Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 15 (September 06, 2019)

Conditional Distribution for DRV

Def: Let (X, Y) be a DRV with JPMF $f_{X,Y}(\cdot,\cdot)$. Suppose the marginal PMF of Y is $f_Y(\cdot)$. The conditional PMF of X, given $Y = y$ is defined by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

provided $f_Y(y) > 0$.

Def: The conditional CDF of X given $Y = y$ is defined by

$$F_{X|Y}(x|y) = P(X \le x | Y = y) = \sum_{\{u \le x : (u,y) \in S_{X,Y}\}} f_{X|Y}(u|y).$$

provided $f_Y(y) > 0$.

Def: The conditional expectation of $h(X)$ given $Y = y$ is defined by

$$E(h(X)|Y = y) = \sum_{x:(x,y)\in S_{X,Y}} h(x) f_{X|Y}(x|y)\,,$$

provided it is absolutely summable.

Remark: Conditional expectation satisfies all the properties of expectation.

Example 1: Let $X \sim P(\lambda_1)$, $Y \sim P(\lambda_2)$ and X and Y are independent. Calculate the conditional expectation of X given $X + Y = n$.

Example 2: Suppose a system has n components. Suppose on a rainy day, component i functions with probability p_i, $i = 1, 2, \ldots, n$ independent of others. Calculate the conditional expected number of components that will function tomorrow given that it will rain tomorrow.

Conditional Distribution for CRV

Let (X, Y) be a CRV. The conditional CDF of X given $Y = y$ is defined as

$$F_{X|Y}(x|y) = \lim_{\epsilon \downarrow 0} P(X \leq x | Y \in (y - \epsilon, y + \epsilon]) \,.$$

provided the limit exists.
Define the conditional PDF of X given $Y = y$, $f_{X|Y}(x|y)$, as the non-negative function satisfying

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(t|y) dt, \qquad \forall x \in \mathbb{R} \,.$$

Theorem: Let $f_{X,Y}$ be the JPDF of (X, Y) and let f_Y be the marginal PDF of Y. If $f_Y(y) > 0$, then the conditional PDF of X given $Y = y$ exists and is given by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Def: The conditional expectation of $h(X)$ given $Y = y$, is defined for all values of y such that $f_Y(y) > 0$, by

$$E(h(X)|Y = y) = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y) dx,$$

provided it is absolutely integrable.

Example 3: Suppose the JPDF of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} 6xy(2 - x - y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional expectation of X given that $Y = y$, where $0 < y < 1$.

Example 4: $f_{X,Y}(x, y) = \frac{1}{2} y e^{-xy}$, $0 < x < \infty, 0 < y < 2$. Find $E(e^{X/2} | Y = 1)$.

Suppose either (X, Y) is a DRV or a CRV. Define $E(X|Y) = g(Y)$, where $g(y) = E(X|Y = y)$. Thus $E(X|Y)$ is again a random variable.

Theorem: $E(X) = E(E(X|Y))$.

Theorem: $E(X - E(X|Y))^2 \leq E(X - f(Y))^2$ for any function f. Thus $E(X|Y)$ is the "best estimate of X given Y".

Example 5: Virat will read either one chapter of his probability book or one chapter of his history book. If the no. of misprints in a chapter of his probability and history book is Poisson with mean 2 and 5 respectively, then assuming that Virat is equally likely to choose either book, what is the expected no. of misprints that he will come across.