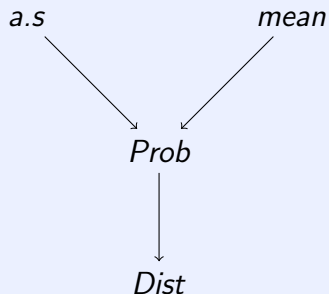


# PROBABILITY THEORY AND RANDOM PROCESSES (MA225)

LECTURE SLIDES

Lecture 19 (September 25, 2019)

# Relation between Modes of Convergence



# Counter Examples

**Example 1:** Let  $\mathcal{S} = [0, 1]$ ,  $\mathcal{F} = \mathcal{B}([0, 1])$  and  $P$  be the uniform measure. Define  $X_n = n1_{[0, \frac{1}{n}]}$ .  $X_n$  converges to 0 in probability and almost surely but not in  $r$ th mean for any  $r \geq 1$ .

**Example 2:** Let  $X_1 = 1_{[0, 1/2]}$ ,  $X_2 = 1_{[1/2, 1]}$   
 $X_3 = 1_{[0, 1/4]}$ ,  $X_4 = 1_{[1/4, 1/2]}$ ,  $X_5 = 1_{[1/2, 3/4]}$ ,  $X_6 = 1_{[3/4, 1]}$  ...  
Then  $X_n$  converges in  $r$ th mean and in probability but not almost surely.

**Example 3:** Let  $X$  be a  $N(0, 1)$  RV defined on some probability space  $(\mathcal{S}, \mathcal{F}, P)$ . Define  $X_n = X$  for all  $n$ . Then  $X_n$  converges in distribution to  $-X$  but not in probability.

**Theorem:** Suppose  $\{X_n\}$  is a sequence of RVs defined on a single probability space and  $X_n$  converges in distribution to some constant  $c$ , then  $X_n$  also converges in probability to  $c$ .

**Theorem:** Let  $\{X_n\}$  be a sequence of random variables with moment generating functions  $M_n(t)$ . Let  $X$  be a random variable with moment generating function  $M(t)$ . If  $M_n(t) \rightarrow M(t)$  for all  $t$  in an open interval containing zero, then  $X_n$  converges to  $X$  in distribution.

**Example 4:** Let  $X_n \sim \text{Bin}(n, p_n)$ , where  $p_n \rightarrow 0$  and  $np_n = \lambda(> 0)$ . Let  $X \sim \text{Poi}(\lambda)$ . Then  $X_n$  converges to  $X$  in distribution.

**Theorem:** Let  $\{X_n\}$  be a sequence of DRV's with PMF  $f_n(\cdot)$ . Let  $X$  be a DRV with PMF  $f(\cdot)$ . If  $f_n(x) \rightarrow f(x)$  for all  $x$ , then  $X_n$  converges to  $X$  in distribution.

**Example 5:** Prove the claim of the previous example using the above Theorem.

**Theorem:** Let  $\{X_n\}$  be a sequence of CRV's with PDF  $f_n(\cdot)$ . Let  $X$  be a CRV with PDF  $f(\cdot)$ . If  $f_n(x) \rightarrow f(x)$  for all  $x$ , then  $X_n$  converges to  $X$  in distribution.

**Example 6:** Let  $X_n \sim U(0, 1 + 1/n)$  and  $X \sim U(0, 1)$ . Then  $X_n$  converges to  $X$  in distribution.