Calculus A (MathA1301)
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الفصل الأول *2021-2020*

Chapter 3
Differentiation

In chapter 2 we defined the slope of a curve at a point as the limit of secant slopes. This limit, called a derivative, measure the rate at which a function changes and is one of the most important ideas in calculus.

3.1 Tangents and Derivative at a Point

Definition. (The derivative of a function at a point)
The derivative of a function f at a point x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

Remark. The following are all interpretations of $f'(x_0)$:

(1) $f'(x_0)$ is the derivative of $f(x)$ at $x = x_0$.

(2) $f'(x_0)$ is the slope of the graph of $y = f(x)$ at $x = x_0$.

(3) $f'(x_0)$ is the slope of the tangent to the curve $y = f(x)$ at $x = x_0$.

(4) $f'(x_0)$ is the rate of change of $f(x)$ with respect to x at $x = x_0$.

3.2 The Derivative as a Function

Definition. (The derivative of a function f as a function f')
The derivative of the function f with respect to the variable x is the function f' given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided this limit exists.

Notation.

(1) There are many ways to denote the derivative of a function $y = f(x)$. The most common notations are: $f'(x) = y' = \dfrac{d}{dx}f(x) = \dfrac{dy}{dx} = D_x f(x) = \dot{f}(x) = \dot{y}$.

(2) The derivative of f at a point x_0 is denoted by: $f'(x_0) = \dfrac{d}{dx}f(x)\Big|_{x_0} = \dfrac{dy}{dx}\Big|_{x_0}$.

Example. Use the definition to find $f'(x)$ for the following functions:

(a) $f(x) = x + 3$.

(b) $f(x) = x/(x + 2)$.

(c) $f(x) = \sqrt{x - 1}$.

Solution.

Differentiable on an interval; one-sided derivatives

Definition.

(1) A function $y = f(x)$ is differentiable at a point x_0 if $f'(x_0)$ exists.

(2) A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval.

(3) A function $y = f(x)$ is differentiable on a closed interval $[a, b]$ if it differentiable on (a, b) and if the right-hand derivative of f at a,

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h},$$

and the left-hand derivative of f at b,

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h},$$

exist.

Example 1. Show that $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but not differentiable at $x = 0$.

Solution.

Example 2. The function $f(x) = \sqrt{x}$ is not differentiable at $x = 0$ since the right-hand derivative of $f(x)$ does not exist at 0.

When does a function not have a derivative at a point?

A function has a derivative at a point x_0 if the slopes of secant lines through $P(x_0, f(x_0))$ and a nearby point $Q(x_0 + h, f(x_0 + h))$ on the graph approach a finite limit as Q approaches P. Whenever the secant fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. Thus, differentiability is a "smoothness" condition on the graph of f.

A function f can fail to have a derivative at a point for many reasons. Some of these reasons are as follows:

(1) If f has a discontinuity at x_0.

(2) If $\displaystyle\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \pm\infty$, then the graph has a vertical tangent at x_0.

(3) If $\displaystyle\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = L \neq \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = M$, then the graph has a corner at x_0

(4) If $\displaystyle\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = \pm\infty \neq \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = \mp\infty$, then the graph has a cusp at x_0

Example 1. Find the points at which f is not differentiable if $f(x) = \lceil x \rceil$.

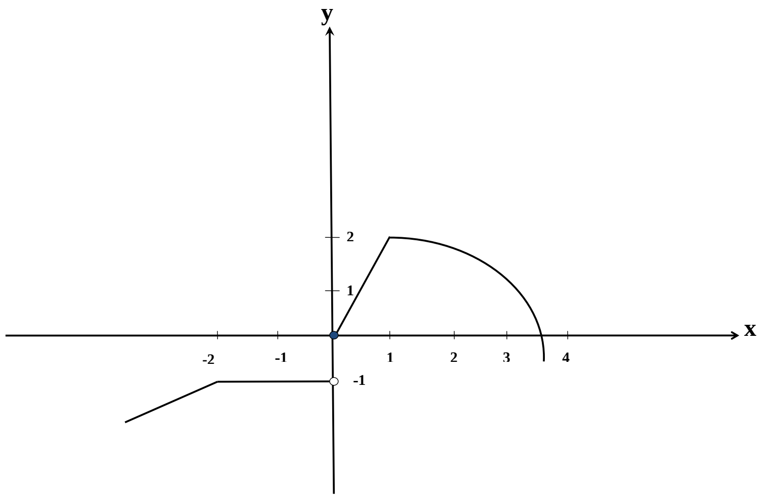
Solution.

Example 2. $f(x) = |x|$ is not differentiable at $x = 0$ since its graph has a corner at $x = 0$.

Example 3. $f(x) = \sqrt{|x|}$ is not differentiable at $x = 0$ since its graph has a cusp at $x = 0$.

