Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 11 (August 23, 2019)

Jointly Distributed Random Variables

Def: A function $\boldsymbol{X} : \mathcal{S} \to \mathbb{R}^n$ is called a random vector.

Def: For any random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, the joint cumulative distribution function (JCDF) is defined by

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) = P\left(X_1 \leq x_1, \ldots, X_n \leq x_n\right),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Remark: $F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y)$.

Remark: $F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y)$.

Properties of JCDF

1. $\lim_{x \to \infty} \lim_{y \to \infty} F_{X,Y}(x, y) = 1$.
2. $\lim_{x \to -\infty} F_{X,Y}(x, y) = 0$ for all $y \in \mathbb{R}$.
3. $\lim_{y \to -\infty} F_{X,Y}(x, y) = 0$ for all $x \in \mathbb{R}$.
4. $F_{X,Y}(\cdot, \cdot)$ is right continuous in each argument keeping other fixed.
5. For $-\infty < a_1 < b_1 < \infty$ and $-\infty < a_2 < b_2 < \infty$,

$$F_{X,Y}(b_1, b_2) - F_{X,Y}(b_1, a_2) - F_{X,Y}(a_1, b_2) + F_{X,Y}(a_1, a_2) \geq 0.$$

Theorem: Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying above properties. Then G is a JCDF of some 2-dimensional random vector.

Discrete Random Vector

Def: A random vector (X, Y) is said to have a discrete distribution if there exists an atmost countable set $S_{X,Y} \in \mathbb{R}^2$ such that $P((X, Y) = (x, y)) > 0$ for all $(x, y) \in S_{X,Y}$ and $P((X, Y) \in S_{X,Y}) = 1$. $S_{X,Y}$ is called the support of (X, Y).

Def: Define a function $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ by

$$f_{X,Y}(x, y) = \begin{cases} P(X = x, Y = y) & \text{if } (x, y) \in S_{X,Y} \\ 0 & \text{otherwise.} \end{cases}$$

The function $f_{X,Y}$ is called joint probability mass function (JPMF) of the DRV (X, Y).

Properties of JPMF

1. $f_{X,\,Y}(x,\,y) \geq 0$ for $(x,\,y) \in \mathbb{R}^2$.
2. $\displaystyle\sum_{(x,y)\in S_{X,\,Y}} f_{X,\,Y}(x,\,y) = 1$.
3. $f_X(x) = \displaystyle\sum_{(x,y)\in S_{X,\,Y}} f_{X,\,Y}(x,\,y)$ for fixed $x \in \mathbb{R}$.

   f_X is called marginal PMF of X in this context.

Theorem: If a function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfy 1 and 2 above for the atmost countable set $D = \{(x,\,y) \in \mathbb{R}^2 : g(x,\,y) > 0\}$ in place of $S_{X,\,Y}$, then g is JPMF of some 2-dimensional DRV.

Expectation of Function of DRV

Def: Let $(X,\ Y)$ be a DRV with JPMF $f_{X,\,Y}$ and support $S_{X,\,Y}$. Let $h : \mathbb{R}^2 \to \mathbb{R}$. Then the expectation of $h(X,\ Y)$ is defined by

$$E\left(h(X,\ Y)\right) = \sum_{(x,\,y) \in S_{X,\,Y}} h(x,\ y) f_{X,\,Y}(x,\ y),$$

provided $\displaystyle\sum_{(x,\,y) \in S_{X,\,Y}} |h(x,\ y)| f_{X,\,Y}(x,\ y) < \infty.$

Continuous Random Vector

Def: A random vector (X, Y) is said to have a continuous distribution if there exists a non-negative integrable function $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t, s) ds dt$$

for all $(x, y) \in \mathbb{R}^2$.

Def: The function $f_{X,Y}$ is called the joint probability density function (JPDF) of (X, Y).

Def: The set $S_{X,Y} = \{(x, y) \in \mathbb{R}^2 : f_{X,Y}(x, y) > 0\}$ is called the support of (X, Y).

Properties of JPDF

1. $f_{X,Y}(x, y) \geq 0$ for $(x, y) \in \mathbb{R}^2$.
2. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1$.
3. $f_X(x) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$ for fixed $x \in \mathbb{R}$.
   f_X is called marginal PDF of X in this context.

Theorem: If a function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfy 1 and 2 above, then g is JPDF of some 2-dimensional CRV.

Expectation of Function of CRV

Def: Let (X, Y) be a CRV with JPDF $f_{X,Y}$. Let $h : \mathbb{R}^2 \to \mathbb{R}$. Then the expectation of $h(X, Y)$ is defined by

$$E\left(h(X, Y)\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) f_{X,Y}(x, y)\,dx\,dy,$$

provided $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(x, y)| f_{X,Y}(x, y)\,dx\,dy < \infty$.

Examples

Example 1: Let (X, Y) be a DRV with JPMF

$$f_{X,Y}(x, y) = \begin{cases} cy & \text{if } x = 1, 2, \ldots, n;\ y = 1, 2, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Example 2: Let (X, Y) be a DRV with JPMF

$$f_{X,Y}(x, y) = \begin{cases} cy & \text{if } x = 1, 2, \ldots, n;\ y = 1, 2, \ldots, n;\ x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Example 3: Let (X, Y) be a CRV with JPDF

$$f_{X,Y}(x, y) = \begin{cases} ce^{-(2x+3y)} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$