Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 19 (September 25, 2019)

Relation between Modes of Convergence

$$
\begin{tikzcd}
a.s \arrow[dr] & & mean \arrow[dl] \\
& Prob \arrow[d] & \\
& Dist &
\end{tikzcd}
$$

Counter Examples

Example 1: Let $\mathcal{S} = [0,1], \mathcal{F} = \mathcal{B}([0,1])$ and P be the uniform measure. Define $X_n = n1_{[0,\frac{1}{n}]}$. X_n converges to 0 in probability and almost surely but not in rth mean for any $r \geq 1$.

Example 2: Let $X_1 = 1_{[0,1/2]}, X_2 = 1_{[1/2,1]}$
$X_3 = 1_{[0,1/4]}, X_4 = 1_{[1/4,1/2]}, X_5 = 1_{[1/2,3/4]}, X_6 = 1_{[3/4,1]} \dots$.
Then X_n converges in rth mean and in probability but not almost surely.

Example 3: Let X be a $N(0,1)$ RV defined on some probability space $(\mathcal{S}, \mathcal{F}, P)$. Define $X_n = X$ for all n. Then X_n converges in distribution to $-X$ but not in probability.

Theorem: Suppose $\{X_n\}$ is a sequence of RVs defined on a single probability space and X_n converges in distribution to some constant c, then X_n also converges in probability to c.

Theorem: Let $\{X_n\}$ be a sequence of random variables with moment generating functions $M_n(t)$. Let X be a random variable with moment generating function $M(t)$. If $M_n(t) \to M(t)$ for all t in an open interval containing zero, then X_n converges to X in distribution.

Example 4: Let $X_n \sim Bin(n, p_n)$, where $p_n \to 0$ and $np_n = \lambda(> 0)$. Let $X \sim Poi(\lambda)$. Then X_n converges to X in distribution.

Theorem: Let $\{X_n\}$ be a sequence of DRVs with PMF $f_n(\cdot)$. Let X be a DRV with PMF $f(\cdot)$. If $f_n(x) \to f(x)$ for all x, then X_n converges to X in distribution.

Example 5: Prove the claim of the previous example using the above Theorem.

Theorem: Let $\{X_n\}$ be a sequence of CRVs with PDF $f_n(\cdot)$. Let X be a CRV with PDF $f(\cdot)$. If $f_n(x) \to f(x)$ for all x, then X_n converges to X in distribution.

Example 6: Let $X_n \sim U(0, 1 + 1/n)$ and $X \sim U(0, 1)$. Then X_n converges to X in distribution.