CHAPTER 15 - LAPLACE TRANSFORM

List of topics for this chapter:
    Definition and Properties of the Laplace Transform
    Inverse Laplace Transform
    Application to Circuits
    Transfer Functions
    Convolution Integral
    Application to Integrodifferential Equations
    Applications

DEFINITION AND PROPERTIES OF THE LAPLACE TRANSFORM

The Laplace transform is an integral transformation of a function f(t) from the time domain into the frequency domain, giving F(s).

Problem 15.1   Find the Laplace transform for

(a) $(10 - 10e^{-3t})\, u(t)$

(b) $(10 - 10e^{-3(t-1)})\, u(t-1)$

(c) $(10 - 10e^{-3t})\, u(t-1)$

(a) $\dfrac{10}{s} - \dfrac{10}{s+3} = \mathbf{10\left(\dfrac{3}{s(s+3)}\right)}$

(b) $\dfrac{10e^{-s}}{s} - \dfrac{10e^{-s}}{s+3} = 10e^{-s}\left(\dfrac{1}{s} - \dfrac{1}{s+3}\right) = \mathbf{10e^{-s}\left(\dfrac{3}{s(s+3)}\right)}$

(c)   Manipulate this to match a transform pair using the time-shifting property.

$$(10 - 10e^{-3t})\, u(t-1) = (10 - 10e^{-3(t-1+1)})\, u(t-1)$$
$$= (10 - 10e^{-3(t-1)}e^{-3})\, u(t-1)$$

The Laplace transform is

$$\frac{10e^{-s}}{s} - \frac{10e^{-3}e^{-s}}{s+3} = 10e^{-s}\left(\frac{1}{s} - \frac{e^{-3}}{s+3}\right)$$
$$= \mathbf{10e^{-s}\left(\frac{(1-e^{-3})s+3}{s(s+3)}\right)}$$

Problem 15.2   Find the Laplace transform for

    (a) $[t e^{-3t} + t^2 e^{-2t}] u(t)$

    (b) $[t + t^2 + t^3] u(t)$

    (c) $[t + t^2] u(t-1)$

    (d) $[t + e^{-t} + (t-4) e^{-t} + t^2 e^{-t}] u(t)$

    (a) $\dfrac{1}{(s+3)^2} + \dfrac{2}{(s+2)^3}$

    (b) $\dfrac{1}{s^2} + \dfrac{2}{s^3} + \dfrac{6}{s^4} = \dfrac{s^2 + 2s + 6}{s^4}$

    (c)   Manipulate this to match a transform pair using the time-shifting property.

$$
\begin{aligned}
[t + t^2] u(t-1) &= [(t-1+1) + (t-1+1)^2] u(t-1) \\
&= [(t-1) + 1 + (t-1)^2 + 2(t-1) + 1] u(t-1) \\
&= [(t-1) + (t-1)^2 + 2(t-1) + 2)] u(t-1) \\
&= [(t-1)^2 + 3(t-1) + 2] u(t-1)
\end{aligned}
$$

    The Laplace transform is

$$
\frac{2e^{-s}}{s^3} + \frac{3e^{-s}}{s^2} + \frac{3e^{-s}}{s} = e^{-s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{3}{s} \right] = \frac{e^{-s}}{s^3}[3s^2 + 3s + 2]
$$

    (d)   First, expand the third term. Then, combine like terms.

$$
\begin{aligned}
[t + e^{-t} + (t-4) e^{-t} + t^2 e^{-t}] u(t) &= [t + e^{-t} + t e^{-t} - 4e^{-t} + t^2 e^{-t}] u(t) \\
&= [t - 3e^{-t} + t e^{-t} + t^2 e^{-t}] u(t)
\end{aligned}
$$

    The Laplace transform is

$$
\frac{1}{s^2} - \frac{3}{(s+1)} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}
$$

INVERSE LAPLACE TRANSFORM

Finding the inverse Laplace transform of F(s) involves two steps:
1.   Decomposing F(s) into simple terms using partial fraction expansion, and
2.   Finding the inverse of each term by matching entries in a Laplace transform table.

Partial Fraction Expansion

Suppose F(s) has the general form $F(s) = \dfrac{N(s)}{D(s)}$. Consider the three possible forms F(s) may take and how to perform partial fraction expansion of F(s) for each form.

Real Poles

If F(s) has only real poles, then the denominator becomes a product of factors so that

$$F(s) = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_n)},$$

where $s = -p_1, -p_2, \cdots, -p_n$ are the real poles and $p_i \neq p_j$ for all $i \neq j$.

Assuming the degree of N(s) is less than the degree of D(s), partial fraction expansion is used to decompose F(s) as

$$F(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n}$$

There are many ways to find the expansion coefficients, $k_1, k_2, \cdots, k_n$, or residues of F(s). One way is the residue method. If both sides of the equation are multiplied by $(s + p_1)$, then

$$(s + p_1)F(s) = k_1 + \frac{k_2(s + p_1)}{s + p_2} + \cdots + \frac{k_n(s + p_1)}{s + p_n}$$

Since $p_i \neq p_j$, set $s = -p_1$ to find k_1.

Hence, $\qquad\qquad\qquad (s + p_1)F(s)\Big|_{s=-p_1} = k_1$

Thus, in general, $\qquad\qquad k_i = (s + p_i)F(s)\Big|_{s=-p_i}$

When the residues of F(s) are known, the inverse of F(s) can be found. Since the inverse transform of each term is

$$L^{-1}\left[\frac{k}{s + a}\right] = k e^{-at} u(t)$$

then

$$f(t) = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t}$$

Repeated Poles

Now, suppose that F(s) has repeated poles at $s = -p$. Then,

$$F(s) = \frac{k_n}{(s + p)^n} + \frac{k_{n-1}}{(s + p)^{n-1}} + \cdots + \frac{k_2}{(s + p)^2} + \frac{k_1}{s + p} + F_1(s),$$

where $F_1(s)$ is the remaining part of F(s) that does not have a pole at $s = -p$.

The expansion coefficient k_n is determined as

$$k_n = (s+p)^n F(s)\Big|_{s=-p}$$

similar to the method for real poles. To determine k_{n-1}, multiply $F(s)$ by $(s+p)^n$, differentiate and evaluate the result at $s = -p$ to isolate k_{n-1}. Thus,

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)]\Big|_{s=-p}$$

and

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]\Big|_{s=-p}$$

In general, the mth term is

$$k_m = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)]\Big|_{s=-p}$$

where $m = 1, 2, \cdots, n-1$.

Once the values $k_1, k_{2,} \cdots, k_n$ are obtained using partial fraction expansion, apply the inverse transform

$$L^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1} e^{-at}}{(n-1)!}$$

to each term in order to obtain

$$f(t) = k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!} t^2 e^{-pt} + \cdots + \frac{k_n}{(n-1)!} t^{n-1} e^{-pt} + f_1(t)$$

Complex Poles

A pair of complex poles may be handled the same as real poles, but the complex algebra may be quite cumbersome. Even so,

$$F(s) = \frac{K_1}{(s+\alpha - j\beta)} + \frac{K_1^*}{(s+\alpha + j\beta)}$$

and

$$K_1 = (s+\alpha - j\beta)F(s)\Big|_{s=-\alpha+j\beta}$$

where $K_1 = |K_1|\angle\theta$ and $K_1^* = |K_1|\angle{-\theta}$.

Then,

$$f(t) = |K_1|e^{j\theta} e^{-(\alpha - j\beta)t} + |K_1|e^{-j\theta} e^{-(\alpha + j\beta)t}$$
$$f(t) = |K_1|e^{-\alpha t} [e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}]$$
$$f(t) = 2|K_1|e^{-\alpha t} \cos(\beta t + \theta)$$

An easier approach is a method known as completing the square. Since $N(s)$ and $D(s)$ always have real coefficients and complex roots occur in conjugate pairs, $F(s)$ may have the general form

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s),$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have complex poles.

Begin completing the square by letting
$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2$$

Also, let
$$A_1 s + A_2 = A_1 (s + \alpha) + B_1 \beta$$

Then,
$$F(s) = \frac{A_1 (s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1 \beta}{(s + \alpha)^2 + \beta^2} + F_1(s)$$

and the inverse transform is $f(t) = A_1 e^{-\alpha t} \cos(\beta t) + B_1 e^{-\alpha t} \sin(\beta t) + f_1(t)$

Problem 15.3   Find the inverse Laplace transform for

(a) $\dfrac{1}{s + 6}$

(b) $\dfrac{1}{s^2 + 9}$

(c) $\dfrac{1}{s^2 + 2s + 1}$

(a) $\underline{e^{-6t}}$

(b)   Manipulate this to match a transform pair.
$$\frac{1}{s^2 + 9} = \left(\frac{1}{3}\right)\left(\frac{3}{s^2 + 9}\right)$$

The inverse Laplace transform is $\underline{(1/3)\sin(3t)}$

(c)   Manipulate this to match a transform pair.
$$\frac{1}{s^2 + 2s + 1} = \frac{1}{(s + 1)^2}$$

The inverse Laplace transform is $\underline{te^{-t}}$

Problem 15.4   Find the inverse Laplace transform for

(a) $\dfrac{1}{s^2 + 6s}$

(b) $\dfrac{s + 2}{s^2 + 4s + 3}$

(c) $\dfrac{s^2 + 2s + 1}{s^3 + s^2 + 4s + 4}$

None of the above match a transform pair. Manipulate these problems, perform partial fraction expansion, and then use a table of Laplace transform pairs to find the inverse Laplace transforms.

Manipulate and perform partial fraction expansion to get the following.

