1

In many physical and Engineering problems, we always seek a solution of the differential equations, whether it is ordinary or partial, which satisfies some specified conditions called the boundary conditions. Any differential equation together with these boundary conditions is called boundary value problem.

Partial Differential Equations

1. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ (Wave equation)

2. $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (One dimensional heat flow equation)

3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Two dimensional heat flow equation)

Classification of Partial Differential Equations of the second order

General PDE of 2^{nd} order

A second order partial differential equation in the function u of the two independent variables x, y is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0. \quad \rightarrow (1)$$

★ If $f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0$ is linear, then equation (1) is said to be linear

* If $f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$ is non-linear, then the above equation is quasi-linear.

Classification of the 2^{nd} order PDE

If
(i) $B^2 - 4AC < 0$ [Elliptic equation]
(ii) $B^2 - 4AC = 0$ [Parabolic equation]
(iii) $B^2 - 4AC > 0$ [Hyperbolic equation]

Example for Elliptic, Parabolic, Hyperbolic differential Equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ Elliptic (Laplace's equation in two dimension)

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$ Hyperbolic (One dimensional wave equation)

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$ Parabolic (One dimensional heat flow equation)

① Classify the p.d.e $x^2 f_{xx} + (1-y^2) f_{yy} = 0$ for $-1 < y < 1$, $-\infty < x < \infty$.

Soln: Second order PDE is

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f(x,y,u,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0.$$

Here
$A \to x^2$
$B \to 0$
$C \to 1-y^2$

$$\therefore B^2 - 4AC = -4x^2(1-y^2)$$
$$= 4x^2(y^2-1)$$

x^2 is always +ve in $-\infty < x < \infty$.

In $-1<y<1$, y^2-1 is negative.

$\therefore B^2-4AC = -ve \quad (x \neq 0)$

$B^2-4AC < 0$

The equation is elliptic.

2) Find the nature of the PDE $xf_{xx}+yf_{yy}=0$ $x>0, y>0$.

Soln Second order PDE is

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0.$$

$\therefore B^2-4AC = -4xy$

$= -ve$ when $x>0, y>0$

$B^2-4AC < 0$

The equation is elliptic.

Here	
	$A \to x$
	$B \to 0$
	$C \to y$

3) classify the p.d.e $f_{xx}-2f_{xy}=0$, $x>0, y>0$.

Soln Second order PDE is

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

$B^2-4AC = 4-0 = +ve$

$B^2-4AC > 0$

$\therefore$ Hyperbolic.

Here	
	$A \to 1$
	$B \to -2$
	$C \to 0$

4) Find the nature of the p.d.e $u_{xx}-2u_{xy}+u_{yy}=0$.

Soln Second order PDE is.

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

$B^2-4AC = 4-4 = 0$

$\therefore$ Parabolic.

	$A \to 1$
	$B \to -2$
	$C \to 1$

5) Find the nature of the p.d.e

$f_{xx} + 2f_{xy} + 4f_{yy} = 0$, $x>0, y>0$

solution

Second order partial differential equations

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

Here $A \to 1$
$B \to 2$
$C \to 4$

$B^2 - 4AC = 4-16 = -ve$

$B^2 - 4AC < 0$

$\therefore$ Elliptic.

Note: To classify the differential equations the region is very important.

For example the PDE $xf_{xx} + f_{yy} = 0$ is

(i) elliptic if $x>0$

(ii) parabolic if $x=0$

(iii) Hyperbolic if $x<0$.

ONE - DIMENSIONAL WAVE EQUATION

THE VIBRATING STRING

As our first physical application of partial differential equation, let us derive the equation governing small transverse vibrations of an elastic string which is stretched to a length 'l' and other fixed at the end point 'O' and 'l' on the x-axis (see fig).

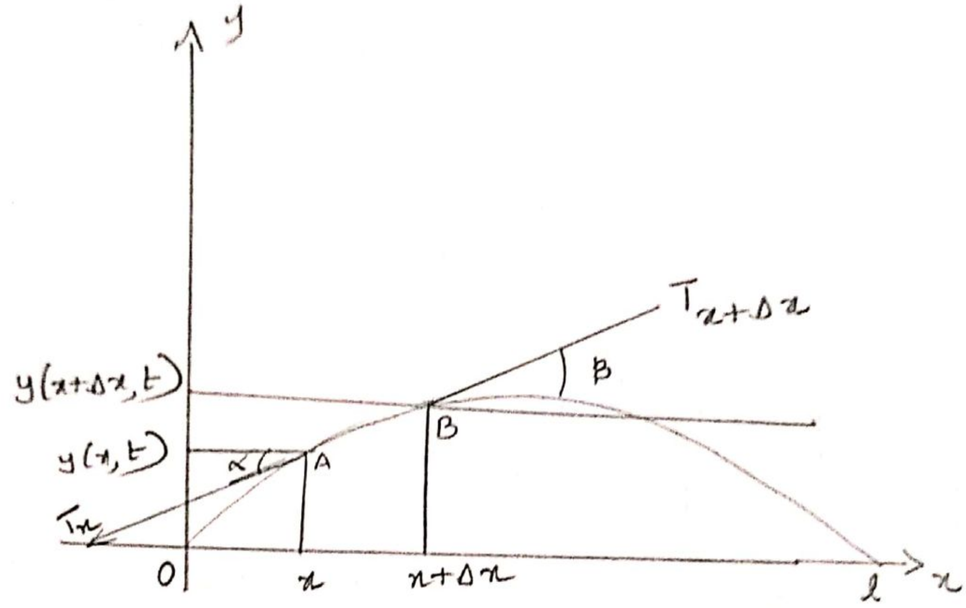
Suppose that the string is pulled back

vertically a distance that is very small compared to the length 'l' and released at time t=0, causing it to vibrate. Our problem is to determine the displacement y(x,t) of the point on the string that is 'x' units away from the end 'O' at any time t>0. When we derive a differential equation, which corresponds to a given physical problem, we usually make some simplifying assumptions in order that the resulting equation does not become too complicated. In our present case we make the following assumptions.

