PART 3

ADVANCED CIRCUIT ANALYSIS

CHAPTER 15

THE LAPLACE TRANSFORM

A man is like a function whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.

—I. N. Tolstroy

Historical Profiles

Pierre Simon Laplace (1749–1827), a French astronomer and mathematician, first presented the transform that bears his name and its applications to differential equations in 1779.

Born of humble origins in Beaumont-en-Auge, Normandy, France, Laplace became a professor of mathematics at the age of 20. His mathematical abilities inspired the famous mathematician Simeon Poisson, who called Laplace the Isaac Newton of France. He made important contributions in potential theory, probability theory, astronomy, and celestial mechanics. He was widely known for his work, *Traite de Mecanique Celeste (Celestial Mechanics)*, which supplemented the work of Newton on astronomy. The Laplace transform, the subject of this chapter, is named after him.

Samuel F. B. Morse (1791–1872), an American painter, invented the telegraph, the first practical, commercialized application of electricity.

Morse was born in Charlestown, Massachusetts and studied at Yale and the Royal Academy of Arts in London to become an artist. In the 1830s, he became intrigued with developing a telegraph. He had a working model by 1836 and applied for a patent in 1838. The U.S. Senate appropriated funds for Morse to construct a telegraph line between Baltimore and Washington, D.C. On May 24, 1844, he sent the famous first message: "What hath God wrought!" Morse also developed a code of dots and dashes for letters and numbers, for sending messages on the telegraph. The development of the telegraph led to the invention of the telephone.

15.1 INTRODUCTION

Our frequency-domain analysis has been limited to circuits with sinusoidal inputs. In other words, we have assumed sinusoidal time-varying excitations in all our non-dc circuits. This chapter introduces the *Laplace transform*, a very powerful tool for analyzing circuits with sinusoidal *or* nonsinusoidal inputs.

The idea of transformation should be familiar by now. When using phasors for the analysis of circuits, we transform the circuit from the time domain to the frequency or phasor domain. Once we obtain the phasor result, we transform it back to the time domain. The Laplace transform method follows the same process: we use the Laplace transformation to transform the circuit from the time domain to the frequency domain, obtain the solution, and apply the inverse Laplace transform to the result to transform it back to the time domain.

The Laplace transform is significant for a number of reasons. First, it can be applied to a wider variety of inputs than phasor analysis. Second, it provides an easy way to solve circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations. Third, the Laplace transform is capable of providing us, in one single operation, the total response of the circuit comprising both the natural and forced responses.

We begin with the definition of the Laplace transform and use it to derive the transforms of some basic, important functions. We consider some properties of the Laplace transform that are very helpful in circuit analysis. We then consider the inverse Laplace transform, transfer functions, and convolution. Finally, we examine how the Laplace transform is applied in circuit analysis, network stability, and network synthesis.

15.2 DEFINITION OF THE LAPLACE TRANSFORM

Given a function $f(t)$, its Laplace transform, denoted by $F(s)$ or $\mathcal{L}[f(t)]$, is given by

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt} \tag{15.1}$$

where s is a complex variable given by

$$s = \sigma + j\omega \tag{15.2}$$

Since the argument st of the exponent e in Eq. (15.1) must be dimensionless, it follows that s has the dimensions of frequency and units of inverse seconds (s^{-1}). In Eq. (15.1), the lower limit is specified as 0^- to indicate a time just before $t = 0$. We use 0^- as the lower limit to include the origin and capture any discontinuity of $f(t)$ at $t = 0$; this will accommodate functions—such as singularity functions—that may be discontinuous at $t = 0$.

For an ordinary function $f(t)$, the lower limit can be replaced by 0.

The Laplace transform is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$.

We assume in Eq. (15.1) that $f(t)$ is ignored for $t < 0$. To ensure that this is the case, a function is often multiplied by the unit step. Thus, $f(t)$ is written as $f(t)u(t)$ or $f(t),\ t \geq 0$.

The Laplace transform in Eq. (15.1) is known as the *one-sided* (or *unilateral*) Laplace transform. The *two-sided* (or *bilateral*) Laplace transform is given by

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt \tag{15.3}$$

The one-sided Laplace transform in Eq. (15.1), being adequate for our purposes, is the only type of Laplace transform that we will treat in this book.

A function $f(t)$ may not have a Laplace transform. In order for $f(t)$ to have a Laplace transform, the integral in Eq. (15.1) must converge to a finite value. Since $|e^{j\omega t}| = 1$ for any value of t, the integral converges when

$|e^{j\omega t}| = \sqrt{\cos^2 \omega t + \sin^2 \omega t} = 1$

$$\int_{0^-}^{\infty} e^{-\sigma t}|f(t)|\,dt \quad < \quad \infty \tag{15.4}$$

for some real value $\sigma = \sigma_c$. Thus, the region of convergence for the Laplace transform is $\text{Re}(s) = \sigma > \sigma_c$, as shown in Fig. 15.1. In this region, $|F(s)| < \infty$ and $F(s)$ exists. $F(s)$ is undefined outside the region of convergence. Fortunately, all functions of interest in circuit analysis satisfy the convergence criterion in Eq. (15.4) and have Laplace transforms. Therefore, it is not necessary to specify σ_c in what follows.

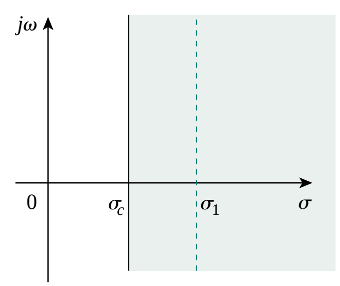

Figure 15.1 Region of convergence for the Laplace transform.

A companion to the direct Laplace transform in Eq. (15.1) is the *inverse* Laplace transform given by

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds \tag{15.5}$$

where the integration is performed along a straight line $(\sigma_1 + j\omega, -\infty < \omega < \infty)$ in the region of convergence, $\sigma_1 > \sigma_c$. See Fig. 15.1. The direct application of Eq. (15.5) involves some knowledge about complex analysis beyond the scope of this book. For this reason, we will not use Eq. (15.5) to find the inverse Laplace transform. We will rather use a look-up table, to be developed in Section 15.3. The functions $f(t)$ and $F(s)$ are regarded as a Laplace transform pair where

$$f(t) \qquad \Longleftrightarrow \qquad F(s) \tag{15.6}$$

meaning that there is one-to-one correspondence between $f(t)$ and $F(s)$. The following examples derive the Laplace transforms of some important functions.

EXAMPLE 15.1

Determine the Laplace transform of each of the following functions: (a) $u(t)$, (b) $e^{-at}u(t)$, $a \geq 0$, and (c) $\delta(t)$.

Solution:

(a) For the unit step function $u(t)$, shown in Fig. 15.2(a), the Laplace transform is

$$\begin{aligned}\mathcal{L}[u(t)] &= \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} \\ &= -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s}\end{aligned} \tag{15.1.1}$$

(b) For the exponential function, shown in Fig. 15.2(b), the Laplace transform is

$$\begin{aligned}\mathcal{L}[e^{-at}u(t)] &= \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt \\ &= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}\end{aligned} \tag{15.1.2}$$

(c) For the unit impulse function, shown in Fig. 15.2(c),

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-0} = 1 \tag{15.1.3}$$

since the impulse function $\delta(t)$ is zero everywhere except at $t = 0$. The sifting property in Eq. (7.33) has been applied in Eq. (15.1.3).

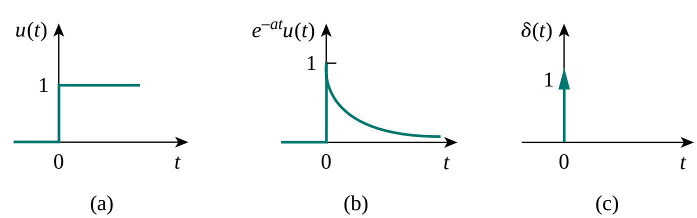

Figure 15.2 For Example 15.1: (a) unit step function, (b) exponential function, (c) unit impulse function.

PRACTICE PROBLEM 15.1

Find the Laplace transforms of these functions: $r(t) = tu(t)$, that is, the ramp function; and $e^{at}u(t)$.

Answer: $1/s^2$, $1/(s-a)$.

EXAMPLE 15.2

Determine the Laplace transform of $f(t) = \sin \omega t u(t)$.

Solution:

Using Eq. (B.26) in addition to Eq. (15.1), we obtain the Laplace transform of the sine function as

$$F(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t) e^{-st}\, dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt$$
$$= \frac{1}{2j} \int_0^\infty (e^{-(s-j\omega)t} - e^{-(s+j\omega)t})\, dt$$
$$= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2}$$

PRACTICE PROBLEM 15.2

Find the Laplace transform of $f(t) = \cos \omega t u(t)$.

Answer: $s/(s^2 + \omega^2)$.

15.3 PROPERTIES OF THE LAPLACE TRANSFORM

The properties of the Laplace transform help us to obtain transform pairs without directly using Eq. (15.1) as we did in Examples 15.1 and 15.2. As we derive each of these properties, we should keep in mind the definition of the Laplace transform in Eq. (15.1).

Linearity

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace transforms of $f_1(t)$ and $f_2(t)$, then

$$\boxed{\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)} \tag{15.7}$$

where a_1 and a_2 are constants. Equation 15.7 expresses the linearity property of the Laplace transform. The proof of Eq. (15.7) follows readily from the definition of the Laplace transform in Eq. (15.1).

For example, by the linearity property in Eq. (15.7), we may write

$$\mathcal{L}[\cos wt] = \mathcal{L}\left[\frac{1}{2}(e^{j\omega t} + e^{-j\omega t}) \right] = \frac{1}{2}\mathcal{L}[e^{j\omega t}] + \frac{1}{2}\mathcal{L}[e^{-j\omega t}] \tag{15.8}$$

But from Example 15.1(b), $\mathcal{L}[e^{-at}] = 1/(s + a)$. Hence,

$$\mathcal{L}[\cos wt] = \frac{1}{2}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega} \right) = \frac{s}{s^2 + \omega^2} \tag{15.9}$$

Scaling

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f(at)] = \int_0^\infty f(at) e^{-st}\, dt \tag{15.10}$$

where a is a constant and $a > 0$. If we let $x = at$, $dx = a\,dt$, then

$$\mathcal{L}[f(at)] = \int_0^\infty f(x) e^{-x(s/a)} \frac{dx}{a} = \frac{1}{a} \int_0^\infty f(x) e^{-x(s/a)}\, dx \tag{15.11}$$

Comparing this integral with the definition of the Laplace transform in Eq. (15.1) shows that s in Eq. (15.1) must be replaced by s/a while the dummy variable t is replaced by x. Hence, we obtain the scaling property as

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)} \tag{15.12}$$

For example, we know from Example 15.2 that

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \tag{15.13}$$

Using the scaling property in Eq. (15.12),

$$\mathcal{L}[\sin 2\omega t] = \frac{1}{2}\frac{\omega}{(s/2)^2 + \omega^2} = \frac{2\omega}{s^2 + 4\omega^2} \tag{15.14}$$

which may also be obtained from Eq. (15.13) by replacing ω with 2ω.

Time Shift

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f(t-a)u(t-a)] = \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt \qquad a \geq 0 \tag{15.15}$$

But $u(t-a) = 0$ for $t < a$ and $u(t-a) = 1$ for $t > a$. Hence,

$$\mathcal{L}[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-st}\,dt \tag{15.16}$$

If we let $x = t - a$, then $dx = dt$ and $t = x + a$. As $t \to a$, $x \to 0$ and as $t \to \infty$, $x \to \infty$. Thus,

$$\begin{aligned}\mathcal{L}[f(t-a)u(t-a)] &= \int_0^\infty f(x)e^{-s(x+a)}\,dx \\ &= e^{-as}\int_0^\infty f(x)e^{-sx}\,dx = e^{-as}F(s)\end{aligned}$$

or

$$\boxed{\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)} \tag{15.17}$$

In other words, if a function is delayed in time by a, the result in the s domain is multiplying the Laplace transform of the function (without the delay) by e^{-as}. This is called the *time-delay* or *time-shift property* of the Laplace transform.

As an example, we know from Eq. (15.9) that

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

Using the time-shift property in Eq. (15.17),

$$\mathcal{L}[\cos \omega(t-a)u(t-a)] = e^{-as}\frac{s}{s^2 + \omega^2} \tag{15.18}$$

Frequency Shift

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-at}f(t)e^{-st}\,dt$$
$$= \int_0^\infty f(t)e^{-(s+a)t}\,dt = F(s+a)$$

or

$$\boxed{\mathcal{L}[e^{-at}f(t)] = F(s+a)} \tag{15.19}$$

That is, the Laplace transform of $e^{-at}f(t)$ can be obtained from the Laplace transform of $f(t)$ by replacing every s with $s+a$. This is known as *frequency shift* or *frequency translation.*

As an example, we know that

$$\cos\omega t \quad \Longleftrightarrow \quad \frac{s}{s^2+\omega^2}$$

and (15.20)

$$\sin\omega t \quad \Longleftrightarrow \quad \frac{\omega}{s^2+\omega^2}$$

Using the shift property in Eq. (15.19), we obtain the Laplace transform of the damped sine and damped cosine functions as

$$\mathcal{L}[e^{-at}\cos\omega t] = \frac{s+a}{(s+a)^2+\omega^2} \tag{15.21a}$$

$$\mathcal{L}[e^{-at}\sin\omega t] = \frac{\omega}{(s+a)^2+\omega^2} \tag{15.21b}$$

Time Differentiation

Given that $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of its derivative is

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^\infty \frac{df}{dt}e^{-st}\,dt \tag{15.22}$$

To integrate this by parts, we let $u = e^{-st}$, $du = -se^{-st}\,dt$, and $dv = (df/dt)\,dt = df(t)$, $v = f(t)$. Then

$$\mathcal{L}\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^\infty f(t)[-se^{-st}]\,dt$$
$$= 0 - f(0^-) + s\int_{0^-}^\infty f(t)e^{-st}\,dt = sF(s) - f(0^-)$$

or

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0^-)} \tag{15.23}$$

The Laplace transform of the second derivative of $f(t)$ is a repeated application of Eq. (15.23) as

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s\mathcal{L}[f'(t)] - f'(0^-) = s[sF(s) - f(0^-)] - f'(0^-)$$
$$= s^2F(s) - sf(0^-) - f'(0^-)$$

or

$$\boxed{\mathcal{L}[f''(t)] = s^2F(s) - sf(0^-) - f'(0^-)} \tag{15.24}$$

Continuing in this manner, we can obtain the Laplace transform of the nth derivative of $f(t)$ as

$$\boxed{\begin{aligned}\mathcal{L}\left[\frac{d^n f}{dt^n}\right] &= s^nF(s) - s^{n-1}f(0^-) \\ &\quad - s^{n-2}f'(0^-) - \cdots - s^0 f^{(n-1)}(0^-)\end{aligned}} \tag{15.25}$$

As an example, we can use Eq. (15.23) to obtain the Laplace transform of the sine from that of the cosine. If we let $f(t) = \cos\omega t$, then $f(0) = 1$ and $f'(t) = -\omega\sin\omega t$. Using Eq. (15.23) and the scaling property,

$$\begin{aligned}\mathcal{L}[\sin\omega t] &= -\frac{1}{\omega}\mathcal{L}[f'(t)] = -\frac{1}{\omega}[sF(s) - f(0^-)] \\ &= -\frac{1}{\omega}\left(s\frac{s}{s^2+\omega^2} - 1\right) = \frac{\omega}{s^2+\omega^2}\end{aligned} \tag{15.26}$$

as expected.

Time Integration

If $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of its integral is

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \int_{0^-}^{\infty}\left[\int_0^t f(x)\,dx\right]e^{-st}\,dt \tag{15.27}$$

To integrate this by parts, we let

$$u = \int_0^t f(x)\,dx, \qquad du = f(t)\,dt$$

and

$$dv = e^{-st}\,dt, \qquad v = -\frac{1}{s}e^{-st}$$

Then

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(t)\,dt\right] &= \left[\int_0^t f(x)\,dx\right]\left(-\frac{1}{s}e^{-st}\right)\Bigg|_{0^-}^{\infty} \\ &\quad - \int_{0^-}^{\infty}\left(-\frac{1}{s}\right)e^{-st}f(t)\,dt\end{aligned}$$

For the first term on the right-hand side of the equation, evaluating the term at $t = \infty$ yields zero due to $e^{-s\infty}$ and evaluating it at $t = 0$ gives $\dfrac{1}{s}\displaystyle\int_0^0 f(x)\,dx = 0$. Thus, the first term is zero, and

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}\,dt = \frac{1}{s}F(s)$$

or simply,

$$\boxed{\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)} \tag{15.28}$$

As an example, if we let $f(t) = u(t)$, from Example 15.1(a), $F(s) = 1/s$. Using Eq. (15.28),

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \mathcal{L}[t] = \frac{1}{s}\left(\frac{1}{s}\right)$$

Thus, the Laplace transform of the ramp function is

$$\mathcal{L}[t] = \frac{1}{s^2} \tag{15.29}$$

Applying Eq. (15.28), this gives

$$\mathcal{L}\left[\int_0^t t\,dt\right] = \mathcal{L}\left[\frac{t^2}{2}\right] = \frac{1}{s}\frac{1}{s^2}$$

or

$$\mathcal{L}[t^2] = \frac{2}{s^3} \tag{15.30}$$

Repeated applications of Eq. (15.28) lead to

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \tag{15.31}$$

Similarly, using integration by parts, we can show that

$$\mathcal{L}\left[\int_{-\infty}^t f(t)\,dt\right] = \frac{1}{s}F(s) + \frac{1}{s}f^{-1}(0^-) \tag{15.32}$$

where

$$f^{-1}(0^-) = \int_{-\infty}^{0^-} f(t)\,dt$$

Frequency Differentiation

If $F(s)$ is the Laplace transform of $f(t)$, then

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Taking the derivative with respect to s,

$$\frac{dF(s)}{ds} = \int_0^\infty f(t)(-te^{-st})\,dt = \int_0^\infty (-tf(t))e^{-st}\,dt = \mathcal{L}[-tf(t)]$$

and the frequency differentiation property becomes

$$\boxed{\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}} \tag{15.33}$$

Repeated applications of this equation lead to

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n} \tag{15.34}$$

For example, we know from Example 15.1(b) that $\mathcal{L}[e^{-at}] = 1/(s+a)$. Using the property in Eq. (15.33),

$$\mathcal{L}[te^{-at}] = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2} \tag{15.35}$$

Note that if $a = 0$, we obtain $\mathcal{L}[t] = 1/s^2$ as in Eq. (15.29), and repeated applications of Eq. (15.33) will yield Eq. (15.31).

Time Periodicity

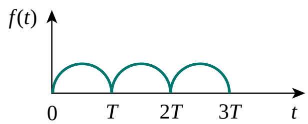

Figure 15.3 A periodic function.