(a) $\dfrac{1}{s^2+6s} = \dfrac{1}{s(s+6)} = \dfrac{1/6}{s} + \dfrac{-1/6}{s+6}$

(b) $\dfrac{s+2}{s^2+4s+3} = \dfrac{s+2}{(s+1)(s+3)} = \dfrac{1/2}{s+1} + \dfrac{1/2}{s+3}$

(c) $\dfrac{s^2+2s+1}{s^3+s^2+4s+4} = \dfrac{(s+1)^2}{(s+1)(s^2+4)} = \dfrac{s+1}{s^2+4} = \dfrac{s}{s^2+4} + \dfrac{1}{2}\left(\dfrac{2}{s^2+4}\right)$

Now, the inverse Laplace transforms are

(a) $\dfrac{1}{6}u(t) - \dfrac{1}{6}e^{-t}$

(b) $\dfrac{1}{2}e^{-t} + \dfrac{1}{2}e^{-3t} = \dfrac{1}{2}[e^{-t}+e^{-3t}]$

(c) $\cos(2t) + 0.5\sin(2t)$

Problem 15.5   Find the inverse Laplace transform for

(a) $\dfrac{s^2+2s+2}{s^3+s^2+4s+4}$

(b) $\dfrac{1}{s^3+3s^2+3s+1}$

The inverse Laplace transforms are

(a) $\dfrac{1}{5}e^{-t} + \cos(2t-36.87°)$ (b) $\dfrac{1}{2}(t^2\,e^{-t})$

APPLICATION TO CIRCUITS

Begin by transforming the circuit from the time domain to the s-domain. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar. Take the inverse Laplace transform of the solution and thus obtain the solution in the time domain.

Solving circuits with initial conditions is a straightforward process following the same basic approach as circuits without initial conditions. Let us start with a capacitor and see how we actually solve such circuits. Start with the defining equation for a capacitor.
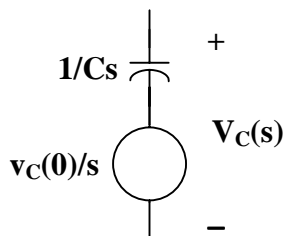
$$i_C(t) = C\frac{dv_C(t)}{dt}$$

Taking the Laplace transform gives
$$I_C(s) = C[s\,V_C(s) - v_C(0)] = Cs\,V_C(s) - Cv_C(0)$$
or
$$V_C(s) = (1/Cs)\,I_C(s) + v_C(0)/s.$$

We now can use the following circuit model for capacitors with initial conditions.

Now let us look at the inductor. The defining equation for the inductor is
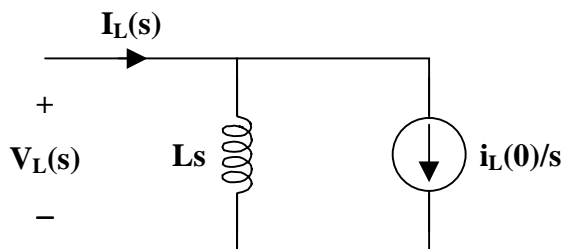$$v_L(t) = L\frac{di_L(t)}{dt}.$$

Taking the Laplace transform gives
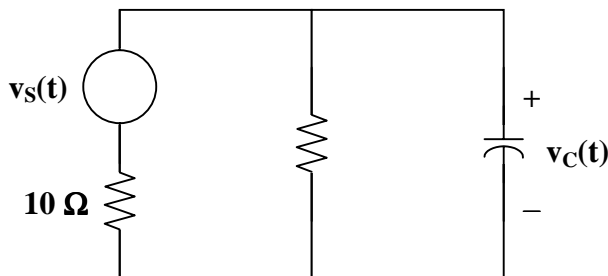$$V_L(s) = L[sI_L(s) - i_L(0)] = Ls\,I_L(s) - Li_L(0)$$
or
$$I_L(s) = V_L(s)/Ls + i_L(0)/s.$$

We can use the following circuit model for inductors with initial conditions.

Problem 15.6   Solve a first-order capacitive circuit with an initial condition using Laplace transforms. In Figure 15.1, solve for $v_C(t)$ for all $t > 0$. The initial value of $v_C(t) = 50$ volts, or $v_C(0) = 50$ volts, and the value of $v_S(t) = 20\,u(t)$ volts.

Figure 15.1

➢ **Carefully DEFINE the problem.**
Each component is labeled completely. The problem is clear.

➢ **PRESENT everything you know about the problem.**
In this circuit, we are expected to find $v_C(t)$ for all $t > 0$ using Laplace transforms. We need to set up a circuit in the frequency domain that will lead us to the desired solution. It should be noted that the initial condition on the capacitor must be included so that the fact that we do not have initial conditions is not violated. This is easy to do. Initial conditions can be represented by impulse functions in the time domain. An impulse function transforms into a constant. Thus, the capacitor can be represented by a capacitor in series with what appears to be a constant DC source. In reality, the Laplace transform of an impulse function is a constant in the frequency domain. Thus, we have the following circuit.
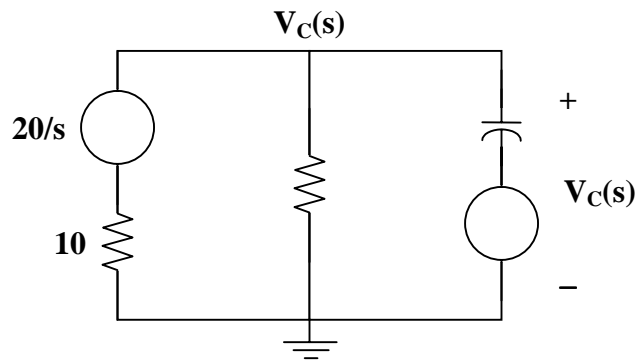
Please note that the actual value of $V_C(s)$ is the voltage across both the initial condition source and the s-domain capacitor. This is in agreement with the model developed earlier. Once we have found $V_C(s)$, then we can take the inverse Laplace transform to find $v_C(t)$.

➢ **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**
Because we have accounted for the initial condition in the frequency domain equivalent circuit, we can use nodal analysis, mesh analysis, or basic circuit analysis to find the voltage across the capacitor for any time greater than zero. Looking at the frequency domain circuit above, it is evident that nodal analysis is the best technique. Because there is only one nonreference node, this could be considered an application of KCL and Ohm's law.

➢ **ATTEMPT a problem solution.**
Summing the currents flowing out of the node gives us
$$\frac{V_C(s) - 20/s}{10} + \frac{V_C(s) - 0}{5} + \frac{V_C(s) - 50/s}{1/0.1s} = 0$$

Simplifying and collecting terms,
$$[1/10 + 1/5 + s/10]\, V_C(s) = 2/s + 5$$

Multiplying both sides by 10,
$$(s + 3)V_C(s) = (20/s) + 50$$

Thus,

$$V_C(s) = \frac{20}{s(s+3)} + \frac{50}{s+3}$$

Let us simplify the first term, using partial fraction expansion.

$$\frac{20}{s(s+3)} = \frac{20/3}{s} + \frac{-20/3}{s+3}$$

Finally,

$$V_C(s) = \frac{20/3}{s} + \frac{-20/3}{s+3} + \frac{50}{s+3}$$

Taking the inverse Laplace transform produces

$$v_C(t) = \{(20/3)[1 - e^{-3t}] + 50e^{-3t}\}\, u(t) \text{ volts}$$

This can also be written as $v_C(t) = \{20/3 + 130/3\, e^{-3t}\}\, u(t)$ volts.

➢ **EVALUATE the solution and check for accuracy.**
How does this compare with our earlier approach, where we used a generic solution for a first-order differential equation?

The value for $\qquad v_C(t) = A + Be^{-(t-t_0)/\tau}$

where $\qquad t_0 = 0$ sec and $\tau = RC = \dfrac{(10)(5)}{(10+5)}(0.1) = \dfrac{1}{3}$,

$v_C(\infty) = 20/3$ volts, and $v_C(0) = 50$ volts.

Thus,

$$A = 20/3 \text{ volts}$$

$$B = v_C(0) - A = 50 - 20/3 = 130/3 \text{ volts}$$

Therefore, $v_C(t) = \{20/3 + 130/3\, e^{-3t}\}\, u(t)$ volts, and our answer is the same as what we got using the Laplace transform approach!

Our check for accuracy was successful.

➢ **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to "ALTERNATIVE solutions" and continue through the process again.**
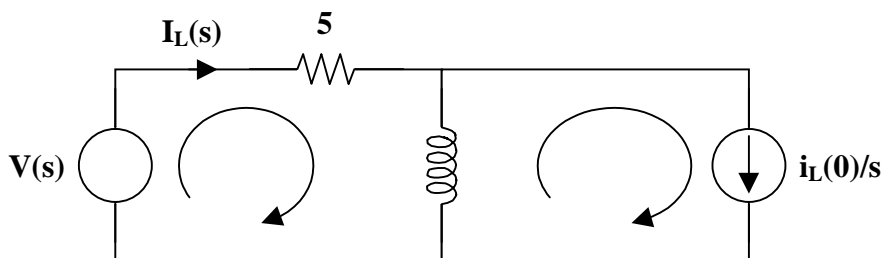This problem has been solved satisfactorily.

$$v_C(t) = \left\{\frac{20}{3} + \frac{130}{3}e^{-3t}\right\} u(t) \text{ V}$$

Problem 15.7   Given the circuit in Figure 15.1, solve for $i_L(t)$ where $v(t) = (10e^{-2t})u(t)$ volts and $i_L(0) = -2$ amps.
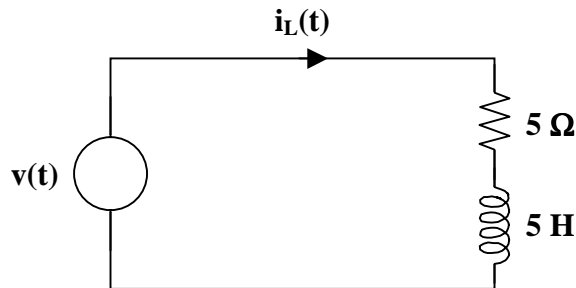
Figure 15.1

We now convert the above circuit into its Laplace equivalent as shown below.