* The mass of the string per unit is constant.
* The string is perfectly elastic and does not offer any resistance to bending.
* The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected.
* The string performs a small transverse motion in a vertical plane that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point

of the string remain small in absolute value.

Derivation of One-dimensional wave equation



To derive the one dimensional wave equation we consider the forces acting on a small portion Δx of the string (see fig).

By Newton's second law of motion, the total force acting on this piece of string is equal to the mass of the string multiplied by its acceleration.

$$\text{Force} = \text{mass} \times \text{acceleration}$$
$$= m \times a$$
$$= (m\Delta x)\frac{\partial^2 y}{\partial t^2} \longrightarrow (1)$$

Here to find acceleration we ~~fo~~ take partial derivative of 'y', w.r.to 't' because 'y' is a function of two variables. We assume in this equation that the string is moving only in the xy-plane and that each particle in the string moves only vertically.

let T_x and $T_{x+\Delta x}$ be the tension vectors at the end points of the given segment $AB(=\Delta x)$. These forces are applied tangentially since the string offers no resistance to bending. Since there is no motion in the x-direction the x-components of the tension vectors must coincide.

But the horizontal components of T_x and $T_{x+\Delta x}$ are $T_x \cos\alpha$ and $T_{x+\Delta x}\cos\beta$ respectively

$$\therefore T_x \cos\alpha = T_{x+\Delta x}\cos\beta = T \rightarrow (2)$$

[$\because$ The horizontal components of the tension must be coincide].

Similarly, in vertical direction we have two forces, namely the vertical components of T_x and $T_{x+\Delta x}$: $T_x \sin\alpha$ and $T_{x+\Delta x}\sin\beta$ here the minus sign appears because that component of A is directed downward. By Newton's Second law the resultant of these two forces is equal to the mass $(m\Delta x)$ of the portion times the acceleration $\frac{\partial^2 y}{\partial x^2}$, evaluated at some point between x and $x+\Delta x$.

Hence $T_{x+\Delta x}\sin\beta - T_x \sin\alpha = m\Delta x \frac{\partial^2 y}{\partial t^2}$ $\rightarrow (3)$

Dividing each term in (3) by the

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corresponding term in (3) we get

$$\frac{T_{x+\delta x}\sin\beta}{T_{x+\delta x}\cos\beta} - \frac{T_x \sin\alpha}{T_x \cos\alpha} = \frac{m}{T}\,\delta x\,\frac{\partial^2 y}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{m}{T}\,\delta x \cdot \frac{\partial^2 y}{\partial t^2} \rightarrow (4)$$

Here $\tan\alpha = \text{slope at } x = \left(\frac{\partial y}{\partial x}\right)_x$

$\tan\beta = \text{slope at } x+\delta x$

$= \left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$ $\rightarrow (5)$

substituting (5) in (4)

$$\frac{1}{\delta x}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \frac{m}{T}\frac{\partial^2 y}{\partial t^2}$$

let $\delta x \to 0$, we obtain in the limit case the equation $\frac{\partial^2 y}{\partial x^2} = \frac{m}{T}\frac{\partial^2 y}{\partial t^2}$.

(or) $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

where $a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass}}$.

This equation is often called "One dimensional Wave equation"

NOTE: In the above derivation we have not made use of the fact that the string is fixed at its end points. We can write these boundary conditions as $y(0,t) = y(l,t) = 0$, $t>0$. In addition to the above, we have not taken into account the initial distortion of the string and the fact that it was at rest when released. These initial conditions may be written as $y(x,0) = f(x),\ 0 \le x \le l$.

$$\left[\frac{\partial y}{\partial t}(x,t)\right]_{t=0} = 0$$ where $f(x)$ is the initial position of the string and $\left[\frac{\partial y}{\partial t}(x,t)\right]_{t=0}$ is the velocity of the point 'x' units away from the origin at time 't'.

The direct method we use to solve this type of problem is due to d'Alembert. Since it is only rarely possible to apply this method, we develop a method which is applicable to us.

The simplest method for solving an ordinary differential equation is variable separable method. Although we have now two independent variables, we can nevertheless adapt the technique to a partial differential equation of the above form.

Solution of the wave equation.

We know that one dimensional wave equation is $$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$$

let $y(x,t) = X(x)T(t)$ be the solution of the given equation where
$X \to$ function of 'x' only
$T \to$ function of 't' only.

