

# **STA 4273H: Statistical Machine Learning**

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## Lecture 2

# Last Class

- In our last class, we looked at:
  - Statistical Decision Theory
  - Linear Regression Models
  - Linear Basis Function Models
  - Regularized Linear Regression Models
  - Bias-Variance Decomposition
- We will now look at the Bayesian framework and Bayesian Linear Regression Models.

# Bayesian Approach

- We formulate our knowledge about the world probabilistically:
  - We **define the model** that expresses our knowledge qualitatively (e.g. independence assumptions, forms of distributions).
  - Our model will have some **unknown parameters**.
  - We capture our assumptions, or prior beliefs, about unknown parameters (e.g. range of plausible values) by **specifying the prior distribution** over those parameters before seeing the data.
- We **observe the data**.
- We compute the **posterior probability distribution** for the parameters, given observed data.
- We use this posterior distribution to:
  - **Make predictions** by averaging over the posterior distribution
  - **Examine/Account for uncertainty** in the parameter values.
  - **Make decisions** by minimizing expected posterior loss.

(See Radford Neal's NIPS tutorial on ``Bayesian Methods for Machine Learning'')

# Posterior Distribution

- The posterior distribution for the model parameters can be found by combining the prior with the likelihood for the parameters given the data.
- This is accomplished by using **Bayes' Rule**:

$$P(\text{parameters} \mid \text{data}) = \frac{P(\text{data} \mid \text{parameters})P(\text{parameters})}{P(\text{data})}$$

$$p(\mathbf{w} \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \mathbf{w})P(\mathbf{w})}{P(\mathcal{D})}$$

Probability of observed data given  $\mathbf{w}$

Prior probability of the weight vector  $\mathbf{w}$

Posterior probability of the weight vector  $\mathbf{w}$  given training data  $\mathcal{D}$

Marginal likelihood (normalizing constant):

$$P(\mathcal{D}) = \int p(\mathcal{D} \mid \mathbf{w})P(\mathbf{w})d\mathbf{w}$$

This integral can be high-dimensional and is often difficult to compute.

# The Rules of Probability

Sum Rule:

$$p(X) = \sum_Y p(X, Y)$$

Product Rule:

$$p(X, Y) = p(Y|X)p(X)$$

# Predictive Distribution

- We can also state Bayes' rule in words:  
posterior  $\propto$  likelihood  $\times$  prior.
- We can make predictions for a new data point  $\mathbf{x}^*$ , given the training dataset by **integrating over the posterior distribution**:

$$p(\mathbf{x}^*|\mathcal{D}) = \int p(\mathbf{x}^*|\mathbf{w}, \mathcal{D})p(\mathbf{w}|\mathcal{D})d\mathbf{w} = \mathbb{E}_{P(\mathbf{w}|\mathcal{D})}[p(\mathbf{x}^*|\mathbf{w}, \mathcal{D})],$$

which is sometimes called **predictive distribution**.

- Note that computing predictive distribution requires knowledge of the posterior distribution:

$$p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})P(\mathbf{w})}{P(\mathcal{D})}, \quad \text{where } P(\mathcal{D}) = \int p(\mathcal{D}|\mathbf{w})P(\mathbf{w})d\mathbf{w}$$

which is usually **intractable**.

# Modeling Challenges

- The first challenge is in specifying suitable model and suitable prior distributions. This can be challenging particularly when dealing with high-dimensional problems we see in machine learning.
  - A suitable model should admit all the possibilities that are thought to be at all likely.
  - A suitable prior should avoid giving zero or very small probabilities to possible events, but should also avoid spreading out the probability over all possibilities.
- We may need to properly model dependencies between parameters in order to avoid having a prior that is too spread out.
- One strategy is to introduce latent variables into the model and hyperparameters into the prior.
- Both of these represent the ways of modeling dependencies in a tractable way.

# Computational Challenges

The other big challenge is computing the posterior distribution. There are several main approaches:

- **Analytical integration**: If we use “conjugate” priors, the posterior distribution can be computed analytically. Only works for simple models and is usually too much to hope for.
- **Gaussian (Laplace) approximation**: Approximate the posterior distribution with a Gaussian. Works well when there is a lot of data compared to the model complexity (as posterior is close to a Gaussian).
- **Monte Carlo integration**: Once we have a sample from the posterior distribution, we can do many things. The dominant current approach is Markov Chain Monte Carlo (MCMC) -- simulate a Markov chain that converges to the posterior distribution. It can be applied to a wide variety of problems.
- **Variational approximation**: A cleverer way to approximate the posterior. It often works much faster compared to MCMC. But often not as general as MCMC.

# Bayesian Linear Regression

- Given observed inputs  $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$ , and corresponding target values  $\mathbf{t} = [t_1, t_2, \dots, t_N]^T$ , we can write down the likelihood function:

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{n=1}^N \mathcal{N}(t_n | \mathbf{w}^T \phi(\mathbf{x}_n), \beta^{-1}),$$

where  $\phi(\mathbf{x}) = (\phi_0(\mathbf{x}), \phi_1(\mathbf{x}), \dots, \phi_{M-1}(\mathbf{x}))^T$  represent our basis functions.

- The corresponding **conjugate prior** is given by a Gaussian distribution:

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} | \mathbf{m}_0, \mathbf{S}_0).$$

- As both the likelihood and the prior terms are Gaussians, the posterior distribution will also be Gaussian.
- If the posterior distributions  $p(\theta|x)$  are in the same family as the prior probability distribution  $p(\theta)$ , the prior and posterior are then called **conjugate distributions**, and the prior is called a **conjugate prior** for the likelihood.

# Bayesian Linear Regression

- Combining the prior together with the likelihood term:

$$p(\mathbf{w}|\mathbf{t}, \mathbf{X}, \mathbf{w}, \beta) \propto \left[ \prod_{n=1}^N \mathcal{N}(t_n | \mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1}) \right] \mathcal{N}(\mathbf{w} | \mathbf{m}_0, \mathbf{S}_0).$$

- The posterior (with a bit of manipulation) takes the following Gaussian form:

$$p(\mathbf{w}|\mathbf{t}) = \mathcal{N}(\mathbf{w} | \mathbf{m}_N, \mathbf{S}_N)$$

where

$$\begin{aligned}\mathbf{m}_N &= \mathbf{S}_N \left( \mathbf{S}_0^{-1} \mathbf{m}_0 + \beta \boldsymbol{\Phi}^T \mathbf{t} \right) \\ \mathbf{S}_N^{-1} &= \mathbf{S}_0^{-1} + \beta \boldsymbol{\Phi}^T \boldsymbol{\Phi}.\end{aligned}$$

- The posterior mean can be expressed in terms of the **least-squares estimator** and the **prior mean**:

$$\mathbf{m}_N = \mathbf{S}_N \left( \mathbf{S}_0^{-1} \mathbf{m}_0 + \beta \boldsymbol{\Phi}^T \mathbf{w}_{ML} \right). \quad \boxed{\mathbf{w}_{ML} = (\boldsymbol{\Phi}^T \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^T \mathbf{t}.}$$

- As we increase our prior precision (decrease prior variance), we place greater weight on the prior mean relative the data.

