Itō Integral

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1 Itō Integral

Important technical tools in analysis of stochastic processes are the so-called Itō integral. Construction of Itō integral is very similar to the definition of the Riemann-Stieltjes integral of functions of real variable. First, we notice that it follows from the feginition of a Wiener process $w(t)$, that the random variable has a normal distribution with zero mean and dispersion t, i.e $w(t) \approx \mathcal{N}(0,t)$. This equality can be rewritten as:

$$\int_0^t dw(\tau) = w(t) - w(0) = w(t) \approx N(0,t) \tag{1}$$

It means that for a constant function $f(\tau) = c$ we have

$$\begin{aligned} \int_0^t f(\tau)dw(\tau) = c\int_0^t dw(\tau) = cw(t) - cw(0) \\ = cw(t) \approx N(0, c^2t) = N(0, \int_0^t f^2(\tau)d\tau) \end{aligned} \tag{2}$$

Gives simple identity gives us and idea, how to define the so called Itō integral of measureable function $f : (0,t) \to \mathbb{R}$ such that $\int_0^t f^2(\tau)d\tau < \infty$. We let

$$\int_0^t f(\tau)dw(\tau) := lim_{v\to 0}\Sigma_{i=0}^{n-1} f(\tau_i)(w(\tau_{i+1}) - w(\tau_i)) \tag{3}$$

where $v = max(\tau_{i+1} - \tau_i)$ is the norm of a partion $0 = \tau_0 < \tau_1 < ... < \tau_n = \tau$ of the interval $(0,t)$. Convergence is meant in probabilty. Let the function f be constant on each subinterval $[\tau_i, \tau_{i+1}]$. Then, for the expected value of the finite sum $\Sigma_{i=1}^n f(\tau_i)(w(t_{i+1}) - w(t_i))$, it holds :

$$E\bigg(\Sigma_{i=0}^{n-1} f(\tau_i)(w(\tau i+1) - w(\tau_i))\bigg) = \Sigma_{i=0}^{n-1} f(\tau_i)E(w(\tau)) = 0, \tag{4}$$

because all increment $w(\tau_{i+1}) - w(\tau_i)$ are normally distributed random variables $w(\tau_{i+1}) - w(\tau_i) \approx N(0, \tau_{i+1} - \tau_i)$. Since these increments are also independet and $w(\tau_{i+1}) - w(\tau_i) =\approx \mathcal{N}(0,1)$ we may conclude for the sum of the independent normally distributed random variables the following identity:

$$E\bigg(\bigg[\Sigma_{i=0}^{n-1} f(\tau_i)(w(\tau_{i+1}) - w(\tau_i))\bigg]^2\bigg) = \Sigma_{i=0}^{n-1} f^2(\tau_i)E(\Phi_i^2)(\tau_{i+1} - \tau_i) \tag{5}$$

$$= \Sigma_{i=1}^{n} f^2(\tau_i)(\tau_{i+1} - \tau_i) \tag{6}$$

2 Itō Lemma

Analysis of functions, representing prices of financial derivatives, whose one or more variable are stochastic random variables satisfying prescribed stochastic differential equations plays a key role in theory of pricing financial derivatives. In this section, we focus our attention on the question whether there exist a stochastic differential equation describing evolution of a smooth functiopn $f(x,t)$ of two variables, where the variable x itself is a solution to prescribed stochastic differential equation. The positive answer to this question is given by Itō lemma. This is a key stone of analysis of stochastic differential equations.

Lemma 2.1 *(Itō lemma). Let $f(x,t)$ be a smooth function of two variables. Assume the variable x is a solution to the stochastic differential equation*

$$dx = \mu(x,t)dt + \sigma(x,t)dw \tag{7}$$

where w is a Wiener process. Then the first differential of the function f is given by

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt \tag{8}$$

and so the function f satisfies the stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial f}{\partial x}dw \tag{9}$$

Now, it follows from the property $dw = \Phi\sqrt{dt}$ where $\Phi \approx \mathcal{N}(0,1)$ that

$$E\left((dw)^2 - dt\right) = 0 \tag{10}$$

and

$$\mathrm{Var}\left((dw)^2 - dt\right) = \left[E\left(\Phi^4\right) - E\left(\Phi^2\right)^2\right](dt)^2 = 2(dt)^2 \tag{11}$$

By negelecting higher order terms in dt we can approximaate the term $(dw)^2$ by dt. We obtain

$$(dx)^2 = \sigma^2(dw)^2 + 2\mu\sigma dwdt + \mu^2(dt)^2 \approx \sigma^2 dt + O\left((dt)^{3/2}\right) + O\left((dt)^2\right) \tag{12}$$

Similar for the term $dxdt = O\left((dt)^{3/2}\right) + O\left((dt)^2\right)$, and cosequently the first order expansion of the differential df with respect to infinitesimal increments dt and dx can be written in the form

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt \tag{13}$$

and then substituting $dx = \mu(x,t)dt + \sigma(x,t)dw$, we will get

$$df = \left(\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial f}{\partial x}dw \tag{14}$$

3 Example of Itō Lemma

Let us consider a Brownian motion $dX = \mu dt + \sigma dw$ and its function $Y(t) = f(X(t), t)$ where $f(X, t) = e^X$. By applying Itō lemma we obtain

$$dY = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial X^2} \right) dt + \sigma \frac{\partial f}{\partial X} dw = \left(\mu + \frac{\sigma^2}{2} \right) Y dt + \sigma Y dw \quad (15)$$

As a cosequences, we obtain for the expected value $E(Y(t))$ the following ordinary differential equation

$$dE(Y(t)) = \left(\mu + \frac{\sigma^2}{2} \right) E(Y(t))dt \quad (16)$$

and easily deduce that $E(Y(t)) = E(Y(0))e^{(\mu + \sigma^2/2)t}$.