Write a mesh equation, where $i_L(0) = -2$ amps.

$$-V(s) + 5I_L(s) + 5s[I_L(s) - (-2/s)] = 0$$

$$5(s+1)I_L(s) = V(s) - 10$$

$$I_L(s) = \frac{V(s) - 10}{5(s+1)}$$

Since $v(t) = (10e^{-2t})u(t)$ volts, $V(s) = 10/(s+2)$. Now, substitute V(s) into the equation.

$$I_L(s) = \frac{10/(s+2) - 10}{5(s+1)}$$

$$I_L(s) = \frac{2}{(s+1)(s+2)} + \frac{-2}{s+1}$$

Simplifying the first term using partial fraction expansion,

$$I_L(s) = \frac{2}{s+1} + \frac{-2}{s+2} + \frac{-2}{s+1}$$

$$I_L(s) = \frac{-2}{s+2}$$

Taking the inverse Laplace transform of $I_L(s)$ gives

$$i_L(t) = \underline{\{-2e^{-2t}\}u(t) \text{ A}}$$

The answer can be easily checked by again summing the voltages around the loop using this value of $i_L(t)$.

$$\sum \text{ of voltages} = 0$$

$$\left\{ -10e^{-2t} + (5)(-2e^{-2t}) + 5\frac{d(-2e^{-2t})}{dt} \right\} u(t) = 0$$

$$\{ -10e^{-2t} + (-10e^{-2t}) + (5)[(-2)(-2)e^{-2t}] \} u(t) = 0$$

$$\{ -20e^{-2t} + 20e^{-2t} \} u(t) = 0$$

The sum does equal zero.

Problem 15.8   Find the current, $i_L(t)$ for all t, given $v(t) = (10 - 10e^{-2t})u(t)$ volts for the circuit in Figure 15.1. Note that there are no initial conditions.
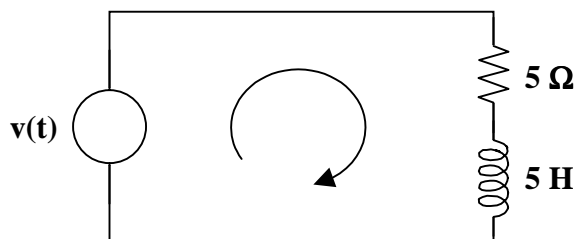
Figure 15.1

This problem is most easily solved using Laplace transforms. The first thing to do is to write a mesh equation since we are looking for a current.

$$-v(t) + 5i_L(t) + 5di_L(t)/dt = 0$$

$$5di_L(t)/dt + 5i_L(t) = v(t)$$

where

$$v(t) = (10 - 10e^{-2t})\, u(t) \text{ volts}$$

Taking the Laplace transform of both sides, noting that $i_L(0) = 0$,

$$(5)(s\,I_L(s) - 0) + 5I_L(s) = \frac{10}{s} + \frac{-10}{s+2}$$

or

$$(s+1)I_L(s) = \frac{2}{s} + \frac{-2}{s+2}$$

and

$$I_L(s) = \left(\frac{2}{s} + \frac{-2}{s+2} \right)\left(\frac{1}{s+1} \right) = 2\left(\frac{1}{s(s+1)} + \frac{-1}{(s+1)(s+2)} \right)$$

Now perform a partial fraction expansion of both terms.

$$I_L(s) = 2\left(\left(\frac{1}{s} + \frac{-1}{s+1} \right) - \left(\frac{1}{s+1} + \frac{-1}{s+2} \right) \right)$$

$$I_L(s) = 2\left(\frac{1}{s} + \frac{-2}{s+1} + \frac{1}{s+2}\right)$$

Now to find $i_L(t)$ all we have to do is to take the inverse transform of $I_L(s)$.

$$i_L(t) = \mathbf{2\{1 - 2e^{-t} + e^{-2t}\}\,u(t)\ A}$$

It is interesting to check and see if our answer is correct. To do that, we need to place this into the original equation.

$$5\,di_L(t)/dt + 5i_L(t) = v(t)$$

where $5\,di_L(t)/dt = 5\{\,(2)[0 - (-1)(2e^{-t})] - 2e^{-2t}\,\}\,u(t) = \{\,20e^{-t} - 20e^{-2t}\,\}\,u(t)$

$5i_L(t) = (5)(2)\{1 - 2e^{-t} + e^{-2t}\}\,u(t) = \{10 - 20e^{-t} + 10e^{-2t}\}\,u(t)$

Clearly,

$\{\,20e^{-t} - 20e^{-2t}\,\}\,u(t) + \{10 - 20e^{-t} + 10e^{-2t}\,\}\,u(t) = \{10 - 10e^{-2t}\,\}\,u(t) = v(t)$

and our answer checks.

===

Problem 15.9   Given the circuit in Figure 15.1, $R_1 = 10\,\Omega$, $R_2 = 10\,\Omega$, $C = 1/10\,F$, and $v(t) = 10\,u(t)$ volts. Calculate $i_{R2}(t)$.
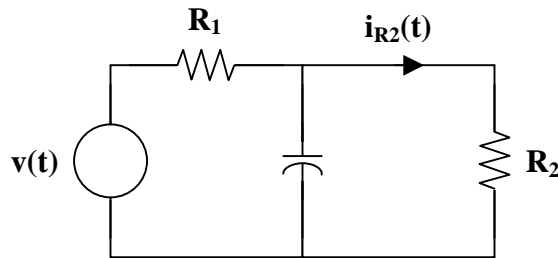
Figure 15.1
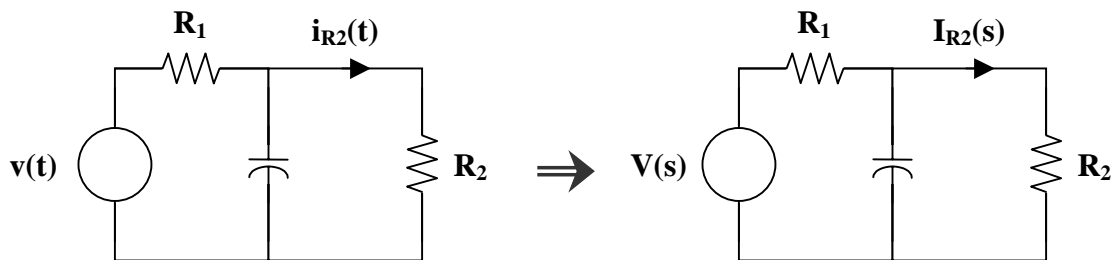
$$V_{R2}(s) = \frac{R_2 \,\|\, (1/Cs)}{R_1 + [R_2 \,\|\, (1/Cs)]}\,V(s)$$

where $R_2 \,\|\, (1/Cs) = \dfrac{R_2/Cs}{R_2 + (1/Cs)} = \dfrac{R_2}{R_2 Cs + 1}$

Then,

$$V_{R2}(s) = \frac{R_2/(R_2Cs+1)}{R_1 + [R_2/(R_2Cs+1)]} V(s) = \frac{R_2}{R_1(R_2Cs+1)+R_2} V(s)$$

where $R_1R_2C = (10)(10)(0.1) = 10$

$$V_{R2}(s) = \frac{10}{10s+10+10} V(s) = \frac{10}{10s+20} V(s) = \frac{1}{s+2} V(s)$$

Taking the Laplace transform of $v(t)$, $V(s) = \frac{10}{s}$.

Then,

$$V_{R2}(s) = \left[\frac{1}{s+2}\right]\left[\frac{10}{s}\right] = \frac{10}{s(s+2)}$$

The partial fraction expansion is

$$V_{R2}(s) = \frac{5}{s} + \frac{-5}{s+2}$$

The inverse Laplace transform is

$$v_{R2}(t) = (5)(1-e^{-2t})\,u(t)$$

Now,

$$i_{R2}(t) = \frac{v_{R2}(t)}{R_2} = \frac{(5)(1-e^{-2t})\,u(t)}{10}$$

Finally,

$$i_{R2}(t) = \underline{\mathbf{(0.5)(1-e^{-2t})\,u(t)\ A}}$$

Problem 15.10   Given $i_1(t) = (1e^{-t})u(t)$ amps, $i_2(t) = (1e^{-t})u(t)$ amps, $v_1(0) = 0$ volts, $v_2(0) = 0$ volts, and the circuit in Figure 15.1, find $v_1(t)$ and $v_2(t)$.
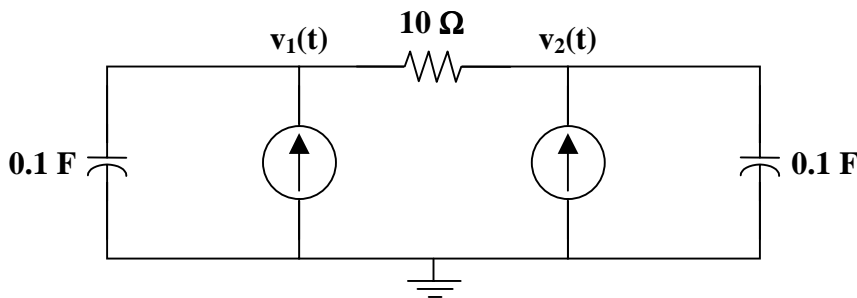
Figure 15.1