(2) $\Rightarrow$ $\frac{\partial y}{\partial x} = X'T$ | $\frac{\partial y}{\partial t} = XT'$

$\frac{\partial^2 y}{\partial x^2} = X''T$ | $\frac{\partial^2 y}{\partial t^2} = XT''$

Substituting these values in (1) we get

$$XT'' = a^2X''T$$

$$\frac{X''}{X} = \frac{T''}{a^2T} = k$$

$\frac{X''}{X} = k$ | $\frac{T''}{a^2T} = k$

$X'' = kX$ | $T'' = ka^2T$

$X'' - kX = 0 \rightarrow (3)$ | $T'' - ka^2T = 0 \rightarrow (4)$

The equations (3) and (4) are ordinary differential equations the solution of which depends on the value of k. Three cases arise

Case (i) let k be positive i.e., $k = p^2$

[Here p^2 is always positive whether p is +ve or -ve].

Now equations (3) and (4) becomes

$X'' - p^2X = 0$ | $T'' - p^2a^2T = 0$

$\frac{d^2X}{dx^2} - p^2X = 0$. | $\frac{d^2T}{dt^2} - p^2a^2T = 0$

A.E is $m^2 - p^2 = 0$ | $m^2 - a^2p^2 = 0$

$m = \pm p$ | $m^2 = a^2p^2$

| $m = \pm ap$

$X = c_1 e^{px} + c_2 e^{-px} \rightarrow (5)$ $\quad T = c_3 e^{pat} + c_4 e^{-pat} \rightarrow (6)$

Substituting (5) & (6) in (2) we get

$$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat})$$

Case (ii) When k is negative, i.e., $k = -p^2$

[Here whether p is +ve or −ve p^2 is always +ve. ∴ $-p^2$ is always −ve]

Now equations (3) & (4) becomes

$$X'' + p^2 X = 0 \qquad T'' + a^2p^2 T = 0$$

$$\frac{d^2X}{dx^2} + p^2 X = 0 \qquad \frac{d^2T}{dt^2} + a^2p^2 T = 0$$

A.E is $m^2 + p^2 = 0$ $\qquad$ A.E is $m^2 + a^2p^2 = 0$

$$m^2 = -p^2 \qquad m^2 = -a^2p^2$$

$$m = \pm ip \qquad m = \pm iap$$

$X = c_5 \cos px + c_6 \sin px \rightarrow (7)$ $\quad T = c_7 \cos pat + c_8 \sin pat \rightarrow (8)$

Substituting (7) & (8) in (2) we get

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat)$$

Case (iii): When $k=0$, the equations (3) and (4) becomes

$$X'' = 0 \qquad T'' = 0$$

$$\frac{d^2X}{dx^2} = 0 \qquad \frac{d^2T}{dt^2} = 0$$

Integrating twice w.r.to 'x' $\qquad$ Integrating twice w.r.to 't'

$X = c_9 x + c_{10} \rightarrow (9)$ $\qquad$ $T = c_{11} t + c_{12} \rightarrow (10)$

Substituting (9) and (10) in (2) we get,

$$y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12}).$$

Thus depending upon the value of k, the various possible solutions of the wave equation are

$$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat}) \rightarrow (11)$$

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat) \rightarrow (12)$$

$$y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12}) \rightarrow (13)$$

Correct solution for vibration of string equation

We can choose the correct solution as follows:

Since we are dealing with problems on vibrations, y must be a periodic function of x and t. Therefore, we choose the solution which contains the trignometric terms since sine and cosine functions are periodic in nature. Hence we select the correct solution as

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$$

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1) In $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, what is '$a^2$'?

soln

$$a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length of the string}}$$

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2) Why 'a^2' (instead of 'a') is used in the vibration of string equation. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$?

soln

$$a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length}} = \text{always } +ve$$

Since $\frac{T}{m}$ is +ve, it is denoted by 'a^2' (not by 'a')

(3) Give the steps to solve one-dimensional wave equation with zero initial velocity.

soln 1. Write the boundary conditions

(a) $y(0,t)=0$

(b) $y(l,t)=0$

(c) $\frac{\partial y}{\partial t}(x,0)=0$

(d) $y(x,0)=f(x)$

2. Choose correct solution

$y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos pat + c_4\sin pat)$,

3. Applying condition (a), we get

$c_1=0$

Applying condition (b), we get

$p=\frac{n\pi}{l}$.

4. Applying condition (c), we get $c_4=0$

The most general solution is

$$y(x,t)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

5. Applying condition (d)

$$y(x,0)=\sum_{n=1}^{\infty} c_n\sin\frac{n\pi x}{l}=f(x) \longrightarrow (A)$$

Expand $f(x)$ in a sine series in $(0,l)$

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \longrightarrow (B)$$

where $c_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$ (using (A) & (B))

6. Substituting c_n in (4) we get solution of the given problem.

4) Explain the initial and boundary value problems.

In ordinary differential equations, first we get the general solution which contains the arbitrary constants and then we determine these constants from the given initial values. This type of problems are called initial value problems. In many physical problems, we always seek a solution of the differential equations which satisfies some specified conditions at the boundaries called boundary conditions. Any differential equations together with these boundary conditions is called boundary value problems.

Problems based on Vibrating string with zero initial velocity.

(1) A string is stretched and fastened at two points $x=0$ and $x=l$ apart. Motion is starting by displacing the string into the form $y=k(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Soln The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

From the given problem, we get the following boundary conditions

(i) $y(0,t) = 0$ for all $t > 0$.

(ii) $y(l,t) = 0$ for all $t > 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$ ($\because$ initial velocity is zero).

