Chapter 4
Applications of Derivatives

In this chapter we use derivatives to find extreme values (قيم قصوى) of functions and to determine and analyze the shapes of graphs.

4.1 Extreme values of functions

This section shows how to find extreme (maximum (عظمى) or minimum (صغرى)) values of a function from its derivative. Once we can do this, we can solve a variety of problems in which we find the optimal (best) way to do something in a given situation. Finding extreme values is one of the most important applications of the derivative.
Definition. (Absolute extreme values)
Let f be a function with domain D.

(1) f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \quad \text{for all} \quad x \in D.$$

(2) f has an absolute minimum value on D at a point c if

$$f(x) \geq f(c) \quad \text{for all} \quad x \in D.$$

Remarks.

(1) Maximum and minimum values are called extreme values.

(2) Absolute extreme values are also called global extreme values.

Example. The function $f(x) = \cos x$ has the absolute maximum value 1 that is assumed at $2n\pi$, and minimum value -1, that is assumed at $x = (2n+1)\pi, \quad n \in \mathbb{Z}$.

Theorem 1. *(The Extreme Value Theorem)*
If f is continuous at every point of a closed interval $[a, b]$, then f has absolute maximum and minimum values at some $x_1, x_2 \in [a, b]$.

Example 1. Show that $f(x) = \dfrac{1}{x}$ has absolute maximum and minimum values in $[2, 3]$ and find them.

Solution.

Example 2. Show that $f(x) = x^3 - 3x$ has absolute maximum and minimum values in $[-2, 4]$. Can you find them?

Solution.

Local Extreme Values

In order to find the absolute extreme values, we should find the local extreme values.
Definition. (Local extreme values)

(1) A function f has a local maximum value at a point c within its domain D if $f(x) \leq f(c)$ for all x in some open interval $I \subseteq D$ containing c.

(2) A function f has a local minimum value at a point c within its domain D if $f(x) \geq f(c)$ for all x in some open interval $I \subseteq D$ containing c.

Remarks.

(1) We can extend the definition of local extreme values to the end points of intervals by replacing open intervals by half-open intervals.

(2) Every absolute extreme value is also a local extreme value. Thus, if we find all local extreme values, we will find the absolute extreme values among these local extreme values.

Finding Extreme Values

Theorem 2. *(The First Derivative Test for Local Extreme Values)*
If f has a local extreme value at an interior point c of its domain, and if $f'(c)$ exists, then $f'(c) = 0$.

Remark. The solutions of $f'(x) = 0$ are candidates for local extreme values.

Example 1. $f(x) = x$ does not have any local extreme values since $f'(x) = 1 \neq 0$ exists for all $x \in \mathbb{R}$.

Example 2. $f(x) = x^2$ has a local minimum at $x = 0$ and $f'(0) = 0$.

Definition. (Critical points)
An interior point c of the domain of a function f is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.

Remark. The only points where a function can have extreme values are critical points and endpoints.

Finding the absolute extreme values of a continuous function f on a finite closed interval

To find the absolute maximum and minimum values of a continuous function $f(x)$ on a closed interval $[a, b]$ use the following process:

(1) Find the critical points of f in $[a, b]$.

(2) Evaluate f at all critical points and endpoints.

(3) Take the largest and smallest of these values.

Example 1. Find the absolute extreme values of $f(x) = \sqrt{x}$ on the interval $[1, 4]$. Then graph the function.

Solution.

Example 2. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x$ on the interval $[-2, 3]$.

Solution.

Example 3. (Exam)
Find the absolute maximum and minimum values of the function $f(x) = 4 - x^2$ on the interval $[-2, 1]$.

Solution.

4.2 The Mean Value Theorem

We know that the derivatives of constant functions are zeros, but could there be a more complicated function whose derivative is zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem.

Rolle's Theorem

Theorem 3. *(Rolle's Theorem)*
Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$.

Example 1. Let $f(x) = \dfrac{1}{3}x^3 - 3x$. Show that there is $c \in (-3, 3)$ such that $f'(c) = 0$.

Solution.

Example 2. Let $f(x) = x^5 + x + 1$. Show that f has exactly one real zero.

Solution.

The Mean Value Theorem

The mean value theorem was first stated by Joseph-Louis Lagrange. It guarantees that there is a point where the tangent line is parallel to the chord AB, where $A = f(a)$ and $B = f(b)$.

Theorem 4. *(The Mean Value Theorem)*
Suppose that $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one number $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Example. Find the value or values of c that satisfy the equation $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for $f(x) = \sqrt{x+1}$ and $[a, b] = [-1, 2]$.

Solution.

Corollary 1. *(Functions with Zero Derivatives Are Constant)*
If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary 2. *(Functions with the Same Derivative Differ by a Constant)*
If $f'(x) = g'(x)$ at each point $x \in (a, b)$, then there is a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$.

Example 1. Find all possible functions with derivative $y' = x^3$.

Solution.

Example 2. Find the function f with derivative $f'(x) = \cot x \csc x - 1$ and whose graph passes through the point $(\pi/2, 2)$.

Solution.

4.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a function it is useful to know where it is increasing and where it is decreasing. This section gives a test to determine where a differentiable function increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions

Recall that

(1) f is said to be increasing on I if

$$x_1 < x_2 \;\Rightarrow\; f(x_1) < f(x_2)$$

for any two points x_1 and x_2 in I.

(2) f is said to be decreasing on I if

$$x_1 < x_2 \;\Rightarrow\; f(x_1) > f(x_2)$$

for any two points x_1 and x_2 in I.

Corollary 3. *(First Derivative Test for Increasing and Decreasing)*
Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).

