\partial^2_{ij}\hat{s}_0) = f_{t_0}Q^{-n-2} \quad \text{in} \quad \Theta.$$

Then, the theory of elliptic PDEs can be used to deduce that $\hat{s}_0$ is, in fact, $C^{k+2,\alpha}(\Theta)$; see [Fig17, Thm. A.42, Prop. A.43].

2.1.2. *Openness.* For smooth functions $f_i \in C^2(\mathbb{S}^n)$ we define

$$Q[f_1,\dots,f_n]=\frac{1}{n!}\sum_{\sigma,\tau\in S_n}(-1)^{\mathrm{sgn}(\tau)+\mathrm{sgn}(\sigma)}B[f_1]^{\sigma(1)}_{\tau(1)}\cdots B[f_n]^{\sigma(n)}_{\tau(n)}.$$

Here

$$B[f]^j_i=g^{jk}(\nabla^2_{i,k}f+fg_{ik}).$$

Note that $Q[s,\dots,s]=\det B[s]$ and $Q[s,\dots,s,1,\dots,1]$ where s appears k times is a multiple of σ_k.

Lemma 2.12.

(1) *Q is independent of the order of its arguments*

$$Q[f_1,\dots,f_n]=Q[f_{\sigma_1},\dots,f_{\sigma_n}]$$

for every permutation $\sigma\in S_n$.

(2) *If $B[f_i]$ for every i is positive-definite, then $Q[f_1,\dots,f_n]$ is positive.*

(3) *If $f_2,\dots,f_n$ are fixed smooth functions, and $B[f_i]$ is positive definite for $i\in\{2,\dots,n\}$, then $Q[f]:=Q[f,f_2,\dots,f_n]$ is a non-degenerate second-order linear elliptic operator, given in local coordinates by an expression of the following form:*

$$Q[f]=Q^{ij}(\nabla^2_{i,j}f+g_{ij}f),$$

where Q^{ij} is a positive definite matrix at each point of $\mathbb{S}^n$, depending only on f_i.

(4) *The following identity for any $f_2,\dots,f_n$ as above holds*

$$\nabla_iQ^{ij}=0.$$

Proof. Proofs of items (2)-(3) are given in [RK96, Ch. VI]. We prove (4). Define

$$Q^j_i=\frac{1}{n!}\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(1)=i,\tau(1)=j}}(-1)^{\mathrm{sgn}(\tau)+\mathrm{sgn}(\sigma)}B[f_2]^{\sigma(2)}_{\tau(2)}\cdots B[f_n]^{\sigma(n)}_{\tau(n)}.$$

Then we have

$$\begin{aligned}n!\,\nabla_jQ^j_i=&\sum_j\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(1)=i,\tau(1)=j}}(-1)^{\mathrm{sgn}(\tau)+\mathrm{sgn}(\sigma)}(\nabla_jB[f_2]^{\sigma(2)}_{\tau(2)})\cdots B[f_n]^{\sigma(n)}_{\tau(n)}\\ &+\cdots+\sum_j\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(1)=i,\tau(1)=j}}(-1)^{\mathrm{sgn}(\tau)+\mathrm{sgn}(\sigma)}B[f_2]^{\sigma(2)}_{\tau(2)}\cdots(\nabla_jB[f_n]^{\sigma(n)}_{\tau(n)})\end{aligned}$$

We will consider the first term.

$$\begin{aligned}
&\sum_j \sum_{\substack{\sigma,\tau\in S_n\\ \sigma(1)=i,\tau(1)=j}} (-1)^{\operatorname{sgn}(\tau)+\operatorname{sgn}(\sigma)}(\nabla_j B[f_2]^{\sigma(2)}_{\tau(2)})\cdots B[f_n]^{\sigma(n)}_{\tau(n)}\\
&=\sum_j \sum_{\substack{\sigma,\tau\in S_n\\ \sigma(1)=i,\tau(1)=j}} (-1)^{\operatorname{sgn}(\tau)+\operatorname{sgn}(\sigma)}(\nabla_{\tau(1)} B[f_2]^{\sigma(2)}_{\tau(2)})\cdots B[f_n]^{\sigma(n)}_{\tau(n)}\\
&=\frac{1}{2}\sum_j \sum_{\substack{\sigma,\tau\in S_n\\ \sigma(1)=i,\tau(1)=j}} (-1)^{\operatorname{sgn}(\tau)+\operatorname{sgn}(\sigma)}(\nabla_{\tau(1)} B[f_2]^{\sigma(2)}_{\tau(2)}-\nabla_{\tau(2)} B[f_2]^{\sigma(2)}_{\tau(1)})\cdots B[f_n]^{\sigma(n)}_{\tau(n)}.
\end{aligned}$$

For any $f\in C^2(\mathbb{S}^n)$, we have $\nabla_j B[f]^i_k=\nabla_k B[f]^i_j$ (cf. Exercise 2.13). Hence, the last term is zero. Moreover, we have

$$Q[f]=Q^i_k B[f]^k_i=Q^i_k g^{jk}(\nabla^2_{i,j}f+fg_{ij}).$$

Now define $Q^{ij}=Q^i_k g^{jk}$. Then from the metric compatibility, $\nabla g=0$, it follows that

$$\nabla_i Q^{ij}=0.$$

□

Exercise 2.13. Show that for any $f\in C^2(\mathbb{S}^n)$, $\nabla_j B[f]^i_k=\nabla_k B[f]^i_j$.

Suppose $s\in C^2(\mathbb{S}^n)$ with $B[s]>0$. Let $f_2=\cdots=f_n=s$ and denote the corresponding $Q[\cdot]=Q[\cdot,s,\ldots,s]$ by $L_s[\cdot]$. Note that we have

$$\begin{aligned}
L_s[u]=Q[u]&=\frac{1}{n}\frac{d}{dt}_{|t=0}Q[s+tu,s+tu,\ldots,s+tu]\\
&=\frac{1}{n}\frac{d}{dt}_{|t=0}\det B[s+tu].
\end{aligned}\tag{2.25}$$

Exercise 2.14. Show that in an orthonormal frame that diagonalizes $B[s]$,

$$L_s[u]=\frac{1}{n}c^{ij}(\nabla^2_{i,j}u+\delta_{ij}u),$$

where $[c^{ij}]$ is the adjugate matrix (transpose of the cofactor matrix) of $B[s]$. Hence by Lemma 2.12, in this case we have

$$Q^{ij}=\frac{1}{n}c^{ij},\quad \nabla_i c^{ij}=0.$$

Therefore,

$$\int uL_s[v]=\int vL_s[u]\tag{2.26}$$

for all C^2-smooth functions u,v. That is, L_s is self-adjoint. Moreover, by approximation, we may allow $B[s]$ to be non-negative definite.

Lemma 2.15. *Suppose $u, s \in C^2(\mathbb{S}^n)$ with $B[s] > 0$. Then*

$$\int uQ[u, s, \dots, s] = \int sQ[u, u, s, \dots, s].$$

Proof. For $\lambda > 0$, large enough we have $B[u + \lambda] > 0$. Hence we obtain

$$\int (u + \lambda)Q[u + \lambda, s, \dots, s] = \int sQ[u + \lambda, u + \lambda, s, \dots, s].$$

On the other hand, we have

$$\begin{aligned}\int (u + \lambda)Q[u + \lambda, s, \dots, s] &= \int uQ[u, s, \dots, s] + \int \lambda^2 Q[1, s, \dots, s] \\ &\quad + \int \lambda uQ[1, s, \dots, s] + \int \lambda Q[u, s, \dots, s] \\ &= \int uQ[u, s, \dots, s] + \int \lambda^2 sQ[1, 1, \dots, s] \\ &\quad + 2 \int \lambda Q[u, s, \dots, s],\end{aligned}$$

as well as

$$\begin{aligned}\int sQ[u + \lambda, u + \lambda, s, \dots, s] &= \int sQ[u, u, s, \dots, s] + \lambda^2 \int sQ[1, 1, s, \dots, s] \\ &\quad + 2 \int \lambda sQ[1, u, s, \dots, s] \\ &= \int sQ[u, u, s, \dots, s] + \lambda^2 \int sQ[1, 1, s, \dots, s] \\ &\quad + 2 \int \lambda uQ[1, s, \dots, s] \\ &= \int sQ[u, u, s, \dots, s] + \lambda^2 \int sQ[1, 1, s, \dots, s] \\ &\quad + 2 \int \lambda Q[u, s, \dots, s].\end{aligned}$$

Comparing the right-hand-sides of these identities the claim follows. $\square$

Lemma 2.16. *Suppose $B[s]$ is non-negative definite. Then for any coordinate function x_i,*

$$\int x_i \det B[s] = 0.$$

Proof. Suppose $B[s]$ is positive definite. The volume of the convex body with support function s is given by

$$V[s] = \frac{1}{n + 1} \int s \det B[s].$$

Moreover, $V[s] = V[s + t\langle \cdot, e_i\rangle]$ and t. For a strictly convex body, by Lemma 2.12, we have

$$0 = \frac{d}{dt}_{|t=0} V[s + t\langle \cdot, e_i\rangle] = \int \langle \cdot, e_i\rangle \det B[s].$$

The general case follows by approximation. $\square$

Let $B_1 = \{f \in C^{k+2,\alpha} : \int x_i f = 0\}$ and $B_2 = \{f \in C^{k,\alpha} : \int x_i f = 0\}$. Now define a transformation between these two Banach spaces,

$$F : B_1 \to B_2, \quad F(u) = \det B[u].$$

We claim that F is locally invertible around any $0 < s \in B_1$ with $B[s] > 0$. That is, if $s \in B_1$ is the support function of some strictly convex hypersurface (hence $s > 0$), then for any f in a neighbourhood of $\det B[s]$ (in the topology of B_2) we can always solve the equation $F(u) = f$, such that u is the support function of some strictly convex hypersurface.

To prove the above claim, in view of the inverse function theorem on Banach spaces, it is sufficient to show that the linearized operator of F at s, $L_s : B_1 \to B_2$, is bounded, one-to-one and onto (i.e. an isomorphism).