(iv) $y(x,0) = k(lx - x^2)$

The correct solution which satisfies our boundary conditions is

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \rightarrow (1)$

Applying condition (i) in (1), we get

$y(0,t) = c_1 (c_3 \cos pat + c_4 \sin pat) = 0$

$c_1 = 0$ and $c_3 \cos pat + c_4 \sin pat \neq 0$

Put $c_1 = 0$ in (1) we get

$y(x,t) = c_2 \sin px (c_3 \cos pat + c_4 \sin pat) \rightarrow (2)$

Applying condition (ii) in (2) we get

$y(l,t) = c_2 \sin pl (c_3 \cos pat + c_4 \sin pat) = 0$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$ [it is defined for all t]

$\therefore$ either $c_2 = 0$ or $\sin pl = 0$

Suppose if we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial solution.

$\therefore$ we take $\sin pl = 0$

$pl = n\pi$ $\quad \because \sin n\pi = 0$.

$p = \frac{n\pi}{l}$ [n being an integer] substituting $p = \frac{n\pi}{l}$ in (2) we get

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi at}{l} + c_4\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Differentiating (3) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = c_2\sin\frac{n\pi x}{l}\left(-c_3\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + c_4\frac{n\pi a}{l}\cos\frac{n\pi at}{l}\right)$$

Applying condition (iii) we get

$$\frac{\partial y}{\partial t}(x,0) = c_2\sin\frac{n\pi x}{l}\left(c_4\frac{n\pi a}{l}\right) = 0.$$

Here $c_2 \neq 0$ (already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

& $\frac{n\pi a}{l} \neq 0$ ($\because$ all are constants)

$\therefore c_4 = 0$.

$\therefore$ Substituting $c_4 = 0$ in (3) we get

$$y(x,t) = c_2c_3\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$= c_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4)$$

where $c_n = c_2c_3$

$\therefore$ The most general solution of (4) is

$$y(x,t) = \sum_{n=1}^{\infty} c_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (5)$$

Applying the boundary condition (iv) in (5) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n\sin\frac{n\pi x}{l} = k(lx - x^2) \rightarrow (6)$$

To find c_n expand $k(lx-x^2)$ in a half-range Fourier sine series in the interval $(0, l)$

$$k(lx-x^2) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \rightarrow (7)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

From (6) & (7) we get $\boxed{b_n = c_n}$

$\therefore\; c_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) - (l-2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n^3\pi^3}{l^3}\right)}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-(lx-x^2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) + (l-2x)\left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right) - 2\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n^3\pi^3}{l^3}\right)}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\left\{0+0-\frac{2\cos n\pi}{\left(\frac{n^3\pi^3}{l^3}\right)}\right\} - \left\{0+0-\frac{2}{\left(\frac{n^3\pi^3}{l^3}\right)}\right\}\right]$$

$$= \frac{2k}{l}\left[-\frac{2\cos n\pi}{\left(\frac{n^3\pi^3}{l^3}\right)} + \frac{2}{\left(\frac{n^3\pi^3}{l^3}\right)}\right] \quad [\because \sin n\pi = 0]$$

$$= \frac{2k}{l}\cdot\frac{l^3}{n^3\pi^3}\left[-2\cos n\pi + 2\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1-\cos n\pi\right]$$

$$c_n = \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

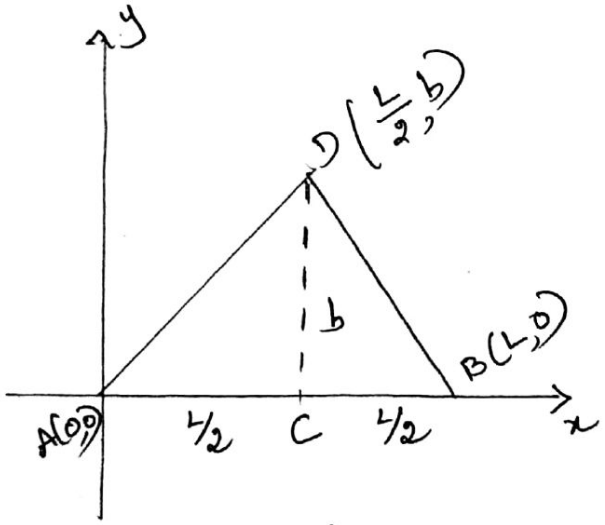
$$\therefore\; c_n = \begin{cases} 0 & \text{when '}n\text{' is even} \\ \frac{8kl^2}{n^3\pi^3} & \text{when '}n\text{' is odd} \end{cases}$$

substituting c_n in (5) we get,

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8kl^2}{\pi^3 n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$= \frac{8kl^2}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}.$$

2) A tightly stretched string of length $2l$ is fastened at both ends. The midpoint of the string is displaced by a distance 'b' transversely and the string is released from rest in this position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.



soln let $2l = L$ (for convenience)

First we find the equation of the string in its initial position ADB (see fig).

