Two Layer Problem

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1 Formulation of the problem as integral equation

1.1 Differential equations and boundary conditions

Concentric ring electrodes are attached to the boundary $\partial\Omega$ of the half space Ω where

$$\Omega \equiv \{\mathbf{x} = (r, \theta, z) \in \mathbb{R}^3 : z < 0\}$$

$$\partial\Omega \equiv \{\mathbf{x} = (r, \theta, z) \in \mathbb{R}^3 : z = 0\}$$

Ω consists of two layers with different conductivities, σ_1 for $-d \leq z < 0$ and σ_2 for $z < -d$. The current driving electrodes are the annuli

$$C_1 \equiv \{\mathbf{x} = (r, \theta, z) \in \mathbb{R}^3 : r_1 \leq r \leq r_2, 0 \leq \theta \leq 2\pi, z = 0\}$$

$$C_2 \equiv \{\mathbf{x} = (r, \theta, z) \in \mathbb{R}^3 : r_5 \leq r \leq r_6, 0 \leq \theta \leq 2\pi, z = 0\}$$

Γ is the region covered by the electrodes, i.e. $\Gamma \equiv C_1 \cup C_2$

Let $\phi_1(r, z)$ and $\phi_2(r, z)$ be the potentials in layer 1 and 2 respectively (independent of θ due to cylindrical symmetry). Then both ϕ_1 and ϕ_2 satisfy the Laplace equation in their respective domains along with boundary conditions at $z = 0$ and $z = -d$. The equations are

$$\nabla^2\phi_1(r, \theta, z) = 0 : -d \leq z < 0 \tag{1}$$

$$\nabla^2\phi_2(r, \theta, z) = 0 : z < -d \tag{2}$$

$$\frac{\partial\phi_1}{\partial z}(r, \theta, 0) = 0 \text{ on } \partial\Omega \setminus \Gamma \tag{3}$$

$$\phi_1(r, \theta, 0) + Z_l\sigma_1\frac{\partial\phi_1}{\partial z}(r, \theta, 0) = V_l \text{ on } C_l, l = 1, 2 \tag{4}$$

$$\int\limits_{C_l} \sigma_1\frac{\partial\phi_1}{\partial z}dA = I_l : l = 1, 2 \tag{5}$$

$$\phi_1(r, \theta, -d) = \phi_2(r, \theta, -d) \tag{6}$$

$$\sigma_1 \frac{\partial \phi_1}{\partial z}(r, \theta, -d) = \sigma_2 \frac{\partial \phi_2}{\partial z}(r, \theta, -d) \tag{7}$$

$$\phi_2 \to 0 \text{ as } z \to -\infty \tag{8}$$

where Z_l is the contact impedance, V_l is the voltage, and I_l is the applied current on the current driving electrode $C_l, l = 1, 2$. $\sigma_1 \frac{\partial \phi_1}{\partial z}$ is the current density on the boundary of the region. (6) is the continuity of the potential while (7) is the continuity of the current density at the boundary between the layers.

1.2 Hankel transform

We try to convert the above equations into integral equations using Hankel transform. Let $\Phi_1(\rho, z)$ and $\Phi_2(\rho, z)$ be the zero-order Hankel transforms of $\phi_1(r, z)$ and $\phi_2(r, z)$ respectively, i.e.,

$$\Phi_1(\rho, z) = \int_0^\infty r\phi_1(r, z) J_0(\rho r) dr \tag{9}$$

$$\Phi_2(\rho, z) = \int_0^\infty r\phi_2(r, z) J_0(\rho r) dr \tag{10}$$

Taking Hankel transform of (1), we get (due to cylindrical symmetry)

$$\frac{\partial^2 \Phi_1}{\partial z^2}(\rho, z) - \rho^2 \Phi_1(\rho, z) = 0$$

which gives

$$\Phi_1(\rho, z) = A(\rho)e^{\rho z} + B(\rho)e^{-\rho z} \tag{11}$$

where A and B are functions of ρ. Similarly (2) gives

$$\Phi_2(\rho, z) = C(\rho)e^{\rho z} \tag{12}$$

where C is a function of ρ. The $e^{-\rho z}$ term is absent since it blows up as $z \to -\infty$.

