Continuity and Differentiability

As a continuation of limits and derivatives studied in the previous years, now we are entering into a very important concept continuity and its graphical peculiarities. We also learn different methods of differentiation and introduce new class of functions such as exponential and logarithmic functions.

Continuity

Continuity of a function at a point

A function f(x) is said to be continuous at a point $'a'$ if the following conditions are satisfied.

1. $f(a)$ should be defined.
2. $f(a)$ should be equal to the limit of the function at $'a'$.

$$lim_{x\to a-}f(x) = lim_{x\to a+}f(x) = f(a)$$

Continuity of a function

A function $f(x)$ is said to be continuous if the function is continuous at every point on its domain. Some standard continuous functions are mentioned below;

1. constant function $f(x) = c$, c is a constant
2. Identity function $f(x) = x$
3. Modulus function $f(x) = |x|$
4. Exponential function $f(x) = e^x$
5. Logarithmic function $f(x) = logx$
6. Polynomial function $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$
7. Rational function $f(x) = \frac{p(x)}{q(x)}$, p(x) and q(x) are Polynomial function and $q(x) \neq 0$
8. Triganometric and Inverse Triganometric Functions.

Graphical approach: If there is a break in the graph of a function then it is not continuous.

Algebra of Continuous functions

Suppose f and g be two real functions in their respective domains then $f+g, g-g, f\times g$ and $\frac{f}{g}, g\neq 0$ are also Continuous.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is defined by

$$lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))|_c$.The function defined by

$$f'(c) = lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

- Every differentiable function is continuous. But the converse need not be.true, eg; $f(x) = |x|$.

Some standard results

1. $\frac{d}{dx}(k) = 0$
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}(x) = 1$
4. $\frac{d}{dx}(\frac{1}{x}) = \frac{-1}{x^2}$
5. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
6. $\frac{d}{dx}(sin(x)) = cos(x)$
7. $\frac{d}{dx}(cos(x)) = -sin(x)$
8. $\frac{d}{dx}tan(x) = sec^2(x)$
9. $\frac{d}{dx}sec(x) = sec(x)tan(x)$
10. $\frac{d}{dx}(cosec(x)) = -cosec(x)cot(x)$
11. $\frac{d}{dx}(cot(x)) = -cosec^2(x)$

Algebra of Derivatives

For differentiable functions f and g

- $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$
- $\frac{d}{dx}[f(x) \times g(x)] = \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x))$
- $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$

Derivative of composite functions(chain rule)

Let f be a real valued function which is a composite of two functions h and g ;i.e $f = hog$,

$$f(x) = h(g(x))$$

Then

$$\frac{d}{dx}f(x) = h'(f(x)) \times g'(x)$$

Derivatives of implicit functions

Sometimes functions may not be given explicitly as $y = f(x)$.For example , consider the following relationship between x and y.

$$x - y - \pi = 0$$

In this case we differentiate both sides of the function with respect to and solve for $\frac{dy}{dx}$.

Derivatives of inverse trigonometric functions

f(x)	$sin^{-1}x$	$cos^{-1}x$	$tan^{-1}x$
f'(x)	$\frac{1}{\sqrt{1-x^2}}$	$-\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

Exponential and Logarithmic Functions

The exponential function with positive base $b > 1$ is the function

$$y = f(x) = b^x$$

Exponential function with base 10 is called the **common exponential function.**

Sum of the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \dots$$

is a number between 2 and 3 and is denoted by e.The exponential function with base e is called **Natural exponential function.**

Let $b > 1$ be a real number. Then we say **logarithm** of a to base b is x if $b^x = a$.If the base b = 10, we say it is common logarithms and if b = e, then we say it is natural logarithms. Often natural logarithm is denoted by ln.logarithmic function is the inverse of exponential function.

Properties of log

- $x = e^{logx}$ for $x \in R^+$
- $log(p^n) = nlog(p)$
- $log(pq) = logp + logq$
- $log(\frac{p}{q}) = logp - logq$
- $\frac{d}{dx}a^x = a^x loga$
- $\frac{d}{dx}e^x = e^x$ (since $loge = 1$)

Logarithmic Differentiation

Function with are complicated Rational functions and of the form $f(x) = u(x)^{v(x)}$ is differentiated using Logarithmic Differentiation method. Here first take logarithm on both sides of the function and proceed as in implicit differentiation.

Parametric Differentiation

Relation between two variable x and y which are expressed in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with parameter t.Here

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Second Order Derivative

If $f'(x)$ is differentiable we may differentiate once again with respect to x.Then $\frac{d}{dx}(\frac{dy}{dx})$ is called the Second Derivate of f with respective to x, denoted by $\frac{d^2y}{dx^2}$ or f"(x) or y".