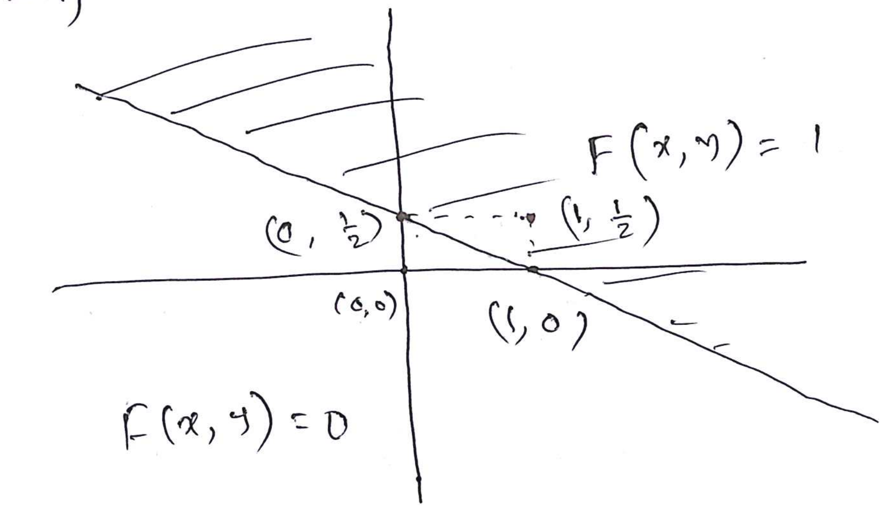
1a)



Let $a_1 = 0,\ b_1 = 1,\ a_2 = 0,\ b_2 = \frac{1}{2}$

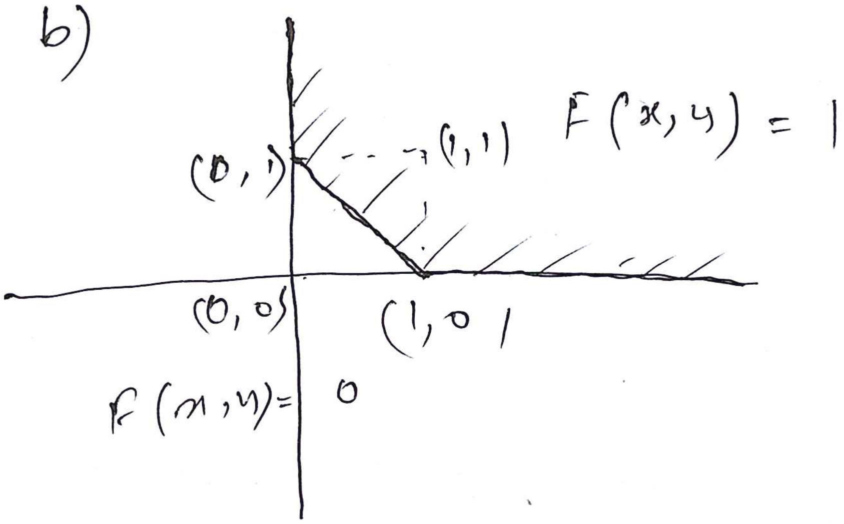
Now $F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$

$= F(1, \frac{1}{2}) - F(0, \frac{1}{2}) - F(1, 0) + F(0, 0)$

$= 1 - 1 - 1 + 0 = -1 \neq 0$.

Thus $F(x, y)$ is not a distribution function.

b)



$a_1 = 0,\ b_1 = 1,\ a_2 = 0,\ b_2 = 1$

Now $F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) - F(a_1, a_2)$

$= F(1, 1) - F(0, 1) - F(1, 0) - F(0, 0) = -1 \neq 0$.

Thus $F(x,y)$ is not a distribution function.

2a)

$$F(x,y) = P(X \le x, Y \le y) \le P(X \le x)$$

$$\le P(Y \le y)$$

$$\le \min\{P(X \le x), P(Y \le y)\}$$

$$= \min(F_1(x), F_2(y)).$$

$$= U(x,y)$$

Now $F(x,y) \geq 0$.