Since there are no initial conditions, the above circuit is easily changed into the s-domain circuit shown below.
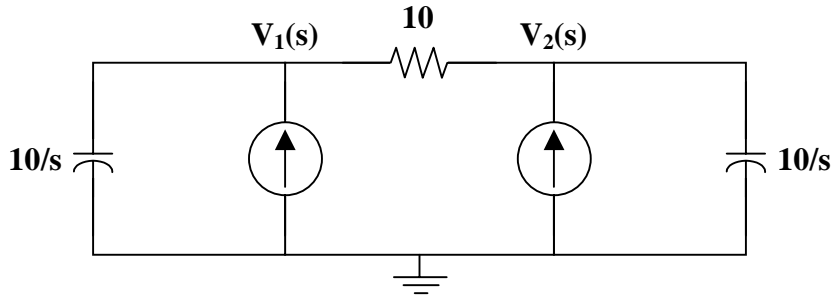
Using nodal analysis,

At node 1 :

$$\frac{V_1(s) - 0}{10/s} - I_1(s) + \frac{V_1(s) - V_2(s)}{10} = 0$$

At node 2 :

$$\frac{V_2(s) - 0}{10/s} - I_2(s) + \frac{V_2(s) - V_1(s)}{10} = 0$$

which leads to

$$\begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} 10I_1(s) \\ 10I_2(s) \end{bmatrix}$$

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \frac{\begin{bmatrix} s+1 & 1 \\ 1 & s+1 \end{bmatrix}}{s^2 + 2s} \begin{bmatrix} 10I_1(s) \\ 10I_2(s) \end{bmatrix}$$

Hence,

$$V_1(s) = \frac{(10)(s+1)}{s^2 + 2s} I_1(s) + \frac{10}{s^2 + 2s} I_2(s)$$

$$V_2(s) = \frac{10}{s^2 + 2s} I_1(s) + \frac{(10)(s+1)}{s^2 + 2s} I_2(s)$$

Taking the Laplace transform of $i_1(t)$ and $i_2(t)$, we find that

$$I_1(s) = I_2(s) = 1/(s+1)$$

Substitute these currents into $V_1(s)$ and $V_2(s)$

$$V_1(s) = \frac{(10)(s+1)}{s(s+2)(s+1)} + \frac{10}{s(s+2)(s+1)}$$

$$V_2(s) = \frac{10}{s(s+2)(s+1)} + \frac{(10)(s+1)}{s(s+2)(s+1)}$$

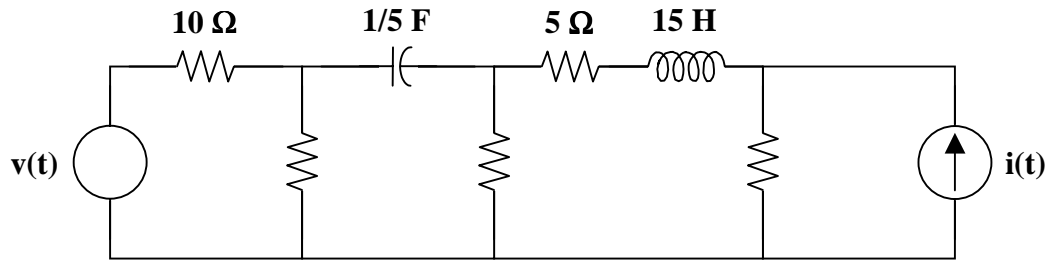$$V_1(s) = V_2(s) = \frac{10}{s(s+2)} + \frac{10}{s(s+1)(s+2)}$$

$$V_1(s) = V_2(s) = \left(\frac{5}{s} + \frac{-5}{s+2} \right) + \left(\frac{5}{s} + \frac{-10}{s+1} + \frac{5}{s+2} \right)$$

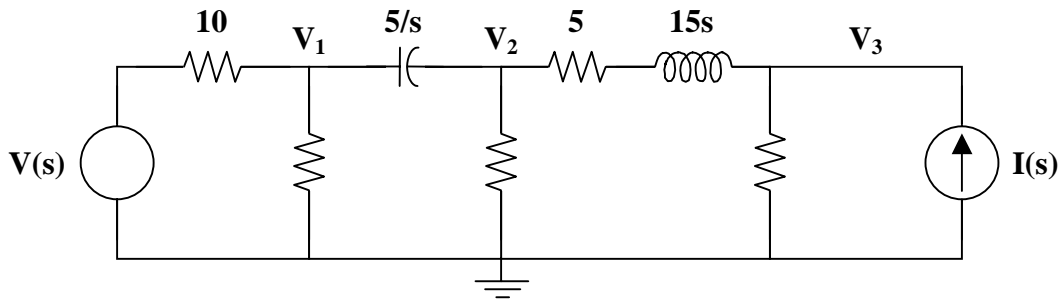$$V_1(s) = V_2(s) = \frac{10}{s} + \frac{-10}{s+1}$$

Therefore,

$$v_1(t) = \{10 - 10e^{-t}\} \, u(t) \, V \qquad \text{and} \qquad v_2(t) = \{10 - 10e^{-t}\} \, u(t) \, V$$

Problem 15.11   Develop the matrix representation for the circuit shown in Figure 15.1, using nodal analysis with three nodes.

Figure 15.1

Transforming Figure 15.1 to the frequency domain,

Using nodal analysis,

$$\frac{V_1 - V(s)}{10} + \frac{V_1 - 0}{20} + \frac{V_1 - V_2}{5/s} = 0$$

$$\frac{V_2 - V_1}{5/s} + \frac{V_2 - 0}{10} + \frac{V_2 - V_3}{5 + 15s} = 0$$

$$\frac{V_3 - V_2}{5 + 15s} + \frac{V_3 - 0}{20} - I(s) = 0$$

Simplifying these equations,

$$2\,V_1 - 2\,V(s) + V_1 + 4s\,V_1 - 4s\,V_2 = 0$$

$$[(2s)(3s+1)](V_2 - V_1) + (3s+1)\,V_2 + (2)(V_2 - V_3) = 0$$

$$(4)(V_3 - V_2) + (3s+1)\,V_3 - (20)(3s+1)\,I(s) = 0$$

Combining like terms,

$$(2+1+4s) V_1 - 4s V_2 = 2 V(s)$$
$$[(-2s)(3s+1)] V_1 + [(2s)(3s+1) + (3s+1) + 2] V_2 - 2 V_3 = 0$$
$$- 4 V_2 + [4 + (3s+1)] V_3 = (20)(3s+1) I(s)$$

Simplifying further,

$$(4s+3) V_1 - 4s V_2 = 2 V(s)$$
$$(-6s^2 - 2s) V_1 + (6s^2 + 5s + 3) V_2 - 2 V_3 = 0$$
$$- 4 V_2 + (3s+5) V_3 = (20)(3s+1) I(s)$$

Therefore, the matrix equation is

$$\begin{bmatrix} 4s+3 & -4s & 0 \\ -6s^2 - 2s & 6s^2 + 5s + 3 & -2 \\ 0 & -4 & 3s+5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 2 V(s) \\ 0 \\ (20)(3s+1) I(s) \end{bmatrix}$$

Problem 15.12   Given the circuit in Figure 15.1, write the s-domain equations for V_1, V_2, V_3, and V_4. DO NOT SOLVE.
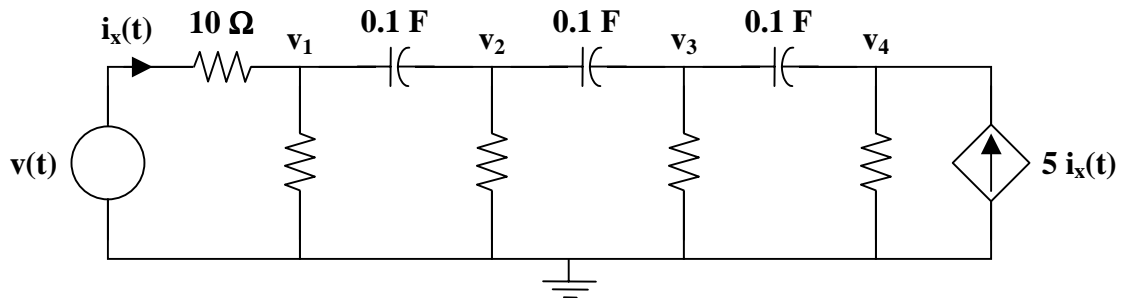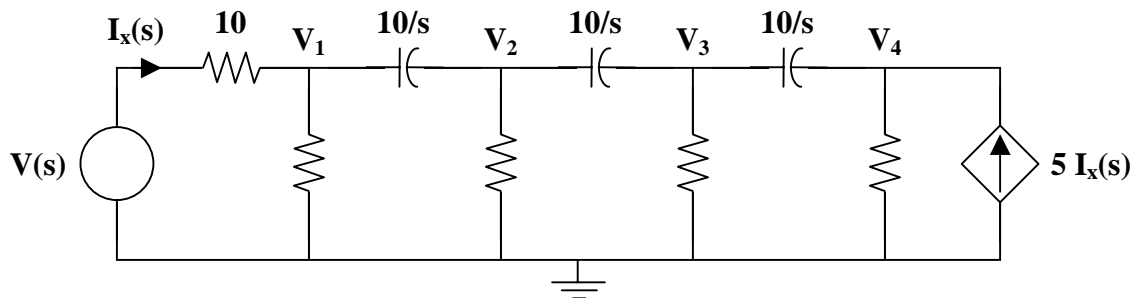

Figure 15.1

Converting this circuit to the s-domain yields,

Using nodal analysis,

At node 1 : $$\frac{V_1 - V(s)}{10} + \frac{V_1 - 0}{10} + \frac{V_1 - V_2}{10/s} = 0$$

At node 2 :
$$\frac{V_2 - V_1}{10/s} + \frac{V_2 - 0}{10} + \frac{V_2 - V_3}{10/s} = 0$$

At node 3 :
$$\frac{V_3 - V_2}{10/s} + \frac{V_3 - 0}{10} + \frac{V_3 - V_4}{10/s} = 0$$

