Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 26 (October 25, 2019)

Irreducibility

Def: A MC is said to be irreducible if all states communicate with each other, *i.e.*, there is a single communicating class.

Example 1:

$$P_1 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hitting Time

Def: For any $A \subset S$, the hitting time T_A is defined by

$$T_A = \inf \{n \geq 1 : X_n \in A\},$$

with the convention that $\inf \phi = \infty$.

Remark: T_A is the first time after 0, when the chain enters A.
Remark: T_A is also called first passage time.
Remark: $T_{\{i\}}$ will be denoted by T_i, $i \in S$.

Classification of States

Def: A state i is called recurrent if $P(T_i < \infty | X_0 = i) = 1$.
Def: A state i is called transient if $P(T_i < \infty | X_0 = i) < 1$.

Remark: Thus i is recurrent if and only if

$$f_{ii} = P\left(X_n = i \text{ for some } n \geq 1 | X_0 = i\right) = 1.$$

Def: A recurrent state i is called null recurrent if $E(T_i | X_0 = i) = \infty$ and positive recurrent if $E(T_i | X_0 = i) < \infty$.

Example

Example 2: (Frog in the Well) $S = \{1, 2, \ldots\}$. For $i \geq 1$ and $0 < \alpha_i < 1$,

$$p_{i,i+1} = \alpha_i, \; p_{i,1} = 1 - \alpha_i.$$

Then $P(T_1 > k | X_0 = 1) = \alpha_1 \alpha_2 \ldots \alpha_k$.

Fact: If $0 \leq q_n < 1$, then $\prod_{n=1}^{\infty}(1 - q_n) \to l \neq 0 \iff \sum_{n=1}^{\infty} q_n$ converges.

As $P(T_1 = \infty | X_0 = 1) = \lim_{k \to \infty} P(T_1 > k | X_0 = 1)$, state 1 is recurrent iff $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$.

The state 1 is positive recurrent iff $\sum_{k=1}^{\infty} \alpha_1 \alpha_2 \ldots \alpha_k < \infty$.

1. $\alpha_i = 1 - \frac{1}{2i^2}$: 1 is transient.
2. $\alpha_i = \alpha$: 1 is positive recurrent.
3. $\alpha_i = 1 - \frac{1}{2i}$: 1 is null recurrent.