Let $P_s : L^2 \to \ker L_s$ be the L^2-projection to the (finite dimensional) kernel of $L_s : C^2(\mathbb{S}^n) \to C(\mathbb{S}^n)$. Hence by (2.26), $L_s + P_s$ has trivial kernel on $C^2(\mathbb{S}^n)$ [3]. Therefore, by elliptic theory

$$L_s + P_s : C^{k+2,\alpha}(\mathbb{S}^n) \to C^{k,\alpha}(\mathbb{S}^n)$$

is an isomorphism. By the next lemma, $L_s : B_1 \to B_2$ is an isomorphism.

Lemma 2.17. *Let $u \in C^2$ satisfies $L_s[u] = 0$, where $s > 0$, $B[s] > 0$. Then for some vector $(a_1, \ldots, a_{n+1})$, we have $u(x) = \sum_{i=1}^{n+1} a_i x_i$.*

Proof. By Lemma 2.15, we have

$$\int sQ[u, u, s, \ldots, s] = \int uL_s[u] = 0. \tag{2.27}$$

Take a local orthonormal frame on $\mathbb{S}^n$ that diagonalizes $B[s]$ and put

$$a_{ij} = \frac{B[u]^i_j}{\sqrt{B[s]^i_i B[s]^j_j}}.$$

By definition of Q, we have

$$L_s[u] = \frac{1}{n!} \sum_i \sum_{\sigma\in S_n, \sigma(1)=i} B[s]^{\sigma(2)}_{\sigma(2)} \cdots B[s]^{\sigma(n)}_{\sigma(n)} B[u]^i_i = \det B[s] \sum_i a_{ii} = 0,$$

[3]Show if $L_s[u] + P_s[u] = 0$, then $P_s[u] = L_s[u] = 0$. Hence $0 = P_s[u] = u$

and

$$Q[u,u,s,\dots,s] = \frac{1}{n!}\sum_{i,j}\sum_{\sigma\in S_n,\sigma(1)=i,\sigma(2)=j} B[s]^{\sigma(3)}_{\sigma(3)}\cdots B[s]^{\sigma(n)}_{\sigma(n)}\left(B[u]^i_i B[u]^j_j - B[u]^i_j B[u]^i_j\right)$$
$$= \det B[s](\sum_{i,j} a_{ii}a_{jj} - a_{ij}^2).$$

Now note that

$$\sum_{i,j} a_{ii}a_{jj} - a_{ij}^2 = \frac{1}{2}\left((\sum_i a_{ii})^2 - \sum_i a_{ii}^2\right) - \sum_{i,j} a_{ij}^2 \le 0.$$

Thus, in view of (2.27), we get $a_{ij} = 0$ for all i,j. Equivalently, we obtain

$$B[u]^i_j \equiv 0, \quad \forall i,j.$$

By Exercise 1.18, $D^2\bar{u} \equiv 0$, where $\bar{u}(x) = |x|u(x/|x|)$, $\forall x \in \mathbb{R}^{n+1}\setminus\{0\}$. Hence, there exists some vector $\vec{a}$ such that $D\bar{u}(x) = \vec{a}$ for all $x \in \mathbb{S}^n$. That is, $u \in \operatorname{Span}\{x_1,\dots,x_{n+1}\}$. □

Therefore, $S_\alpha = [0,1]$ for $k \ge 3$. That is, for any $f \in C^{k,\alpha}$ with $k \ge 3$ and $\int x_i f = 0$ for all i, there exists a solution to

$$\det B[s] = f$$

with $s \in C^{k+2,\alpha}(\mathbb{S}^n)$. Due to Exercise 1.22, s is the support function of a convex body of class C^k_+.

Remark 2.18. Suppose $0 < f \in C^{1,1}$ satisfies (2.2) . In this case, we may approximate f in $C^{1,1}$-norm with positive functions $f_l \in C^{k,\alpha}$ for k sufficiently large. To ensure the necessary sufficient condition is satisfied we replace f_l by $\bar{f}_l$ given by

$$\bar{f}_l = f_l - \sum_{i=1}^n \left(\int x_i f_l\right)\left(\int x_i^2\right)^{-1} x_i + 2\varepsilon_l,$$

where

$$-\varepsilon_l = \min\left\{0, \inf_{\mathbb{S}^n}\left(f_l - \sum_{i=1}^n \left(\int x_i f_l\right)\left(\int x_i^2\right)^{-1} x_i\right)\right\}.$$

Now $\det B[s_l] = \bar{f}_l$ have solutions in $C^{k+2,\alpha}$. By Proposition 2.19, a subsequence of $\{s_l\}_l$ converges in $C^{2,\beta}$ ($\forall \beta \in (0,1)$) to a solution of

$$\det B[s] = f.$$

It can be shown that the solution s is, in fact, of class $C^{3,\beta}$, for all $\beta \in (0,1)$; cf. [Fig17, Thm. A.42, Prop. A.43].

Proposition 2.19. [GM03] *For each integer $\ell \ge 1$ and $0 < \beta < 1$, there exists a constant C depending only on $\ell, \beta, \min f$ and $\|f\|_{C^{\ell,1}}$ such that*

$$\|s\|_{C^{\ell+1,\beta}} \le C$$

for all strictly convex solutions of (2.1) *satisfying the condition* (2.2).

2.2. **Uniqueness.** Suppose $s_i \in C^2$ with $\det B[s_i] > 0$. Then by the Minkowski inequality [Sch14, Thm. 7.2.1] we have

$$\left(\int s_1 \det B[s_2]\right)^{n+1} \geq \int s_1 \det B[s_1] \left(\int s_2 \det B[s_2]\right)^n, \tag{2.28}$$

and equality holds if and only if for some vector $\vec{a}$ and constant $c > 0$, we have

$$s_2(x) - cs_1(x) = \langle x, \vec{a} \rangle, \quad \forall x \in \mathbb{S}^n.$$

Now suppose $\det B[s_i] = f$, for $i = 1, 2$. Then

$$\left(\int s_1 f\right)^{n+1} \geq \int s_1 f \left(\int s_2 f\right)^n \Rightarrow \int s_1 f \geq \int s_2 f.$$

Similarly, we have

$$\int s_2 f \geq \int s_1 f.$$

Hence equality holds in (2.28), for some vector $\vec{a}$, we have

$$s_2(x) - s_1(x) = \langle x, \vec{a} \rangle.$$

3. L_p-Minkowski problem

We say $f \in C^\infty(\mathbb{S}^n)$ is even if $f(x) = f(-x)$ for all $x \in \mathbb{S}^n$. In the section, we prove the following theorem due to Lutwak-Oliker [LO95].

Theorem 3.1 (even L_p-Minkowski problem)**.** *Suppose $1 < p \neq n+1$. Let $f \in C^\infty(\mathbb{S}^n)$ be an even positive function. Then there exists an even solution $0 < s \in C^\infty(\mathbb{S}^n)$ to*

$$\det B[s] = s^{p-1} f. \tag{3.1}$$

Therefore, we can find an origin-symmetric, strictly convex hypersurface in $\mathbb{R}^{n+1}$ whose support function is s and Gauss curvature is $f^{-1}s^{1-p}$.

3.1. **Existence of solution.** Let $f_t := 1 - t + tf$ and $S_\alpha = \{t \in [0,1]$: the equation $\det B[s] = s^{p-1} f_t$ has a $C^{k+2,\alpha}$ solution $s_t > 0$ such that $B[s_t] > 0$ and s_t is even$\}$. For $k \geq 3$, using the method of continuity we show that $S_\alpha = [0,1]$. Note that $0 \in S_\alpha$, hence S_α is not empty. Recall from the previous section (case $p = 1$) that the continuity method consists of two steps. To show closedness, we must prove for any sequence $t_k \in S_\alpha$ such that $t_k \to t_0$, we have $t_0 \in S_\alpha$. To show openness, we need to show for any $t \in S_\alpha \setminus \{1\}$ there is $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \in S_\alpha$.

3.1.1. *Closedness.* Let $\mathcal{M}^n = \partial K$ be an origin-symmetric, smooth, strictly convex hypersurface with support function s.

Lemma 3.2. *Let $1 < p \neq n+1$ and $\det B[s] = s^{p-1} f$. Then $c < s < C$ with c, C depending only on f. Moreover, we have $|\nabla s| \leq C$.*

Proof. *Case* $1 < p < n+1$: The volume of K is given by

$$V = \frac{1}{n+1}\int s \det B[s] = \frac{1}{n+1}\int s^p f.$$

Hence we have

$$V \geq c_1 \int s^p$$

where $c_1 := \frac{1}{n+1}\min f$. Moreover, by the Alexandrov-Fenchel inequality,

$$\int s \geq c_2 V^{\frac{1}{n+1}} \Rightarrow \int s^p \geq c_3 V^{\frac{p}{n+1}},$$

where we used the Hölder inequality to get the r.h.s. Hence we obtain

$$V \geq c_4 V^{\frac{p}{n+1}} \Rightarrow V^{\frac{(n+1)-p}{n+1}} \geq c.$$

That is, the volume has a lower bound depending only on f.