At node 4 :
$$\frac{V_4 - V_3}{10/s} + \frac{V_4 - 0}{10} - 5I_x(s) = 0 \qquad \text{where} \quad I_x(s) = \frac{V(s) - V_1}{10}$$

Simplifying,
$$(s + 2) V_1 - s V_2 = V(s)$$
$$- s V_1 + (2s + 1) V_2 - s V_3 = 0$$
$$- s V_2 + (2s + 1) V_3 - s V_4 = 0$$
$$5 V_1 - s V_3 + (s + 1) V_4 = 5 V(s)$$

Finally, collect the equations and place them into a matrix form.

$$\begin{bmatrix} s+2 & -s & 0 & 0 \\ -s & 2s+1 & -s & 0 \\ 0 & -s & 2s+1 & -s \\ 5 & 0 & -s & s+1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \\ 0 \\ 5V(s) \end{bmatrix}$$

Problem 15.13   *[15.49]*   For the circuit in Figure 15.1 find $v_o(t)$ for $t > 0$.
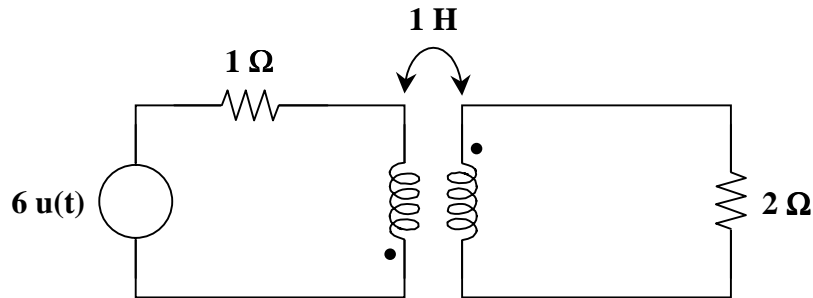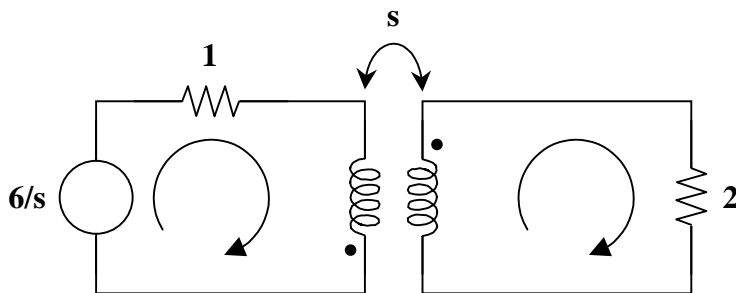

Figure 15.1

Consider the circuit shown below.

For mesh 1,

$$\frac{6}{s} = (1+2s)\,I_1 + s\,I_2 \tag{1}$$

For mesh 2,

$$0 = s\,I_1 + (2+s)\,I_2$$

$$I_1 = -\left(1+\frac{2}{s}\right)I_2 \tag{2}$$

Substituting (2) into (1) gives

$$\frac{6}{s} = -(1+2s)\left(1+\frac{2}{s}\right)I_2 + s\,I_2 = \frac{-(s^2+5s+2)}{s}I_2$$

or $\quad I_2 = \dfrac{-6}{s^2+5s+2}$

$$V_o = 2\,I_2 = \frac{-12}{s^2+5s+2} = \frac{-12}{(s+0.438)(s+4.561)}$$

Since the roots of $s^2 + 5s + 2 = 0$ are -0.438 and -4.561,

$$V_o = \frac{A}{s+0.438} + \frac{B}{s+4.561}$$

$$A = \frac{-12}{4.123} = -2.91, \qquad\qquad B = \frac{-12}{-4.123} = 2.91$$

$$V_o(s) = \frac{-2.91}{s+0.438} + \frac{2.91}{s+4.561}$$

$$v_o(t) = \underline{\mathbf{2.91[e^{-4.561t} - e^{0.438t}]\,u(t)\ V}}$$

Problem 15.14　　Given the circuit shown in Figure 15.1 and $v_{in}(t) = (10 + 10e^{-t})\,u(t)$ volts, find $v_{out}(t)$, $\forall\,t$.
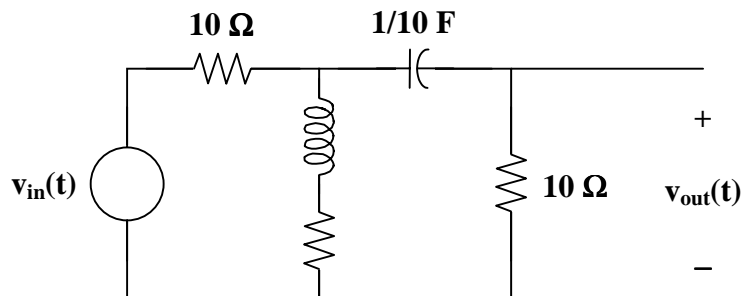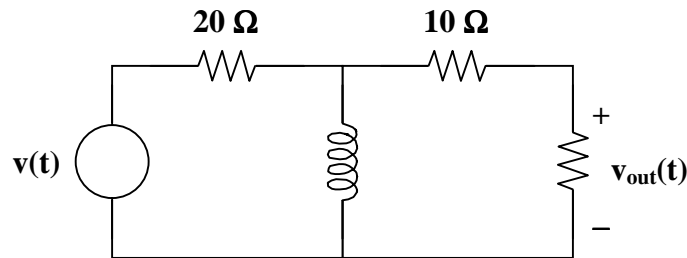

10 Ω　　　**1/10 F**

$v_{in}(t)$　　　　　　　　　**10 Ω**　　$v_{out}(t)$　　+　　−

Figure 15.1

$$v_{out}(t) = \underline{(10e^{-t} - 5te^{-t})\ V}$$

Problem 15.15   Given the circuit shown in Figure 15.1 and $v_{out}(t) = \{5e^{-2t}\,u(t)\}$ volts.
Find $v(t)$.

Figure 15.1

$$v(t) = \underline{20\,u(t)\ V}$$

TRANSFER FUNCTIONS

The transfer function $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$,
assuming all initial conditions are zero.

Problem 15.16   *[15.51]*   The transfer function of a system is

$$H(s) = \frac{s^2}{3s+1}$$

Find the output when the system has an input of $4e^{-t/3}\,u(t)$.

$$Y(s) = H(s)\,X(s), \qquad\qquad X(s) = \frac{4}{s+1/3} = \frac{12}{3s+1}$$

$$Y(s) = \frac{12\,s^2}{(3s+1)^2} = \frac{4}{3} - \frac{8s+4/3}{(3s+1)^2}$$

$$Y(s) = \frac{4}{3} - \frac{8}{9}\cdot\frac{s}{(s+1/3)^2} - \frac{4}{27}\cdot\frac{1}{(s+1/3)^2}$$

Let $G(s) = \dfrac{-8}{9}\cdot\dfrac{s}{(s+1/3)^2}$

Using the time differentiation property,

$$g(t) = \frac{-8}{9} \cdot \frac{d}{dt}(t e^{-t/3}) = \left(\frac{-8}{9}\right)\left(\frac{-1}{3} t e^{-t/3} + e^{-t/3}\right)$$

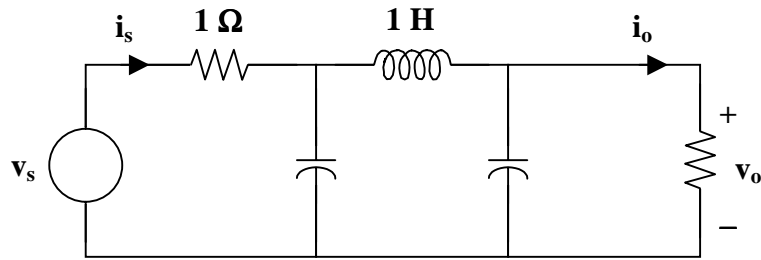$$g(t) = \frac{8}{27} t e^{-t/3} - \frac{8}{9} e^{-t/3}$$

Hence,

$$y(t) = \frac{4}{3} u(t) + \frac{8}{27} t e^{-t/3} - \frac{8}{9} e^{-t/3} - \frac{4}{27} t e^{-t/3}$$

$$\underline{y(t) = \frac{4}{3} u(t) - \frac{8}{9} e^{-t/3} + \frac{4}{27} t e^{-t/3}}$$

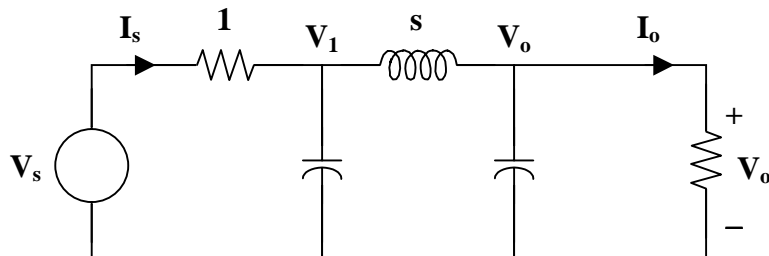
Problem 15.17   *[15.59]*   Refer to the network in Figure 15.1. Find the following transfer functions:

(a) $H_1(s) = V_o(s)/V_s(s)$

(b) $H_2(s) = V_o(s)/I_s(s)$

(c) $H_3(s) = I_o(s)/I_s(s)$

(d) $H_4(s) = I_o(s)/V_s(s)$

Figure 15.1

(a)   Consider the following circuit.

