Appendix

D Mathematical Derivations

Equations 7.5, 11.9A–B, 12.7–12.9, 12.14, and 12.15 have been derived specifically for this text. This appendix contains the details of these derivations, which were deemed too complicated for inclusion within the body of the text.

EQUATION 7.5

The random variance of the resistor is assumed to consist of an areal component and a peripheral component. If we assume the random fluctuations occur on an infinitesimal scale of distance and the variance of resistivity of a unit area element equals σ_a^2, then the areal variance σ_A^2 of a resistor having width W and length L equals

$$\sigma_A^2 = \frac{\sigma_a^2}{WL} \qquad \text{[D.1]}$$

since $R \equiv R_s W / L$, $L = R_s W / R$. Substituting this into equation D.1 gives

$$\sigma_A^2 = \frac{\sigma_a^2 R}{W^2 R_s} \qquad \text{[D.2]}$$

If the variance of width of a section of the resistor having unit length equals σ_w^2, then this section's resistance will have a variance of σ_w^2 / W^2 and a resistor of length L will have a peripheral variance σ_P^2 of

$$\sigma_P^2 = \frac{\sigma_w^2}{W^2 L} = \frac{\sigma_w^2 R}{W^3 R_s} \qquad \text{[D.3]}$$

Summing the areal and peripheral variances and taking the square root of the resulting sum, one obtains the standard deviation of resistance σ_R

$$\sigma_R = \sqrt{\frac{\sigma_w^2 R}{W^3 R_s} + \frac{\sigma_a^2 R}{W^2 R_s}} \qquad \text{[D.4]}$$

Let $k_p \equiv \sigma_w^2 / R_s$ and $k_a \equiv \sigma_a^2 / R_s$. Normalizing σ_R by dividing by R results in

$$\frac{\sigma_R}{R} = \frac{1}{W\sqrt{R}}\sqrt{k_a + \frac{k_p}{W}} \qquad \text{[D.5]}$$

EQUATION 11.9A–B

In a circularly symmetric device, current flows radially from source to drain. The length of the channel therefore equals the difference between the radii of the outer and inner edges of the channel, or $L = 1/2\ (B–A)$. The width of the channel W can be determined using the Shichman-Hodges equation for the linear region. Rearranging this equation in terms of L gives

$$L = k'\frac{W}{I_D}\left(V_{gst} - \frac{V_{DS}}{2}\right)V_{DS} \qquad \text{[D.6]}$$

Differentiating to find dL/dV_{DS} gives

$$\frac{dL}{dV_{DS}} = k'\frac{W}{I_D}(V_{gst} - V_{DS}) \qquad \text{[D.7]}$$

For any infinitesimal length of channel dL the corresponding width W equals $2\pi L$, where $L = 0$ denotes the center of the annular structure. Separating terms and integrating gives

$$\int_0^{V_{DS}} (V_{gst} - V_{DS})dV_{DS} = \frac{I_D}{k'}\int_{A/2}^{B/2} \frac{dL}{2\pi L} \qquad \text{[D.8]}$$

The limits of these integrals assume that the source is at the inner edge of the channel ($L = A/2$) and the drain is at the outer edge of the channel ($L = B/2$). Integrating gives

$$V_{gst}V_{DS} - \frac{V_{DS}}{2} = \frac{I_D}{2\pi k'}\ln(B/A) \qquad \text{[D.9]}$$

Gathering terms,

$$I_D = \frac{2\pi k'}{\ln(B/A)}\left(V_{gst} - \frac{V_{DS}}{2}\right)V_{DS} \qquad \text{[D.10]}$$

This equation is analogous to the Shichman-Hodges equation for the linear region with a W/L ratio equal to

$$\frac{W}{L} = \frac{2\pi}{\ln(B/A)} \qquad \text{[D.11]}$$

Substituting the previously determined value of L gives

$$W = \frac{\pi(B - A)}{\ln(B/A)} \qquad \text{[D.12]}$$

EQUATION 12.7

Let a source/drain finger consist of a uniform rectangular strip having width W, length L, and sheet resistance, R_s. Let variable x define position along the length of the finger. Assume that an equal amount of current flows into the finger at every point along its length, so the current $I(x)$ increases linearly from $x = 0$ to $x = L$. Let I_{max} equal the current $I(x)$ at $x = L$. The voltage drop V from $x = 0$ to $x = L$ equals