Put $Q[\cdot] = Q[\cdot, s, \dots, s]$. By Lemma 2.12-(3),

$$\sigma_n = \det B[s] = Q^{ij}(\nabla^2_{i,j}s + sg_{ij}) \Rightarrow Q^{ij}\nabla^2_{i,j}s = \sigma_n - Q^{ij}g_{ij}s,$$

where Q^{ij} is the corresponding elliptic operator introduced in Lemma 2.12. Multiplying both sides by s^{1-p}, and using $s^{1-p}\sigma_n = f$ we get

$$s^{1-p}Q^{ij}\nabla^2_{i,j}s = f - Q^{ij}g_{ij}s^{2-p}.$$

Recall that by Exercise 2.14 in an orthonormal frame that diagonalizes $B[s]$,

$$Q^{ii} = \frac{1}{n}\frac{\det B[s]}{B[s]_i^i},$$

Hence $Q^{ij}g_{ij} \geq \sigma_n^{\frac{n-1}{n}}$. Now using $\nabla_i Q^{ij} = 0$ (cf. Lemma 2.12) we obtain

$$\int s^{\frac{n+1-p}{n}} f^{\frac{n-1}{n}} = \int \sigma_n^{\frac{n-1}{n}} s^{2-p} \leq \int Q^{ij}g_{ij}s^{2-p} \leq \int f.$$

Let $p, q \in \mathcal{M}^n$ such that the line segment joining p and q has length L (the extrinsic diameter of $\mathcal{M}^n$). Since $\mathcal{M}^n$ is origin symmetric, 0 is in the middle of the line segment. Hence $\pm\frac{L}{2}u \in \mathcal{M}^n$ for some $u \in \mathbb{S}^n$. For any $x \in \mathbb{S}^n$ we have

$$s(x) = \sup_{p\in K}\langle p, x\rangle \geq \frac{L}{2}|\langle u, x\rangle|.$$

Thus

$$\int f \geq \int s^{\frac{n+1-p}{n}} f^{\frac{n-1}{n}} \geq \left(\frac{L}{2}\right)^{\frac{n+1-p}{n}} \int f^{\frac{n-1}{n}}(x)|\langle u, x\rangle|^{\frac{n+1-p}{n}}.$$

Therefore, L has an upper bound depending only on f. Since K is origin-symmetric, this implies that s is positive and bounded. Moreover, since the volume is bounded below, the lower bound of s depends only on f.

Case $p > n+1$. Let x_0 be a point at which s attain its maximum. Then

$$s^n(x_0) \geq \det B[s](x_0) = s^{p-1}(x_0)f(x_0).$$

Therefore, $(\max s)^{n+1-p} \geq \min f$. Similarly $(\min s)^{n+1-p} \leq \max f$.

In either case the bound on the gradient of s follows as in Lemma 2.6. $\square$

By the previous lemma, $|\nabla s_{t_k}|$ is uniformly bounded above in k. Thus a subsequence of $\{s_{t_k}\}$ converges to a function s_{t_0} in $C^{\alpha}(\mathbb{S}^n)$. Then, as in the previous section, we may choose the sets P_k containing strictly a compact set Θ such that

$$\hat{s}_k := Q(x)s_{t_k}(\varphi(x)) - \lambda,$$

satisfies

$$\begin{cases} \det[\partial^2_{ij}\hat{s}_k] = f_{t_k}(\hat{s}_k + \lambda)^{p-1}Q^{-n-1-p} & \text{in } P_k, \\ \hat{s}_k = 0 & \text{on } \partial P_k. \end{cases} \tag{3.2}$$

Moreover, the r.h.s. of (3.2) is bounded away from 0 uniformly in k.

Applying Theorem 2.7 and Theorem 2.11, we deduce that $\{\hat{s}_k\}$ has uniform C^3 bound in Θ. Thus $\{\hat{s}_k\}$ converges in Θ in $C^{2,\alpha}$-norm (for any $\alpha \in (0,1)$) to $\hat{s}_0$. Hence $\hat{s}_0$ is a $C^{2,\alpha}$ solution of

$$\det[\partial^2_{ij}\hat{s}_0] = f_{t_0}(\tilde{s}_0 + \lambda)^{p-1}Q^{-n-1-p} \quad \text{in} \quad \Theta.$$

It can be shown that, in fact, $\hat{s}_0 \in C^{k+2,\alpha}(\Theta)$. This in particular implies that s_0 is a $C^{k+2,\alpha}$ solution of

$$\det B[s_0] = s_0^{p-1}f_{t_0} \quad \text{in} \quad \mathbb{S}^n.$$

Hence $t_0 \in S_\alpha$.

3.1.2. *Openness.* Define

$$F(u) = u^{1-p}\det B[u].$$

To show that S_α is open, we prove that F is locally invertible around any $s \in B_1$ with $B[s] > 0$. Note that

$$\mathcal{L}_s[u] := \frac{d}{dt}_{|t=0} F(s+tu) = (1-p)s^{-p}u\det B[s] + ns^{1-p}L_s[u],$$

where L_s was defined in (2.25). Suppose now $\mathcal{L}_s[u] = 0$ for some $u \in B_1$. Then $s^p\mathcal{L}_s[u] = 0$. That is,

$$(1-p)u\det B[s] + nsL_s[u] = 0.$$

Integrating this over $\mathbb{S}^n$ and using that L_s is self-adjoint we obtain

$$\begin{aligned} \int (1-p)u\det B[s] + nuL_s[s] = (n+1-p)\int u\det B[s] = 0 \\ \Rightarrow \int u\det B[s] = 0. \end{aligned} \tag{3.3}$$

On the other hand, by a theorem of Hilbert-Alexandrov (cf. [RK96, p. 138]) or the local Aleksandrov-Fenchel inequality, (3.3) implies that

$$\int uL_s[u] \leq 0,$$

and equality holds if $u(x) = \langle \vec{a}, x\rangle$ for some vector $\vec{a}$. Since $s^{p-1}u\mathcal{L}_s[u] = 0$, we have

$$(1-p)\int s^{-1}u^2 \det B[s] + nuL_s[u] = 0.$$

Hence, we arrive at

$$\int u^2 s^{-1} \det B[s] \leq 0.$$

That is, the kernel of $\mathcal{L}_s$ is trivial. Therefore by the inverse function theorem there is $\delta > 0$ such that for all functions $\|\tilde{f} - F(s)\|_{C^{k,\alpha}} < \delta$, the equation $F(\cdot) = \tilde{f}$ has a solution that is even and $C^{k+2,\alpha}$. In particular, for any $t \in S_\alpha$, the equation $F(\cdot) = f_{\tilde{t}}$ for $\tilde{t}$ sufficiently close to t has a solution such that $s_{\tilde{t}}$ and $\det B[s_{\tilde{t}}] > 0$. Therefore, $\tilde{t} \in S_\alpha$.

3.2. **Uniqueness.** Suppose $0 < s_i \in C^2$ with $\det B[s_i] > 0$. Now, by the L_p-Minkowski inequality [Lut93] for $p > 1$,

$$\left(\int s_1^p s_2^{1-p} \det B[s_2]\right)^{n+1} \geq \left(\int s_1 \det B[s_1]\right)^p \left(\int s_2 \det B[s_2]\right)^{n+1-p}, \tag{3.4}$$

and equality holds if and only if for some constant $c > 0$, we have

$$s_2 = cs_1.$$

Suppose and $1 < p < n+1$ and $\det B[s_i] = s_i^{p-1}f$, for $i = 1, 2$. Then

$$\left(\int s_1^p f\right)^{n+1} \geq \left(\int s_1^p f\right)^p \left(\int s_2^p f\right)^{n+1-p} \Rightarrow \int s_1^p f \geq \int s_2^p f.$$

Similarly, we have

$$\int s_2^p f \geq \int s_1^p f.$$

Hence, equality holds in (3.4) and we have

$$s_1 = s_2.$$

The argument for the case $p > n+1$ is similar.

4. Christoffel-Minkowski Problem

The main reference for this section is Guan-Ma [GM03]. The regular Christoffel-Minkowski problem asks the following question: Given a positive smooth function f on the unit sphere, what are the necessary and sufficient conditions on f that ensure the existence of a smooth, closed, strictly convex hypersurface whose σ_k(radii of curvature), as a function of the outer unit normal, is f? The problem is equivalent to finding a smooth function $s : \mathbb{S}^n \to [0, \infty)$ such that $B[s] > 0$ and

$$\sigma_k(B[s]) = f \tag{4.1}$$

A necessary condition for the existence of a strictly convex solution (hypersurface) is

$$\int x_i f = 0 \tag{4.2}$$

for all coordinate functions x_i. However, this is no longer a sufficient condition when $k < n$, and we need to impose another condition on f to guarantee the existence of solutions.

Theorem 4.1 (Christoffel-Minkowski problem)**.** [GM03] *Let $1 \le k < n$ and $f \in C^\infty(\mathbb{S}^n)$ be a positive function. Suppose $\int x_i f = 0$ for all coordinate functions x_i and $B[f^{-\frac{1}{k}}]$ is non-negative definite. Then, we can find a closed, smooth, strictly convex hypersurface in $\mathbb{R}^{n+1}$ whose support function s satisfies* (4.1)*. Moreover, any two such hypersurface must coincide after a translation.*

4.1. **Continuity method.** Let f_t, $t \in [0,1]$, be a suitable path that $f_0 = 1$ and $f_1 = f$. Let $S_\alpha = \{t \in [0,1] :$ the equation $\sigma_k(B[s]) = f_t$ has a $C^{\ell+2,\alpha}$ solution s_t such that $B[s_t] > 0$ and $\int s_t x_i = 0$ for all $i\}$. We want to show that for large ℓ, this set is both open and closed in $[0,1]$ and hence $S_\alpha = [0,1]$. Our suitable path is defined so that

$$B[f_t^{-\frac{1}{k}}] = \nabla^2 f_t^{-\frac{1}{k}} + g f_t^{-\frac{1}{k}} > 0, \quad t \in [0,1) \tag{4.3}$$

and $\int f_t x_i = 0$ for all i. The existence of such a path is proved in [BIS23b].

This particular, choice of “path” allows us to find a strictly convex solution via the full rank theorem Theorem 4.19.

