Inverse Trigonometric Functions

Recall some basic results from class XI

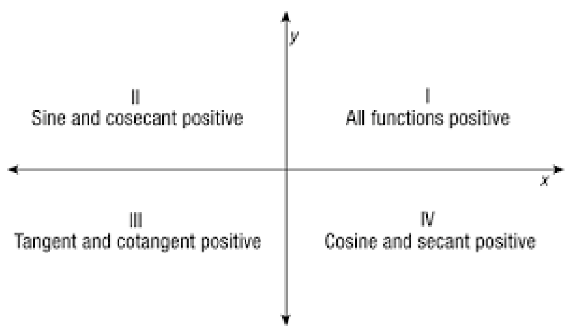
Phythagoras' relations

$sin^2\theta + cos^2\theta = 1, sec^2\theta - tan^2\theta = 1, cosec^2\theta - cot^2\theta = 1$

Trigonometric Ratios of particular angles

	$0°$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

Signs of Trigonometric functions in Quadrants



Domain and range of trigonometric functions

Functions	Domain	Range
(i) sine	R	[–1, 1]
(ii) cosine	R	[–1, 1]
(iii) tangent	$R - \{x : x = (2n+1)\frac{\pi}{2}, n \in Z\}$	R
(iv) cosecant	$R - \{x : x = n\pi, n \in Z\}$	R – [–1, 1]
(v) secant	$R - \{x : x = (2n+1)\frac{\pi}{2}, n \in Z\}$	R – [–1, 1]
(vi) cotangent	$R - \{x : x = n\pi, n \in Z\}$	R

Triganometric Functions of larger angles

For an angle θ and integer n

$$sin(n\pi \pm \theta) = sin\theta$$
$$cos(n\pi \pm \theta) = cos\theta$$
$$tan(n\pi \pm \theta) = tan\theta$$
$$cosec(n\pi \pm \theta) = cosec\theta$$
$$sec(n\pi \pm \theta) = sec\theta$$
$$cot(n\pi \pm \theta) = cot\theta$$

(even multiple of $90^0 + \theta$)

$$sin((2n+1)\frac{\pi}{2} \pm \theta) = cos\theta$$
$$cos((2n+1)\frac{\pi}{2} \pm \theta) = sin\theta$$
$$tan((2n+1)\frac{\pi}{2} \pm \theta) = cot\theta$$
$$cot((2n+1)\frac{\pi}{2} \pm \theta) = tan\theta$$
$$cosec((2n+1)\frac{\pi}{2} \pm \theta) = sec\theta$$
$$sec((2n+1)\frac{\pi}{2} \pm \theta) = cosec\theta$$

(even multiple of $90^0 + \theta$)

Compound Angle Formulas

- $sin(x+y) = sin\,x cosy + cosx\, siny$
- $sin(x-y) = sinx\, cosy - cosx\, siny$
- $cos(x+y) = cosx\, cosy - sinx\, siny$
- $cos(x-y) = cosx\, cosy + sinx\, siny$
- $tan(x+y) = \frac{tanx+tany}{1-tanx\, tany}$
- $tan(x-y) = \frac{tanx-tany}{1+tanx\, tany}$
- $cot(x+y) = \frac{cotx\, coty-1}{cotx+coty}$
- $cot(x-y) = \frac{cotx\, coty+1}{cotx-coty}$

Multiple Angle Formulas

- $sin2x = 2sinx\,cosx = \frac{2tanx}{1+tan^2x}$
- $sin3x = 3sinx – 4sin^3x$
- $cos2x = cos^2x - sin^2x = 1 - 2sin^2x = 2cos^2x - 1 = \frac{1-tan^2x}{1+tan^2x}$
- $cos3x = 4cos^3x - 3cosx$
- $tan2x = \frac{2tanx}{1-tan^2x}$
- $tan3x = \frac{3tanx - tan^3x}{1-3tan^2(x)}$

Trigonometric functions are real functions which are not objective and thus its inverse does not exist. In this chapter we study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverse and observe its graphical peculiarities.