At $z = -d$,

$$\Phi_1(\rho, -d) = \int_0^\infty r\phi_1(r, -d) J_0(\rho r) dr$$

$$\Phi_2(\rho, -d) = \int_0^\infty r\phi_2(r, -d) J_0(\rho r) dr$$

Hence from (6),

$$\Phi_1(\rho, -d) = \Phi_2(\rho, -d)$$

Using (11) and (12),

$$A(\rho)e^{-\rho d} + B(\rho)e^{\rho d} = C(\rho)e^{-\rho d} \tag{13}$$

Differentiating (9) and (10) w.r.t. z, substituting $z = -d$ and multiplying both sides by the respective conductivities,

$$\sigma_1 \frac{\partial \Phi_1}{\partial z}(\rho, -d) = \int_0^\infty r\sigma_1 \frac{\partial \phi_1}{\partial z}(r, -d) J_0(\rho r) dr$$

$$\sigma_2 \frac{\partial \Phi_2}{\partial z}(\rho, -d) = \int_0^\infty r\sigma_2 \frac{\partial \phi_2}{\partial z}(r, -d) J_0(\rho r) dr$$

Hence from (7),

$$\sigma_1 \frac{\partial \Phi_1}{\partial z}(\rho, -d) = \sigma_2 \frac{\partial \Phi_2}{\partial z}(\rho, -d)$$

Using (11) and (12),

$$\sigma_1 [A(\rho)\rho e^{-\rho d} - B(\rho)\rho e^{\rho d}] = \sigma_2 C(\rho)\rho e^{-\rho d}$$

$$\sigma_1 A(\rho) e^{-\rho d} - \sigma_1 B(\rho) e^{\rho d} = \sigma_2 C(\rho) e^{-\rho d} \tag{14}$$

Solving (13) and (14),

$$A(\rho) = \frac{\sigma_1 + \sigma_2}{2\sigma_1} C(\rho)$$

$$B(\rho) = \frac{\sigma_1 - \sigma_2}{2\sigma_1} C(\rho) e^{-2\rho d}$$

Substituting in (11),

$$\Phi_1(\rho, z) = \frac{\sigma_1 + \sigma_2}{2\sigma_1} C(\rho) e^{\rho z} + \frac{\sigma_1 - \sigma_2}{2\sigma_1} C(\rho) e^{-2\rho d} e^{-\rho z} \tag{15}$$

Taking the inverse Hankel transform of (15) and (12),

$$\phi_1(r, z) = \frac{\sigma_1 + \sigma_2}{2\sigma_1} \int_0^\infty \rho C(\rho) e^{\rho z} J_0(\rho r) d\rho + \frac{\sigma_1 - \sigma_2}{2\sigma_1} \int_0^\infty \rho C(\rho) e^{-2\rho d} e^{-\rho z} J_0(\rho r) d\rho \tag{16}$$

$$\phi_2(r, z) = \int_0^\infty \rho C(\rho) e^{\rho z} J_0(\rho r) d\rho \tag{17}$$

1.3 Boundary conditions at the surface

Now we need to bring in the boudary conditions at $z = 0$. Putting $z = 0$ in (16) and its derivative w.r.t. z, we get

$$\phi_1(r, 0) = \int_0^\infty \left(\frac{\sigma_1 + \sigma_2}{2\sigma_1} + \frac{\sigma_1 - \sigma_2}{2\sigma_1} e^{-2\rho d} \right) \rho C(\rho) J_0(\rho r) d\rho \tag{18}$$

$$\frac{\partial \phi_1}{\partial z}(r, 0) = \int_0^\infty \left(\frac{\sigma_1 + \sigma_2}{2\sigma_1} - \frac{\sigma_1 - \sigma_2}{2\sigma_1} e^{-2\rho d} \right) \rho^2 C(\rho) J_0(\rho r) d\rho \tag{19}$$

Define

$$D(\rho) \equiv \rho C(\rho) \left(\frac{\sigma_1 + \sigma_2}{2\sigma_1} - \frac{\sigma_1 - \sigma_2}{2\sigma_1} e^{-2\rho d} \right) \tag{20}$$

Then,

$$\frac{\partial \phi_1}{\partial z}(r, 0) = \int_0^\infty \rho D(\rho) J_0(\rho r) d\rho$$