If function $f(t)$ is a periodic function such as shown in Fig. 15.3, it can be represented as the sum of time-shifted functions shown in Fig. 15.4. Thus,

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) + \cdots \\ &= f_1(t) + f_1(t-T)u(t-T) \\ &\quad + f_1(t-2T)u(t-2T) + \cdots \end{aligned} \tag{15.36}$$

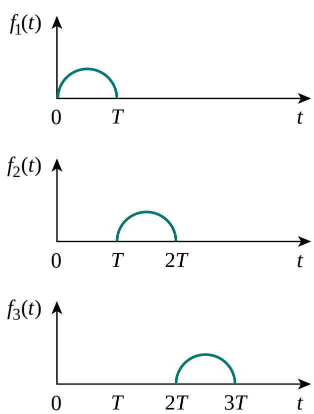

Figure 15.4 Decomposition of the periodic function in Fig. 15.2.

where $f_1(t)$ is the same as the function $f(t)$ gated over the interval $0 < t < T$, that is,

$$f_1(t) = f(t)[u(t) - u(t-T)] \tag{15.37a}$$

or

$$f_1(t) = \begin{cases} f(t), & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \tag{15.37b}$$

We now transform each term in Eq. (15.36) and apply the time-shift property in Eq. (15.17). We obtain

$$\begin{aligned} F(s) &= F_1(s) + F_1(s)e^{-Ts} + F_1(s)e^{-2Ts} + F_1(s)e^{-3Ts} + \cdots \\ &= F_1(s)[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots] \end{aligned} \tag{15.38}$$

But

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{15.39}$$

if $|x| < 1$. Hence,

$$\boxed{F(s) = \frac{F_1(s)}{1 - e^{-Ts}}} \tag{15.40}$$

where $F_1(s)$ is the Laplace transform of $f_1(t)$; in other words, $F_1(s)$ is the transform $f(t)$ defined over its first period only. Equation (15.40) shows that the Laplace transform of a periodic function is the transform of the first period of the function divided by $1 - e^{-Ts}$.

Initial and Final Values

The initial-value and final-value properties allow us to find the initial value $f(0)$ and the final value $f(\infty)$ of $f(t)$ directly from its Laplace transform $F(s)$. To obtain these properties, we begin with the differentiation property in Eq. (15.23), namely,

$$sF(s) - f(0^+) = \mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}\,dt \tag{15.41}$$

If we let $s \to \infty$, the integrand in Eq. (15.41) vanishes due to the damping exponential factor, and Eq. (15.41) becomes

$$\lim_{s\to\infty} \left[sF(s) - f(0^+)\right] = 0$$

or

$$\boxed{f(0^+) = \lim_{s\to\infty} sF(s)} \tag{15.42}$$

This is known as the *initial-value theorem*. For example, we know from Eq. (15.21a) that

$$f(t) = e^{-2t}\cos 10t \qquad \Longleftrightarrow \qquad F(s) = \frac{s+2}{(s+2)^2 + 10^2} \tag{15.43}$$

Using the initial-value theorem,

$$\begin{aligned} f(0^+) = \lim_{s\to\infty} sF(s) &= \lim_{s\to\infty} \frac{s^2 + 2s}{s^2 + 4s + 104} \\ &= \lim_{s\to\infty} \frac{1 + 2/s}{1 + 4/s + 104/s^2} = 1 \end{aligned}$$

which confirms what we would expect from the given $f(t)$.

In Eq. (15.41), we let $s \to 0$; then

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \int_0^\infty \frac{df}{dt} e^{0t}\, dt = \int_0^\infty df = f(\infty) - f(0^-)$$

or

$$\boxed{f(\infty) = \lim_{s\to 0} sF(s)} \tag{15.44}$$

This is referred to as the *final-value theorem*. In order for the final-value theorem to hold, all poles of $F(s)$ must be located in the left half of the s plane (see Fig. 15.1 or Fig. 15.9); that is, the poles must have negative real parts. The only exception to this requirement is the case in which $F(s)$ has a simple pole at $s = 0$, because the effect of $1/s$ will be nullified by $sF(s)$ in Eq. (15.44). For example, from Eq. (15.21b),

$$f(t) = e^{-2t}\sin 5t \qquad \Longleftrightarrow \qquad F(s) = \frac{5}{(s+2)^2 + 5^2} \tag{15.45}$$

Applying the final-value theorem,

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{5s}{s^2 + 4s + 29} = 0$$

as expected from the given $f(t)$. As another example,

$$f(t) = \sin t \qquad \Longleftrightarrow \qquad f(s) = \frac{1}{s^2 + 1} \tag{15.46}$$

so that

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s^2 + 1} = 0$$

This is incorrect, because $f(t) = \sin t$ oscillates between $+1$ and -1 and does not have a limit as $t \to \infty$. Thus, the final-value theorem cannot be used to find the final value of $f(t) = \sin t$, because $F(s)$ has poles at $s = \pm j$, which are not in the left half of the s plane. In general, the final-value theorem does not apply in finding the final values of sinusoidal functions—these functions oscillate forever and do not have final values.

The initial-value and final-value theorems depict the relationship between the origin and infinity in the time domain and the s domain. They serve as useful checks on Laplace transforms.

Table 15.1 provides a list of the properties of the Laplace transform. The last property (on convolution) will be proved in Section 15.7. There are other properties, but these are enough for present purposes. Table 15.2 summarizes the Laplace transforms of some common functions. We have omitted the factor $u(t)$ except where it is necessary.

TABLE 15.1 Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)\,dt$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_1(t)$	$F_1(s)F_2(s)$

TABLE 15.2 Laplace transform pairs.

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

EXAMPLE 15.3

Obtain the Laplace transform of $f(t) = \delta(t) + 2u(t) - 3e^{-2t}$, $t \geq 0$.

Solution:

By the linearity property,

$$F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}]$$
$$= 1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2+s+4}{s(s+2)}$$

PRACTICE PROBLEM 15.3

Find the Laplace transform of $f(t) = \cos 2t + e^{-3t}$, $t \geq 0$.

Answer: $\dfrac{2s^2+3s+4}{(s+3)(s^2+4)}$.

EXAMPLE 15.4

Determine the Laplace transform of $f(t) = t^2 \sin 2t\, u(t)$.

Solution:

We know that

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2+2^2}$$

Using frequency differentiation in Eq. (15.34),

$$F(s) = \mathcal{L}[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$
$$= \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right) = \frac{12s^2-16}{(s^2+4)^3}$$

PRACTICE PROBLEM 15.4

Find the Laplace transform of $f(t) = t^2 \cos 3t\, u(t)$.

Answer: $\dfrac{2s(s^2-27)}{(s^2+9)^3}$.

EXAMPLE 15.5

Find the Laplace transform of the gate function in Fig. 15.5.

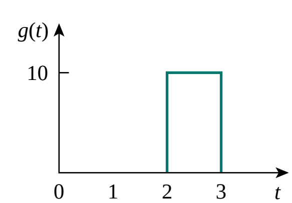

Figure 15.5 The gate function; for Example 15.5.

Solution:

We can express the gate function in Fig. 15.5 as

$$g(t) = 10[u(t-2) - u(t-3)]$$

Since we know the Laplace transform of $u(t)$, we apply the time-shift property and obtain

$$G(s) = 10\left(\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right) = \frac{10}{s}(e^{-2s} - e^{-3s})$$

PRACTICE PROBLEM 15.5

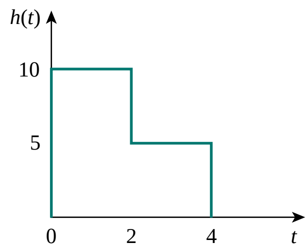

Figure 15.6 For Practice Prob. 15.5.

Find the Laplace transform of the function $h(t)$ in Fig. 15.6.

Answer: $\frac{5}{s}(2 - e^{-2s} - e^{-4s})$.

EXAMPLE 15.6

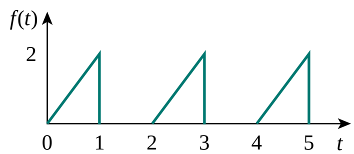

Figure 15.7 For Example 15.6.

Calculate the Laplace transform of the periodic function in Fig. 15.7.

Solution:

The period of the function is $T = 2$. To apply Eq. (15.40), we first obtain the transform of the first period of the function.

$$\begin{aligned} f_1(t) &= 2t[u(t) - u(t-1)] = 2tu(t) - 2tu(t-1) \\ &= 2tu(t) - 2(t-1+1)u(t-1) \\ &= 2tu(t) - 2(t-1)u(t-1) - 2u(t-1) \end{aligned}$$

Using the time-shift property,

$$F_1(s) = \frac{2}{s^2} - 2\frac{e^{-s}}{s^2} - \frac{2}{s}e^{-s} = \frac{2}{s^2}(1 - e^{-s} - se^{-s})$$

Thus, the transform of the periodic function in Fig. 15.7 is

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \frac{2}{s^2(1 - e^{-2s})}(1 - e^{-s} - se^{-s})$$

PRACTICE PROBLEM 15.6

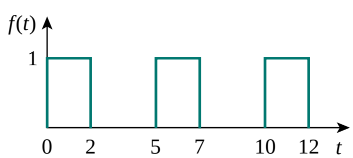

Figure 15.8 For Practice Prob. 15.6.

Determine the Laplace transform of the periodic function in Fig. 15.8.

Answer: $\frac{1 - e^{-2s}}{s(1 - e^{-5s})}$.

EXAMPLE 15.7

Find the initial and final values of the function whose Laplace transform is

$$H(s) = \frac{20}{(s+3)(s^2 + 8s + 25)}$$

Solution:

Applying the initial-value theorem,

$$h(0) = \lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{20s}{(s+3)(s^2+8s+25)}$$
$$= \lim_{s\to\infty} \frac{20/s^2}{(1+3/s)(1+8/s+25/s^2)} = \frac{0}{(1+0)(1+0+0)} = 0$$

To be sure that the final-value theorem is applicable, we check where the poles of $H(s)$ are located. The poles of $H(s)$ are $s = -3, -4 \pm j3$, which all have negative real parts: they are all located on the left half of the s plane (Fig. 15.9). Hence the final-value theorem applies and

$$h(\infty) = \lim_{s\to 0} sH(s) = \lim_{s\to 0} \frac{20s}{(s+3)(s^2+8s+25)}$$
$$= \frac{0}{(0+3)(0+0+25)} = 0$$

Both the initial and final values could be determined from $h(t)$ if we knew it. See Example 15.11, where $h(t)$ is given.

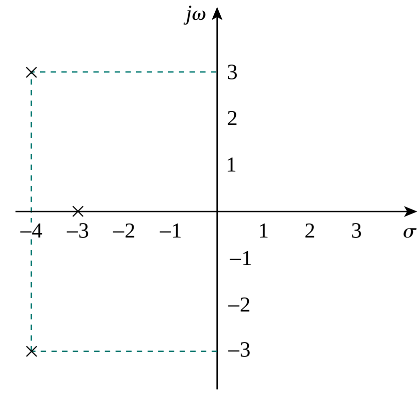

Figure 15.9 For Example 15.7: Poles of $H(s)$.

PRACTICE PROBLEM 15.7

Obtain the initial and the final values of

$$G(s) = \frac{s^3+2s+6}{s(s+1)^2(s+3)}$$

Answer: 1, 2.

15.4 THE INVERSE LAPLACE TRANSFORM

Given $F(s)$, how do we transform it back to the time domain and obtain the corresponding $f(t)$? By matching entries in Table 15.2, we avoid using Eq. (15.5) to find $f(t)$.

Suppose $F(s)$ has the general form of

$$F(s) = \frac{N(s)}{D(s)} \tag{15.47}$$

where $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial. The roots of $N(s) = 0$ are called the *zeros* of $F(s)$, while the roots of $D(s) = 0$ are the *poles* of $F(s)$. Although Eq. (15.47) is similar in form to Eq. (14.3), here $F(s)$ is the Laplace transform of a function, which is not necessarily a transfer function. We use *partial fraction expansion* to break $F(s)$ down into simple terms whose inverse transform we obtain from Table 15.2. Thus, finding the inverse Laplace transform of $F(s)$ involves two steps.

Software packages such as Matlab, Mathcad, and Maple are capable of finding partial fraction expansions quite easily.

Steps to Find the Inverse Laplace Transform:

1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Table 15.2.

Let us consider the three possible forms $F(s)$ may take and how to apply the two steps to each form.

15.4.1 Simple Poles

Recall from Chapter 14 that a simple pole is a first-order pole. If $F(s)$ has only simple poles, then $D(s)$ becomes a product of factors, so that

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{15.48}$$

where $s = -p_1, -p_2, \ldots, -p_n$ are the simple poles, and $p_i \neq p_j$ for all $i \neq j$ (i.e., the poles are distinct). Assuming that the degree of $N(s)$ is less than the degree of $D(s)$, we use partial fraction expansion to decompose $F(s)$ in Eq. (15.48) as

Otherwise, we must first apply long division so that $F(s) = N(s)/D(s) = Q(s) + R(s)/D(s)$, where the degree of $R(s)$, the remainder of the long division, is less than the degree of $D(s)$.

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n} \tag{15.49}$$

The expansion coefficients $k_1, k_2, \ldots, k_n$ are known as the *residues* of $F(s)$. There are many ways of finding the expansion coefficients. One way is using the *residue method*. If we multiply both sides of Eq. (15.49) by $(s + p_1)$, we obtain

$$(s+p_1)F(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \cdots + \frac{(s+p_1)k_n}{s+p_n} \tag{15.50}$$

Since $p_i \neq p_j$, setting $s = -p_1$ in Eq. (15.50) leaves only k_1 on the right-hand side of Eq. (15.50). Hence,

$$(s+p_1)F(s)\big|_{s=-p_1} = k_1 \tag{15.51}$$

Thus, in general,

$$\boxed{k_i = (s+p_i)F(s)\big|_{s=-p_i}} \tag{15.52}$$

This is known as *Heaviside's theorem*. Once the values of k_i are known, we proceed to find the inverse of $F(s)$ using Eq. (15.49). Since the inverse transform of each term in Eq. (15.49) is $\mathcal{L}^{-1}[k/(s+a)] = ke^{-at}u(t)$, then, from Table 15.1,

Historical note: Named after Oliver Heaviside (1850–1925), an English engineer, the pioneer of operational calculus.

$$\boxed{f(t) = \left(k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t}\right)} \tag{15.53}$$

15.4.2 Repeated Poles

Suppose $F(s)$ has n repeated poles at $s = -p$. Then we may represent $F(s)$ as

$$\begin{aligned} F(s) = {} & \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} \\ & + \frac{k_1}{s+p} + F_1(s) \end{aligned} \tag{15.54}$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have a pole at $s = -p$. We determine the expansion coefficient k_n as

$$k_n = (s+p)^n F(s)\big|_{s=-p} \tag{15.55}$$

as we did above. To determine k_{n-1}, we multiply each term in Eq. (15.54) by $(s+p)^n$ and differentiate to get rid of k_n, then evaluate the result at $s=-p$ to get rid of the other coefficients except k_{n-1}. Thus, we obtain

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)]\,\big|_{s=-p} \tag{15.56}$$

Repeating this gives

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]\,\big|_{s=-p} \tag{15.57}$$

The mth term becomes

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)]\,\big|_{s=-p} \tag{15.58}$$

where $m = 1, 2, \ldots, n-1$. One can expect the differentiation to be difficult to handle as m increases. Once we obtain the values of $k_1, k_2, \ldots, k_n$ by partial fraction expansion, we apply the inverse transform

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!} \tag{15.59}$$

to each term in the right-hand side of Eq. (15.54) and obtain

$$\boxed{\begin{aligned} f(t) &= k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!}t^2 e^{-pt} \\ &\quad + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt} + f_1(t) \end{aligned}} \tag{15.60}$$

15.4.3 Complex Poles

A pair of complex poles is simple if it is not repeated; it is a double or multiple pole if repeated. Simple complex poles may be handled the same as simple real poles, but because complex algebra is involved the result is always cumbersome. An easier approach is a method known as *completing the square*. The idea is to express each complex pole pair (or quadratic term) in $D(s)$ as a complete square such as $(s+\alpha)^2 + \beta^2$ and then use Table 15.2 to find the inverse of the term.

Since $N(s)$ and $D(s)$ always have real coefficients and we know that the complex roots of polynomials with real coefficients must occur in conjugate pairs, $F(s)$ may have the general form

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s) \tag{15.61}$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have this pair of complex poles. If we complete the square by letting

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s+\alpha)^2 + \beta^2 \tag{15.62}$$

and we also let

$$A_1 s + A_2 = A_1(s+\alpha) + B_1\beta \tag{15.63}$$

then Eq. (15.61) becomes

$$F(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{B_1\beta}{(s+\alpha)^2+\beta^2} + F_1(s) \tag{15.64}$$

From Table 15.2, the inverse transform is

$$\boxed{f(t) = A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t + f_1(t)} \tag{15.65}$$

The sine and cosine terms can be combined using Eq. (9.12).

Whether the pole is simple, repeated, or complex, a general approach that can always be used in finding the expansion coefficients is the *method of algebra*, illustrated in Examples 15.9 to 15.11. To apply the method, we first set $F(s) = N(s)/D(s)$ equal to an expansion containing unknown constants. We multiply the result through by a common denominator. Then we determine the unknown constants by equating coefficients (i.e., by algebraically solving a set of simultaneous equations for these coefficients at like powers of s).

Another general approach is to substitute specific, convenient values of s to obtain as many simultaneous equations as the number of unknown coefficients, and then solve for the unknown coefficients. We must make sure that each selected value of s is not one of the poles of $F(s)$. Example 15.11 illustrates this idea.

EXAMPLE 15.8

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

The inverse transform is given by

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left(\frac{3}{s}\right) - \mathcal{L}^{-1}\left(\frac{5}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2+4}\right) \\ &= 3u(t) - 5e^{-t} + 3\sin 2t, \qquad t \ge 0 \end{aligned}$$

where Table 15.2 has been consulted for the inverse of each term.

PRACTICE PROBLEM 15.8

Determine the inverse Laplace transform of

$$F(s) = 1 + \frac{4}{s+3} - \frac{5s}{s^2+16}$$

Answer: $\delta(t) + 4e^{-3t} - 5\cos 4t$, $t \ge 0$.

EXAMPLE 15.9

Find $f(t)$ given that

$$F(s) = \frac{s^2+12}{s(s+2)(s+3)}$$

Solution:

Unlike in the previous example where the partial fractions have been provided, we first need to determine the partial fractions. Since there are three poles, we let

$$\frac{s^2+12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} \tag{15.9.1}$$

where A, B, and C are the constants to be determined. We can find the constants using two approaches.

METHOD 1 **Residue method:**

$$A = sF(s)\Big|_{s=0} = \frac{s^2+12}{(s+2)(s+3)}\Bigg|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s+2)F(s)\Big|_{s=-2} = \frac{s^2+12}{s(s+3)}\Bigg|_{s=-2} = \frac{4+12}{(-2)(1)} = -8$$

$$C = (s+3)F(s)\Big|_{s=-3} = \frac{s^2+12}{s(s+2)}\Bigg|_{s=-3} = \frac{9+12}{(-3)(-1)} = 7$$

METHOD 2 **Algebraic method:** Multiplying both sides of Eq. (15.9.1) by $s(s+2)(s+3)$ gives

$$s^2+12 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

or

$$s^2+12 = A(s^2+5s+6) + B(s^2+3s) + C(s^2+2s)$$

Equating the coefficients of like powers of s gives

Constant: $12 = 6A \implies A = 2$

s: $0 = 5A + 3B + 2C \implies 3B + 2C = -10$

s^2: $1 = A + B + C \implies B + C = -1$

Thus $A = 2$, $B = -8$, $C = 7$, and Eq. (15.9.1) becomes

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

By finding the inverse transform of each term, we obtain

$$f(t) = 2u(t) - 8e^{-2t} + 7e^{-3t}, \qquad t \geq 0.$$

PRACTICE PROBLEM 15.9

Find $f(t)$ if

$$F(s) = \frac{6(s+2)}{(s+1)(s+3)(s+4)}$$

Answer: $f(t) = e^{-t} + 3e^{-3t} - 4e^{-4t}$, $t \geq 0$.

