1) Suppose X & Y are independent then we know that

$$f(x,y) = f_X(x)\, f_Y(y)$$ where $f_X(\cdot)$ is the marginal PDF of X and $f_Y(y)$ is the marginal PDF of Y.

Now suppose $$f(x,y) = g(x)\, h(y)$$

Thus $$f_Y(y) = h(y) \int_{\mathbb{R}} g(x)\, dx = c_1 h(y)$$

Sim. $$f_X(x) = g(x) \int_{\mathbb{R}} g(y)\, dy = c_2\, g(x).$$

Also since $\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\, dx\, dy = 1$, we have $c_1 c_2 = 1$.

Thus $$f(x,y) = c_1 c_2\, h(y)\, g(x) = f_X(x)\, f_Y(y).$$

Hence X & Y are independent.

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2. $$M_{Y,Z}(t_1, t_2) = E\left[e^{t_1 Y + t_2 Z}\right] = E\left[e^{t_1(X_1+X_2) + t_2(X_1^2+X_2^2)}\right]$$

$$= E\left[e^{t_1 X_1 + t_2 X_1^2}\right] E\left[e^{t_1 X_2 + t_2 X_2^2}\right]$$

$$= \left(E\left[e^{t_1 X + t_2 X^2}\right]\right)^2 \quad \text{where } X \sim N(0,1)$$

$$E\left[e^{t_1 X + t_2 X^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t_1 u + t_2 u^2} e^{-\frac{u^2}{2}}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(1-2t_2)}{2}\left\{u^2 - 2u\frac{t_1}{1-2t_2} + \frac{t_1^2}{(1-2t_2)^2} - \frac{t_1^2}{(1-2t_2)^2}\right\}}\, du$$

The integral exists for $t_2 < \frac{1}{2}$. Thus

$$E\left[e^{t_1 X + t_2 X^2}\right] = \frac{1}{(1-2t_2)^{-1/2}} e^{\frac{t_1^2}{2(1-2t_2)}}$$

Thus $M_{Y,Z}(t_1,t_2) = \frac{1}{1-2t_2} e^{\frac{t_1^2}{1-2t_2}}$ for $t_2 < \frac{1}{2}$.

$$EY = \frac{\partial}{\partial t_1} M_{Y,Z}(t_1,t_2)\Big|_{t_1=0,t_2=0} = \frac{1}{1-2t_2} e^{\frac{t_1^2}{1-2t_2}} \frac{2t_1}{1-2t_2}\Big|_{t_1=0,t_2=0} = 0$$

$$EZ = \frac{\partial}{\partial t_2} M_{Y,Z}(t_1,t_2)\Big|_{t_1=0,t_2=0} =$$

$$EY^2 = \frac{\partial^2}{\partial t_1^2} M_{Y,Z}(t_1,t_2)\Big|_{t_1=0,t_2=0} = 2$$

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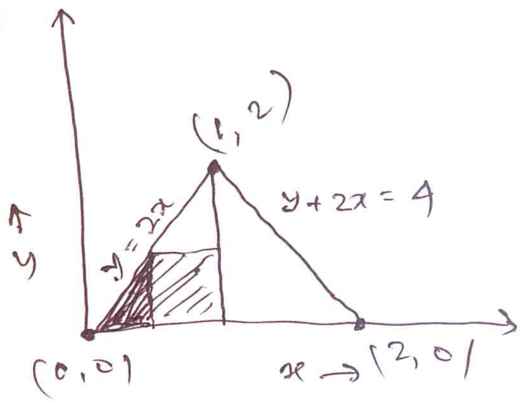
2)



The area of the triangle is $\frac{1}{2}\cdot 2\cdot 2 = 2$.

Thus (X, Y) has JPDF given by

$$f_{X,Y}(x,y) = \frac{1}{2} \quad \text{if } (x,y)\in R$$
$$= 0 \quad \text{otherwise}$$

~~where~~

where R is the interior of the triangle.

$$\therefore P(X\le 1, Y\le 1) = \frac{1}{2}\text{ Area of shaded region}$$

$$= \frac{1}{2}(\text{Area of triangle}) + \frac{1}{2}(\text{Area of rectangle})$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{2}\cdot 1\right) + \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

3) Thus JPDF of (X_1, X_2) is

$$f_{X_1,X_2}(x_1, x_2) = 4x_1 x_2 e^{-(x_1^2 + x_2^2)} \qquad \begin{array}{l} 0 < x < \infty \\ 0 < y < \infty \end{array}$$

Let $U = \sqrt{X_1^2 + X_2^2}$, $V = \dfrac{X_1}{X_2}$.

