Indian Institute of Technology Guwahati
Probability Theory and Random Processes (MA225)
Problem Set 10

1. Show that if P is the transition matrix of an irreducible chain with finitely many states, then $Q = \frac{1}{2}(I + P)$ is the transition matrix of an irreducible and aperiodic chain. Also show that P and Q have the same stationary distributions.

2. Prove that for an irreducible Markov chain with N states it is possible to go from any state to any other state in at most $N - 1$ steps.

3. Consider a Markov chain on the vertices of a triangle: the chain moves from one vertex to another with probability 1/2. Find the probability that, in n steps, the chain returns to the vertex it started from.

4. I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

   (a) If the probability of rain is p, what is the probability that I get wet? Ans: $\frac{pq}{q+4}$, where $q = 1 - p$.

   (b) Current estimates show that $p = 0.6$ in Guwahati. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.01? Ans: 24

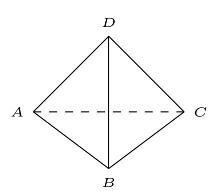
5. Consider a Markov chain with states $S = \{0, \ldots, N\}$ and transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$, for $1 \le i \le N - 1$, $p_{0,1} = p$, $p_{0,0} = q$, $p_{N,N} = p$, and $p_{N,N-1} = q$ where $p + q = 1$, $0 < p < 1$.

   (a) Is the Markov chain irreducible? Ans: Yes

   (b) Is it aperiodic? Ans: Yes

   (c) Find the stationary distribution. Ans: For $p = 0.5$, $\pi_i = \frac{1}{N+1}$ for all $i \in S$. For $p \neq 0.5$, $\pi_i = \frac{\left(\frac{p}{q}\right)-1}{\left(\frac{p}{q}\right)^{N+1}-1}\left(\frac{p}{q}\right)^i$.

6. A particle performs random walk on the vertices of a tetrahedron. At each step it remains where it is with probability 0.25 or moves to one of its neighboring vertices each with probability 0.25. Suppose that the particle starts at A (see the following figure). Find the mean number of steps until its first return to A. Ans: 4.



7. A particle moves among $n+1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise or counter clockwise direction with probability 1/2 each. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the probability that all states have been visited by time T.

8. Consider the Markov chain with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

   (a) Show that this is irreducible and aperiodic.

   (b) Find the matrix which is the limit of P^n as $n \to \infty$.

9. Show that a Markov chain with transition probability matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

has more than one stationary distributions. Find the matrix that P^n converges to, as $n \to \infty$, and verify that it is not a matrix all of whose rows are the same. You should work out this exercise by direct methods, without appealing to the general limiting theory of Markov chains.

10. The President of the United States tells person A his or her intention to contest or not to contest in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability α that a person will change the answer from yes to no when transmitting it to the next person and a probability β that he or she will change it from no to yes. We choose as states the message, either yes or no. The initial state represents the Presidents choice. Suppose $\alpha = 0.5$, $\beta = 0.75$.

    (a) Assume that the President says that he or she will contest. Find the expected length of time before the first time the answer is passed on incorrectly.

    (b) Find $\lim_{n\to\infty} P^n$, where P is the transition probability matrix.

11. A DNA nucleotide has any four values. A standard model for a mutational change of the nucleotide at a specific location is a MC model that supposes that in going from period to period the nucleotide does not change with probabilities $1 - 3\alpha$ and if it does change then it is equally likely to change to any of the other three values, for some $0 < \alpha < 1/3$.

    (a) Show that $p_{11}^{(n)} = 1/4 + \frac{3}{4}(1 - 4\alpha)^n$.

    (b) What is the long run proportion of time that the chain is in each state.

12. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are not shoes at the door from which he departed). On his return, he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of K pairs of running shoes, what proportion of time does he run barefooted?

13. Three out of every four trucks on the road are followed by a car, while only one out of five car is followed by a truck. What fraction of vehicles on the road are trucks?

14. Each of two switches is either on or off during a day. On day n, each switch will be independently on with probability (1+No. of on switches during day $n - 1$)/4. What fraction of days are both switches on? What fraction are both off?

15. Assume that an experiment has m equally probable outcomes. Show that the expected number of independent trials before the first occurrence of k consecutive occurrences of one of these outcomes is

$$\frac{m^k - 1}{m - 1}.$$

16. Consider a MC with state space $\{0, 1, 2, \ldots\}$ and transition probabilities $p_{01} = 1$, $p_{2i,2i-1} = p$, $p_{2i,2i+1} = q$ for $i = 1, 2, \ldots$, $p_{2i+1,2i+2} = p$, $p_{2i+1,2i} = q$ for $i = 0, 1, \ldots$, where $p + q = 1$. Is it positive recurrent?

17. Toss a fair die repeatedly. Let S_n denote the sum of the first n outcomes. Define

$$D_n = \#\{k : k \leq n,\ S_k \text{ is divisible by } 7\}.$$

Show that $\frac{D_n}{n}$ has a limit with probability one and find the limit. Hint: The desired limit is a stationary distribution for an appropriate Markov chain with 7 states. Ans: 1/7.