Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 01

Books

- Text Books
  - *Introduction to Probability Models* by *Seldon M. Ross*.
- Reference Books
  - *An Introduction to Probability Theory and its Applications* by *W. Feller*.
  - *Probability and Random Processes* by *G. R. Grimmett and D. R. Stirzaker*.

Grading Policy

- Weights in different examination are as follows:
    - Quiz I: 15%
    - Mid-semester Examination: 30%
    - Quiz II: 15%
    - End-semester Examination: 40%
- For each examination, linear scaling will be used.
- An **F** grade will be awarded if you obtain less than 20% of total marks after the end semester examination.

Course Website

http://www.iitg.ac.in/aganguly/ma225.php

Classical Probability

- S: Set of all possible outcomes.
- Def: $P(A) = \frac{\text{Favourable number of cases to } A}{\text{Total number of cases}} = \frac{\#A}{\#S}$.
- Example 1: A die is rolled. What is the probability of getting 3 on upper face?
  - Ans: $1/6$.
- Example 2: Consider a target comprising of three concentric circles of radii $1/3$, 1, and $\sqrt{3}$ feet. What is the probability that a shooter hits inside the inner circle?
  - Both $\#A$ as well as $\#S$ are infinite, the classical probability can not be used here.

Remarks

- The classical definition works in the first example but does not work in the second.
- Need a better definition which works for wider class of models.
- Start with classical definition and take three key properties to give more general definition of probability.
- Define the probability as a set function.
- Define the domain properly.

Countability and Uncountability

Def: We say that two sets A and B are equivalent if there exists a bijection from A to B. We denote it by $A \sim B$.

Def: For any set A we say:

1. A is countable if $A \sim \mathbb{N}$
2. A is atmost countable if either A is finite or A is countable.
3. A is uncountable if A is not atmost countable.

Example 1: $\mathbb{Z}$ is countable.

Remark: If a set is countable, then it can be written as sequence $\{x_n\}$ of distinct terms.

Summary of Results

Theorem: Every subset of an atmost countable set is again atmost countable.

Theorem: Let $\{E_n\}_{n\geq 1}$ be a sequence of atmost countable sets and put $S = \cup_{n=1}^{\infty} E_n$. Then S is again atmost countable.

Theorem: Let $A_1, A_2, \ldots, A_n$ be atmost countable sets. Then $B = A_1 \times A_2 \times \ldots \times A_n$ is also atmost countable.

Corollary: The set of rationals is countable.

Theorem: The set of all binary sequences is uncountable.

Corollary: $[0, 1]$ is uncountable.

Corollary: $\mathbb{R}$ is uncountable.

Corollary: Q^c is uncountable.

Corollary: Any interval is uncountable.

Random Experiment

Def: An experiment is called a random experiment if it satisfies the following three properties:

1. All the out comes of the experiment is known in advance.
2. The outcome of a particular performance of an experiment is not known in advance.
3. The experiment can be repeated under identical conditions.

Example 1: Toss a coin.
Example 2: Toss a coin until the first head appears.
Example 3: Measuring the height of a student.

Sample Space

Def: The collection of all possible outcomes of a random experiment is called the sample space of the random experiment. It will be denoted by $\mathcal{S}$.

Example 1: $\mathcal{S} = \{H,\ T\}$.
Example 2: $\mathcal{S} = \{H,\ TH,\ TTH,\ \ldots\}$
Example 3: $\mathcal{S} = (0,\ \infty)$

σ-algebra

Def: A non-empty collection, $\mathcal{F}$, of subsets of $\mathcal{S}$ is called a $\sigma-$algebra (or $\sigma-$field) if

1. $\mathcal{S} \in \mathcal{F}$
2. $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$
3. $A_1, A_2, \ldots \in \mathcal{F}$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Example 1: $\mathcal{F}_1 = \{\phi, \mathcal{S}, \{H\}, \{T\}\}$, $\mathcal{F}_2 = \{\phi, \mathcal{S}\}$,
$\mathcal{F}_3 = \{\phi, \mathcal{S}, \{H\}\}$
Example 2: $\mathcal{F} = \mathcal{P}(\mathcal{S})$
Example 3: $\mathcal{F} = \{\phi, \mathcal{S}, (4, 5), (4, 5)^c\}$