1). Sin

S = sample space

A, B, C events

Only A occurs = $A \cap B^c \cap C^c$

Atleast two of A, B, C occurs = $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

The event that both A, B but not C occurs = $A \cap B \cap C^c$

The event of atmost one of A, B, C occurs = $(A^c \cup B^c) \cap (A^c \cup C^c) \cap (B^c \cup C^c)$

(i) only A occurs = $A \cap B^c \cap C^c$

(ii) Atleast two A, B, C occurs = $(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C) \cup (A \cap B \cap C)$

=

(iii) Both A, B but not C = $A \cap B \cap C^c$

(iv) At most one of A, B, C occurs = $A\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{A}\bar{B}\bar{C}$

2). (a.

$S = \{0, 1, 2, \ldots\}$

$$P(\phi) = \sum_{x \in \phi} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P(S) = \sum_{x \in S} e^{-\lambda} \frac{\lambda^x}{x!} = 1$$

$P(A) \geq 0$

let $A_1, A_2, \ldots$, $A_i \cap A_j = \phi$; $i \neq j$

Now, to S.T.

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

i.e to show,

$$\sum_{x \in \bigcup_{i=1}^{\infty} A_i} \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{i=1}^{\infty} \sum_{x \in A_i} e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{--- (1)}$$

but $x \in \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow x \in A_i$ for exactly one value of i.

$\because A_i \cap A_j = \phi$, $i \neq j$

so, (1) is true.

14). b). For $a<b<c$ in GP, we have $b = ar$, $c = ar^2$ where $r > 1$

$1 \le a < ar < ar^2 \le 50 \Rightarrow 1 \le r^2 \le 50 \Rightarrow 1 < r \le \sqrt{50}$

Case - I

r is an int.

$r \in \{2, 3, \dots, 7\}$ for each r, $1 \le a \le 50/r^2$

$ar > 1$, $r > 1$
$ar^2 \le 50$, $a > 1$
$r^2 \le \frac{50}{a} \le 50$

r	Possible values of a	#
2	1, 2, 3, . . . , 12	12
3	1, 2, 3, 4, 5	5
4	1, 2, 3	3
5	1, 2	2
6	1	1
7	1	1
	Total	24

Case-II

r is not an int.

but $ar \in \mathbb{Q}$ & $a \in \mathbb{Q}$ so, $r \in \mathbb{Q}$

let $r = \frac{m}{n}$, $\gcd(m,n) = 1$, $m > n > 1$

$1 \le a < a \cdot \frac{m}{n} < a \frac{m^2}{n^2} \le 50$

$\boxed{1 < r \le \sqrt{50}}$

$\because a \frac{m^2}{n^2} \in \mathbb{Z} \Rightarrow a = k \cdot n^2$

for fixed $r = \frac{m}{n}$, we have, $1 \le a \le 50 \frac{n^2}{m^2}$

& $\boxed{a = k \cdot n^2}$ $\boxed{1 \le a \le 50}$

r	Range of a	Possible values of a	#
3/2	[1, 22.22]	4, 8, 12, 16, 20	5
5/2	[1, 8]	4, 8	2
7/2	[1, 4.08]	4	1
4/3	[1, 28.125]	9, 18, 27	3
5/3	[1, 18]	9, 18	2
7/3	[1, 9.18]	9	1
5/4	[1, 32]	16, 32	2
7/4	[1, 16.33]	16	1
6/5	[1, 34.72]	25	1
7/5	[1, 25.51]	25	1
7/6	[1, 36.73]	36	1
		Total	20

$r > 1$

if $r = 3/2$, $a = 4k$,
if $r = \frac{13}{12}$, $a = 12^2 k$, not possible.