Thus $u = \sqrt{x_1^2 + x_2^2}$ and $v = \dfrac{x_1}{x_2}$

$x_1 = \dfrac{uv}{\sqrt{1+v^2}}$, $x_2 = \dfrac{u}{\sqrt{1+v^2}}$.

as x_1, x_2 varies from 0 to ∞, u, v also varies from 0 to ∞.

Now $J^{-1} = \begin{vmatrix} \dfrac{x_1}{u} & \dfrac{x_2}{u} \\ \dfrac{1}{x_2} & -\dfrac{x_1}{x_2^2} \end{vmatrix} = -\dfrac{1}{u}\left(1 + \dfrac{x_1^2}{x_2^2}\right)$

$$= -\frac{1}{u}(1+v^2).$$

Thus $f_{U,V}(u, v) = 4\dfrac{u^2 v}{1+v^2} e^{-u^2} \dfrac{u}{1+v^2} \qquad 0 < u, v < \infty.$

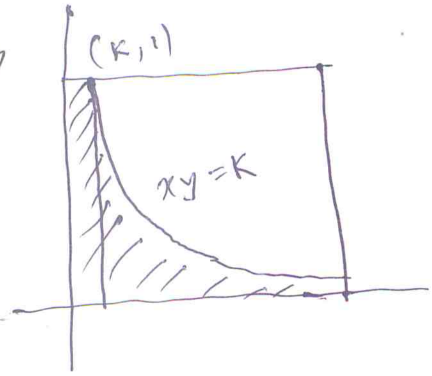
$$= \frac{4u^3 v}{(1+v^2)^2} e^{-u^2} \qquad 0 < u, v < \infty.$$

Thus $f_U(u) = u^3 e^{-u^2} \displaystyle\int_0^\infty \frac{4v}{(1+v^2)^2} dv = 2u^3 e^{-u^2}, \; 0 < u < \infty.$

4) Let X and Y be the two numbers. We are interested in calculating the probability $P(XY < K)$.

Now JPDF of (X, Y) is $f_{X,Y}(x,y) = 1$ if $0 < x, y < 1$
$= 0$ otherwise.

Thus $P(XY < k)$ = area of shaded region



$$\cancel{= \int_0^1 \int_0^{k/x} dy\,dx}$$

$$= k + \int_k^1 \frac{k}{x}\,dx = k + k\log 1 - k\log k$$

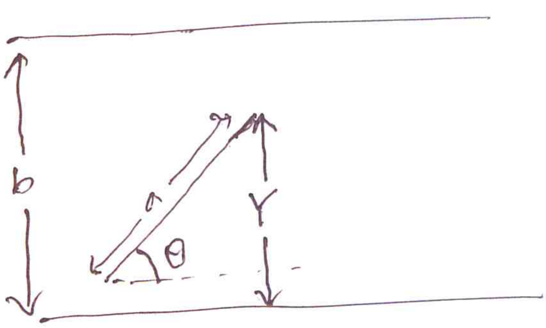
$$= k(1 - \log k).$$

5) We denote the inclination of the needle with the horizonal by θ and let Y be the distance of the upper tip of the needle from the nearest ruling below it.

Thus $\theta \sim U(0, \pi)$ and

$Y \sim U(0, b)$ and

θ & Y are independent.



Thus (θ, Y) has JPDF given by

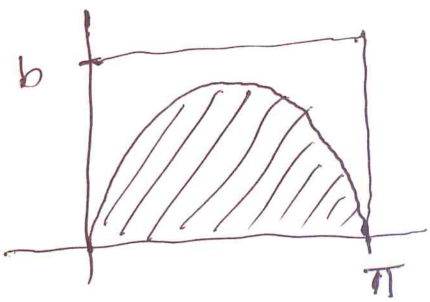
$$f_{\theta,Y}(\theta, y) = \frac{1}{\pi b} \quad \text{for } \begin{array}{l} 0 < \theta < \pi \\ 0 < y < b \end{array}$$

$$= 0 \quad \text{otherwise.}$$

The needle will intersect a line iff $Y \leq a\sin\theta$.

