$$\begin{array}{l} \qquad\quad 3 \qquad\quad 1 \qquad 2 \\ \text{Ex: } x_1 - 2x_2 + x_3 = 3 \\ \qquad 2x_1 + x_2 - x_3 = 5 \\ \qquad 3x_1 - x_2 + 2x_3 = 12 \end{array}$$

solution

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix} \end{array}$$

$$= 1(2-1) + 2(4+3) + 1(-2-3) = 1 + 14 - 5 = 10$$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \Delta x_1 = \begin{vmatrix} 3 & -2 & 1 \\ 5 & 1 & -1 \\ 12 & -1 & 2 \end{vmatrix} \end{array} = 3(2-1) + 2(10+12) + 1(-5-12)$$

$$= 3 + 44 - 17 = 30$$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \Delta x_2 = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & -1 \\ 3 & 12 & 2 \end{vmatrix} \end{array} = 1(10+12) - 3(4+3) + 1(24-15)$$

$$= 22 - 21 + 9 = 10$$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \Delta x_3 = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & 12 \end{vmatrix} \end{array} = 1(12+5) + 2(24-15) + 3(-2-3)$$

$$= 17 + 18 - 15 = 20$$

$x_1 = \frac{\Delta x_1}{\Delta} = \frac{30}{10} = 3$, $x_2 = \frac{\Delta x_2}{\Delta} = \frac{10}{10} = 1$, $x_3 = \frac{\Delta x_3}{\Delta} = \frac{20}{10} = 2$

Ex: Solve the following linear system using Cramer's rule:

$$2x + y = 7$$

$$3x - 4y = 5$$

Solution

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -8 - 3 = -11$$

$$\Delta x = \begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix} = -28 - 5 = -33$$

$$\Delta y = \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} = 10 - 21 = -11$$

$$x = \frac{\Delta x}{\Delta} = \frac{-33}{-11} = 3$$

$$y = \frac{\Delta y}{\Delta} = \frac{-11}{-11} = 1$$

Solution of linear equations System

* Cramer's Rule

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

* System of two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Solution

Determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\Delta x_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = ?$$

$$\Delta x_2 = \begin{vmatrix} a_{11} & b_1 \\ a & b_2 \end{vmatrix} = ?$$

$$x_1 = \frac{Dx_1}{\Delta} =$$

$$x_2 = \frac{\Delta x_2}{\Delta}$$

Ex: Express the matrix

$$B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

as a sum of symmetric and skew symmetric matrices.

Solution

$$B = \underbrace{P}_{\text{symm}} + \underbrace{Q}_{\text{skew}}$$

$$P = \frac{1}{2}(B+B') = \frac{1}{2}\left(\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}$$

$$Q = \frac{1}{2}(B-B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

$$A = \underbrace{P}_{\text{symmetric}} + \underbrace{Q}_{\text{skew}}$$

$$= \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

$$A = \frac{1}{2}A + \frac{1}{2}A$$

$$= \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A' - \frac{1}{2}A'$$

$$= \underbrace{\frac{1}{2}(A+A')}_{\text{symm}} + \underbrace{\frac{1}{2}(A-A')}_{\text{skew}}$$

$$B = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$$

Solution

$$B' = \begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -7 \\ -4 & 7 & 0 \end{bmatrix}$$

$$= - \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$$

$= -B$ Skew symmetric

* for any square matrix A

$A + A'$ is symmetric

$A - A'$ is skew symmetric

$$(A+A')' = A' + A$$

$$(A-A')' = A' - A = -(-A' + A)$$

* Symmetric and Skew symmetric matrix.

$A' = A$ Symmetric

$A' = -A$ Skew symmetric

Ex: Determine if A is symmetric or skew symmetric

$$A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

Solution

$$A' = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix} = A$$

Symmetric

* Transpose of a matrix:

Let $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \\ -1 & 7 \end{bmatrix}$

Find the transpose of A

Solution

$$A^t \text{ or } A' = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 5 & 7 \end{bmatrix}$$

* $(A')' = A$
* $(A \pm B)^t = A^t \pm B^t$
* $(A \pm B)' = A' \pm B'$
* $(AB)' = B'A'$