Ito Integral

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Summarised: DATE

1 Ito Integral

Important technical tools in analysis of stochastic processes are the so-called Ito integral. Construction of Ito integral is very similar to the definition of the Riemann-Stieltjes integral of functions of real variable. First, we notice that it follows from the feginition of a Wiener process $w(t)$, that the random variable has a normal distribution with zero mean and dispersion t, i.e $w(t) \approx \mathcal{N}(0,t)$. This equality can be rewritten as:

$$\int_0^t dw(\tau) = w(t) - w(0) = w(t) \approx N(0,t) \tag{1}$$

It means that for a constant function $f(\tau) = c$ we have

$$\begin{aligned} \int_0^t f(\tau)dw(\tau) &= c\int_0^t dw(\tau) = cw(t) - cw(0) \qquad (2) \\ &= cw(t) \approx N(0,c^2 t) = N(0, \int_0^t f^2(\tau)d\tau) \end{aligned}$$

Gives simple identity gives us and idea, how to define the so called Ito integral of measureable function $f : (0,t) \to \mathbb{R}$ such that $\int_0^t f^2(\tau)d\tau < \infty$. We let

$$\int_0^t f(\tau)dw(\tau) := lim_{v\to 0}\Sigma_{i=0}^{n-1} f(\tau_i)(w(\tau_{i+1}) - w(\tau_i)) \tag{3}$$

where $v = max(\tau_{i+1} - \tau_i)$ is the norm of a partion $0 = \tau_0 < \tau_1 < ... < \tau_n = \tau$ of the interval $(0,t)$. Convergence is meant in probabilty. Let the function f be constant on each subinterval $[\tau_i, \tau_{i+1}]$. Then, for the expected value of the finite sum $\Sigma_{i=1}^n f(\tau_i)(w(t_{i+1}) - w(t_i))$, it holds :

$$E\Bigg(\Sigma_{i=0}^{n-1} f(\tau_i)(w(\tau i + 1) - w(\tau_i))\Bigg) = \Sigma_{i=0}^{n-1} f(\tau_i)E(w(\tau)) = 0, \tag{4}$$

because all increment $w(\tau_{i+1}) - w(\tau_i)$ are normally distributed random variables $w(\tau_{i+1}) - w(\tau_i) \approx N(0, \tau_{i+1} - \tau_i)$. Since these increments are also independet and $w(\tau_{i+1}) - w(\tau_i) =\approx \mathcal{N}(0,1)$ we may conclude for the sum of the independent normally distributed random variables the following identity:

$$E\Bigg(\bigg[\Sigma_{i=0}^{n-1} f(\tau_i)(w(\tau_{i+1}) - w(\tau_i))\bigg]^2\Bigg) = \Sigma_{i=0}^{n-1} f^2(\tau_i)E(\Phi_i^2)(\tau_{i+1} - \tau_i) \tag{5}$$

$$= \Sigma_{i=1}^{n} f^2(\tau_i)(\tau_{i+1} - \tau_i) \tag{6}$$

2 Ito Lemma

Analysis of functions, representing prices of financial derivatives, whose one or more variable are stochastic random variables satisfying prescribed stochastic differential equations plays a key role in theory of pricing financial derivatives. In this section, we focus our attention on the question whether there exist a stochastic differential equation describing evolution of a smooth functiopn $f(x,t)$ of two variables, where the variable x itself is a solution to prescribed stochastic differential equation. The positive answer to this question is given by Ito lemma. This is a key stone of analysis of stochastic differential equations.

Lemma 2.1 *(Ito lemma). Let $f(x,t)$ be a smooth function of two variables. Assume the variable x is a solution to the stochastic differential equation*

$$dx = \mu(x,t)dt + \sigma(x,t)dw \tag{7}$$

where w is a Wiener process. Then the first differential of the function f is given by

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt \tag{8}$$

and so the function f satisfies the stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial f}{\partial x}dw \tag{9}$$