EXAMPLE 15.10

Calculate $v(t)$ given that

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

Solution:

While the previous example is on simple roots, this example is on repeated roots. Let

$$\begin{aligned} V(s) &= \frac{10s^2 + 4}{s(s+1)(s+2)^2} \\ &= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2} \end{aligned} \tag{15.10.1}$$

METHOD 1 **Residue method:**

$$A = sV(s)\Big|_{s=0} = \frac{10s^2+4}{(s+1)(s+2)^2}\Bigg|_{s=0} = \frac{4}{(1)(2)^2} = 1$$

$$B = (s+1)V(s)\Big|_{s=-1} = \frac{10s^2+4}{s(s+2)^2}\Bigg|_{s=-1} = \frac{14}{(-1)(1)^2} = -14$$

$$C = (s+2)^2V(s)\Big|_{s=-2} = \frac{10s^2+4}{s(s+1)}\Bigg|_{s=-2} = \frac{44}{(-2)(-1)} = 22$$

$$\begin{aligned} D &= \frac{d}{ds}[(s+2)^2V(s)]\Bigg|_{s=-2} = \frac{d}{ds}\left(\frac{10s^2+4}{s^2+s}\right)\Bigg|_{s=-2} \\ &= \frac{(s^2+s)(20s) - (10s^2+4)(2s+1)}{(s^2+s)^2}\Bigg|_{s=-2} = \frac{52}{4} = 13 \end{aligned}$$

METHOD 2 **Algebraic method:** Multiplying Eq. (15.10.1) by $s(s+1)(s+2)^2$, we obtain

$$\begin{aligned} 10s^2 + 4 = {} & A(s+1)(s+2)^2 + Bs(s+2)^2 \\ & + Cs(s+1) + Ds(s+1)(s+2) \end{aligned}$$

or

$$\begin{aligned} 10s^2 + 4 = {} & A(s^3 + 5s^2 + 8s + 4) + B(s^3 + 4s^2 + 4s) \\ & + C(s^2 + s) + D(s^3 + 3s^2 + 2s) \end{aligned}$$

Equating coefficients,

Constant:	$4 = 4A$	$\Longrightarrow$	$A = 1$
s:	$0 = 8A + 4B + C + 2D$	$\Longrightarrow$	$4B + C + 2D = -8$
s^2:	$10 = 5A + 4B + C + 3D$	$\Longrightarrow$	$4B + C + 3D = 5$
s^3:	$0 = A + B + D$	$\Longrightarrow$	$B + D = -1$

Solving these simultaneous equations gives $A = 1$, $B = -14$, $C = 22$, $D = 13$, so that

$$V(s) = \frac{1}{s} - \frac{14}{s+1} + \frac{13}{s+2} + \frac{22}{(s+2)^2}$$

Taking the inverse transform of each term, we get

$$v(t) = u(t) - 14e^{-t} + 13e^{-2t} + 22te^{-2t}, \qquad t \geq 0$$

PRACTICE PROBLEM 15.10

Obtain $g(t)$ if

$$G(s) = \frac{s^3 + 2s + 6}{s(s+1)^2(s+3)}$$

Answer: $2u(t) - 3.25e^{-t} - 1.5te^{-t} + 2.25e^{-3t}, t \geq 0$.

EXAMPLE 15.11

Find the inverse transform of the frequency-domain function in Example 15.7:

$$H(s) = \frac{20}{(s+3)(s^2 + 8s + 25)}$$

Solution:

In this example, $H(s)$ has a pair of complex poles at $s^2 + 8s + 25 = 0$ or $s = -4 \pm j3$. We let

$$H(s) = \frac{20}{(s+3)(s^2 + 8s + 25)} = \frac{A}{s+3} + \frac{Bs + C}{(s^2 + 8s + 25)} \tag{15.11.1}$$

We now determine the expansion coefficients in two ways.

METHOD 1 **Combination of methods:** We can obtain A using the method of residue,

$$A = (s+3)H(s)\Big|_{s=-3} = \frac{20}{s^2 + 8s + 25}\Bigg|_{s=-3} = \frac{20}{10} = 2$$

Although B and C can be obtained using the method of residue, we will not do so, to avoid complex algebra. Rather, we can substitute two specific values of s [say $s = 0, 1$, which are not poles of $F(s)$] into Eq. (15.11.1). This will give us two simultaneous equations from which to find B and C. If we let $s = 0$ in Eq. (15.11.1), we obtain

$$\frac{20}{75} = \frac{A}{3} + \frac{C}{25}$$

or

$$20 = 25A + 3C \tag{15.11.2}$$

Since $A = 2$, Eq. (15.11.2) gives $C = -10$. Substituting $s = 1$ into Eq. (15.11.1) gives

$$\frac{20}{(4)(34)} = \frac{A}{4} + \frac{B + C}{34}$$

or

$$20 = 34A + 4B + 4C \tag{15.11.3}$$

But $A = 2$, $C = -10$, so that Eq. (15.11.3) gives $B = -2$.

METHOD 2 **Algebraic method:** Multiplying both sides of Eq. (15.11.1) by $(s+3)(s^2+8s+25)$ yields

$$\begin{aligned} 20 &= A(s^2+8s+25) + (Bs+C)(s+3) \\ &= A(s^2+8s+25) + B(s^2+3s) + C(s+3) \end{aligned} \tag{15.11.4}$$

Equating coefficients,

$$\begin{array}{llll} s^2: & 0 = A + B & \Longrightarrow & A = -B \\ s: & 0 = 8A + 3B + C = 5A + C & \Longrightarrow & C = -5A \\ \text{Constant:} & 20 = 25A + 3C = 25A - 15A & \Longrightarrow & A = 2 \end{array}$$

That is, $B = -2$, $C = -10$. Thus

$$\begin{aligned} H(s) &= \frac{2}{s+3} - \frac{2s+10}{(s^2+8s+25)} = \frac{2}{s+3} - \frac{2(s+4)+2}{(s+4)^2+9} \\ &= \frac{2}{s+3} - \frac{2(s+4)}{(s+4)^2+9} - \frac{2}{3}\frac{3}{(s+4)^2+9} \end{aligned}$$

Taking the inverse of each term, we obtain

$$h(t) = 2e^{-3t} - 2e^{-4t}\cos 3t - \frac{2}{3}e^{-4t}\sin 3t \tag{15.11.5}$$

It is alright to leave the result this way. However, we can combine the cosine and sine terms as

$$h(t) = 2e^{-3t} - Ae^{-4t}\cos(3t - \theta) \tag{15.11.6}$$

To obtain Eq. (15.11.6) from Eq. (15.11.5), we apply Eq. (9.12). Next, we determine the coefficient A and the phase angle θ:

$$A = \sqrt{2^2 + \left(\tfrac{2}{3}\right)^2} = 2.108, \qquad \theta = \tan^{-1}\frac{\frac{2}{3}}{2} = 18.43^\circ$$

Thus,

$$h(t) = 2e^{-3t} - 2.108e^{-4t}\cos(3t - 18.43^\circ)$$

PRACTICE PROBLEM 15.11

Find $g(t)$ given that

$$G(s) = \frac{10}{(s+1)(s^2+4s+13)}$$

Answer: $e^{-t} - e^{-2t}\cos 3t + \dfrac{1}{3}e^{-2t}\sin 3t$, $t \geq 0$.

15.5 APPLICATION TO CIRCUITS

Having mastered how to obtain the Laplace transform and its inverse, we are now prepared to employ the Laplace transform to analyze circuits. This usually involves three steps.

Steps in applying the Laplace transform:

1. Transform the circuit from the time domain to the s domain.
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Only the first step is new and will be discussed here. As we did in phasor analysis, we transform a circuit in the time domain to the frequency or s domain by Laplace transforming each term in the circuit.

As one can infer from step 2, all the circuit analysis techniques applied for dc circuits are applicable to the s domain.

For a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t) \tag{15.66}$$

Taking the Laplace transform, we get

$$\boxed{V(s) = RI(s)} \tag{15.67}$$

For an inductor,

$$v(t) = L\frac{di(t)}{dt} \tag{15.68}$$

Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-) \tag{15.69}$$

or

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}} \tag{15.70}$$

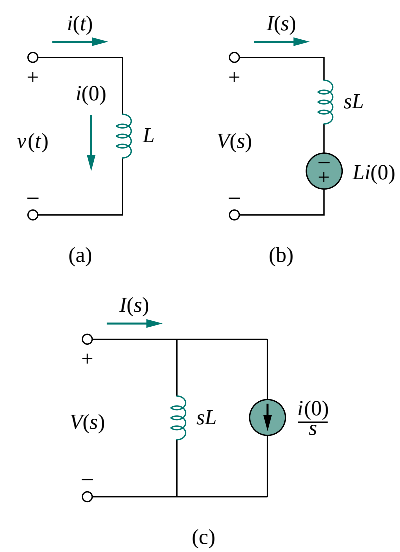

Figure 15.10 Representation of an inductor: (a) time-domain, (b,c) s-domain equivalents.

The s-domain equivalents are shown in Fig. 15.10, where the initial condition is modeled as a voltage or current source.

For a capacitor,

$$i(t) = C\frac{dv(t)}{dt} \tag{15.71}$$

which transforms into the s domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-) \tag{15.72}$$

or

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}} \tag{15.73}$$

The s-domain equivalents are shown in Fig. 15.11. With the s-domain equivalents, the Laplace transform can be used readily to solve first- and second-order circuits such as those we considered in Chapters 7 and 8. We should observe from Eqs. (15.68) to (15.73) that the initial conditions are part of the transformation. This is one advantage of using

The elegance of using the Laplace transform in circuit analysis lies in the automatic inclusion of the initial conditions in the transformation process, thus providing a complete (transient and steady-state) solution.

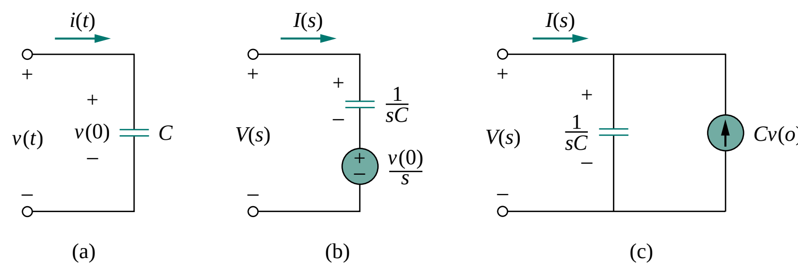

Figure 15.11 Representation of a capacitor: (a) time-domain, (b,c) s-domain equivalents.

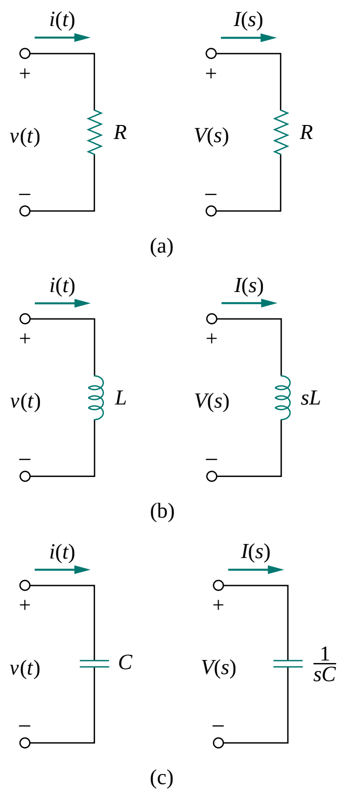

Figure 15.12 Time-domain and s-domain representations of passive elements under zero initial conditions.

TABLE 15.3 Impedance of an element in the s domain.*

Element	$Z(s) = V(s)/I(s)$
Resistor	R
Inductor	sL
Capacitor	$1/sC$

*Assuming zero initial conditions

the Laplace transform in circuit analysis. Another advantage is that a complete response—transient and steady state—of a network is obtained. We will illustrate this with Examples 15.13 and 15.14. Also, observe the duality of Eqs. (15.70) and (15.73), confirming what we already know from Chapter 8 (see Table 8.1), namely, that L and C, $I(s)$ and $V(s)$, and $v(0)$ and $i(0)$ are dual pairs.

If we assume zero initial conditions for the inductor and the capacitor, the above equations reduce to:

$$\begin{aligned} \text{Resistor:} \quad & V(s) = RI(s) \\ \text{Inductor:} \quad & V(s) = sLI(s) \\ \text{Capacitor:} \quad & V(s) = \frac{1}{sC}I(s) \end{aligned} \tag{15.74}$$

The s-domain equivalents are shown in Fig. 15.12.

We define the impedance in the s-domain as the ratio of the voltage transform to the current transform under zero initial conditions, that is,

$$Z(s) = \frac{V(s)}{I(s)} \tag{15.75}$$

Thus the impedances of the three circuit elements are

$$\begin{aligned} \text{Resistor:} \quad & Z(s) = R \\ \text{Inductor:} \quad & Z(s) = sL \\ \text{Capacitor:} \quad & Z(s) = \frac{1}{sC} \end{aligned} \tag{15.76}$$

Table 15.3 summarizes these. The admittance in the s domain is the reciprocal of the impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \tag{15.77}$$

The use of the Laplace transform in circuit analysis facilitates the use of various signal sources such as impulse, step, ramp, exponential, and sinusoidal.

EXAMPLE 15.12

Find $v_o(t)$ in the circuit in Fig. 15.13, assuming zero initial conditions.

Solution:

We first transform the circuit from the time domain to the s domain.

$$u(t) \implies \frac{1}{s}$$

$$1\text{ H} \implies sL = s$$

$$\frac{1}{3}\text{ F} \implies \frac{1}{sC} = \frac{3}{s}$$

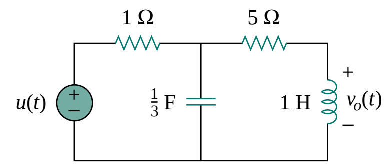

Figure 15.13 For Example 15.12.

The resulting s-domain circuit is in Fig. 15.14. We now apply mesh analysis. For mesh 1,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2 \tag{15.12.1}$$

For mesh 2,

$$0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2$$

or

$$I_1 = \frac{1}{3}(s^2 + 5s + 3) I_2 \tag{15.12.2}$$

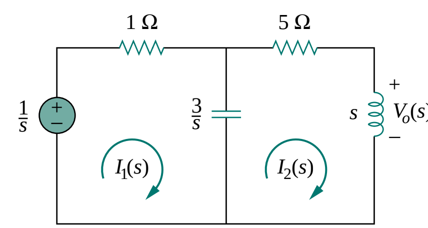

Figure 15.14 Mesh analysis of the frequency-domain equivalent of the same circuit.

Substituting this into Eq. (15.12.1),

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) \frac{1}{3}(s^2 + 5s + 3) I_2 - \frac{3}{s} I_2$$

Multiplying through by $3s$ gives

$$3 = (s^3 + 8s^2 + 18s) I_2 \qquad \implies \qquad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin \sqrt{2}t \text{ V}, \qquad t \ge 0$$

PRACTICE PROBLEM 15.12

Determine $v_o(t)$ in the circuit of Fig. 15.15, assuming zero initial conditions.

Answer: $8(1 - e^{-2t} - 2te^{-2t})u(t)$ V.

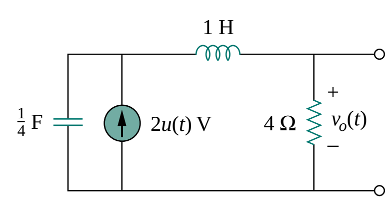

Figure 15.15 For Practice Prob. 15.12.

EXAMPLE 15.13

Find $v_o(t)$ in the circuit of Fig. 15.16. Assume $v_o(0) = 5$ V.

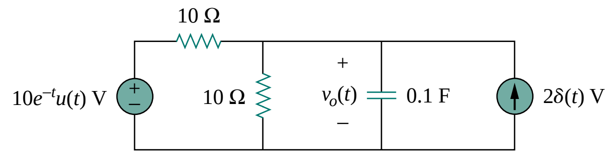

Figure 15.16 For Example 15.13.

Solution:

We transform the circuit to the s domain as shown in Fig. 15.17. The initial condition is included in the form of the current source $Cv_o(0) = 0.1(5) = 0.5$ A. [See Fig. 15.11(c).] We apply nodal analysis. At the top node,

$$\frac{10/(s+1) - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}$$

or

$$\frac{1}{s+1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10}V_o(s+2)$$

Multiplying through by 10,

$$\frac{10}{s+1} + 25 = V_o(s+2)$$

or

$$V_o = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

where

$$A = (s+1)V_o(s)\Big|_{s=-1} = \frac{25s+35}{(s+2)}\Bigg|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2)V_o(s)\Big|_{s=-2} = \frac{25s+35}{(s+1)}\Bigg|_{s=-2} = \frac{-15}{-1} = 15$$

Thus,

$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t)$$

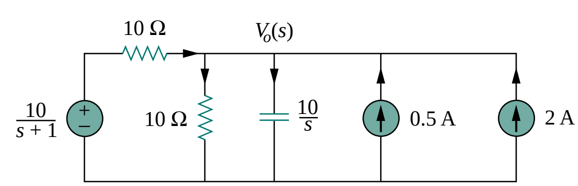

Figure 15.17 Nodal analysis of the equivalent of the circuit in Fig. 15.16.

PRACTICE PROBLEM 15.13

Find $v_o(t)$ in the circuit shown in Fig. 15.18.

Answer: $(\frac{4}{5}e^{-2t} + \frac{8}{15}e^{-t/3})u(t)$.

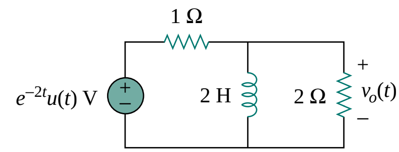

Figure 15.18 For Practice Prob. 15.13.

EXAMPLE 15.14

In the circuit in Fig. 15.19(a), the switch moves from position a to position b at $t = 0$. Find $i(t)$ for $t > 0$.

Solution:

The initial current through the inductor is $i(0) = I_o$. For $t > 0$, Fig. 15.19(b) shows the circuit transformed to the s domain. The initial condition is incorporated in the form of a voltage source as $Li(0) = LI_o$. Using mesh analysis,

$$I(s)(R + sL) - LI_o - \frac{V_o}{s} = 0 \qquad (15.14.1)$$

or

$$I(s) = \frac{LI_o}{R + sL} + \frac{V_o}{s(R + sL)} = \frac{I_o}{s + R/L} + \frac{V_o/L}{s(s + R/L)} \qquad (15.14.2)$$

Applying partial fraction expansion on the second term on the right-hand side of Eq. (15.14.2) yields

$$I(s) = \frac{I_o}{s + R/L} + \frac{V_o/R}{s} - \frac{V_o/R}{(s + R/L)} \qquad (15.14.3)$$

The inverse Laplace transform of this gives

$$i(t) = \left(I_o - \frac{V_o}{R}\right)e^{-t/\tau} + \frac{V_o}{R}, \qquad t \geq 0 \qquad (15.14.4)$$

where $\tau = R/L$. The term in fences is the transient response, while the second term is the steady-state response. In other words, the final value is $i(\infty) = V_o/R$, which we could have predicted by applying the final-value theorem on Eq. (15.14.2) or (15.14.3); that is,

$$\lim_{s\to 0} sI(s) = \lim_{s\to 0}\left(\frac{sI_o}{s + R/L} + \frac{V_o/L}{s + R/L}\right) = \frac{V_o}{R} \qquad (15.14.5)$$

Equation (15.14.4) may also be written as

$$i(t) = I_o e^{-t/\tau} + \frac{V_o}{R}(1 - e^{-t/\tau}), \qquad t \geq 0 \qquad (15.14.6)$$

The first term is the natural response, while the second term is the forced response. If the initial condition $I_o = 0$, Eq. (15.14.6) becomes

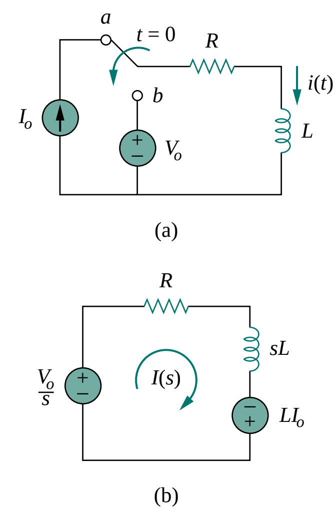

Figure 15.19 For Example 15.14.

$$i(t) = \frac{V_o}{R}(1 - e^{-t/\tau}), \qquad t \geq 0 \tag{15.14.7}$$

which is the step response, since it is due to the step input V_o with no initial energy.

