# Determinants

## Determinant

Determinant is a real number associated with a square matrix $A = [a_{ij}]$ of order n. The determinant of matrix A is denoted by $|A|$ or $det()$

If A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then determinant of A is denoted as

$$det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

### Determinant of matrix of order 1

For matrix $A = \begin{bmatrix} a \end{bmatrix}$ of order 1 $|A| = a$

### Determinant of matrix of order 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2 then

$$det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

### Determinant of matrix of order 3

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3

**Expansion along first Row ($\mathbf{R_1}$)**

$$|A| = (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

1+1, 1+2 ,1+3 are sum of positions of $a_{11}, a_{12}, a_{13}$ respectively, matrix of order 2 is chosen by removing row and column containing $a_{11}, a_{12}, a_{13}$ respectively

similarly we can find Determinants with other rows and columns too

**Remarks**

- For easier calculations choose the row/columns with maximum number of zeroes
- If $A = kB$ then $|A| = k|B|$
- If $A = k^nB$ then $|A| = k^n|B|$

## Area of a Triangle

Area of a Triangle with vertices $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ is

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

# Minors and Cofactors

**Minor** of an element $a_{ij}$ of a determinant is the determinant obtained by deleting its ith row and jth column in which element $a_{ij}$ lies. Minor of an element $a_{ij}$ is denoted by $M_{ij}$.

**Cofactor** of an element $a_{ij}$ , denoted by $A_{ij}$ is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where $M_{ij}$ is the minor of $a_{ij}$

# Adjoint and Inverse of a matrix

The **adjoint** of a matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $A = [a_{ij}]_{n\times n}$, where $A_{ij}$ is the cofactor of the element $a_{ij}$ . Adjoint of the matrix A is denoted by adj A

$$adj \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- $adj \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

- **Theorem 1** - If A be any given square matrix of order n, then

$$A(adj A) = (adj A)A = |A|I$$

  where I is the identity matrix

## Singular Matrix

A square matrix A is said to be **singular** if $|A| = 0$, **non singular** if $|A| \neq 0$

- **Theorem 2** - If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

- **Theorem 3** - The determinant of the product of matrices is equal to product of their respective determinants, that is,$|AB| = |A||B|$ , where A and B are square matrices of the same order

- $|(adj A)A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$

- **Theorem 4** - A square matrix A is invertible if and only if A is nonsingular matrix.

- $A^{-1} = \frac{adj A}{|A|}$

# Applications of Determinants and Matrices

## Consistent and inconsistent System

A system of equations is said to be **consistent** if its solution (one or more) exists.

A system of equations is said to be **inconsistent** if its solution does not exist.

## Solution of system of linear equations using inverse of a matrix

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equations can be written as, $AX = B$, i.e.,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

**case 1** If A is a nonsingular matrix, then its inverse exists, so solution is $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

**case 2**If A is a singular matrix, then $|A| = 0$. then solution does not exist and the system of equations is called inconsistent.