European Options

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1 Pricing Plain Vanilla Call and Put Options

With regard to the Black-Scholes model for pricing derivative securies, the partial differential equation describing the evolution of the price of an an option written on a stock paying continous dividengs has the following form

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0 \\ &V(S,T) = \bar{V}(S), S > 0, t \in [0,T] \end{aligned} \tag{1}$$

The meaning of respective variables is the following $V = V(S,t)$ is the price of a European option written on the underlying asset having $S > 0$ its present spot price at a time $t \in [0,T]$ and $T > 0$ is the expiration time of the option. The remaining model parameters are $\sigma > 0$ is volatily of the stock, i.e, the standard deviation of the estochastic time evolution of the underlying asset, $r > 0$ is an interest rate continuosly compounded riskless zero coupon bond and q is continous annualized dividend yield paid by the stock.

Finally, we recall that, in the case of Europena call option, the terminal condition $\bar{V}(S) = V(S,T)$ at the expiration time is given by the function

$$\bar{V}(S) = (S-E)^+ = \begin{cases} S-E, & \text{for } S \geq E \\ 0, & \text{for } 0 < S < E \end{cases} \tag{2}$$

where E is the expiration (strike) price at which the option contract is signed. in the case of a put option, the terminal condition teads as follows:

$$\bar{V}(S) = (E-S)^+ = \begin{cases} E-S, & \text{for } 0 < S \leq E \\ 0, & \text{for } E < S \end{cases} \tag{3}$$

The main idea of construction of an explicit solution to equation 1 with given terminal condition consists in sequence of transformations of this equation into basic form of parabolic equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in (-\infty,\infty) \times [0,T] \tag{4}$$

with the prescribed the initial condition.

1 Transformation of time. We transform the time $t \in [0,T]$ such that it flows just in opposite direction, i.e., from the expiration time T to the initial time $t = 0$. To this end, we introduce a new variable $\tau = T - t$ and set

$$W(S,\tau) = V(S,T-\tau), \qquad \text{and so} \qquad V(S,t) = W(S,T-t) \tag{5}$$

Using the relation $dt = -d\tau$ equation (1) is transformed into:

$$\begin{aligned} &\frac{\partial W}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} - (r-q)S\frac{\partial W}{\partial S} + rW = 0 \\ &W(S,0) = \bar{V}(S), \quad S > 0, \tau \in [0,T] \end{aligned} \tag{6}$$

2 The logarithmic transformations of the underling stock price. It consist in the substitution $S = e^x$, $x = lnS$ and introducing a new function

$$Z(x,\tau) = W(e^x,\tau), \quad \text{and so } W(S,\tau) = Z(\ln S, \tau) \tag{7}$$

Notice that $S \in (0,\infty)$ if and only if $x \in (-\infty,\infty)$. Using the chain rule for the differentiation we obtain

$$\frac{\partial Z}{\partial x} = S\frac{\partial W}{\partial S}, \quad \frac{\partial^2 Z}{\partial x^2} = S^2\frac{\partial^2 W}{\partial S^2} + S\frac{\partial W}{\partial S} = S^2\frac{\partial^2 W}{\partial S^2} + \frac{\partial Z}{\partial x} \tag{8}$$

Equation (6) can be then rewritten in the form :

$$\begin{aligned} &\tfrac{\partial Z}{\partial \tau} - \tfrac{1}{2}\sigma^2\tfrac{\partial^2 Z}{\partial x^2} + \left(\tfrac{\sigma^2}{2} - r + q\right)\tfrac{\partial Z}{\partial x} + rZ = 0 \\ &Z(x,0) = \bar{V}(e^x), \quad -\infty < x < \infty, \tau \in [0,T] \end{aligned} \tag{9}$$