$$V = \int_0^L \frac{R_s I(x)}{W} dx \qquad \text{[D.13]}$$

Since $I(x) = I_{max} x / L$, this reduces to

$$V = \frac{R_s I_{max}}{WL} \int_0^L x\,dx = \frac{R_s I_{max} L}{2W} \qquad \text{[D.14]}$$

$$V = \frac{R_s L}{2W} I_{max} \qquad \text{[D.15]}$$

The resistance of any source/drain finger may be divided into three components: (1) the portion of the finger consisting only of metal-1, (2) the portion of the finger consisting of a sandwich of metal-1 and metal-2, and (3) the portion of the finger under a plate of metal-2. If we assume that the metal-2 buses are equipotential across their width, then the resistance of the finger consists of only components (1) and (2). The voltage drop V_1 across component (1) equals

$$V_1 = \frac{R_{s1} B}{2W} I_1 \qquad \text{[D.16]}$$

where I_1 equals the current flowing through component (1) of a single source/drain finger. Since component (1) has a length of B, while the full length of one finger equals L, and since there are N_D pairs of source/drain fingers, the current I_1 equals

$$I_1 = \frac{B}{LN_D} I_D \qquad \text{[D.17]}$$

where I_D equals the total drain current of all fingers. Substituting equation D.17 into equation D.16 yields

$$V_1 = \frac{R_{s1} B^2}{2WLN_D} I_D \qquad \text{[D.18]}$$

The voltage drop V_2 across component (2) can likewise be computed

$$R_2 = \frac{R_{s12} A}{2W} I_2 + \frac{R_{s12} A}{W} I_1 \qquad \text{[D.19]}$$

where I_2 equals the total current entering the finger within component (2) and I_1 equals the current flowing into component (2) from component (1). I_2 can be computed in a similar manner to that used to determine I_1. Substituting I_1 and I_2 into equation D.19,

$$V_2 = \frac{R_{s12} A}{2W}\left(\frac{A}{LN_D}\right) I_D + \frac{R_{s12} A}{W}\left(\frac{B}{LN_D}\right) I_D \qquad \text{[D.20]}$$

$$V_2 = \frac{R_{s12} A}{2W}\left(\frac{A + 2B}{LN_D}\right) I_D \qquad \text{[D.21]}$$

But since $L \equiv A + 2B$,

$$R_2 = \frac{R_{s12} A}{2WN_D} I_D \qquad \text{[D.22]}$$

The total voltage drop across the fingers of the transistor equals twice the sum of V_1 and V_2 because there are both source and drain fingers involved. To this must be added the voltage drop across the metal-2 buses V_B

$$V_B = \frac{HR_{s2}}{2B} I_D \qquad \text{[D.23]}$$

The total voltage drop V_M therefore equals

$$V_M = \left(\frac{R_{s1}B^2}{WLN_D} + \frac{R_{s12}A}{WN_D} + \frac{HR_{s2}}{2B}\right)I_D \qquad \text{[D.24]}$$

Since the metallization resistance $R_M \equiv V_M / I_D$,

$$R_M\frac{R_{s1}B^2}{WLN_D} + \frac{R_{s12}A}{WN_D} + \frac{HR_{s2}}{2B} \qquad \text{[D.25]}$$

EQUATION 12.9 AND 12.10

To determine the optimum value of B, take $\partial R_M / \partial B$ and set this equal to zero. This determines an inflection point in the function $R_M(B)$. Begin by replacing A by $(L - 2B)$ and differentiating

$$\frac{\partial R_M}{\partial B} = \frac{2BR_{s1}}{WN_DL} - \frac{2R_{s12}}{WN_D} \qquad \text{[D.26]}$$

Setting this equal to zero, we obtain

$$\frac{BR_{s1}}{L} = R_{s12} \qquad \text{[D.27]}$$

$$\frac{B}{L} = \frac{R_{s12}}{R_{s1}} \qquad \text{[D.28]}$$

Assuming both metal layers have resistivity ρ, $R_{s1} = \rho / t_1$ where t_1 is the thickness of metal-1, and $R_{s12} = \rho / (t_1 + t_2)$, where t_2 is the thickness of metal-2. Substituting these relationships into equation D.28 gives

$$\frac{B}{L} = \frac{t_1}{t_1 + t_2} \qquad \text{[D.29]}$$

EQUATION 12.14

This equation is derived from the Shichman-Hodges equation for saturation. Let transistor M_1 have drain current I_{D1}, transconductance k_1, and effective gate voltage V_{gst1}. Let transistor M_2 have drain current I_{D2}, transconductance k_2, and effective gate voltage V_{gst2}. Since $I_{D1} \equiv I_{D2}$,

$$k_1V_{gst1}^2 = k_2V_{gst2}^2 \qquad \text{[D.30]}$$

Rearranging

$$\frac{k_1}{k_2} = \left(\frac{V_{gst2}}{V_{gst1}}\right)^2 \qquad \text{[D.31]}$$