PRACTICE PROBLEM 15.14

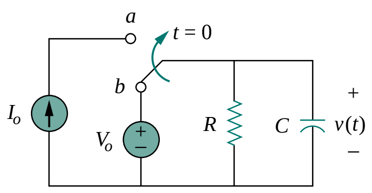

Figure 15.20 For Practice Prob. 15.14.

The switch in Fig. 15.20 has been in position b for a long time. It is moved to position a at $t = 0$. Determine $v(t)$ for $t > 0$.

Answer: $v(t) = (V_o - I_oR)e^{-t/\tau} + I_oR$, $t > 0$, where $\tau = RC$.

15.6 TRANSFER FUNCTIONS

For electrical networks, the transfer function is also known as the *network function.*

The *transfer function* is a key concept in signal processing because it indicates how a signal is processed as it passes through a network. It is a fitting tool for finding the network response, determining (or designing for) network stability, and network synthesis. The transfer function of a network describes how the output behaves in respect to the input. It specifies the transfer from the input to the output in the s domain, assuming no initial energy.

> The transfer function $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

Thus,

$$\boxed{H(s) = \frac{Y(s)}{X(s)}} \tag{15.78}$$

The transfer function depends on what we define as input and output. Since the input and output can be either current or voltage at any place in the circuit, there are four possible transfer functions:

Some authors would not consider Eqs. (15.79c) and (15.79d) transfer functions.

$$H(s) = \text{Voltage gain} = \frac{V_o(s)}{V_i(s)} \tag{15.79a}$$

$$H(s) = \text{Current gain} = \frac{I_o(s)}{I_i(s)} \tag{15.79b}$$

$$H(s) = \text{Impedance} = \frac{V(s)}{I(s)} \tag{15.79c}$$

$$H(s) = \text{Admittance} = \frac{I(s)}{V(s)} \tag{15.79d}$$

Thus, a circuit can have many transfer functions. Note that $H(s)$ is dimensionless in Eqs. (15.79a) and (15.79b).

Each of the transfer functions in Eq. (15.79) can be found in two ways. One way is to assume any convenient input $X(s)$, use any circuit analysis technique (such as current or voltage division, nodal or mesh analysis) to find the output $Y(s)$, and then obtain the ratio of the two. The other approach is to apply the *ladder method*, which involves walking our way through the circuit. By this approach, we assume that the output is 1 V or 1 A as appropriate and use the basic laws of Ohm and Kirchhoff (KCL only) to obtain the input. The transfer function becomes unity divided by the input. This approach may be more convenient to use when the circuit has many loops or nodes so that applying nodal or mesh analysis becomes cumbersome. In the first method, we assume an input and find the output; in the second method, we assume the output and find the input. In both methods, we calculate $H(s)$ as the ratio of output to input transforms. The two methods rely on the linearity property, since we only deal with linear circuits in this book. Example 15.16 illustrates these methods.

Equation (15.78) assumes that both $X(s)$ and $Y(s)$ are known. Sometimes, we know the input $X(s)$ and the transfer function $H(s)$. We find the output $Y(s)$ as

$$Y(s) = H(s)X(s) \tag{15.80}$$

and take the inverse transform to get $y(t)$. A special case is when the input is the unit impulse function, $x(t) = \delta(t)$, so that $X(s) = 1$. For this case,

$$Y(s) = H(s) \qquad \text{or} \qquad y(t) = h(t) \tag{15.81}$$

where

$$h(t) = \mathcal{L}^{-1}[H(s)] \tag{15.82}$$

The term $h(t)$ represents the *unit impulse response*—it is the time-domain response of the network to a unit impulse. Thus, Eq. (15.82) provides a new interpretation for the transfer function: $H(s)$ is the Laplace transform of the unit impulse response of the network. Once we know the impulse response $h(t)$ of a network, we can obtain the response of the network to *any* input signal using Eq. (15.80) in the s domain or using the convolution integral (see next section) in the time domain.

The unit impulse response is the output response of a circuit when the input is a unit impulse.

EXAMPLE 15.15

The output of a linear system is $y(t) = 10e^{-t}\cos 4tu(t)$ when the input is $x(t) = e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

Solution:

If $x(t) = e^{-t}u(t)$ and $y(t) = 10e^{-t}\cos 4tu(t)$, then

$$X(s) = \frac{1}{s+1} \qquad \text{and} \qquad Y(s) = \frac{10(s+1)}{(s+1)^2 + 4^2}$$

Hence,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10}{(s+1)^2 + 16} = \frac{10}{s^2 + 2s + 17}$$

To find $h(t)$, we write $H(s)$ as

$$H(s) = \frac{10}{4} \frac{4}{(s+1)^2 + 4^2}$$

From Table 15.2, we obtain

$$h(t) = 2.5e^{-t} \sin 4t$$

PRACTICE PROBLEM 15.15

The transfer function of a linear system is

$$H(s) = \frac{2s}{s+6}$$

Find the output $y(t)$ due to the input $e^{-3t}u(t)$ and its impulse response.

Answer: $-2e^{-3t} + 4e^{-6t}$, $t \geq 0$, $2\delta(t) - 12e^{-6t}u(t)$.

EXAMPLE 15.16

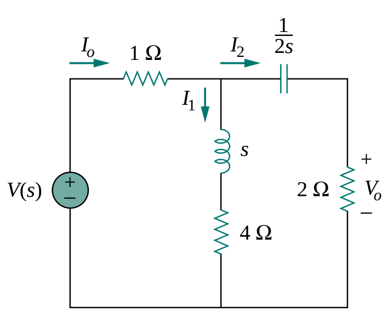

Figure 15.21 For Example 15.16.

Determine the transfer function $H(s) = V_o(s)/I_o(s)$ of the circuit in Fig. 15.21.

Solution:

METHOD 1 By current division,

$$I_2 = \frac{(s+4)I_o}{s + 4 + 2 + 1/2s}$$

But

$$V_o = 2I_2 = \frac{2(s+4)I_o}{s + 6 + 1/2s}$$

Hence,

$$H(s) = \frac{V_o(s)}{I_o(s)} = \frac{4s(s+4)}{2s^2 + 12s + 1}$$

METHOD 2 We can apply the ladder method. We let $V_o = 1$ V. By Ohm's law, $I_2 = V_o/2 = 1/2$ A. The voltage across the $(2 + 1/2s)$ impedance is

$$V_1 = I_2 \left(2 + \frac{1}{2s}\right) = 1 + \frac{1}{4s} = \frac{4s+1}{4s}$$

This is the same as the voltage across the $(s+4)$ impedance. Hence,

$$I_1 = \frac{V_1}{s+4} = \frac{4s+1}{4s(s+4)}$$

Applying KCL at the top node yields

$$I_o = I_1 + I_2 = \frac{4s+1}{4s(s+4)} + \frac{1}{2} = \frac{2s^2+12s+1}{4s(s+4)}$$

Hence,

$$H(s) = \frac{V_o}{I_o} = \frac{1}{I_o} = \frac{4s(s+4)}{2s^2+12s+1}$$

as before.

PRACTICE PROBLEM 15.16

Find the transfer function $H(s) = I_1(s)/I_o(s)$ in the circuit of Fig. 15.21.

Answer: $\dfrac{4s+1}{2s^2+12s+1}$.

EXAMPLE 15.17

For the s-domain circuit in Fig. 15.22, find: (a) the transfer function $H(s) = V_o/V_i$, (b) the impulse response, (c) the response when $v_i(t) = u(t)$ V, (d) the response when $v_i(t) = 8\cos 2t$ V.

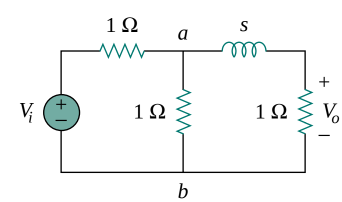

Figure 15.22 For Example 15.17.

Solution:

(a) Using voltage division,

$$V_o = \frac{1}{s+1} V_{ab} \tag{15.17.1}$$

But

$$V_{ab} = \frac{1 \parallel (s+1)}{1 + 1 \parallel (s+1)} V_i = \frac{(s+1)/(s+2)}{1 + (s+1)/(s+2)} V_i$$

or

$$V_{ab} = \frac{s+1}{2s+3} V_i \tag{15.17.2}$$

Substituting Eq. (15.17.2) into Eq. (15.17.1) results in

$$V_o = \frac{V_i}{2s+3}$$

Thus, the impulse response is

$$H(s) = \frac{V_o}{V_i} = \frac{1}{2s+3}$$

(b) We may write $H(s)$ as

$$H(s) = \frac{1}{2} \frac{1}{s + \frac{3}{2}}$$

Its inverse Laplace transform is the required impulse response:

$$h(t) = \frac{1}{2} e^{-3t/2} u(t)$$

(c) When $v_i(t) = u(t)$, $V_i(s) = 1/s$, and

$$V_o(s) = H(s)V_i(s) = \frac{1}{2s(s+\frac{3}{2})} = \frac{A}{s} + \frac{B}{s+\frac{3}{2}}$$

where

$$A = sV_o(s)\Big|_{s=0} = \frac{1}{2(s+\frac{3}{2})}\Bigg|_{s=0} = \frac{1}{3}$$

$$B = \left(s+\frac{3}{2}\right)V_o(s)\Bigg|_{s=-3/2} = \frac{1}{2s}\Bigg|_{s=-3/2} = -\frac{1}{3}$$

Hence, for $v_i(t) = u(t)$,

$$V_o(s) = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+\frac{3}{2}}\right)$$

and its inverse Laplace transform is

$$v_o(t) = \frac{1}{3}(1 - e^{-3t/2})u(t) \text{ V}$$

(d) When $v_i(t) = 8\cos 2t$, then $V_i(s) = \dfrac{8s}{s^2+4}$, and

$$\begin{aligned} V_o(s) = H(s)V_i(s) &= \frac{4s}{(s+\frac{3}{2})(s^2+4)} \\ &= \frac{A}{s+\frac{3}{2}} + \frac{Bs+C}{s^2+4} \end{aligned} \tag{15.17.3}$$

where

$$A = \left(s+\frac{3}{2}\right)V_o(s)\Bigg|_{s=-3/2} = \frac{4s}{s^2+4}\Bigg|_{s=-3/2} = -\frac{24}{25}$$

To get B and C, we multiply Eq. (15.17.3) by $(s+3/2)(s^2+4)$. We get

$$4s = A(s^2+4) + B\left(s^2+\frac{3}{2}s\right) + C\left(s+\frac{3}{2}\right)$$

Equating coefficients,

$$\begin{aligned} &\text{Constant:} \quad 0 = 4A + \frac{3}{2}C \quad \Longrightarrow \quad C = -\frac{8}{3}A \\ &s: \quad 4 = \frac{3}{2}B + C \\ &s^2: \quad 0 = A + B \quad \Longrightarrow \quad B = -A \end{aligned}$$

Solving these gives $A = -24/25$, $B = 24/25$, $C = 64/25$. Hence, for $v_i(t) = 8\cos 2t$ V,

$$V_o(s) = \frac{-\frac{24}{25}}{s+\frac{3}{2}} + \frac{24}{25}\frac{s}{s^2+4} + \frac{32}{25}\frac{2}{s^2+4}$$

and its inverse is

$$v_o(t) = \frac{24}{25}\left(-e^{-3t/2} + \cos 2t + \frac{4}{3}\sin 2t\right)u(t) \text{ V}$$

PRACTICE PROBLEM 15.17

Rework Example 15.17 for the circuit shown in Fig. 15.23.

Answer: (a) $2/(s+4)$, (b) $2e^{-4t}u(t)$, (c) $\frac{1}{2}(1-e^{-4t})u(t)$ V, (d) $\frac{3}{2}(e^{-4t}+\cos 2t+\frac{1}{2}\sin 2t)u(t)$ V.

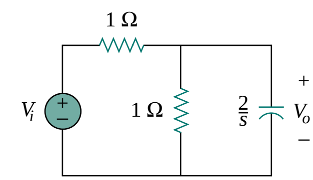

Figure 15.23 For Practice Prob. 15.17.

15.7 THE CONVOLUTION INTEGRAL

The term *convolution* means "folding." Convolution is an invaluable tool to the engineer because it provides a means of viewing and characterizing physical systems. For example, it is used in finding the response $y(t)$ of a system to an excitation $x(t)$, knowing the system impulse response $h(t)$. This is achieved through the *convolution integral*, defined as

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda \tag{15.83}$$

or simply

$$y(t) = x(t) * h(t) \tag{15.84}$$

where λ is a dummy variable and the asterisk denotes convolution. Equation (15.83) or (15.84) states that the output is equal to the input convolved with the unit impulse response. The convolution process is commutative:

$$y(t) = x(t) * h(t) = h(t) * x(t) \tag{15.85a}$$

or

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\,d\lambda \tag{15.85b}$$

This implies that the order in which the two functions are convolved is immaterial. We will see shortly how to take advantage of this commutative property when performing graphical computation of the convolution integral.

> The convolution of two signals consists of time-reversing one of the signals, shifting it, and multiplying it point by point with the second signal, and integrating the product.

The convolution integral in Eq. (15.83) is the general one; it applies to any linear system. However, the convolution integral can be simplified if we assume that a system has two properties. First, if $x(t) = 0$ for $t < 0$, then

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda = \int_{0}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda \tag{15.86}$$

Second, if the system's impulse response is *causal* (i.e., $h(t) = 0$ for $t < 0$), then $h(t - \lambda) = 0$ for $t - \lambda < 0$ or $\lambda > t$, so that Eq. (15.86) becomes

$$\boxed{y(t) = h(t) * x(t) = \int_0^t x(\lambda)h(t-\lambda)\,d\lambda} \tag{15.87}$$

Here are some properties of the convolution integral.

1. $x(t) * h(t) = h(t) * x(t)$ (Commutative)
2. $f(t) * [x(t) + y(t)] = f(t) * x(t) + f(t) * y(t)$ (Distributive)
3. $f(t) * [x(t) * y(t)] = [f(t) * x(t)] * y(t)$ (Associative)
4. $f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\lambda)\delta(t-\lambda)\,d\lambda = f(t)$
5. $f(t) * \delta(t - t_o) = f(t - t_o)$
6. $f(t) * \delta'(t) = \int_{-\infty}^{\infty} f(\lambda)\delta'(t-\lambda)\,d\lambda = f'(t)$
7. $f(t) * u(t) = \int_{-\infty}^{\infty} f(\lambda)u(t-\lambda)\,d\lambda = \int_{-\infty}^{t} f(\lambda)\,d\lambda$

Before learning how to evaluate the convolution integral in Eq. (15.87), let us establish the link between the Laplace transform and the convolution integral. Given two functions $f_1(t)$ and $f_2(t)$ with Laplace transforms $F_1(s)$ and $F_2(s)$, respectively, their convolution is

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(\lambda) f_2(t-\lambda)\,d\lambda \tag{15.88}$$

Taking the Laplace transform gives

$$F(s) = \mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s) \tag{15.89}$$

To prove that Eq. (15.89) is true, we begin with the fact that $F_1(s)$ is defined as

$$F_1(s) = \int_0^\infty f_1(\lambda)e^{-s\lambda}\,d\lambda \tag{15.90}$$

Multiplying this with $F_2(s)$ gives

$$F_1(s)F_2(s) = \int_0^\infty f_1(\lambda)[F_2(s)e^{-s\lambda}]\,d\lambda \tag{15.91}$$

We recall from the time shift property in Eq. (15.17) that the term in brackets can be written as

$$\begin{aligned} F_2(s)e^{-s\lambda} &= \mathcal{L}[f_2(t-\lambda)u(t-\lambda)] \\ &= \int_0^\infty f_2(t-\lambda)u(t-\lambda)e^{-s\lambda}\,dt \end{aligned} \tag{15.92}$$

Substituting Eq. (15.92) into Eq. (15.91) gives

$$F_1(s)F_2(s) = \int_0^\infty f_1(\lambda)\left[\int_0^\infty f_2(t-\lambda)u(t-\lambda)e^{-s\lambda}\,dt\right]d\lambda \tag{15.93}$$

Interchanging the order of integration results in

$$F_1(s)F_2(s) = \int_0^\infty \left[\int_0^t f_1(\lambda)f_2(t-\lambda)\,d\lambda\right]e^{-s\lambda}\,dt \tag{15.94}$$

The integral in brackets extends only from 0 to t because the delayed unit step $u(t-\lambda) = 1$ for $\lambda < t$ and $u(t-\lambda) = 0$ for $\lambda > t$. We notice that the integral is the convolution of $f_1(t)$ and $f_2(t)$ as in Eq. (15.88). Hence,

$$\boxed{F_1(s)F_s(s) = \mathcal{L}[f_1(t) * f_2(t)]} \tag{15.95}$$

as desired. This indicates that convolution in the time domain is equivalent to multiplication in the s domain. For example, if $x(t) = 4e^{-t}$ and $h(t) = 5e^{-2t}$, applying the property in Eq. (15.95), we get

$$\begin{aligned} h(t) * x(t) &= \mathcal{L}^{-1}[H(s)X(s)] = \mathcal{L}^{-1}\left[\left(\frac{5}{s+2}\right)\left(\frac{4}{s+1}\right)\right] \\ &= \mathcal{L}^{-1}\left[\frac{20}{s+1} + \frac{-20}{s+2}\right] \\ &= 20(e^{-t} - e^{-2t}), \qquad t \geq 0 \end{aligned} \tag{15.96}$$

Although we can find the convolution of two signals using Eq. (15.95), as we have just done, if the product $F_1(s)F_2(s)$ is very complicated, finding the inverse may be tough. Also, there are situations in which $f_1(t)$ and $f_2(t)$ are available in the form of experimental data and there are no explicit Laplace transforms. In these cases, one must do the convolution in the time domain.

The process of convolving two signals in the time domain is better appreciated from a graphical point of view. The graphical procedure for evaluating the convolution integral in Eq. (15.87) usually involves four steps.

Steps to evaluate the convolution integral:

1. Folding: Take the mirror image of $h(\lambda)$ about the ordinate axis to obtain $h(-\lambda)$.
2. Displacement: Shift or delay $h(-\lambda)$ by t to obtain $h(t-\lambda)$.
3. Multiplication: Find the product of $h(t-\lambda)$ and $x(\lambda)$.
4. Integration: For a given time t, calculate the area under the product $h(t-\lambda)x(\lambda)$ for $0 < \lambda < t$ to get $y(t)$ at t.

The folding operation in step 1 is the reason for the term *convolution*. The function $h(t-\lambda)$ scans or slides over $x(\lambda)$. In view of this superposition procedure, the convolution integral is also known as the *superposition integral*.

To apply the four steps, it is necessary to be able to sketch $x(\lambda)$ and $h(t-\lambda)$. To get $x(\lambda)$ from the original function $x(t)$ involves merely replacing t with λ. Sketching $h(t-\lambda)$ is the key to the convolution process. It involves reflecting $h(\lambda)$ about the vertical axis and shifting it by t. Analytically, we obtain $h(t-\lambda)$ by replacing every t in $h(t)$ by $t-\lambda$. Since convolution is commutative, it may be more convenient to apply steps 1 and 2 to $x(t)$ instead of $h(t)$. The best way to illustrate the procedure is with some examples.