3 Transformations into the basic parabolic partial differential equation. Terms containing the lower order derivatives Z and $\frac{\partial Z}{\partial x}$ can be eliminated by an exponentail transformations

$$u(x,\tau) = e^{\alpha x + \beta\tau}Z(x,\tau), \quad \text{i.e.} \quad Z(x,\tau) = e^{-\alpha x - \beta\tau}u(x,\tau) \tag{10}$$

where constants α, β will be specified later. We obtain

$$\begin{aligned} \frac{\partial Z}{\partial x} &= e^{-\alpha x - \beta\tau}\left(\frac{\partial u}{\partial x} - \alpha u\right) \\ \frac{\partial^2 Z}{\partial x^2} &= e^{-\alpha x - \beta\tau}\left(\frac{\partial^2 u}{\partial x^2} - 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u\right) \\ \frac{\partial Z}{\partial \tau} &= e^{-\alpha x - \beta\tau}\left(\frac{\partial u}{\partial \tau} - \beta u\right) \end{aligned} \tag{11}$$

For the new tra function u we may therefore conclude that it is a solution to the partial differential equation

$$\begin{aligned} &\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu = 0 \\ &u(x,0) = e^{\alpha x \bar{V}(e^x)} \end{aligned} \tag{12}$$

For the new transformed function u we may therefore conclude that it is a solution to the partial differential equation

$$\begin{aligned} &\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu = 0 \\ &u(x,0) = e^{\alpha x \bar{V}(e^x)} \end{aligned} \tag{13}$$

where the coefficients A, B satisfy

$$A = \alpha\sigma^2 + \frac{\sigma^2}{2} - r + q, \qquad \text{a } B = (1+\alpha)r - \beta - \alpha q - \frac{\alpha^2\sigma^2 + \alpha\sigma^2}{2} \tag{14}$$

By simple algebraic computation, we find that constants α, β can be chosen in such way that the terms A, B are vanishing. Indeed

$$\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{r+q}{2} + \frac{\sigma^2}{8} + \frac{(r-q)^2}{2\sigma^2} \tag{15}$$

With this choice of coefficients α, β the resulting equation for the function u has the form

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0 \tag{16}$$

$$u(x,0) = e^{\alpha x}\bar{V}\left(e^x\right), \quad -\infty < x < \infty, \tau \in [0,T] \tag{17}$$

4 Applying the Green formula for a solution to the heat equation. The explicit solution for equation (17) has the form of an integral:

$$u(x,\tau) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2\tau}} u(s,0)ds \tag{18}$$

Now, by a sequence of backward substituions $u \mapsto Z \mapsto W \mapsto V$, we finally obtain:

$$V(S, T-\tau) = e^{-\beta\tau} e^{-\alpha \ln S} u(\ln S, \tau) \tag{19}$$

and hence

$$V(S, T-\tau) = \frac{e^{-\beta\tau}}{\sqrt{2\sigma^2\pi\tau}} S^{-\alpha} \int_{-\infty}^{\infty} e^{-\frac{(\ln S - s)^2}{2\sigma^2\tau}} e^{\alpha s} \bar{V}\left(e^s\right) ds \tag{20}$$

For thee European call option we have $\bar{V}(S) = (S-E)^+$ and so the realtion can be further simplified as follows :

$$V(S, T-\tau) = \frac{e^{-\beta\tau}}{\sqrt{2\sigma^2\pi\tau}} S^{-\alpha} \int_{lnE}^{\infty} e^{-\frac{(\ln S - s)^2}{2\sigma^2\tau}} e^{\alpha s} \left(e^s - E\right) ds \tag{21}$$

The substituions $y = s - lnS$ leads to :

$$V(S, T-\tau) = \frac{e^{-\beta\tau}}{\sqrt{2\sigma^2\pi\tau}} S^{-\alpha} \int_{-ln\frac{S}{E}}^{\infty} e^{-\frac{(y)^2}{2\sigma^2\tau}} \left(Se^{(1+\alpha)y} - Ee^{\alpha y}\right) ds \tag{22}$$

