Countability and Uncountability

Definition 1. *We say that two sets A and B are equivalent if there exists a bijection from A to B. We denote it by $A \sim B$.*

Definition 2. *For any positive integer n, let $J_n = \{1, 2, \ldots, n\}$ and $\mathbb{N}$ be the set of all positive integers (natural numbers). For any set A, we say:*

(a) A is finite if $A = \phi$ or $A \sim J_n$ for some $n \in \mathbb{N}$. n is said to be the cardinality of A or number of elements in A.

(b) A is infinite if A is not finite.

(c) A is countable if $A \sim \mathbb{N}$.

(d) A is atmost countable if A is finite or countable.

(e) A is uncountable if A is neither finite not countable.

Example 1. *The set of all integers, $\mathbb{Z}$, is countable. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by*

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd.} \end{cases}$$

Remark 1. *A finite set cannot be equivalent to any of its proper subset. However, this is possible for an infinite set. For example consider the bijection $f : \mathbb{N} \to 2\mathbb{N}$ defined by*

$$f(n) = 2n.$$

Remark 2. *If a set is countable, then it can be written as a sequence $\{x_n\}_{n\geq 1}$ of distinct terms.*

Theorem 1. *Every infinite subset of a countable set A is countable.*

Proof. Suppose that $E \subset A$ and E is infinite. We can write the elements of A as $\{x_n\}$, sequence of distinct elements. Construct a sequence $\{n_k\}_{k\geq 1}$ as follows: Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, n_2, \ldots, n_{k-1}$, let n_k be the smallest positive integer grater than n_{k-1} such that $x_{n_k} \in E$. Now $f(k) = x_{n_k}$ is a bijection from $\mathbb{N}$ to E. $\square$

Theorem 2. *Let $\{E_n\}_{n\geq 1}$ be a sequence of atmost countable sets and put $S = \cup_{n=1}^{\infty} E_n$. Then S is atmost countable.*

Theorem 3. *Let $A_1, A_2, \ldots, A_n$ be atmost countable sets. Then $B_n = A_1 \times A_2 \times \ldots \times A_n$ is atmost countable.*

Proof. We will prove it by induction. $B_1 = A_1$ is atmost countable. Now assume that B_{n-1} is atmost countable. The elements of B_n are of the form (b, a) where $b \in B_{n-1}$ and $a \in A_n$. For every fixed b, the set, A_b, of pairs (b, a) is equivalent to A_n and hence atmost countable. Then $B_n = \cup_{b \in B_{n-1}} A_b$. Hence B_n is atmost countable by Theorem 2. □

Corollary 1. *The set of rationals, $\mathbb{Q}$, is countable.*

Theorem 4. *The set, A, of all binary sequences is uncountable.*

Proof. We will prove it by contradiction. If possible, suppose that A is countable. Then we can write it as a sequence of distinct elements $\{s_n\}_{n \geq 1}$. Now consider the sequence s, whose nth term is 1 if the nth term of s_n is 0 and 0 if the nth term of s_n is 1. Then $s \neq s_n$ for all $n \in \mathbb{N}$. It is a contradiction to the fact that A is countable. □

Corollary 2. $[0, 1]$ *is uncountable.*

Corollary 3. $\mathbb{R}$ *is uncountable.*

Corollary 4. $\mathbb{Q}^c$ *is uncountable.*

Corollary 5. *Any interval is uncountable.*