The equation of the string (or line) AD is

$$\frac{x-0}{0-L/2} = \frac{y-0}{0-b} \quad [\because CD = b]$$

The equation of the string (or line) DB is

$$\frac{x-L/2}{L/2-L} = \frac{y-b}{b-0}$$

[Using $\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$]

$-bx = -\frac{L}{2}y$

$y = \frac{2bx}{L}, \quad 0<x<\frac{L}{2}$

$b\left(x-\frac{L}{2}\right) = \frac{L}{2}(y-b)$

$y-b = \frac{bL-2bx}{L}$

$y = \frac{bL-2bx}{L}+b$

$y = \frac{2b}{L}(L-x)$

Initial displacement of the string is in the form

$$y(x,0) = \begin{cases} \frac{2bx}{L}, & 0<x<L/2 \\ \frac{2b}{L}(L-x), & \frac{L}{2}<x<L \end{cases}$$

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

the boundary conditions are

(i) $y(0,t)=0$ for all $t>0$

(ii) $y(L,t)=0$ for all $t>0$

(iii) $\frac{\partial y}{\partial t}(x,0)=0$ [$\because$ initial velocity is zero]

(iv) $y(x,0) = \begin{cases} \frac{2bx}{L}, & 0<x<L/2 \\ \frac{2b}{L}(L-x), & \frac{L}{2}<x<L. \end{cases}$

The solution of the wave equation (1) satisfying the boundary conditions (i), (ii), (iii) is

$$y(x,t) = c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L} \rightarrow (2)$$

Applying condition (iv) in (2) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x) \text{ (say)} \rightarrow (3)$$

where $f(x) = \begin{cases} \frac{2bx}{L}, & 0<x<L/2 \\ \frac{2b}{L}(L-x), & \frac{L}{2}<x<L \end{cases}$

To find 'c_n' expand $f(x)$ in a half-range Fourier sine series in the interval $0<x<L$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \rightarrow (6)$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$.

From (3) and (4) we get $\boxed{c_n = b_n}$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\int_0^{L/2} f(x)\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{2}{L}\left[\int_0^{L/2} \frac{2bx}{L}\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}\frac{2b}{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{4b}{L^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)}\right) - 1\left(\frac{-\sin\frac{n\pi x}{L}}{\left(\frac{n^2\pi^2}{L^2}\right)}\right)\right\}_0^{L/2} + \left\{(L-x)\left(\frac{-\cos\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{L}}{\left(\frac{n^2\pi^2}{L^2}\right)}\right)\right\}_{L/2}^{L}\right]$$

$$= \frac{4b}{L^2}\left[\left\{-x\left(\frac{\cos\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)}\right) + \frac{\sin\frac{n\pi x}{L}}{\left(\frac{n^2\pi^2}{L^2}\right)}\right\}_0^{L/2} + \left\{-(L-x)\left(\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - \left(\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_{L/2}^{L}\right]$$

$$= \frac{4b}{L^2}\left[\left\{\left(\frac{-L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - (0,0)\right\}\right.$$

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$$+ \left\{ -(L-L)\left(\frac{\cos n\pi}{\left(\frac{n\pi}{L}\right)}\right) - \left(\frac{\sin n\pi}{\left(\frac{n^2\pi^2}{L^2}\right)}\right)\right\}$$

$$- \left\{ -\left(L-\frac{L}{2}\right)\left(\frac{\cos \frac{n\pi}{2}}{\left(\frac{n\pi}{L}\right)}\right) - \left(\frac{\sin \frac{n\pi}{2}}{\left(\frac{n^2\pi^2}{L^2}\right)}\right)\right\}\Bigg]$$

$$= \frac{4b}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore\ C_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2} \longrightarrow (5)$$

Substituting (5) in (2), we get.

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

$$y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l} \quad [\because L=2l]$$

---

(3) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0)= y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position find the displacement y at any distance x from one end at any time t.

Sol: The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem we have the following boundary conditions

(i) $y(0,t)=0$, for all $t>0$

(ii) $y(l,t)=0$, for all $t>0$

(iii) $\frac{\partial y}{\partial t}(x,0)=0$ ($\because$ initial velocity is zero)

(iv) $y(x,0) = y_0\sin^3\frac{\pi x}{l}$

The most general solution after applying first three conditions is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l} \rightarrow (1)$$

Applying condition (iv) in (1) we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l} \rightarrow (2)$$

We know that

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\sin^3\frac{\pi x}{l} = \frac{1}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) \rightarrow (3)$$

From (2) & (3) we get

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

$$c_1\sin\frac{\pi x}{l} + c_2\sin\frac{2\pi x}{l} + c_3\sin\frac{3\pi x}{l} + \cdots = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

Equating like coefficients on either side, we get

$c_1 = \frac{3y_0}{4}$, $c_2 = 0$, $c_3 = \frac{-y_0}{4}$, $c_4 = 0$, $c_5 = 0$, $c_6 = 0 \cdots$

$$(1) \Rightarrow y(x,t) = c_1\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} + c_2\sin\frac{2\pi x}{l}\cos\frac{2\pi a t}{l} + c_3\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l} + c_4\sin\frac{4\pi x}{l}\cos\frac{4\pi a t}{l} + \cdots \rightarrow (4)$$

Substituting the above values of $c_1, c_2, c_3 \cdots$ in (4)

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}.$$

Note: $\sin^3\frac{\pi x}{l}$ contains only two terms. Hence we need not expand $\sin^3\frac{\pi x}{l}$ in a half range sine series.

