My notes while reading:
An Introduction To Computational Learning Theory

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Contents

Chapter 1

The Probably Approximately Correct Learning Model

1.1 A Rectangle Learning Game

The objective of this game is to learn an unknown *target* (axis-aligned) rectangle $\mathcal{R} = [a, b] \times [c, d] \subset \mathbb{R}^2$.

The player can gain information about $\mathcal{R}$ only by chosing random points according to some distribution $\mathcal{D}$, and asking the game whether they are inside $\mathcal{R}$. By convention, points inside $\mathcal{R}$ are considered positive.

Figure 1.1 shows an example of a possible target rectangle $\mathcal{R}$ along with some points labeled using it.

The player's goal is to find a hypothesis rectangle $\mathcal{R}'$ that "approximates" $\mathcal{R}$ "as closely as possible". To measure the quality of this approximation, we will consider the region $\mathcal{R}\Delta\mathcal{R}'$ of points that $\mathcal{R}$ and $\mathcal{R}'$ label differntly. More precisely, we will consider the probability $\mathbb{P}(\mathcal{R}\Delta\mathcal{R}')$ of falling with this region according to $\mathcal{D}$ and try to minimize this quantity.

Since $\mathcal{R}$ is unknown in practice, $\mathcal{R}'$ is evaluated by sampling a number of points from $\mathcal{D}$, labeling them using $\mathcal{R}'$ and comparing to their true labels by asking the game. We then take the number of falsely labeled points devided by the total of chosen points as an estimation of $\mathbb{P}(\mathcal{R}\Delta\mathcal{R}')$. Note that this score is the same as $1 - \text{accuracy}$.

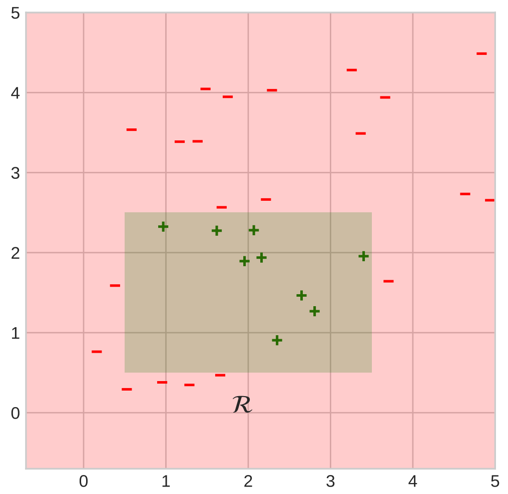

Figure 1.1: The target rectangle $\mathcal{R}$ along with a labeled sample of points

A simple strategy for the player is to sample a sufficiently large number m of points form $\mathcal{D}$ and request their labels, then chose as hypothesis $\mathcal{R}'$ the smallest rectangle containing all positive examples and no negative ones[1]. If all m points are negative, we chose $\mathcal{R}' = \emptyset$. Figure 1.2 ilustrates this for one example.

We will now prove that this strategy works, more specifically, we will show the following theorem:

Theorem 1.1.
Let $\mathcal{D}$ be a distribution over $\mathbb{R}^2$, $\mathcal{R}$ a rectangle, and $\varepsilon, \delta > 0$ be positive real numbers. There exisits an integer $m \in \mathbb{N}$, such that the hypothesis rectangle $\mathcal{R}'$ generated by m sampled points has with probability $\geq 1 - \delta$ an error $\leq \varepsilon$.

Proof.
Consider a distribution $\mathcal{D}$, a rectangle $\mathcal{R} = [a, b] \times [c, d]$, a hypothesis rectangle $\mathcal{R}' = [a', b'] \times [c', d']$ constructed using the startegy we described, and positive real numbers $\varepsilon, \delta > 0$.

Note that $\mathcal{R}'$ cannot have false positives. In fact, $\mathcal{R}' \subset \mathcal{R}$ and therefore

[1]Note that such at least one rectangle with this property is garenteed to exist, because $\mathcal{R}$ is one such a rectangle

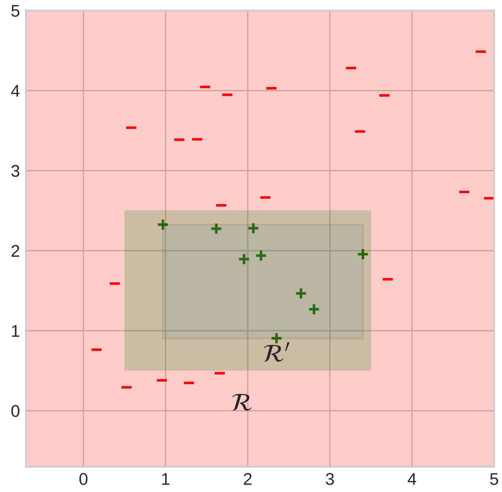

Figure 1.2: The tightest fit rectangle for the example of Figure 1.1

$\mathcal{R}' \,\Delta \mathcal{R} = \mathcal{R} \backslash \mathcal{R}'$. This region can be expressed as the union of four rectangles:

$$\begin{aligned} \mathcal{R} \backslash \mathcal{R}' = \quad & [a, a'] \times [c', d'] & \cup \\ & [a, b] \times [c, c'] & \cup \\ & [b', b] \times [c', d'] & \cup \\ & [a, b] \times [d, d'] & \end{aligned}$$

We denote the union of four rectangles respectively by L, B, R, T. It immediately follows that:

$$\mathbb{P}(\mathcal{R} \backslash \mathcal{R}') = \mathbb{P}(L \cup B \cup R \cup T) \leq \mathbb{P}(L) + \mathbb{P}(B) + \mathbb{P}(R) + \mathbb{P}(T)$$

Consequently, to show that $\mathbb{P}(\mathcal{R} \backslash \mathcal{R}') \leq \varepsilon$, it suffices to show that $\mathbb{P}(X) \leq \frac{\varepsilon}{4}$ for all $X \in \{L, B, R, T\}$.

$\square$