Let $\Delta V_{gs} \equiv V_{gs1} - V_{gs2}$; then $V_{gs2} = V_{gs1} - \Delta V_{gs}$. Let $\Delta V_t \equiv V_{t1} - V_{t2}$; then $V_{t2} = V_{t1} - \Delta V_t$. Substituting these relationships into equation D.31 yields

$$\frac{k_1}{k_2} = \left(\frac{V_{gs2} - V_{t2}}{V_{gst1}}\right)^2 = \left(\frac{V_{gs1} - \Delta V_{gs} - V_{t1} + \Delta V_t}{V_{gst1}}\right)^2 \qquad \text{[D.32]}$$

$$\frac{k_1}{k_2} = \left(1 + \frac{\Delta V_t - \Delta V_{gs}}{V_{gst1}}\right)^2 \qquad \text{[D.33]}$$

$$\sqrt{\frac{k_1}{k_2}} = 1 + \frac{\Delta V_t - \Delta V_{gs}}{V_{gst1}} \qquad [D.34]$$

$$\left(\sqrt{\frac{k_1}{k_2}} - 1\right)V_{gst1} = \Delta V_t - \Delta V_{gs} \qquad [D.35]$$

Collecting ΔV_{gs}

$$\Delta V_{gs} = \Delta V_t - V_{gst1}\left(\sqrt{\frac{k_1}{k_2}} - 1\right) \qquad [D.36]$$

Let $\Delta k \equiv k_1 - k_2$; then $k_1 = k_2 + \Delta k$. Substituting this into equation D.36 gives

$$\Delta V_{gs} = \Delta V_t - V_{gst1}\left(\sqrt{1 + \frac{\Delta k}{k_2}} - 1\right) \qquad [D.37]$$

By the binominal expansion

$$\sqrt{1 + x} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{3x^3}{48} + \ldots \qquad [D.38]$$

If x is sufficiently small, then only the first two terms are significant. Applying this expansion to equation D.37 gives

$$\Delta V_{gs} \cong \Delta V_t - V_{gst1}\frac{\Delta k}{2k_2} \qquad [D.39]$$

EQUATION 12.15

This equation is derived from the Shichman-Hodges equation for saturation. Let transistor M_1 have drain current I_{D1}, transconductance k_1, and effective gate voltage V_{gst1}. Let transistor M_2 have drain current I_{D2}, transconductance k_2, and effective gate voltage V_{gst2}. The ratio of the two drain currents I_{D2} / I_{D1} equals

$$\frac{I_{D2}}{I_{D1}} = \frac{k_2}{k_1}\left(\frac{V_{gst2}}{V_{gst1}}\right)^2 \qquad [D.40]$$

Let $\Delta V_t \equiv V_{t1} - V_{t2}$; then $V_{t2} = V_{t1} - \Delta V_t$. Substituting this into equation D.40

$$\frac{I_{D2}}{I_{D1}} = \frac{k_2}{k_1}\left(\frac{V_{gs2} - V_{t2}}{V_{gst1}}\right)^2 = \frac{k_2}{k_1}\left(\frac{V_{gs2} - V_{t1} + \Delta Vt}{V_{gst1}}\right)^2 \qquad [D.41]$$

But since $V_{gs1} \equiv V_{gs2}$,

$$\frac{I_{D2}}{I_{D1}} = \frac{k_2}{k_1}\left(\frac{V_{gs1} - V_{t1} + \Delta V_t}{V_{gst1}}\right)^2 = \frac{k_2}{k_1}\left(\frac{V_{gst1} + \Delta V_t}{V_{gst1}}\right)^2 \qquad [D.42]$$

Expanding

$$\frac{I_{D2}}{I_{D1}} = \frac{k_2}{k_1}\left(\frac{V_{gst1}^2 + 2\Delta V_t V_{gst1} + \Delta V_t^2}{V_{gst1}^2}\right) \qquad [D.43]$$

So long as $\Delta V_t << V_{gst1}$,

$$\frac{I_{D2}}{I_{D1}} \cong \frac{k_2}{k_1}\left(1 + \frac{2\Delta V_t}{V_{gst1}}\right) \qquad [D.44]$$