Thus the reqd. prob. is = area of shaded region

$$= \frac{a}{\pi b}\int_0^{\pi} \sin\theta\, d\theta = \frac{2a}{\pi b}.$$



6)

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = 48\,x_1x_2x_3 \qquad 0<x_1<x_2<x_3<1$$
$$= 0 \qquad \text{otherwise.}$$

$$Y_1 = \frac{X_1}{X_2}, \quad Y_2 = \frac{X_2}{X_3}, \quad Y_3 = X_3.$$

Thus $x_3 = y_3, \quad x_2 = y_2y_3, \quad x_1 = y_1y_2y_3$

$$0<y_3<1, \quad 0<y_2<1, \quad 0<y_3<1.$$

$$J = \begin{vmatrix} y_2y_3 & y_1y_3 & y_1y_2 \\ 0 & y_3 & y_2 \\ 0 & 0 & 1 \end{vmatrix} = y_2y_3^2$$

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = 48\,y_1y_2^2y_3^3 \times y_2y_3^2 \qquad 0<y_1,y_2,y_3<1$$
$$= 48\,y_1y_2^3y_3^5.$$

Thus
$$f_{Y_1}(y_1) = 2y_1, \quad 0<y_1<1$$
$$f_{Y_2}(y_2) = 4y_2^3, \quad 0<y_2<1$$
$$f_{Y_3}(y_3) = 6y_3^5, \quad 0<y_3<1.$$

$\boxed{Q7}$

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} e^{-(x_1+x_2+x_3)} & \text{if } x_1>0,\ x_2>0,\ x_3>0 \\ 0 & \text{O.W.} \end{cases}$$

Now $x_1 = y_1 y_3$, $x_2 = y_2 y_3$ and $x_3 = y_3(1-y_1-y_2)$

Hence $x_1>0,\ x_2>0,\ x_3>0 \Rightarrow y_1>0,\ y_2>0,\ y_3>0$ and $y_1+y_2<1$.

$$J = \begin{vmatrix} y_3 & 0 & -y_3 \\ 0 & y_3 & -y_3 \\ y_1 & y_2 & 1-y_1-y_2 \end{vmatrix} = y_3^2$$

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \begin{cases} y_3^2\, e^{-y_3} & \text{if } y_1>0,\ y_2>0,\ y_3>0,\ y_1+y_2<1 \\ 0 & \text{O.W.} \end{cases}$$

$$f_{Y_1}(y_1) = \begin{cases} 2(1-y_1) & \text{if } 0<y_1<1 \\ 0 & \text{O.W.} \end{cases}$$

$$f_{Y_2}(y_2) = \begin{cases} 2(1-y_2) & \text{if } 0<y_2<1 \\ 0 & \text{O.W.} \end{cases}$$

$$f_{Y_3}(y_3) = \begin{cases} y_3^2\, e^{-y_3} & \text{if } y_3>0 \\ 0 & \text{O.W.} \end{cases}$$

II

$\boxed{Q8}$

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} e^{-(x_1+x_2+x_3)} & \text{if } x_1>0,\ x_2>0,\ x_3>0. \\ 0 & \text{o.w.} \end{cases}$$

Now $x_1 = y_1 y_3$, $x_2 = y_3(y_2 - y_1)$ and $x_3 = y_3(1-y_2)$.