Thus, $D(\rho)$ is the Hankel transform of $\frac{\partial \phi_1}{\partial z}(r,0)$,

$$D(\rho) = \int_0^\infty r \frac{\partial \phi_1}{\partial z}(r,0) J_0(\rho r) dr$$

Using the fact that current density is 0 in $\partial\Omega \setminus \Gamma$,

$$D(\rho) = \int_{r_1}^{r_2} r \frac{\partial \phi_1}{\partial z}(r,0) J_0(\rho r) dr + \int_{r_5}^{r_6} r \frac{\partial \phi_1}{\partial z}(r,0) J_0(\rho r) dr \tag{21}$$

To simplify notations, define

$$\mu \equiv \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}$$

Then from (18) and (20),

$$\phi_1(r,0) = \int_0^\infty \frac{\frac{\sigma_1+\sigma_2}{2\sigma_1} + \frac{\sigma_1-\sigma_2}{2\sigma_1} e^{-2\rho d}}{\frac{\sigma_1+\sigma_2}{2\sigma_1} - \frac{\sigma_1-\sigma_2}{2\sigma_1} e^{-2\rho d}} D(\rho) J_0(\rho r) d\rho$$

$$\phi_1(r,0) = \int_0^\infty \frac{1 + \mu e^{-2\rho d}}{1 - \mu e^{-2\rho d}} D(\rho) J_0(\rho r) d\rho$$

Substituting $D(\rho)$ from (21),

$$\phi_1(r,0) = \int_0^\infty \frac{1 + \mu e^{-2\rho d}}{1 - \mu e^{-2\rho d}} \left[\int_{r_1}^{r_2} s \frac{\partial \phi_1}{\partial z}(s,0) J_0(\rho s) ds + \int_{r_5}^{r_6} s \frac{\partial \phi_1}{\partial z}(s,0) J_0(\rho s) ds \right] J_0(\rho r) d\rho$$

Interchanging the integrals,

$$\begin{aligned} \phi_1(r,0) = & \int_{r_1}^{r_2} s \frac{\partial \phi_1}{\partial z}(s,0) \left[\int_0^\infty \frac{1 + \mu e^{-2\rho d}}{1 - \mu e^{-2\rho d}} J_0(\rho r) J_0(\rho s) d\rho \right] ds \\ & + \int_{r_5}^{r_6} s \frac{\partial \phi_1}{\partial z}(s,0) \left[\int_0^\infty \frac{1 + \mu e^{-2\rho d}}{1 - \mu e^{-2\rho d}} J_0(\rho r) J_0(\rho s) d\rho \right] ds \end{aligned} \tag{22}$$

Define kernel $\widetilde{K}$ as the integral

$$\widetilde{K}(r,s) \equiv \int_0^\infty \frac{1 + \mu e^{-2\rho d}}{1 - \mu e^{-2\rho d}} J_0(\rho r) J_0(\rho s) d\rho$$

Then,

$$\widetilde{K}(r,s) = \int_0^\infty J_0(\rho r) J_0(\rho s) d\rho + 2 \int_0^\infty \frac{\mu e^{-2\rho d}}{1 - \mu e^{-2\rho d}} J_0(\rho r) J_0(\rho s) d\rho \tag{23}$$

Now [1],

$$\int_0^\infty J_0(\rho r) J_0(\rho s) d\rho = \frac{1}{2\pi} k(r,s) \tag{24}$$

where $k(r,s)$ is

$$k(r,s) = \begin{cases} \frac{4}{s} K\left(\frac{r}{s}\right), r < s \\ \frac{4}{r} K\left(\frac{s}{r}\right), s < r \end{cases} \tag{25}$$

where $K(x)$ is the complete elliptic integral of the first kind. The second term can be expanded as a geometric series since $|\mu| < 1$ by definition (since the conductivities are positive) and $|e^{-2\rho d}| < 1$,

$$2\int_0^\infty \frac{\mu e^{-2\rho d}}{1-\mu e^{-2\rho d}} J_0(\rho r) J_0(\rho s) d\rho = 2\sum_{n=1}^{\infty}\left[\int_0^\infty \mu^n e^{-2n\rho d} J_0(\rho r) J_0(\rho s) d\rho\right]$$