$\therefore$ req. prob $= \frac{24+20}{{}^{50}C_3}$

10). (i) $P((A^c \cup B^c) \cap C^c) = P(A^c \cup B^c) + P(C^c) - P(A^c \cup B^c \cup C^c)$

$= 1 - P(A \cap B) + 1 - P(C) - [1 - P(A \cap B \cap C)]$

(ii). $P((A^c \cap B^c) \cup C^c) = P(A^c \cap B^c) + P(C^c) - P(A^c \cap B^c \cap C^c)$

$= P(A \cup B)^c + 1 - P(C) - P(A \cup B \cup C)^c$

$= 1 - [P(A) + P(B) - P(A \cap B)] + 1 - P(C) - 0.1$

(d) $0 \leq P(D \cap B \cap C) \leq P(D \cap B) = 0$

$0 \leq P(A \cap C \cap D) \leq P(C \cap D) = 0$

(e). $P(A \cup B \cup D) = P(A) + P(B) + P(D) - P(A \cap B) - P(A \cap D) - P(B \cap D) + P(A \cap B \cap D)$

(f). $P((A \cap B) \cup (C \cap D)) = P(A \cap B) + P(C \cap D) - P(A \cap B \cap C \cap D)$

11). $P(A)P(B^c) - P(A \cap B^c)$

$= P(A)(1 - P(B)) - P(A \cap B^c) = P(A) - P(A \cap B^c) - P(A)P(B)$

$= P(A \cap B) - P(A)P(B)$ $[P(A) = P(A \cap B) + P(A B^c)]$

Again, $P(A^c)P(B) - P(A^c \cap B)$

$= (1 - P(A))P(B) - P(A^c \cap B)$

$= P(B) - P(A)P(B) - P(A^c \cap B) = P(B) - P(A^c \cap B) - P(A)P(B)$

$= P(A \cap B) - P(A)P(B)$ $[P(B) = P(B \cap A) + P(B \cap A^c)]$

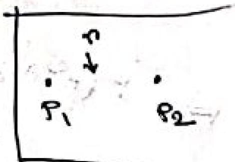
Lastly $P((A \cup B)^c) - P(A^c)P(B^c)$

$= P((A \cup B)^c) - (1 - P(A))(1 - P(B))$

$= P((A \cup B)^c) - 1 + P(A) + P(B) - P(A)P(B)$

$= P(A) + P(B) - P(A \cup B) - P(A)P(B) = P(A \cap B) - P(A)P(B)$

12) Total number of ways in which n persons stand in a row is $n!$.



The r persons can be chosen is $\binom{n-2}{r}$ ways from $(n-2)$ persons [Leaving P_1 & P_2 there are $(n-2)$ persons]

These r persons can be arranged in $r!$ ways between P_1 & P_2.

Now, consider the unit from P_1 to P_2 (including r persons in between) as one person then we need to arrange $(n-2-r+1)$ persons

$\because i_1 < i_2 < \cdots < i_m$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}) = \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{(n-m+1)}$$

Now, no. of events of like A_i is $^nC_1$

no. of " of like $A_i \cap A_j$, $i<j$ is $^nC_2$

etc.

So, the req. prob. is

$$1 - \left[n \times \frac{1}{n} - {}^nC_2 \cdot \frac{1}{n(n-1)} + {}^nC_3 \cdot \frac{1}{n(n-1)(n-2)} + \cdots + (-1)^n \cdot \frac{1}{n!}\right]$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \cdot \frac{1}{n!}$$

17). Three ~~distinct~~ digits can be drawn ~~in~~ (with repeatation) in 10^3 ways.

Three same digits can be drawn in 10 ways.

Three distinct digits can be drawn in $10 \times 9 \times 8$, ways.

Hence exactly two different digits can be drawn in

$$10^3 - (10 \times 9 \times 8 + 10)$$ [$\because$ there are 10 digits at all]

$$= 10^3 - 730$$

So, the req. prob. $= \frac{10^3 - 730}{10^3} = 0.27$

□

Pick up 2 distinct digit from 10 digit is $\binom{10}{2}$

Pick up another digit ~~from~~

Now, pick up digit of previous two digit is $\binom{2}{1}$ from the same

Now, arrange them in $\frac{3!}{2!}$ ways [(a, a, b) etc]

So, req. prob. $\dfrac{\binom{10}{2} \times \binom{2}{1} \times \frac{3!}{2!}}{10 \times 10 \times 10}$

