Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 12 (August 26, 2019)

Examples

Example 1: $E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$ for real constants a_i.

Example 2: At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the expected number of men who selects their own hat.

Some Remarks

Remark: (X, Y) is discrete random vector iff X and Y are discrete random variables.

Remark: If (X, Y) is continuous random vector, then X and Y are continuous random variables.

Remark: If (X, Y) is continuous random vector, then

$$P((X, Y) \in A)) = \int \int_{(x,y) \in A} f(x, y) dx dy,$$

for all $A \subseteq \mathbb{R}^2$ such that the integration is possible.

Some Remarks

Remark: (X, Y) may not be a continuous random vector even if X and Y are continuous random variables.

Remark: In general, if there exists a set A in $\mathbb{R}^2$ whose area is zero and $P((X, Y) \in A) > 0$, then (X, Y) does not have a JPDF.

Remark: If the joint distribution is known, then the marginal distributions can be recovered. However, the converse is not true.

Example 3: Let f and g be two PDFs and F and G be the corresponding CDFs. Define, for $-1 < \alpha < 1$,

$$h(x, y) = f(x)g(y)\left\{1 + \alpha(1 - 2F(x))(1 - 2G(y))\right\}.$$

Then h is a JPDF whose marginals are f and g.

Independent Random Variables

Def: The random variables $X_1, X_2, \ldots, X_n$ are said to be independent if

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i),$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Remark: X and Y are independent iff the events $E_x = \{X \leq x\}$ and $F_y = \{Y \leq y\}$ are independent for all $(x, y) \in \mathbb{R}^2$.

Remark: For DRV/CRV (X, Y), the condition of independence is equivalent to

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \text{ for all } (x, y) \in \mathbb{R}^2.$$

Independent Random Variables

Theorem: If X and Y are independent, then

$$E\left(g(X)h(Y)\right) = E\left(g(X)\right)E\left(h(Y)\right),$$

provided all the expectations exist.

Def: The covariance of two random variables X and Y is defined by

$$Cov(X,\ Y) = E\left[(X - E(X))(Y - E(Y))\right] = E(XY) - E(X)E(Y).$$

Def: The correlation coefficient of X and Y is defined by

$$\rho(X,\ Y) = \frac{Cov(X,\ Y)}{\sqrt{Var(X)Var(Y)}}.$$

Remark: If X and Y are independent, then $Cov(X, Y) = 0$. The converse is not true in general.

Remark: $|\rho(X, Y)| \leq 1$.

Remark: $Cov(X, X) = Var(X)$.

Remark: $Cov(X, Y) = Cov(Y, X)$.

Remark: $Cov(aX, Y) = aCov(X, Y)$.

Remark: $Cov(X + Z, Y) = Cov(X, Y) + Cov(Z, Y)$.

Remark: $Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$.

Remark: $Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, Y_j)$.

Remark: If X_i's are independent, then $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$.