The integral in the summation can be written in terms of Legendre function of the second kind [1],

$$\int_0^\infty e^{-2n\rho d} J_0(\rho r) J_0(\rho s) d\rho = \frac{1}{\pi\sqrt{rs}} Q_{-\frac{1}{2}}\left(\frac{(2nd)^2 + r^2 + s^2}{2rs}\right)$$

which can be written in terms of complete elliptic integral of the first kind [1],

$$Q_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{z+1}} K\left(\sqrt{\frac{2}{z+1}}\right)$$

Thus,

$$\begin{aligned} k_5(r,s) &= 2\int_0^\infty \frac{\mu e^{-2\rho d}}{1-\mu e^{-2\rho d}} J_0(\rho r) J_0(\rho s) d\rho \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty} \mu^n \sqrt{\frac{4}{2rs + (2nd)^2 + r^2 + s^2}} K\left(\sqrt{\frac{4rs}{2rs + (2nd)^2 + r^2 + s^2}}\right) \end{aligned} \tag{26}$$

This decays quite fast due to the μ^n term and so relatively few terms of the series give a good approximation for the integral. The first term in the kernel $\widetilde{K}(r,s)$ has a weak (logarithmic) singularity when $r = s$ which necessitates the use of special methods like subtraction technique for the numerical solution. The second term $k_5(r,s)$ is well-defined for all r, s.

1.4 Integral equations

Substituting the partial derivates from (4) into (22), we get

$$\phi_1(r,0) = \int_{r_1}^{r_2} \frac{V_1 - \phi_1(s,0)}{\sigma_1 Z_1} s\widetilde{K}(r,s) ds + \int_{r_5}^{r_6} \frac{V_2 - \phi_1(s,0)}{\sigma_1 Z_2} s\widetilde{K}(r,s) ds$$

$$\begin{aligned} \phi_1(r,0) = &\frac{V_1}{\sigma_1 Z_1}\int_{r_1}^{r_2} s\widetilde{K}(r,s) ds - \frac{1}{\sigma_1 Z_1}\int_{r_1}^{r_2} s\widetilde{K}(r,s)\phi_1(s,0) ds \\ &+ \frac{V_2}{\sigma_1 Z_2}\int_{r_5}^{r_6} s\widetilde{K}(r,s) ds - \frac{1}{\sigma_1 Z_2}\int_{r_5}^{r_6} s\widetilde{K}(r,s)\phi_1(s,0) ds \end{aligned} \tag{27}$$

Substituting the partial derivates from (4) into the total current equation (5), we get

$$I_1 = 2\pi\int_{r_1}^{r_2} s\sigma_1 \frac{\partial \phi_1}{\partial z}(s,0) ds = 2\pi\int_{r_1}^{r_2} \frac{V_1 - \phi_1(s,0)}{Z_1} ds \tag{28}$$

$$I_2 = -I_1 = 2\pi \int_{r_5}^{r_6} \frac{V_2 - \phi_1(s,0)}{Z_2} ds \tag{29}$$

Equation (27) is a Fredholm integral equation of the second kind (with a weakly singular kernel). We need to solve (27) subject to (28) and (29) to get $\phi_1(r,0)$, V_1 and V_2. Once we have these, we can find $D(\rho)$ using (21) and (4),

$$D(\rho) = \int_{r_1}^{r_2} \frac{V_1 - \phi_1(r,0)}{\sigma_1 Z_1} r J_0(\rho r) dr + \int_{r_5}^{r_6} \frac{V_2 - \phi_1(r,0)}{\sigma_1 Z_2} r J_0(\rho r) dr \tag{30}$$

$\rho C(\rho)$ can be calculated from $D(\rho)$ using (20), and then the potential at any point, i.e. $\phi_1(r,z)$ and $\phi_2(r,z)$ can be calculated from (16) and (17) respectively. The activation for the nerves parallel to the xy-plane can be calculated as the second derivative of the potential w.r.t. r at the given depth [7].