---

(4) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially displaced in a sinusoidal arc of height y_0 and then released from rest. Find the displacement 'y' at any

$$+\left\{-(L-L)\left(\frac{\cos n\pi}{\left(\frac{n\pi}{L}\right)}\right)-\left(\frac{\sin n\pi}{\left(\frac{n^2\pi^2}{L^2}\right)}\right)\right\}$$

$$-\left\{-(L-L/2)\left(\frac{\cos n\pi/2}{\left(\frac{n\pi}{L}\right)}\right)-\left(\frac{\sin \frac{n\pi}{2}}{\left(\frac{n^2\pi^2}{L^2}\right)}\right)\right\}\Bigg]$$

$$=\frac{4b}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore\ c_n=\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\longrightarrow(5)$$

Substituting (5) in (2), we get.

$$y(x,t)=\sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

$$y(x,t)=\frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}\quad[\because L=2l]$$

---

(3) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0)=y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position find the displacement y at any distance x from one end at any time t.

Sol: The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem we have the following boundary conditions

(i) $y(0,t)=0$, for all $t>0$

(ii) $y(l,t)=0$, for all $t>0$

(iii) $\frac{\partial y}{\partial t}(x,0)=0$ [$\because$ initial velocity is zero]

(iv) $y(x,0)=y_0\sin^3\frac{\pi x}{l}$

The most general solution after applying first three conditions is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l} \rightarrow (1)$$

Applying condition (iv) in (1) we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l} \rightarrow (2)$$

We know that

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\sin^3\frac{\pi x}{l} = \frac{1}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) \rightarrow (3)$$

From (2) & (3) we get

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

$$c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + c_3 \sin\frac{3\pi x}{l} + \cdots = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

Equating like coefficients on either side, we get

$$c_1 = \frac{3y_0}{4},\ c_2 = 0,\ c_3 = \frac{-y_0}{4},\ c_4 = 0,\ c_5 = 0,\ c_6 = 0 \ldots$$

$$(1) \Rightarrow y(x,t) = c_1 \sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} + c_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi a t}{l} + c_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l} + c_4 \sin\frac{4\pi x}{l}\cos\frac{4\pi a t}{l} + \cdots \rightarrow (4)$$

Substituting the above values of $c_1, c_2, c_3 \ldots$ in (4)

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}.$$

Note: $\sin^3\frac{\pi x}{l}$ contains only two terms. Hence we need not expand $\sin^3\frac{\pi x}{l}$ in a half range sine series.

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(4) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially displaced in a sinusoidal arc of height y_0 and then released from rest. Find the displacement 'y' at any

distance 'x' from one end at time t.

soln The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ → ①

From the given problem we have the following boundary conditions.

(i) $y(0,t) = 0, \quad t > 0$

(ii) $y(l,t) = 0, \quad t > 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$ [∵ initial velocity is zero]

(iv) $y(x,0) = y_0 \sin\frac{\pi x}{l}$ [sinusoidal arc of height y_0]

The most general solution of ① after applying conditions (i), (ii), (iii) is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \quad \longrightarrow ②$$

Applying condition (iv) in ②, we get,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin\frac{\pi x}{l}$$

$$c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + \cdots = y_0 \sin\frac{\pi x}{l}$$

Equating like coefficients we get

$$c_1 = y_0, \quad c_2 = c_3 = \cdots = 0.$$

$$① \Rightarrow y(x,t) = c_1 \sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} + c_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi a t}{l} + c_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l} + c_4 \sin\frac{4\pi x}{l}\cos\frac{4\pi a t}{l} \longrightarrow ③$$

substituting $c_1 = y_0$ & $c_2 = c_3 = \cdots = 0$ in ③ we get

$$y(x,t) = y_0 \sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} + 0 + 0$$

$$y(x,t) = y_0 \sin\frac{\pi x}{l}\cos\frac{\pi a t}{l}$$

Home Work sums

1. A string is tightly stretched and its ends are fastened at two points $x=0$ and $x=l$. The mid point of the string is displaced transversely through a small distance 'b' and the string is released from rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

2) A taut string of length l has its ends $x=0$ and $x=l$ fixed. The point where $x=l/3$ is drawn aside a small distance h, the displacement $y(x,t)$ satisfies $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$. Determine $y(x,t)$ at any time t.

Ans: $y(x,t) = \frac{9h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$

3) The points of trisection of a string of length 'l' are pulled aside through a distance 'h' on opposite sides from the position of equilibrium and the string is released from rest. Find the displacement of the string at any time 't'. Show that the mid point of the string remains at rest.

(i) $y(x,t) = \frac{9h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{2n\pi}{3} \sin\frac{2n\pi x}{l} \cos\frac{2n\pi a t}{l}$. (ii) $y(l/2,t) = 0$

4) A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $f(x) = kx^2(l-x)$ where 'k' is a constant, and then released from

rest. Find the displacement of any point 'x' of the string at any time $t>0$.

$$y(x,t) = \frac{-4kl^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1+2(-1)^n\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

5) Find the displacement of any point of a string, if it is of length $2l$ and vibrating between fixed end points with initial velocity zero and initial displacement given by

$$f(x) = \frac{kx}{l} \quad \text{in } 0<x<l$$
$$= 2k - \frac{kx}{l} \quad \text{in } l<x<2l.$$

Ans: $$y(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}.$$

6) An elastic string is stretched between two points at a distance π apart. In its equilibrium position the string is in the shape of the curve $f(x) = k(\sin x - \sin^3 x)$. Obtain $y(x,t)$ the vertical displacement if y satisfies the equation $\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$.