Also $F(x,y) = P(X \le x, Y \le y)$

$$= P(X \le x) + P(Y \le y) - P(X \le x \text{ or } Y \le y)$$

$$\geq F_1(x) + F_2(x) - 1.$$

Thus $F(x,y) \geq \max\{F_1(x) + F_2(x) - 1, 0\}$

$$= L(x,y).$$

b) $\lim_{x \to \infty} \lim_{y \to \infty} U(x,y) = \lim_{x \to \infty} \lim_{y \to \infty} \min\{F_1(x), F_2(y)\}$

$$= \lim_{x \to \infty} \min\{F_1(x), 1\}$$

$$= \lim_{x \to \infty} F_1(x) = 1.$$

$\lim_{x \to -\infty} U(x,y) = \lim_{x \to -\infty} \min\{F_1(x), F_2(y)\}$

$$= \min\{0, F_2(y)\} = 0.$$

Sim, $\lim_{y \to -\infty} U(x,y) = 0$.

$$\lim_{h \downarrow 0} L(x+h, y) = \lim_{h \downarrow 0} \min\{F_1(x+h), F_2(y)\}$$

$$= \min\{F_1(x), F_2(y)\} = L(x, y).$$

Sim. right cont. in y.

Suppose $F_1(a_1) \leq F_1(b_1) \leq F_2(a_2) \leq F_2(b_2)$ $[a_1 < b_1,\ a_2 < b_2]$

Then $U(b_1, b_2) - U(a_1, b_2) - U(b_1, a_2) - U(a_1, a_2)$

$$= \min\{F_1(b_1), F_2(b_2)\} - \min\{F_1(a_1), F_2(b_2)\}$$

$$- \min\{F_1(b_1), F_2(a_2)\} + \min\{F_1(a_1), F_2(a_2)\}$$

$$= F_1(b_1) - F_1(a_1) - F_1(b_1) + F_1(a_1)$$

$$= 0$$

Suppose $F_1(a_1) \leq F_2(a_2) \leq F_1(b_1) \leq F_2(b_2)$

Then we get, $U(b_1, b_2) - U(a_1, b_2) - U(b_1, a_2) + U(a_1, a_2)$

$$= F_1(b_1) - \cancel{F_1(a_1)} - F_2(a_2) + \cancel{F_1(a_1)}$$

$$\geq 0.$$

Similarly for other cases. (Remember $F_1(a_1) \leq F_1(b_1)$ & $F_2(a_2) \leq F_2(b_2)$ by prop. of dist. function).

Now $\lim_{x \to \infty} \lim_{y \to \infty} L(x, y) = \lim_{x \to \infty} \lim_{y \to \infty} \max\{F_1(x) + F_2(y) - 1, 0\}$

$$= \lim_{x \to \infty} \max\{F_1(x), 0\} = \lim_{x \to \infty} F_1(x) = 1.$$

$$\lim_{y \to -\infty} L(x,y) = \lim_{y \to -\infty} \max\{F_1(x) + F_2(y) - 1, 0\}$$

$$= \cancel{\lim} \max\{F_1(x) - 1, 0\} = 0.$$

Sim. $\lim_{x \to -\infty} L(x,y) = 0$.

$$\lim_{h \downarrow 0} L(x, y+h) = \lim_{h \downarrow 0} \max\{F_1(x) + F_2(y+h) - 1, 0\}$$

$$= \max\{F_1(x) + F_2(y) - 1, 0\}$$

$$= L(x,y).$$

Sim. right cont. in x.

If $F_1(b_1) + F_2(b_2) < 1$, trivial $\quad \begin{bmatrix} a_1 < b_1, \\ a_2 < b_2 \end{bmatrix}$

If $F_1(a_1) + F_2(a_2) \geq 1$, trivial.

Suppose $F_1(b_1) + F_2(b_2) \geq 1$ and

$F_1(a_1) + F_2(a_2) < 1$.

Suppose $F_1(a_1) + F_2(b_2) < 1$

$F_1(b_1) + F_2(a_2) \geq 1$.

