Ordered Sets and Ordered Fields

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There are several approaches one can adopt to construct the various number systems. We are familiar with the method involving Dedekind Cuts. We will now focus on the axiomatic formalisation of the arithmetic associated with these number systems.

1 Definition

Let $(\mathbb{R}, +, \times, \leq)$ be a *Dedekind complete, ordered field.* Then $\mathbb{R}$ is defined as the field of Real Numbers.

What is a *field*? What does it mean for a field or a set to be *ordered*? We will explore these concepts in the following sections.

2 Motivation

The notion of *order* is quite fundamental, especially to our deterministic model of reality. It features everywhere, from the alphabetical ordering of words in a dictionary, to algorithms in computer science. Orders arise from the necessity to make *comparisons* between measurements, but they extend to contexts beyond the relative quantities in a sequential structure. Take the subset relation; it is a more nuanced manifestation of order, which brings with its analysis the ideas of *inclusion* and *inequality.* Essentially, order is required to explore the properties of a number system. It is quite tedious, and at times even impossible, to examine the characteristics of a set which has no order on it.

In the following animation, it is evident that once the set has been ordered by the less-than relation, it is much easier to study the properties of the set.

3 Partial Order

A partial ordering (or simply ordering) of the set S is a binary relation, say $\preceq$, which is

reflexive: $a \preceq a$ for every $a \preceq\in S$,

transitive: if $a \preceq b$and $b \preceq c$, then $a \preceq c$,

antisymmetric: if $a \preceq b$ and $b \preceq a$, then $a = b$

A set with a partial order defined on it is called a partially ordered set, just an ordered set. A partially ordered set is also called a poset. Consider a group of people as an example for a set. Consider the relation "is friends with". This binary relation allows us to compare the elements of the set, although not all elements, as the relation "is friends with" might satisfy only certain pairs of the set. Hence, this relation brings partial order to the set.

If all the elements in the set, i.e. all people in the group, are friends, satisfying the relation "is friends with", and therefore "comparable", the set becomes *totally* or *linearly ordered*.

Take a look at the following animation -

4 What is a field?

A set A is defined as a *field* if

1. it has at least two elements and,

2. the two operations - addition and multiplication - that have been defined on F satisfy the following conditions -

   (a) **commutativity** $x + y = y + x$, and $xy = yx$ for all $x, y \in F$

   (b) **associativity** $(x + y) + z = x + (y + z)$, and $(xy)z = x(yz)$ for all $x, y, z \in F$

   (c) **identity** There exists a unique additive identity element and a unique multiplicative identity element for all $x \in F$, such that $x + 0 = x$ and $x.1 = x$.

   (d) **inverse** There exist additive inverses and multiplicative inverses for all $non - zerox \in F$ with respect to the neutral elements 0 and 1 respectively, such that $x + (-x) = 0$, and $x.(1/x) = 1$

   (e) **distributivity** $x(y + z) = xy + xz$, for all $x, y, z \in F$

Given a set F and two elements $x, y \in F$,an operation on F is a function that takes the ordered pair (x, y) to a third element $z \in F$.

(animation)

An *Ordered Field* enjoys three additional properties -

1. **transitivity** $x < y < z$ implies $x < z$ for all $x, y, z \in F$

2. **trichotomy** For all $x, y \in F$, only one of the statements $x < y$, $y < x$, $x = y$ is true.

3. **translation** $x < y$ implies $x + z < y + z$ and $xz < yz$ for all $x, y, z \in F$.

The real numbers $\mathbb{R}$ is defined as a *totally or linearly ordered set.*

4.1 Does an order exist on the Complex Numbers?

It must have struck you by now, that comparing two complex numbers is not as straightforward as comparing two real numbers or natural numbers. This is because the complex numbers are not ordered.

Proof: (By Contradiction) Assumption: The complex numbers are comparable, i.e., they have an ordering on them. Since $i \neq 0$, trichotomy ensures that either $i \in \mathbb{C}^+$ or $i \in \mathbb{C}^-$, where $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the positive and negative complex numbers respectively.

Case 1: If $i \in \mathbb{C}^+$, then $i^3 \in \mathbb{C}^+$, as the product of n positive numbers is positive. However, $i^3 = -i$. This is a contradiction.

Case 2: If $i \in \mathbb{C}^+$, then $-i^3 \in \mathbb{C}^+$. Since $-i^3 = i$, this is a contradiction again.

Hence, our initial assumption that there exists an ordering on the complex numbers is wrong. The order between two complex numbers is not complete.

5 Applications

Indeed, a solid understanding of order is considered vital in the related areas of Category Theory, Lattice Theory, Graph Theory, and Algebraic Topology.

"Order Theory" is a branch of mathematics that deals extensively with order and comparison. It delves deeper into the concept of ordered sets, building on top of the ideas established by set theory.

Hasse Diagrams are an integral part of Order Theory, used to visualise finite posets. Take a look at the following animation.

Additionally, the notion of order has been referenced by several physicists and metaphysicists. In the search for a theory that captures quantum gravity, one notable hypothesis is that of Causal Sets, which proposes that spacetime events are related by a partial order.

6 History

Although order is ubiquitous in mathematics, its formal study seems to have picked up pace only in the 19th century. Since then, it has proved to be a fundamental characteristics in various fields, central to the development of discrete mathematics in the 20th century. Dedekind, Stone, Tarski, Birkhoff, Malcev, Schutzenberger, and Dilworth are some of the prominent names that one comes across, while reading about the history of order.

7 Pause and Ponder

The subset, lesser-than, and more-than are popular examples of relations that impose order upon sets. Can you come up with other relations that would help compare elements of a set in a different light?

Is there a way to geometrically prove that the complex numbers are not ordered? Is order as fundamental to geometry as it is to algebra?

Our understanding of *time* is and *causality* is inseparable from the notion of order. How different do you think our perception of reality would be, had order not been as inherent a concept as it is now?

8 References

1. Pugh, Charles C.(2002). *Real Mathematical Analysis*
2. Rota, GC. (1997). *The Many Lives of Lattice Theory*
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