EXAMPLE 15.18

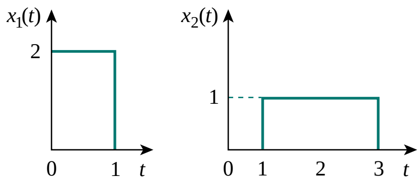

Figure 15.24 For Example 15.18.

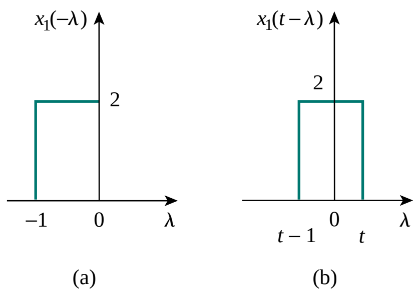

Figure 15.25 (a) Folding $x_1(\lambda)$, (b) shifting $x_1(-\lambda)$ by t.

Find the convolution of the two signals in Fig. 15.24.

Solution:

We follow the four steps to get $y(t) = x_1(t) * x_2(t)$. First, we fold $x_1(t)$ as shown in Fig. 15.25(a) and shift it by t as shown in Fig. 15.25(b). For different values of t, we now multiply the two functions and integrate to determine the area of the overlapping region.

For $0 < t < 1$, there is no overlap of the two functions, as shown in Fig. 15.26(a). Hence,

$$y(t) = x_1(t) * x_2(t) = 0, \qquad 0 < t < 1 \tag{15.18.1}$$

For $1 < t < 2$, the two signals overlap between 1 and t, as shown in Fig. 15.26(b).

$$y(t) = \int_1^t (2)(1)\,d\lambda = 2\lambda \Big|_1^t = 2(t-1), \qquad 1 < t < 2 \tag{15.18.2}$$

For $2 < t < 3$, the two signals completely overlap between $(t-1)$ and t, as shown in Fig. 15.26(c). It is easy to see that the area under the curve is 2. Or

$$y(t) = \int_{t-1}^t (2)(1)\,d\lambda = 2\lambda \Big|_{t-1}^t = 2, \qquad 2 < t < 3 \tag{15.18.3}$$

For $3 < t < 4$, the two signals overlap between $(t-1)$ and 3, as shown in Fig. 15.26(d).

$$\begin{aligned} y(t) &= \int_{t-1}^3 (2)(1)\,d\lambda = 2\lambda \Big|_{t-1}^3 \\ &= 2(3 - t + 1) = 8 - 2t, \qquad 3 < t < 4 \end{aligned} \tag{15.18.4}$$

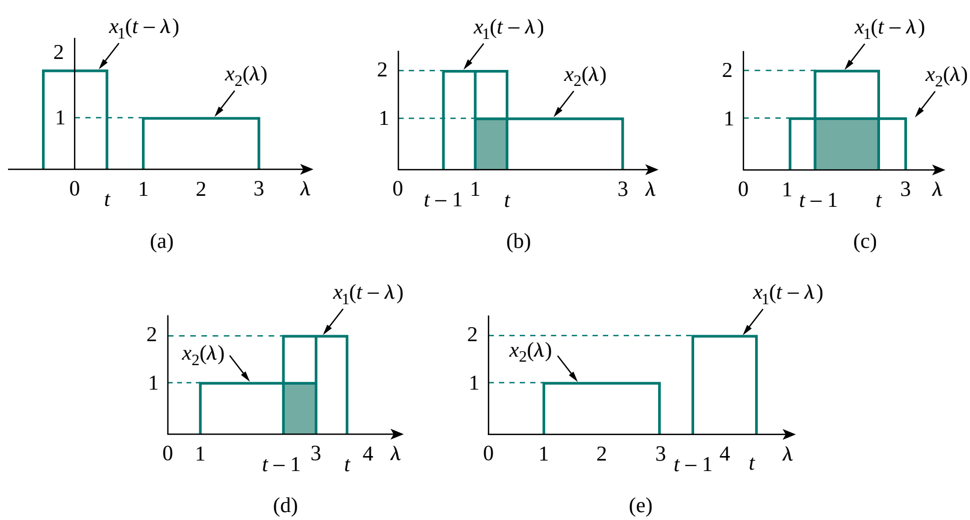

Figure 15.26 Overlapping of $x_1(t-\lambda)$ and $x_2(\lambda)$ for: (a) $0 < t < 1$, (b) $1 < t < 2$, (c) $2 < t < 3$, (d) $3 < t < 4$, (e) $t > 4$.

For $t > 4$, the two signals do not overlap [Fig. 15.26(e)], and

$$y(t) = 0, \qquad t > 4 \tag{15.18.5}$$

Combining Eqs. (15.18.1) to (15.18.5), we obtain

$$y(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 2t - 2, & 1 \le t \le 2 \\ 2, & 2 \le t \le 3 \\ 8 - 2t, & 3 \le t \le 4 \\ 0, & t \ge 4 \end{cases} \tag{15.18.6}$$

which is sketched in Fig. 15.27. Notice that $y(t)$ in this equation is continuous. This fact can be used to check the results as we move from one range of t to another. The result in Eq. (15.18.6) can be obtained without using the graphical procedure—by directly using Eq. (15.87) and the properties of step functions. This will be illustrated in Example 15.20.

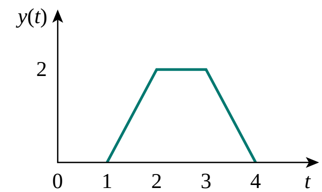

Figure 15.27 Convolution of signals $x_1(t)$ and $x_2(t)$ in Fig. 15.24.

PRACTICE PROBLEM 15.18

Graphically convolve the two functions in Fig. 15.28.

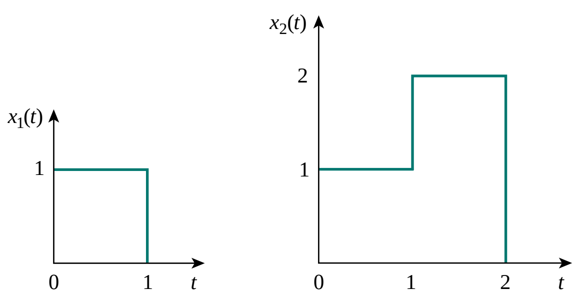

Figure 15.28 For Practice Prob. 15.18.

Answer: The result of the convolution $y(t)$ is shown in Fig. 15.29, where

$$y(t) = \begin{cases} t, & 0 \le t \le 2 \\ 6 - 2t, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

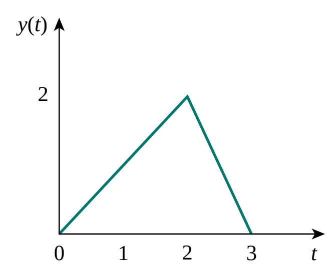

Figure 15.29 Convolution of the signals in Fig. 15.28.

EXAMPLE 15.19

Graphically convolve $g(t)$ and $u(t)$ shown in Fig. 15.30.

Solution:

Let $y(t) = g(t) * u(t)$. We can find $y(t)$ in two ways.

METHOD 1 Suppose we fold $g(t)$, as in Fig. 15.31(a), and shift it by t, as in Fig. 15.31(b). Since $g(t) = t, 0 < t < 1$ originally, we expect that $g(t - \lambda) = t - \lambda$, $0 < t - \lambda < 1$ or $t - 1 < \lambda < t$. There is no overlap of the two functions when $t < 0$ so that $y(0) = 0$ for this case.

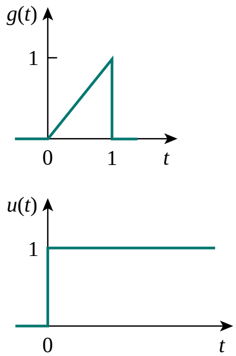

Figure 15.30 For Example 15.19.

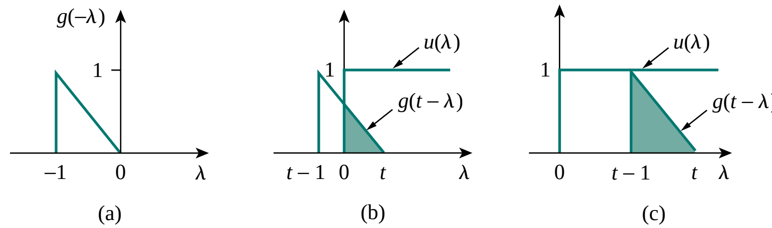

Figure 15.31 Convolution of $g(t)$ and $u(t)$ in Fig. 15.30 with $g(t)$ folded.

For $0 < t < 1$, $g(t-\lambda)$ and $u(\lambda)$ overlap from 0 to t, as evident in Fig. 15.31(b). Therefore,

$$\begin{aligned} y(t) &= \int_0^t (1)(t-\lambda)\,d\lambda = \left(t\lambda - \frac{1}{2}\lambda^2\right)\Bigg|_0^t \\ &= t^2 - \frac{t^2}{2} = \frac{t^2}{2}, \qquad 0 \le t \le 1 \end{aligned} \tag{15.19.1}$$

For $t > 1$, the two functions overlap completely between $(t-1)$ and t [see Fig. 15.31(c)]. Hence,

$$\begin{aligned} y(t) &= \int_{t-1}^t (1)(t-\lambda)\,d\lambda \\ &= \left(t\lambda - \frac{1}{2}\lambda^2\right)\Bigg|_{t-1}^t = \frac{1}{2}, \qquad t \ge 1 \end{aligned} \tag{15.19.2}$$

Thus, from Eqs. (15.19.1) and (15.19.2),

$$y(t) = \begin{cases} \dfrac{1}{2}t^2, & 0 \le t \le 1 \\ \dfrac{1}{2}, & t \ge 1 \end{cases}$$

METHOD 2 Instead of folding g, suppose we fold the unit step function $u(t)$, as in Fig. 15.32(a), and then shift it by t, as in Fig. 15.32(b). Since $u(t) = 1$ for $t > 0$, $u(t-\lambda) = 1$ for $t - \lambda > 0$ or $\lambda < t$, the two functions overlap from 0 to t, so that

$$y(t) = \int_0^t (1)\lambda\,d\lambda = \frac{1}{2}\lambda^2\Bigg|_0^t = \frac{t^2}{2}, \qquad 0 \le t \le 1 \tag{15.19.3}$$

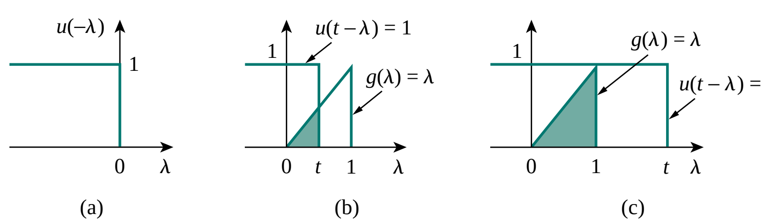

Figure 15.32 Convolution of $g(t)$ and $u(t)$ in Fig. 15.30 with $u(t)$ folded.

For $t > 1$, the two functions overlap between 0 and 1, as shown in Fig. 15.32(c). Hence,

$$y(t) = \int_0^1 (1)\lambda\, d\lambda = \frac{1}{2}\lambda^2 \bigg|_0^1 = \frac{1}{2}, \qquad t \geq 1 \tag{15.19.4}$$

And, from Eqs. (15.19.3) and (15.19.4),

$$y(t) = \begin{cases} \frac{1}{2}t^2, & 0 \leq t \leq 1 \\ \frac{1}{2}, & t \geq 1 \end{cases}$$

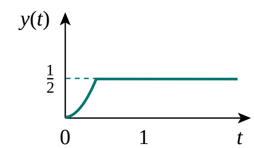

Figure 15.33 Result of Example 15.19.

Although the two methods give the same result, as expected, notice that it is more convenient to fold the unit step function $u(t)$ than fold $g(t)$ in this example. Figure 15.33 shows $y(t)$.

PRACTICE PROBLEM 15.19

Given $g(t)$ and $f(t)$ in Fig. 15.34, graphically find $y(t) = g(t) * f(t)$.

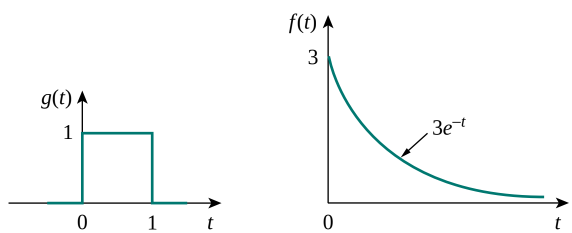

Figure 15.34 For Practice Prob. 15.19.

Answer: $y(t) = \begin{cases} 3(1 - e^{-t}), & 0 \leq t \leq 1 \\ 3(e - 1)e^{-t}, & t \geq 1 \\ 0, & \text{elsewhere} \end{cases}$

EXAMPLE 15.20

For the RL circuit in Fig. 15.35(a), use the convolution integral to find the response $i_o(t)$ due to the excitation shown in Fig. 15.35(b).

Solution:

This problem can be solved in two ways: directly using the convolution integral or using the graphical technique. To use either approach, we first need the unit impulse response $h(t)$ of the circuit. In the s domain, applying the current division principle to the circuit in Fig. 15.36(a) gives

$$I_o = \frac{1}{s+1} I_s$$

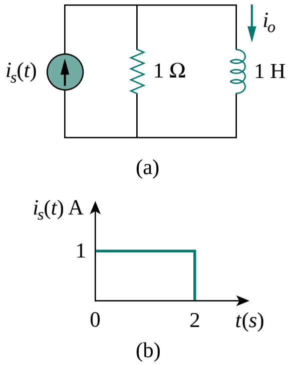

Figure 15.35 For Example 15.20.

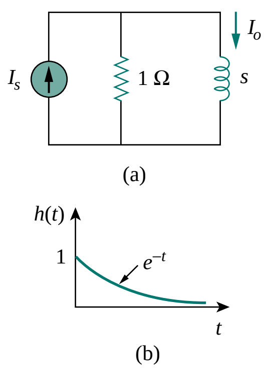

Figure 15.36 For the circuit in Fig. 15.35: (a) its s-domain equivalent, (b) its impulse response.

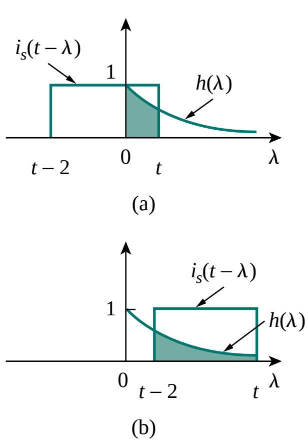

Figure 15.37 For Example 15.20.

Hence,

$$H(s) = \frac{I_o}{I_s} = \frac{1}{s+1} \tag{15.20.1}$$

and the inverse Laplace transform of this gives

$$h(t) = e^{-t}u(t) \tag{15.20.2}$$

Figure 15.36(b) shows the impulse response $h(t)$ of the circuit.

METHOD 1 To use the convolution integral directly, recall that the response is given in the s domain as

$$I_o(s) = H(s)I_s(s)$$

With the given $i_s(t)$ in Fig. 15.35(b),

$$i_s(t) = u(t) - u(t-2)$$

so that

$$\begin{aligned} i_o(t) = h(t) * i_s(t) &= \int_0^t i_s(\lambda)h(t-\lambda)\,d\lambda \\ &= \int_0^t [u(\lambda) - u(\lambda-2)]e^{-(t-\lambda)}\,d\lambda \end{aligned} \tag{15.20.3}$$

Since $u(\lambda-2) = 0$ for $0 < \lambda < 2$, the integrand involving $u(\lambda)$ is nonzero for all $\lambda > 0$, whereas the integrand involving $u(\lambda-2)$ is nonzero only for $\lambda > 2$. The best way to handle the integral is to do the two parts separately. For $0 < t < 2$,

$$\begin{aligned} i_o'(t) &= \int_0^t (1)e^{-(t-\lambda)}\,d\lambda = e^{-t}\int_0^t (1)e^{\lambda}\,d\lambda \\ &= e^{-t}(e^t - 1) = 1 - e^{-t}, \qquad 0 < t < 2 \end{aligned} \tag{15.20.4}$$

For $t > 2$,

$$\begin{aligned} i_o''(t) &= \int_2^t (1)e^{-(t-\lambda)}\,d\lambda = e^{-t}\int_2^t e^{\lambda}\,d\lambda \\ &= e^{-t}(e^t - e^2) = 1 - e^2e^{-t}, \qquad t > 2 \end{aligned} \tag{15.20.5}$$

Substituting Eqs. (15.20.4) and (15.20.5) into Eq. (15.20.3) gives

$$\begin{aligned} i_o(t) &= i_o'(t) - i_o''(t) \\ &= (1 - e^{-t})[u(t-2) - u(t)] - (1 - e^2e^{-t})u(t-2) \\ &= \begin{cases} 1 - e^{-t}, & 0 < t < 2 \\ (e^2 - 1)e^{-t}, & t > 2 \end{cases} \end{aligned} \tag{15.20.6}$$

METHOD 2 To use the graphical technique, we may fold $i_s(t)$ in Fig. 15.35(a) and shift by t, as shown in Fig. 15.37(a). For $0 < t < 2$, the overlap between $i_s(t-\lambda)$ and $h(\lambda)$ is from 0 to t, so that

$$i_o(t) = \int_0^t (1)e^{-\lambda}\,d\lambda = -e^{-\lambda}\Big|_0^t = 1 - e^{-t}, \qquad 0 \le t \le 2 \tag{15.20.7}$$

For $t > 2$, the two functions overlap between $(t-2)$ and t, as in Fig. 15.37(b). Hence

$$i_o(t) = \int_{t-2}^{t} (1)e^{-\lambda}\, d\lambda = -e^{-\lambda}\Big|_{t-2}^{t} = -e^{-t} + e^{-(t-2)} \\ = (e^2 - 1)e^{-t}, \qquad t \ge 0 \tag{15.20.8}$$

From Eqs. (15.20.7) and (15.20.8), the response is

$$i_o(t) = \begin{cases} 1 - e^{-t}, & 0 \le t \le 2 \\ (e^2 - 1)e^{-t}, & t \ge 2 \end{cases} \tag{15.20.9}$$

which is the same as in Eq. (15.20.6). Thus, the response $i_o(t)$ along the excitation $i_s(t)$ is as shown in Fig. 15.38.

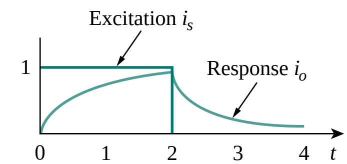

Figure 15.38 For Example 15.20; excitation and response.

PRACTICE PROBLEM 15.20

Use convolution to find $v_o(t)$ in the circuit of Fig. 15.39(a) when the excitation is the signal shown in Fig. 15.39(b).

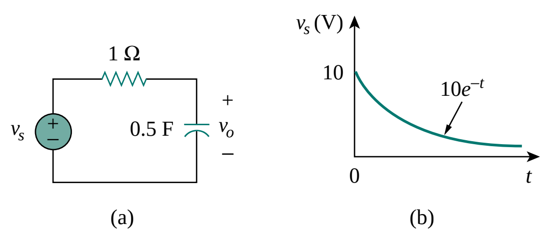

Figure 15.39 For Practice Prob. 15.20.

Answer: $20(e^{-t} - e^{-2t})$ V.

†15.8 APPLICATION TO INTEGRODIFFERENTIAL EQUATIONS

The Laplace transform is useful in solving linear integrodifferential equations. Using the differentiation and integration properties of Laplace transforms, each term in the integrodifferential equation is transformed. Initial conditions are automatically taken into account. We solve the resulting algebraic equation in the s domain. We then convert the solution back to the time domain by using the inverse transform. The following examples illustrate the process.

EXAMPLE 15.21

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

subject to $v(0) = 1$, $v'(0) = -2$.

