DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

MA225 Probability Theory and Random Processes July - November 2017

Duration: 60 Minutes **Quiz – I : Solutions** Maximum Marks: 15

**NS**

Answer all the questions. Support your conclusions with precise explanations.

In the sequel, all the conventions and notations are as used in the class. Answers given without proper justification will be awarded zero marks.

1. You roll a (six-sided) fair die repeatedly until either a 2 shows up or an odd number shows up. Write down the sample space for this experiment. What is the probability of rolling a 2 before rolling an odd number? 3

**Solution:** The sample space for the given experiment is given by

$$\Omega = \{\omega | \omega = (\omega_1, \omega_2, \ldots, \omega_k), k \in \mathbb{N}, \text{ with } \omega_i \in \{4,6\}, i = 1,2,\ldots,k-1, \text{ and } \omega_k \in \{1,2,3,5\}\}.$$

1 mark

Now, to compute the probability of rolling a 2 before rolling an odd number, realize that the only way we can have a 2 rolled before an odd number is to have $m$ rolls that result in either a 4 or 6 followed by a roll that results in a 2. After that, we don't care what happens because the condition has been satisfied. Of course, we need to sum this over all $m$'s from 0 to $\infty$ to obtain all such sequences of interest. That is, if $A$ is the roll number corresponding to a 2 being rolled and $B$ is the roll number corresponding to an odd number being rolled, then

$$P(B > A) = \sum_{m=0}^{\infty} \left(\frac{2}{6}\right)^m \frac{1}{6} = \frac{1}{4}.$$

2 marks

2. (a) If the events $A$ and $B$ are independent then show that $A$ and $B^c$ are independent.

   (b) Suppose that the events $A$ and $B$ are independent, the event $A$ is not a null event, the probability that event $A$ occurs is twice the probability that event $B$ occurs, and the probability that at least one of events $A$ and $B$ occurs is 8 times the probability that both events $A$ and $B$ occur. What is the probability that event $A$ occurs? 1+2

**Solution:**

(a) Given that $A$ and $B$ are independent and this means that $P(A \cap B) = P(A)P(B)$. We can write $A = (A \cap B) \cup (A \cap B^c)$ and $(A \cap B) \cap (A \cap B^c) = \emptyset$. Hence $P(A) = P(A \cap B) + P(A \cap B^c) \implies P(A \cap B^c) = P(A) - P(A \cap B) = P(A)(1 - P(B)) = P(A)P(B^c)$. 1 mark

(b) Given that $P(A \cap B) = P(A)P(B)$, $P(A) \neq 0$, $P(A) = 2P(B)$, $P(A \cup B) = 8P(A \cap B)$.
Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\implies P(A) + P(B) = 9P(A \cap B) = 9P(A)P(B)$
$\implies 3P(B) = 18P(B) \implies P(B) = 1/6$ since $P(B) \neq 0$.
Hence, $P(A) = 2P(B) = 1/3$. 2 marks

3. If $X$ is a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that for some $a \in \mathbb{R}$, we have

$$P\left\{X \le a + \frac{1}{2^{n+1}}\right\} = P\left\{X \le a - \frac{10}{n+1}\right\} = \frac{1}{2}, \quad \forall n \in \mathbb{N}.$$

Obtain $P\{X \leq a\}$ and $P\{X = a\}$. $\boxed{3}$

**Solution:**

Let $A_n = \{\omega | X(\omega) \leq a + \frac{1}{2^{n+1}}\} = X^{-1}(-\infty, a + \frac{1}{2^{n+1}}]$. Then $A_n \supseteq A_{n+1}$ for all $n$, i.e., $A_n$ is a sequence of contracting events. By the continuity property of $P$, we then have

$$\begin{aligned}
\lim_{n\to\infty} F\Big(a + \frac{1}{2^{n+1}}\Big) &= \lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n) = P(\bigcap_{n=1}^{\infty} X^{-1}(-\infty, a + \frac{1}{2^{n+1}}]) \\
&= P\Big(X^{-1}(\bigcap_{n=1}^{\infty}(-\infty, a + \frac{1}{2^{n+1}}])\Big) \\
&= P(X^{-1}(-\infty, a]) = F(a).
\end{aligned}$$

Now, since $F\Big(a + \frac{1}{2^{n+1}}\Big) = \frac{1}{2}$ for $n \in \mathbb{N}$, we have $F(a) = P\{X \leq a\} = \frac{1}{2}$. $\boxed{\text{1 mark}}$

Similarly, let $B_n = \{\omega | X(\omega) \leq a - \frac{10}{n+1}\}$. Then $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Again using the continuity property, we get

$$\begin{aligned}
\lim_{n\to\infty} F\Big(a - \frac{10}{n+1}\Big) &= \lim_{n\to\infty} P(B_n) = P(\bigcup_{n=1}^{\infty} B_n) = P(\bigcup_{n=1}^{\infty} X^{-1}(-\infty, a - \frac{10}{n+1}]) \\
&= P\Big(X^{-1}(\bigcup_{n=1}^{\infty}(-\infty, a - \frac{10}{n+1}])\Big) \\
&= P(X^{-1}(-\infty, a)) = \lim_{x\uparrow a} F(x) = F(a-).
\end{aligned}$$