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9). To show,

$$\sum_{i=1}^{n} P(A_i) - \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right)$$

<u>Basis step</u> $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

<u>Induction hypothesis :-</u>

let This is true for $n = k$,

so, $P\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n} \sum_{j=1}^{n} P(A_i \cap A_j)$

<u>Inductive step :-</u>

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)$$
$$= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right)$$
$$\geq P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right)$$
$$\geq \sum_{i=1}^{k} P(A_i) - \sum_{\substack{i=1 \\ i<j}}^{k} \sum_{j=1}^{k} P(A_i \cap A_j) - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right) + P(A_{k+1}) \quad -①$$

Now, $P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right) = P\left((A_1 \cap A_{k+1}) \cup \cdots \cup (A_k \cap A_{k+1})\right)$

$$\le P(A_1 \cap A_{k+1}) + \cdots + P(A_k \cap A_{k+1}) \quad -②$$

$\therefore$ $P\left(\bigcup_{i=1}^{k+1} A_i\right) \geq \sum_{i=1}^{k+1} P(A_{i+1}) - \sum_{\substack{i=1 \\ i<j}}^{k+1} \sum_{j=1}^{k+1} P(A_i \cap A_j)$ by ① & ②

10).

<u>(a).</u> $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

~~(a)~~ $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C)$

(b) $P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C)) - P(A \cap C \cap B)$

$= P(A \cap C) + P(B \cap C) - P(A \cap C \cap B)$

$= 0.2 + 0.2 - 0.1 = 0.3$

(b)(ii) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap B \cap C)$

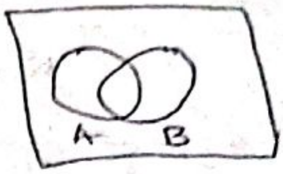
(5) (a).

$$\sigma\langle A\rangle = \bigcap_{A \subset \mathcal{F},\ \mathcal{F} \text{ is a } \sigma \text{ algebra on } S} \mathcal{F}$$

$$= \{\phi, A, A^c, S\}$$

(b) Sln

$A, B \neq \phi,$
$A \cup B = S$
$A \cap B \neq \phi$



Step1 :- Form a partition of S using A & B.
This is $A \cap B^c$, $A \cap B$, $B \cap A^c$, $A^c \cap B^c$.

Step2 :- Take all possible unions of the above sets.
There will be $2^{2^2} = 16$ sets.

$$\sigma\langle A,B\rangle = \{ A\cap B^c,\ A\cap B,\ B\cap A^c,\ A^c \cap B^c,\ A,\ (A\cap B^c) \cup (B \cap A^c),\ B^c,\ B,\ (A\cap B) \cup (A^c \cap B^c),\ A^c,\ A\cup B,\ (A\cap B)^c,\ B^c \cup A,\ A^c \cup B \}$$

6). We prove by induction.

- Basis step :- $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$
$\leq P(A_1) + P(A_2)$

- Induction hypothesis: let the result be true for $K = n-1$

- Inductive step:

$P(A_1 \cup A_2 \cup \cdots \cup A_n)$
$\leq P(A_1 \cup \cdots \cup A_{n-1}) + P(A_n)$
$\leq \sum_{i=1}^{n} P(A_i)$

$$= \sum_{i_1=1}^{n-1} P(A_{i_1} \cap A_n) - \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n-1} P(A_{i_1} A_{i_2} A_n) + \sum_{\substack{i_1,i_2,i_3=1 \\ i_1<i_2<i_3}}^{n-1} P(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_n)$$

$$- \cdots + (-1)^n P(A_1 A_2 \cdots A_{n-1} A_n) \quad \text{--- (2)}$$

Now, from (1) & (2) and by inclusion and exclusion,

$$P(A_1 \cup \cdots \cup A_n)$$

$$= \sum_{i_1=1}^{n-1} P(A_{i_1}) - \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n-1} P(A_{i_1} \cap A_{i_2}) + \cdots + (-1)^n P(A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

$$+ P(A_n) - \Big[\sum_{i_1=1}^{n-1} P(A_{i_1} A_n) - \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n-1} P(A_{i_1} \cap A_{i_2} \cap A_n)$$