# Bayesian Linear Regression

- Consider a zero mean isotropic Gaussian prior, which is governed by a single precision parameter  $\alpha$ :

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} | \mathbf{0}, \alpha^{-1} \mathbf{I})$$

for which the posterior is Gaussian with:

$$\begin{aligned}\mathbf{m}_N &= \beta \mathbf{S}_N \boldsymbol{\Phi}^T \mathbf{t} \\ \mathbf{S}_N^{-1} &= \alpha \mathbf{I} + \beta \boldsymbol{\Phi}^T \boldsymbol{\Phi}.\end{aligned}$$

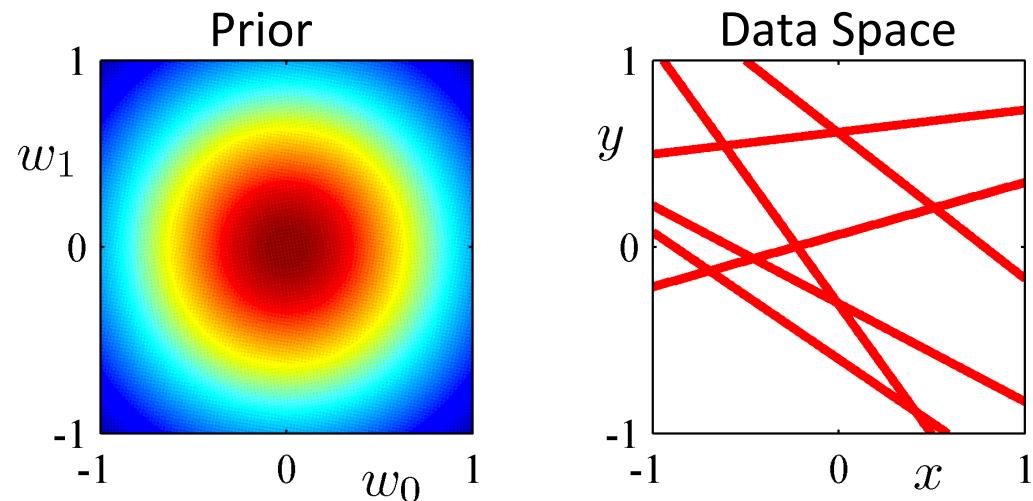
$$\mathbf{w}_{ML} = (\boldsymbol{\Phi}^T \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^T \mathbf{t}.$$

- If we consider an infinitely broad prior,  $\alpha \rightarrow 0$ , the mean  $\mathbf{m}_N$  of the posterior distribution reduces to **maximum likelihood value**  $\mathbf{w}_{ML}$ .
  - The log of the posterior distribution is given by the sum of the log-likelihood and the log of the prior:
- $$\ln p(\mathbf{w} | \mathcal{D}) = -\frac{\beta}{2} \sum_{n=1}^N (t_n - \mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}_n))^2 - \frac{\alpha}{2} \mathbf{w}^T \mathbf{w} + \text{const.}$$
- Maximizing this posterior with respect to  $\mathbf{w}$  is equivalent to **minimizing the sum-of-squares error function** with a quadratic regulation term  $\lambda = \alpha / \beta$ .

# Bayesian Linear Regression

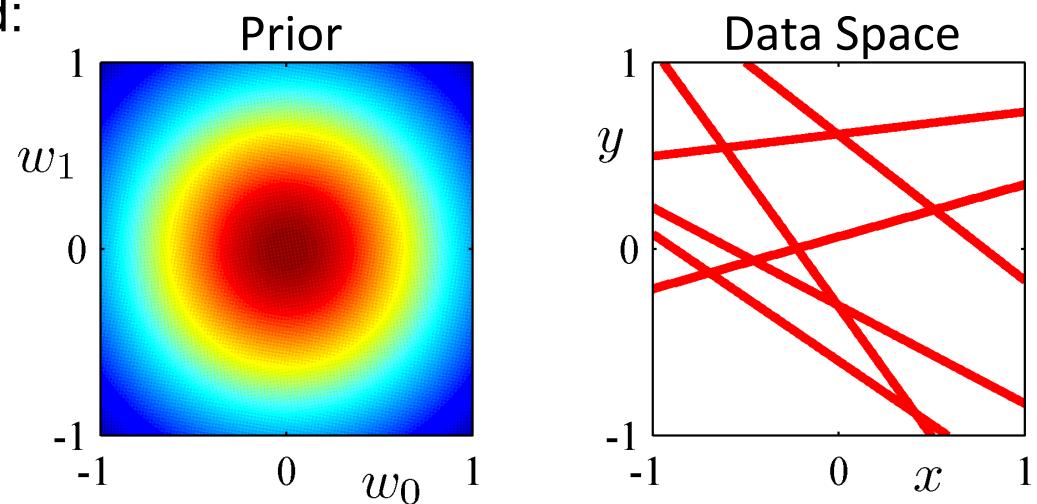
- Consider a linear model of the form:  $y(x, \mathbf{w}) = w_0 + w_1 x$ .
- The training data is generated from the function  $f(x, \mathbf{a}) = a_0 + a_1 x$  with  $a_0 = 0.3; a_1 = 0.5$ , by first choosing  $x_n$  uniformly from  $[-1;1]$ , evaluating  $f(x, \mathbf{a})$ , and then adding a small Gaussian noise.
- **Goal:** recover the values of  $a_0, a_1$  from such data.

0 data points are observed:

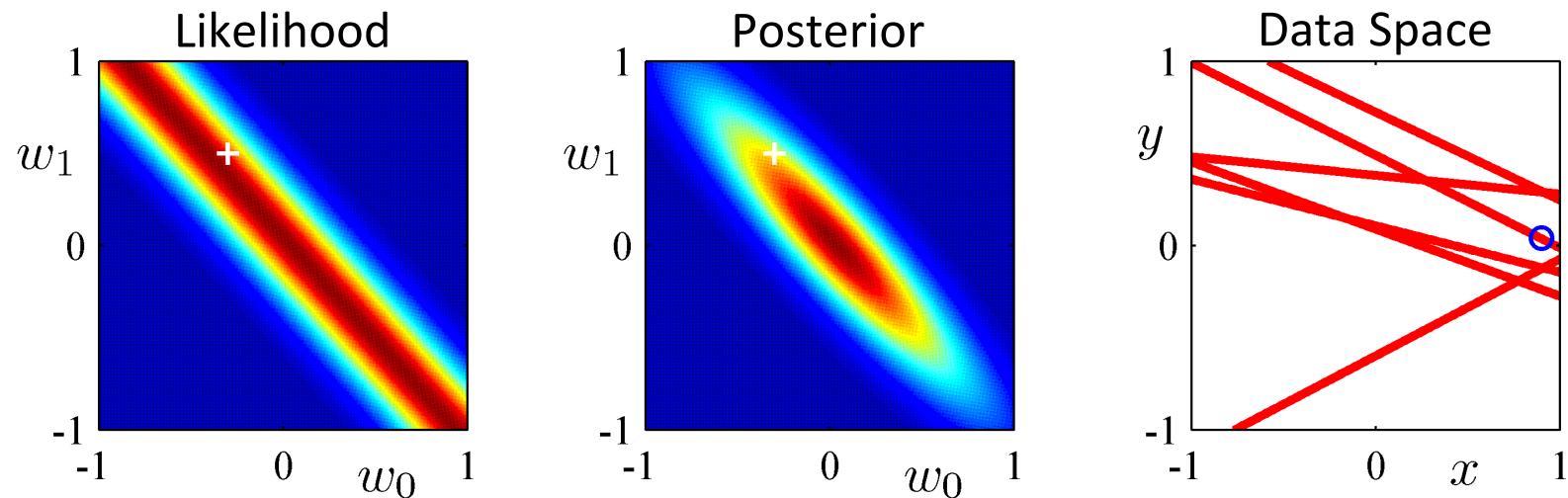


# Bayesian Linear Regression

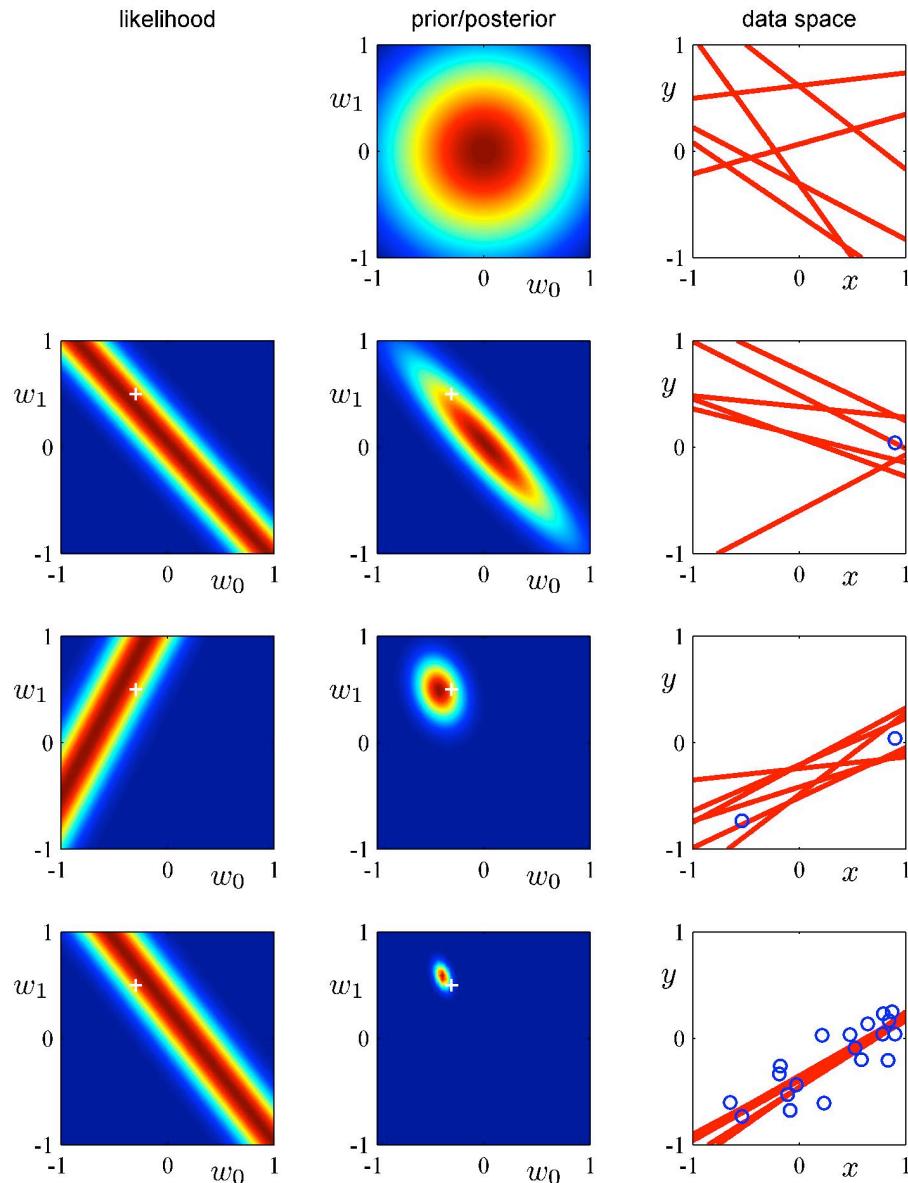
0 data points are observed:



1 data point is observed:



# Bayesian Linear Regression



0 data points are observed.

1 data point is observed.

2 data points are observed.

20 data points are observed.

# Predictive Distribution

- We can make predictions for a new input vector  $\mathbf{x}$  by integrating over the posterior distribution:

$$\begin{aligned} p(t|\mathbf{t}, \mathbf{x}, \mathbf{X}, \alpha, \beta) &= \int p(t|\mathbf{x}, \mathbf{w}, \beta)p(\mathbf{w}|\mathbf{t}, \mathbf{X}, \alpha, \beta)d\mathbf{w} \\ &= \mathcal{N}(t|\mathbf{m}_N^T \boldsymbol{\phi}(\mathbf{x}), \sigma_N^2(\mathbf{x})), \end{aligned}$$

where

$$\sigma_N^2(\mathbf{x}) = \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}).$$

Noise in the target values

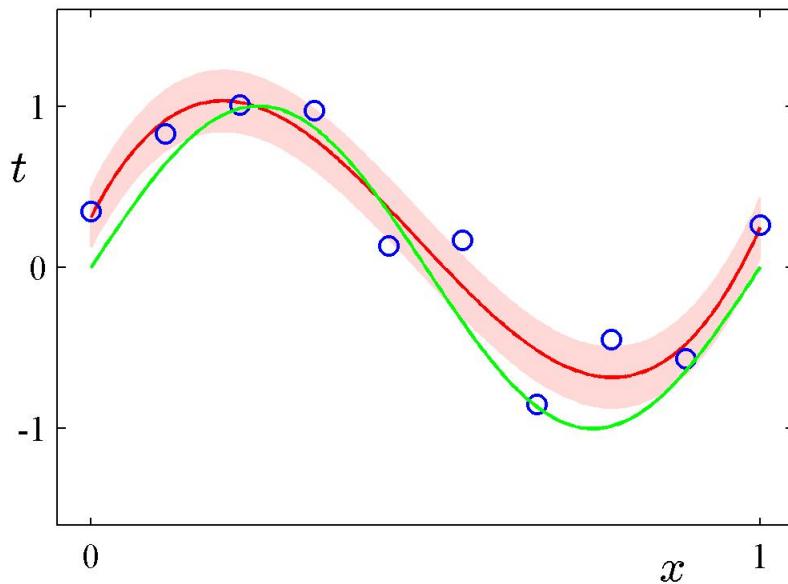
Uncertainty associated with parameter values.

$$\begin{aligned} \mathbf{m}_N &= \beta \mathbf{S}_N \boldsymbol{\Phi}^T \mathbf{t} \\ \mathbf{S}_N^{-1} &= \alpha \mathbf{I} + \beta \boldsymbol{\Phi}^T \boldsymbol{\Phi}. \end{aligned}$$

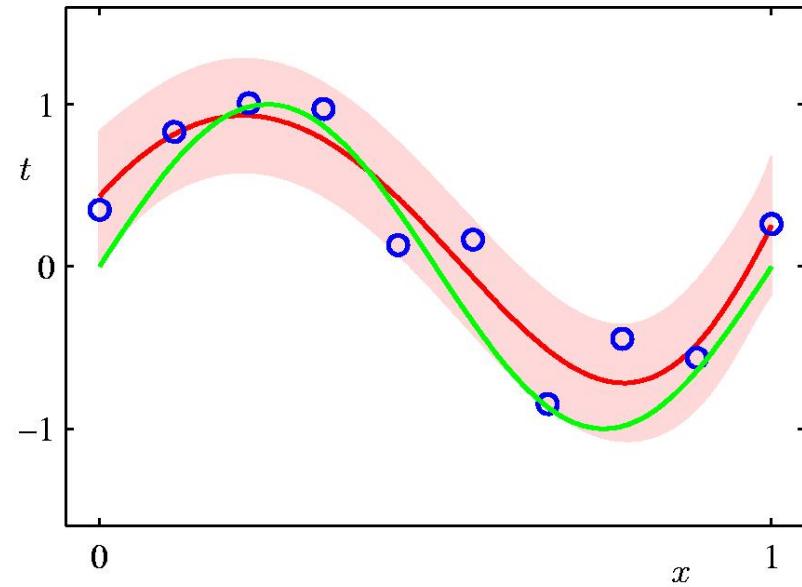
- In the limit, as  $N \rightarrow \infty$ , the second term goes to zero.
- The variance of the predictive distribution arises only from the additive noise governed by the parameter  $\beta$ .

