Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 30 (November 07, 2019)

Theorem: Let $\{X_n\}$ be a irreducible MC. Then the following are equivalent.

1. All states are positive recurrent.
2. One state is positive recurrent.
3. $\{X_n\}$ has a stationary distribution.

If any of the above holds, then

1. the stationary distribution is unique and is given by $\pi_i = \frac{1}{E(T_i | X_0 = i)}$.
2. for any initial distribution μ and for any $j \in S$, $L_n(j) \to \pi_j$ with porbability 1.
3. for any initial distribution μ and for any $j \in S$, $\frac{1}{n+1} \sum_{k=0}^{n} p_{jj}^{(n)} \to \pi_j$.
4. if $\{X_n\}$ is aperiodic (i.e. $d(i) = 1$), then for any initial distribution μ and for any $j \in S$, $\lim\limits_{n \to \infty} P_\mu (X_n = j) = \pi_j$.

Some Problems

Example 1: A problem of interest to sociologists is to determine the proportion of society that belongs to upper class, middle class and lower class(in terms of wealth). One possible model would be to assume that transitions between economic classes of successive generations in a family happens according to a MC, i.e., we assume that economic condition of the child depends only on his or her parents economic condition. If such a model is true and the TPM is given by

$$\begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}$$

then in the long run what proportion of the society will be in each class?

$$\pi_0 = 0.07,\ \pi_1 = 0.62,\ \pi_2 = 0.31.$$

Some Problems

Example 2: Suppose the number of families that check into a hotel on successive days are independent Poisson random variables with mean λ. Also suppose that the number of days a family stays in the hotel is a geometric random variable with parameter $p, 0 < p < 1$. That means, a family who spent the previous night in the hotel will, independently of how long they have already spent in the hotel, check out the next day with probability p. Also suppose that the families act independently. Under these conditions it is easy to see that if X_n denotes the number of families that are staying in the hotel on day n then $\{X_n\}$ is a MC. Find

1. the TPM of the MC;
2. $E(X_n|X_0 = i)$;
3. the stationary distribution of the MC.

Example 3: [A Gambling Model] Consider a gambler who at each play of the game either wins Re. 1 with probability p or losses Re. 1 with probability $1 - p$. Suppose that the gambler quits play either when he goes broke or he attains a fortune of Rs. N. What is the probability that starting with i units of wealth the gambler will go home a winner.