Integrals

The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.In this chapter we study different method of find indefinite integral and definite integrals of certain functions and its properties.

Integration

Let $\frac{d}{dx}F(x) = f(x)$. then we write

$$\int f(x)dx = F(x) + C$$

These integrals are called indefinite integrals and C is the constant of integration.

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the Table

Symbols/Terms/Phrases	**Meaning**
$\int f(x)dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)dx$	Integrand
x in $\int f(x)dx$	Variable of Integration
Integrate	Find the integral
An integral of F	A function F such that $F'(x) = f(x)$
Integration	The process of finding integral
constant of Integration	Any real number C, considered as constant function

Some properties

- Indefinite integral is a collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upward or downwards along the y-axis.
- $\int f(x) \pm g(x) = \int f(x) \pm \int g(x)$
- For any real number k,$\int kf(x) = k\int f(x)$

Some Standard Results

Derivatives	Integrals (Anti derivatives)
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
Particularly, we note that	
$\frac{d}{dx}(x) = 1$;	$\int dx = x + C$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x\, dx = \sin x + C$
(iii) $\frac{d}{dx}(-\cos x) = \sin x$;	$\int \sin x\, dx = -\cos x + C$
(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$;	$\int \sec^2 x\, dx = \tan x + C$
(v) $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$;	$\int \operatorname{cosec}^2 x\, dx = -\cot x + C$
(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$;	$\int \sec x \tan x\, dx = \sec x + C$
(vii) $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$;	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
(ix) $\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
(x) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
(xi) $\frac{d}{dx}(-\cot^{-1} x) = \frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xiv) $\frac{d}{dx}(e^x) = e^x$; $\int e^x dx = e^x + C$

(xv) $\frac{d}{dx}(\log|x|) = \frac{1}{x}$; $\int \frac{1}{x} dx = \log|x| + C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$; $\int a^x dx = \frac{a^x}{\log a} + C$

Methods of Integration

Integration by substitution

The given integral $I = \int f(x)dx$ is transformed into another form by changing the independent variable x to t by substituting $x = g(t)$ So that $\frac{dx}{dt} = g'(t) \Rightarrow dx = g'(t)dt$

$$I = \int f(x)dx = \int f(g(t))g'(t)dt$$

Some more Standard Results derived using substitution

- $\int tan\, x\, dx = log|sec\, x| + C$
- $\int cot\, x\, dx = log|sin\, x| + C$
- $\int sec\, x\, dx = log|sec\, x + tan\, x| + C$
- $\int cosec\, x\, dx = log|cosec\, x - cot\, x| + C$

Integrals of some particular Functions

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} log|\frac{x-a}{x+a}| + C$
2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} log|\frac{a+x}{a-x}| + C$
3. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} tan^{-1}\frac{x}{a} + C$
4. $\int \frac{dx}{\sqrt{x^2-a^2}} = log|x + \sqrt{x^2-a^2}| + C$
5. $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$
6. $\int \frac{dx}{\sqrt{^2+a^2}} = log|x + \sqrt{x^2+a^2} + C|$
7. **To find the integral of the type** $\int \frac{dx}{ax^2+bx+c}$ **and** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, we write

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Now put $x+\frac{b}{2a}=t$ so that $dx=dt$ and writing $\frac{c}{a}-\frac{b^2}{4a^2}=\pm k^2$. We find the integral reduced to the form

$$\frac{1}{a}\int\frac{dt}{t^2\pm k^2} \ or \ \frac{1}{a}\int\frac{dt}{\sqrt{t^2\pm k^2}}$$

depending upon the sign of $\frac{c}{a}-\frac{b^2}{4a^2}=\pm k^2$ and hence can be evaluated.

8. **To find the integral of the type** $\frac{px+q}{ax^2+bx+c}$ **and** $\frac{px+q}{\sqrt{ax^2+bx+c}}$ where p , q , a , b , c are constants , we are to find real numbers A, B such that

$$px+q=A\frac{d}{dx}(ax^2+bx+c)+B=A(2ax+b)+B$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

Integration by Partial Fractions

Consider integrals of the form $\int\frac{P(x)}{Q(x)}dx$ whre P and Q are polinomials in x and $Q(x)\neq 0$. If the degree of $P(x)$ is less than $Q(x)$, then the rational function is **Proper rational function** otherwise it is called **Improper rational function**

If $\frac{P(x)}{Q(x)}$ is improper function, first it should be converted to proper by long division and now it takes the form $\frac{P(x)}{Q(x)}=T(x)+\frac{P_1(x)}{Q(x)}$ Where T(x) is polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper function.

Now if $\frac{P(x)}{Q(x)}$ is proper function we factorise the denominator Q(x) into simpler polynomials and decompose into simpler rational function. For this we use the following table.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a\neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$,
	where x^2+bx+c cannot be factorised further	

Integration by Parts

$$\int f(x)g(x)dx=f(x)\int g(x)dx-\int\left(f'(x)\left[\int g(x)dx\right]\right)dx$$

Here the priority of taking first function and second function is more important, for this use order of the letters in words ILATE, where

- I- Inverse Triganometric function
- L - Logarithmic function
- A - Algebraic function
- T - Triganometric function
- E - Exponential function

Integrals of some more types

- $\int \sqrt{x^2-a^2}dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}log|x+\sqrt{x^2-a^2}| + c$
- $\int \sqrt{x^2+a^2}dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}log|x+\sqrt{x^2+a^2}| + c$
- $\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\frac{x}{a} + c$

Definite Integral

A definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)dx$, where a is the upper limit and b is the lower limit of the integral. If $\frac{d}{dx}F(x) = f(x)$ and $\int f(x)dx = F(x)+C$, then

$$\int_a^b f(x)dx = F(a) - F(b)$$

Definite integral as the limit of a sum

Let $f(x)$ be continuous function defined on a closed interval $[a, b]$. Then $\int_a^b f(x)dx$ is area bounded by the curve y = f(x), the ordinates x = a, x = b and the x-axis.

$$\int_a^b f(x)dx = lim_{h\to 0}h\left[f(a) + f(a+h) + \cdots + f(a+(n-1)h)\right]$$

OR

$$\int_a^b f(x)dx = (b-a)lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + \cdots + f(a+(n-1)h)\right]$$

where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

Fundamental Theorem of Calculus

We have defined $\int_a^b f(x)dx$ as the area of the region bounded by the curve y = f (x), the ordinates $x = a$ and $x = b$ and x-axis.We denote this function of x by A(x).

$$A(x) = \int_a^b f(x)dx$$

First fundamental theorem of integral calculus

Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.

Second fundamental theorem of integral calculus

Let f be continuous function defined on the closed interval [a, b] and F be an anti derivative of f. Then

$$\int_a^b f(x)dx = [F(x)]_b^a = F(b) - F(a)$$

Some Properties of Definite Integrals

$\mathbf{P_0}$: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$\mathbf{P_1}$: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

$\mathbf{P_2}$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\mathbf{P_3}$: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$\mathbf{P_4}$: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(Note that P_4 is a particular case of P_3)

$\mathbf{P_5}$: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$\mathbf{P_6}$: $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and

0 if $f(2a-x) = -f(x)$

$\mathbf{P_7}$: (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.