Basic Concepts

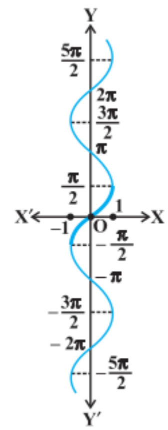
In Class XI, we have studied trigonometric functions, which are defined as follows:

- sine :$R \to [-1, 1]$
- cos :$R \to [-1, 1]$
- tan :$R - \{x : x = (2n+1)\frac{\pi}{2}, n \in Z\} \to R$
- cot :$R - \{x : x = n\pi, n \in Z\} \to R$
- sec :$R - \{x : x = (2n+1)\frac{\pi}{2}, n \in Z\} \to R - (-1, 1)$
- cosec :$R - \{x : x = n\pi, n \in Z\} \to R - (-1, 1)$

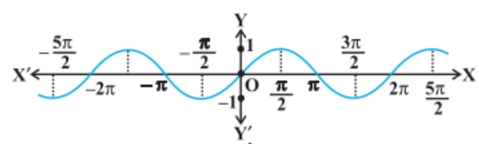
These trigonometric functions are onto but not one-one , To define inverse of these trigonometric functions we restrict the domains to *principal value branches* and define as follows .

Graphs

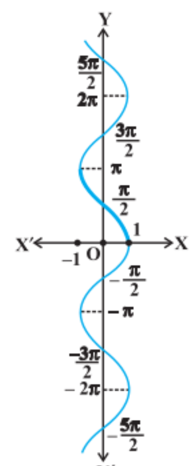
$\sin^{-1}$	:	$[-1, 1]$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\to$	$[0, \pi]$
cosec^{-1}	:	$\mathbf{R} - (-1,1)$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$\mathbf{R} - (-1, 1)$	$\to$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$	:	$\mathbf{R}$	$\to$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	$\mathbf{R}$	$\to$	$(0, \pi)$



