

European Options

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1 Pricing Plain Vanilla Call and Put Options

With regard to the Black-Scholes model for pricing derivative securities, the partial differential equation describing the evolution of the price of an option written on a stock paying continuous dividends has the following form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0 \quad (1)$$
$$V(S, T) = \bar{V}(S), S > 0, t \in [0, T]$$

The meaning of respective variables is the following $V = V(S, t)$ is the price of a European option written on the underlying asset having $S > 0$ its present spot price at a time $t \in [0, T]$ and $T > 0$ is the expiration time of the option. The remaining model parameters are $\sigma > 0$ is volatility of the stock, i.e., the standard deviation of the stochastic time evolution of the underlying asset, $r > 0$ is an interest rate continuously compounded riskless zero coupon bond and q is continuous annualized dividend yield paid by the stock.

Finally, we recall that, in the case of European call option, the terminal condition $\bar{V}(S) = V(S, T)$ at the expiration time is given by the function

$$\bar{V}(S) = (S - E)^+ = \begin{cases} S - E, & \text{for } S \geq E \\ 0, & \text{for } 0 < S < E \end{cases} \quad (2)$$

where E is the expiration (strike) price at which the option contract is signed. in the case of a put option, the terminal condition reads as follows:

$$\bar{V}(S) = (E - S)^+ = \begin{cases} E - S, & \text{for } 0 < S \leq E \\ 0, & \text{for } E < S \end{cases} \quad (3)$$

The main idea of construction of an explicit solution to equation ?? with given terminal condition consists in sequence of transformations of this equation into basic form of parabolic equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (x, t) \in (-\infty, \infty) \times [0, T] \quad (4)$$

with the prescribed the initial condition.

- 1 Transformation of time. We transform the time $t \in [0, T]$ such that it flows just in opposite direction, i.e., from the expiration time T to the initial time $t = 0$. To this end, we introduce a new variable $\tau = T - t$ and set

$$W(S, \tau) = V(S, T - \tau), \quad \text{and so} \quad V(S, t) = W(S, T - t) \quad (5)$$

Using the relation $dt = -d\tau$ equation (??) is transformed into:

$$\begin{aligned} \frac{\partial W}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} - (r - q)S \frac{\partial W}{\partial S} + rW &= 0 \\ W(S, 0) &= \bar{V}(S), \quad S > 0, \tau \in [0, T] \end{aligned} \quad (6)$$

- 2 The logarithmic transformations of the underling stock price. It consist in the substitution $S = e^x$, $x = \ln S$ and introducing a new function

$$Z(x, \tau) = W(e^x, \tau), \quad \text{and so } W(S, \tau) = Z(\ln S, \tau) \quad (7)$$

Notice that $S \in (0, \infty)$ if and only if $x \in (-\infty, \infty)$. Using the chain rule for the differentiation we obtain

$$\frac{\partial Z}{\partial x} = S \frac{\partial W}{\partial S}, \quad \frac{\partial^2 Z}{\partial x^2} = S^2 \frac{\partial^2 W}{\partial S^2} + S \frac{\partial W}{\partial S} = S^2 \frac{\partial^2 W}{\partial S^2} + \frac{\partial Z}{\partial x} \quad (8)$$

Equation (??) can be then rewritten in the form :

$$\begin{aligned} \frac{\partial Z}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 Z}{\partial x^2} + \left(\frac{\sigma^2}{2} - r + q \right) \frac{\partial Z}{\partial x} + rZ &= 0 \\ Z(x, 0) &= \bar{V}(e^x), \quad -\infty < x < \infty, \tau \in [0, T] \end{aligned} \quad (9)$$