Now, since $F\big(a - \frac{10}{n+1}\big) = \frac{1}{2}$ for $n \in \mathbb{N}$, we have $F(a-) = P\{X < a\} = \frac{1}{2}$. $\boxed{\text{1 mark}}$

Finally, $P(X = a) = P(X \leq a) - P(X < a) = F(a) - F(a-) = 0$. $\boxed{\text{1 mark}}$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{x}{2}, & 0 < x \leq 1 \\ \frac{1}{2}, & 1 < x \leq 2 \\ \frac{3-x}{2}, & 2 < x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Does $f$ satisfy the properties of a probability density function of any random variable? If yes, find the distribution function. Also, compute $P\{1.5 \leq X \leq 2.5\}$ $\boxed{3}$

**Solution:** To show $f$ is probability density function of some random variable, we need to check whether $f$ is non-negative and satisfies $\int_{-\infty}^{\infty} f(x)dx = 1$. Here, since $f(x) \geq 0$ for $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 \frac{x}{2}dx + \int_1^2 \frac{1}{2}dx + \int_2^3 \frac{3-x}{2}dx = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1,$$

the given $f$ is a PDF of some continuous random variable $X$ (say). $\boxed{\text{1 mark}}$

The distribution function corresponding to the given PDF is given by

$$F(x) = P\{X \le x\} = \begin{cases} 0, \quad x \le 0 \\ \int_0^x \frac{x}{2} dx = \frac{1}{4}x^2, \quad 0 < x \le 1 \\ \frac{1}{4} + \int_1^x \frac{1}{2} dx = \frac{2x-1}{4}, \quad 1 < x \le 2 \\ \frac{3}{4} + \int_2^x \frac{3-x}{2} dx = \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, \quad 2 < x \le 3 \\ 1, \quad x > 3 \end{cases}$$

1 mark

Now, since $P\{X = 1.5\} = 0$ as $X$ is continuous, we have that

$$P\{1.5 \le x \le 2.5\} = F(2.5) - F(1.5) = \frac{15}{16} - \frac{1}{2} = \frac{7}{16}.$$

1 mark

5. Suppose a box has 5 balls labelled $1, 2, \ldots, 5$. Two balls are selected at random and with replacement. Let $X$ denote the larger of the two numbers on the balls selected. Find the smallest and largest $\sigma$-fields with respect to which $X$ is measurable. If possible, find another $\sigma$-field with respect to which $X$ is measurable. Also, determine the probability mass function of $X$. 3

**Solution:** For the given experiment, we have $\Omega = \{(i,j) | i,j = 1,2,3,4,5\}$ and $X((i,j)) = \max\{i,j\}, 1 \le i,j \le 5$ with $range(X) = \{1,2,3,4,5\}$. Since $X$ is a discrete random variable, the smallest $\sigma$-field with respect to which $X$ would be measurable is given by

$$\mathcal{F}^X = \sigma(\{X^{-1}\{1\}, X^{-1}\{2\}, X^{-1}\{3\}, X^{-1}\{4\}, X^{-1}\{5\}\}),$$

where, as usual, $\sigma(\mathcal{C})$ means the $\sigma$-field generated by the class of events $\mathcal{C}$ and

$$\begin{aligned} X^{-1}\{1\} &= \{(1,1)\}, \\ X^{-1}\{2\} &= \{(1,2),(2,1),(2,2)\}, \\ X^{-1}\{3\} &= \{(1,3),(3,1),(2,3),(3,2),(3,3)\}, \\ X^{-1}\{4\} &= \{(1,4),(4,1),(2,4),(4,2),(3,4),(4,3),(4,4)\}, \\ X^{-1}\{5\} &= \{(1,5),(5,1),(2,5),(5,2),(3,5),(5,3),(4,5),(5,4),(5,5)\}. \end{aligned}$$

This $\sigma$-field $\mathcal{F}^X$ consists of $2^5 = 32$ elements. 1 mark

The largest $\sigma$-field with respect to which $X$ is measurable is of course the power set of $\Omega$, $\mathcal{P}(\Omega)$, and will consist of $2^{25}$ elements.

The function $X$ is measurable with respect to any $\sigma$-field (over $\Omega$) that contains $\mathcal{F}^X$ given above. One such $\sigma$-field would be $\mathcal{F}^0 = \sigma(\{\mathcal{F}^X \cup \{(2,2)\}\})$ which will have $2^6 = 64$ elements. 1 mark

The probability mass function (PMF) is obtained as

$$\begin{aligned} f(x) = P\{X = x\} &= P\{X = x, \text{both the balls have the same label}\} \\ &\quad + P\{X = x, \text{both the balls do not have the same label}\} \\ &= \frac{1}{25} + 2\frac{x-1}{25} = \frac{2x-1}{25}, \quad x = 1, 2, \ldots, 5 \end{aligned}$$

and $f(x)$ is zero otherwise. 1 mark

(That is, $P\{X = 1\} = \frac{1}{25}, P\{X = 2\} = \frac{3}{25}, P\{X = 3\} = \frac{5}{25}, P\{X = 4\} = \frac{7}{25}, P\{X = 5\} = \frac{9}{25}$)