# Predictive Distribution: Bayes vs. ML

Predictive distribution based on  
the maximum likelihood estimate



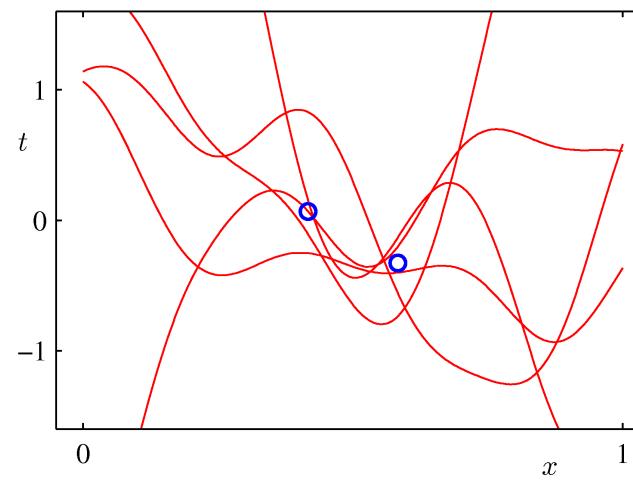
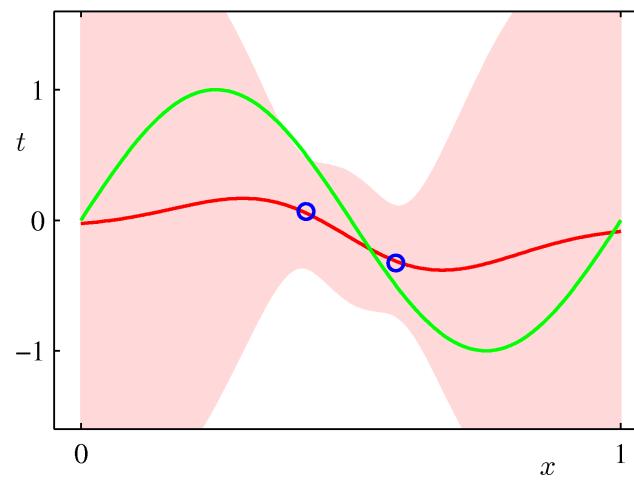
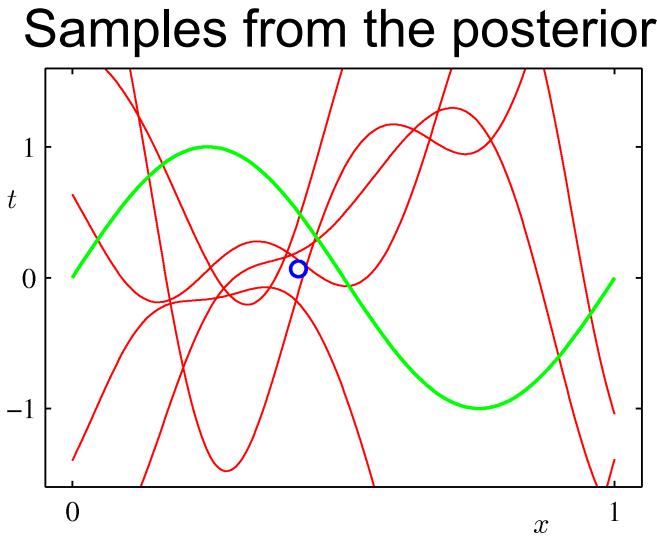
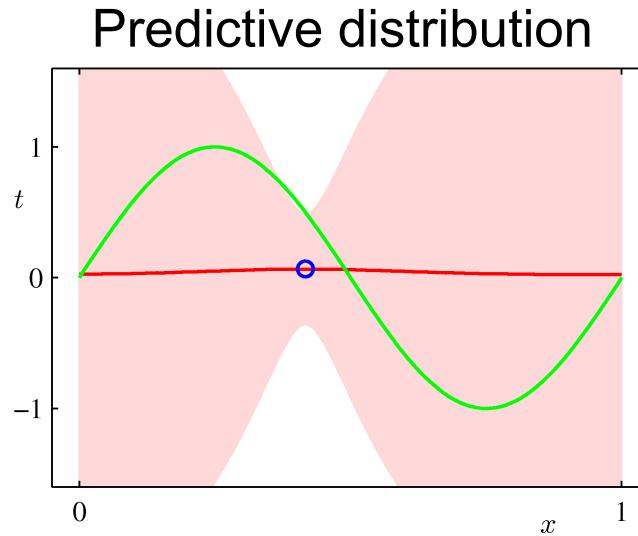
Bayesian predictive distribution



$$p(t|x, \mathbf{w}_{\text{ML}}, \beta_{\text{ML}}) = \mathcal{N}(t|y(x, \mathbf{w}_{\text{ML}}), \beta_{\text{ML}}^{-1}) \quad p(t|x, \mathbf{t}, \mathbf{X}) = \mathcal{N}(t|\mathbf{m}_N^T \boldsymbol{\phi}(x), \sigma_N^2(x))$$

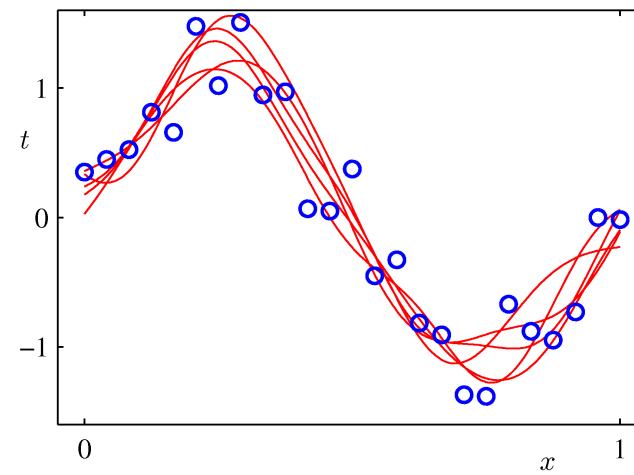
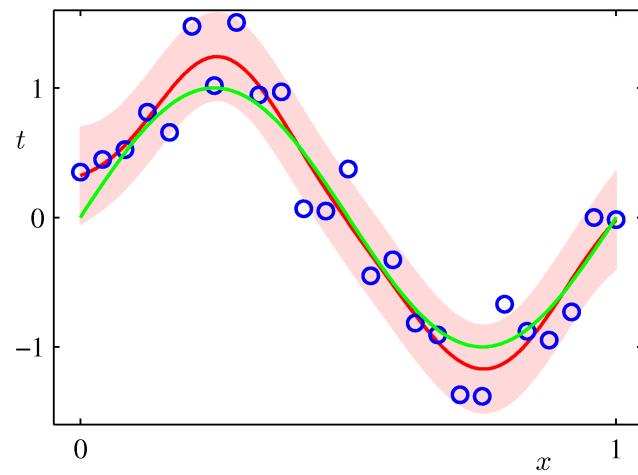
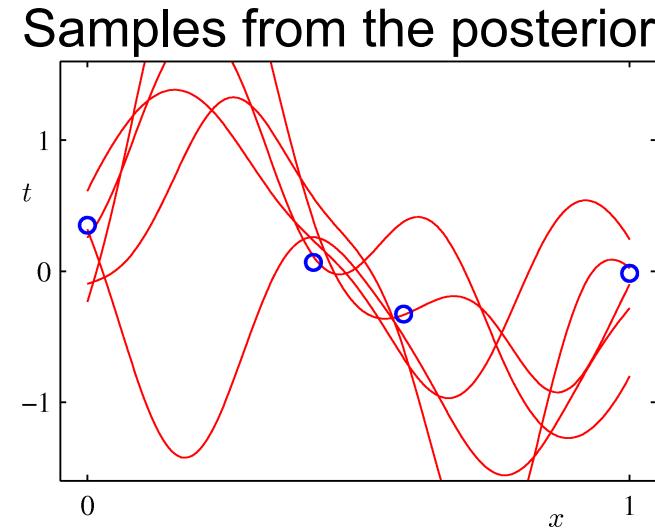
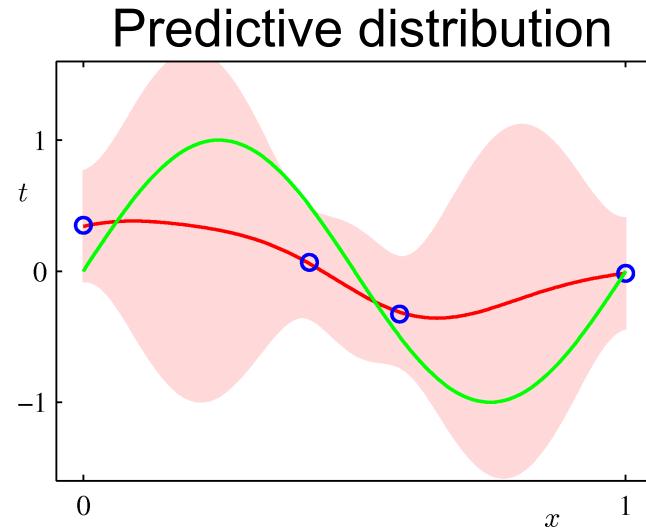
# Predictive Distribution

Sinusoidal dataset, 9 Gaussian basis functions.



# Predictive Distribution

Sinusoidal dataset, 9 Gaussian basis functions.



