Relations and Functions

Cartesian Products of Sets

The Cartesian product between two sets A and B is denoted by A × B is the set of all ordered pairs of elements from A and B

ie; $A \times B = \{(a, b) : a \in A, b \in B\}$

Example: consider the two sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, b_4\}$

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_1)$
$, (a_2, b_2), (a_2, b_3), (a_2, b_4)\}$.

Properties:

1. Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.
2. In general $A \times B \neq B \times A, but\, if\, A = B, A \times B = B \times A$.
3. n(A × B) = n(A) × n(B)
4. n(A × B) = n(B × A)
5. If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$. $A \times A \times A = (a, b, c) : a, b, c \in A$. Here (a, b, c) is called an ordered triplet.

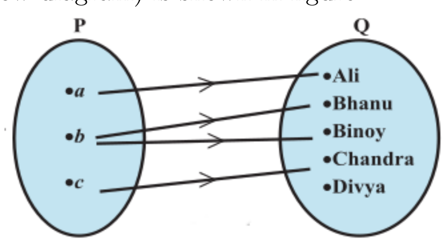
Relations

A relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product A × B.

Example: Let P = {a, b, c} and Q ={ Ali, Bhanu, Binoy, Chandra, Divya}. Then

R = {(a, Ali), (b, Bhanu), (b, Binoy), (c, Chandra)} is a relation between sets P and Q

A visual representation of this relation R (called an arrow diagram) is shown in figure



- The second element in the ordered pair is called the **image** of the first element
- The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the **domain** of the relation R.
- The set of all second elements in a relation R from a set A to a set B is called the **range** of the relation R. The whole set B is called the **codomain** of the relation R. Note that range ⊂ codomain.
- The number of relation that can be written from A to B if n(A) = p, n(B) = q is 2^{pq}.

A relation may be represented algebraically either by the **Roster method** or by the **Set-builder method**.An **arrow diagram** is a visual representation of a relation.

Functions

A relation f from A to B (f : A → B) is said to be a function if every element of set A has one and only one image in set B.

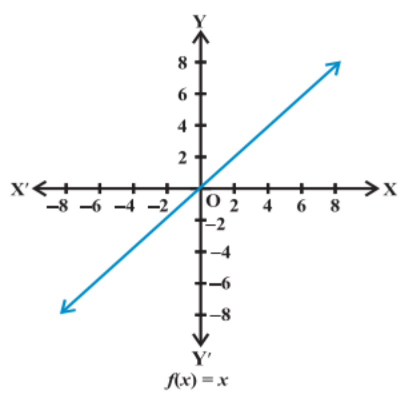
If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the **image** of a under f and a is called the **preimage** of b under b under f.

- A function which has either R or one of its subsets as its range is called a **real valued function**. Further, if its domain is also either R or a subset of R, it is called a real function.

Some functions and their graphs

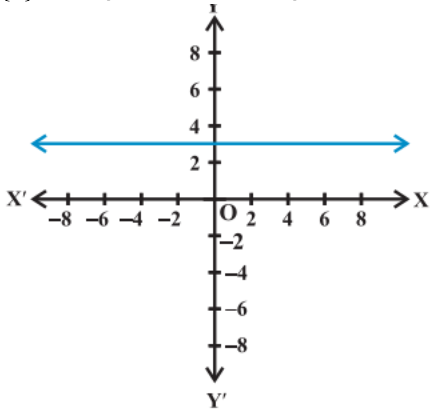
Identity function

A function f : R → R defined by f(x) = x. Here the domain and range of f are R. The graph is a straight line passing through the origin which makes 45 degrees with the positive direction of the x-axis.



Constant function

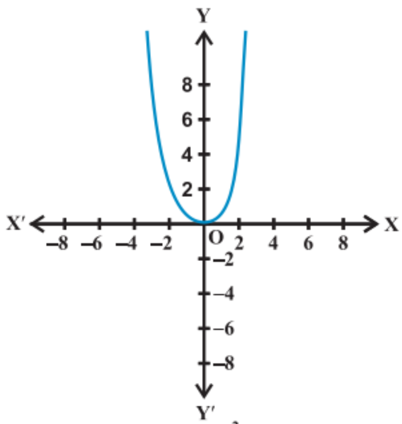
A function $f : R \to R$ defined by $f(x) = c$, where c is a constant.Here domain of f is R and its range is {c}.The graph is a straight line parallel to the x-axis.