Example 4. $f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0. \end{cases}$ is not differentiable at $x = 0$ since its graph has a vertical tangent at $x = 0$.

Example 5. If f has the graph in the accompanying figure, then find the points at which f is not differentiable. Give reasons for your answer.



Solution.

Differentiable functions are continuous

Theorem 1. *(Differentiability implies continuity)*
If f has a derivative at $x = x_0$, then f is continuous at $x = x_0$.

Remark. The converse of Theorem 1 is not true. That is, if f is continuous at x_0, then f may or may not be differentiable at x_0.

Example 1. The function $f(x) = |x|$ is is continuous at $x = 0$ but not differentiable at $x = 0$.

Example 2. (Exam) Let $f(x) = \begin{cases} x^2 - 1, & x < 3 \\ \dfrac{8x}{3}, & x \geq 3. \end{cases}$

Show that f is continuous at $x = 3$ but not differentiable at $x = 3$.

Solution.

3.3 Differentiation Rules

In this section we introduces several rules that allow us to differentiate many functions directly, without using the definition.

Differentiation Rules

1. If k is a constant, then $\boxed{\dfrac{d}{dx}(k) = 0.}$

2. If n is a real number, then $\boxed{\dfrac{d}{dx}x^n = nx^{n-1}}$ for all x where x^n and x^{n-1} are defined.

3. If f is a differentiable function of x and k is a constant, then $\boxed{\dfrac{d}{dx}(kf(x)) = k\dfrac{d}{dx}f(x).}$

4. If f and g are differentiable functions of x, then $\boxed{\dfrac{d}{dx}(f(x) \pm g(x)) = \dfrac{d}{dx}f(x) \pm \dfrac{d}{dx}g(x).}$

5. If f and g are differentiable functions of x, then $\boxed{\dfrac{d}{dx}(f(x)g(x)) = g(x)\dfrac{d}{dx}f(x) + f(x)\dfrac{d}{dx}g(x).}$

6. If f and g are differentiable functions of x and $g(x) \neq 0$, then $\boxed{\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.}$

Definition. (Normal line)
The normal line to a curve $y = f(x)$ at a point $P(x_0, f(x_0))$ is the line through P that is perpendicular to the tangent line of the curve at P.

Example 1. Find equations for the tangent and normal lines to the curve $y = x^4 - 2x^3 + 2x - 5$ at the point $(2, -1)$.

Solution.

Example 2. Find equations for the tangent and normal lines to the curve $y = x^3 - 3x^2 - x + 7$ at the points where the slope of the curve is 8.

Solution.

Example 3. Find the value of a that makes $f(x) = \begin{cases} 2x^2 - 2, & x \leq -1 \\ a(x+1), & x > -1. \end{cases}$

differentiable for all $x-$values.

Solution.

Second and Higher-Order Derivatives

$y' = f'(x)$ is called the first (order) derivative of $y = f(x)$. Since $f'(x)$ is also a function, we can differentiate it to obtain the second derivative of $y = f(x)$, $y'' = \dfrac{d}{dx}y' = \dfrac{d^2y}{dx^2}$.

In general, the $n - th$ derivative of $y = f(x)$ is $y^{(n)} = \dfrac{d}{dx}y^{(n-1)} = \dfrac{d^ny}{dx^n}$.

Example 1. Find $f'''(x)$ if $f(x) = 2x^3 + 9x^2 + 5x^{-3}$.

Solution.

Example 2. Find $y''(x)$ if $y = \dfrac{x^2 - 3x}{2x^4 + 5}$.

Solution.

3.5 Derivatives of Trigonometric Functions

$\dfrac{d}{dx}\sin x = \cos x$	$\dfrac{d}{dx}\cos x = -\sin x$

Proof. $\dfrac{d}{dx}\sin x = \lim\limits_{h\to 0}\dfrac{\sin(x+h) - \sin x}{h} = \lim\limits_{h\to 0}\dfrac{\sin x(\cos h - 1) + \sin h\cos x}{h} = \cos x.$ □

Example 1. Prove the following rules:
$\dfrac{d}{dx}\tan x = \sec^2 x, \quad \dfrac{d}{dx}\cot x = -csc^2 x, \quad \dfrac{d}{dx}\sec x = \tan x\sec x, \quad \dfrac{d}{dx}\csc x = -\cot x\csc x.$

Solution.

Example 2. Find y' if $y = x^2\cot x - \dfrac{1}{x^4}$.

Solution.

Example 3. Find y' if $y = \dfrac{\sqrt{x} + \cos x}{\tan x - \cot x}$.

Solution.

Example 4. Find y'' if $y = \csc x$.

Solution.