Then $L(b_1,b_2) - L(a_1,b_2) - L(a_2,b_1) + L(a_1,a_2)$

$$= \cancel{F_1(b_1)} + F_2(b_2) - \cancel{1} - 0 - \cancel{F_1(b_1)} - F_2(a_2) + \cancel{1} + 0$$

≥ 0. Sim. consider other cases.

c) $\lim_{x\to\infty} L(x,y) = \lim_{x\to\infty} \max\{F_1(x)+F_2(y)-1,\ 0\}$

$= \max\{F_1(x),\ 0\} = F_1(x).$

$\lim_{y\to\infty} L(x,y) = \lim_{y\to\infty} \max\{F_1(x)+F_2(y)-1,\ 0\}$

$= \max\{F_2(y),\ 0\} = F_2(y).$

$\lim_{x\to\infty} U(x,y) = \lim_{x\to\infty} \min\{F_1(x),\ F_2(y)\}$

$= \min\{1,\ F_2(y)\} = F_2(y).$

Sim. $\lim_{y\to\infty} U(x,y) = F_1(x)$.

(3) Suppose $F_1(x) \le F_2(y)$

$\Rightarrow P(X \le x) \le P(X \le g^{-1}(y))$

$\Rightarrow x \le g^{-1}(y)$

Now $P(X \le x, Y \le y) = P(X \le x, X \le g^{-1}(y))$

$= P(X \le x) = F_1(x)$

Suppose $F_1(x) > F_2(y)$

$\Rightarrow P(X \le x) > P(X \le g^{-1}(y))$

$\Rightarrow x > g^{-1}(y)$. Thus

$P(X \le x, Y \le y) = P(X \le g^{-1}(y)) = P(g(X) \le y)$
$= F_2(y).$

b) If $F_1(x) + F_2(y) < 1 \Rightarrow P(X \leq x) + P(g(X) \leq y) < 1$

$\Rightarrow P(X \leq x) + P(X \geq g^{-1}(y)) < 1 \Rightarrow x < g^{-1}(y)$

Thus $F_1(x) + F_2(y) \geq 1 \Rightarrow x \geq g^{-1}(y)$.

Now $P(X \leq x,\ g(X) \leq y) = P(X \leq x,\ X \geq g^{-1}(y))$

$= P(g^{-1}(y) \leq X \leq x) = 0$ if $F_1(x) + F_2(y) < 1$

$= F_1(x) - (1 - P(X \geq g^{-1}(y)))$

$= F_1(x) + F_2(y) - 1$ if $F_1(x) + F_2(y) \geq 1$

4a) $P(X + Y < 8) = P(X=4, Y=1) + P(X=4, Y=2)$

$+ P(X=4, Y=3) + P(X=5, Y=1) + P(X=5, Y=2)$

$+ P(X=6, Y=1)$

$= 0.08 + 0.11 + 0.09 + 0.04 + 0.12 + 0.09$

$= 0.53$

$P(X + Y > 7) = 1 - P(X + Y \leq 8)$

$= 1 - 0.53 = 0.47$

$P(XY \leq 14) = 1 - P(X \geq 15)$

$= 1 - P(X=4, Y=4) - P(X=5, Y=3)$

$- P(X=5, Y=4) - P(X=6, Y=3) - P(X=6, Y=4)$

$= 1 - 0.03 - 0.21 - 0.05 - 0.08 - 0.04$

$= 0.59$.

~~b) [illegible]~~

b) Do the calculations.

