Real Numbers

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1. Definition

A Real number is a cut in $\mathbb{Q}$

2. Motivation

It is a mind-boggling, yet rewarding task to contemplate the relationship between geometric length and arithmetic magnitude. Have you ever given it much thought? In the time of the Greeks, the prevailing number system was limited to the rational numbers. With this numerical structure, they were able to satisfactorily carry out measurements, but every once in a while, they would hit a wall. Take for example a square with area 2. We are aware that the area of a square is equal to its side multiplied to itself. However, there is no rational number whose square is equal to 2.

Essentially, the Greeks realised that not all ratios were commensurable, i.e. it was not always possible to find a third length that could be used to express the lengths that made up the ratio. Their number system was irreconcilable with incommensurability, and hence, they abandoned algebra for geometry, for compared to number, length was a better established, more concrete concept in their minds.

But how is it possible for $\sqrt{2}$ to exist as a geometric length and not a number?

3. Bird's Eye View

The existence of $\sqrt{2}$, π, , proved that there were *gaps* in the rational numbers. You could picture them as immeasurably small holes. These *gaps* indicate to us that the rational numbers are indeed insufficient and *incomplete.*

4. Context of the definition

How does one work around this limitation of the rationals? By constructing the real numbers, of course! To do so, we will use the concept of Dedekind Cuts. Using a Dedekind Cut, a real number is represented as a spot at which the number line has been *cut* with an infinitely thin knife. Neat, isn't it?! Note, however, that this cut is not a gap between two individual rational numbers, rather, it is a gap between *two sets of rational numbers.* What's remarkable about the Dedekind Cuts is the fact that it allows us to define a real number purely in terms of the rationals.

4.1. Definition of a cut:

A cut in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that

1. A $\cup B = \mathbb{Q}, A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset$
2. If a $\in A$ and $b \in B$ then $a < b$
3. A contains no largest element

$\sqrt{2}$ corresponds to the Dedekind cut $A \mid B$ such that

1. A $= \left\{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\right\}$
2. B $= \left\{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \geq 2\right\}$

5. Applications

The construction of real numbers comes in handy for more than just finding diagonals of unit squares or the sides of squares with areas of 2, 3, 5, etc. The rationals are inadequate when we deal with geometry, and more so when it comes to trigonometry, where numbers like ı⊐ make frequent appearances. Since differential and integral calculus are all about slopes, and tangents, and areas under curves, we require the real numbers to accurately describe them as well.

6. History

The first mention of irrational numbers can be found in certain Sutras of the Vedic period. Aryabhatta used the word "aasanna", which roughly translates to "approaching", while determining the value ı⊐. Our modern theory of real numbers, seems to derive more or less from Eudoxus's theory of proportions. Perhaps the most famous, and arguably one of the most elegant proofs in pure mathematics is the one that proves that $\sqrt{2}$ is not rational, as it appears in Book X of Euclid's Elements.

7. Pause and Ponder

- 1. What does it mean for a number to have a square root? We've seen that $\sqrt{2}$ is real; can -2 have a square root?
- 2. While defining a cut, why shouldn't set A of the cut have a largest element? Would it be equivalent to set B containing no smallest element? Why is this condition significant?
- 3. We just constructed the real numbers. How would you suppose the natural numbers were constructed? What about the other number systems?
- 4. Is ∞ a real number? How many real numbers are there? Are there more reals, or more rationals? And what about 0?
- 5. If you were to add infinitely many rational numbers together, would it be possible to arrive at a non-rational number?
- 6. How many prime numbers are there?

8. References

1. Pugh, Charles C.(2002). *Real Mathematical Analysis*
2. Tao, Terence. (2006). *Analysis 1*