3.6 The Chain Rule

If $h(x) = \sin(x^2 + 5)$, then $h'(x) =?$
Let $g(x) = x^2 + 5$ and $f(x) = \sin x$. Then $h(x) = (f \circ g)(x)$. We know $f'(x) = \cos x$ and $g'(x) = 2x$. Thus, if we can write the derivative of h in terms of the derivatives of g and f, then we can find $h'(x)$.

In this section we will develop a rule to differentiate composite function $(f \circ g)(x)$.

Theorem 2. *(The Chain Rule)*
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x)$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))\ g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Example 1. Let $f \circ g$ be a composite of the differentiable functions $y = f(u)$ and $u = g(x)$. If $f'(4) = -3,\ g(1) = 4,\ g'(1) = 7$, then find $(f \circ g)'(1)$.

Solution.

Example 2. Given $y = \tan u$ and $u = x^2 + \sin x$, find dy/dx.

Solution.

Example 3. Find y' at $x = 1$ if $y = \cos(5x^3 - 3x + 6)$.

Solution.

Example 4. Find y' if $y = \sec^3(\sin(3x))$.

Solution.

Example 5. Find y' if $y = f\left(\frac{1}{x}\right)$.

Solution.

Example 6. (Exam)
Find $\frac{dy}{dx}$ if $y = \sec(\sqrt{x} + x)\tan\left(\frac{1}{x^2}\right)$.

Solution.

Example 7. (Exam)
Find $\frac{dy}{dx}$ if $y = \left[x^3 + \cos(2x)\right]^{-3}$.

Solution.

Example 8. Find y'' if $y = (3x^5 - 5)\cot(3x^2 - 1)$.

Solution.

Example 9. Let $y = f(x)g^4(x)$ and $f(3) = 5,\ g(3) = 2,\ f'(3) = -3,\ g'(3) = 4$. Find dy/dx at $x = 3$.

Solution.

Example 10. (Exam)
Let f and g be two functions such that $g(x) = f(\sqrt{x}) + \sqrt{f(x)}$. If $f(1) = 4$ and $f'(1) = 8$, then find $g'(1)$.

Solution.

Example 11. Let f be a differentiable function. Show that if f is even, then f' is odd.

Solution.

3.7 Implicit Differentiation

When we can not put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate in the usual way, we may still be able to find y' by implicit differentiation.

Implicit defined functions

The graph of of the equation $x = y^2$ has a well-defined slope at nearly every point because it is the union of the graphs of the two differentiable functions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$. In order to find the slope we treat y as a differentiable function of x and differentiate both sides of the equation $y^2 = x$ with respect to x, using the differentiation rules.

Example 1. Find $\dfrac{dy}{dx}$ if $x^4 - 4xy + y^2 = 8$.

Solution.

Example 2. Find $\dfrac{dy}{dx}$ if $y = 3x^3 + 2xy^{3/2} + \cos(x^2y)$.

Solution.

Example 3. If $y^2 + 3y - 2x = 4$, find $\dfrac{d^2y}{dx^2}$ at the point $(3, 2)$

Solution.

Example 4. Find $\dfrac{d^2y}{dx^2}$ if $x^{5/2} + 4y^{5/2} = y$.

Solution.

Example 5. Verify that the point $(-1, 1)$ is on the curve $x^2 - xy + 2y^3 = 4$ and find the tangent and normal lines to the curve at this point.

Solution.

Example 6. Find the tangent and normal lines to the curve $x\sin(2y) = y\cos(2x)$ at the point $(\pi/4, \pi/2)$.

Solution.

3.9 Linearization and Differentials

Linearization is a method to approximate complicated functions with simpler ones.

Definition. (Linearization)
Let f be differentiable function at $x = x_0$.

(1) The linearization of f at x_0 is the approximating function

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

(2) The approximation $f(x) \approx L(x)$ of f by L is the standard linear approximation of f at x_0.

Remark. If L is the linear approximation of f at x_0, then x_0 is the center of the approximation.

Example 1. Find the linearization of $f(x) = \sqrt{x^2 + 9}$ at $x = -4$ and use it to approximate $f(-4.5)$.

Solution.

Example 2. Find the linearization of $f(x) = 2x^2 + x^{-2}$ at $x = 1$.

Solution.

Example 3. Find a linearization of $f(x) = x + \sqrt{x}$ at suitably chosen integer near $x = 2.1$ at which $f(x)$ and $f'(x)$ are easy to evaluate. Then use the linearization to approximate $f(2.1)$.

Solution.

Differentials

We sometimes use the Leibniz notation $\dfrac{dy}{dx}$ to represent the derivative of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative.

Definition. (Differential)

Let $y = f(x)$ be differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = f'(x)dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx. If dx is given a specific value and x is a particular number in the domain of the function f, then these values determine the numerical value of dy.

Example. Let $y = x\sqrt{4 - x^2}$. Find dy and the value of dy when $x = 0$ and $dx = 0.1$

Solution.