5. $$P(X=x) = \frac{(1-\theta_1-\theta_2)^K \, \theta_1^x \, (x+K-1)!}{x!\,(K-1)!\,(1-\theta_2)^{x+K}} \sum_{y=0}^{\infty} \frac{(x+K-1+y)!}{y!\,(x+K-1)!}\, \theta_2^y (1-\theta_2)^{x+K}$$

$$= \frac{(x+K-1)!}{x!\,(K-1)!} \left(\frac{\theta_1}{1-\theta_2}\right)^x \left(1-\frac{\theta_1}{1-\theta_2}\right)^K$$

for $x = 0, 1, 2, \ldots$

Thus $X \sim NB\left(K,\ 1-\frac{\theta_1}{1-\theta_2}\right)$

$$P(Y=y) = \frac{(1-\theta_1-\theta_2)^K \, \theta_2^y \, (y+K-1)!}{y!\,(K-1)!\,(1-\theta_1)^{y+K}} \sum_{x=0}^{\infty} \frac{(y+K-1+x)!}{x!\,(y+K-1)!}\, \theta_2^x (1-\theta_1)^{y+K}$$

$$= \binom{y+K-1}{K-1} \left(\frac{\theta_2}{1-\theta_1}\right)^y \left(1-\frac{\theta_2}{1-\theta_1}\right)^K$$

for $y = 0, 1, 2, \ldots$

Thus $Y \sim NB\left(K,\ 1-\frac{\theta_2}{1-\theta_1}\right)$.

6. The possible configurations are

3, 0, 0 $\quad N=1$

2, 1, 0 $\quad N=2$

1, 1, 1 $\quad N=3$

0 3, 0 $\quad N=1$

0, 2, 1 $\quad N=2$

1, 2, 0 $\quad N=2$

0, 0, 3 $\quad N=1$

0, 1, 2 $\quad N=2$

1, 0, 2 $\quad N=2$

2, 0, 1 $\quad N=2$

a) $P(N=1, X_1=0) = \frac{1}{5}$

$P(N=1, X_1=3) = \frac{1}{10}$, $\quad P(N=2, X_1=2) = \frac{1}{5}$

$P(N=2, X_1=0) = \frac{1}{5}$ $\quad P(N=3, X_1=1) = \frac{1}{10}$

$P(N=2, X_1=1) = \frac{1}{5}$.

b) $P(X_1=0, X_2=0) = \frac{1}{10}$ $\quad P(X_1=1, X_2=2) = \frac{1}{10}$

$P(X_1=0, X_2=1) = \frac{1}{10}$ $\quad P(X_1=2, X_2=0) = \frac{1}{10}$

$P(X_1=0, X_2=2) = \frac{1}{10}$ $\quad P(X_1=2, X_2=1) = \frac{1}{10}$

$P(X_1=0, X_2=3) = \frac{1}{10}$ $\quad P(X_1=3, X_2=0) = \frac{1}{10}$

$P(X_1=1, X_2=0) = \frac{1}{10}$

$P(X_1=1, X_2=1) = \frac{1}{10}$

c) $P(N=1) = \frac{3}{10}$ $\qquad$ $P(X_2=0) = \frac{4}{10}$

$P(N=2) = \frac{6}{10}$ $\qquad$ $P(X_2=1) = \frac{3}{10}$

$P(N=3) = \frac{1}{10}$ $\qquad$ $P(X_2=2) = \frac{2}{10}$

$P(X_2=3) = \frac{1}{10}$.

d)

$$P(X_1=0) = \sum_{k=0}^{3} P(X_1=0, X_2=k) = \frac{4}{10}$$

$$P(X_1=1) = \sum_{k=0}^{2} P(X_1=0, X_2=k) = \frac{3}{10}$$

$$P(X_1=2) = \sum_{k=0}^{1} P(X_1=0, X_2=k) = \frac{2}{10}$$

$$P(X_1=3) = P(X_1=3, X_2=0) = \frac{1}{10}.$$

Q7

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu .$$

$$Var(\bar{X}) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n} Var(X_i) + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} Cov(X_i, X_j)\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2$$

