

Indian Institute of Technology Guwahati
Probability Theory and Stochastic Processes (MA225)
Problem Set 05

1. Let (X, Y) be a continuous random vector with JPDP $f(\cdot, \cdot)$. Show that X and Y are independent if and only if $f(x, y) = g(x)h(y)$ for all $(x, y) \in \mathbb{R}^2$.
2. Let (X, Y) be uniform over the interior of the triangle with vertices $(0, 0)$, $(2, 0)$ and $(1, 2)$. Find $P(X \leq 1, Y \leq 1)$.
3. If X_1 and X_2 are independent random variables each having PDF $2xe^{-x^2}$ ($0 < x < \infty$), then find the PDF of the random variable $\sqrt{X_1^2 + X_2^2}$.
4. Two numbers are independently chosen at random between 0 and 1. What is the probability that their product is less than a constant k ($0 < k < 1$)?
5. A vertical board is ruled with horizontal parallel lines at constant distance b apart. A needle of length a ($a < b$) is thrown at random on the board. Find the probability that it will intersect one of the lines.
6. Let X_1, X_2, X_3 have the joint PDF

$$f(x_1, x_2, x_3) = \begin{cases} 48x_1x_2x_3 & \text{if } 0 < x_1 < x_2 < x_3 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal distributions of $Y_1 = \frac{X_1}{X_2}$, $Y_2 = \frac{X_2}{X_3}$, and $Y_3 = X_3$.

7. Let X_1, X_2, X_3 be i.i.d. $Exp(1)$ random variables. Find the joint PDF of $Y_1 = \frac{X_1}{X_1+X_2+X_3}$, $Y_2 = \frac{X_2}{X_1+X_2+X_3}$, and $Y_3 = X_1 + X_2 + X_3$. Also find the marginal PDF of Y_1, Y_2 , and Y_3 .
8. Let X_1, X_2, X_3 be i.i.d. $Exp(1)$ random variables. Find the joint PDF of $Y_1 = \frac{X_1}{X_1+X_2+X_3}$, $Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$, and $Y_3 = X_1 + X_2 + X_3$. Also find the marginal PDF of Y_1, Y_2 , and Y_3 .
9. Let X and Y be two independent random variables having $Gamma(\alpha_1, \beta)$ and $Gamma(\alpha_2, \beta)$ distributions, respectively. Find the PDF of $\frac{X}{X+Y}$.
10. Let X be a random variable of continuous type. The integral part, Y , of X has a $P(\lambda)$ distribution and the fractional part, Z , has a $U(0, 1)$ distribution. Find the CDF of X , assuming that Y and Z are independent. Using the CDF find the PDF of X .
11. Let X_1, X_2, \dots, X_n be i.i.d. $U(0, 1)$ random variables. Define $X_{(n)} = \max\{X_1, \dots, X_n\}$ and $X_{(1)} = \min\{X_1, \dots, X_n\}$. Find the joint and marginal distributions of $X_{(1)}$ and $X_{(n)}$.
12. Let X_1 and X_2 be i.i.d. $P(\lambda)$ random variables. Find the PMF of $X_{(2)} = \max\{X_1, X_2\}$.