$$+ \sum_{i_1,i_2,i_3=1}^{n-1} P(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_n) - \cdots$$

$$+ (-1)^n P(A_1 A_2 \cdots A_n) \Big]$$

$$= \Big[\sum_{i_1=1}^{n-1} P(A_{i_1}) + P(A_n) \Big] - \Big[\sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n-1} P(A_{i_1} \cap A_{i_2}) + \sum_{i=1}^{n-1} P(A_i A_n) \Big]$$

$$+ \Big[\sum_{\substack{i_1,i_2,i_3=1 \\ i_1<i_2<i_3}}^{n-1} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n-1} P(A_{i_1} \cap A_{i_2} \cap A_n) \Big]$$

$$- \Big[\sum_{i_1,i_2,i_3,i_4=1}^{n-1} P(A_{i_1} A_{i_2} \cap A_{i_3} \cap A_{i_4}) + \sum_{\substack{i_1,i_2,i_3=1 \\ i_1<i_2<i_3}} P(A_{i_1} A_{i_2} A_{i_3} A_n) \Big]$$

$$+ \Big[(-1)^n P(A_1 \cap A_2 \cap \cdots A_{n-1}) - (-1)^{n-1} \sum_{\substack{i_1,\ldots,i_{n-2}=1 \\ i_1<\cdots<i_{n-2}}} P(A_{i_1} \cap \cdots \cap A_{i_{n-2}} \cap A_n) \Big]$$

$$- (-1)^n P(A_1 \cap \cdots \cap A_n)$$

$=$

3) let $\mathcal{F} = \bigcap_{\alpha \in I} \mathcal{F}_\alpha$

$\Omega \in \mathcal{F}$ $\because$ $\Omega \in \mathcal{F}_\alpha \ \forall \alpha$

let $A \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha \Rightarrow A \in \mathcal{F}_\alpha \ \forall \alpha$

Now, since, $\forall \alpha$, $\mathcal{F}_\alpha$ is a σ-algebra,

$A^c \in \mathcal{F}_\alpha \ \forall \alpha$

so, $A^c \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha = \mathcal{F}$

of $\{A_i\}_{i=1}^{\infty} \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha \Rightarrow \{A_i\}_{i=1}^{\infty} \in \mathcal{F}_\alpha \ \forall \alpha$

$\because$ each $\mathcal{F}_\alpha$ is a σ-algebra,

$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\alpha \ \forall \alpha$

$\Rightarrow$ $\bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha = \mathcal{F}$

So, $\mathcal{F}$ is σ-algebra.

(4)

Sin let $S = \mathbb{R}$, let $\mathcal{F}_1 = \{\phi, \mathbb{R}, (0,2), (0,2)^c\}$

and let, $\mathcal{F}_2 = \{\phi, \mathbb{R}, (1,3), (1,3)^c\}$

Then $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \mathbb{R}, (0,2), (1,3), (0,2)^c, (1,3)^c\}$

Now, $(0,2), (1,3) \in \mathcal{F}_1 \cup \mathcal{F}_2$

so, $(0,2)^c \cup (1,3)^c$ must belong to $\mathcal{F}_1 \cup \mathcal{F}_2$

so. $(0,2) \cap (1,3)$ must belong to $\mathcal{F}_1 \cup \mathcal{F}_2$

but $(0,2) \cap (1,3) = (1,2) \notin \mathcal{F}_1 \cup \mathcal{F}_2$

so, $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-algebra.

☐ let $S = \{1, 2, 3, 4\}$

$\mathcal{F}_1 = \{\phi, S, \{1,2\}, \{3,4\}\}$

$\mathcal{F}_2 = \{\phi, S, \{1,2,3\}, \{4\}\}$

Now, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \Omega, \{1,2\}, \{3,4\}, \{1,2,3\}, \{4\}\}$

$\{1,2\} \cup \{4\} \notin \mathcal{F}$ but $\{1,2\}, \{4\} \in \mathcal{F}$

in a row and it can be performed in $(n-r-1)!$ ways.

