Axiom of Completeness

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1 Definition

1.1 The Axiom of Completeness

Every non-empty set of real numbers that is bounded above has a least upper bound.

1.1.1 Definition of a least upper bound:

A real number u_0 is the *least upper bound* for a set $S \subseteq R$ if -

1. u_0 is an upper bound for S,
2. if u is any upper bound for S, then $u_0 \leq u$

   (animation 1)

2 Motivation

The Euclidean conception of continuum provides us with a fair understanding of the implications of the Axiom of Completeness. Consider a unit circle constructed on the rational number plane. The circle is centered at the origin O, (0,0). Let H be the point (0,1). We know that (1,0) is a point *outside*(or rather, not inside) the circle. Therefore, line segment OH is starts inside the circle, and ends outside it, without ever crossing or intersecting the circle itself!

Logically, this is an absurdity. It it equivalent to claiming that the circle has *holes* in it, which is similar to what we encountered while constructing the real numbers.

The completeness axiom helps us avoid such absurdities; it ensures that the number line is *continuous*, devoid of any *gaps*, and that every convergent sequence of real numbers has a real limit.

animation2

3 Bird's Eye View

The following animation should give you an intuition for what upper bounds, lower bounds, least upper bound, greatest bound represent. animation3

4 Context of the Definition

animation 4 and proof

5 Applications

5.1 Nested Interval Property

Consider a closed interval $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$ for $n \in N$. Assume that each $I_{n+1} \subseteq I_n$, giving us the sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq I_5...$. We note that the sequence has a nonempty intersection, i.e. $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

We know this to be true because the Axiom of Completeness allows for a $x \in I_n foreachn \in N$.

5.2 Infimums and Supremums

Assume set S is a non-empty subset of R. Its supremum is its least upper bound if it is bounded above, and its infimum is its greatest lower bound if it is bounded below. Otherwise, its supremum is $+\infty$ and its infimum is $-\infty$.

Supremums and infimums play an extremely important role in the field of calculus. The power function,x^m, is defined as $x^m = \sup x^{\frac{a}{b}}$, where $\frac{a}{b} \in Q$ and $\frac{a}{b} \leq m$. $\frac{a}{b}$ is a value that *limits* to the value m. (animation)

5.3 History

Mathematics before the 19th century was not as much about number systems and numerical models as much as it was about being able to work with numbers and expect consistent results. The Greeks' method of exhaustion used to estimate volumes, Newton's Method of Fluxions, and Leibniz's idea of derivatives were all constructed upon the concept of repeating a process recursively to inch closer to a particular result, which today, we call the concept of *limits*. *Limits* and consequently *least upper bounds* are quite fundamental to Analysis and Calculus. Bolzano, in the 1800s, is credited with formally recognising the importance of the Axiom of Completeness. It is only following that that contemporary mathematicians were able to deal with limits and, lower and upper bounds rigorously.

5.4 Points to Ponder

The Axiom of Completeness has established its significance in the field of analysis, but its implications go far beyond the obvious existence of roots and continuity of the real line. Here are a few questions to ask yourself -

1. What are the supremum and infimum of an empty set, and why?
2. Is the Axiom of Completeness an axiom or can we prove it?

3. What are the supremum and infimum of an empty set, and why?

4. How many irrational numbers can you find between two rational numbers? And how many irrationals do you think are between two rationals? Do you think perhaps the completeness axiom might be able to answer this question?

6 References

1. Abbott, Stephen(2000). *Understanding Analysis*

2. Tao, Terence. (2006). *Analysis 1*

3. https://math.stanford.edu/ feferman/papers/ConceptContin.pdf