At node 1,

$$\frac{V_s - V_1}{1} = s V_1 + \frac{V_1 - V_o}{s}$$

$$V_s = \left(1 + s + \frac{1}{s}\right) V_1 - \frac{1}{s} V_o \qquad (1)$$

At node o,

$$\frac{V_1 - V_o}{s} = s\, V_o + V_o = (s+1)\, V_o$$

$$V_1 = (s^2 + s + 1)\, V_o \qquad\qquad\qquad (2)$$

Substituting (2) into (1)

$$V_s = (s + 1 + 1/s)(s^2 + s + 1)V_o - 1/s\, V_o$$

$$V_s = (s^3 + 2s^2 + 3s + 2)V_o$$

$$H_1(s) = \frac{V_o}{V_s} = \underline{\frac{1}{s^3 + 2s^2 + 3s + 2}}$$

(b)
$$I_s = V_s - V_1 = (s^3 + 2s^2 + 3s + 2)V_o - (s^2 + s + 1)V_o$$

$$I_s = (s^3 + s^2 + 2s + 1)V_o$$

$$H_2(s) = \frac{V_o}{I_s} = \underline{\frac{1}{s^3 + s^2 + 2s + 1}}$$

(c)
$$I_o = \frac{V_o}{1}$$

$$H_3(s) = \frac{I_o}{I_s} = \frac{V_o}{I_s} = H_2(s) = \underline{\frac{1}{s^3 + s^2 + 2s + 1}}$$

(d)
$$H_4(s) = \frac{I_o}{V_s} = \frac{V_o}{V_s} = H_1(s) = \underline{\frac{1}{s^3 + 2s^2 + 3s + 2}}$$

CONVOLUTION INTEGRAL

The convolution of two signals consists of time-reversing one signal, shifting it, multiplying it point by point with the second signal, and integrating the product.

Steps to evaluate the convolution integral :
1. Folding :   Take the mirror image of $h(\lambda)$ about the ordinate axis to obtain $h(-\lambda)$.
2. Displacement :   Shift or delay $h(-\lambda)$ by t to obtain $h(t - \lambda)$.
3. Multiplication : Find the product of $h(t - \lambda)$ and $x(\lambda)$.
4. Integration :   For a given time t, calculate the area under the product $h(t - \lambda)x(\lambda)$ for $0 < \lambda < t$ to get $y(t)$ at t.

Problem 15.18   *[15.69]*   Given that $F_1(s) = F_2(s) = s/(s^2 + 1)$, find $L^{-1}[F_1(s)F_2(s)]$ by convolution.

$$f_1(t) = f_2(t) = \cos(t)$$

$$L^{-1}[F_1(s)F_2(s)] = \int_0^t \cos(\lambda)\cos(t-\lambda)\,d\lambda$$

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$L^{-1}[F_1(s)F_2(s)] = \frac{1}{2}\int_0^t \cos(t-2\lambda) - \cos(t)\,d\lambda$$

$$L^{-1}[F_1(s)F_2(s)] = \frac{1}{2}\cos(t)\cdot\lambda\Big|_0^t + \frac{1}{2}\cdot\frac{\sin(t-2\lambda)}{-2}\Big|_0^t$$

$$L^{-1}[F_1(s)F_2(s)] = \mathbf{1/2\,t\cos(t) + 1/2\sin(t)}$$

APPLICATION TO INTEGRODIFFERENTIAL EQUATIONS

Problem 15.19   Given the following is a matrix representation of a circuit in the frequency domain. Determine the value of $v_1(t)$ and $v_2(t)$.

$$\begin{bmatrix} s+2 & -s \\ -s & s+2 \end{bmatrix}\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} 8/s \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \frac{\begin{bmatrix} s+2 & s \\ s & s+2 \end{bmatrix}}{s^2+4s+4-s^2}\begin{bmatrix} 8/s \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} s+2 & s \\ s & s+2 \end{bmatrix}}{(4)(s+1)}\begin{bmatrix} 8/s \\ 0 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{(s+2)}{(4)(s+1)} & \dfrac{s}{(4)(s+1)} \\ \dfrac{s}{(4)(s+1)} & \dfrac{(s+2)}{(4)(s+1)} \end{bmatrix}\begin{bmatrix} 8/s \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{(8)(s+2)}{(4s)(s+1)} \\ \dfrac{8s}{(4s)(s+1)} \end{bmatrix} = \begin{bmatrix} \dfrac{4}{s} + \dfrac{-2}{s+1} \\ \dfrac{2}{s+1} \end{bmatrix}$$

Therefore, $v_1(t) = \mathbf{\{4 - 2e^{-t}\}\,u(t)\,V}$ and $v_2(t) = \mathbf{\{2e^{-t}\}\,u(t)\,V}$.

Problem 15.20   Solve for $v_1(t)$ and $v_2(t)$ given the following matrix equation.

$$\begin{bmatrix} s+1 & -s \\ -3 & s+2 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} 2/s \\ 4/(s+3) \end{bmatrix}$$

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \frac{\begin{bmatrix} s+2 & s \\ 3 & s+1 \end{bmatrix}}{(s+1)(s+2)-3s} \begin{bmatrix} 2/s \\ 4/(s+3) \end{bmatrix} = \frac{\begin{bmatrix} s+2 & s \\ 3 & s+1 \end{bmatrix}}{s^2+2} \begin{bmatrix} 2/s \\ 4/(s+3) \end{bmatrix}$$

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{s^2+2} & \dfrac{s}{s^2+2} \\ \dfrac{3}{s^2+2} & \dfrac{s+1}{s^2+2} \end{bmatrix} \begin{bmatrix} \dfrac{2}{s} \\ \dfrac{4}{s+3} \end{bmatrix}$$

$$V_1(s) = \left(\frac{s+2}{s^2+2}\right)\left(\frac{2}{s}\right) + \left(\frac{s}{s^2+2}\right)\left(\frac{4}{s+3}\right) = \frac{(2)(s+2)(s+3)+(4s)(s)}{s(s^2+2)(s+3)} = \frac{6s^2+10s+12}{s(s^2+2)(s+3)}$$

$$V_2(s) = \left(\frac{3}{s^2+2}\right)\left(\frac{2}{s}\right) + \left(\frac{s+1}{s^2+2}\right)\left(\frac{4}{s+3}\right) = \frac{(6)(s+3)+(4)(s+1)(s)}{s(s^2+2)(s+3)} = \frac{4s^2+10s+18}{s(s^2+2)(s+3)}$$

Now, we find the inverse Laplace transforms of $V_1(s)$ and $V_2(s)$.

Partial fraction expansion yields

$$V_1(s) = \frac{2}{s} + \frac{-0.4545 - 0.9642j}{s - \sqrt{2}j} + \frac{-0.4545 + 0.9642j}{s + \sqrt{2}j} + \frac{-1.0909}{s+3}$$

$$V_2(s) = \frac{3}{s} + \frac{-1.1364 - 0.6428j}{s - \sqrt{2}j} + \frac{-1.1364 + 0.6428j}{s + \sqrt{2}j} + \frac{-0.7273}{s+3}$$

The inverse Laplace transform is

$$v_1(t) = 2u(t) + 1.066e^{-115.24j}e^{j\sqrt{2}t} + 1.066e^{115.24j}e^{-j\sqrt{2}t} - 1.0909e^{-3t}$$

$$v_2(t) = 3u(t) + 1.3056e^{-150.51j}e^{j\sqrt{2}t} + 1.3056e^{150.51j}e^{-j\sqrt{2}t} - 0.7273e^{-3t}$$

or

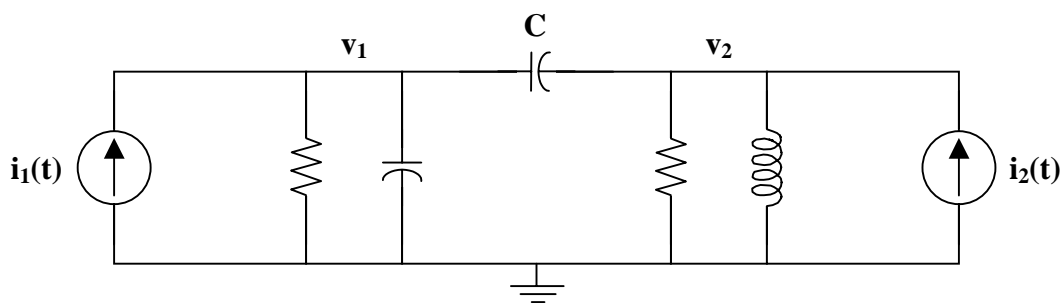$$v_1(t) = 2u(t) + 1.066(e^{j(\sqrt{2}t-115.24)} + e^{-j(\sqrt{2}t-115.24)}) - 1.0909e^{-3t}$$

$$v_2(t) = 3u(t) + 1.3056(e^{j(\sqrt{2}t-150.51)} + e^{-j(\sqrt{2}t-150.51)}) - 0.7273e^{-3t}$$

Finally,

$$\underline{v_1(t) = 2u(t) + 2.132\cos(\sqrt{2}t - 115.24°) - 1.0909e^{-3t}}$$

$$\underline{v_2(t) = 3u(t) + 2.611\cos(\sqrt{2}t - 150.51°) - 0.7273e^{-3t}}$$

Problem 15.21   Given the circuit in Figure 15.1 and the following s-domain matrix nodal equations, determine the values for R, C, and L.
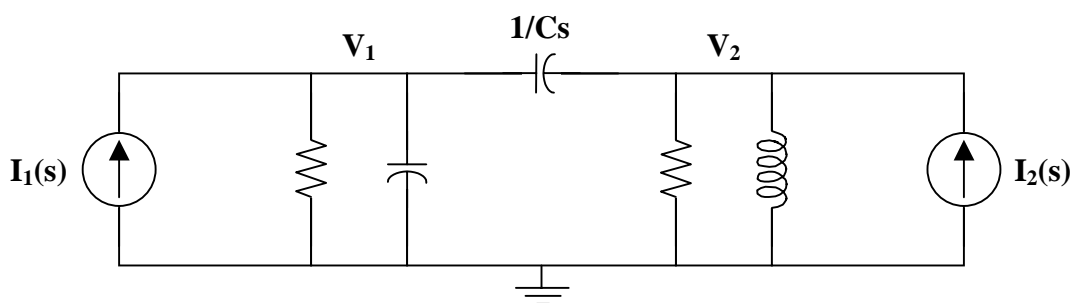

Figure 15.1

The matrix representation of the s-domain nodal equations is

$$\begin{bmatrix} \dfrac{5s+4}{20} & \dfrac{-s}{20} \\ \dfrac{-s}{20} & \dfrac{s^2+2s+4}{20s} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

So, the two equations are

$$\frac{5s+4}{20} V_1 + \frac{-s}{20} V_2 = I_1(s)$$

$$\frac{-s}{20} V_1 + \frac{s^2+2s+4}{20s} V_2 = I_2(s)$$

Draw the s-domain circuit.

