Three Dimensional Geometry

To refer a point in space we require a third axis (say z-axis) which leads to the concept of three-dimensional geometry. In this chapter we study the concept of direction cosines, direction ratios, equation of a line and a plane, angle between two lines and two planes, angle between a line and a plane, shortest distance between two skew lines, distance of a point from a plane

Direction Cosines and direction ratios

Consider a directed line passing through the origin makes angles α, β and γ with the positive direction x-axis, y-axis, and z-axis. Then α, β, and γ are called direction angles. The cosine of α, β, and γ are called direction cosines. Generally $cos\alpha = l$, $cos\beta = m$ and $cos\gamma = n$. Any scalar multiple of direction cosines are called direction ratios.

- If (a, b, c) is the coordinate of a point P then a,b,c is a direction ratio of the directed line passing along P and origin. Direction cosines will be

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- $l^2 + m^2 + n^2 = 1$

Direction cosines of a line passing through two points

Direction ratios of a line segmnent passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$x_2 - x_1, y_2 - y_1, z_2 - z_1$$

Direction angles are

$$\frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB}, \frac{z_2 - z_1}{AB}$$

Equation of a line in space

Equation of a line through a given point $\vec{a}$ and parallel to given vector $\vec{b}$

Let $\vec{a}$ be the position vector of the given point A with respect to the origin O of the coordinate system.Let l be the line which passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vector of an arbitary point P on the line then the equation of the line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

where λ be any be any real number.

Cartesian form

Let (x_1, y_1, z_1) be coordinates of a given point in line and a, b and c be the direction ratios of the line then

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

This is the **Cartesian** equation of the line

Angle between two lines

Let L_1 and L_2 be two lines passing through the origin and with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively. then the angle between them is

$$cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are

- perpedicular if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Shortest distance between two lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance.

Furthermore there are lines which are neither intersecting nor parallel, such pair of lines are non coplanar and are called **skew lines**.

Distance between two skew lines

Let l_1 and l_2 be two skew lines with equations

$$\vec{r} = \vec{a_1} + \lambda\vec{b_1}, \vec{r} = \vec{a_2} + \lambda\vec{b_2}$$

Then the distance between them will be

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$