Lemma 4.2. *Suppose $1 \le k \le n$ and $0 < f \in C^\infty(\mathbb{S}^n)$ satisfies*

$$\nabla^2 f^{-\frac{1}{k}} + f^{-\frac{1}{k}} g \ge 0, \quad \int x f(x) = 0.$$

Then for each $t \in [0,1)$, there exists $z_t \in \mathbb{R}^{n+1}$, such that

$$f_t(x) := (1 - t + t f^{-\frac{1}{k}}(x) - \langle x, z_t \rangle)^{-k}$$

satisfies

$$\nabla^2 f_t^{-\frac{1}{k}} + f_t^{-\frac{1}{k}} \bar{g} > 0, \quad \int x f_t(x) = 0.$$

Moreover, we have

$$|z_t| \le 1 + \max f^{-\frac{1}{k}}. \tag{4.4}$$

Proof. Note that for $0 \le t \le 1$,

$$s_{L_t} := 1 - t + t f^{-\frac{1}{k}}$$

is the support function of a convex body L_t, which is smooth and strictly convex for $t < 1$. By [Iva16, Lem. 3.1] (see the proof of Remark 2.4 here), there exists a unique point z_t in the interior of L_t such that

$$\begin{cases} \min\limits_{v \in L_t} \int -\log(s_{L_t - v}), & k = 1 \\ \min\limits_{v \in L_t} \int s_{L_t - v}^{-k+1}, & k > 1 \end{cases}$$

is attained. Hence the support function of $L_t - z_t$ given by $f_t^{-\frac{1}{k}}$ is positive and satisfies the required integral condition, see [Iva16, Lem. 3.1]. Since z_t is in the interior of L_t, the upper bound on the norm of z_t follows. ∎

4.1.1. *Closedness.* To show the closedness part of the continuity argument, we need some introduction to curvature functions. We follow the exposition in [ACGL20].

Definition 4.3. A function $q : \mathbb{R}^n \to \mathbb{R}$ is said to be S_n-invariant (or symmetric) if

$$q(z_1, \dots, z_n) = q(z_{\sigma(1)}, \dots, z_{\sigma(n)}), \quad \forall \sigma \in S_n,$$

where S_n is the group of permutations of the set $\{1, \dots, n\}$.

Let $S(n)$ denote the set of symmetric $n \times n$ matrices. Then a function $q : S(n) \to \mathbb{R}$ is said to be $SO(n)$-invariant (or symmetric) if $q(P) = q(Q^{-1}PQ)$ for all $Q \in SO(n)$. Here $SO(n)$ is the special orthogonal group of degree n.

Remark 4.4. Let λ denote the eigenvalue map. This is a multivalued map that assigns to a symmetric matrix P the set of n-tuples $\lambda(P)$ with components given by its eigenvalues. Now, for any S_n-invariant function q, there is a corresponding $SO(n)$-invariant function $\hat{q} : S(n) \to \mathbb{R}$ defined by $\hat{q}(P) = q(z_1, \dots, z_n)$ for any $(z_1, \dots, z_n) \in \lambda(P)$. Since q is symmetric, $\hat{q}$ is well-defined. Now suppose $\hat{q}$ is $SO(n)$-invariant. Define $q(z_1, \dots, z_n) = \hat{q}(P)$, where $P \in \lambda^{-1}([(z_1, \dots, z_n)])$ is the orbit of $(z_1, \dots, z_n)$ under the S_n action. Since $\hat{q}$ is symmetric, it takes the same value on any two symmetric matrices with equal eigenvalues. Hence, q is well-defined. In summary, every S_n-invariant function gives rise to a canonical $SO(n)$-invariant function and vice versa. Hence, from now on, we may use the letter q for both a function of matrix variables and as a function of eigenvalue variables.

Let q be a symmetric function. Define

$$\dot{q}_z^i v_i = \frac{d}{dt}_{|t=0} q(z + tv), \quad \ddot{q}_z^{ij} v_i v_j = \frac{d^2}{dt^2}_{|t=0} q(z + tv), \quad \forall z, v \in \mathbb{R}^n$$

and

$$\dot{q}_Z^{ij} V_{ij} = \frac{d}{dt}_{|t=0} q(Z + tV), \quad \ddot{q}_Z^{kl,rs} V_{kl} V_{rs} = \frac{d^2}{dt^2}_{|t=0} q(Z + tV) \quad \forall Z, V \in S(n).$$

Theorem 4.5. *Let q be a symmetric function. Then q is smooth with respect to the matrix variables if and only if it is smooth with respect to the eigenvalues variables. Moreover, the first and the second derivatives are related by the following formulae: For any* <u>*diagonal*</u> *matrix Z in the matrix domain of q with eigenvalue n-tuple $z \in \lambda(Z)$,*

$$\dot{q}_Z^{kl} = \begin{cases} \dot{q}_z^k, & k = l, \\ 0, & k \neq l, \end{cases}$$

and if the eigenvalues are all distinct, then

$$\ddot{q}_Z^{kl,rs} V_{pq} V_{rs} = \sum_{k,l} \ddot{q}_z^{kl} V_{kk} V_{ll} + \sum_{k \neq l} \frac{\dot{q}_z^k - \dot{q}_z^l}{z_k - z_l} (V_{kl})^2, \tag{4.5}$$

for any $V \in SO(n)$.

Remark 4.6. Let s be the support function of a smooth, strictly convex body K and $f = q(\lambda)$, where as usual $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the principal radii of curvature K, and q is a symmetric function. By the previous remark, we may write $f = q(B[s])$.

Choose an orthonormal frame of $T\mathbb{S}^n$ such that it diagonalizes $\alpha_{ij} := B[s]_i^j$ at a given point. Due to Theorem 4.5, assuming the eigenvalues are distinct:

$$\begin{aligned}\nabla_i f &= \dot{q}_\alpha^{kl} \nabla_i \alpha_{kl},\\ \nabla^2_{i,i} f &= \dot{q}_\alpha^{kl} \nabla^2_{i,j} \alpha_{kl} + \ddot{q}_\alpha^{kl,rs} \nabla_i \alpha_{kl} \nabla_i \alpha_{rs}\\ &= \dot{q}_\lambda^k \nabla^2_{i,i} \alpha_{kk} + \sum_{p,q} \ddot{q}_\lambda^{kl} \nabla_i \alpha_{kk} \nabla_i \alpha_{ll} + \sum_{k \neq l} \frac{\dot{q}_\lambda^k - \dot{q}_\lambda^l}{\lambda_k - \lambda_l} (\nabla_i \alpha_{kl})^2.\end{aligned}$$

We say $z \in \mathbb{R}^n$ is *simple* if its components are pairwise distinct.

Lemma 4.7 (Concavity)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open convex set. Suppose $q : \Omega \to \mathbb{R}$ is smooth and symmetric. If q is concave, then for every simple $z \in \Omega$,*

$$\frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} \leq 0$$

for each pair (i, j) with $i \neq j$.

Proof. Suppose q is concave. Then, for any vector $v \in \mathbb{R}^n$ and any $t \geq 0$ such that $z + tv \in \Omega$, we have

$$\frac{d}{dt} \dot{q}^i(z + tv) v_i = \frac{d^2}{dt^2} q(z + tv) \leq 0.$$

Hence,

$$\dot{q}^k(z + tv) v_k \leq \dot{q}^k(z) v_k, \quad \forall k.$$

Set $v = -(e_i - e_j)$, where e_i is the basis vector in the direction of the i-th coordinate. Then

$$(\dot{q}^i - \dot{q}^j)|_z \leq (\dot{q}^i - \dot{q}^j)|_{z - t(e_i - e_j)}.$$

Suppose $z_i > z_j$. Then there is some $t_0 \geq 0$, such that

$$(z - t_0(e_i - e_j))_i = (z - t_0(e_i - e_j))_j.$$

Let

$$w := (z_1, \ldots, z_i - t_0, \ldots, z_j + t_0, \ldots, z_n) = (z_1, \ldots, z_j + t_0, \ldots, z_i - t_0, \ldots, z_n).$$

Note that $z - t_0(e_i - e_j)$ lies on the line segment joining z and the point obtained from z by switching its i-th and j-th coordinates; hence, $z - t_0(e_i - e_j) \in \Omega$. Since q is symmetric,

$$\begin{aligned}(4.6) \qquad & q(z_1, \ldots, z_i - (t_0 + r), \ldots, z_j + t_0 + r, \ldots, z_n)\\ &= q(z_1, \ldots, z_j + t_0 + r, \ldots, z_i - (t_0 + r), \ldots, z_n).\end{aligned}$$

for all t sufficiently small. Taking derivative of both sides of (4.6) with respect to r yields

$$-\dot{q}_w^i + \dot{q}_w^j = \dot{q}_w^i - \dot{q}_w^j.$$

That is, $\dot{q}^i = \dot{q}^j$ at w, and hence the claim follows. ∎

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be an open convex set. A smooth symmetric function $q : \Omega \to \mathbb{R}$ is concave with respect to the eigenvalue variables if and only if it is concave with respect to the matrix variables.*

Proof. Due to identity (4.5), for any symmetric matrix V we have

$$\ddot{q}_Z^{ij,kl} V_{ij} V_{kl} = \ddot{q}_z^{ij} V_{ii} V_{jj} + 2 \sum_{i>j} \frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} (V_{ij})^2$$

at any diagonal matrix Z with distinct eigenvalues z_i. Hence, the concavity of q at Z with respect to the matrix component implies the concavity of q at z with respect to the eigenvalues. The converse follows from the previous lemma.

To see if the claim holds at any diagonal matrix Z, observe that this is the limiting case along a sequence $Z^{(k)}$ of diagonal matrices with distinct eigenvalues, which limits to Z. The general case follows from the invariance of q with respect to similarity transformations. $\square$

4.2. **Hyperbolic polynomials.** We write $\mathbb{R}[x_1, \dots, x_m]$ for the set of polynomials in $x_1, \dots, x_m$ with real coefficients. We say a homogeneous polynomial $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic in direction $\xi \in \mathbb{R}^n$ if and only if

$$\begin{gathered} \mathcal{P}(\xi) > 0, \quad \text{and} \\ \forall x \in \mathbb{R}^n \quad \mathcal{P}(x + t\xi) \in \mathbb{R}[t] \quad \text{is real rooted.} \end{gathered} \tag{4.7}$$

Define the cone

$$\mathcal{C}(\mathcal{P}, \xi) = \{x \in \mathbb{R}^n : \mathcal{P}(x - t\xi) \in \mathbb{R}[t] \quad \text{has positive roots}\}. \tag{4.8}$$

Now we collect a few standard facts about the hyperbolic polynomials.