Writing the node equations gives,
$$(1/R + s/5 + Cs)\, V_1 - Cs\, V_2 = I_1(s)$$
and
$$- Cs\, V_1 + (Cs + 1/10 + 1/Ls)\, V_2 = I_2(s)$$

Clearly, $$\frac{1}{R} + \frac{s}{5} + Cs = \frac{5s+4}{20} = \frac{s}{4} + \frac{1}{5}$$

Thus,   1/R = 1/5   or   R = **5 Ω**

s(0.2 + C) = s(1/4)   or   C = 0.25 − 0.2 = **0.05 F**

Also, $Cs + 1/10 + 1/Ls = \dfrac{s^2 + 2s + 4}{20s}$

Finally, $0.05s + 0.1 + 1/Ls = 0.05s + 0.1 + 1/5s$ or   L = **5 H**

APPLICATIONS

Problem 15.22   Given the op-amp circuit in Figure 15.1, determine the value of $v_{out}(t)$, where the value of $v_C(t)$ is equal to 2 volts at t = 0. In other words, $v_C(0) = 2$ volts.
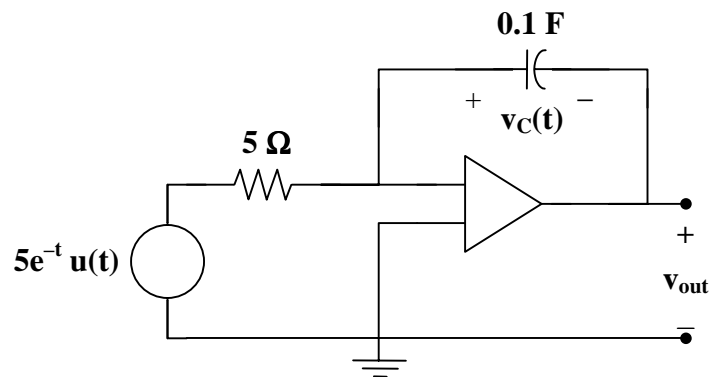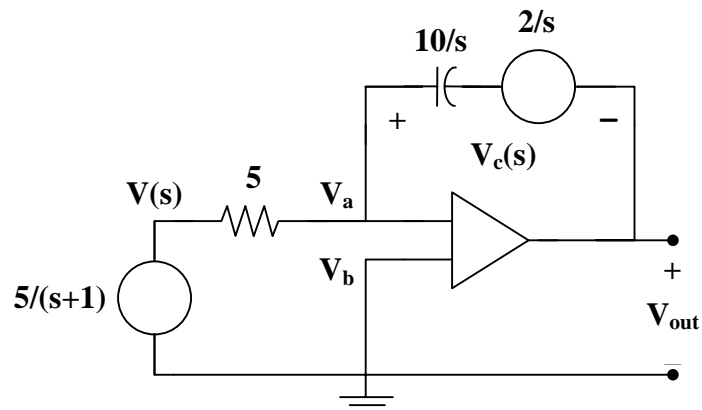

Figure 15.1

Begin by transforming the circuit from the time domain to the frequency domain.

Writing a node equation at a,

$$\frac{V_a - V(s)}{5} + \frac{V_a - [(2/s) + V_{out}(s)]}{10/s} = 0$$

which is only one equation with two unknowns.

We need a constraint equation.

$$V_a = V_b = 0.$$

Then,

$$\frac{-V(s)}{5} + \frac{-[(2/s) + V_{out}(s)]}{10/s} = 0$$

$$-2V(s) - [2 + s\,V_{out}(s)] = 0$$

$$V_{out}(s) = \frac{-2V(s)}{s} + \frac{-2}{s} = \left(\frac{-2}{s}\right)\left(\frac{5}{s+1}\right) + \frac{-2}{s} = \frac{-10}{s(s+1)} + \frac{-2}{s}$$
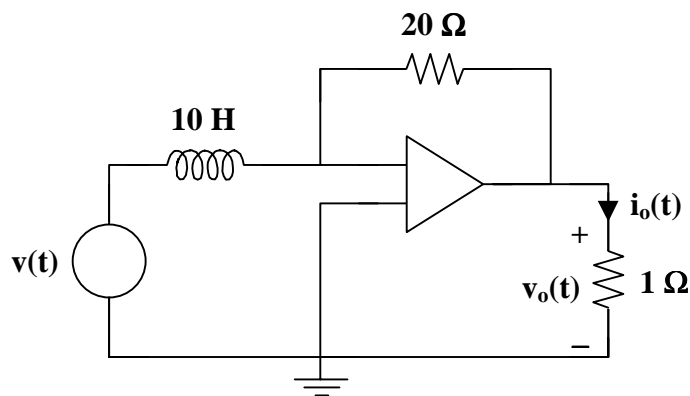
Partial fraction expansion of the first term leads to

$$V_{out}(s) = \frac{-10}{s} + \frac{10}{s+1} + \frac{-2}{s} = \frac{-12}{s} + \frac{10}{s+1}$$

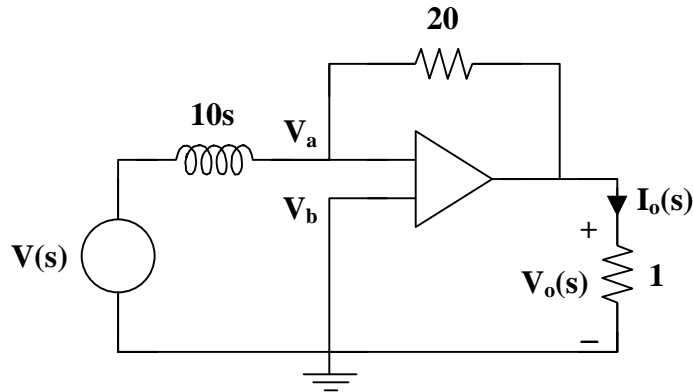Taking the inverse Laplace transform of both sides produces

$$v_{out}(t) = \{-12 + 10e^{-t}\}\,u(t)\ \mathbf{V}$$

====

Problem 15.23   In Figure 15.1, find $v_o(t)$ and $i_o(t)$, given that $v(t) = (10e^{-t})u(t)$ volts.

20 Ω

10 H

$i_o(t)$

v(t)

$v_o(t)$ **1 Ω**

Figure 15.1

Begin by transforming the time domain circuit to the frequency domain.

At node a :

$$\frac{V_a - V(s)}{10s} + \frac{V_a - V_o(s)}{20} = 0$$

We cannot help ourselves by writing any more node equations. So we have one equation and two unknowns. We need another equation without generating any additional unknowns. So we go to the constraint equation.

$$V_a = V_b = 0$$

So, now our node equation becomes

$$\frac{-V(s)}{10s} + \frac{-V_o(s)}{20} = 0$$

$$\frac{V_o(s)}{20} = \frac{-V(s)}{10s}$$

which leads to

$$V_o(s) = \frac{-2}{s} V(s)$$

Taking the Laplace transform of $v(t)$ gives

$$V(s) = \frac{10}{s+1}$$

Substituting for $V(s)$ yields

$$V_o(s) = \left(\frac{-2}{s}\right)\left(\frac{10}{s+1}\right) = \frac{-20}{s(s+1)}$$

After partial fraction expansion,
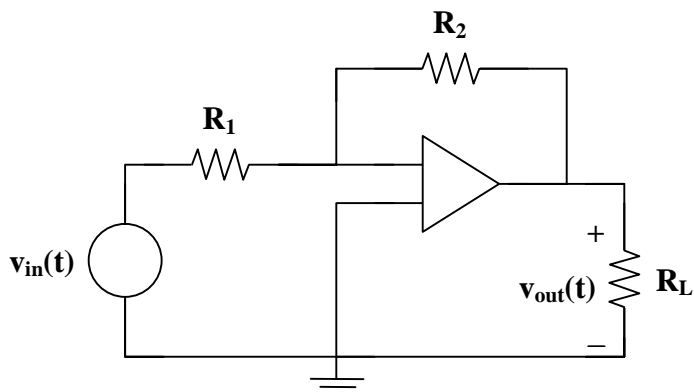
$$V_o(s) = \frac{-20}{s} + \frac{20}{s+1}$$

Taking the inverse Laplace transform produces

$$v_o(t) = \underline{(-20 + 20e^{-t})\, u(t)\ V}$$

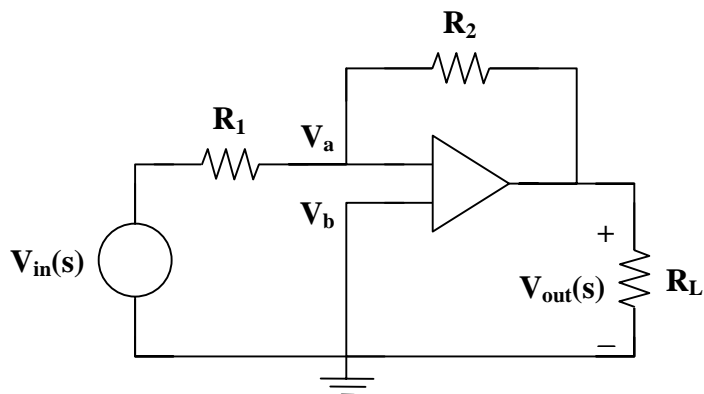Since $v = iR$ in either domain, we do not need to find $I_o(s)$.

$$i_o(t) = v_o(t)/1 = \mathbf{(-20 + 20e^{-t})\, u(t)\ A}$$

Problem 15.24   Given the circuit in Figure 15.1, calculate $V_{out}(s)$ in terms of $V_{in}(s)$.