# Gamma-Gaussian Conjugate Prior

- So far we have assumed that the noise parameter  $\beta$  is known.
- If both  $\mathbf{w}$  and  $\beta$  are treated as unknown, then we can introduce a conjugate prior distribution that will be given by the **Gaussian-Gamma distribution**:

$$p(\mathbf{w}, \beta) = \mathcal{N}(\mathbf{w} | \mathbf{m}_0, \beta^{-1} \mathbf{S}_0) \text{Gam}(\beta | a_0, b_0),$$

where the Gamma distribution is given by:

$$\text{Gam}(\beta | a, b) = \frac{1}{\Gamma(a)} b^a \beta^{a-1} \exp(-b\beta), \quad \Gamma(a) = \int_0^\infty u^{a-1} e^{-u} du.$$

- The posterior distribution takes the same functional form as the prior:

$$\mathcal{N}(\mathbf{w} | \mathbf{m}_N, \beta^{-1} \mathbf{S}_N) \text{Gam}(\beta | a_N, b_N).$$

# Equivalent Kernel

- The predictive mean can be written as:

$$\begin{aligned}y(\mathbf{x}, \mathbf{m}_N) &= \mathbf{m}_N^T \boldsymbol{\phi}(\mathbf{x}) = \beta \boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\Phi}^T \mathbf{t} \\&= \sum_{n=1}^N \underbrace{\beta \boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_n)}_{k(\mathbf{x}, \mathbf{x}_n)} t_n \\&= \sum_{n=1}^N k(\mathbf{x}, \mathbf{x}_n) t_n.\end{aligned}$$

$$\begin{aligned}\mathbf{m}_N &= \beta \mathbf{S}_N \boldsymbol{\Phi}^T \mathbf{t} \\ \mathbf{S}_N^{-1} &= \alpha \mathbf{I} + \beta \boldsymbol{\Phi}^T \boldsymbol{\Phi}.\end{aligned}$$

Equivalent kernel  
or smoother  
matrix.

- The mean of the predictive distribution at a time  $\mathbf{x}$  can be written as a linear combination of the training set target values.
- Such regression functions are called linear smoothers.

# Equivalent Kernel

- The kernel as a covariance function:

$$\begin{aligned}\text{cov}[y(\mathbf{x}), y(\mathbf{x}')] &= \text{cov}[\phi(\mathbf{x})^T \mathbf{w}, \mathbf{w}^T \phi(\mathbf{x}')] \\ &= \phi(\mathbf{x})^T \mathbf{S}_N \phi(\mathbf{x}') = \beta^{-1} k(\mathbf{x}, \mathbf{x}').\end{aligned}$$

- We can avoid the use of basis functions and define the kernel function directly, leading to *Gaussian Processes*.

# Bayesian Model Comparison

- The Bayesian view of model comparison involves the use of **probabilities to represent uncertainty** in the choice of the model.
- We would like to compare a set of L models  $\{\mathcal{M}_i\}$ , where  $i = 1, 2, \dots, L$ , using a training set D.
- We **specify the prior distribution** over the different models  $p(\mathcal{M}_i)$ .
- Given a training set D, we **evaluate the posterior**:

$$p(\mathcal{M}_i | \mathcal{D}) \propto p(\mathcal{M}_i) p(\mathcal{D} | \mathcal{M}_i).$$

Posterior

Prior

*Model evidence or  
marginal likelihood*

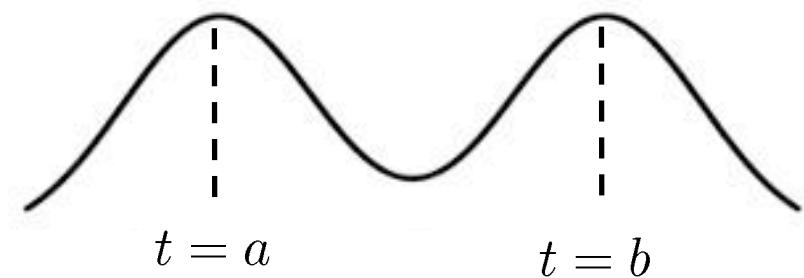
- For simplicity, we will assume that all models are a-priori equal.
- The model evidence expresses the preference shown by the data for different models.
- The ratio of the two model evidences for two models is known as **Bayes factor**: 
$$\frac{p(\mathcal{D} | \mathcal{M}_i)}{p(\mathcal{D} | \mathcal{M}_j)}$$

# Bayesian Model Comparison

- Once we compute the posterior  $p(M_i|\mathcal{D})$ , we can compute the predictive (mixture) distribution:

$$p(t|\mathbf{x}, \mathcal{D}) = \sum_{i=1}^L p(t|\mathbf{x}, M_i, \mathcal{D})p(M_i|\mathcal{D}).$$

- The overall predictive distribution is obtained by averaging the predictive distributions of individual models, weighted by the posterior probabilities.
- For example, if we have two models, and one predicts a narrow distribution around  $t=a$  while the other predicts a narrow distribution around  $t=b$ , then the overall predictions will be bimodal:
- A simpler approximation, known as model selection, is to use the model with the highest evidence.



# Bayesian Model Comparison

- Remember, the posterior is given by

$$p(\mathcal{M}_i | \mathcal{D}) \propto p(\mathcal{M}_i) p(\mathcal{D} | \mathcal{M}_i).$$

For a model governed by a set of parameters  $\mathbf{w}$ , the **model evidence** can be computed as follows:

$$p(\mathcal{D} | \mathcal{M}_i) = \int p(\mathcal{D} | \mathbf{w}, \mathcal{M}_i) p(\mathbf{w} | \mathcal{M}_i) d\mathbf{w}.$$

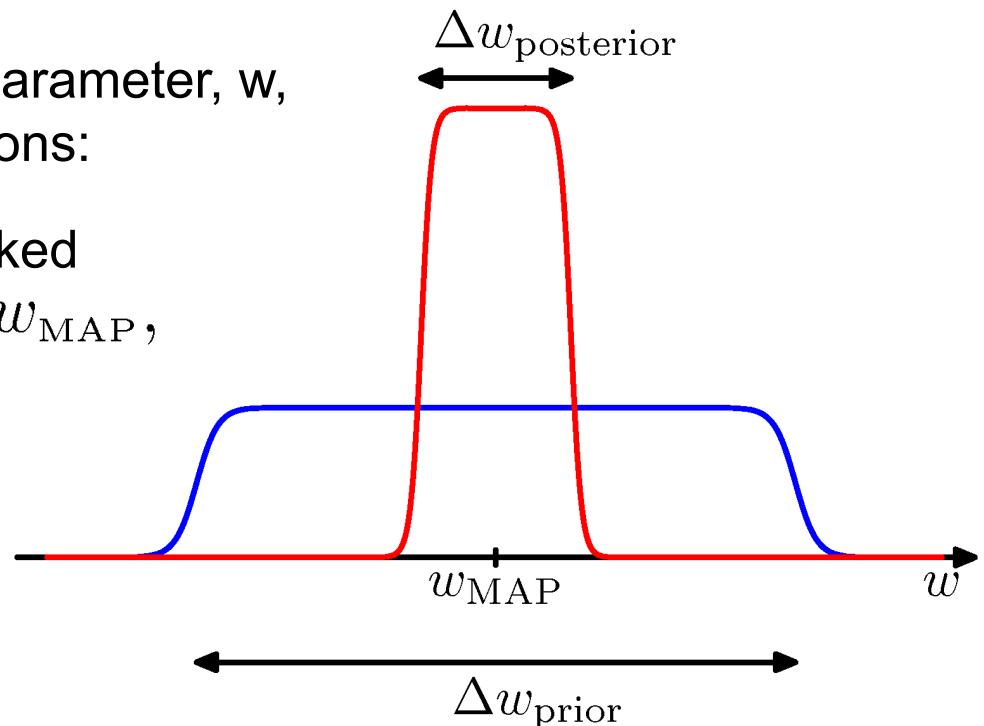
- Observe that **the evidence is the normalizing term** that appears in the denominator of Bayes' rule:

$$p(\mathbf{w} | \mathcal{D}, \mathcal{M}_i) = \frac{p(\mathcal{D} | \mathbf{w}, \mathcal{M}_i) p(\mathbf{w} | \mathcal{M}_i)}{p(\mathcal{D} | \mathcal{M}_i)}$$

- The model evidence is also often called **marginal likelihood**.