A practical computation using above formula requires rewritting the price V into a form containing elementary or special functions. Recall that the cumulative distribution function $N(X)$ and the error function $erf(x)$ of the normal distribution are defined by means of the Euler integral as follows:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-\frac{\xi^2}{2}d\xi}, \quad \frac{1 - erf(x)}{2} = \frac{1}{\pi} \int_{x}^{\infty} e^{\xi^2} d\xi \tag{23}$$

The following identities are useful:

$$erf(-x) = -erf(x) \quad \frac{1}{2}\left(1 + erf\left(\frac{x}{\sqrt{2}}\right)\right) = \frac{1}{\sqrt{2}}\int_{\infty}^{x} e^{-\xi^2/2} d\xi = N(x) \tag{24}$$

for each $x \in \mathbb{R}$. Let us consider the integral

$$I_1 = \frac{e^{\beta\tau}}{\sqrt{2\sigma^2\tau}}\frac{1}{\sqrt{\pi}}\int_{-ln\frac{S}{E}}^{\infty} e^{-\frac{y^2}{2\sigma^2\tau}+(1+\alpha)y} dy \tag{25}$$

By introducing another substitution $\xi = \frac{y}{\sqrt{2\sigma^2\tau}} - \frac{1+\alpha}{2}\sqrt{2\sigma^2\tau}$ and using realtion (15) we obtain

$$-\frac{y^2}{2\sigma^2\tau} + (1+\alpha)y = -xi^2 + (1+\alpha)^2\frac{\sigma^2\tau}{2} = -\xi^2 + (\beta - q)\tau \tag{26}$$

Hence

$$\begin{aligned} I_1 &= e^{-q\tau}\frac{1}{\sqrt{\pi}}\int_{-\frac{1+\alpha}{2}\sqrt{2\sigma^2\tau}-\frac{ln\frac{S}{E}}{\sqrt{2*\sigma^2\tau}}}^{\infty} e^{-xi^2} d\xi & (27)\\ &= e^{-q\tau}/2\left[1 - erf\left(-\frac{1+\alpha}{2}\sqrt{2\sigma^2\tau} - \frac{ln\frac{S}{E}}{\sqrt{2\sigma^2\tau}}\right)\right] & (28)\\ &= e^{-q\tau}/2\left[1 + erf\left(\frac{1}{\sqrt{2}}\frac{(r - q + \frac{\sigma^2}{2})\tau + ln\frac{S}{E}}{\sigma\sqrt{\tau}}\right)\right] & (29)\\ & & (30) \end{aligned}$$

By introducing another substituion $\xi = \frac{y}{\sqrt{2\sigma^2\tau}} - \frac{\alpha}{\sigma 2\sigma^2\tau}$ we can compute the integral

$$I_2 = \frac{e^{-r\tau}}{2}\left[1 + erf\left(\frac{1}{\sqrt{2}}\frac{(r - q - \frac{\sigma^2}{2})\tau + ln\frac{S}{E}}{\sigma\sqrt{\tau}}\right)\right] \tag{31}$$

Substituting above results for the integrals I_1 and I_2 we can finally get the formula for the **Call European Options** $V(S, 0)$:

$$\begin{aligned} V(S, T-\tau) = &\frac{Se^{-q\tau}}{2}\left[1 + erf\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{(r - q + \frac{\sigma^2}{2})\tau + ln\frac{S}{E}}{\sigma\sqrt{\tau}}\right)\right] & (32)\\ &- \frac{Ee^{-r\tau}}{2}\left[1 + erf\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{(r - q - \frac{\sigma^2}{2})\tau + ln\frac{S}{E}}{\sigma\sqrt{\tau}}\right)\right] & (33) \end{aligned}$$

Using the relations between functions $N(x)$ and $erf(x)$ we finally conclude

$$V(S, t) = Se^{-q(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2) \tag{34}$$

where

$$d_1 = \frac{(r - q + \frac{\sigma^2}{2})(T-t) + ln\frac{S}{E}}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t} \tag{35}$$

Expression (34) is called the Black-Scholers formula for prcing European call options.