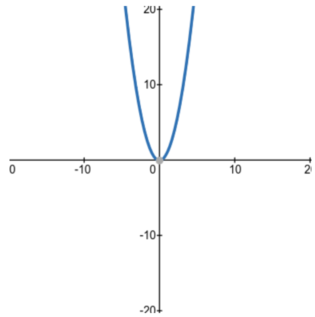
LIMITS AND DERIVATIVES

This chapter is an introduction to Calculus. Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change.

Limit of a function

Consider the function $f(x) = x^2$. Observe that as x takes values very close to 0, the value of $f(x)$ also moves towards 0



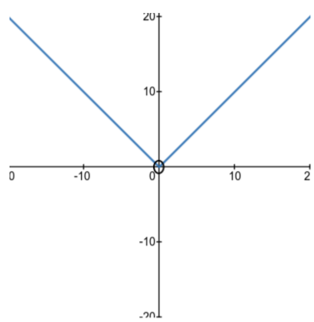
We say

$$lim_{x \to 0} f(x) = 0$$

(to be read as limit of f (x) as x tends to zero equals zero). In general as $x \to a$, $f(x) \to l$, then l is called **limit of the function** $f(x)$ which is symbolically written as

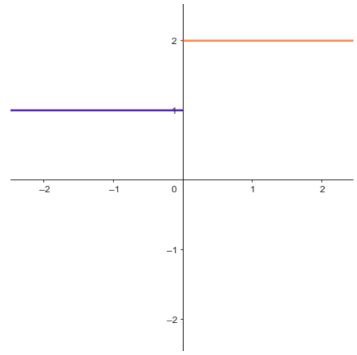
$$lim_{x \to a} f(x) = l$$

Consider the following function $g(x) = |x|, x \neq 0$. Observe that $g(0)$ is not defined. Computing the value of$g(x)$ for values of x very near to 0, we see that the value of $g(x)$ moves towards 0.So $lim_{x \to 0} g(x) = 0$



- $\lim_{x \to a^-} f(x) = A$ Read as **left limit** of $f(x)$ is 'A', means that $f(x) \to A$ as $x \to a-$. To evaluate the left limit we use the following substitution $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$
- $\lim_{x \to a^+} f(x) = B$ Read as **right limit** of $f(x)$ is 'B', means that $f(x) \to B$ as $x \to a+$. To evaluate the left limit we use the following substitution $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$

If left limit and right limit of $f(x)$at $x = a$ are equal, then we say that the limit of the function $f(x)$ exists at $x = a$ and is denoted $lim_{x \to 0} f(x)$ Otherwise we say that $lim_{x \to 0} f(x)$ does not exists.



Consider the function $f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$

The left hand limit of $f(x)$ at 0 is $lim_{x \to 0-} f(x) = 1$ and the right hand limit of $f(x)$ at 0 is $; im_{x \to 0+} f(x) = 2$. So limit does not exists for this function at 0

Algebra of limits

For functions f and g

- $lim_{x \to a}(kf(x)) = k\, lim_{x \to a} f(x)$
- $lim_{x \to a}(f(x) \pm g(x)) = lim_{x \to a} f(x) \pm lim_{x \to a} g(x)$
- $lim_{x \to a}(f(x) \times g(x)) = lim_{x \to a} f(x) \times lim_{x \to a} g(x)$
- $lim_{x \to a}(\frac{f(x)}{g(x)}) = \frac{lim_{x \to a} f(x)}{lim_{x \to a} g(x)}$

Some standard results

- $lim_{x \to a} k = k$, where k is a constant .
- $lim_{x \to a} f(x) = f(a)$, if f s a polynomial function.
- For rational function of the form $\frac{0}{0}$ f possible we can factorise the numerator and denominator and then, cancel the common factors and again put x = a.
- $lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$
- Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition, For some a, if both $lim_{x \to a} f(x)$ and $lim_{x \to a} g(x)$ exist, then $lim_{x \to a} f(x) \leq lim_{x \to a} g(x)$.
- (Sandwich Theorm) Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For some real number a, if $lim_{x \to a} f(x) = l = lim_{x \to a} h(x)$, then $lim_{x \to a} g(x) = l$.
- $lim_{x \to \infty} \frac{1}{x} = 0$
- $lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- $lim_{x \to 0} \frac{log(1+x)}{x} = 1$

Limits of Trigonometric Functions

- $lim_{x\to 0}\frac{sin(x)}{x} = 1$
- $lim_{x\to 0}\frac{1-cos(x)}{x} = 0$

Derivatives

Suppose f is a real valued function and a is a point in its domain of definition. The **derivative** of f at a is defined by

$$lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided this limit exists. Derivative of $f(x)$ at a is denoted by $f'(a)$.

First principle of derivative

Suppose f is a real valued function, the function defined by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Wherever this limit exists is defined as the derivative of f at x and is denoted by $f'(x)$ or $\frac{dy}{dx}$ or y'

Algebra of Derivatives

For differentiable functions f and g

- $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$
- $\frac{d}{dx}[f(x) \times g(x)] = \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x))$
- $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{d}{dx}(f(x))-f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$

Some standard results

- $\frac{d}{dx}(k) = 0$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(sin(x)) = cos(x)$
- $\frac{d}{dx}(cos(x)) = -sin(x)$
- $\frac{d}{dx}tan(x) = sec^2(x)$
- $\frac{d}{dx}sec(x) = sec(x)tan(x)$
- $\frac{d}{dx}(cosec(x)) = -cosec(x)cot(x)$
- $\frac{d}{dx}(cot(x)) = -cosec^2(x)$