Solution:

We take the Laplace transform of each term in the given differential equation and obtain

$$[s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

Substituting $v(0) = 1$, $v'(0) = -2$,

$$s^2V(s) - s + 2 + 6sV(s) - 6 + 8V(s) = \frac{2}{s}$$

or

$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s}$$

Hence,

$$V(s) = \frac{s^2 + 4s + 2}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

where

$$A = sV(s)\Big|_{s=0} = \left.\frac{s^2 + 4s + 2}{(s+2)(s+4)}\right|_{s=0} = \frac{2}{(2)(4)} = \frac{1}{4}$$

$$B = (s+2)V(s)\Big|_{s=-2} = \left.\frac{s^2 + 4s + 2}{s(s+4)}\right|_{s=-2} = \frac{-2}{(-2)(2)} = \frac{1}{2}$$

$$C = (s+4)V(s)\Big|_{s=-4} = \left.\frac{s^2 + 4s + 2}{s(s+2)}\right|_{s=-4} = \frac{2}{(-4)(-2)} = \frac{1}{4}$$

Hence,

$$V(s) = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{4}}{s+4}$$

By the inverse Laplace transform,

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

PRACTICE PROBLEM 15.21

Solve the following differential equation using the Laplace transform method.

$$\frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 4v(t) = e^{-t}$$

if $v(0) = v'(0) = 1$.

Answer: $(e^{-t} + 2te^{-2t})u(t)$.

EXAMPLE 15.22

Solve for the response $y(t)$ in the following integrodifferential equation.

$$\frac{dy}{dt} + 5y(t) + 6\int_0^t y(\tau)\,d\tau = u(t), \qquad y(0) = 2$$

Solution:

Taking the Laplace transform of each term, we get

$$[sY(s) - y(0)] + 5Y(s) + \frac{6}{s}Y(s) = \frac{1}{s}$$

Substituting $y(0) = 2$ and multiplying through by s,

$$Y(s)(s^2 + 5s + 6) = 1 + 2s$$

or

$$Y(s) = \frac{2s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

where

$$A = (s+2)Y(s)\left.\right|_{s=-2} = \left.\frac{2s+1}{s+3}\right|_{s=-2} = \frac{-3}{1} = -3$$

$$B = (s+3)Y(s)\left.\right|_{s=-3} = \left.\frac{2s+1}{s+2}\right|_{s=-3} = \frac{-5}{-1} = 5$$

Thus,

$$Y(s) = \frac{-3}{s+2} + \frac{5}{s+3}$$

Its inverse transform is

$$y(t) = (-3e^{-2t} + 5e^{-3t})$$

PRACTICE PROBLEM 15.22

Use the Laplace transform to solve the integrodifferential equation

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(\tau)\,d\tau = 2e^{-3t}, \qquad y(0) = 0$$

Answer: $(-e^{-t} + 4e^{-2t} - 3e^{-3t})u(t)$.

†15.9 APPLICATIONS

So far we have considered three applications of Laplace's transform: circuit analysis in general, obtaining transfer functions, and solving linear integrodifferential equations. The Laplace transform also finds application in other areas in circuit analysis, signal processing, and control systems. Here we will consider two more important applications: network stability and network synthesis.

15.9.1 Network Stability

A circuit is *stable* if its impulse response $h(t)$ is bounded (i.e., $h(t)$ converges to a finite value) as $t \to \infty$; it is *unstable* if $h(t)$ grows without bound as $t \to \infty$. In mathematical terms, a circuit is stable when

$$\lim_{t\to\infty} |h(t)| = \text{finite} \qquad (15.97)$$

Since the transfer function $H(s)$ is the Laplace transform of the impulse response $h(t)$, $H(s)$ must meet certain requirements in order for Eq. (15.97) to hold. Recall that $H(s)$ may be written as

$$H(s) = \frac{N(s)}{D(s)} \tag{15.98}$$

where the roots of $N(s) = 0$ are called the *zeros* of $H(s)$ because they make $H(s) = 0$, while the roots of $D(s) = 0$ are called the *poles* of $H(s)$ since they cause $H(s) \to \infty$. The zeros and poles of $H(s)$ are often located in the s plane as shown in Fig. 15.40(a). Recall from Eqs. (15.47) and (15.48) that $H(s)$ may also be written in terms of its poles as

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{15.99}$$

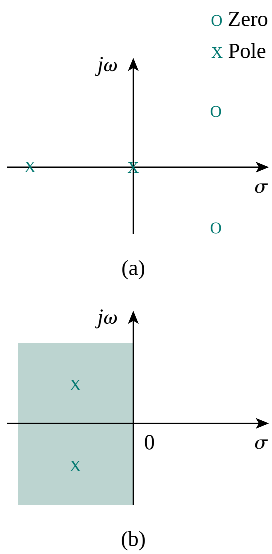

Figure 15.40 The complex s plane: (a) poles and zeros plotted, (b) left-half plane.

$H(s)$ must meet two requirements for the circuit to be stable. First, the degree of $N(s)$ must be less than the degree of $D(s)$; otherwise, long division would produce

$$H(s) = k_n s^n + k_{n-1}s^{n-1} + \cdots + k_1 s + k_0 + \frac{R(s)}{D(s)} \tag{15.100}$$

where the degree of $R(s)$, the remainder of the long division, is less than the degree of $D(s)$. The inverse of $H(s)$ in Eq. (15.99) does not meet the condition in Eq. (15.97). Second, all the poles of $H(s)$ in Eq. (15.98) (i.e., all the roots of $D(s) = 0$) must have negative real parts; in other words, all the poles must lie in the left half of the s plane, as shown typically in Fig. 15.40(b). The reason for this will be apparent if we take the inverse Laplace transform of $H(s)$ in Eq. (15.98). Since Eq. (15.98) is similar to Eq. (15.48), its partial fraction expansion is similar to the one in Eq. (15.49) so that the inverse of $H(s)$ is similar to that in Eq. (15.53). Hence,

$$h(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t}) \tag{15.101}$$

We see from this equation that each pole p_i must be positive (i.e., pole $s = -p_i$ in the left-half plane) in order for $e^{-p_i t}$ to decrease with increasing t. Thus,

> A circuit is **stable** when all the poles of its transfer function $H(s)$ lie in the left half of the s plane.

An unstable circuit never reaches steady state because the transient response does not decay to zero. Consequently, steady-state analysis is only applicable to stable circuits.

A circuit made up exclusively of passive elements (R, L, and C) and independent sources cannot be unstable, because that would imply that some branch currents or voltages would grow indefinitely with sources set to zero. Passive elements cannot generate such indefinite growth. Passive circuits either are stable or have poles with zero real parts. To show that this is the case, consider the series RLC circuit in Fig. 15.41. The transfer function is given by

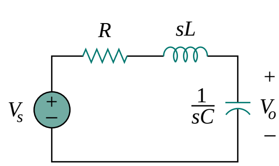

Figure 15.41 A typical RLC circuit.

$$H(s) = \frac{V_o}{V_s} = \frac{1/sC}{R + sL + 1/sC}$$

or

$$H(s) = \frac{1/LC}{s^2 + sR/L + 1/LC} \tag{15.102}$$

Notice that $D(s) = s^2 + sR/L + 1/LC = 0$ is the same as the characteristic equation obtained for the series RLC circuit in Eq. (8.8). The circuit has poles at

$$p_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2} \tag{15.103}$$

where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{LC}$$

For $R, L, C > 0$, the two poles always lie in the left half of the s plane, implying that the circuit is always stable. However, when $R = 0$, $\alpha = 0$ and the circuit becomes unstable. Although ideally this is possible, it does not really happen, because R is never zero.

On the other hand, active circuits or passive circuits with controlled sources can supply energy, and they can be unstable. In fact, an oscillator is a typical example of a circuit designed to be unstable. An oscillator is designed such that its transfer function is of the form

$$H(s) = \frac{N(s)}{s^2 + \omega_0{}^2} = \frac{N(s)}{(s + j\omega_0)(s - j\omega_0)} \tag{15.104}$$

so that its output is sinusoidal.

EXAMPLE 15.23

Determine the values of k for which the circuit in Fig. 15.42 is stable.

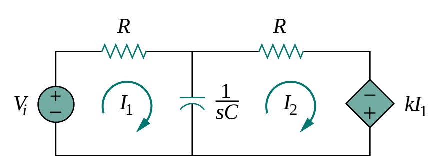

Figure 15.42 For Example 15.23.

Solution:

Applying mesh analysis to the first-order circuit in Fig. 15.42 gives

$$V_i = \left(R + \frac{1}{sC}\right) I_1 - \frac{I_2}{sC} \tag{15.23.1}$$

and

$$0 = -kI_1 + \left(R + \frac{1}{sC}\right) I_2 - \frac{I_1}{sC}$$

or

$$0 = -\left(k + \frac{1}{sC}\right) I_1 + \left(R + \frac{1}{sC}\right) I_2 \tag{15.23.2}$$

We can write Eqs. (15.23.1) and (15.23.2) in matrix form as

$$\begin{bmatrix} V_i \\ 0 \end{bmatrix} = \begin{bmatrix} \left(R + \frac{1}{sC}\right) & -\frac{1}{sC} \\ -\left(k + \frac{1}{sC}\right) & \left(R + \frac{1}{sC}\right) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The determinant is

$$\Delta = \left(R + \frac{1}{sC}\right)^2 - \frac{k}{sC} - \frac{1}{s^2C^2} = \frac{sR^2C + 2R - k}{sC} \tag{15.23.3}$$

The characteristic equation ($\Delta = 0$) gives the single pole as

$$p = \frac{k - 2R}{R^2 C}$$

which is negative when $k < 2R$. Thus, we conclude the circuit is stable when $k < 2R$ and unstable for $k > 2R$.

PRACTICE PROBLEM 15.23

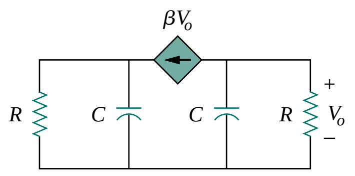

Figure 15.43 For Practice Prob. 15.23.

For what value of β is the circuit in Fig. 15.43 stable?

Answer: $\beta > 1/R$.

EXAMPLE 15.24

An active filter has the transfer function

$$H(s) = \frac{k}{s^2 + s(4 - k) + 1}$$

For what values of k is the filter stable?

Solution:

As a second-order circuit, $H(s)$ may be written as

$$H(s) = \frac{N(s)}{s^2 + bs + c}$$

where $b = 4 - k$, $c = 1$, and $N(s) = k$. This has poles at $p^2 + bp + c = 0$, that is,

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

For the circuit to be stable, the poles must be located in the left half of the s plane. This implies that $b > 0$.

Applying this to the given $H(s)$ means that for the circuit to be stable, $4 - k > 0$ or $k < 4$.

PRACTICE PROBLEM 15.24

A second-order active circuit has the transfer function

$$H(s) = \frac{1}{s^2 + s(10 + \alpha) + 25}$$

Find the range of the values of α for which the circuit is stable. What is the value of α that will cause oscillation?

Answer: $\alpha > -10$, $\alpha = -10$.

15.9.2 Network Synthesis

Network synthesis may be regarded as the process of obtaining an appropriate network to represent a given transfer function. Network synthesis is easier in the s domain than in the time domain.

In network analysis, we find the transfer function of a given network. In network synthesis, we reverse the approach: given a transfer function, we are required to find a suitable network.

> **Network synthesis** is finding a network that represents a given transfer function.

Keep in mind that in synthesis, there may be many different answers—or possibly no answers—because there are many circuits that can be used to represent the same transfer function; in network analysis, there is only one answer.

Network synthesis is an exciting field of prime engineering importance. Being able to look at a transfer function and come up with the type of circuit it represents is a great asset to a circuit designer. Although network synthesis constitutes a whole course by itself and requires some experience, the following examples are meant to whet your appetite.

EXAMPLE 15.25

Given the transfer function

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{10}{s^2 + 3s + 10}$$

realize the function using the circuit in Fig. 15.44(a). (a) Select $R = 5\ \Omega$, and find L and C. (b) Select $R = 1\ \Omega$, and find L and C.

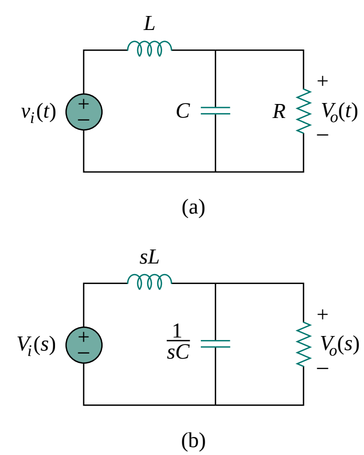

Figure 15.44 For Example 15.25.

Solution:

The s-domain equivalent of the circuit in Fig. 15.44(a) is shown in Fig. 15.44(b). The parallel combination of R and C gives

$$R \left\| \frac{1}{sC} \right. = \frac{R/sC}{R + 1/sC} = \frac{R}{1 + sRC}$$

Using the voltage division principle,

$$V_o = \frac{R/(1 + sRC)}{sL + R/(1 + sRC)} V_i = \frac{R}{sL(1 + sRC) + R} V_i$$

or

$$\frac{V_o}{V_i} = \frac{R}{s^2 RLC + sL + R} = \frac{1/LC}{s^2 + s/RC + 1/LC}$$

Comparing this with the given transfer function $H(s)$ reveals that

$$\frac{1}{LC} = 10, \qquad \frac{1}{RC} = 3$$

There are several values of R, L, and C that satisfy these requirements. This is the reason for specifying one element value so that others can be determined.

(a) If we select $R = 5\ \Omega$, then

$$C = \frac{1}{3R} = 66.67\text{ mF}, \qquad L = \frac{1}{10C} = 1.5\text{ H}$$

(b) If we select $R = 1\ \Omega$, then

$$C = \frac{1}{3R} = 0.333\text{ F}, \qquad L = \frac{1}{10C} = 0.3\text{ H}$$

Making $R = 1\ \Omega$ can be regarded as *normalizing* the design.

In this example we have used passive elements to realize the given transfer function. The same goal can be achieved by using active elements, as the next example demonstrates.

PRACTICE PROBLEM 15.25

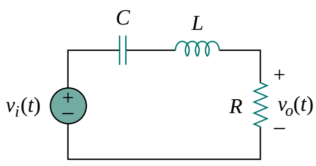

Figure 15.45 For Practice Prob. 15.25.

Realize the function

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{4s}{s^2 + 4s + 20}$$

using the circuit in Fig. 15.45. Select $R = 2\ \Omega$, and determine L and C.

Answer: 0.5 H, 0.1 F.

EXAMPLE 15.26

Synthesize the function

$$T(s) = \frac{V_o(s)}{V_s(s)} = \frac{10^6}{s^2 + 100s + 10^6}$$

using the topology in Fig. 15.46.

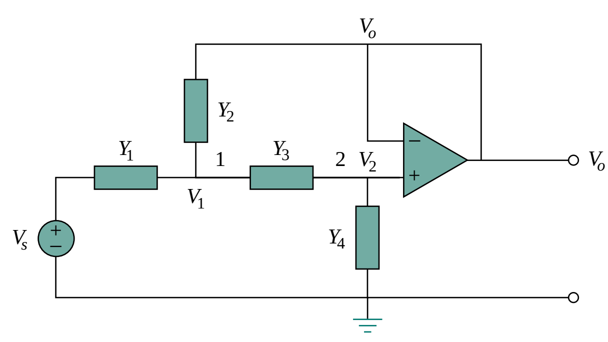

Figure 15.46 For Example 15.26.

Solution:

We apply nodal analysis to nodes 1 and 2. At node 1,

$$(V_s - V_1)Y_1 = (V_1 - V_o)Y_2 + (V_1 - V_2)Y_3 \tag{15.26.1}$$

At node 2,

$$(V_1 - V_2)Y_3 = (V_2 - 0)Y_4 \tag{15.26.2}$$

But $V_2 = V_o$, so Eq. (15.26.1) becomes

$$Y_1 V_s = (Y_1 + Y_2 + Y_3)V_1 - (Y_2 + Y_3)V_o \tag{15.26.3}$$

and Eq. (15.26.2) becomes

$$V_1 Y_3 = (Y_3 + Y_4)V_o$$

or

$$V_1 = \frac{1}{Y_3}(Y_3 + Y_4)V_o \tag{15.26.4}$$

Substituting Eq. (15.26.4) into Eq. (15.26.3) gives

$$Y_1 V_s = (Y_1 + Y_2 + Y_3)\frac{1}{Y_3}(Y_3 + Y_4)V_o - (Y_2 + Y_3)V_o$$

or

$$Y_1 Y_3 V_s = [Y_1 Y_3 + Y_4(Y_1 + Y_2 + Y_3)]V_o$$

Thus,

$$\frac{V_o}{V_s} = \frac{Y_1 Y_3}{Y_1 Y_3 + Y_4(Y_1 + Y_2 + Y_3)} \tag{15.26.5}$$

To synthesize the given transfer function $T(s)$, compare it with the one in Eq. (15.26.5). Notice two things: (1) $Y_1 Y_3$ must not involve s because the numerator of $T(s)$ is constant; (2) the given transfer function is second-order, which implies that we must have two capacitors. Therefore, we must make Y_1 and Y_3 resistive, while Y_2 and Y_4 are capacitive. So we select

$$Y_1 = \frac{1}{R_1}, \qquad Y_2 = sC_1, \qquad Y_3 = \frac{1}{R_2}, \qquad Y_4 = sC_2 \tag{15.26.6}$$

Substituting Eq. (15.26.6) into Eq. (15.26.5) gives

$$\begin{aligned}\frac{V_o}{V_s} &= \frac{1/(R_1 R_2)}{1/(R_1 R_2) + sC_2(1/R_1 + 1/R_2 + sC_1)}\\ &= \frac{1/(R_1 R_2 C_1 C_2)}{s^2 + s(R_1 + R_2)/(R_1 R_2 C_1) + 1/(R_1 R_2 C_1 C_2)}\end{aligned}$$

Comparing this with the given transfer function $T(s)$, we notice that

$$\frac{1}{R_1 R_2 C_1 C_2} = 10^6, \qquad \frac{R_1 + R_2}{R_1 R_2 C_1} = 100$$

If we select $R_1 = R_2 = 10\ \text{k}\Omega$, then

$$C_1 = \frac{R_1 + R_2}{100 R_1 R_2} = \frac{20 \times 10^3}{100 \times 100 \times 10^6} = 2\ \mu\text{F}$$

$$C_2 = \frac{10^{-6}}{R_1 R_2 C_1} = \frac{10^{-6}}{100 \times 10^6 \times 2 \times 10^{-6}} = 5\ \text{nF}$$

Thus, the given transfer function is realized using the circuit shown in Fig. 15.47.

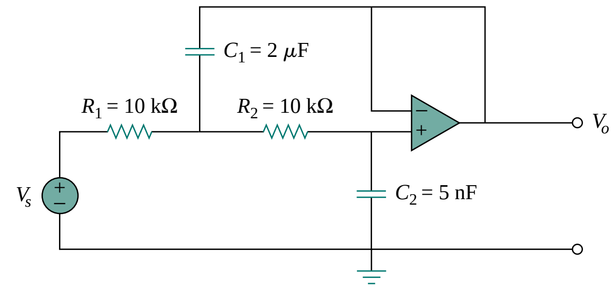

Figure 15.47 For Example 15.26.

PRACTICE PROBLEM 15.26

Synthesize the function

$$\frac{V_o(s)}{V_{in}} = \frac{-2s}{s^2 + 6s + 10}$$

using the op amp circuit shown in Fig. 15.48. Select

$$Y_1 = \frac{1}{R_1}, \qquad Y_2 = sC_1, \qquad Y_3 = sC_2, \qquad Y_4 = \frac{1}{R_2}$$

Let $R_1 = 1\ k\Omega$, and determine C_1, C_2, and R_2.