2 Numerical solution of the integral equations

2.1 Subtraction technique

To use the Nyström method for solving the integral equations, we need to remove the singularity in the kernel. Since the singularity is only at $r = s$, the subtraction technique [6] was used for this purpose. Only the first term of the kernel $\frac{1}{2\pi}k(r,s)$ is considered in the following discussion as the second term $k_5(r,s)$ is well-defined everywhere.

$$\int_a^b sk(r,s)\phi_1(s,0)ds = \int_a^b sk(r,s)(\phi_1(s,0) - \phi_1(r,0))ds + \phi_1(r,0)\int_a^b sk(r,s)ds$$

In the first term, we can assume that the integrand is zero when $r = s$ since the singularity is weak. The integral in the second term can be evaluated analytically (expression given below).

The integrals consisting of the second term $k_5(r,s)$ in the kernel were evaluated using numerical methods but analytical expressions were obtained for the integral of $sk(r,s)$ using [4] and some basic subtstitution. Here K and E denote complete elliptic integrals of the first and the second kind respectively.

$$\int_a^b sk(r,s)ds = \begin{cases} 0, & a = b \\ 4(b-a)K(0), & r = 0 \\ 4r\left[\frac{E\left(\frac{r}{b}\right)}{\left(\frac{r}{b}\right)} - \frac{E\left(\frac{r}{a}\right)}{\left(\frac{r}{a}\right)}\right], & 0 < r < a \\ 4r\left[-E\left(\frac{a}{r}\right) + \frac{E\left(\frac{r}{b}\right)}{\left(\frac{r}{b}\right)}\right], & r = a \\ 4r\left[-E\left(\frac{a}{r}\right) + \left(1 - \left(\frac{a}{r}\right)^2\right)K\left(\frac{a}{r}\right) + \frac{E\left(\frac{r}{b}\right)}{\left(\frac{r}{b}\right)}\right], & a < r \le b \\ 4r\left[-E\left(\frac{a}{r}\right) + \left(1 - \left(\frac{a}{r}\right)^2\right)K\left(\frac{a}{r}\right) + E\left(\frac{b}{r}\right) - \left(1 - \left(\frac{b}{r}\right)^2\right)K\left(\frac{b}{r}\right)\right], & r > b \end{cases}$$

Thus, the integral equation (27) becomes

$$\begin{aligned}
\phi_1(r,0) = {} & \frac{V_1}{\sigma_1 Z_1}\int_{r_1}^{r_2} sk_5(r,s)ds - \frac{1}{\sigma_1 Z_1}\int_{r_1}^{r_2} sk_5(r,s)\phi_1(s,0)ds \\
& + \frac{V_2}{\sigma_1 Z_2}\int_{r_5}^{r_6} sk_5(r,s)ds - \frac{1}{\sigma_1 Z_2}\int_{r_5}^{r_6} sk_5(r,s)\phi_1(s,0)ds \\
& + \frac{V_1}{2\pi\sigma_1 Z_1}\int_{r_1}^{r_2} sk(r,s)ds + \frac{V_2}{2\pi\sigma_1 Z_2}\int_{r_5}^{r_6} sk(r,s)ds \\
& - \frac{1}{2\pi\sigma_1 Z_1}\int_{r_1}^{r_2} sk(r,s)\left(\phi_1(s,0)-\phi_1(r,0)\right)ds \\
& - \frac{1}{2\pi\sigma_1 Z_2}\int_{r_5}^{r_6} sk(r,s)\left(\phi_1(s,0)-\phi_1(r,0)\right)ds \\
& - \frac{\phi_1(r,0)}{2\pi\sigma_1 Z_1}\int_{r_1}^{r_2} sk(r,s)ds - \frac{\phi_1(r,0)}{2\pi\sigma_1 Z_2}\int_{r_5}^{r_6} sk(r,s)ds
\end{aligned} \tag{31}$$