Again, P_1 & P_2 can change their position giving 2 ways to make the row.

So, the required prob. is

$$\frac{2(n-r-1)!\binom{n-2}{r}r!}{n!} = \frac{2(n-r-1)}{n(n-1)}$$

1 person

13). There are 6^3 ways to choose three numbers from $\{1,\cdots,6\}$.

Root will be real if $b^2 \geq 4ac$. We count in how many ways it can happen.

(a,c)	Possible values of $b \ni b^2 \geq 4ac$	No. of ways
(1,1)	2, 3, 4, 5, 6	5
(2,1) & (1,2)	3, 4, 5, 6	8
(3,1) & (1,3)	4, 5, 6	6
(5,1) & (1,5)	5, 6	4
(6,1) & (1,6)	5, 6	4
(2,2)	4, 5, 6	3
(3,2) & (2,3)	5, 6	4
(4,2) & (2,4)	6	2
(3,3)	6	1
	Total	43

So, required prob. $= \frac{43}{6^3}$

14). (a). without replacement 3 numbers from the set $\{1,\cdots,50\}$ can be chosen in $\binom{50}{3}$ ways.

Three real numbers will be in A.P if $a+c=2b$, $(a<b<c)$

It means that if we choose a & c such that the sum of a+c is an even integer, then b can be found automatically, (Taking $b=\frac{a+c}{2}$)

Now, a+c will be even if both a & c are even or both a & c are odd.

there are 25 even & 25 odd in $\{1,\cdots,50\}$

So, total no. of ways a, b & c can be drawn from $\{1,\cdots,50\}$ such that they are in A.P is $\binom{25}{2}+\binom{25}{2}$. So, req. prob. is $= \frac{2\times {}^{25}C_2}{{}^{50}C_3}$

20).

There are $2n$ parts.
So, we can arrange them in $2n!$ ways.

Now, if we take the parts of the respective stick together, we can arrange them in $n!$ ways.

and for each stick, the two parts can arrange themselves in $2!$ ways.

So, that the original stick can be formed.

So, the req. prob. $= \dfrac{2^n \, n!}{(2n)!}$

18). r = indistinguishable balls
(b). n = distinct cells.

As the balls are indistinguishable, it follows that the outcome of the experiment of distributing the r balls into n urns can be described by a vector $(x_1, \cdots, x_n)$, x_i = number of balls that are distributed into the ith urn.

Hence the problem reduces to finding the number of distinct non-negative integer valued vectors $(x_1, \cdots, x_n)$ such that

$$x_1 + x_2 + \cdots + x_n = r$$

To compute this, let us start by considering the number of positive integer-valued solutions.

let we have r indistinguishable objects, lined up and we want to divide them into n non-empty groups. To do this we can select $n-1$ of the $r-1$ spaces $(r > n)$ between adjacent objects as our dividing points.

For instance, if $r = 8$ and $n = 3$, and choose 2 divisors as

$$0\ 0\ 0 \mid 0\ 0\ 0 \mid 0\ 0$$

$$x_1 = 3,\ x_2 = 3,\ x_3 = 2$$

15). The no. of ways 4 groups (each of size 4) can be made is $\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} \Big/ 4!$ —— ①

The no. of ways 4 groups each having one graduate student can be made is

$$\frac{4!\,\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{4!} \quad \text{—— ②}$$

So, the req. prob. $= \frac{②}{①} = \frac{2\times3\times7}{15\times14\times13}$

☐ 4 groups each of size 4

$$= \binom{16}{4\ 4\ 4\ 4}$$

each group contains one graduate student

$$= \binom{12}{3\ 3\ 3\ 3} \times \binom{4}{1\ 1\ 1\ 1}$$

$\therefore$ Req. prob. $=$ ______

Q.16

Define the event that the ith letter is inserted in the ith envelope by A_i, $i = 1, \cdots, n$

we need to find,

$$P\left[A_1^c \cap A_2^c \cap \cdots \cap A_n^c\right] = P\left(A_1 \cup \cdots \cup A_n\right)^c$$

$$= 1 - P\left(A_1 \cup \cdots \cup A_n\right) \quad \text{by Demorg}$$

Now, $P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{i<j} P(A_i \cap A_j) + \cdots + (-1)^n P(A_1 A \cdots \cap A_n)$