Figure 15.1

Begin by transforming the circuit from the time domain to the frequency domain.

Using nodal analysis,

$$\frac{V_a - V_{in}(s)}{R_1} + \frac{V_a - V_{out}(s)}{R_2} = 0$$

This is one equation with two unknowns. Thus, we need the following constraint equation.
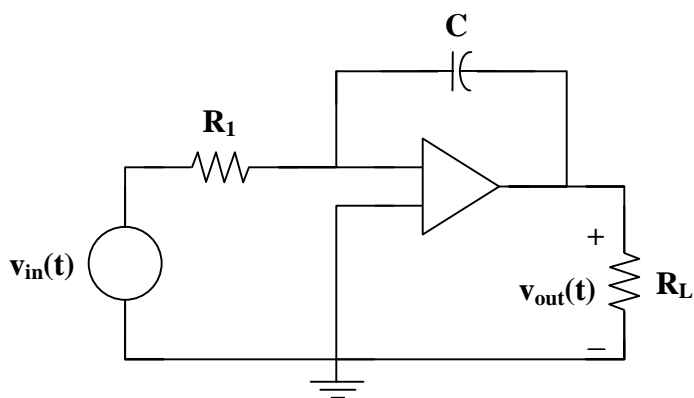
$$V_a = V_b = 0$$

Then the node equation becomes

$$\frac{-V_{in}(s)}{R_1} + \frac{-V_{out}(s)}{R_2} = 0$$
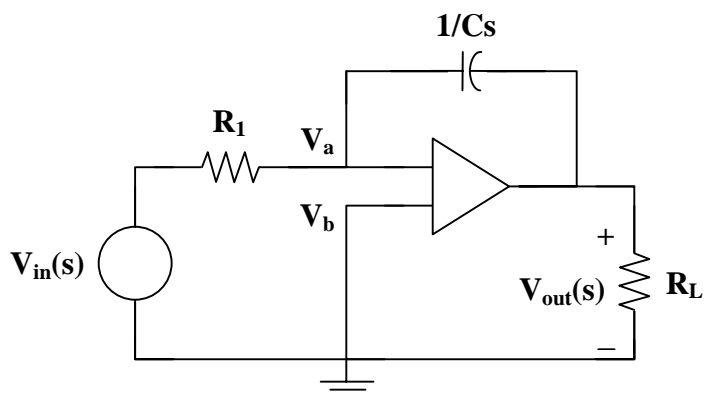
$$\frac{V_{out}(s)}{R_2} = \frac{-V_{in}(s)}{R_1}$$

Therefore,

$$V_{out}(s) = \frac{-R_2}{R_1} V_{in}(s)$$

Problem 15.25   Given the circuit in Figure 15.1, calculate $V_{out}(s)$ in terms of $V_{in}(s)$.

Figure 15.1

Begin by transforming the circuit from the time domain to the frequency domain.

Using nodal analysis,

$$\frac{V_a - V_{in}(s)}{R_1} + \frac{V_a - V_{out}(s)}{1/Cs} = 0$$

This is one equation with two unknowns. Thus, we need the following constraint equation.

$$V_a = V_b = 0$$

Then, the node equation becomes

$$\frac{-V_{in}(s)}{R_1} + \frac{-V_{out}(s)}{1/Cs} = 0$$

$$\frac{V_{out}(s)}{1/Cs} = \frac{-V_{in}(s)}{R_1}$$

Therefore,

$$V_{out}(s) = \underline{\frac{-1}{R_1 Cs} V_{in}(s)}$$

Problem 15.26   Given $R_1 = 100\text{ k}\Omega$. What value of C, in Figure 15.1, yields

$$V_{out}(t) = -\int V_{in}(t)\, dt$$

Taking the Laplace transform,

$$V_{out}(s) = \frac{-1}{s} V_{in}(s)$$

From Problem 15.25, we know that

$$V_{out}(s) = \frac{-1}{R_1 Cs} V_{in}(s)$$

Thus,

$$R_1 C = 1 \qquad \text{or} \qquad C = \frac{1}{R_1} = \frac{1}{100 \times 10^3} = 1 \times 10^{-5} = \underline{\mathbf{10\ \mu F}}$$

Problem 15.27   Given $R_1 = 10\text{ k}\Omega$, $R_L = 100\ \Omega$, $C = 50\ \mu F$, and $V_{in}(t) = 10e^{-2t} u(t)$. Calculate the total energy that v_{in} delivers to the circuit shown in Figure 15.1. Also, find the total energy delivered to R_L.

$$w(t) = \int p(t)\, dt = 1/R \int v^2(t)\, dt$$

To find the total energy that v_{in} delivers to the circuit,

$$w_{v_{in}}(t) = 1/R_1 \int v_{in}^2(t)\, dt$$

$$w_{v_{in}}(t) = 1/10{,}000 \int_0^t (10e^{-2\tau})(10e^{-2\tau})\, d\tau$$

$$= 100/10{,}000 \int_0^t e^{-4\tau}\, d\tau$$

$$= (1/100)(-1/4 e^{-4\tau}\big|_0^t)$$

$$= (1/100)(-1/4 e^{-4t} + 1/4)$$

$$= \underline{\mathbf{(1/400)(1 - e^{-4t})}}$$

To find the total energy delivered to R_L,

$$w_{R_L}(t) = 1/R_L \int v_{out}^2(t)\, dt$$

First, find $v_{out}(t)$. From Problem 15.25, we know that

$$V_{out}(s) = \frac{-1}{R_1 Cs} V_{in}(s)$$

where $R_1 C = (10 \times 10^3)(50 \times 10^{-6}) = 500 \times 10^{-3} = 0.5$

Then,

$$V_{out}(s) = \frac{-1}{R_1 Cs} V_{in}(s) = \frac{-2}{s} V_{in}(s)$$

Taking the Laplace transform of $v_{in}(t)$,

$$V_{in}(s) = \frac{10}{s+2}$$

Substitute $V_{in}(s)$ into the equation for $V_{out}(s)$.

$$V_{out}(s) = \frac{-2}{s} V_{in}(s) = \left(\frac{-2}{s}\right)\left(\frac{10}{s+2}\right) = \frac{-20}{s(s+2)}$$

Take the partial fraction expansion of $V_{out}(s)$.

$$V_{out}(s) = \frac{-10}{s} + \frac{10}{s+2}$$

Hence,

$$V_{out}(t) = (-10 + 10e^{-2t})\, u(t)$$

Now,

$$
\begin{aligned}
w_{R_L}(t) &= 1/R_L \int v_{out}^2(t)\, dt \\
&= 1/100 \int_0^t (-10+10e^{-2\tau})(-10+10e^{-2\tau})\, d\tau \\
&= 1/100 \int_0^t (100e^{-4\tau} - 200e^{-2\tau} + 100)\, d\tau \\
&= \int_0^t e^{-4\tau}\, d\tau - 2\int_0^t e^{-2\tau}\, d\tau + \int_0^t 1\, d\tau \\
&= [(-1/4)e^{-4t} + 1/4] - (2)[(-1/2)e^{-2t} + 1/2] + t \\
&= \underline{\mathbf{(1/4)(-e^{-4t} + 1) + (-e^{-2t} + 1) + t}}
\end{aligned}
$$

Problem 15.28   *[15.89]*   A gyrator is a device for simulating an inductor in a network. A basic gyrator is shown in Figure 15.1. By finding $V_o(s)/I_o(s)$, show that the inductance produced by the gyrator is $L = CR^2$.
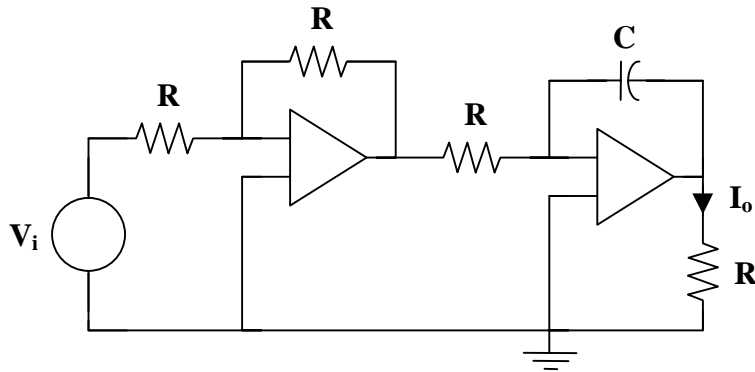
Figure 15.1

The gyrator is equivalent to two cascaded inverting amplifiers. Let V_1 be the voltage at the output of the first op amp.

$$V_1 = \frac{-R}{R}V_i = -V_i$$

$$V_o = \frac{-1/sC}{R}V_1 = \frac{1}{sCR}V_i$$

$$I_o = \frac{V_o}{R} = \frac{V_o}{sR^2C}$$

$$\frac{V_o}{I_o} = sR^2C$$

$$\frac{V_o}{I_o} = sL, \quad \text{when } L = R^2C$$