Line and the Plane

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1 Definition

Euclidean Space, $\mathbb{X}$ is an n-dimensional space $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}$ (Cartesian product), of all ordered n-tuples of real numbers, $(x_1, x_2, x_3, ..., x_n)$. animation 1

2 Motivation

The geometrical concept of *motion*, roughly speaking, gives us the connection between distances and relative positions of objects. The motions we are most familiar with, which we see n the physical world, are *rotation* and *translation.* It is possible to compare two objects geometrically using these two operations; object A is simply a fixed number of rotations and translations applied to object B in a specified order. However, to be able to do so, we must have defined a *space*, which is independent of units of measurement, reference frames, and physical locations. This is what the Euclidean Space essentially is. Note, however, that these motions are *linear.* They follow the axioms of Euclidean Geometry.

(animation on Euclidean Geometry)

3 Bird's Eye View

From the definition of Euclidean Space, we learn that an element $x \in (x_1, x_2, x_3, ..., x_n)$ can be represented by a scalar or a vector. This brings us to the concept of a *vector space.* A **vector space** is a set V, together with well-defined operations of vector addition and scalar multiplication in V, in accordance with the natural laws of linear algebra (assosiativity, commutativity, etc.) Vector spaces can be defined in numerous ways; set $A = \{0\}$ is the simplest example of a vector space, and the set of all continuous functions over a well defined interval is another instance. Similarly, the Euclidean Space $\mathbb{R}^n$ is a vector space.

4 Context of the Definition

4.1 Cartesian Product

Cartesian Product is the set $A \times B$ of all ordered pairs (a, b) such that $a \in A$ and $b \in B$

cartesian product animation

There are certain operations defined on the Euclidean Space which allow us to explore its properties to a deeper extent -

4.2 Inner Product

Also called the *dot product*, the *inner product* $\langle x, y\rangle$ of two vectors $x = (x_1, x_2, x_3, ..., x_n), y = (y_1, y_2, y_3, ..., y_n) \in \mathbb{R}^n$ is defined as $\langle x, y\rangle = x_1y_1 + x_2y_2 + ... + x_ny_n$. Note that the inner product exhibits three important properties; it is bilinear, symmetric, and positive-definite. For any $x, y, z \in \mathbb{R}^n$ and any $c \in \mathbb{R}$ - *(bilinear)* $\langle x, y + cz\rangle = \langle x, y\rangle + c\,\langle x, y\rangle$,

(symmetric) $\langle x, y\rangle = \langle y, x\rangle$,

(positive definite) $\langle x, x\rangle \geq 0\,\langle y, x\rangle$ and $\langle x, x\rangle = 0$ if and only x is the zero vector.

4.3 Euclidean Norm

The Euclidean *norm* of a vector $x \in \mathbb{R}^n$ is defined as

$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{(x_1^2 + x_2^2 + ... + x_n^2)}$

The norm of a vector is a *scalar*, i.e., *real number*. In simpler terms, the norm is just the *magnitude* or *length* of the vector.

4.4 Euclidean Metric

The notion of *distance* when dealing with Euclidean spaces is defined in the following manner -

$d\,(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$

A vector space that satisfies the properties that follow from this definition is known as a *metric space.* A metric space (X, d) is defined a set X equipped with a metric d. The Euclidean space, for which these properties hold, is a metric space.

5 Consequences

Of the properties that apply to the inner product of a vector space, an especially interesting one is the Cauchy-Schwarz Inequality

5.1 Cauchy-Schwarz Inequality

For all $x, y \in \mathbb{R}^n$, $\langle x, y\rangle \leq |x|\,|y|$

The following animation provides the proof.

animation on CS and triangle inequality

6 History

In the time of the Greeks, numbers, rather than having a purely algebraic interpretation, were physically significant as lengths and distances. In order to understand how these lengths and distances behaved, the Greeks axiomatised certain obvious properties of physical space. We find these axioms in Euclid's *Elements*, and hence the name. Euclidean space is simply the abstraction of the geometry of the real world.

7 Pause and Ponder

We've come from understanding what a real number truly is, and constructing the number line, to developing a plane, and understanding a three dimensional vector space. Speaking of dimensions, what does the Cauchy-Schwarz inequality tell us when applied in higher dimensions? The triangle inequality follows from the Cauchy-Schwarz inequality, but how exactly?

And lastly, if Euclidean Geometry is important because it models the world around us, what is the significance of Non-Euclidean Geometry?

8 References