$$= \sigma^2/n .$$

$$Cov(\bar{X}, X_i - \bar{X}) = Cov(\bar{X}, X_i) - Var(\bar{X})$$

$$= \frac{1}{n} Cov(X_i, X_i) - Var(\bar{X})$$

$$= \frac{\sigma^2}{n} - \frac{\sigma^2}{n}$$

$$= 0. \qquad \square$$

Q8

$$P(X=x) = P(X=x \mid X_1=1)P(X_1=1) + P(X=x \mid X_1=0)P(X_1=0).$$

Now for $x = 1, 2, \dots, n$,

$$P(X=x \mid X_1=1) = \binom{n-1}{x-1} p^{x-1}(1-p)^{n-x}$$

For $x = 1, 2, \dots, n-1$, $P(X=x \mid X_1=0) = 0$

For $x = n$, $P(X=x \mid X_1=0) = 1.$

Hence for $x = 1, 2, \dots, n-1$, $P(X=x) = \binom{n-1}{x-1} p^{x}(1-p)^{n-x}$

For $x = n$, $P(X=n) = p^n + 1-p.$

As $\sum_{x=1}^{n} P(X=x) = 1$, the PMF of X is

$$f_X(x) = \begin{cases} \binom{n-1}{x-1} p^x (1-p)^{n-x} & \text{if } x = 1, 2, \ldots, n-1 \\ p^n + 1 - p & \text{if } x = n \\ 0 & \text{o.w.} \end{cases}$$

$\boxed{Q9}$ The marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

$$= \begin{cases} \dfrac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1-1} (1-\cancel{\theta}x)^{\theta_3-1} \displaystyle\int_0^{1-x} y^{\theta_2-1} \left(1-\frac{y}{1-x}\right)^{\theta_3-1} dy & \text{if } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1-1} (1-x)^{\theta_3-1+\theta_2-1+1} \int_0^1 y^{\theta_2-1} (1-y)^{\theta_3-1}\,dy$$

if $0<x<1$.

$$= \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1-1} (1-x)^{\theta_2+\theta_3-1} \frac{\Gamma(\theta_2)\Gamma(\theta_3)}{\Gamma(\theta_2+\theta_3)}$$

if $0<x<1$

$$= \begin{cases} \dfrac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2+\theta_3)} x^{\theta_1-1} (1-x)^{\theta_2+\theta_3-1} & \text{if } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

∎

Similarly, the marginal PDF of ~~X~~ Y is

$$f_Y(y) = \begin{cases} \dfrac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_2)\,\Gamma(\theta_1+\theta_3)}\, y^{\theta_2-1}(1-y)^{\theta_1+\theta_3-1} & \text{if } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$\square$

$\boxed{Q10}$ (a) The marginal PDF of X is

$$f_X(x) = \begin{cases} 4x\int_0^1 y\,dy & \text{if } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 2x & \text{if } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

The marginal PDF of Y is

$$f_Y(y) = \begin{cases} 2y & \text{if } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$\square$

$\boxed{Q10(b)}$ As $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ for all $(x,y)\in\mathbb{R}^2$, X & Y are independent. $\square$

$\boxed{Q10(c)}$ $P(0<X<0.5,\ 0.25<Y<1)$

$= P(0<X<0.5)\,P(0.25<Y<1)$

$= \left(\frac{1}{2}\right)^2 \times \left\{1-\left(\frac{1}{4}\right)^2\right\} = \frac{15}{64}.$ $\square$

$$P(X+Y<1) = \iint_{x+y<1} f_{X,Y}(x,y)\,dx\,dy$$

$$= \underset{x+y<1}{\int_0^1\int_0^1} 4xy\,dx\,dy$$

$$= 4\int_0^1 y \int_0^{1-y} x\,dx\,dy$$

$$= \frac{4}{2}\int_0^1 y(1-y)^2\,dy$$

$$= \frac{4}{2}\cdot B(2,3)$$

$$= \frac{1}{2}\times\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{2}{2\times 24} = \frac{1}{24} \qquad \square.$$

