Application of Derivatives

In this chapter we analyses the physical and geometrical applications of derivatives in real life such as to determine rate of change, to find tangents and normal to a curve, to find turning points, intervals in which the curve is increasing and decreasing, to find approximate value of certain quantities.

Rate of Change

By $\frac{dy}{dx}$,we mean the rate of change of y with respect to x. If s is the displacement function in terms of t and v the velocity at that time. Then

$$velocity = \frac{ds}{dt}$$

$$acceleration = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Increasing and Decreasing Functions

Let I be an interval contained in the domain of a real valued function f. Then f is said to be

1. **increasing** on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in I$.
2. **strictly increasing** on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.
3. **decreasing** on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$.
4. **strictly decreasing** on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.
5. **constant** on I, if $f(x) = c$ for all $x \in I$, where c is a constant.

Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing, decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, decreasing, respectively, in I.

Theorem 1

Let f be continuous on $[a, b]$ and differentiable on the open interval (a, b). Then

1. f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$
2. f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$
3. f is constant in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Maxima and Minima

Let f be a function defined on an interval I. Then

1. f is said to have a **maximum value** in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$

   The number $f(c)$ is called the **maximum value** of f in I and the point c is called a **point of maximum value** of f in I.

2. f is said to have a **minimum value** in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.

   The number $f(c)$,is called the **minimum value** of f in I and the point c, is called a **point of minimum value** of f in I.

3. f is said to have an **extreme value** in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

   The number $f(c)$, in this case, is called an extreme value of f in I and the point c is called an **extreme point**.

Local Minma and Local Maxima

Let f be a real valued function and let c be an interior point in the domain of f. Then

- c is called a point of **local maxima** if there is an $h > 0$ such that

  $$f(c) \geq f(x) \text{ for all } x \text{ in } (c-h, c+h), x \neq c$$

  The value of $f(c)$ is called **local maximum** value of f

- c is called a point of **local minima** if there is an $h > 0$ such that

  $$f(c) \leq f(x) \text{ for all } x \text{ in } (c-h, c+h), x \neq c$$

  The value of $f(c)$ is called **local minimum** value of f

Theorem 2

Let f be a function defined on an open interval I. Suppose $c \in I$ be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

Theorem 3 (First Derivative Test)

1. If $f'(c) = 0$ and $f'(x)$ changes its sign from positive to negative from left to right of $x = c$, then the point is a local maximum point.

2. If $f'(c) = 0$ and $f'(x)$ changes its sign from negative to positive from left to right of $x = c$, then the point is a local minimum point.

3. If $f'(c) = 0$ and if there is no change of sign for $f'(x)$ from left to right of $x = c$, then the point is a inflexion point.

Theorem 4 (Second Derivative Test)

Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

1. If $f'(c) = 0$ and $f"(c) < 0$, then $x = c$ is a local maximum point.

2. If $f'(c) = 0$ and $f"(c) > 0$, then $x = c$ is a local minimum point.

3. If $f'(c) = 0$ and $f"(c) = 0$, then the test fails and go to first derivate test for checking maxima and minima.

Maximum and Minimum Values of a Function in a Closed Interval

Consider $f(x) = x + 1, x \in (0, 1)$ .the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of f to the closed interval $[0, 1]$, then f still may not have a local maximum (minimum) values but it certainly does have maximum value $3 = f(1)$ and minimum value $2 = f(0)$. The maximum value 3 of f at $x = 1$ is called **absolute maximum value** (global maximum or greatest value) of f on the interval $[0, 1]$. Similarly, the minimum value 2 of f at $x = 0$ is called the **absolute minimum value** (global minimum or least value) of f on $[0, 1]$.

Theorem 5

Let f be a continuous function on an interval $I = [a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I

Theorem 6

Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

1. $f'(c) = 0$ if f attains its absolute maximum value at c
2. $f'(c) = 0$ if f attains its absolute minimum value at c.

Working rule

1. Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.
2. Take the end points of the interval
3. At all these points (listed in Step 1 and 2), calculate the values of f
4. Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f .