Archimedean Property

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1 Definition

1.1 Archimedean Property

1. Given any $x \in R$, there exists $n \in N$ such that $n > x$
2. Given any $y > 0 \in R$, there exists $n \in N$ such that $\frac{1}{n} < y$.

2 Motivation

The Archimedean Property for R is one of the most important consequences of the Axiom of Completeness. This is so because there are various results in calculus and analysis that derive from the Archimedean Property. When the property is generalised to ordered fields, these fields are studied as Archimedean fields.

The Archimedean Property sets the basis of the distinction between the real analysis we are studying and non-standard models of analysis; Non-standard analysis deals with the extended Real numbers, which include infinitesimals.

3 Bird's Eye View

The geometric interpretation of the Archimedean Property is surprisingly intuitive - *Any line segment, no matter how long, may be covered by a finite number of line segments of a given positive length, no matter how small.*

There are two basic implications of the property. Let us consider the first part of our definition - *Given any $x \in R$, there exists $n \in N$ such that $n > x$*

This proposes that *it is impossible to find infinite elements in the real line.*

The second part of the definition is mathematically equivalent to the first part. *Given any $y > 0 \in R$, there exists $n \in N$ such that $\frac{1}{n} < y$.* can be interpreted to mean *there are no infinitesimals in the real line.*

4 Context of the definition

Consider a line segment AB of length y, and another line segment PQ of length x. The Archimedean property allows us to cover, and even exceed AB with n number of PQs, regardless of how big AB might be.

How do we prove this?

We know that the natural numbers are not bounded above. For the sake of the argument, let us assume they are.

Assumption: N is bounded above

This means that the set N has a supremum. Let $\alpha = supN$.

$\alpha - 1$ is not an upper bound for N, and therefore there exists an $n \in N$ that satisfies $\alpha - 1 < n$. $\alpha - 1 < n$ is equivalent to $\alpha < n + 1$. By induction, we know that $n + 1 \in mathbbN$. This contradicts the fact that α is an upper bound of N. Hence, our assumption was wrong.

5 Applications

The Archimedean Property is of great significance when it comes to proving a fundamental result, that the sequence $\frac{1}{n}$ converges to 0.

Additionally, it is useful in determining the nature of Q as a set inside R.

5.1 Density of Q in R

Theorem: For any two real numbers a and b such that $a < b$, there exists a rational number r such that $a < r < b$.

Proof: Let us take $r = \frac{m}{n}, m, n \in N$, the standard way in which rational numbers are represented.

$$a < \frac{m}{n} < b \tag{1}$$

Now all that matters to satisfy $a < r < b$, is the manner in which we choose m and n.

1. We choose $n \in N$ large enough to satisfy $\frac{1}{n} < b - a$.
   Multiplying n to equation, we have the inequality

$$na < m < nb \tag{2}$$

2. Now we choose $m \in N$ so that it is the smallest natural number greater than na; $m \in N$ satisfies -

$$m - 1 \leq na < m \tag{3}$$

From equation 2, we have -

$$a < \frac{m}{n} \tag{A}$$

Considering equation 3,

$$m \leq na + 1$$

$$m < n(b - 1/n) + 1$$

$$m < nb$$

$$\frac{m}{n} < b \tag{B}$$

Combining A and B, we have a rational number of the form $\frac{m}{n}$ in between any two real numbers a and b.

6 History

The Archimidean Property, also known as the Exodus Theorem (since Archimedes credits it to Exodus), is defined in Book V of Euclid's Elements, as -

Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.

The number systems used by the pioneers of calculus like Leibniz included the infinitesimals as a part of the reals. Back in the time, calculus using infinitesimals was quite popular. The reals were a Non-Archimedean field. However, with the development of analysis, and contributions to it from Cantor, Dedekind, Bolzano, and others, the Archimedean property became prominent, and marked the exit of the infinitesimals from the real numbers.

7 Pause and Ponder

Equipped with the Archimedean Property, we are in a better position to contemplate some very elementary questions we thought of while dealing with the construction of the real numbers -

1. Is infinity a part of the real number?
2. We have proved that Q is dense in R. Is the corollary, "*For any two real numbers a and b such that $a < b$, there exists an irrational number z such that $a < z < b$.*" also true?
3. Can there exist ordered fields that are Non-Archimedean in nature?

8 References

Rudin, Walter. (1976). *Principles of Mathematical Analysis*

Abbott, Stephen (2002). *Understanding Analysis*