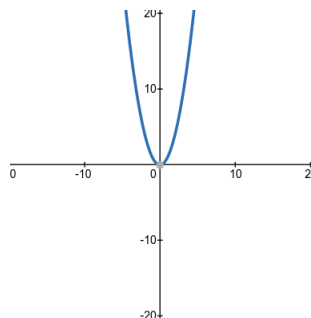


LIMITS AND DERIVATIVES

This chapter is an introduction to Calculus. Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change.

Limit of a function

Consider the function $f(x) = x^2$. Observe that as x takes values very close to 0, the value of $f(x)$ also moves towards 0



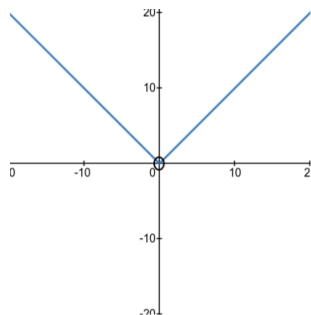
We say

$$\lim_{x \rightarrow 0} f(x) = 0$$

(to be read as limit of $f(x)$ as x tends to zero equals zero). In general as $x \rightarrow a$, $f(x) \rightarrow l$, then l is called **limit of the function** $f(x)$ which is symbolically written as

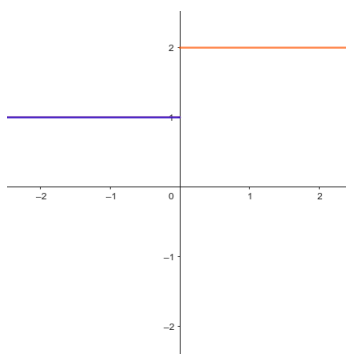
$$\lim_{x \rightarrow a} f(x) = l$$

Consider the following function $g(x) = |x|, x \neq 0$. Observe that $g(0)$ is not defined. Computing the value of $g(x)$ for values of x very near to 0, we see that the value of $g(x)$ moves towards 0. So $\lim_{x \rightarrow 0} g(x) = 0$



- $\lim_{x \rightarrow a^-} f(x) = A$ Read as **left limit** of $f(x)$ is 'A', means that $f(x) \rightarrow A$ as $x \rightarrow a^-$. To evaluate the left limit we use the following substitution $\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a - h)$
- $\lim_{x \rightarrow a^+} f(x) = B$ Read as **right limit** of $f(x)$ is 'B', means that $f(x) \rightarrow B$ as $x \rightarrow a^+$. To evaluate the right limit we use the following substitution $\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a + h)$

If left limit and right limit of $f(x)$ at $x = a$ are equal, then we say that the limit of the function $f(x)$ exists at $x = a$ and is denoted $\lim_{x \rightarrow a} f(x)$. Otherwise we say that $\lim_{x \rightarrow a} f(x)$ does not exist.



Consider the function $f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$

The left hand limit of $f(x)$ at 0 is $\lim_{x \rightarrow 0^-} f(x) = 1$ and the right hand limit of $f(x)$ at 0 is $\lim_{x \rightarrow 0^+} f(x) = 2$. So limit does not exist for this function at 0

Algebra of limits

For functions f and g

- $\lim_{x \rightarrow a} (kf(x)) = k \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x) \times g(x)) = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$

Some standard results

- $\lim_{x \rightarrow a} k = k$, where k is a constant .
- $\lim_{x \rightarrow a} f(x) = f(a)$, if f is a polynomial function.
- For rational function of the form $\frac{0}{0}$ if possible we can factorise the numerator and denominator and then, cancel the common factors and again put $x = a$.
- $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = n$
- Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition, For some a , if both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$.
- (Sandwich Theorem) Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For some real number a , if $\lim_{x \rightarrow a} f(x) = l = \lim_{x \rightarrow a} h(x)$, then $\lim_{x \rightarrow a} g(x) = l$.
- $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$
- $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

Limits of Trigonometric Functions

- $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 0$
- $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0$

Derivatives

Suppose f is a real valued function and a is a point in its domain of definition. The **derivative** of f at a is defined by

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. Derivative of $f(x)$ at a is denoted by $f'(a)$.

First principle of derivative

Suppose f is a real valued function, the function defined by

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Wherever this limit exists is defined as the derivative of f at x and is denoted by $f'(x)$ or $\frac{dy}{dx}$ or y'

Algebra of Derivatives

For differentiable functions f and g

- $\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$
- $\frac{d}{dx}[f(x) \times g(x)] = \frac{d}{dx}(f(x))g(x) + f(x) \frac{d}{dx}(g(x))$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \frac{d}{dx}(f(x)) - f(x) \frac{d}{dx}(g(x))}{(g(x))^2}$

Some standard results

- $\frac{d}{dx}(k) = 0$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(\sin(x)) = \cos(x)$
- $\frac{d}{dx}(\cos(x)) = -\sin(x)$
- $\frac{d}{dx}\tan(x) = \sec^2(x)$
- $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
- $\frac{d}{dx}(\operatorname{cosec}(x)) = -\operatorname{cosec}(x)\cot(x)$
- $\frac{d}{dx}(\cot(x)) = -\operatorname{cosec}^2(x)$