- 3 Transformations into the basic parabolic partial differential equation. Terms containing the lower order derivatives Z and $\frac{\partial Z}{\partial x}$ can be eliminated by an exponentail transformations

$$u(x, \tau) = e^{\alpha x + \beta \tau} Z(x, \tau), \quad \text{i.e.} \quad Z(x, \tau) = e^{-\alpha x - \beta \tau} u(x, \tau) \quad (10)$$

where constants α, β will be specified later. We obtain

$$\begin{aligned} \frac{\partial Z}{\partial x} &= e^{-\alpha x - \beta \tau} \left(\frac{\partial u}{\partial x} - \alpha u \right) \\ \frac{\partial^2 Z}{\partial x^2} &= e^{-\alpha x - \beta \tau} \left(\frac{\partial^2 u}{\partial x^2} - 2\alpha \frac{\partial u}{\partial x} + \alpha^2 u \right) \\ \frac{\partial Z}{\partial \tau} &= e^{-\alpha x - \beta \tau} \left(\frac{\partial u}{\partial \tau} - \beta u \right) \end{aligned} \quad (11)$$

For the new tra function u we may therefore conclude that it is a solution to the partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + A \frac{\partial u}{\partial x} + Bu &= 0 \\ u(x, 0) &= e^{\alpha x \bar{V}(e^x)} \end{aligned} \quad (12)$$

For the new transformed function u we may therefore conclude that it is a solution to the partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + A \frac{\partial u}{\partial x} + Bu &= 0 \\ u(x, 0) &= e^{\alpha x \bar{V}(e^x)} \end{aligned} \quad (13)$$

where the coefficients A, B satisfy

$$A = \alpha\sigma^2 + \frac{\sigma^2}{2} - r + q, \quad a \quad B = (1 + \alpha)r - \beta - \alpha q - \frac{\alpha^2\sigma^2 + \alpha\sigma^2}{2} \quad (14)$$

By simple algebraic computation, we find that constants α, β can be chosen in such way that the terms A, B are vanishing. Indeed

$$\alpha = \frac{r - q}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{r + q}{2} + \frac{\sigma^2}{8} + \frac{(r - q)^2}{2\sigma^2} \quad (15)$$

With this choice of coefficients α, β the resulting equation for the function u has the form

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0 \quad (16)$$

$$u(x, 0) = e^{\alpha x} \bar{V}(e^x), \quad -\infty < x < \infty, \tau \in [0, T] \quad (17)$$

4 Applying the Green formula for a solution to the heat equation. The explicit solution for equation (??) has the form of an integral:

$$u(x, \tau) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2\tau}} u(s, 0) ds \quad (18)$$

Now, by a sequence of backward substitutions $u \mapsto Z \mapsto W \mapsto V$, we finally obtain:

$$V(S, T - \tau) = e^{-\beta\tau} e^{-\alpha \ln S} u(\ln S, \tau) \quad (19)$$

and hence

$$V(S, T - \tau) = \frac{e^{-\beta\tau}}{\sqrt{2\sigma^2\pi\tau}} S^{-\alpha} \int_{-\infty}^{\infty} e^{-\frac{(\ln S - s)^2}{2\sigma^2\tau}} e^{\alpha s} \bar{V}(e^s) ds \quad (20)$$

For the European call option we have $\bar{V}(S) = (S - E)^+$ and so the relation can be further simplified as follows :

$$V(S, T - \tau) = \frac{e^{-\beta\tau}}{\sqrt{2\sigma^2\pi\tau}} S^{-\alpha} \int_{\ln E}^{\infty} e^{-\frac{(\ln S - s)^2}{2\sigma^2\tau}} e^{\alpha s} (e^s - E) ds \quad (21)$$

The substitutions $y = s - \ln S$ leads to :

$$V(S, T - \tau) = \frac{e^{-\beta\tau}}{\sqrt{2\sigma^2\pi\tau}} S^{-\alpha} \int_{-\ln \frac{S}{E}}^{\infty} e^{-\frac{y^2}{2\sigma^2\tau}} \left(S e^{(1+\alpha)y} - E e^{\alpha y} \right) dy \quad (22)$$

