Integrals

The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.In this chapter we study different method of find indefinite integral and definite integrals of certain functions and its properties.

Integration

Let $\frac{d}{dx}F(x) = f(x)$. then we write

$$\int f(x)dx = F(x) + C$$

These integrals are called indefinite integrals and C is the constant of integration.

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the Table

Symbols/Terms/Phrases	**Meaning**
$\int f(x)dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)dx$	Integrand
x in $\int f(x)dx$	Variable of Integration
Integrate	Find the integral
An integral of F	A function F such that $F'(x) = x$
Integration	The process of finding integral
constant of Integration	Any real number C, considered as constant function

Some properties

- Indefinite integral is a collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upward or downwards along the y-axis.
- $\int f(x) \pm g(x) = \int f(x) \pm \int g(x)$
- For any real number k,$\int kf(x) = k\int f(x)$

Some Standard Results

	Derivatives	**Integrals (Anti derivatives)**
(i)	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$;	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C$, $n\neq -1$
	Particularly, we note that	
	$\frac{d}{dx}(x)=1$;	$\int dx=x+C$
(ii)	$\frac{d}{dx}(\sin x)=\cos x$;	$\int \cos x\,dx=\sin x+C$
(iii)	$\frac{d}{dx}(-\cos x)=\sin x$;	$\int \sin x\,dx=-\cos x+C$
(iv)	$\frac{d}{dx}(\tan x)=\sec^2 x$;	$\int \sec^2 x\,dx=\tan x+C$
(v)	$\frac{d}{dx}(-\cot x)=\operatorname{cosec}^2 x$;	$\int \operatorname{cosec}^2 x\,dx=-\cot x+C$
(vi)	$\frac{d}{dx}(\sec x)=\sec x\tan x$;	$\int \sec x\tan x\,dx=\sec x+C$
(vii)	$\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x\cot x$;	$\int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+C$
(viii)	$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1} x+C$
(ix)	$\frac{d}{dx}(-\cos^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1} x+C$
(x)	$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=\tan^{-1} x+C$
(xi)	$\frac{d}{dx}(-\cot^{-1} x)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=-\cot^{-1} x+C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xiv) $\frac{d}{dx}(e^x) = e^x$; $\int e^x dx = e^x + C$

(xv) $\frac{d}{dx}(\log|x|) = \frac{1}{x}$; $\int \frac{1}{x} dx = \log|x| + C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$; $\int a^x dx = \frac{a^x}{\log a} + C$

Methods of Integration

Integration by substitution

The given integral $I = \int f(x)dx$ is transformed into another form by changing the independent variable x to t by substituting $x = g(t)$ So that $\frac{dx}{dt} = g'(t) \Rightarrow dx = g'(t)dt$

$$I = \int f(x)dx = \int f(g(t))g'(t)dt$$

Some more Standard Results derived using substitution

- $\int tan\,x\,dx = log|sec\,x| + C$
- $\int cot\,x\,dx = log|sin\,x| + C$
- $\int sec\,x\,dx = log|sec\,x + tan\,x| + C$
- $\int cosec\,x\,dx = log|cosec\,x - cot\,x| + C$

Integrals of some particular Functions

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}log|\frac{x-a}{x+a}| + C$
2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}log|\frac{a+x}{a-x}| + C$
3. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + C$
4. $\int \frac{dx}{\sqrt{x^2-a^2}} = log|x + \sqrt{x^2-a^2}| + C$
5. $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$
6. $\int \frac{dx}{\sqrt{^2+a^2}} = log|x + \sqrt{x^2+a^2} + C|$
7. **To find the integral of the type** $\int \frac{dx}{ax^2+bx+c}$ **and** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, we write

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Now put $x + \frac{b}{2a} = t$ so that $dx = dt$ and writing $\frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$. We find the integral reduced to the form

$$\frac{1}{a}\int \frac{dt}{t^2 \pm k^2} \ or\ \frac{1}{a}\int \frac{dt}{\sqrt{t^2 \pm k^2}}$$

depending upon the sign of $\frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$ and hence can be evaluated.

8. **To find the integral of the type** $\frac{px+q}{ax^2+bx+c}$ **and** $\frac{px+q}{\sqrt{ax^2+bx+c}}$ where p , q , a , b , c are constants , we are to find real numbers A, B such that

$$px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A(2ax + b) + B$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

Integration by Partial Fractions

Consider integrals of the form