# Bayesian Model Comparison

- We next get some insight into the model evidence by making simple approximations.
- For a given model with a single parameter,  $w$ , consider the following approximations:
  - Assume that **the posterior** is picked around the most probable value  $w_{\text{MAP}}$ , with width  $\Delta w_{\text{posterior}}$
  - Assume that **the prior** is flat with width  $\Delta w_{\text{prior}}$



$$\begin{aligned} p(\mathcal{D}) &= \int p(\mathcal{D}|w)p(w) dw \\ &\simeq p(\mathcal{D}|w_{\text{MAP}}) \frac{\Delta w_{\text{posterior}}}{\Delta w_{\text{prior}}} \end{aligned}$$

# Bayesian Model Comparison

- Taking the logarithms, we obtain:

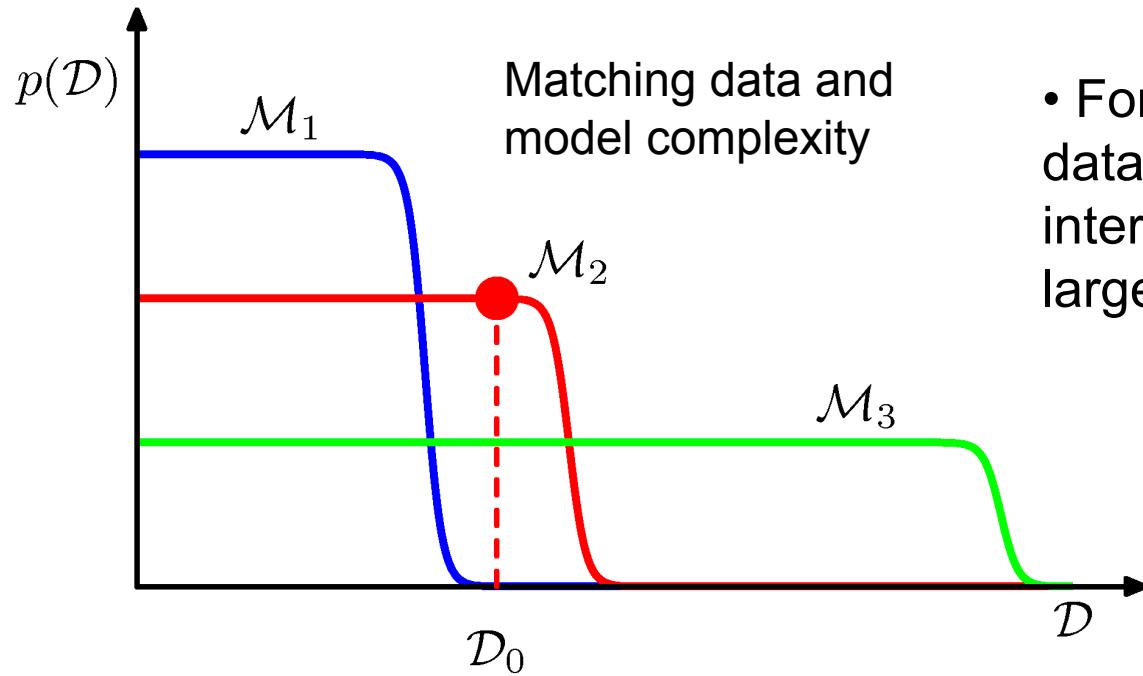
$$\ln p(\mathcal{D}) \simeq \ln p(\mathcal{D}|w_{\text{MAP}}) + \ln \underbrace{\left( \frac{\Delta w_{\text{posterior}}}{\Delta w_{\text{prior}}} \right)}_{\text{Negative}}.$$

- With  $M$  parameters, all assumed to have the same  $\Delta w_{\text{posterior}}/\Delta w_{\text{prior}}$  ratio:

$$\ln p(\mathcal{D}) \simeq \ln p(\mathcal{D}|w_{\text{MAP}}) + M \ln \underbrace{\left( \frac{\Delta w_{\text{posterior}}}{\Delta w_{\text{prior}}} \right)}_{\text{Negative and linear in } M}.$$

- As we **increase the complexity** of the model (increase the number of adaptive parameters  $M$ ), **the first term will increase**, whereas the second term will decrease due to the dependence on  $M$ .
- The optimal model complexity: trade-off between these two competing terms.

# Bayesian Model Comparison



- For the particular observed dataset  $\mathcal{D}_0$ , the model  $\mathcal{M}_2$  with intermediate complexity has the largest evidence.
- The simple model cannot fit the data well, whereas the more complex model spreads its predictive probability and so assigns relatively small probability to any one of them.
- The marginal likelihood is **very sensitive to the prior used!**
- Computing the marginal likelihood makes sense only if you are certain about the choice of the prior.

# Evidence Approximation

- In the fully Bayesian approach, we would also specify a prior distribution over the hyperparameters  $p(\alpha, \beta)$ .
- The **fully Bayesian predictive distribution** is then given by marginalizing over model parameters as well as hyperparameters:

$$p(t^* | \mathbf{x}^*, \mathcal{D}) = \iiint p(t^* | \mathbf{x}^*, \mathbf{w}, \beta) p(\mathbf{w} | \mathcal{D}, \alpha, \beta) p(\alpha, \beta | \mathcal{D}) d\mathbf{w} d\alpha d\beta.$$

Likelihood      posterior      posterior over  
                          over weights      hyperparameters

↑                         ↑                         ↑  
target and input      precision of      precision      training data:  
on test case            output noise      of the prior      inputs and targets

- However, this integral is intractable (even when everything is Gaussian). Need to approximate.
- Note: the fully Bayesian approach is to integrate over the posterior distribution for  $\{\alpha, \beta, \mathbf{w}\}$ . This can be done by MCMC, which we will consider later. For now, we will use evidence approximation: much faster.

# Evidence Approximation

- The fully Bayesian predictive distribution is given by:

$$p(t^* | \mathbf{x}^*, \mathcal{D}) = \iiint p(t^* | \mathbf{x}^*, \mathbf{w}, \beta) p(\mathbf{w} | \mathcal{D}, \alpha, \beta) p(\alpha, \beta | \mathcal{D}) d\mathbf{w} d\alpha d\beta.$$

- If we assume that the posterior over hyperparameters  $\alpha$  and  $\beta$  is sharply picked, we can approximate:

$$p(t^* | \mathbf{x}^*, \mathcal{D}) \approx p(t^* | \mathbf{x}^*, \mathcal{D}, \hat{\alpha}, \hat{\beta}) = \int p(t^* | \mathbf{x}^*, \mathcal{D}, \hat{\alpha}, \hat{\beta}) p(\mathbf{w} | \mathcal{D}, \hat{\alpha}, \hat{\beta}) d\mathbf{w}.$$

where  $(\hat{\alpha}, \hat{\beta})$  is the mode of the posterior  $p(\alpha, \beta | \mathcal{D})$ .

- So we **integrate out parameters** but **maximize over hyperparameters**.
- This is known as **empirical Bayes, Type II Maximum Likelihood, Evidence Approximation**.

# Evidence Approximation

- From Bayes' rule we obtain:

$$p(\alpha, \beta | \mathbf{t}, \mathbf{X}) \propto p(\mathbf{t} | \mathbf{X}, \alpha, \beta) p(\alpha, \beta).$$

- If we assume that the prior over hyperparameters  $p(\alpha, \beta)$  is flat, we get:

$$p(\alpha, \beta | \mathbf{t}, \mathbf{X}) \propto p(\mathbf{t} | \mathbf{X}, \alpha, \beta).$$

- The values  $(\hat{\alpha}, \hat{\beta})$  are obtained by maximizing the marginal likelihood  $p(\mathbf{t} | \mathbf{X}, \alpha, \beta)$ .
- This will allow us to determine the values of these hyperparameters from the training data.
- Recall that the ratio  $\alpha/\beta$  is analogous to the regularization parameter.