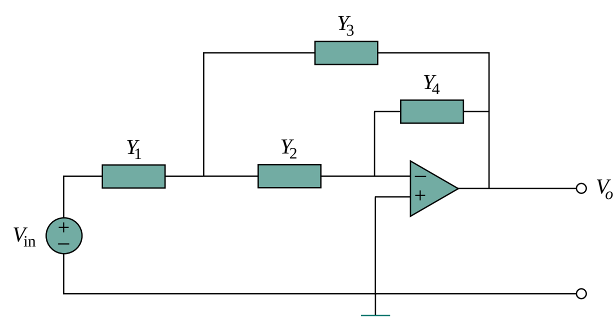

Figure 15.48 For Practice Prob. 15.26.

Answer: 0.1 mF, 0.5 mF, 2 kΩ.

15.10 SUMMARY

1. The Laplace transform allows a signal represented by a function in the time domain to be analyzed in the s domain (or complex frequency domain). It is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

2. Properties of the Laplace transform are listed in Table 15.1, while the Laplace transforms of basic common functions are listed in Table 15.2.
3. The inverse Laplace transform can be found using partial fraction expansions and using the Laplace transform pairs in Table 15.2 as a look-up table. Real poles lead to exponential functions and complex poles to damped sinusoids.
4. The Laplace transform can be used to analyze a circuit. We convert each element from the time domain to the s domain, solve the problem using any circuit technique, and convert the result to the time domain using the inverse transform.
5. In the s domain, the circuit elements are replaced with the initial condition at $t = 0$ as follows:

Resistor:	v_R	$\Longrightarrow$	$V_R = RI$
Inductor:	v_L	$\Longrightarrow$	$V_L = sLI - Li(0^-)$
Capacitor:	v_C	$\Longrightarrow$	$V_C = \dfrac{I}{sC} - \dfrac{v(0^-)}{s}$

6. Using the Laplace transform to analyze a circuit results in a complete (both natural and forced) response, as the initial conditions are incorporated in the transformation process.
7. The transfer function $H(s)$ of a network is the Laplace transform of the impulse response $h(t)$.
8. In the s domain, the transfer function $H(s)$ relates the output response $Y(s)$ and an input excitation $X(s)$; that is, $H(s) = Y(s)/X(s)$.
9. The convolution of two signals consists of time-reversing one of the signals, shifting it, multiplying it point by point with the second signal, and integrating the product. The convolution integral relates the convolution of two signals in the time domain to the inverse of the product of their Laplace transforms:

$$\mathcal{L}^{-1}[F_1(s)F_2(s)] = f_1(t) * f_2(t) = \int_0^t f_1(\lambda) f_2(t-\lambda)\, d\lambda$$

10. In the time domain, the output $y(t)$ of the network is the convolution of the impulse response with the input $x(t)$,

$$y(t) = h(t) * x(t)$$

11. The Laplace transform can be used to solve a linear integrodifferential equation.
12. Two other typical areas of applications of the Laplace transform are circuit stability and synthesis. A circuit is stable when all the poles of its transfer function lie in the left half of the s plane. Network synthesis is the process of obtaining an appropriate network to represent a given transfer function for which analysis in the s domain is well suited.

REVIEW QUESTIONS

15.1 Every function $f(t)$ has a Laplace transform.
(a) True (b) False

15.2 The variable s in the Laplace transform $H(s)$ is called
(a) complex frequency (b) transfer function
(c) zero (d) pole

15.3 The Laplace transform of $u(t-2)$ is:
(a) $\dfrac{1}{s+2}$ (b) $\dfrac{1}{s-2}$
(c) $\dfrac{e^{2s}}{s}$ (d) $\dfrac{e^{-2s}}{s}$

15.4 The zero of the function

$$F(s) = \frac{s+1}{(s+2)(s+3)(s+4)}$$

is at
(a) -4 (b) -3 (c) -2 (d) -1

15.5 The poles of the function

$$F(s) = \frac{s+1}{(s+2)(s+3)(s+4)}$$

are at
(a) -4 (b) -3 (c) -2 (d) -1

15.6 If $F(s) = 1/(s+2)$, then $f(t)$ is
(a) $e^{2t}u(t)$ (b) $e^{-2t}u(t)$
(c) $u(t-2)$ (d) $u(t+2)$

15.7 Given that $F(s) = e^{-2s}/(s+1)$, then $f(t)$ is
(a) $e^{-2(t-1)}u(t-1)$ (b) $e^{-(t-2)}u(t-2)$
(c) $e^{-t}u(t-2)$ (d) $e^{-t}u(t+1)$
(e) $e^{-(t-2)}u(t)$

15.8 The initial value of $f(t)$ with transform

$$F(s) = \frac{s+1}{(s+2)(s+3)}$$

is:
(a) nonexistent (b) ∞ (c) 0
(d) 1 (e) $\frac{1}{6}$

15.9 The inverse Laplace transform of

$$\frac{s+2}{(s+2)^2+1}$$

is:
(a) $e^{-t}\cos 2t$ (b) $e^{-t}\sin 2t$ (c) $e^{-2t}\cos t$
(d) $e^{-2t}\sin 2t$ (e) none of the above

15.10 A transfer function is defined only when all initial conditions are zero.
(a) True (b) False

Answers: 15.1b, 15.2a, 15.3d, 15.4d, 15.5a,b,c, 15.6b, 15.7b, 15.8d, 15.9c, 15.10b.

PROBLEMS

Sections 15.2 and 15.3 Definition and Properties of the Laplace Transform

15.1 Find the Laplace transform of:
(a) $\cosh at$ (b) $\sinh at$
[*Hint:* $\cosh x = \frac{1}{2}(e^x + e^{-x})$,
$\sinh x = \frac{1}{2}(e^x - e^{-x})$.]

15.2 Determine the Laplace transform of:
(a) $\cos(\omega t + \theta)$ (b) $\sin(\omega t + \theta)$

15.3 Obtain the Laplace transform of each of the following functions:
(a) $e^{-2t}\cos 3tu(t)$ (b) $e^{-2t}\sin 4tu(t)$
(c) $e^{-3t}\cosh 2tu(t)$ (d) $e^{-4t}\sinh tu(t)$
(e) $te^{-t}\sin 2tu(t)$

15.4 Find the Laplace transform of each of the following functions:
(a) $t^2\cos(2t+30^\circ)u(t)$ (b) $3t^4e^{-2t}u(t)$
(c) $2tu(t) - 4\dfrac{d}{dt}\delta(t)$ (d) $2e^{-(t-1)}u(t)$
(e) $5u(t/2)$ (f) $6e^{-t/3}u(t)$
(g) $\dfrac{d^n}{dt^n}\delta(t)$

15.5 Calculate the Laplace transforms of these functions:
(a) $2\delta(t-1)$ (b) $10u(t-2)$
(c) $(t+4)u(t)$ (d) $2e^{-t}u(t-4)$

15.6 Obtain the Laplace transform of
(a) $10\cos 4(t-1)u(t)$ (b) $t^2e^{-2t}u(t) + u(t-3)$

15.7 Find the Laplace transforms of the following functions:

(a) $2\delta(3t) + 6u(2t) + 4e^{-2t} - 10e^{-3t}$
(b) $te^{-t}u(t-1)$
(c) $\cos 2(t-1)u(t-1)$
(d) $\sin 4t[u(t) - u(t-\pi)]$

15.8 Determine the Laplace transforms of these functions:

(a) $f(t) = (t-4)u(t-2)$
(b) $g(t) = 2e^{-4t}u(t-1)$
(c) $h(t) = 5\cos(2t-1)u(t)$
(d) $p(t) = 6[u(t-2) - u(t-4)]$

15.9 In two different ways, find the Laplace transform of

$$g(t) = \frac{d}{dt}(te^{-t}\cos t)$$

15.10 Find $F(s)$ if:

(a) $f(t) = 6e^{-t}\cosh 2t$ (b) $f(t) = 3te^{-2t}\sinh 4t$
(c) $f(t) = 8e^{-3t}\cosh tu(t-2)$

15.11 Calculate the Laplace transform of the function in Fig. 15.49.

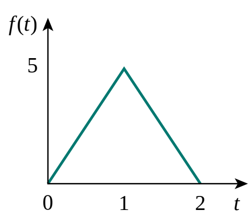

Figure 15.49 For Prob. 15.11.

15.12 Find the Laplace transform of the function in Fig. 15.50.

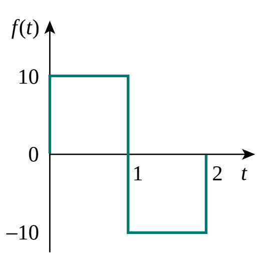

Figure 15.50 For Prob. 15.12.

15.13 Obtain the Laplace transform of $f(t)$ in Fig. 15.51.

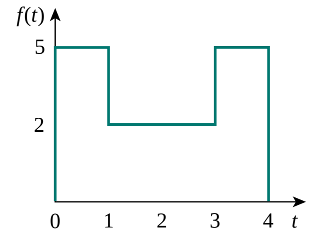

Figure 15.51 For Prob. 15.13.

15.14 Determine the Laplace transforms of the function in Fig. 15.52.

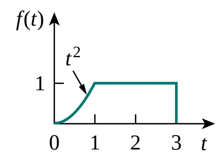

Figure 15.52 For Prob. 15.14.

15.15 Obtain the Laplace transforms of the functions in Fig. 15.53.

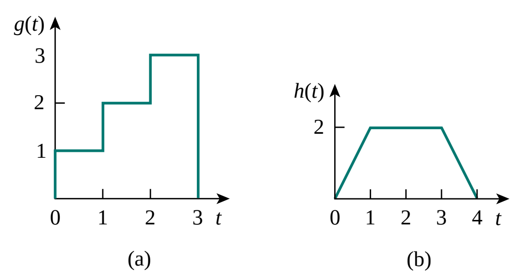

Figure 15.53 For Prob. 15.15.

15.16 Calculate the Laplace transform of the train of unit impulses in Fig. 15.54.

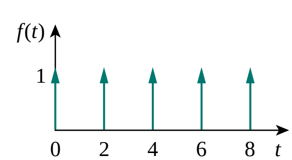

Figure 15.54 For Prob. 15.16.

15.17 The periodic function shown in Fig. 15.55 is defined over its period as

$$g(t) = \begin{cases} \sin \pi t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

Find $G(s)$.

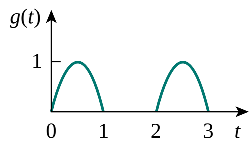

Figure 15.55 For Prob. 15.17.

15.18 Obtain the Laplace transform of the periodic waveform in Fig. 15.56.

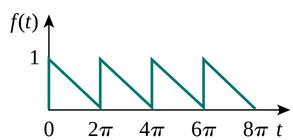

Figure 15.56 For Prob. 15.18.

15.19 Find the Laplace transforms of the functions in Fig. 15.57.

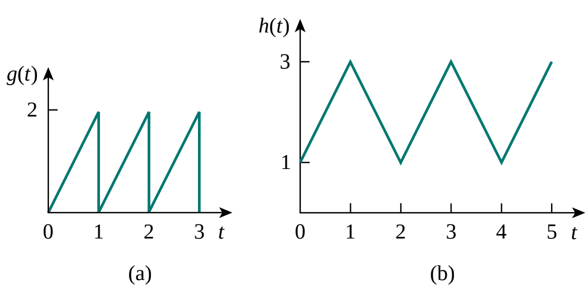

Figure 15.57 For Prob. 15.19.

15.20 Determine the Laplace transforms of the periodic functions in Fig. 15.58.

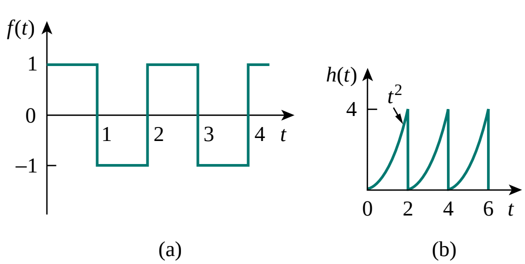

Figure 15.58 For Prob. 15.20.

15.21 Find the initial and final values, if they exist, of the following Laplace transforms:

(a) $F(s) = \dfrac{10s^3 + 1}{s^2 + 6s + 5}$

(b) $F(s) = \dfrac{s + 1}{s^2 - 4s + 6}$

(c) $F(s) = \dfrac{2s^2 + 7}{(s + 1)(s + 2)(s^2 + 2s + 5)}$

15.22 Find $f(0)$ and $f(\infty)$, if they exist, when:

(a) $F(s) = \dfrac{8(s + 1)(s + 3)}{s(s + 2)(s + 4)}$

(b) $F(s) = \dfrac{6(s - 1)}{s^4 - 1}$

15.23 Determine the initial and final values of $f(t)$, if they exist, given that:

(a) $F(s) = \dfrac{s^2 + 3}{s^3 + 4s^2 + 6}$

(b) $F(s) = \dfrac{s^2 - 2s + 1}{(s - 2)(s^2 + 2s + 4)}$

Section 15.4 The Inverse Laplace Transform

15.24 Determine the inverse Laplace transform of each of the following functions:

(a) $F(s) = \dfrac{1}{s} + \dfrac{2}{s + 1}$

(b) $G(s) = \dfrac{3s + 1}{s + 4}$

(c) $H(s) = \dfrac{4}{(s + 1)(s + 3)}$

(d) $J(s) = \dfrac{12}{(s + 2)^2(s + 4)}$

15.25 Find $f(t)$ for each $F(s)$:

(a) $\dfrac{10s}{(s + 1)(s + 2)(s + 3)}$

(b) $\dfrac{2s^2 + 4s + 1}{(s + 1)(s + 2)^3}$

(c) $\dfrac{s + 1}{(s + 2)(s^2 + 2s + 5)}$

15.26 Determine the inverse Laplace transform of each of the following functions:

(a) $\dfrac{8(s + 1)(s + 3)}{s(s + 2)(s + 4)}$ (b) $\dfrac{s^2 - 2s + 4}{(s + 1)(s + 2)^2}$

(c) $\dfrac{s^2 + 1}{(s + 3)(s^2 + 4s + 5)}$

15.27 Calculate the inverse Laplace transform of:

(a) $\dfrac{6(s-1)}{s^4-1}$ (b) $\dfrac{se^{-\pi s}}{s^2+1}$ (c) $\dfrac{8}{s(s+1)^3}$

15.28 Find the time functions that have the following Laplace transforms:

(a) $F(s) = 10 + \dfrac{s^2+1}{s^2+4}$

(b) $G(s) = \dfrac{e^{-s}+4e^{-2s}}{s^2+6s+8}$

(c) $H(s) = \dfrac{(s+1)e^{-2s}}{s(s+3)(s+4)}$

15.29 Obtain $f(t)$ for the following transforms:

(a) $F(s) = \dfrac{(s+3)e^{-6s}}{(s+1)(s+2)}$

(b) $F(s) = \dfrac{4-e^{-2s}}{s^2+5s+4}$

(c) $F(s) = \dfrac{se^{-s}}{(s+3)(s^2+4)}$

15.30 Obtain the inverse Laplace transforms of the following functions:

(a) $X(s) = \dfrac{1}{s^2(s+2)(s+3)}$

(b) $Y(s) = \dfrac{1}{s(s+1)^2}$

(c) $Z(s) = \dfrac{1}{s(s+1)(s^2+6s+10)}$

15.31 Obtain the inverse Laplace transforms of these functions:

(a) $\dfrac{12e^{-2s}}{s(s^2+4)}$ (b) $\dfrac{2s+1}{(s^2+1)(s^2+9)}$

(c) $\dfrac{9s^2}{(s^2+4s+13)}$

15.32 Find $f(t)$ given that:

(a) $F(s) = \dfrac{s^2+4s}{s^2+10s+26}$

(b) $F(s) = \dfrac{5s^2+7s+29}{s(s^2+4s+29)}$

***15.33** Determine $f(t)$ if:

(a) $F(s) = \dfrac{2s^3+4s^2+1}{(s^2+2s+17)(s^2+4s+20)}$

(b) $F(s) = \dfrac{s^2+4}{(s^2+9)(s^2+6s+3)}$

Section 15.5 Application to Circuits

15.34 Determine $i(t)$ in the circuit of Fig. 15.59 by means of the Laplace transform.

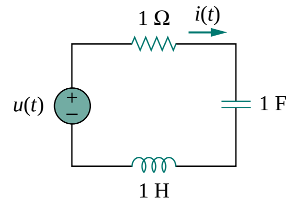

Figure 15.59 For Prob. 15.34.

15.35 Find $v_o(t)$ in the circuit in Fig. 15.60.

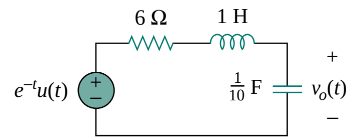

Figure 15.60 For Prob. 15.35.

15.36 Find the input impedance $Z_{in}(s)$ of each of the circuits in Fig. 15.61.

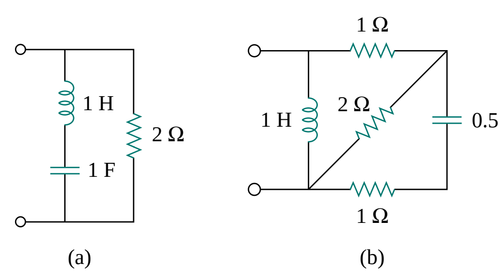

Figure 15.61 For Prob. 15.36.

15.37 Obtain the mesh currents in the circuit of Fig. 15.62.

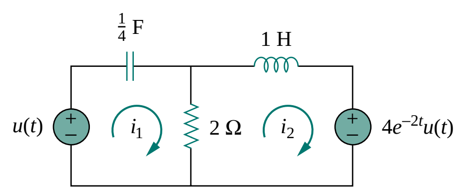

Figure 15.62 For Prob. 15.37.

15.38 Find $v_o(t)$ in the circuit in Fig. 15.63.

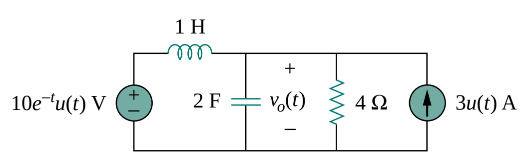

Figure 15.63 For Prob. 15.38.

15.39 Determine $i_o(t)$ in the circuit in Fig. 15.64.

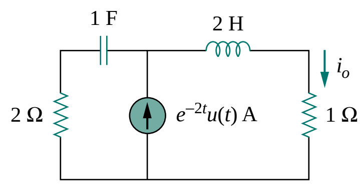

Figure 15.64 For Prob. 15.39.

***15.40** Determine $i_o(t)$ in the network shown in Fig. 15.65.

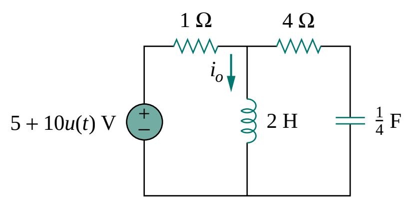

Figure 15.65 For Prob. 15.40.

***15.41** Find $i_o(t)$ for $t > 0$ in the circuit in Fig. 15.66.

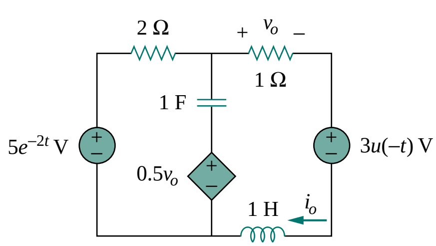

Figure 15.66 For Prob. 15.41.

15.42 Calculate $i_o(t)$ for $t > 0$ in the network of Fig. 15.67.

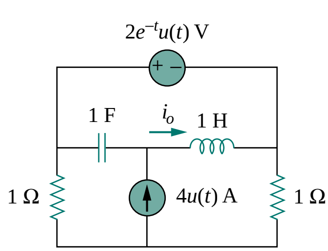

Figure 15.67 For Prob. 15.42.