2.2 Formulation of the problem as a set of linear equations using quadrature

We used the Nyström method [6] for solving the integral equations (31), (28) and (29). MATLAB was used for all the coding. Gauss-Legendre quadrature [6] was used to select n_1 points $x_1 \dots x_{n_1}$ and weights $w_1 \dots w_{n_1}$ in $[r_1, r_2]$ and n_2 points $x_{n_1+1} \dots x_{n_1+n_2}$ and weights $w_{n_1+1} \dots w_{n_1+n_2}$ in $[r_5, r_6]$. Thus we have $n1+n2+2$ variables $\phi_1(x_1,0)\dots\phi_1(x_{n_1+n_2},0), V_1, V_2$ and $n1+n2+2$ equations ($n1+n2$ from (31)) and the remaining two from (28) and (29). On replacing the integrals in these equations by the quadrature sums, we get

$$\begin{aligned}
\phi_1(x_i,0) = {} & \frac{V_1}{\sigma_1 Z_1}\sum_{j=1}^{n_1} w_j x_j k_5(x_i,x_j) - \frac{1}{\sigma_1 Z_1}\sum_{j=1}^{n_1} w_j x_j k_5(x_i,x_j)\phi_1(x_j,0) \\
& + \frac{V_2}{\sigma_1 Z_2}\sum_{j=n_1+1}^{n_1+n_2} w_j x_j k_5(x_i,x_j) - \frac{1}{\sigma_1 Z_2}\sum_{j=n_1+1}^{n_1+n_2} w_j x_j k_5(x_i,x_j)\phi_1(x_j,0) \\
& + \frac{V_1}{2\pi\sigma_1 Z_1}\int_{r_1}^{r_2} sk(r,s)ds + \frac{V_2}{2\pi\sigma_1 Z_2}\int_{r_5}^{r_6} sk(r,s)ds \\
& - \frac{1}{2\pi\sigma_1 Z_1}\sum_{j=1}^{n_1} w_j x_j k(x_i,x_j)\left(\phi_1(x_j,0)-\phi_1(x_i,0)\right) \\
& - \frac{1}{2\pi\sigma_1 Z_2}\sum_{j=n_1+1}^{n_1+n_2} w_j x_j k(x_i,x_j)\left(\phi_1(x_j,0)-\phi_1(x_i,0)\right) \\
& - \frac{\phi_1(x_i,0)}{2\pi\sigma_1 Z_1}\int_{r_1}^{r_2} sk(r,s)ds - \frac{\phi_1(x_i,0)}{2\pi\sigma_1 Z_2}\int_{r_5}^{r_6} sk(r,s)ds \\
& \text{for } i = 1\dots n_1+n_2
\end{aligned} \tag{32}$$

$$I_1 = \frac{\pi}{Z_1}(r_2^2 - r_1^2)V_1 - \frac{2\pi}{Z_1}\sum_{j=1}^{n_1} w_j \phi_1(x_j, 0) \tag{33}$$

$$-I_1 = \frac{\pi}{Z_2}(r_6^2 - r_5^2)V_2 - \frac{2\pi}{Z_2}\sum_{j=n_1+1}^{n_1+n_2} w_j \phi_1(x_j, 0) \tag{34}$$

Note that the first $n_1 + n_2$ equations are homogenous in the variables. These equations can be solved by any standard procedure to obtain the surface potentials at the quadrature points and the electrode voltages. Since evaluation of $k_5(r, s)$ is computationally intensive, in cases where the electrode radii remain constant (e.g. while studying effects of contact impedance), the values of $k_5(r, s)$ on the quadrature grid should be stored in the memory.

The choice of the number of quadrature points represents a tradeoff between accuracy and speed. In cases with low contact impedances or where potential and activation function values close to the surface are desired, more quadrature points should be used. The evaluation of $k_5(r, s)$ can be done by using the series in (26). The series can be truncated when the terms get smaller than some threshold. In cases where μ is closer to 1, convergence is slower. It was observed that adding many terms of the sum at a time was faster than adding terms one-by-one (in MATLAB).

3 Calculating potential and activation function

3.1 Potential on the surface $\partial\Omega$ and current density

To calculate the potential at any point $(r, \theta, 0)$ on the surface, put r in place of x_i in (32). This gave better results as compared to doing the same substitution in (27). For low values of contact impedances, the potential is almost constant underneath the electrodes.

The current densities below the electrodes can calculated using (4),

$$j(r, \theta, 0) = \begin{cases} \frac{V_1 - \phi_1(r,\theta,0)}{Z_1}, & (r, \theta, 0) \in C_1 \\ \frac{V_2 - \phi_1(r,\theta,0)}{Z_2}, & (r, \theta, 0) \in C_2 \\ 0, & (r, \theta, 0) \in \partial\Omega \setminus \Gamma \end{cases}$$

For high values of contact impedances, the current density is almost constant underneath the electrodes. For low values of contact impedances, the current density is higher close to the electrode edges. But very high contact impedances lead to very high values of V_1 and V_2 for the same current levels.

