Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 27 (October 31, 2019)

Strong Markov Property

Theorem: (The Strong Markov Property) For any state i, and any initial distribution $\mu = \{\mu_i\}$ and any $k < \infty$, any states $i_1, i_2, \ldots, i_k$,

$$\begin{aligned} & P_\mu\left(X_{T_i+j} = i_j, j = 1, 2, \ldots, k, \ T_i < \infty\right) \\ & = P_\mu\left(T_i < \infty\right) P\left(X_j = i_j, j = 1, 2, \ldots, k | X_0 = i\right). \end{aligned}$$

Def: Let i be a state. Define $T_i^{(0)} = 0$ and for $k \geq 0$

$$T_i^{(k+1)} = \begin{cases} \inf\left\{n : n > T_i^{(k)}, X_n = i\right\} & \text{if } T_i^{(k)} < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Theorem: Let i be a recurrent state. Then for all $k \geq 0$,

$$P\left(T_i^{(k)} < \infty | X_0 = i\right) = 1.$$

Cycles

Def: Let $\eta_r = \left\{ X_j : \; T_i^{(r)} \leq j < T_i^{(r+1)}, \; T_i^{(r+1)} - T_i^{(r)} \right\}$ for $r = 0, 1, \ldots$. The η_r's are called cycles or excursions.

Theorem: Let i be a recurrent state. Under $X_0 = i$, the sequence $\{\eta_r\}_{r=0}^{\infty}$ are i.i.d. as random vectors with a random number of components, i.e., for any $k \in \mathbb{N}$,

$$P(\eta_r = (x_{r_0}, x_{r_1}, \ldots, x_{r_{j_r}}, j_r), r = 0, 1, \ldots, k | X_0 = i)$$
$$= \prod_{r=0}^{k} P(\eta_1 = (x_{r_0}, x_{r_1}, \ldots, x_{r_{j_r}}, j_r) | X_0 = i),$$

for any states $x_{r_0}, x_{r_1}, \ldots, x_{r_{j_r}}$ and time j_r, $r = 0, 1, \ldots, k$.

Number of Visits

Theorem: For any state i, let N_i be the number of visits to state i. Then,

- (a) i recurrent implies $P(N_i = \infty | X_0 = i) = 1$.
- (b) i transient implies $P(N_i = n | X_0 = i) = f_{ii}^n (1 - f_{ii})$ for $n = 0, 1, 2, \ldots$, where $f_{ii} = P(T_i < \infty | X_0 = i)$ is the probability of returning to i starting from i. Thus $N_i | X_0 = i \sim Geo(1 - f_{ii})$.

Corollary:

1. A state i is recurrent iff $\displaystyle\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$.
2. A state i is transient iff $\displaystyle\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

Remark: $P(X_n = i \text{ for infinitely many } n | X_0 = i) = 1 \text{ or } 0$ iff recurrent or transient.

Some Theorems

Theorem: If the state space S is finite, then at least one state must be recurrent.

Theorem: Let $i \leftrightarrow j$. Then

1. If i is recurrent, then j is recurrent.
2. If i is transient, then j is transient.

Theorem: Let i be recurrent and $i \to j$. Then $f_{ij} = P\left(T_j < \infty | X_0 = i\right) = 1$, $f_{ji} = P\left(T_i < \infty | X_0 = j\right) = 1$ and j is recurrent.

Remark: Above theorem is not true if i is transient.

Theorem: Suppose that $\{X_n\}$ is irreducible and recurrent. Then for all $i \in S$, $P_\mu\left(T_i < \infty\right) = 1$ for any initial distribution μ.