Lemma 4.9. *Assume $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic in direction $\xi \in \mathbb{R}^n$. Then $D_\xi p$ is hyperbolic in direction ξ as well, unless $D_\xi \mathcal{P} = 0$.*

Lemma 4.10. [Nui68] *Assume $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic in direction $\xi \in \mathbb{R}^n$. Then for any $v \in \mathcal{C}(p, \xi)$, the polynomial*

$$\mathcal{H}(x_1, \dots, x_n, x_{n+1}) = \mathcal{P}(x) - x_{n+1} D_v \mathcal{P}(x) \in \mathbb{R}[x_1, \dots, x_n, x_{n+1}] \tag{4.9}$$

is hyperbolic in direction $(v, 0)$. Moreover, if $x \in \mathcal{C}(\mathcal{P}, v) = \mathcal{C}(\mathcal{P}, \xi)$ and $\mathcal{H}(x_1, \dots, x_n, y) > 0$, then $(x, y) \in \mathcal{C}(\mathcal{H}, (v, 0))$.

Lemma 4.11. *If $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic in direction $\xi \in \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ is a linear subspace containing ξ, then the restriction $\mathcal{P} : V \to \mathbb{R}$ is also hyperbolic in direction ξ.*

Theorem 4.12. [Gül, Thm. 6.1], [BGLS01, Cor. 4.7] *Let $\mathcal{P} \in \mathbb{R}[x_1, \dots, x_n]$ be hyperbolic in direction $\xi \in \mathbb{R}^n$. For $v \in \mathcal{C}(p, \xi)$, define the barrier in direction v as*

$$D_v \log \mathcal{P} = \frac{D_v \mathcal{P}}{\mathcal{P}}. \tag{4.10}$$

Then $\frac{\mathcal{P}}{D_v \mathcal{P}}$ is concave on $\mathcal{C}(\mathcal{P}, \xi)$.

Example 4.13. Let $\Gamma_+^n := \{z \in \mathbb{R}^n : z_i > 0 \quad \text{for each } i\}$. Consider $q = \sigma_k^{\frac{1}{k}} : \Gamma_+^n \to \mathbb{R}$, defined by $q(z) = \left(\sum_{i_1 < \cdots < i_k} z_{i_1} \cdots z_{i_k}\right)^{\frac{1}{k}}$. Then q is concave.

Before delving into the C^2 estimate, let us explain the necessary condition (4.2) that f has to satisfy.

Lemma 4.14. *Let $s \in C^2(\mathbb{S}^n)$ and $B[s] \geq 0$. Suppose $\sigma_k(B[s]) = f$. Then*

$$\int x_i f = 0$$

Proof. We may first assume $B[s] > 0$, and then approximate. Due to Lemma 2.12,

$$\begin{aligned} c_k \int x_i f &= \int x_i Q[s, \ldots, s, 1, \ldots, 1] \\ &= \int x_i Q^{ij}(\nabla^2_{i,j} s + g_{ij} s) \\ &= \int s Q^{ij}(\nabla^2 x_i + g x_i). \end{aligned}$$

Here, s appears k times in $Q[s, \ldots, s, 1, \ldots, 1]$ and $c_k > 0$. Since

$$\nabla^2 x_i + g x_i = \operatorname{Hess}_{\mathbb{R}^{n+1}} x_i = 0,$$

the claim follows. □

Exercise 4.15. Let $q = \sigma_k(B[s])$. Show that $\dot{q}^{ij}\delta_{ij} \geq c_k q^{\frac{k-1}{k}}$ for some constant $c_k > 0$. *Hint*: In an orthonormal frame that diagonalizes $B[s]$:

$$\dot{q}^{ii} = \dot{q}^i_\lambda = \frac{\partial \sigma_k}{\partial \lambda_i}$$

and $\dot{q}^{ij} = 0$ for $i \neq j$.

Lemma 4.16. *Suppose $s > 0$ is a smooth, strictly convex solution of $\sigma_k(B[s]) = f$. Then* $\max s$ *is bounded, and the bound depends only on* $\min f, \max f$.

Proof. Let $q = \sigma_k(B[s])$ and $\alpha_{ij} = B[s]_i^j$. Note that

$$\dot{q}^{ij}\alpha_{ij} = \frac{\partial \sigma_k}{\partial \lambda_i}\lambda_i = kf.$$

Using $\dot{q}^{ij}\delta_{ij} \geq c_k q^{\frac{k-1}{k}}$ (see Exercise 4.15), $\nabla_i q^{ij} = 0$ (cf. Lemma 2.12), and integration by parts, we obtain

$$\int s f^{\frac{k-1}{k}} \leq c_1 \int f.$$

We may assume $s(1, 0, \ldots, 0) = \max s$. By convexity, we have $s(x) \geq (\max s)x_1$. Hence,

$$\max s \leq \frac{c_2 \int f}{\int_{\{x : x_1 > \frac{1}{2}\}} x_1 f^{\frac{k-1}{k}}}. \tag{4.11}$$

□

Exercise 4.17. Show that $g^{jk}\nabla^2_{i,k}\sigma_1 = \Delta B[s]_i^j - nB[s]_i^j + \sigma_1\delta_{ij}$.

Hint: The Riemannian curvature of $\mathbb{S}^n$ is given by

$$R_{kjil} = g_{ik}g_{jl} - g_{ij}g_{lk}.$$

Apply formula (1.8) to $h = \nabla^2 s + gs$ and use $\nabla g = 0$.

Lemma 4.18. *Suppose $s > 0$ is a smooth, strictly convex solution of $\sigma_k(B[s]) = f$. Then for each $\ell \geq 1$ and $\gamma \in (0,1)$, $\|s\|_{C^{\ell+1,\gamma}} \leq C_{\ell,\gamma}$ for some constant depending only on f.*

Proof. We only need to consider the case $k > 1$. Define $q = \sigma_k^{\frac{1}{k}}(B[s])$ and $\alpha_{ij} = B[s]_i^j$. By Exercise 4.17,

$$g^{kj}\nabla^2_{i,k}\sigma_1 = \Delta\alpha_{ij} - n\alpha_{ij} + \sigma_1\delta_{ij}.$$

Hence,

$$\begin{aligned}\sum_{i,j}\dot{q}^{ij}g^{kj}\nabla^2_{i,k}\sigma_1 &= \dot{q}^{ij}\Delta\alpha_{ij} - n\dot{q}^{ij}\alpha_{ij} + \dot{q}^{ij}\delta_{ij}\sigma_1 \\ &= \dot{q}^{ij}\Delta\alpha_{ij} - nf^{\frac{1}{k}} + \dot{q}^{ij}\delta_{ij}\sigma_1 \\ &= \Delta q - g^{mn}\ddot{q}^{ij,kl}\nabla_m\alpha_{ij}\nabla_n\alpha_{kl} - nf^{\frac{1}{k}} + \dot{q}^{ij}\delta_{ij}\sigma_1.\end{aligned}$$

Therefore, in an orthonormal frame that diagonalizes α, using the concavity of q (cf. Theorem 4.8), we obtain

$$\begin{aligned}\dot{q}^{ij}\delta_{ij}\sigma_1 &= \dot{q}^{ij}\nabla^2_{i,j}\sigma_1 + \ddot{q}^{ij,kl}\nabla_m\alpha_{ij}\nabla_m\alpha_{kl} + nf^{\frac{1}{k}} - \Delta f^{\frac{1}{k}} \\ &\leq \dot{q}^{ij}\nabla^2_{i,j}\sigma_1 + nf^{\frac{1}{k}} - \Delta f^{\frac{1}{k}}.\end{aligned} \tag{4.12}$$

Moreover, we have $\dot{q}^{ij}\delta_{ij} \geq c_k$. Thus, at the maximum of σ_1 there holds

$$c_k\sigma_1 \leq nf^{\frac{1}{k}} - \Delta f^{\frac{1}{k}}. \tag{4.13}$$

From this we obtain $\|s\|_{C^2} \leq C$, where C depends only on n, $\min f$ and $\|f\|_{C^2}$. The higher order estimate, follows from Evans-Krylov and Schauder regularity theory. ☐

The following theorem completes the closedness of the set S_α.

Theorem 4.19 (Full rank theorem)**.** *Let $0 < f \in C^\infty(\mathbb{S}^n)$ and $s : \mathbb{S}^n \to \mathbb{R}$ be a smooth function, with $B[s] \geq 0$ and $\sigma_k(B[s]) = f$. If $B[f^{-\frac{1}{k}}] \geq 0$, then $B[s] > 0$. That is, s is the support function of a closed, smooth, strictly convex hypersurface.*

4.2.1. *Openness.* The argument to show openness is similar to the one we have seen for the Minkowski problem. Here we only show the kernel of the corresponding linearized operator is "trivial".

Let s be the support function of a smooth, strictly convex body. Let $q = \sigma_k(B[s])$. Define the linearized operator

$$L_{s,k}[u] = \frac{d}{dt}_{|t=0}\sigma_k(B[s+tu]) = \dot{q}^{ij}(\nabla^2_{i,j}u + \delta_{ij}u), \quad u \in C^2(\mathbb{S}^n).$$

Lemma 4.20. *Suppose $L_{s,k}[u] = 0$, then $u(x) = \langle \vec{a}, x\rangle$ for some vector $\vec{a}$.*

Proof. Let us define $Q_k[u] = Q[u, s, \ldots, s, 1 \ldots, 1]$, where s appears $k-1$ times. Then, by the (local) Aleksandrov-Fenchel inequality,

$$\left(\int uQ_k[s]\right)^2 \geq \int uQ_k[u] \int sQ_k[s]. \tag{4.14}$$

Moreover, equality holds if and only if $u(x) = as + \langle \vec{a}, x\rangle$ for some vector $\vec{a}$ and constant a.

Now note that

$$\sigma_k(B[s + tu]) = c_k Q[s + tu, \ldots, s + tu, 1, \ldots, 1],$$

where $s + tu$ appears k times. Hence,

$$0 = L_{s,k}[u] = kc_k Q_k[u].$$

Since $\int uQ_k[s] = \int sQ_k[u] = 0$ (cf. Lemma 2.12), we have equality in (4.14) and the claim follows (from $Q_k[u] = 0$ we deduce that $a = 0$). $\square$

Having Lemma 4.20 in hand, we can then proceed as in the case of the Minkowski problem to show the openness of S_α in $[0, 1]$.