3.2 Potential and activation function in Ω

As mentioned at the end of section 1, to find out the potential, we should first find out $D(\rho)$ from (30). We can approximate the integral using the potential values at the quadrature points (as obtained above). Thus,

$$D(\rho) = \sum_{j=1}^{n_1} w_j \frac{V_1 - \phi_1(x_j, 0)}{\sigma_1 Z_1} x_j J_0(\rho x_j) + \sum_{j=n_1+1}^{n_1+n_2} w_j \frac{V_2 - \phi_1(x_j, 0)}{\sigma_1 Z_2} x_j J_0(\rho x_j)$$

More number of quadrature points can be used to obtain $D(\rho)$ to higher accuracies. In any case, the values obtained may not be reliable at very high values of ρ due to rapid oscillations of $J_0(\rho r)$. But that does not pose any problem in finding the potentials due to the rapid decay of the exponential term.

$\rho C(\rho)$ can be calculated from $D(\rho)$ using (20). Now evaluation of the potentials (16) and (17) involves a highly oscillatory integral - an inverse Hankel transform. If only Bessel's function was oscillatory we could've used special quadrature as suggested in [3]. But $D(\rho)$, being a Hankel transform, is also oscillatory. Thus $C(\rho)$ is also oscillatory and since it is not available in closed form, most methods of finding the Hankel transform numerically fail. MATLAB *integral* function, which uses adaptive quadrature, gives accurate results (as compared to FEM results) but is slow.

We used a method suggested in [2, 5] which is based on the Fourier-Bessel series expansion of a function. We assume that the potentials are zero for $r > R$ where R is chosen outside the range where we are interested in finding the potentials. Then we make use of the sampling theorem - we expand the potential as a Fourier-Bessel series, the coefficients are given by the Hankel transform which is known. The potential can be approximated by truncating the series. The number of terms required can be as less as 50 deep down in the body but can be higher near the surface or in cases of low contact impedance. Also R should be chosen carefully, if it higher than optimal then more terms of the series are required for similar accuracy, if it is lower then results are inaccurate. Thus, optimal values of R and the number of terms in the series can be found out with a bit of trial and error.

$$\phi_i(r, z) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{1}{J_1(\alpha_n)^2} \Phi_i \left(\frac{\alpha_n}{R}, z \right) J_0 \left(\frac{\alpha_n r}{R} \right), i = 1, 2 \tag{35}$$

where α_n is the n^{th} positive zero of the Bessel's function of order zero. The Hankel transform $\Phi_i(\rho, z)$ is known in terms of $\rho C(\rho)$ from (15) and (12).

To find the activation function we need to numerically differentiate the potentials w.r.t. r. The results are sufficiently smooth hence finite differences can be used for this process.

References

[1] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables.* Number 55. Courier Dover Publications, 1972.

[2] Richard Barakat, Elaine Parshall, and Barbara H. Sandler. Zero-order hankel transformation algorithms based on filon quadrature philosophy for diffraction optics and beam propagation. *J. Opt. Soc. Am. A*, 15(3):652–659, Mar 1998.

[3] K.C. Chung, G.A. Evans, and J.R. Webster. A method to generate generalized quadrature rules for oscillatory integrals. *Applied Numerical Mathematics*, 34(1):85 – 93, 2000.

[4] I. M. Gradshteyn, I. S.; Ryzhik. *Table of Integrals, Series and Products, 4th Edition*. Academic Press, 1965.

[5] Manuel Guizar-Sicairos and Julio C. Gutiérrez-Vega. Computation of quasi-discrete hankel transforms of integer order for propagating optical wave fields. *J. Opt. Soc. Am. A*, 21(1):53–58, Jan 2004.

[6] W Press, S Teukolsky, W Vetterling, and B Flannery. *Numerical Recipes in Fortran 77: The Art of Scientific Computing*. Cambridge Univ. Press, New York, 1992.

[7] Frank Rattay. *Electrical nerve stimulation*. Springer, 1990.