Analysis of Dependence of Options Prices on Model Parameters

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1 A Useful Identity for Black-Scholes Option Prices

Before we begin analysis parameters of option, we derive some useful identity for Black-Scholes equation. Note from this we denote V^{ec} as European option call and V^{ep} as European option put :

$$V^{ec}(S,t) = Se^{-q(T-t)}N(d_1) - Ee^{-r(T-t)}N(d2) \tag{1}$$

$$V^{ep}(S,t) = Ee^{-r(T-t)}N(-d2) - Se^{-q(T-t)}N(-d1) \tag{2}$$

where

$$d_1 = \frac{ln\frac{S}{E} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t} \tag{3}$$

We begin with computing the difference $(d_1^2 - d_2^2)/2$. Since $d_2 = d_1 - \sigma\sqrt{T-t}$ we obtain

$$\frac{d_1^2 - d_2^2}{2} = \frac{(d_1 + d_2)(d_1 - d_2)}{2} = \frac{2ln\frac{S}{E} + 2(r-q)(T-t)\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}2} \tag{4}$$

$$= ln\frac{S}{E} + (r-q)(T-t) \tag{5}$$

and hence

$$\frac{d_1^2}{2} = \frac{d_2^2}{2} + ln\frac{S}{E} + (r-q)(T-t) \tag{6}$$

For derivative of cumulative distribution function $N'(d)$ of the standardized normal distribution we have

$$N'(d) = \frac{1}{\sqrt{2\pi}}exp(-d^2/2) \tag{7}$$

Using the above identity for the difference $(d_1^2 - d_2^2)$ we finally obtain an important identity:

$$Se^{-q(T-t)}N'(d_1) - Ee^{-r(T-t)}N'(d_2) = 0 \tag{8}$$

2 Delta of an Option

The basic sensitivity factor, which is often evaluated when analyzing the market data, is dependece of a change of derivative price with respect to a change of the price with respect to a change of the price og the underlying asset stock. In the infinitesimal form this factor can be written as partial derivative:

$$\Delta = \frac{\partial V}{\partial S} \tag{9}$$

For European call and put options we are able to derive an explicit formulae for the factor Δ. We differentiate the functions V^{ec} and V^{ep} with respect to S. For a call option, it the relationship $\partial d_1/\partial S = \partial d_2/\partial S$ and using the identity (8) we got:

$$\Delta^{ec} = \frac{\partial V^{ec}}{\partial S} = Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} \tag{10}$$
$$+ e^{-q(T-t)}N(d_1) = e^{-q(T-t)N(d_1)} \tag{11}$$

for a European put we obtain:

$$\Delta^{ep} = \frac{\partial V^{ec}}{\partial S} = Se^{-q(T-t)}N'(-d_1)\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(-d_2)\frac{\partial d_2}{\partial S} \tag{12}$$
$$- e^{-q(T-t)}N(-d_1) = -e^{-q(T-t)}N(-d_1) \tag{13}$$

3 Gamma of an Option

If we could consider option as a particle which move in space (in this context every point in space associate with stock price, or maybe we can say as (Price Space)[1]) we can say that Δ as velocity. And we define another factor called Gamma Γ defined by

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2} \tag{14}$$

From eq.(14) we can say that Γ is a acceleration of option. It represent sensitivity of the price changes of option. Similarly as for the factor Δ, we are able to derive an explicit formula for the factor Γ for European call and put options. Differentiating formula Δ respect to S we obtain

$$\Gamma^{ec} = \Gamma^{ep} = \frac{\partial \Delta^{ec}}{\partial S} = e^{-q(T-t)N'(d_1)\frac{\partial d_1}{\partial S}} \tag{15}$$
$$= e^{-q(T-t)}\frac{exp(-\frac{1}{2}d_1^2)}{\sigma\sqrt{2\pi(T-t)}S} \tag{16}$$

4 Rho of an Option

Factor ρ shows the sensitivity of the derivative price with respect to a change of interest rate of riskless bond $r > 0$. Factor ρ is therfore given as a derivative

$$\rho = \frac{\partial V}{\partial r} \tag{17}$$

[1]This concept it just intuitevly by me, is interesting though, to analyze deep down of algebra or calculus property of this concept

An analytical expression of the factor ρ can be obtained by differentiating the explicit formula for prices of European call and put option. Using the eq.(8) and the fact that $d_2 = d_1 - \sigma\sqrt{T-t}$ we obtain:

$$\rho^{ec} = \frac{\partial V^{ec}}{\partial r} = Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial r} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r} \tag{18}$$
$$+ E(T-t)e^{-r(T-t)}N(d_2) = E(T-t)e^{-r(T-t)}N(d_2) \tag{19}$$

$$\rho^{ep} = \frac{\partial V^{ep}}{\partial r} = -Ee^{-q(T-t)}N'(-d_2)\frac{\partial d_2}{\partial r} + Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial r} \tag{20}$$
$$- E(T-t)e^{-r(T-t)}N(-d_2) = E(T-t)e^{-r(T-t)}N(-d_2) \tag{21}$$

5 Theta of Option

The factor Θ represent sensitivity of a derivative price with respect to the expiration time of the derivative T and it is defined as the derivative. The factor Θ can be expreseed as

$$\Theta = \frac{\partial V}{\partial t} \tag{22}$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$, we obtain

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}} \tag{23}$$

After some rearrangements, we finally obtain:

$$\Theta^{ec} = \frac{\partial V^{ec}}{\partial t} = Sqe^{q(T-t)}N(d_1) - Ere^{-r(T-t)}N(d_2) \tag{24}$$
$$- \frac{E\sigma}{2\sqrt{T-t}}e^{-r(T-t)}N'(d_2) \tag{25}$$

$$\Theta^{ep} = \frac{\partial V^{ep}}{\partial t} = Ere^{q(T-t)}N(-d_2) - Sqe^{-q(T-t)}N(-d_1) \tag{26}$$
$$- \frac{E\sigma}{2\sqrt{T-t}}e^{-r(T-t)}N'(-d_2) \tag{27}$$

6 Vega of Option

The factor Vega γ [2] describes sensitivity of a derivative secuirty with respect to change if the volatility og the underlying asset price and hence it is defined as derivative

$$\gamma = \frac{\partial \gamma}{\partial \sigma} \tag{28}$$

Thus for vega we've got:

$$\gamma^{ec} = \gamma^{ep} = Ee^{-r(T-t)}N'(d_2)\sqrt{T-t} \tag{29}$$

[2]Vega is not Greek letter