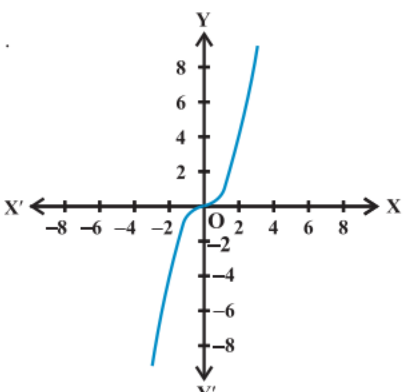
$f(x) = 3$

Polynomial Function

A function f : R → R defined by $f(x) = a_0 + a_1x + \ldots.. + a_nx$, where n is a no-negative integer and $a_0, a_1, \ldots., a_n \in R$



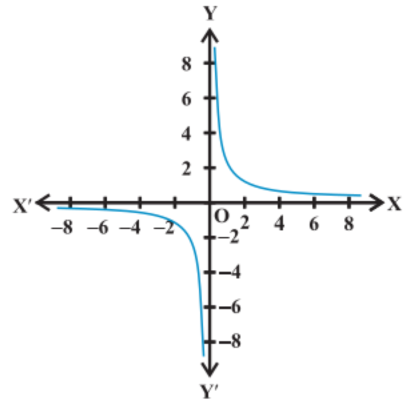
$f(x) = x^2$



$f(x) = x^3$

Rational functions

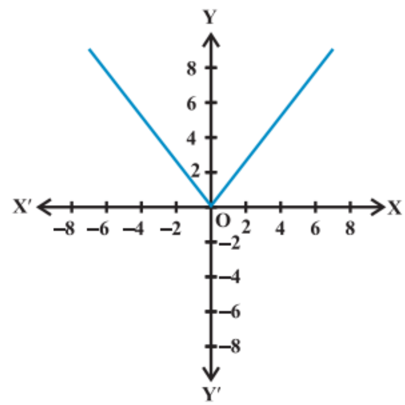
A function $f : R \to R$ defined by $f(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are functions of x defined in a domain, where $q(x) \neq 0$



$f(x) = \frac{1}{x}$

Modulus function

A function $f : R \to R$ defined by $f(x) = |x|$



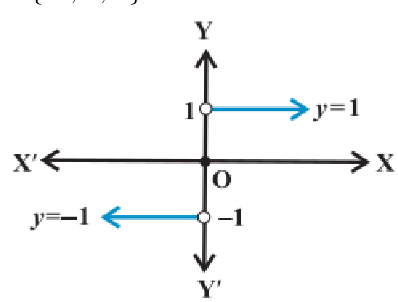
$f(x) = |x|$

Signum function

A function f: R → R defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x > 0 \end{cases}$$

The domain of the signum function is R and the range is the set {−1, 0, 1}

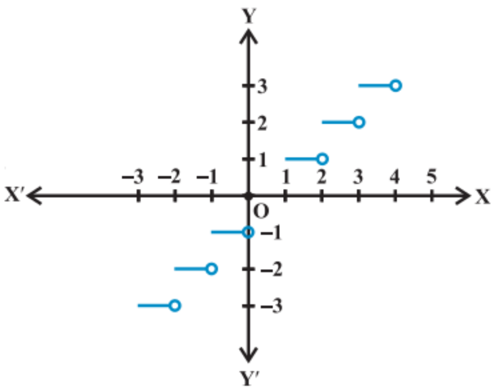


$f(x) = \frac{|x|}{x}, x$ ‘ 0 and 0 for $x = 0$

Greatest integer function

f: R $\rightarrow$ R defined by

$$f(x) = \begin{cases} \dots \\ 1 & \text{if } 1 \leq x < 2 \\ 0 & \text{if } 1 \leq x < 1 \\ -1 & \text{if } -1 \leq x < 0 \\ \dots \end{cases}$$



$f(x) = [x]$

Algebra of real functions

1. **Addition of two real functions**:Let f : X $\rightarrow$ R and g : X $\rightarrow$ R be any two real functions, where $X \subset R$. Then, we define $(f+g) : X \rightarrow R$ by $(f+g)(x) = f(x) + g(x)$ for all $x \in X$
2. **Substraction of two real functions**:Let f : X $\rightarrow$ R and g : X $\rightarrow$ R be any two real functions, where $X \subset R$. Then, we define $(f-g) : X \rightarrow R$ by $(f-g)(x) = f(x) - g(x)$ for all $x \in X$
3. **Multiplication by a scalar**:Let f : X $\rightarrow$ R be a real-valued function and k be a scalar. Then, the product kf : X $\rightarrow$ R by (kf)(x) = kf (x) for all $x \in X$
4. **Multiplication of two real functions**:Let f : X $\rightarrow$ R and g : X $\rightarrow$ R be any two real functions, where $X \subset R$. Then, we define fg : X $\rightarrow$ R by fg(x) = f(x) $\times$ g(x) for all $x \in X$
5. **Quotient of two real functions**: Let f and g be two real functions defined from X$\rightarrow$R, where $X \subset R$. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined by

   $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ Provided $g(x) \neq 0$ for $x \in X$