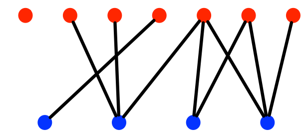
Graph fundamentals

Bipartite graph characterization



Lemma. *If a graph contains an odd closed walk, then it contains an odd cycle.*

Proof strategy: Consider a shortest closed odd walk W. If W is not a cycle, then it has a repeated vertex x that is not the first/last one, i.e., W is $u, \ldots, x, \ldots, x, \ldots, u$. Notice there are two closed walks shorter than W (namely, $x, \ldots, x$ on inside and $u, \ldots, x, \ldots, u$ using outside portions). ■

Can you say something analogous to the above lemma for a closed even walk? How about a closed even trail?

Definition For vertices $u, v \in V(G)$, the *distance* from u to v, denoted $d(u, v)$, in G is the length of a shortest u, v-path.[1]

Theorem (König). *A graph is bipartite if and only if it has no odd cycle.*

Proof. ($\Rightarrow$) If G is bipartite with bipartition X, Y of the vertices, then any cycle C has vertices that must alternately be in X and Y. Thus, since a cycle is closed, C must have an even number of vertices and hence is an even cycle.

($\Leftarrow$) First, we prove the result in the case where G is connected. Assume G has no odd cycle. Let $u \in V(G)$ be an arbitrary vertex. Partition all other vertices based on the parity of distance (even or

[1]Note: If no u, v-path exists, we say that $d(u, v) = \infty$ (or sometimes it is said to be undefined).

odd) from u. That is, let

$$\begin{aligned} X &= \{v \in V(G) : d(u,v) \text{ is even}\}, \\ Y &= \{v \in V(G) : d(u,v) \text{ is odd}\}. \end{aligned}$$

Clearly, $X \cap Y = \emptyset$ and $X \cup Y = V(G)$ since G is connected.

We claim that X, Y is a bipartition of G. Suppose not – then there exists an edge incident to two vertices of X or an edge incident to two vertices of Y. Without loss of generality, assume the former. Let $x_1, x_2 \in X$ and $x_1 \sim x_2$. It follows that

$$\begin{aligned} x_1 \in X \quad &\Longrightarrow \quad \exists\ u, x_1\text{-path } P_1 \text{ of even length}, \\ x_2 \in X \quad &\Longrightarrow \quad \exists\ u, x_2\text{-path } P_2 \text{ of even length}. \end{aligned}$$

Concatenate u, x_1-path P_1, the edge x_1x_2, and the x_2, u-path ${P_2}^{-1}$ to obtain a closed odd walk. By the previous lemma, the graph must contain an odd cycle. $\Rightarrow\Leftarrow$ Hence, it must be the case the X, Y is a valid bipartition, so G is bipartite.

Exercise: Finish proof in the case where G is a disconnected graph.

■

Corollary. *A graph G is bipartite if and only if the adjacency matrix A of G satisfies* trace $A^k = 0$ *for* $k = 3, 5, 7, \ldots, 2\lceil \frac{n}{2} \rceil - 1$.

We can make the corollary stronger by simply requiring trace $A^r = 0$ where $r = 2\lceil \frac{n}{2} \rceil - 1$ (the largest odd integer less than or equal to n).

Induction trap

The following result follows immediately from the degree-sum formula.

Lemma. *Every graph has an even number of vertices of odd degree.*

We will use the above lemma on our way to proving a property of 3-regular connected graphs.

Consider the following 3-regular graphs (*cubic graphs*):

Note that none of them have any cut edges. (Also, all have an even # of vtcs.)

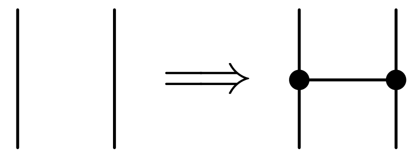
Claim: No 3-regular connected graph has a cut edge.

Proof. We give a proof by induction on k, where $2k$ is the # of vtcs in the graph.

Base case: The smallest 3-regular graph is K_4, which is connected. Since every edge of K_4 appears in a cycle, by the previous thm, K_4 has no cut edges, and so the base case holds.

Induction hypothesis: Assume that a 3-regular, connected graph G with $2k$ vertices, for $k \geq 2$, has no cut edges.

We obtain a 3-regular, connected graph G' with $2(k+1)$ vertices by an "expansion" operation: subdivide two edges of G (i.e., replace the edges by paths of length 2 through new vtcs.), and add an edge joining the two new vertices.



Since G was connected, the expanded graph G' is also connected. Any u, v-path that existed in G that traversed a subdivided edge simply became longer in G', and a path to a new vertex x in G' can be built from a path in G to a neighbor of x.