Again $x_1>0, x_2>0, x_3>0 \Rightarrow y_1>0,\ y_3>0,\ y_2-y_1>0,\ 1-y_2>0.$

$$J = \begin{vmatrix} y_3 & -y_3 & 0 \\ 0 & y_3 & -y_3 \\ y_1 & y_2-y_1 & 1-y_2 \end{vmatrix} = y_3^2$$

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \begin{cases} y_3^2\, e^{-y_3} & \text{if } y_1>0,\ y_3>0,\ y_2-y_1>0, \\ & \quad 1-y_2>0 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{Y_1}(y_1) = \begin{cases} 2(1-y_1) & \text{if } 0<y_1<1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{Y_2}(y_2) = \begin{cases} 2y_2 & \text{if } 0<y_2<1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{Y_3}(y_3) = \begin{cases} \frac{1}{2} y_3^2\, e^{-y_3} & \text{if } y_3>0 \\ 0 & \text{o.w.} \end{cases}$$

$\square$

$\boxed{Q9}$

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1} y^{\alpha_2-1} e^{-\beta(x+y)} & \text{if } x>0, y>0 \\ 0 & \text{o.w.} \end{cases}$$

~~Now~~ Take $U = \dfrac{X}{X+Y}$, $V = X+Y$

Proceed as Q7 or Q8.

$$f_U(u) = \begin{cases} \dfrac{1}{B(\alpha_1,\alpha_2)} u^{\alpha_1-1} (1-u)^{\alpha_2-1} & \text{if } 0<u<1 \\ 0 & \text{o.w.} \end{cases}$$

$\boxed{Q10}$ For $x<0$, $F_X(x) = 0$.

For $x>0$, $F_X(x) = P(X \le x)$

$$= P(Y+Z \le x)$$

$$= \sum_{y=0}^{\infty} P(Y+Z \le x \mid Y=y)\, P(Y=y)$$

$$= \sum_{y=0}^{\infty} P(Z \le x-y)\, P(Y=y)$$

$$= \sum_{y=0}^{[x]-1} \frac{e^{-\lambda}\lambda^y}{y!} + \frac{e^{-\lambda}\lambda^{[x]}}{[x]!}(x-y).$$

Hence

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \displaystyle\sum_{y=0}^{[x]-1} \frac{e^{-\lambda}\lambda^y}{y!} + \frac{e^{-\lambda}\lambda^{[x]}}{[x]!}(x-[x]) & x \geq 0 \end{cases}$$

$$f_X(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{[x]}}{[x]!} & \text{if } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$\square$.

$\boxed{Q11}$

$$F_{X_{(1)},X_{(n)}}(x,y) = P(X_{(1)} \leq x,\ X_{(n)} \leq y)$$

$$= P(X_{(n)} \leq y) - P(X_{(1)} > x,\ X_{(n)} \leq y)$$

~~= P(All of $X_1, \dots, X_n$ are ≤~~

$$= P(X_1 \leq y, \dots, X_n \leq y) - P(x < X_1 \leq y, \dots, x < X_n \leq y)$$

$$= \begin{cases} 0 & \text{if } \cancel{x}\ y < 0. \\ y^n - y^n & \text{if } x < 0 < y < 1 \\ 1^n - 1^n & \text{if } x < 0,\ y > 1 \\ y^n - (y-x)^n & \text{if } 0 < x < y < 1 \\ 1^n - (1-x)^n & \text{if } 0 < x < 1 < y \end{cases}$$

$$= \begin{cases} 1^n - 0^n & \text{if} \quad 0 < 1 < x < y < \infty \\ y^n - 0^n & \text{if} \quad 0 < y < x < 1 \\ y^n - 0^n & \text{if} \quad 0 < y < 1 < x \\ 1^n - 0^n & \text{if} \quad 0 < 1 < y < x < \infty \end{cases}$$

$$f_{X_{(1)}, X_{(n)}}(x, y) = \begin{cases} n(n-1)(y-x)^{n-2} & \text{if } 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X_{(n)}}(y) = \begin{cases} n\, y^{n-1} & \text{if } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X_{(1)}}(x) = \begin{cases} n(1-x)^{n-1} & \text{if } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \qquad \square$$

$\boxed{Q12}$ For $x = 0, 1, 2, \ldots$

$$\begin{aligned} P(X_{(2)} = x) &= P(X_1 = x,\ X_2 < x) \\ &\quad + P(X_1 < x,\ X_2 = x) \\ &\quad + P(X_1 = x,\ X_2 = x) \\ &= 2\,\frac{e^{-\lambda}\lambda^x}{x!} \sum_{i=0}^{x-1} \frac{e^{-\lambda}\lambda^i}{i!} + \frac{e^{-2\lambda}\lambda^{2x}}{x!\,x!} \end{aligned} \qquad \square$$