Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

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MA 102 (RA, RKS, MGPP, KVK)

Linear Transformations

Topics:

- Linear Transformation
- Kernel and Range
- Matrix of a Linear Transformation

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Such functions are called linear transformations (LT).

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Example: Is $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T([x, y]^\top) = [2x, x+y]^\top$ an LT? Yes. Note that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

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$$B\mathbf{x} := x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n.$$

Thus $B : \mathbb{F}^n \longrightarrow \mathbb{V}, \mathbf{x} \longmapsto B\mathbf{x}$, is an LT.

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Theorem: Let $B := [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be an ordered basis of $\mathbb{V}$. Then the LT $B : \mathbb{F}^n \longrightarrow \mathbb{V}, \mathbf{x} \longmapsto B\mathbf{x}$, is bijective.

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Example: The LT $S : \mathbb{V} \to \mathbb{F}^n, \mathbf{v} \longrightarrow [\mathbf{v}]_B$, is an isomorphism. Thus, any VS over $\mathbb{F}$ of dimension n is isomorphic to $\mathbb{F}^n$.

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Corollary: Let $\mathbb{V}$ be a VS with an ordered basis B and $T, S \in \mathcal{L}(V)$. Then $[S \circ T]_B = [S]_B[T]_B$.

Change of Basis

Let $B := [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ and $C := [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ be ordered bases of $\mathbb{V}$.

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Exercise: Find the change of basis matrix $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases $B = [1, x, x^2]$ and $C = [1 + x, x + x^2, 1 + x^2]$ of $\mathbb{R}_2[x]$.

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Exercises

- Check whether the following LTs are one-one and onto.
  - $T : \mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = [x, 0]^\top$, $x \in \mathbb{R}$.
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- Let $T : \mathbb{V} \to \mathbb{W}$ be an LT and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{V}$ be such that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ are linearly independent. Are $\mathbf{v}_1, \ldots, \mathbf{v}_k$ linearly dependent? Justify?

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- Let $T : \mathbb{V} \to \mathbb{W}$ be linear and one-one. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an LI subset of $\mathbb{V}$ then show that $T(S) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is LI in $\mathbb{W}$.

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- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
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  - If $T : \mathbb{V} \to \mathbb{W}$ is an one-one and onto (i.e., invertible). linear transformation, then show that $T^{-1} : \mathbb{W} \to \mathbb{V}$ is an LT.
  - Argue that if $\mathbb{V}$ is isomorphic to $\mathbb{W}$, then $\mathbb{W}$ is isomorphic to $\mathbb{V}$.

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- Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$.
  - If $T : \mathbb{V} \to \mathbb{W}$ is an one-one and onto (i.e., invertible). linear transformation, then show that $T^{-1} : \mathbb{W} \to \mathbb{V}$ is an LT.
  - Argue that if $\mathbb{V}$ is isomorphic to $\mathbb{W}$, then $\mathbb{W}$ is isomorphic to $\mathbb{V}$.
- Let $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Then show that a linear transformation $T : \mathbb{V} \to \mathbb{W}$ is one-one iff T is onto.

Exercises

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T[x, y]^\top = [x - y, -3x + 4y]^\top$ and $S[x, y]^\top = [4x + y, 3x + y]^\top$ for $[x, y]^\top \in \mathbb{R}^2$. Compute $T \circ S$ and $S \circ T$. What is your observation?
- Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$.
  - If $T : \mathbb{V} \to \mathbb{W}$ is an one-one and onto (i.e., invertible). linear transformation, then show that $T^{-1} : \mathbb{W} \to \mathbb{V}$ is an LT.
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- Let $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Then show that a linear transformation $T : \mathbb{V} \to \mathbb{W}$ is one-one iff T is onto.
- Let $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Then a one-one linear transformation $T : \mathbb{V} \to \mathbb{W}$ maps a basis for $\mathbb{V}$ onto a basis for $\mathbb{W}$.

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- Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional vector spaces. Then $\mathbb{V}$ is isomorphic to $\mathbb{W}$ iff $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

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- Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $\mathbb{V}$, and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{V}$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent in $\mathbb{V}$ if and only if $\{[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, \ldots, [\mathbf{u}_k]_B\}$ is linearly independent in $\mathbb{R}^n$.

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- Suppose $T, S : \mathbb{V} \to \mathbb{W}$ are LT's, and B and C are ordered bases of $\mathbb{V}$ and $\mathbb{W}$, resp. Show that

$$[T + S]_{C \leftarrow B} = [T]_{C \leftarrow B} + [S]_{C \leftarrow B},$$

$$[\alpha T]_{C \leftarrow B} = \alpha [T]_{C \leftarrow B}.$$

Exercises

- Let $\mathbb{V}, \mathbb{W}$ be n dimensional with bases B and C, resp., and $T : V \to W$ an LT. Then T is invertible if and only if the matrix $[T]_{C \leftarrow B}$ is invertible. In that case,

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- Let $T : \mathbb{R}^2 \to \mathbb{R}_1[x]$ be defined by $T([a, b]^\top) = a + (a + b)x$ for $[a, b]^\top \in \mathbb{R}^2$. Find $[T]_{C \leftarrow B}$ w.r.t. standard bases, show that T is invertible, and thus find T^{-1}.

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