

Indian Institute of Technology Guwahati
Probability Theory and Stochastic Processes (MA225)
Problem Set 04

1. Check whether the following functions are CDFs of 2-dim random vector or not.

$$(a) F(x, y) = \begin{cases} 1 & \text{if } x + 2y \geq 1 \\ 0 & \text{if } x + 2y < 1. \end{cases}$$

$$(b) F(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1 & \text{otherwise.} \end{cases}$$

2. Let $F(\cdot, \cdot)$ be the CDFs of a two-dimensional random vector (X, Y) , and let $F_1(\cdot)$ and $F_2(\cdot)$, respectively, be the marginal CDFs of X and Y . Define $U(x, y) = \min\{F_1(x), F_2(y)\}$ and $L(x, y) = \max\{F_1(x) + F_2(y) - 1, 0\}$. Prove the followings.

- (a) $L(x, y) \leq F(x, y) \leq U(x, y)$.
(b) $L(x, y)$ and $U(x, y)$ are CDFs of 2-dimensional random vector.
(c) The marginal distributions of $L(\cdot, \cdot)$ and $U(\cdot, \cdot)$ are same as that of $F(\cdot, \cdot)$.

3. Let the random variable X have CDF $F_1(\cdot)$ and let $Y = g(X)$ have distribution function $F_2(\cdot)$, where $g(\cdot)$ is some function. Prove that

- (a) If $g(\cdot)$ is increasing, $F_{X,Y}(x, y) = \min\{F_1(x), F_2(y)\}$.
(b) If $g(\cdot)$ is decreasing, $F_{X,Y}(x, y) = \max\{F_1(x) + F_2(y) - 1, 0\}$.

4. Consider the following joint PMF of the random vector (X, Y) .

$\begin{array}{c c} x & y \end{array}$	1	2	3	4
4	0.08	0.11	0.09	0.03
5	0.04	0.12	0.21	0.05
6	0.09	0.06	0.08	0.04

- (a) Find the probabilities $P(X + Y < 8)$, $P(X + Y > 7)$, $P(XY \leq 14)$.
(b) Find the $Corr(X, Y)$

5. For the bivariate negative binomial distribution, the PMF is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{(x+y+k-1)!}{x!y!(k-1)!} \theta_1^x \theta_2^y (1 - \theta_1 - \theta_2)^k & \text{if } x \in \{0, 1, 2, \dots\}, y \in \{0, 1, 2, \dots\} \\ 0 & \text{otherwise,} \end{cases}$$

k is a positive integer, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, and $0 < \theta_1 + \theta_2 < 1$. Find both the marginal distributions.

6. Three balls are randomly placed in three empty boxes B_1 , B_2 , and B_3 . Let N denote the total number of boxes which are occupied and let X_i denote the number of balls in the box B_i , $i = 1, 2, 3$.

- (a) Find the joint PMF of (N, X_1) .
(b) Find the joint PMF of (X_1, X_2) .
(c) Find the marginal distributions of N and X_2 .
(d) Find the marginal PMF of X_1 from the joint PMF of (X_1, X_2) .

7. Let X_1, \dots, X_n be *i.i.d.* random variables with mean μ and variance σ^2 . Then $E(\bar{X}) = \mu$, $Var(\bar{X}) = \frac{\sigma^2}{n}$, and $Cov(\bar{X}, X_i - \bar{X}) = 0$ for all $i = 1, 2, \dots, n$, where $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$.

8. Suppose that X_1, \dots, X_n are independent and identically distributed random variables such that $P(X_i = 0) = 1 - p = 1 - P(X_i = 1)$, $i = 1, \dots, n$, for some $p \in (0, 1)$. Let X be the number of X_1, \dots, X_n that are as large as X_1 . Find the PMF of X .

9. For the bivariate beta random vector (X, Y) having PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{\Gamma(\theta_1 + \theta_2 + \theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1-1} y^{\theta_2-1} (1-x-y)^{\theta_3-1} & \text{if } x > 0, y > 0, x+y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_i > 0$, $i = 1, 2, 3$. Find both the marginal PDFs.

10. The joint PDF of (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} 4xy & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Find the marginal PDFs of X and Y .
- (b) Verify whether X and Y are independent.
- (c) Find $P(\{0 < X < 0.5, 0.25 < Y < 1\})$ and $P(\{X + Y < 1\})$.

11. Let $X = (X_1, X_2, X_3)$ be a random vector with joint PDF

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)} \left(1 + x_1 x_2 x_3 e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)}\right) \quad \text{if } (x_1, x_2, x_3) \in \mathbb{R}^3$$

- (a) Are X_1 , X_2 , and X_3 independent?
- (b) Are X_1 , X_2 , and X_3 pairwise independent?

12. Let X and Y be jointly distributed random variables with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 2$, and $\text{Corr}(X, Y) = 1/3$. Find $\text{Corr}\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right)$.

13. Suppose that the random vector (X, Y) is uniformly distributed over the region $A = \{(x, y) : 0 < x < y < 1\}$. Find $\text{Cov}(X, Y)$.