Ans: $$y(x,t) = \frac{k}{4}\sin x\cos t + \frac{k}{4}\sin 3x\cos 3t.$$

7) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = k\sin\frac{3\pi x}{l}\cos\frac{2\pi x}{l}$. If it is released from rest from this position, determine the displacement $y(x,t)$.

Ans: $$y(x,t) = \frac{k}{2}\left(\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + \sin\frac{5\pi x}{l}\cos\frac{5\pi at}{l}\right)$$

PROBLEMS ON VIBRATING STRING WITH NON-ZERO INITIAL VELOCITY

In the previous article we allow the string to vibrate by taking it to some position say $f(x)$ and then released from rest. Therefore in that case the initial velocity $\frac{\partial y(x,t)}{\partial t}$ at $t=0$ is zero. We may also allow the string to vibrate by giving some velocity to the string in its equilibrium position. This initial velocity is given to each and every point in the string from 0 to l and hence it may be a function of x say $g(x)$. Because the string is in its equilibrium position and hence there is no displacement at time $t=0$ i.e., $y(x,0)=0$ for every x. we have $y(x,t)=0$ at $t=0$

Boundary conditions when the string is given initial velocity.

(i) $y(0,t)=0$ for all $t>0$.

(ii) $y(l,t)=0$ for all $t>0$.

(iii) $y(x,0)=0$ for all x in $(0,l)$

(iv) $\frac{\partial y}{\partial t}(x,0)=g(x)$ for all x in $(0,l)$.

Steps to solve the wave equation when initial velocity is given

1. Write boundary conditions

(i). $y(0,t)=0$, $t>0$

(ii) $y(l,t)=0$, $t>0$

(iii) $y(x,0)=0$, $0\le x\le l$.

(iv) $\frac{\partial y}{\partial t}(x,0) = g(x)$ for all x in $(0,l)$

2. Write the general solution after applying conditions (i), (ii), (iii)

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}.$$

3. Applying condition (iv), we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x) \rightarrow (1)$$

Expand $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \rightarrow (2)$$

From (1) & (2), $b_n = c_n$

But $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$

$\therefore\ c_n = \frac{1}{n\pi a} b_n$.

4. Substituting 'c_n' in (2) we get the required solution.

---

1) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$, then show that

$$y(x,t) = \frac{8\lambda l^3}{\pi^4 a} \sum_{n=1,3,5}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}.$$

Soln The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0,t) = 0$ for all $t>0$.

(ii) $y(l,t) = 0$ for all $t>0$.

(iii) $y(x,0) = 0$ for all x in $(0,l)$

[∵ The string when $t=0$ is in its equilibrium position and hence there is no displacement]

(iv) $\frac{\partial y}{\partial t}(x,0) = \lambda x(l-x)$ for every x in $(0,l)$

Applying the first three boundary conditions in the correct solution of (1) we get the most general solution as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \rightarrow (2)$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \rightarrow (3)$$

Put $t=0$ in (3) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = \lambda x(l-x)$$

[By condition (iv)] $\rightarrow$ (4)

Now to find c_n expand $\lambda x(l-x)$ in a half-range Fourier sine series, we get

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (5)$$

From (4) and (5) we get

$$\sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Equating like co-efficients we get

$$c_n \frac{n\pi a}{l} = b_n$$

$$\boxed{c_n = b_n \frac{l}{n\pi a}} \rightarrow (6)$$

But $$b_n = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n^3\pi^3}{l^3}\right)}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[-(lx-x^2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) + (l-2x)\left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right) - 2\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n^3\pi^3}{l^3}\right)}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left\{-0+0-\frac{2\cos n\pi}{\left(\frac{n^3\pi^3}{l^3}\right)}\right\} - \left\{-0+0-\frac{2}{\left(\frac{n^3\pi^3}{l^3}\right)}\right\}\right]$$

$$= \frac{2\lambda}{l}\left[\frac{-2\cos n\pi}{\left(\frac{n^3\pi^3}{l^3}\right)} + \frac{2}{\left(\frac{n^3\pi^3}{l^3}\right)}\right]$$

$$b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore b_n = \begin{cases} 0, & \text{when 'n' is even} \\ \frac{8\lambda l^2}{n^3\pi^3}, & \text{when 'n' is odd} \end{cases} \rightarrow (7)$$

substituting (7) in (6) we get

$$c_n = b_n \frac{l}{n\pi a}$$

$$= \frac{8\lambda l^2}{n^3\pi^3}\frac{l}{n\pi a}$$

$$c_n = \frac{8\lambda l^3}{n^4\pi^4 a} \rightarrow (8)$$

Substituting (8) in (2) we get

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8\lambda l^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8\lambda l^3}{\pi^4 a}\sum_{n=1,3,5}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

2) If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that

$$v = \begin{cases} kx, & 0 < x < l/2 \\ k(l-x), & l/2 < x \le l \end{cases}$$ Determine the displacement function $y(x,t)$ at any time t

Sol: The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

the boundary conditions are

(i) $y(0,t) = 0$ for all $t > 0$

(ii) $y(l,t) = 0$ for all $t > 0$

(iii) $y(x,0) = 0$ for all x in $(0,l)$

(iv) $\frac{\partial y}{\partial t}(x,0) = \begin{cases} kx, & 0 < x \le l/2 \\ k(l-x), & l/2 < x \le l. \end{cases}$