4.3. **Full rank theorem.** In this section, we prove Theorem 4.19. The original proof by Guan-Ma in [GM03] is very complicated. We outline a simple proof from [BIS23a].

Definition 4.21 (Inverse concavity)**.** Suppose $q : \Gamma_+^n \to \mathbb{R}$ is positive, S_n-invariant. We say q is inverse-concave if the dual function $q_* : \Gamma_+^n \to \mathbb{R}$ defined by

$$q_*(z_1^{-1}, \ldots, z_n^{-1}) = q(z_1, \ldots, z_n)^{-1}$$

is concave. Similarly, let $q : S_+(n) \to \mathbb{R}$ be a positive $SO(n)$-invariant function if the function

$$q_* : S_+(n) \to \mathbb{R}$$

defined by $q_*(Z^{-1}) = q(Z)^{-1}$ is concave, then we say q is inverse-concave.

Theorem 4.22. *Let q be a positive symmetric function. Then q is inverse concave if and only if the quadratic form $Q_Z : S(n) \times S(n) \to \mathbb{R}$ defined by*

$$Q_Z(V, V) := \ddot{q}_Z(V, V) + 2\dot{q}_Z(VZ^{-1}V) - \frac{2\dot{q}_Z(V)\dot{q}_Z(V)}{q(Z)}$$

is nonnegative definite for all $Z \in S_+(n)$, where juxtaposition of matrix variables denotes matrix multiplication. Equivalently, q is inverse-concave if and only if the quadratic form $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q_z(v, v) = \ddot{q}_z(v, v) - 2\frac{(\dot{q}_z(v))^2}{q(v)} + 2\dot{q}_z(vz^{-1}v)$$

is nonnegative definite for all $z \in \Gamma_+^n$ and, in addition,

$$\frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} + \frac{\dot{q}_z^i}{z_j} + \frac{\dot{q}_z^j}{z_i} \geq 0$$

for each $i \neq j$ and each simple z, where $z^{-1} := (z_1^{-1}, \dots, z_n^{-1})$ and the juxtaposition of eigenvalue variables denotes components-wise multiplication.

Proof. The first derivative of $q_*(Z^{-1})$ with respect to $Z \in S_+(n)$ in the direction $V \in S(n)$ is given by

$$q(Z)^{-2}\dot{q}_Z(V) = \dot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}), \tag{4.15}$$

while the second derivative is given by

$$2\frac{(\dot{q}_Z(V))^2}{q(Z)^3} - \frac{\ddot{q}_Z(V,V)}{q(Z)^2} = \ddot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}, Z^{-1}VZ^{-1}) + 2\dot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}VZ^{-1}).$$

Here we used that the derivative of Z^{-1} in the direction V is $-Z^{-1}VZ^{-1}$. Now the first claim follows from (4.15):

$$\begin{aligned} q(Z)^{-2}\left(\ddot{q}_Z(V,V) - 2\frac{(\dot{q}_Z(V))^2}{q(Z)} + 2\dot{q}_Z(VZ^{-1}V)\right) \\ = -\ddot{q}_{*Z^{-1}}(Z^{-1}VZ^{-1}, Z^{-1}VZ^{-1}). \end{aligned}$$

To prove the second claim, we differentiate one $q_*(z^{-1})$ with respect to $z \in \Gamma_+$ in the direction $v \in \mathbb{R}^n$ to obtain

$$\dot{q}_{*z^{-1}}(z^{-1}vz^{-1}) = q(z)^{-2}\dot{q}_z(v). \tag{4.16}$$

Differentiating once more and using (4.16), we obtain

$$-\ddot{q}_{*z^{-1}}(z^{-1}vz^{-1}, z^{-1}vz^{-1}) = q(z)^{-2}\left(\ddot{q}_z(v,v) - 2\frac{(\dot{q}_z(v))^2}{q(v)} + 2\dot{q}_z(vz^{-1}v)\right).$$

Moreover, we have

$$\begin{aligned} \frac{\dot{q}_{*z^{-1}}^i - \dot{q}_{*z^{-1}}^j}{z_i - z_j} &= \frac{1}{q(z)^2(z_i - z_j)}(\dot{q}_z^i z_i^2 - \dot{q}_z^j z_j^2) \\ &= \frac{z_i z_j}{q(z)^2}\left(\frac{\dot{q}_z^i - \dot{q}_z^j}{z_i - z_j} + \frac{\dot{q}_z^i}{z_j} + \frac{\dot{q}_z^j}{z_i}\right). \end{aligned}$$

□

Remark 4.23. Note that $q : \Gamma_+ \to \mathbb{R}$ defined by $q(z) = \left(\sum_{i_1 < \dots < i_k} z_{i_1} \dots z_{i_k}\right)^{\frac{1}{k}}$ is inverse concave. Let $s \in C^2(\mathbb{S}^n)$ and $B[s] > 0$. Now in Theorem 4.22 take $Z_{ij} = \alpha_{ij} = B[s]_i^j$ and $V = \nabla_1 \alpha$. Therefore,

$$\sum_{i,j,k,l} \ddot{q}_\alpha^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} + 2\sum_j \dot{q}_\alpha^{ii} \frac{(\nabla_1 \alpha_{ij})^2}{\lambda_j} \geq 2\frac{(\nabla_1 q(\alpha))^2}{q(\alpha)}. \tag{4.17}$$

Corollary 4.24. *Let $u \in C^2(\mathbb{S}^n)$ with $B[u] \geq 0$ and $\lambda_1 \leq \cdots \leq \lambda_n$ denote the eigenvalues of $\alpha_{ij} = B[u]^j_i$. Let $\{e_i\}_i$ be a local orthonormal frame of $T\mathbb{S}^n$ with $B[u]|_{x_0}(e_i, e_j) = \delta_{ij}\lambda_i$. Suppose μ is the multiplicity of the smallest eigenvalue $\lambda_1 = 0$ at x_0. If $q(B[u])$ is inverse-concave and $q(\alpha) = f > 0$, then at x_0:*

$$\sum_{i,j,k,l>\mu} \ddot{q}^{ij,kl}_{\alpha} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} + 2\sum_{j>\mu} \dot{q}^{ii}_{\alpha} \frac{(\nabla_1 \alpha_{ij})^2}{\lambda_j} \geq 2\frac{(\nabla_1 f)^2}{f}.$$

Proof. Define

$$V_{ij} = \begin{cases} \nabla_1 \alpha_{ij} & i, j > \mu \\ 0 & i \leq \mu \text{ or } j \leq \mu. \end{cases}$$

Moreover, define $\alpha_\varepsilon = B[u] + \varepsilon\,\mathrm{id}$. By the previous lemma,

$$Q_{\alpha_\varepsilon}(V, V) \geq 0.$$

Hence, in view of (4.17),

$$\sum_{i,j,k,l} \ddot{q}^{ij,kl}_{\alpha_\varepsilon} V_{ij} V_{kl} + 2\sum_{j} \dot{q}^{ii}_{\alpha_\varepsilon} \frac{(V_{ij})^2}{\lambda_j + \varepsilon} \geq 2\frac{(\nabla_1 q(\alpha_\varepsilon))^2}{q(\alpha_\varepsilon)^2}.$$

This implies that

$$\sum_{i,j,k,l>\mu} \ddot{q}^{ij,kl}_{\alpha_\varepsilon} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} + 2\sum_{j>\mu} \dot{q}^{ii}_{\alpha_\varepsilon} \frac{(\nabla_1 \alpha_{ij})^2}{\lambda_j + \varepsilon} \geq 2\frac{(\nabla_1 q(\alpha_\varepsilon))^2}{q(\alpha_\varepsilon)^2}.$$

Now let $\varepsilon \to 0$. ☐

Lemma 4.25. [BCD17] *Let $u \in C^2(\mathbb{S}^n)$ and $\lambda_1 \leq \cdots \leq \lambda_n$ denote the eigenvalues of $B[u]$ and $\tau = \nabla^2 u + gu$. Let $\{e_i\}_i$ be a local orthonormal frame of $T\mathbb{S}^n$ with $\tau|_{x_0}(e_i, e_j) = \delta_{ij}\lambda_i$. Suppose μ is the multiplicity of the smallest eigenvalue at x_0. Let ψ be a smooth function such that $\psi \leq \lambda_1$ everywhere and $\psi(x_0) = \lambda_1(x_0)$. Then, at x_0, we have*

$$\nabla^2_{i,i}\psi \leq \nabla^2_{i,i}\tau_{11} - 2\sum_{j>\mu} \frac{(\nabla_i \tau_{1j})^2}{\lambda_j - \lambda_1}.$$

Moreover, for any lower support of ψ at x_0 we have $\nabla_i \tau_{kl} = \nabla_i \psi \delta_{kl}$ for $1 \leq k, l \leq \mu$.

