Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 05 (August 02, 2019)

Random Variables

Def: A function $X : \mathcal{S} \to \mathbb{R}$ is called a random variable.

Example 1: Tossing a fair coin n times. Assume that the tosses are independent. Let $X : \mathcal{S} \to \mathbb{R}$ be defined by the no. of heads.

Example 2: Throwing a fair die twice. Assume the throws are independent. Let $X : \mathcal{S} \to \mathbb{R}$ be defined by the sum of the outcomes.

Example 3: Suppose we are testing the reliability of a battery. Define $X : \mathcal{S} \to \mathbb{R}$ by $X_1(\omega) = \omega$. Now suppose we are mainly interested in whether the battery would last more than 2 years or not. Then $X_2 = 1_{(2,\infty)}$.

Example 1: Take n=2. $P(X = 0) = P(X = 2) = 1/4$, $P(X = 1) = 1/2$.

Example 2: $P(X = 2) = 1/36, P(X = 3) = 2/36, P(X = 4) = 3/36, P(X = 5) = 4/36, P(X = 6) = 5/36, P(X = 7) = 6/36, P(X = 8) = 5/36, P(X = 9) = 4/36, P(X = 10) = 3/36, P(X = 11) = 2/36, P(X = 12) = 1/36$.

Example 3: $P(I) = \int_I e^{-t} dt$, defines a probability on $\mathcal{B}(0, \infty)$. $P(X_2 = 1) = e^{-2}, P(X_2 = 0) = 1 - e^{-2}$.

Cumulative Distribution Function

Def: The cumulative distribution function (CDF) of a random variable X is a function $F_X : \mathbb{R} \to [0, \infty)$ defined by

$$F_X(x) = P(X \leq x)\,.$$

Example 1:

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1/4 & \text{if} \quad 0 \leq x < 1, \\ 3/4 & \text{if} \quad 1 \leq x < 2, \\ 1 & \text{if} \quad x \geq 2\,. \end{cases}$$

Example 2:

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 2, \\ 1/36 & \text{if} \quad 2 \le x < 3, \\ 3/36 & \text{if} \quad 3 \le x < 4, \\ 6/36 & \text{if} \quad 4 \le x < 5, \\ 10/36 & \text{if} \quad 5 \le x < 6, \\ 15/36 & \text{if} \quad 6 \le x < 7, \\ 21/36 & \text{if} \quad 7 \le x < 8, \\ 26/36 & \text{if} \quad 8 \le x < 9, \\ 30/36 & \text{if} \quad 9 \le x < 10, \\ 33/36 & \text{if} \quad 10 \le x < 11, \\ 35/36 & \text{if} \quad 11 \le x < 12, \\ 1 & \text{if} \quad x \ge 12\,. \end{cases}$$

Example 3:

$$F_{X_1}(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - e^{-x} & \text{if} \quad x \geq 0\,. \end{cases}$$

$$F_{X_2}(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - e^{-2} & \text{if} \quad 0 \leq x < 1, \\ 1 & \text{if} \quad x \geq 1\,. \end{cases}$$

Proposition: The CDF of a random variable has the following properties:

(1) $F_X(\cdot)$ is non-decreasing and hence has only jump discontinuities.

(2) $\lim_{x \uparrow \infty} F_X(x) = 1$, $\lim_{x \downarrow -\infty} F_X(x) = 0$.

(3) $\lim_{h \downarrow 0} F_X(x+h) = F_X(x), \forall x \in \mathbb{R}$, thus CDF is right continuous.

(4) $\lim_{h \downarrow 0} F_X(x-h) = F_X(x) - P(X = x), \forall x \in \mathbb{R}$.

Theorem: Let F be a function satisfying properties (1)-(3). Then F is a CDF.

Remarks

- Random variable is just a function and does not depend on the probability. But the distribution of the random variable depends on the probability. So keeping the function same if we change the probability then the random variable will remain the same but its distribution will change. Consider Example 1, but with the probabilities
  $P(HH) = 9/16, P(TT) = 1/16, P(HT) = P(TH) = 3/16.$
  What will be the distribution function in this case?
- If $x \in \mathbb{R}$ is such that $P(X = x) > 0$, then x is said to be an atom of the distribution function of X. Thus if the distribution function of a random variable has no atoms then it is continuous.
- $P(a < X \leq b) = F_X(b) - F_X(a)$.
- $P(a \leq X \leq b) = F_X(b) - F_X(a-)$.
- $P(a < X < b) = F_X(b-) - F_X(a)$.
- $P(a \leq X < b) = F_X(b-) - F_X(a-)$.