After applying the first three boundary conditions in the correct solution of (1), we get the most general solution as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \rightarrow (2)$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \rightarrow (3)$$

Put $t=0$ in (3) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = f(x) \rightarrow (4).$$

where $f(x) = \begin{cases} kx, & 0 < x \le l/2 \\ k(l-x), & l/2 < x \le l. \end{cases}$

To find c_n expand $f(x)$ in a half-range Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (5)$$

From (4) & (5), we get

$$\sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Equating like coefficients we get

$$c_n \frac{n\pi a}{l} = b_n$$

$$c_n = \frac{b_n l}{n\pi a} \rightarrow (6)$$

Now $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx$.

$$= \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2} kx\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} k(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2k}{l}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2k}{l}\left[\left\{-x\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) + \left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right\}_0^{l/2} + \left\{-(l-x)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) - \left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2k}{l}\left[\left\{-\frac{1}{2}\left(\frac{\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)}\right) + \left(\frac{\sin\frac{n\pi}{2}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right\} - \left\{0\cancel{\left(\frac{\cos 0}{\left(\frac{n\pi}{l}\right)}\right)} + \cancel{\frac{\sin 0}{\left(\frac{n^2\pi^2}{l^2}\right)}}\right\}\right.$$

$$+\left\{-(l-l)\left(\frac{\cos n\pi}{\left(\frac{n\pi}{l}\right)}\right)-\frac{\sin n\pi}{\left(\frac{n^2\pi^2}{l^2}\right)}\right\}-\left\{-\left(l-\frac{l}{2}\right)\left(\frac{\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)}\right)-\left(\frac{\sin\frac{n\pi}{2}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right\}\Bigg]$$

$$=\frac{2k}{l}\left[\cancel{\frac{-\frac{l}{2}\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)}}+\frac{\sin\frac{n\pi}{2}}{\left(\frac{n^2\pi^2}{l^2}\right)}+\cancel{\frac{\frac{l}{2}\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)}}+\frac{\sin\frac{n\pi}{2}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right]$$

$$=\frac{2k}{l}\left[\frac{2\sin\frac{n\pi}{2}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right]$$

$$b_n=\frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}\longrightarrow (7)$$

substituting (7) in (6), we get

$$c_n=\frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\frac{l}{n\pi a}$$

$$\Rightarrow\quad c_n=\frac{4kl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}\longrightarrow (8)$$

substituting (8) in (2), we get

$$y(x,t)=\sum_{n=1}^{\infty}\frac{4kl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,t)=\frac{4kl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

Homework sums and Assignment sums

1) A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities v where

$$v = \begin{cases} \frac{cx}{l}, & 0 < x < l \\ \frac{c}{l}(2l - x), & l < x < 2l \end{cases}$$

x being the distance from an end point. Find the displacement of the string at any time.

$$y(x,t) = \frac{16cl}{\pi^3 a} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2l} \sin\frac{n\pi a t}{2l}.$$

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2) If a string of length 'l' is initially at rest in its equilibrium position and each of its points is given the velocity

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\frac{\pi x}{l}, \quad 0 < x < l$$

Determine the displacement function $y(x,t)$.

$$y(x,t) = \frac{3v_0 l}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a t}{l}.$$

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3) A uniform string of length 'l' is struck in such a way that an initial velocity of v_0 is imparted to the portion of the string between $l/4$ and $3l/4$ while the string is in its equilibrium position. Find the displacement of the string at any time.

$$y(x,t) = \frac{4v_0 l}{a\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin\frac{n\pi}{4} \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}.$$

---

4) Solve $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$, $y(0,t) = y(10,t) = 0$, $y(x,0) = 0$,

$\frac{\partial y}{\partial t}(x,0) = 3\sin 2\pi x - 2\sin 5\pi x$. $y(x,t) = \frac{3}{4\pi}\sin 2\pi x \sin 4\pi t$

$- \frac{1}{[illegible]} \sin 5\pi x \sin 10\pi t$

(3) If $y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$ is the solution of wave motion satisfying certain conditions, then what will be the solution satisfying

$y(x,0) = A\sin\left(\frac{\pi x}{l}\right)$, $0 \le x \le l$.

Soln

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$A\sin\frac{\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \dots \quad \text{(Given)}$$

$$b_1 = A, \quad b_2 = b_3 = \dots = 0$$

$$y(x,t) = A\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l}$$

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4) Write down the boundary conditions for the following boundary value problem "If a string of length 'l' initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\frac{\pi x}{l}$ $0<x<l$, determine the displacement function $y(x,t)$?

Soln The boundary conditions are

(i) $y(0,t) = 0$, $t>0$ (ii) $y(l,t) = 0$, $t>0$

(iii) $y(x,0) = 0$, $0<x<l$ (iv) $\frac{\partial y}{\partial t}(x,0) = v_0 \sin^3\frac{\pi x}{l}$, $0<x<l$

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(5) Write the boundary conditions for solving the string equation, if the string is subjected to initial displacement $f(x)$ and initial velocity $g(x)$

Soln (i) $y(0,t) = 0$, (ii) $y(l,t) = 0$,

(iii) $\frac{\partial y}{\partial t}(x,0) = g(x)$, (iv) $y(x,0) = f(x)$