Proof of the full rank theorem; Theorem 4.19. Recall that

$$\begin{aligned} \nabla_m f^{\frac{1}{k}} &= \dot{q}^{ij} \nabla_m \alpha_{ij}, \\ \nabla^2_{m,n} f^{\frac{1}{k}} &= \dot{q}^{ij} \nabla^2_{m,n} \alpha_{ij} + \ddot{q}^{ij,kl} \nabla_m \alpha_{ij} \nabla_n \alpha_{kl}. \end{aligned}$$

Let $\tau = \nabla^2 s + gs$. We have

$$\nabla^2_{m,n}\alpha_{ij} = g^{pj} \nabla^2_{m,n} r_{ip} = g^{pj} \left(\nabla^2_{i,p} \tau_{mn} - \tau_{mn} g_{ip} + \tau_{ip} g_{mn} \right).$$

In particular, for $m = n = 1$:

$$\begin{aligned} \nabla^2_{1,1}\alpha_{ij} &= g^{pj} \nabla^2_{i,p} \tau_{11} - \tau_{11}\delta_{ij} + \alpha_{ij}, \\ \nabla^2_{1,1} f^{\frac{1}{k}} &= \dot{q}^{ij} g^{pj} \nabla^2_{i,p} \tau_{11} + \ddot{q}^{ij,kl} \nabla_1 \alpha_{ij} \nabla_1 \alpha_{kl} - \tau_{11} \dot{q}^{ij} \delta_{ij} + f^{\frac{1}{k}}. \end{aligned}$$

Hence, at x_0:

$$\nabla^2_{1,1} f^{\frac{1}{k}} = \dot{q}^{ii}\nabla^2_{i,i}\tau_{11} + \ddot{q}^{ij,kl}\nabla_1\alpha_{ij}\nabla_1\alpha_{kl} - \tau_{11}\dot{q}^{ij}\delta_{ij} + f^{\frac{1}{k}}.$$

In view of Lemma 4.25, at x_0 we have

$$\dot{q}^{ii}\nabla^2_{i,i}\psi \leq -\ddot{q}^{ij,kl}\nabla_1\alpha_{ij}\nabla_1\alpha_{kl} - 2\sum_{l>\mu}\frac{\dot{q}^{ii}(\nabla_i r_{1l})^2}{\lambda_l} + \nabla^2_{1,1}f^{\frac{1}{k}} - f^{\frac{1}{k}} + \lambda_1\dot{q}^{ij}\delta_{ij}.$$

Also, note that

$$\begin{aligned}\dot{q}^{ii}\nabla^2_{i,i}\psi \leq &- \sum_{i,j,k,l>\mu}\ddot{q}^{ij,kl}\nabla_1\alpha_{ij}\nabla_1\alpha_{kl} - 2\sum_{l>\mu}\frac{\dot{q}^{ii}(\nabla_i r_{1l})^2}{\lambda_l}\\ &- \sum_{(i,j,k,l)\in\Lambda}\ddot{q}^{ij,kl}\nabla_1\alpha_{ij}\nabla_1\alpha_{kl} + \nabla^2_{1,1}f^{\frac{1}{k}} - f^{\frac{1}{k}} + \lambda_1\dot{q}^{ij}\delta_{ij},\end{aligned}$$

where Λ is the complement of the set $\{i,j,k,l>\mu\}$ in $\{1,\ldots,n\}^4$. Due to Corollary 4.24, that $\nabla\alpha$ is fully symmetric, Lemma 4.25, and our assumption that $B[f^{-\frac{1}{k}}]\geq 0$, we have

$$\begin{aligned}\dot{q}^{ii}\nabla^2_{i,i}\psi &\leq c|\nabla\psi| + \nabla^2_{1,1}f^{\frac{1}{k}} - 2f^{-\frac{1}{k}}|\nabla_1 f^{\frac{1}{k}}|^2 - f^{\frac{1}{k}} + \psi\dot{q}^{ij}\delta_{ij}\\ &= c|\nabla\psi| - f^{\frac{2}{k}}B[f^{-\frac{1}{k}}]_1^1 + \psi\dot{q}^{ij}\delta_{ij}\\ &\leq c|\nabla\psi| + \psi\dot{q}^{ij}\delta_{ij}.\end{aligned}$$

The strong maximum principle shows if λ_1 is zero at some point, then $\lambda_1\equiv 0$. However, at a point where the support function attains its maximum, we have $\lambda_1>0$. ☐

4.4. **Uniqueness.** Suppose we have

$$\sigma_k(B[s_1]) = \sigma_k(B[s_2]) = f.$$

Then, by (4.14), we have

$$\left(\int s_2 Q_k[s_1]\right)^2 \geq \int s_2 Q_k[s_2]\int s_1 Q_k[s_1].$$

Therefore, $\int s_2 Q_k[s_2] = \int s_2 Q_k[s_1] \geq \int s_1 Q_k[s_1]$. Similarly, $\int s_2 Q_k[s_1] \leq \int s_1 Q_k[s_1]$. That is, we have equality in (4.14); hence, uniqueness holds up to a linear function.

5. L_p-Christoffel-Minkowski problem

In this section, we study the even L_p-Christoffel-Minkowski problem.

$$\sigma_k = f s^{p-1}. \tag{5.1}$$

For the case $p\geq k+1$ see [HMS04]. Here, we focus on the (most interesting) case $1<p<k+1$. We prove the following theorem.

Theorem 5.1. [GX18] *Let $1\leq k<n$ and $1<p<k+1$ and $f\in C^\infty(\mathbb{S}^n)$ be a positive, even function. Suppose $B[f^{-\frac{1}{p+k-1}}]$ is non-negative definite. Then, a unique origin-symmetric, smooth, strictly convex hypersurface exists in $\mathbb{R}^{n+1}$ satisfying (5.1).*

Again, our primary method of establishing this existence result is the continuity method in combination with a constant rank theorem. Regarding the continuity method, we only establish the closedness part, i.e., the C^0 and C^2 a priori estimates (verifying the openness part is similar to the case of the L_p-Minkowski problem). We follow the approach in [HI23].

Lemma 5.2. [CW00] *Let $\mathcal{M}^n$ be a smooth, origin-symmetric, strictly convex hypersurface. Let $R = \max s$ and $r = \min s$. We have either $\frac{R}{r} \le \sqrt{n+1}$ or $\frac{R^2}{r} \le C_n \max_{\mathbb{S}^n} \lambda_n$, where C_n is a constant depending only on n.*

Proof. Suppose $R > r\sqrt{n+1}$. Due to convexity, we may find two perpendicular directions, say e_1, e_2, such that $s(e_2) = r$ and

$$s(e_1) > \frac{R}{\sqrt{n+1}}.$$

Now project $\mathcal{M}$ to the x_1x_2-plane and denote the corresponding convex body by P. Since P is origin-symmetric, $(\pm R/2\sqrt{n+1}, 0)$ are in the interior of P and $(0, \pm r) \in \partial P$.

For simplicity, let $P_r := \frac{1}{r}P$, and write D for the disk of radius $1/2$ centered at the origin. We have $D \subset P_r$. Let $\tilde{D}$ denote the convex hull of D and $(\pm R/(2r\sqrt{n+1}), 0)$. Then $\partial\tilde{D}$ is the union of four tangential segments to the circle ∂D and two closed arcs of ∂D. The four tangent lines are given by

$$y = \pm \frac{1}{\sqrt{\left(\frac{R}{r\sqrt{n+1}}\right)^2 - 1}} \left(x \pm \frac{R}{2r\sqrt{n+1}}\right).$$

Now it is easy to verify that the rectangle

$$-\frac{R}{4r\sqrt{n+1}} \le x_1 \le \frac{R}{4r\sqrt{n+1}}, \quad -\frac{1}{4} \le x_2 \le \frac{1}{4}$$

is contained in the interior of $\tilde{D}$. Since the ellipse

$$E_0 = \left\{(x_1, x_2) : 16r^2(n+1)\frac{x_1^2}{R^2} + 16x_2^2 \le 1\right\}$$

lies in this rectangle, rE_0 is contained in the interior of P. Therefore, for some $r/4 \le h \le r$,

$$E_1 := \left\{(x_1, x_2) : 16(n+1)\frac{x_1^2}{R^2} + \frac{x_2^2}{h^2} \le 1\right\} \subset P,$$

while touching P at

$$\vec{a} := \left(\frac{R}{4\sqrt{n+1}}\cos\theta, h\sin\theta\right)$$

for some $-\frac{\pi}{2} \le \theta \ne 0 \le \frac{\pi}{2}$. We may assume $-\frac{\pi}{2} \le \theta < 0$. Hence, by comparing the slope of the tangent line of E_1 at $\vec{a}$ with the one joining $\vec{a}$ and $(R/2\sqrt{n+1}, 0)$, and in view of the convexity of P, we have

$$\frac{-h\sin\theta}{\frac{R}{2\sqrt{n+1}} - \frac{R}{4\sqrt{n+1}}\cos\theta} \ge -\frac{16(n+1)h^2}{R^2}\frac{\frac{R}{4\sqrt{n+1}}\cos\theta}{h\sin\theta}.$$

This gives

$$\sin^2\theta \geq \frac{3}{4}.$$

Now the claim follows from estimating the radius of curvature of E_1 at $\vec{a}$ from below:

$$\frac{4\sqrt{n+1}\left(\frac{R^2}{16(n+1)}\sin^2\theta + h^2\cos^2\theta\right)^{\frac{3}{2}}}{hR} \geq \frac{|\sin\theta|^3 R^2}{16(n+1)h} \geq \frac{1}{C_n}\frac{R^2}{r},$$

where $C_n := \frac{128(n+1)}{\sqrt{27}}$. □

From (5.1), we have the following basic estimates:

$$R^{p-k-1} \leq \frac{c_{n,k}}{\min f}, \quad r^{p-k-1} \geq \frac{c_{n,k}}{\max f}. \tag{5.2}$$

Therefore, there is a lower bound on R and an upper bound on r. To obtain the C^0 and C^2 estimates, the following gradient estimate plays a crucial role in our argument.

