COMPLEX NUMBERS AND QUADRATIC EQUATIONS

we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution since the root of a negative number does not exist in a real number. So, we need to extend the real number system to a larger number system to accommodate such numbers.

Complex Numbers

A number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$. Usually, a complex number is denoted by z, a is the real part of z denoted by $Re(z)$ and b is the imaginary part of z denoted by $Im(z)$.Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

For example, $2 + i3, (-1) + i\sqrt{3}, 4 + i\frac{-1}{11}$ are complex numbers.

Algebra of Complex Numbers

1. **Addition of two complex numbers:**

   Let $z_1 = a+ib$ and $z_2 = c+id$ be two complex numbers. Then the sum $z_1 + z_2$ is obtained by adding the real and imaginary parts.

   For example $(2+i3)+(-6+i5) = (2-6)+i(3+5) = -4 + i8$.

   The addition of complex numbers satisfy the following properties:

   (a) $z_1 + z+_2$ is a complex number (Closure)

   (b) $z_1 + z_2 = z_2 + z_1$ (commutative)

   (c) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (associative)

   (d) $0 + i0$ is the identity element.

   (e) $-z$ is the inverse of z.

2. **Difference of two complex numbers:** Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows: $z_1 - z_2 = z_1 + (-z_2)$

3. **Multiplication of two complex numbers:** Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers. Then the product z_1z_2 is defined as follows: $z_1z_2 = (ac-bd) + i(ad + bc)$.

   For example, $(3+i5)(2+i6) = (3\times 2-5\times 6)+i(3\times 6 + 5 \times 2) = -24 + i28$

   The multiplication of complex numbers possesses the following properties

   (a) Product of two complex numbers is a complex number(closure)

   (b) $z_1z_2 = z_2z_1$ (commutative).

   (c) $z_1(z_2z_3) = (z_1z_2)z_3$ (associative).

   (d) $1 + i0$ is the identity element.

   (e) $\frac{1}{z}$ is the inverse of z.

   (f) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (distributive law)

4. **Division of two complex numbers:** Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2}$$

5. **Power of i:**In General $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i, i^{-1} = -i$

6. **The square roots of a negative real number:**

   We have

$$(\sqrt{3}i)^2 = \sqrt{3}^2 i^2 = 3 \times -1 = -3$$

$$(\sqrt{-3}i)^2 = \sqrt{-3}^2 i^2 = 3 \times -1 = -3$$

   Therefor the square roots or -3 are $\sqrt{3}i$ and $-\sqrt{3}i$. In general if a is a positive real number then $\sqrt{-a} = i\sqrt{a}$ and $-i\sqrt{a}$.

   Therefore $\sqrt{a}\times\sqrt{b} \neq ab$ if both a and b are negative real numbers.

7. **Identities:**

   - $(z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$
   - $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$
   - $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_2^2z_1 + z_2^3$
   - $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_2^2z_1 - z_2^3$

The Modulus and the Conjugate of a Complex Number:

Consider a complex number $z = a + ib$. Then, the conjugate of z is denoted by $\bar{z}$, defined as $\bar{z} = a-ib$ and the modulus of z is denoted by $|z|$, defined as $\sqrt{a^2 + b^2}$.

the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

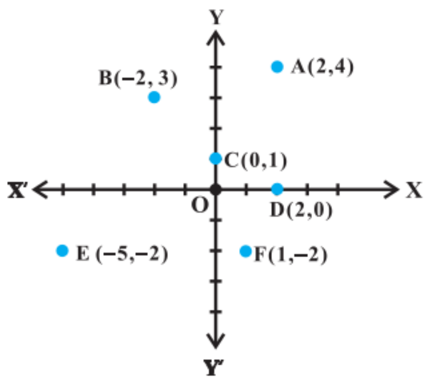
Properties:

1. $|z_1z_2| = |z_1||z_2|$
2. $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$
3. $\overline{z_1z_2} = \bar{z_1}\bar{z_2}$
4. $\overline{z_1 \pm z_2} = \bar{z_1} \pm \bar{z_2}$
5. $\overline{(\frac{z_1}{z_2})} = \frac{\bar{z_1}}{\bar{z_2}}$
6. $z\bar{z} = |z|^2$

Argand Plane and Polar Representation

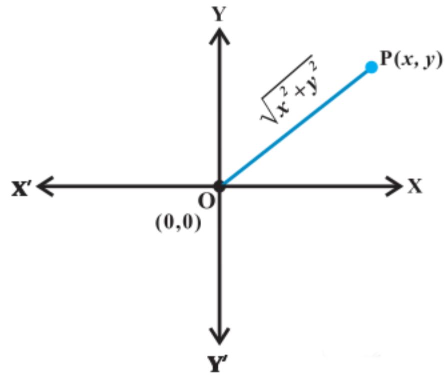
A complex number $z = a + ib$ which corresponds to the ordered pair (a, b) can be represented geometrically as the unique point $P(a, b)$ in the $XY-$plane, where the real part is taken along the $x-$axis and the imaginary part along the $y-$axis. Such a plane is called the *Argand Plane or Complex plane.*

Some complex numbers such as $2 + 4i, -2 + 3i, 0 + 1i, 2+0i, -5-2i$ and $1-2i$ which correspond to the ordered pairs $(2, 4), (-2, 3), (0, 1), (2, 0), (-5, -2)$, and $(1, -2)$, respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Figure



Let$P(x, y)$ be a point in the complex plane representing the non-zero complex number $z = x + iy$. Let θ radian be the angle made by the directed line segment OP with the positive real axis OX in the anticlockwise direction. Let $OP = r$, is known as modulus or absolute value of the complex number z denoted by $|z|$ or $mod(z)$.

The pair (r, θ) is known as polar coordinates of the point P.θ is called amplitude or argument of the complex number, denoted by $argz$ or $ampz$.



For a complex number $a + bi$

$$|z| = \sqrt{a^2 + b^2}$$

$$tan\theta = \frac{b}{a}$$

$$\theta = tan^{-1}(\frac{b}{a})$$

Shortcut method to find arg z

x	y	arg z lies in	arg z =
$+ve$	$+ve$	Quadrant I	θ
$-ve$	$+ve$	Quadrant II	$\pi - \theta$
$-ve$	$-ve$	Quadrant III	$\theta - \pi$
$+ve$	$-ve$	Quadrant IV	$-\theta$