# Evidence Approximation

- The marginal likelihood is obtained by integrating out parameters:

$$p(\mathbf{t}|\mathbf{X}, \alpha, \beta) = \int p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta)p(\mathbf{w}|\alpha)d\mathbf{w}.$$

$$\begin{aligned}\mathbf{m}_N &= \beta \mathbf{S}_N \boldsymbol{\Phi}^T \mathbf{t} \\ \mathbf{S}_N^{-1} &= \alpha \mathbf{I} + \beta \boldsymbol{\Phi}^T \boldsymbol{\Phi}.\end{aligned}$$

- We can write the evidence function in the form:

$$p(\mathbf{t}|\mathbf{X}, \alpha, \beta) = \left(\frac{\beta}{2\pi}\right)^{N/2} \left(\frac{\alpha}{2\pi}\right)^{M/2} \int \exp(-E(\mathbf{w}))d\mathbf{w},$$

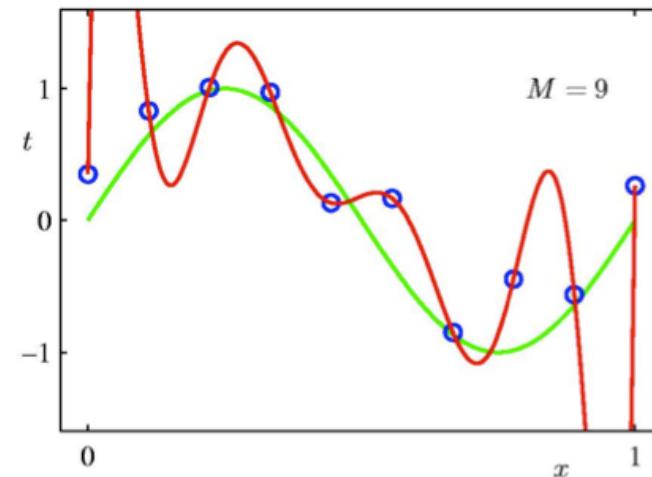
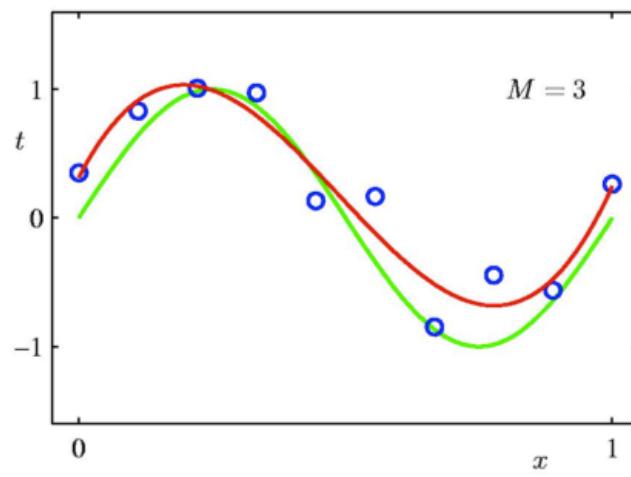
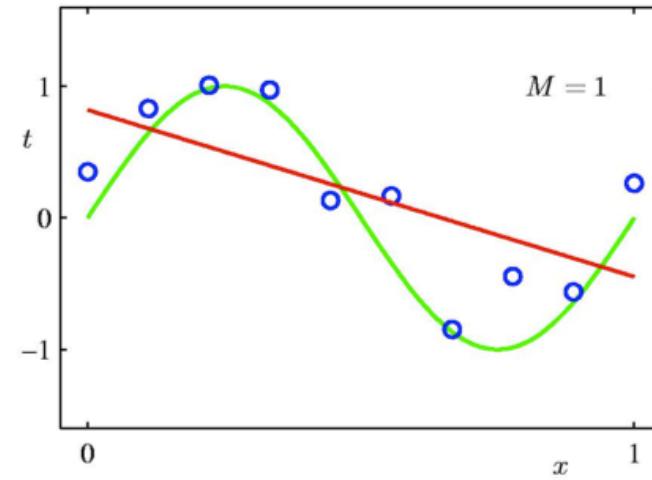
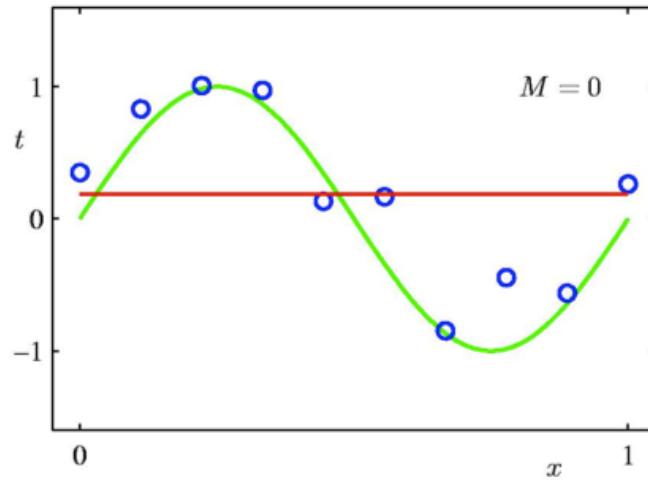
where

$$E(\mathbf{w}) = \beta E_{\mathcal{D}}(\mathbf{w}) + \alpha E_W(\mathbf{w}) = \frac{\beta}{2} \|\mathbf{t} - \boldsymbol{\Phi} \mathbf{w}\|^2 + \frac{\alpha}{2} \mathbf{w}^T \mathbf{w}.$$

- Using standard results for the Gaussian distribution, we obtain:

$$\ln p(\mathbf{t}|\alpha, \beta) = \frac{M}{2} \ln \alpha + \frac{N}{2} \ln \beta - E(\mathbf{m}_N) + \frac{1}{2} \ln |\mathbf{S}_N| - \frac{N}{2} \ln(2\pi).$$

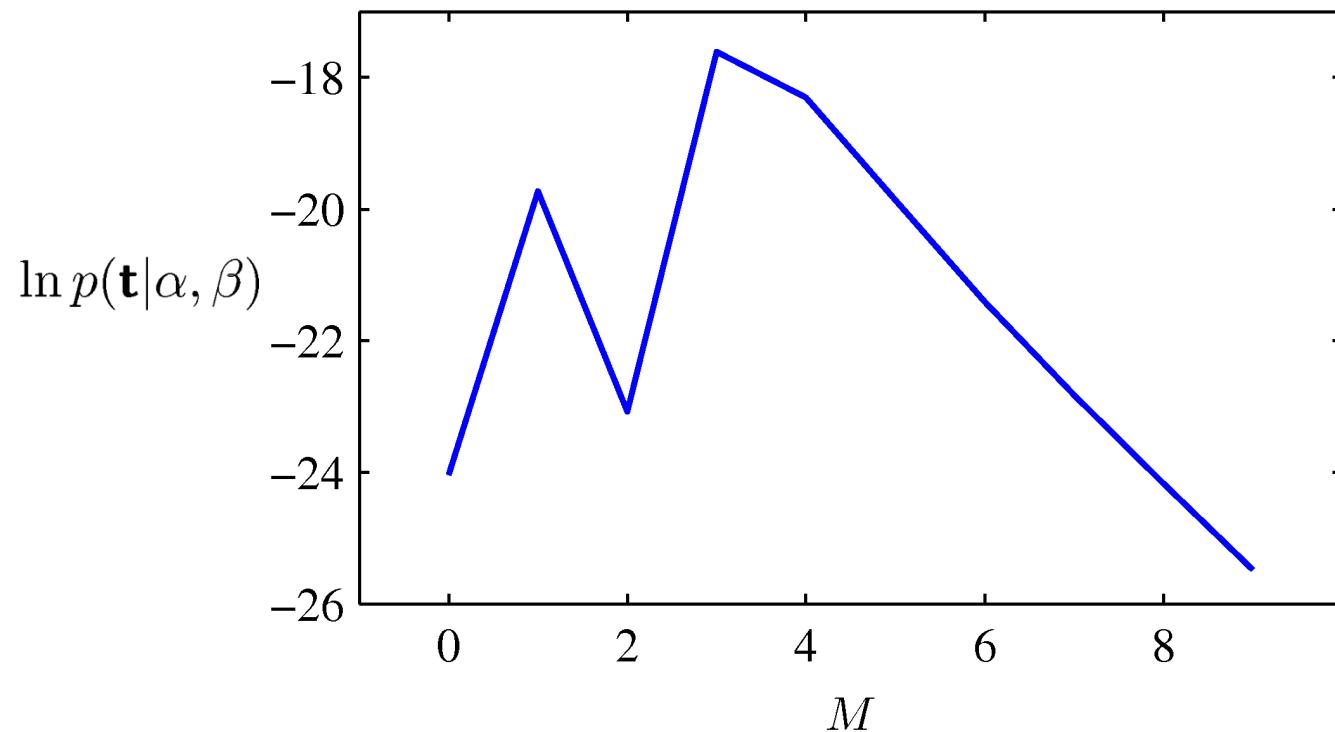
# Some Fits to the Data



For  $M=9$ , we have fitted the training data perfectly.