15.43 In the circuit of Fig. 15.68, let $i(0) = 1$ A, $v_o(0) = 2$ V, and $v_s = 4e^{-2t}u(t)$ V. Find $v_o(t)$ for $t > 0$.

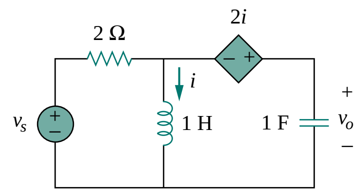

Figure 15.68 For Prob. 15.43.

15.44 Find $v_o(t)$ in the circuit in Fig. 15.69 if $v_x(0) = 2$ V and $i(0) = 1$ A.

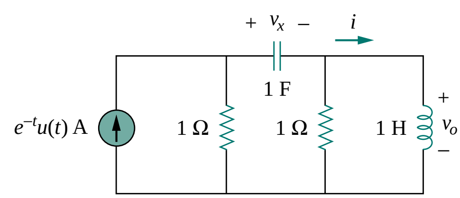

Figure 15.69 For Prob. 15.44.

15.45 Consider the parallel RLC circuit of Fig. 15.70. Find $v(t)$ and $i(t)$ given that $v(0) = 5$ and $i(0) = -2$ A.

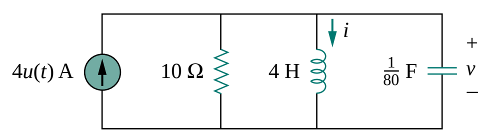

Figure 15.70 For Prob. 15.45.

15.46 For the RLC circuit shown in Fig. 15.71, find the complete response if $v(0) = 2$ V when the switch is closed.

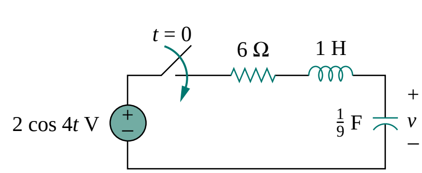

Figure 15.71 For Prob. 15.46.

15.47 For the op amp circuit in Fig. 15.72, find $v_o(t)$ for $t > 0$. Take $v_s = 3e^{-5t}u(t)$ V.

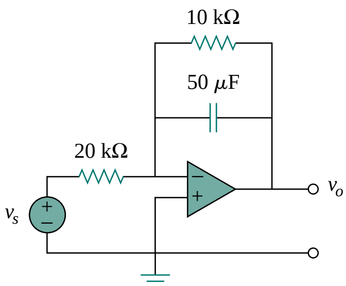

Figure 15.72 For Prob. 15.47.

15.48 Find $I_1(s)$ and $I_2(s)$ in the circuit of Fig. 15.73.

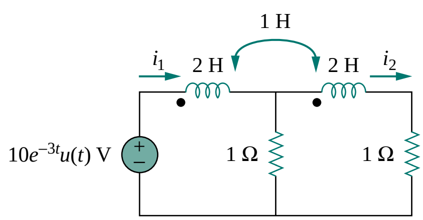

Figure 15.73 For Prob. 15.48.

15.49 For the circuit in Fig. 15.74, find $v_o(t)$ for $t > 0$.

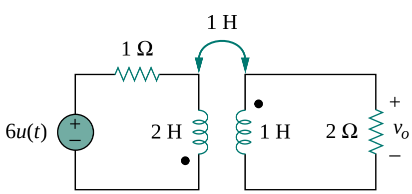

Figure 15.74 For Prob. 15.49.

15.50 For the ideal transformer circuit in Fig. 15.75, determine $i_o(t)$.

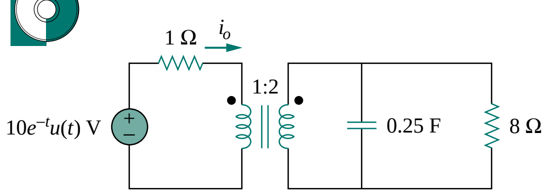

Figure 15.75 For Prob. 15.50.

Section 15.6 Transfer Functions

15.51 The transfer function of a system is

$$H(s) = \frac{s^2}{3s + 1}$$

Find the output when the system has an input of $4e^{-t/3}u(t)$.

15.52 When the input to a system is a unit step function, the response is $10 \cos 2t$. Obtain the transfer function of the system.

15.53 A circuit is known to have its transfer function as

$$H(s) = \frac{s + 3}{s^2 + 4s + 5}$$

Find its output when:

(a) the input is a unit step function

(b) the input is $6te^{-2t}u(t)$.

15.54 When a unit step is applied to a system at $t = 0$, its response is

$$y(t) = 4 + \frac{1}{2}e^{-3t} - e^{-2t}(2\cos 4t + 3\sin 4t)$$

What is the transfer function of the system?

15.55 For the circuit in Fig. 15.76, find $H(s) = V_o(s)/V_s(s)$. Assume zero initial conditions.

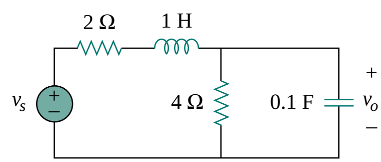

Figure 15.76 For Prob. 15.55.

15.56 Obtain the transfer function $H(s) = V_o/V_s$ for the circuit of Fig. 15.77.

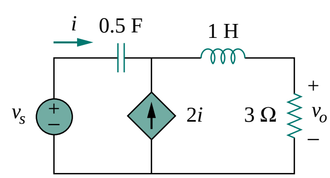

Figure 15.77 For Prob. 15.56.

15.57 Repeat the previous problem for $H(s) = V_o/I$.

15.58 For the circuit in Fig. 15.78, find:

(a) I_1/V_s (b) I_2/V_x

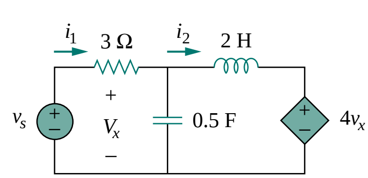

Figure 15.78 For Prob. 15.58.

15.59 Refer to the network in Fig. 15.79. Find the following transfer functions:

(a) $H_1(s) = V_o(s)/V_s(s)$

(b) $H_2(s) = V_o(s)/I_s(s)$

(c) $H_3(s) = I_o(s)/I_s(s)$

(d) $H_4(s) = I_o(s)/V_s(s)$

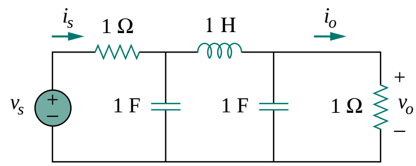

Figure 15.79 For Prob. 15.59.

15.60 Calculate the gain $H(s) = V_o/V_s$ in the op amp circuit of Fig. 15.80.

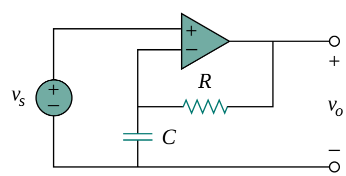

Figure 15.80 For Prob. 15.60.

15.61 Refer to the RL circuit in Fig. 15.81. Find:

(a) the impulse response $h(t)$ of the circuit

(b) the unit step response of the circuit.

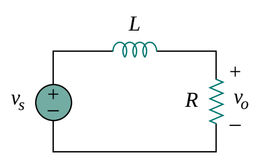

Figure 15.81 For Prob. 15.61.

15.62 A network has the impulse response $h(t) = 2e^{-t}u(t)$. When the input signal $v_i(t) = 5u(t)$ is applied to it, find its output.

15.63 Obtain the impulse response of a system modeled by the differential equation

$$2\frac{dy}{dt} + y(t) = x(t)$$

where $x(t)$ is the input and $y(t)$ is the output.

Section 15.7 The Convolution Integral

15.64 Graphically convolve the pairs of functions in Fig. 15.82.

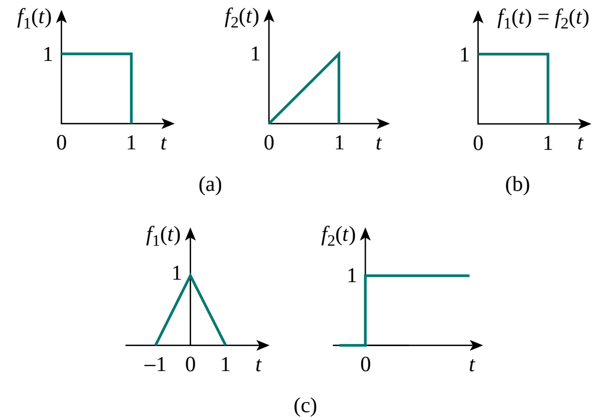

Figure 15.82 For Prob. 15.64.

15.65 Find $y(t) = x(t) * h(t)$ for each paired $x(t)$ and $h(t)$ in Fig. 15.83.

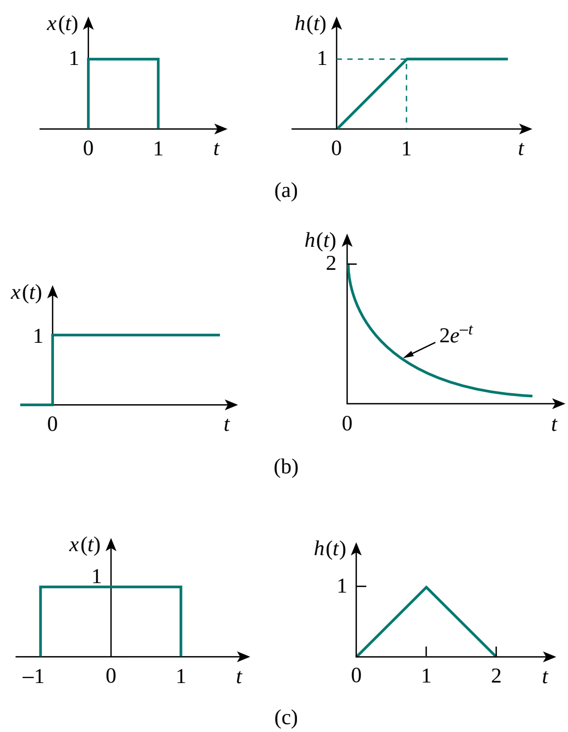

Figure 15.83 For Prob. 15.65.

15.66 Obtain the convolution of the pairs of signals in Fig. 15.84.

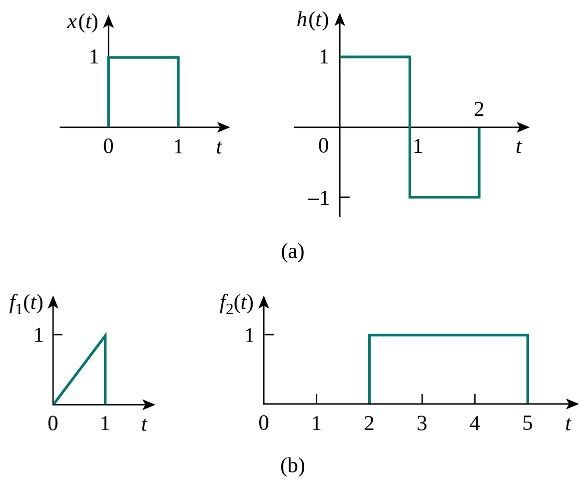

Figure 15.84 For Prob. 15.66.

15.67 Show that:

(a) $x(t) * \delta(t) = x(t)$

(b) $f(t) * u(t) = \int_0^t f(\lambda)\, d\lambda$

15.68 Determine the convolution for each of the following pairs of continuous signals:

(a) $x_1(t) = e^{-t}$, $t > 0$, $x_2(t) = 4e^{-2t}$, $0 < t < 3$

(b) $x_1(t) = u(t-1) - u(t-3)$,
$x_2(t) = u(t) - u(t-1)$

(c) $x_1(t) = 4e^{-t}u(t)$,
$x_2(t) = u(t+1) - 2u(t) + u(t-1)$

15.69 Given that $F_1(s) = F_2(s) = s/(s^2+1)$, find $\mathcal{L}^{-1}[F_1(s)F_2(s)]$ by convolution.

15.70 Find $f(t)$ using convolution given that:

(a) $F(s) = \dfrac{4}{(s^2+2s+5)^2}$

(b) $F(s) = \dfrac{2s}{(s+1)(s^2+4)}$

Section 15.8 Application to Integrodifferential Equations

15.71 Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2} + 2\frac{dv(t)}{dt} + 10v(t) = 3\cos 2t$$

subject to $v(0) = 1$, $dv(0)/dt = -2$.

15.72 Use the Laplace transform to find $i(t)$ for $t > 0$ if

$$\frac{d^2i}{dt^2} + 3\frac{di}{dt} + 2i + \delta(t) = 0,$$

$$i(0) = 0, \qquad i'(0) = 3$$

15.73 Solve the following equation by means of the Laplace transform:

$$y'' + 5y' + 6y = \cos 2t$$

Let $y(0) = 1$, $y'(0) = 4$.

15.74 The voltage across a circuit is given by

$$v'' + 3v' + 2v = 5e^{-3t}$$

Find $v(t)$ if the initial conditions are $v(0) = 0$, $v'(0) = -1$.

15.75 Solve for $y(t)$ in the following differential equation if the initial conditions are zero.

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 8\frac{dy}{dt} = e^{-t}\cos 2t$$

15.76 Solve for $v(t)$ in the integrodifferential equation

$$4\frac{dv}{dt} + 12\int_{-\infty}^{t} v\, dt = 0$$

given that $v(0) = 2$.

15.77 Solve the following integrodifferential equation using the Laplace transform method:

$$\frac{dy(t)}{dt} + 9\int_0^t y(\tau)\, d\tau = \cos 2t, \qquad y(0) = 1$$

15.78 Solve the integrodifferential equation

$$\frac{dy}{dt} + 4y + 3\int_0^t y\, dt = 6e^{-2t}, \qquad y(0) = -1$$

15.79 Solve the following integrodifferential equation

$$2\frac{dx}{dt} + 5x + 3\int_0^t x\, dt + 4 = \sin 4t, \qquad x(0) = 1$$

Section 15.9 Applications

15.80 Show that the parallel RLC circuit shown in Fig. 15.85 is stable.

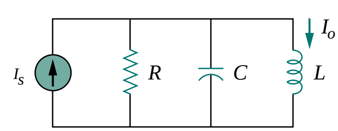

Figure 15.85 For Prob. 15.80.

15.81 A system is formed by cascading two systems as shown in Fig. 15.86. Given that the impulse response of the systems are

$$h_1(t) = 3e^{-t}u(t), \qquad h_2(t) = e^{-4t}u(t)$$

(a) Obtain the impulse response of the overall system.

(b) Check if the overall system is stable.

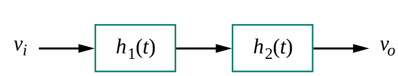

Figure 15.86 For Prob. 15.81.

15.82 Determine whether the op amp circuit in Fig. 15.87 is stable.

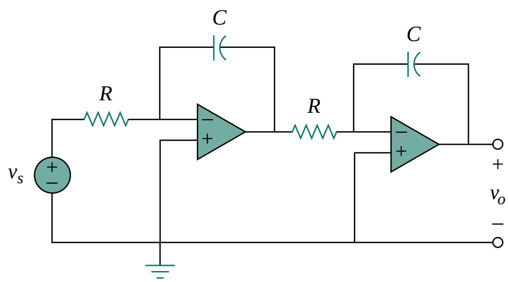

Figure 15.87 For Prob. 15.82.

15.83 It is desired to realize the transfer function

$$\frac{V_2(s)}{V_1(s)} = \frac{2s}{s^2 + 2s + 6}$$

using the circuit in Fig. 15.88. Choose $R = 1\ \text{k}\Omega$ and find L and C.

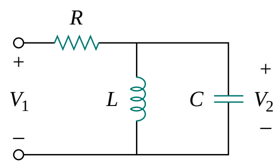

Figure 15.88 For Prob. 15.83.

15.84 Realize the transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{5}{s^2 + 6s + 25}$$

using the circuit in Fig. 15.89. Choose $R_1 = 4\ \Omega$ and $R_2 = 1\ \Omega$, and determine L and C.

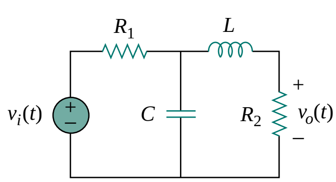

Figure 15.89 For Prob. 15.84.

15.85 Realize the transfer function

$$\frac{V_o(s)}{V_s(s)} = -\frac{s}{s + 10}$$

using the circuit in Fig. 15.90. Let $Y_1 = sC_1$, $Y_2 = 1/R_1$, $Y_3 = sC_2$. Choose $R_1 = 1\ \text{k}\Omega$ and determine C_1 and C_2.

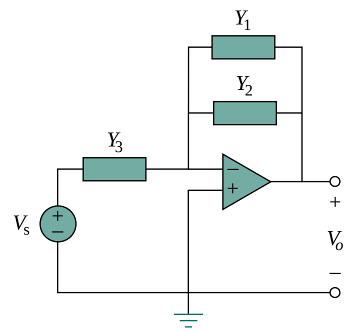

Figure 15.90 For Prob. 15.85.

15.86 Synthesize the transfer function

$$\frac{V_o(s)}{V_{\text{in}}(s)} = \frac{10^6}{s^2 + 100s + 10^6}$$

using the topology of Fig. 15.91. Let $Y_1 = 1/R_1$, $Y_2 = 1/R_2$, $Y_3 = sC_1$, $Y_4 = sC_2$. Choose $R_1 = 1$ kΩ and determine C_1, C_2, and R_2.

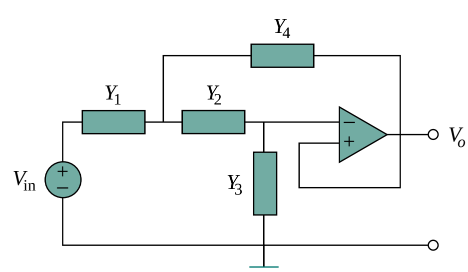

Figure 15.91 For Prob. 15.86.

COMPREHENSIVE PROBLEMS

15.87 Obtain the transfer function of the op amp circuit in Fig. 15.92 in the form of

$$\frac{V_o(s)}{V_i(s)} = \frac{as}{s^2 + bs + c}$$

where a, b, and c are constants. Determine the constants.

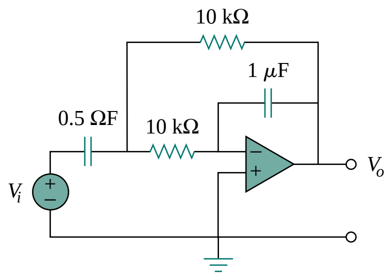

Figure 15.92 For Prob. 15.87.

15.88 A certain network has an input admittance $Y(s)$. The admittance has a pole at $s = -3$, a zero at $s = -1$, and $Y(\infty) = 0.25$ S.

(a) Find $Y(s)$.

(b) An 8-V battery is connected to the network via a switch. If the switch is closed at $t = 0$, find the current $i(t)$ through $Y(s)$ using the Laplace transform.

15.89 A gyrator is a device for simulating an inductor in a network. A basic gyrator circuit is shown in Fig. 15.93. By finding $V_i(s)/I_o(s)$, show that the inductance produced by the gyrator is $L = CR^2$.

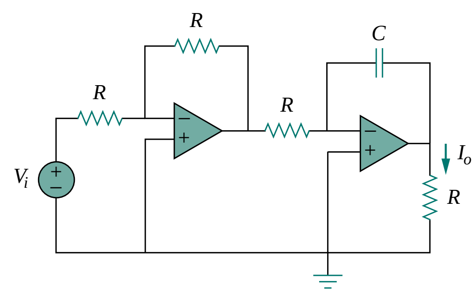

Figure 15.93 For Prob. 15.89.