(1) If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on $[a,b]$.

(2) If $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on $[a,b]$.

Proof. The mean value theorem says that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ □

Example. Find the intervals on which the function $f(x) = x^2$ is increasing and decreasing.

Solution.

The First Derivative Test for Local Extreme Values

Theorem 5. *(The First Derivative Test for Local Extreme Values)*
Suppose that f is a continuous function and let c be a critical point of f. Assume that f is differentiable on an open interval containing c except possibly at c itself. Then

(1) if f' changes from positive to negative at c, then f has a local maximum value at c;

(2) if f' changes from negative to positive at c, then f has a local minimum value at c;

(3) if f' does not change sign at c, then f has no local extreme value at c.

Example 1. Let $f(x) = x^{1/5}(2x - 3)$.

(a) Find the critical points of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Identify the function's local extreme values.

(d) Which of the extreme values, if any, are absolute?

Solution.

Example 2. Let $f(x) = x^{2/3}(4 - x^2)$.

(a) Find the critical points of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) At what points, if any, does f assume local extreme values.

(d) Which of the extreme values, if any, are absolute?

Solution.

Example 3. Let $f'(x) = (2-x)(x+1)^2$.

(a) Find the critical points of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values, if any, of f.

Solution.

Example 4. Find the local extreme values of $f(x) = \cos x$ on the interval $0 \le x \le 2\pi$.

Solution.

Example 5. Find the extreme values of the function $f(x) = \begin{cases} x^2 + 2x + 3, & x \leq 0 \\ 2x^2 - 8x + 3, & x > 0. \end{cases}$

Solution.

4.4 Concavity and Curve Sketching

In this section, we will use the second derivative to find how the graph of a function bends or turns. Then we will use this knowledge and the previous knowledge about increasing and decreasing and asymptotes to draw accurate graph of a function.

Concavity

Definition. (Concavity)
The graph of a differentiable function $y = f(x)$ is concave up on an open interval I if f' is increasing on I and it is concave down on I if f' is decreasing on I.

If $y = f(x)$ has a second derivative, then we can apply the following test for concavity.

The Second Derivative test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

(1) If $f''(x) > 0$ on I, then the graph of f is concave up on I.

(2) If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Example 1. Show that $f(x) = x^2$ is concave up on $\mathbb{R}$ and $g(x) = -x^2$ is concave down on $\mathbb{R}$.

Solution.

Example 2. Show that $f(x) = \cos x$ is concave down on $(0, \pi/2)$ and concave up on $(\pi/2, 3\pi/2)$.

Solution.

Points of inflection

Definition. (Point of inflection)
Let f be a continuous function on an open interval containing c. If f changes concavity at the point $(c, f(c))$, then $(c, f(c))$ is called a point of inflection.

Example. The function $f(x) = x^3$ has an inflection point $(0, 0)$.

Second derivative test for inflection points

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.
Example 1. Show that $x = 0$ is a point of inflection of $f(x) = x^3$.

Solution.

Example 2. Show that $x = 0$ is not a point of inflection of $f(x) = x^4$.

Solution.

Second derivative test for local extreme values

Theorem 5. *(Second derivative test for local extreme values)*
Suppose f'' is continuous on an open interval that contains $x = c$.

(1) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

(2) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

(3) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

Example. Show that $f(x) = x^2 + 3$ has a local minimum at $x = 0$.

Solution.

Curve Sketching

To graph $y = f(x)$ we follow the following steps:

1. Identify the domain of f.
2. Find the asymptotes that may exist.
3. Find the critical points of f, if any, and classify them. Find where the function is increasing and where it is decreasing.
4. Find the points of inflection, if any, and determine the concavity of the curve.
5. Plot key points, such as the intercepts and the points found in previous steps.
6. Sketch the graph of f together with any asymptotes that exits.

Example 1. Sketch the graph of $f(x) = 2x^2 - x^4$.

Solution.

Example 2. Let $f(x) = x^{3/5}$.

(a) Find the critical points of f and identify the function's local extreme values.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the inflection points of f and intervals on which f is concave down and up.

(d) Sketch the graph of f.

Solution.

Example 3. Let $f(x) = \dfrac{x-4}{2x+4}$.

(a) Find the critical points of f and identify the function's local extreme values.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the inflection points of f and intervals on which f is concave down and up.

(d) Find the asymptotes.

(e) Sketch the graph of f.

Solution.

Example 4. (Exam)
Let $f(x) = \dfrac{x^2 - 8}{x - 3}$.

(a) Find the critical points of f and identify the function's local extreme values.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the inflection points of f and intervals on which f is concave down and up.

(d) Find the asymptotes.

(e) Sketch the graph of f.

Solution.

Example 5. Let $f(x) = \dfrac{x^2 - 49}{x^2 + 5x - 14}$.

(a) Find the critical points of f and identify the function's local extreme values.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the inflection points of f and intervals on which f is concave down and up.

(d) Find the asymptotes.

(e) Sketch the graph of f.

Solution.

Example 6. (Exam)
Let $f(x) = \dfrac{9x}{(x-1)^2}$.

(a) Find the critical points of f and identify the function's local extreme values.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the inflection points of f and intervals on which f is concave down and up.

(d) Find the asymptotes.

(e) Sketch the graph of f.

Solution.

Example 7. (Exam)

Let $f(x) = \dfrac{x^2+4}{x}$.

(i) Find the asymptotes of f.

(ii) Find the intervals on which f is increasing and decreasing.

(iii) Find the local extreme values.

(iv) Find intervals on which f is concave down and up.

(v) Find the inflection points if it exists.

(vi) Sketch the graph of f.

Solution.