Furthermore, since G was assumed to have no cut edges, we know every edge of G lies on a cycle. These cycles are still present in G' (some may be longer). The new edge xy also belongs to a cycle in G' that uses a path in G between edges that were subdivided. It follows from our previous thm that G' has no cut edges.

Thus, by induction, every 3-regular connected graph has no cut edges.

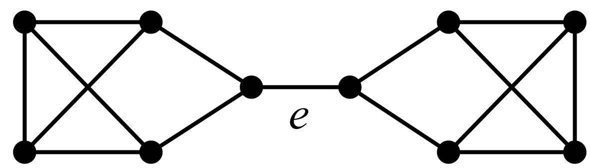
$\blacksquare$

HOWEVER, THE CLAIM IS FALSE!!

Where is the flaw in the proof by induction?

The previous proof fails because not every 3-regular connected graph can be constructed from K_4 via "expansions".

Counterexample to claim:



Induction trap: if the inductive step builds an instance with the new value of parameter from a smaller instance, then we must prove that all instances with the new value have been considered.

New topic (moving on from graph fundamentals) to... trees!

Trees

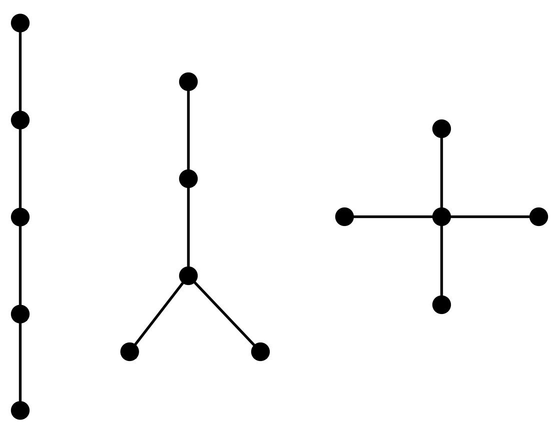
Definitions A *tree* is a connected, acyclic graph.

A *forest* is an acyclic graph, i.e., a disjoint union of trees.

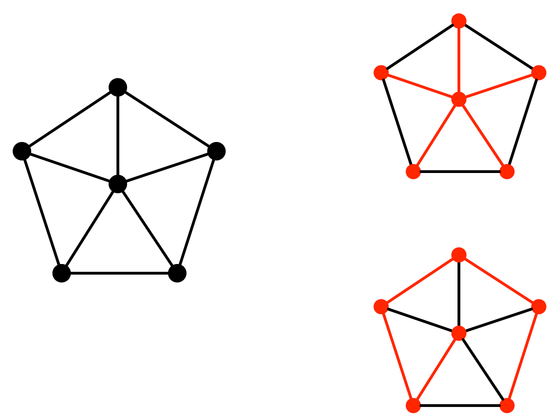
A *leaf* is a vertex of degree one.

A *spanning tree* of a graph G is a spanning subgraph of G that is a tree.

Example All trees on 5 vertices



Example Spanning trees of wheel graph on 6 vertices



Properties of trees

Have students brainstorm and generate properties of trees.

Do all trees have leaves? (Only exception is K_1.)

Proposition. *Let T be a tree with at least two vtcs. Then T has a leaf.*

Proof. Let $P = (u, v_1, v_2, \ldots, v_{k-1}, v)$ be a longest path in T. We claim that u and v are leaves. Suppose, BWOC, that u is not a leaf.

Then u has a neighbor in T other than v_1, say w.

Since T is acyclic, $w \notin V(P)$. So $(w, u, v_1, \ldots, v_{k-1}, v)$ is a longer path than P. $\Rightarrow\Leftarrow$ Thus, u and, by the same argument, v are leaves of T. ■

Proposition. *Let T be a tree with leaf v. Then $T - v$ is a tree.*

Proof sketch. Acyclic: T acyclic $\Rightarrow T - v$ acyclic
Connected: For $u, w \in V(T) - v$, if a u, w-path in T contains v, then v has degree ≥ 2. $\Rightarrow\Leftarrow$ ■

These two propositions are the foundation for the standard proof by induction for trees. This proof technique is used for the next result.

Proposition. *If T is a tree with n vertices, then T has $n-1$ edges.*

Proof. Proof is by induction on n.

Base case ($n = 1$): The only tree with one vertex is K_1, which has 0 edges.

IH: Suppose any tree with $n = k$ vtcs has $k - 1$ edges, where $k \geq 1$.

Consider a tree with $n = k + 1$ vtcs. Let v be a leaf of T, and let $T' = T - v$. Note that T' is a tree on k vtcs so, by IH, T' has $k - 1$ edges. Since $d_T(v) = 1$, T has $(k - 1) + 1 = k$ edges. ■