Lemma 5.3. *For any $0 < \gamma < \frac{2(p-1)}{k}$, there exists a constant $\beta \geq 2$, depending on γ, k, p, $\min f$, and $\|f\|_{C^1}$, such that*

$$\frac{s^2 + |\nabla s|^2}{s^\gamma} \leq \beta R^{2-\gamma}. \tag{5.3}$$

Proof. Let $\rho^2 = s^2 + |\nabla s|^2$ and $\zeta = \frac{\rho^2}{s^\gamma}$, where $0 < \gamma < \frac{2(p-1)}{k}$. Let $\tau = \tau[s]$. Assume $\max\zeta > R^{2-\gamma}$ (i.e., $\beta > 1$). Therefore, at a point x_0 where ζ attains its maximum, we have

$$(\nabla^2 s + sg)\nabla s = \frac{\gamma}{2}\frac{\rho^2}{s}\nabla s.$$

At x_0, $\nabla s \neq 0$ is an eigenvector of $\alpha = \tau^{\sharp_g}$. Hence, find an orthonormal basis $\{e_i\}$ for $T_{x_0}\mathbb{S}^n$ such that $e_1 = \frac{\nabla s}{|\nabla s|}$ and $\tau|_{x_0}$ is diagonal. In particular, $\tau_{1i} = 0$ for $i = 2, \ldots, n$, while

$$\tau_{11} = \frac{\gamma}{2}\frac{\rho^2}{s}. \tag{5.4}$$

Moreover, at x_0 we have

$$\begin{aligned} \zeta_{;ii} = {} & \frac{2}{s^\gamma}(\tau_{\ell ii}s_\ell + \tau_{ii}^2 - s\tau_{ii}) - 4\gamma\frac{\tau_{\ell i}s_\ell s_i}{s^{\gamma+1}} \\ & - \frac{\gamma\rho^2(\tau_{ii} - s\delta_{ii})}{s^{\gamma+1}} + \gamma(\gamma+1)\frac{\rho^2 s_i^2}{s^{\gamma+2}}, \end{aligned}$$

and

$$\begin{aligned} \frac{\zeta_{;ii}}{\zeta} = {} & \frac{2}{\rho^2}\left(\tau_{\ell ii}s_\ell + \tau_{ii}^2 - s\tau_{ii}\right) - 4\gamma\frac{\tau_{ii}s_i^2}{s\rho^2} \\ & - \frac{\gamma(\tau_{ii} - s\delta_{ii})}{s} + \gamma(\gamma+1)\frac{s_i^2}{s^2}. \end{aligned}$$

Let $q = \sigma_k$. Note that $\nabla\tau$ is fully symmetric, and q is k-homogeneous. Using (5.4), $\zeta_{;ii} \le 0$ and $\zeta_i = 0$ for $i = 1, \dots, n$, we have

$$
\begin{aligned}
0 &\ge \frac{2}{\rho^2}\dot{q}^{ii}\left(\tau_{\ell ii}s_\ell + \tau_{ii}^2 - s\tau_{ii}\right) - 4\gamma\dot{q}^{11}\frac{\tau_{11}s_1^2}{s\rho^2} \\
&\quad - \gamma\dot{q}^{ii}\frac{\tau_{ii}}{s} + \gamma\dot{q}^{ii} + \gamma(\gamma+1)\dot{q}^{11}\frac{s_1^2}{s^2} \\
&= \frac{2}{\rho^2}\left((s^{p-1}f)_1 s_1 + \dot{q}^{ii}\tau_{ii}^2 - ks^p f\right) - 2\gamma^2\dot{q}^{11}\frac{s_1^2}{s^2} \\
&\quad - \gamma k s^{p-2} f + \gamma\dot{q}^{ii} + \gamma(\gamma+1)\dot{q}^{11}\frac{s_1^2}{s^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 \ge{}& 2(p-1)\frac{s^{p-2}s_1^2 f}{\rho^2} + 2\frac{s^{p-1}s_1 f_1}{\rho^2} - 2k\frac{s^p f}{\rho^2} \\
&+ \left(\frac{\gamma^2\rho^2}{2s_1^2} - \gamma(\gamma-1)\right)\dot{q}^{11}\frac{s_1^2}{s^2} - \gamma k s^{p-2} f.
\end{aligned}
$$

For $0 < \gamma \le 2$,

$$
\frac{\gamma^2\rho^2}{2s_1^2} - \gamma(\gamma-1) \ge \frac{\gamma^2}{2} - \gamma(\gamma-1) = \gamma\left(1 - \frac{\gamma}{2}\right) \ge 0.
$$

Now if for some $\beta \ge 2$ we had

$$
\left.\frac{\rho^2}{s^\gamma}\right|_{x_0} \ge \beta R^{2-\gamma},
$$

then

$$
s_1^2 \ge \beta R^{2-\gamma}s^\gamma - s^2 \ge \frac{\beta}{2}R^{2-\gamma}s^\gamma,
$$

and hence

$$
\begin{aligned}
0 &\ge \frac{2s^{p-2}f}{\rho^2}\left((p-1)s_1^2 + ss_1(\log f)_1 - ks^2 - \frac{k\gamma}{2}(s^2 + s_1^2)\right) \\
&= \frac{2s^{p-2}f}{\rho^2}\left(\left(p - 1 - \frac{k\gamma}{2}\right)s_1^2 + ss_1(\log f)_1 - \left(k + \frac{k\gamma}{2}\right)s^2\right) \\
&\ge \frac{2s^{p-2}f}{\rho^2}\left(\left(p - 1 - \frac{k\gamma}{2}\right)\frac{\beta}{2}R^{2-\gamma}s^\gamma - c_1\beta^{\frac{1}{2}}R^{1-\frac{\gamma}{2}}s^{1+\frac{\gamma}{2}} - c_2 s^2\right) \\
&= \frac{2s^{p+\gamma-2}f}{\rho^2}\left(\left(p - 1 - \frac{k\gamma}{2}\right)\frac{\beta}{2}R^{2-\gamma} - c_1\beta^{\frac{1}{2}}R^{1-\frac{\gamma}{2}}s^{1-\frac{\gamma}{2}} - c_2 s^{2-\gamma}\right) \\
&\ge \frac{2s^{p+\gamma-2}R^{2-\gamma}f}{\rho^2}\left(\left(p - 1 - \frac{k\gamma}{2}\right)\frac{\beta}{2} - c_1\beta^{\frac{1}{2}} - c_2\right).
\end{aligned}
$$

Here, we used $0 < \gamma < \frac{2(p-1)}{k}$ on the last three lines. Moreover, the constant c_1 depends on $\min f$ and $\|f\|_{C^1}$, and the constant c_2 depends on k. However, we would obtain a contradiction for β large enough. ☐

Proposition 5.4. *Let $1 < p < k+1$ Suppose $s > 0$ is an even, smooth, strictly convex solution of $\sigma_k(B[s]) = fs^{p-1}$. Then for each $\ell \geq 1$ and $\gamma \in (0,1)$, we have*

$$1/C \leq s \leq C, \quad \|s\|_{C^{\ell+1,\gamma}} \leq C_{\ell,\gamma}$$

where $C > 0$ is a constant depending only on n, γ, k, p, $\min f$ and $\|f\|_{C^2}$.

Proof. Let $q := \sigma_k^{\frac{1}{k}}$. Note that $q^k = fs^{p-1}$. In view of the identity

$$\nabla^2_{i,i}\sigma_1 = \Delta\tau_{ii} - n\tau_{ii} + \sigma_1$$

and concavity of q, there holds

$$0 \leq \dot{q}^{ij}\delta_{ij}\sigma_1 \leq \dot{q}^{ij}g^{pj}\nabla^2_{i,p}\sigma_1 + n(s^{p-1}f)^{\frac{1}{k}} - \Delta(s^{p-1}f)^{\frac{1}{k}}. \tag{5.5}$$

We calculate

$$\begin{aligned} -k\Delta(s^{p-1}f)^{\frac{1}{k}} &= (1-p)s^{\frac{p-1}{k}-1}f^{\frac{1}{k}}\sigma_1 - n(1-p)(s^{p-1}f)^{\frac{1}{k}} \\ &\quad + \frac{1}{k}(1-p)(p-k-1)s^{\frac{p-1}{k}-2}|\nabla s|^2 f^{\frac{1}{k}} \\ &\quad + 2(1-p)s^{\frac{p-1}{k}-1}g(\nabla s, \nabla f^{\frac{1}{k}}) - ks^{\frac{p-1}{k}}\Delta f^{\frac{1}{k}}. \end{aligned} \tag{5.6}$$

Thus, for $p > 1$, at a point where σ_1 attains its maximum we have

$$\sigma_1 \leq c_1\left(\frac{|\nabla s|^2}{s} + R\right).$$

Due to Lemma 5.3,

$$\sigma_1 \leq c_1\left(\frac{|\nabla s|^2}{s} + R\right) \leq c_2(s^{\gamma-1}R^{2-\gamma} + R) \leq c_3 R\left(\frac{R}{r}\right)^{1-\gamma}.$$

By Lemma 5.2, for some constant C depending on n, γ, k, p, $\min f$ and $\|f\|_{C^2}$ we have

$$\left(\frac{R}{r}\right)^{\gamma} \leq C.$$

Now, the uniform lower and upper bounds on the support function and the uniform upper bound on the principal radii of curvature follow from (5.2). The higher-order estimates follow from the Evans-Krylov and the Schauder regularity theory. ☐

5.1. **Full rank theorem.**

Theorem 5.5 (Full rank theorem)**.** *Let $0 < f \in C^\infty(\mathbb{S}^n)$ and $s : \mathbb{S}^n \to (0,\infty)$ be a smooth function, with $B[s] \geq 0$ and $\sigma_k(B[s]) = fs^{p-1}$. If $B[f^{-\frac{1}{p+k-1}}] \geq 0$, then $B[s] > 0$. That is, s is the support function of a closed, smooth, strictly convex hypersurface.*

Proof. ☐

5.2. **Uniqueness.**

6. Prescribed L_p curvature problem

The prescribed curvature problem asks the following question.

Question 6.1. Given a positive, smooth function $f : \mathbb{S}^n \to \mathbb{R}$, is there a closed, smooth, strictly convex hypersurface whose k-th elementary symmetric function of the principal curvatures, S_k, as a function of the unit normal vector, is f?

The L_p version of this problem is stated as follows.

Question 6.2. Given a smooth function $f : \mathbb{S}^n \to (0, \infty)$, is there a closed, smooth, strictly convex hypersurface with the support function s such that, for some constant c,

$$fs^{p-1}S_k = c \quad ? \tag{6.1}$$

In this section, we give the solutions to these two problems for the case $1 \leq p < k+1$ and provided f is even:

Theorem 6.3. [GG02] *Let $1 \leq p < k+1$, $1 \leq k < n$, and $\ell \geq 2$. Let $f \in C^{\ell}(\mathbb{S}^n)$ be a positive, even function. Then there exists an origin-symmetric, $C^{\ell+1,\alpha}$-smooth (for all $0 < \alpha < 1$), strictly convex hypersurface with the support function s such that*

$$fs^{p-1}S_k = 1.$$

To do so, we need to establish the C^0 and C^2 a priori estimates (our method from the previous section also works here) and apply a degree-theoretic argument; the continuity method does not work.

7. Open questions

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