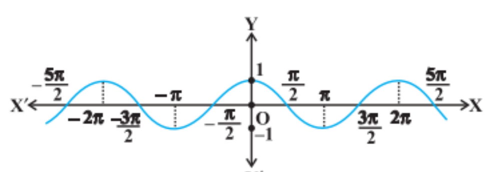
$y = \sin^{-1}x$



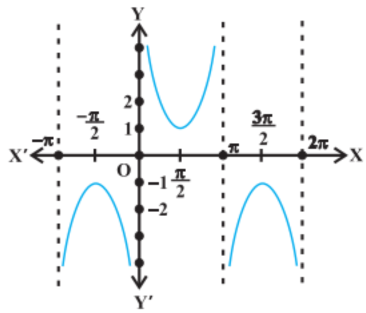
$y = \sin x$



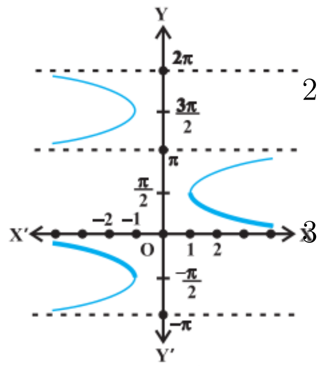
$y = \cos^{-1}x$



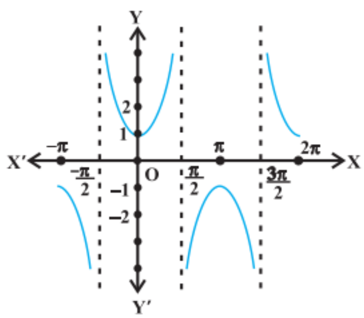
$y = \cos x$



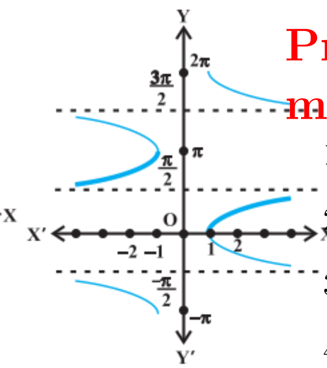
$y = \operatorname{cosec} x$



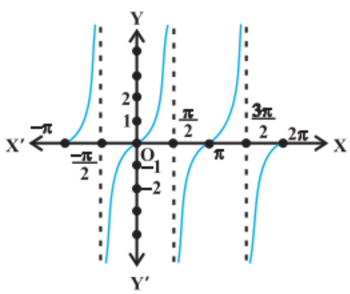
$y = \operatorname{cosec}^{-1} x$



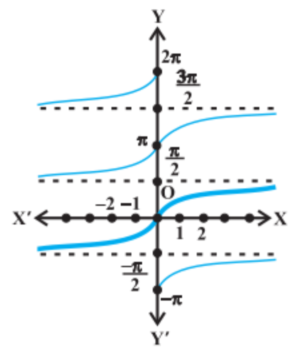
$y = \sec x$



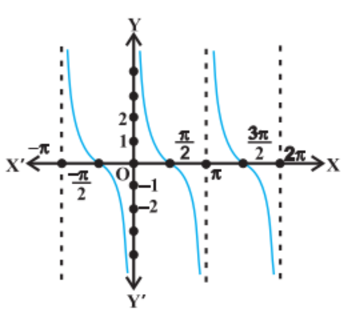
$y = \sec^{-1} x$



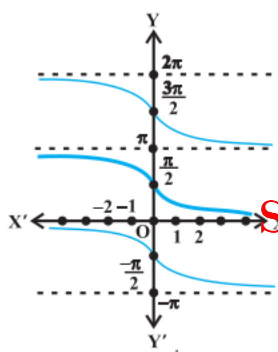
$y = \tan x$



$y = \tan^{-1} x$



$y = \cot x$



$y = \cot^{-1} x$

Note

1. $sin^{-1}x$ should not be confused with $(sinx)^{-1}$. In fact $(sinx)^{-1} = \frac{1}{sin(x)}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

Properties of Inverse Trigonometric Functions

1. $sin(sin^{-1}x) = x, x \in [-1, 1]$
2. $cos(cos^{-1}x) = x, x \in [-1, 1]$
3. $tan(tan^{-1}x) = x, x \in R$
4. $cosec(cosec^{-1}x) = x, x \in R - [-1, 1]$
5. $sec(sec^{-1}x) = x, x \in R - [-1, 1]$
6. $cot(cot^{-1}x) = x, x \in R$
7. $sin^{-1}(sinx) = x, x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$
8. $cos^{-1}(cosx) = x, x \in [0, \pi]$
9. $tan^{-1}(tanx) = x, x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$
10. $cosec^{-1}(cosecx) = x, x \in [\frac{-\pi}{2}, \frac{\pi}{2}] - \{0\}$
11. $cos^{-1}(cosx) = x, x \in [0, \pi] - \{\frac{\pi}{2}\}$
12. $cot^{-1}(cotx) = x, x \in [0, \pi]$

Some Tricks

- For $\sqrt{1 - x^2}$ Put $x = sin(\theta)$ or $x = cos(\theta)$
- For $\sqrt{1 + x^2}$ Put $x = tan(\theta)$ or $x = cot(\theta)$
- For $\sqrt{x^2 - 1}$ Put $x = sec(\theta)$ or $x = cosec(\theta)$
- $1 - sin(x) = sin^2\frac{x}{2} + cos^2\frac{x}{2} - 2sin\frac{x}{2}cos\frac{x}{2}$
  $1 + sin(x) = sin^2\frac{x}{2} + cos^2\frac{x}{2} + 2sin\frac{x}{2}cos\frac{x}{2}$