# Evidence Approximation

Using sinusoidal data,  $M^{\text{th}}$  degree polynomial.



The evidence favours the model with  $M=3$ .

# Maximizing the Evidence

- Remember:

$$\ln p(\mathbf{t}|\alpha, \beta) = \frac{M}{2} \ln \alpha + \frac{N}{2} \ln \beta - E(\mathbf{m}_N) + \frac{1}{2} \ln |\mathbf{S}_N| - \frac{N}{2} \ln(2\pi).$$

- To maximize the evidence  $p(\mathbf{t}|\mathbf{X}, \alpha, \beta)$  with respect to  $\alpha$  and  $\beta$ , define the following eigenvector equation:

$$(\beta \Phi^T \Phi) \mathbf{u}_i = \lambda_i \mathbf{u}_i.$$

Precision matrix of the Gaussian posterior distribution

- Therefore the matrix:

$$\mathbf{A} = \mathbf{S}_N^{-1} = \alpha \mathbf{I} + \beta \Phi^T \Phi$$

has eigenvalues  $\alpha + \lambda_i$ .

- The derivative:

$$\frac{d}{d\alpha} \ln |\mathbf{A}| = \frac{d}{d\alpha} \ln \prod_i (\alpha + \lambda_i) = \frac{d}{d\alpha} \sum_i \ln(\alpha + \lambda_i) = \sum_i \frac{1}{\alpha + \lambda_i}.$$

# Maximizing the Evidence

- Remember:

$$\ln p(\mathbf{t}|\alpha, \beta) = \frac{M}{2} \ln \alpha + \frac{N}{2} \ln \beta - E(\mathbf{m}_N) + \frac{1}{2} \ln |\mathbf{S}_N| - \frac{N}{2} \ln(2\pi).$$

where

$$E(\mathbf{m}_N) = \frac{\beta}{2} \|\mathbf{t} - \Phi \mathbf{m}_N\|^2 + \frac{\alpha}{2} \mathbf{m}_N^T \mathbf{m}_N.$$

- Differentiating  $\ln p(\mathbf{t}|\alpha, \beta)$ , the stationary points with respect to  $\alpha$  satisfy:

$$\frac{M}{2\alpha} - \frac{1}{2} \mathbf{m}_N^T \mathbf{m}_N - \frac{1}{2} \sum_i \frac{1}{\alpha + \lambda_i} = 0.$$

$$\alpha \mathbf{m}_N^T \mathbf{m}_N = M - \alpha \sum_i \frac{1}{\alpha + \lambda_i} = \gamma,$$

where the quantity  $\gamma$ , effective number of parameters, can be defined as:

$$\gamma = \sum_i \frac{\lambda_i}{\lambda_i + \alpha}.$$

# Maximizing the Evidence

- The stationary points with respect to  $\alpha$  satisfy:

$$\alpha \mathbf{m}_N^T \mathbf{m}_N = M - \alpha \sum_i \frac{1}{\alpha + \lambda_i} = \gamma,$$

where the quantity  $\gamma$ , **effective number of parameters**, is defined as:

$$\gamma = \sum_i \frac{\lambda_i}{\lambda_i + \alpha}.$$

Note that the eigenvalues need to be computed only once.

- Iterate until convergence:

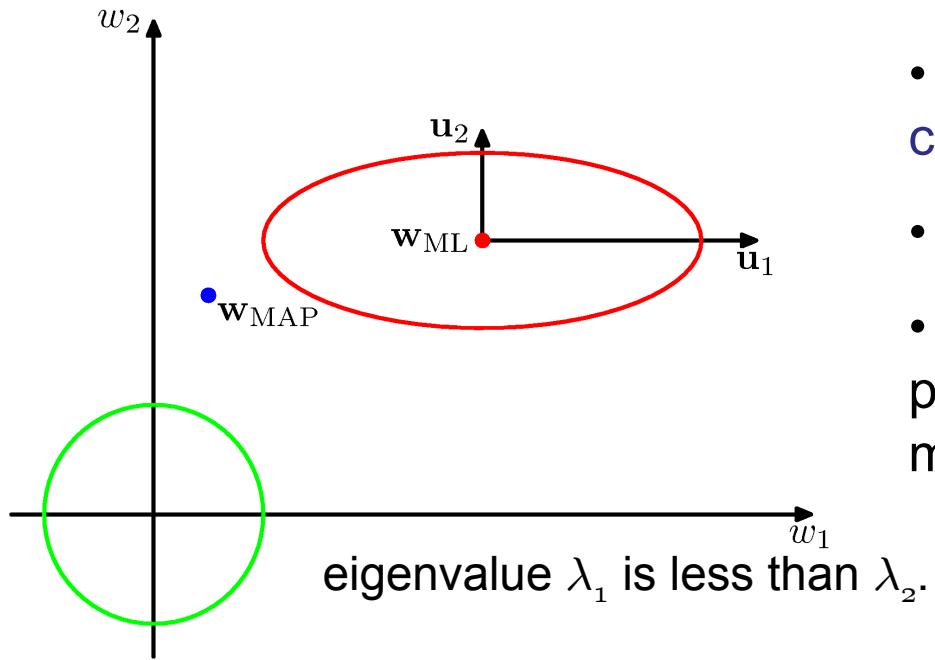
$$\alpha = \frac{\gamma}{\mathbf{m}_N^T \mathbf{m}}; \quad \gamma = \sum_i \frac{\lambda_i}{\lambda_i + \alpha}; \quad \begin{aligned} \mathbf{m}_N &= \beta \mathbf{S}_N \Phi^T \mathbf{t} \\ \mathbf{S}_N^{-1} &= \alpha \mathbf{I} + \beta \Phi^T \Phi. \end{aligned}$$

- Similarly:

$$\frac{1}{\beta} = \frac{1}{N - \gamma} \sum_{n=1}^N \{t_n - \mathbf{m}_N^T \phi(\mathbf{x}_n)\}^2$$

# Effective Number of Parameters

- Consider the contours of the likelihood function and the prior.



- The eigenvalue  $\lambda_i$  measures the curvature of the log-likelihood function.
- The quantity  $\gamma$  will lie  $0 \leq \gamma \leq M$ .
- For  $\lambda_i \gg \alpha$ , the corresponding parameter  $w_i$  will be close to its maximum likelihood. The ratio:  $\frac{\lambda_i}{\lambda_i + \alpha}$  will be close to one.

- Such parameters are called well determined, as their values are highly constrained by the data.
- For  $\lambda_i \ll \alpha$ , the corresponding parameters will be close to zero (pulled by the prior), as will the ratio  $\lambda_i/(\lambda_i + \alpha)$ .
- We see that  $\gamma$  measures the effective total number of well determined parameters.

# Quick Approximation

- In the limit  $N \gg M$ ,  $\gamma = M$ , and we consider to use the easy to compute approximations:

$$\begin{aligned}\alpha &= \frac{M}{\mathbf{m}_N^T \mathbf{m}_N} \\ \frac{1}{\beta} &= \frac{1}{N} \sum_{n=1}^N \{t_n - \mathbf{m}_N^T \phi(\mathbf{x}_n)\}^2.\end{aligned}$$

# Limitations

- $M$  basis function along each dimension of a  $D$ -dimensional input space requires  $M^D$  basis functions: **the curse of dimensionality**.
- Fortunately, we can get away with fewer basis functions, by choosing these using the training data (e.g. adaptive basis functions), which we will see later.
- Second, the data vectors typically lie close to a nonlinear low-dimensional manifold, whose intrinsic dimensionality is smaller than that of the input space.