$\boxed{Q\ 11}$

$\boxed{Q11(a)}$ The marginal PDF of ~~X_1 is~~ (X_1, X_2) is

$$\cancel{f_{X_1}(x_1) = \frac{1}{(2\pi)^{3/2}}\left[e^{-\frac{1}{2}x_1^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x_2^2+x_3^2)}\,dx_2\,dx_3 + x_1 e^{-\frac{1}{2}x_1^2}\iint x_2 x_3 e\right.}$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{(2\pi)^{3/2}}\left[e^{-\frac{1}{2}(x_1^2+x_2^2)}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x_3^2}\,dx_3 + x_1x_2\,e^{-\frac{1}{2}(x_1^2+x_2^2)}\int_{-\infty}^{+\infty} x_3\,e^{-\frac{1}{2}x_3^2}\,dx_3\right]$$

$$= \frac{1}{(2\pi)}\,e^{-\frac{1}{2}(x_1^2+x_2^2)} \qquad (x_1,x_2)\in\mathbb{R}^2.$$

The marginal PDF of X_1 is

$$f_{X_1}(x_1) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}x_1^2} \qquad x_1 \in \mathbb{R}$$

The marginal PDF of X_2 is

$$f_{X_2}(x_2) = \frac{1}{(2\pi)^{1/2}} e^{-x_2^2/2} \qquad \text{if } x_2 \in \mathbb{R}$$

The marginal PDF of X_3 is

$$f_{X_3}(x_3) = \frac{1}{\sqrt{2\pi}} e^{-x_3^2/2} \qquad \text{if } x_3 \in \mathbb{R}.$$

Hence for any $(x_1, x_2, x_3) \neq (0,0,0)$,

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) \neq f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3).$$

$\Rightarrow X_1, X_2$ & X_3 are not independent. $\square$

$\boxed{\text{Q11 (b)}}$ As $f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2$,

X_1 & X_2 are independent.

Similarly X_1 & X_3 are independent.

Similarly X_2 & X_3 are independent. $\square$

$\boxed{\text{Q12}}$ $$\text{Corr}\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) = \frac{\text{Cov}\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right)}{\sqrt{\text{Var}\left(\frac{X}{3} + \frac{2Y}{3}\right)\text{Var}\left(\frac{2X}{3} + \frac{Y}{3}\right)}}$$

Now $$\text{Cov}\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) = \frac{2}{9}\text{Cov}(X,X) + \frac{1}{9}\text{Cov}(X,Y) + \frac{4}{9}\text{Cov}(X,Y) + \frac{2}{9}\text{Var}(Y) \quad \text{—— } (*)$$

$\text{Var}(X) = 2, \quad \text{Var}(Y) = 2$

$\text{Corr}(X,Y) = \frac{1}{3} \Rightarrow \text{Cov}(X,Y) = \frac{1}{3} \times 2 = \frac{2}{3}.$

From ⊛, $\mathrm{Cov}\left(\frac{X}{3}+\frac{2Y}{3}, \frac{2X}{3}+\frac{Y}{3}\right)$

$$= \frac{4}{9} + \frac{2}{27} + \frac{8}{27} + \frac{4}{9}$$

$$= \frac{34}{27}.$$

$$\mathrm{Var}\left(\frac{X}{3}+\frac{2Y}{3}\right) = \frac{1}{9}\mathrm{Var}(X) + \frac{4}{9}\mathrm{Cov}(X,Y) + \frac{4}{9}\mathrm{Var}(Y)$$

$$= \frac{2}{9} + \frac{8}{27} + \frac{8}{9}$$

$$= \frac{38}{27}.$$

$$\mathrm{Var}\left(\frac{2X}{3}+\frac{Y}{3}\right) = \frac{4}{9}\mathrm{Var}(X) + \frac{4}{9}\mathrm{Cov}(X,Y) + \frac{1}{9}\mathrm{Var}(Y)$$

$$= \frac{8}{9} + \frac{8}{27} + \frac{2}{9}$$

$$= \frac{38}{27}$$

□

Required correlation is $\frac{17}{19}$.

$\boxed{Q13}$ The JPDF of (X,Y) is

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = 2\int_0^1\int_0^y xy\,dx\,dy = \int_0^1 y^3\,dy = \frac{1}{4}$$

$$E(X) = 2\int_0^1\int_0^y x\,dx\,dy = \frac{1}{3}, \quad E(Y) = 2\int_0^1\int_0^y y\,dx\,dy = \frac{2}{3}$$

□.

$$\mathrm{Cov}(X,Y) = 1/36$$