A practical computation using above formula requires rewriting the price V into a form containing elementary or special functions. Recall that the cumulative distribution function $N(X)$ and the error function $erf(x)$ of the normal distribution are defined by means of the Euler integral as follows:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{\xi^2}{2}} d\xi, \quad \frac{1 - erf(x)}{2} = \frac{1}{\pi} \int_x^{\infty} e^{-\xi^2} d\xi \quad (23)$$

The following identities are useful:

$$\operatorname{erf}(-x) = -\operatorname{erf}(x) \quad \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right) \right) = \frac{1}{\sqrt{2}} \int_{-\infty}^x e^{-\xi^2/2} d\xi = N(x) \quad (24)$$

for each $x \in \mathbb{R}$. Let us consider the integral

$$I_1 = \frac{e^{\beta\tau}}{\sqrt{2\sigma^2\tau}} \frac{1}{\sqrt{\pi}} \int_{-\ln \frac{S}{E}}^{\infty} e^{-\frac{y^2}{2\sigma^2\tau} + (1+\alpha)y} dy \quad (25)$$

By introducing another substitution $\xi = \frac{y}{\sqrt{2\sigma^2\tau}} - \frac{1+\alpha}{2}\sqrt{2\sigma^2\tau}$ and using relation (??) we obtain

$$-\frac{y^2}{2\sigma^2\tau} + (1+\alpha)y = -xi^2 + (1+\alpha)^2 \frac{\sigma^2\tau}{2} = -\xi^2 + (\beta - q)\tau \quad (26)$$

Hence

$$I_1 = e^{-q\tau} \frac{1}{\sqrt{\pi}} \int_{-\frac{1+\alpha}{2}\sqrt{2\sigma^2\tau} - \frac{\ln \frac{S}{E}}{\sqrt{2\sigma^2\tau}}}^{\infty} e^{-xi^2} d\xi \quad (27)$$

$$= e^{-q\tau} / 2 \left[1 - \operatorname{erf} \left(-\frac{1+\alpha}{2}\sqrt{2\sigma^2\tau} - \frac{\ln \frac{S}{E}}{\sqrt{2\sigma^2\tau}} \right) \right] \quad (28)$$

$$= e^{-q\tau} / 2 \left[1 + \operatorname{erf} \left(\frac{1}{\sqrt{2}} \frac{(r - q + \frac{\sigma^2}{2})\tau + \ln \frac{S}{E}}{\sigma\sqrt{\tau}} \right) \right] \quad (29)$$

$$(30)$$

By introducing another substitution $\xi = \frac{y}{\sqrt{2\sigma^2\tau}} - \frac{\alpha}{2\sigma^2\tau}$ we can compute the integral

$$I_2 = \frac{e^{-r\tau}}{2} \left[1 + \operatorname{erf} \left(\frac{1}{\sqrt{2}} \frac{(r - q - \frac{\sigma^2}{2})\tau + \ln \frac{S}{E}}{\sigma\sqrt{\tau}} \right) \right] \quad (31)$$

Substituting above results for the integrals I_1 and I_2 we can finally get the formula for the **Call European Options** $V(S, 0)$:

$$V(S, T - \tau) = \frac{Se^{-q\tau}}{2} \left[1 + \operatorname{erf} \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{(r - q + \frac{\sigma^2}{2})\tau + \ln \frac{S}{E}}{\sigma\sqrt{\tau}} \right) \right] \quad (32)$$

$$- \frac{Ee^{-r\tau}}{2} \left[1 + \operatorname{erf} \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{(r - q - \frac{\sigma^2}{2})\tau + \ln \frac{S}{E}}{\sigma\sqrt{\tau}} \right) \right] \quad (33)$$

Using the relations between functions $N(x)$ and $\operatorname{erf}(x)$ we finally conclude

$$V(S, t) = Se^{-q(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2) \quad (34)$$

where

$$d_1 = \frac{(r - q + \frac{\sigma^2}{2})(T - t) + \ln \frac{S}{E}}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t} \quad (35)$$

Expression (??) is called the Black-Scholers formula for pricing European call options. For the pricing European put options

$$V(S, t) = Ee^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(d_1) \quad (36)$$