Here, $P(A_i) = \frac{1}{n}$ [$\because$ There is exactly one letter corresponding to its own envelope and there are n - envelope & each events are equally likely]

$i<j$, $P(A_i \cap A_j) = P(A_j | A_i)\, P(A_i)$

$$= \frac{1}{n-1} \times \frac{1}{n}$$

when, ith letter is inserted into its own envelope the jth letter, $(i \neq j)$ can be inserted with prob. $\frac{1}{n-1}$.

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7) Sin

Define, $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $\ldots$, $B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$

So, B_i's are disjoint

& $B_i \subset A_i \; \forall i$ and

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

So, $P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) \le \sum_{i=1}^{\infty} P(A_i)$

$\because B_i \subseteq A_i$

$\Rightarrow P(B_i) \le P(A_i)$

by monotonicity property of prob.

□

$B_1 = A_1$, $B_2 = A_1 \cup A_2$, $\ldots$

$P(A_1 \cup A_2 \cup \cdots) = P(B_1 \cup B_2 \cup \cdots)$

$= P(\lim_n B_n)$ $\quad \because \lim B_n = B_1 \cup B_2 \cup \cdots$

$B_1 \cup \cdots \cup B_n = B_n$

$B_i \uparrow$

$= \lim_n P(B_n)$

$= \lim_n P(A_1 \cup \cdots \cup A_n)$

$\le \lim_n [P(A_1) + \cdots + P(A_n)]$

$= \sum_{i=1}^{\infty} P(A_i)$

8). We prove by induction.

<u>Basis step</u> :- $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

<u>Induction hypothesis</u> :- Let this is true for $K = n-1$

<u>Inductive step</u> :-

$P(A_1 \cup A_2 \cup \cdots \cup A_n)$

$= P(A_1 \cup \cdots \cup A_{n-1}) + P(A_n) - P\left(\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n\right)$

$= P(A_1 \cup \cdots \cup A_{n-1}) + P(A_n) - P\left(\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right)$ —(1)

$= \sum_{i=1}^{n} \cancel{P(A_i)} - \sum$

Now, $P[(A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cap A_n]$

$= P[(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)]$ $\leftarrow$ $(n-1)$ events

$= \cancel{\sum_{i=1}^{n-1}}$

13).
Sol

r - indistinguishable balls

n - distinguishable shells

If we arrange r balls in a row and then insist (n−1) bars among the balls, we will get n groups of balls. These groups can be considered as the balls in different cells. For example in the following picture there are n = 3 cells and r = 4 balls.

$\cdot \; \cdot \, | \, \cdot \, | \, \cdot$ $\quad r_1 = 2, \; r_2 = 1, \; r_3 = 1$

$\cdot \; \cdot \; \cdot \; \cdot \; | \, |$ $\quad r_1 = 4, \; r_2 = 0, \; r_3 = 0$

a). To obtain the distinguishable distⁿ, we have r balls and (n−1) bars. Hence if we have (n−1+r) places and if we choose r places to put the balls and rest (n−1) places to put the bars, we will have distinguishable distⁿ. clearly it can be done in $\binom{n+r-1}{r}$ ways.

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$\therefore$ If $r \geq n$,
there are $\binom{r-1}{n-1}$ distinct positive integer valued vectors $(x_1, \cdots, x_n)$ satisfying $x_1 + \cdots + x_n = r$
$x_i > 0$,
$i = 1, \cdots, n$

b). To obtain the number of non-negative (as opposed to positive) solutions, note that the number of non-negative solutions of $x_1 + \cdots + x_n = r$ is the same as the number of positive solutions of $y_1 + \cdots + y_n = n + r$
since, $y_i = x_i + 1$, $i = 1, \cdots, n$

So, there are $\binom{r+n-1}{n-1}$ distinct non-neg. int. valued vectors $(x_1, \cdots, x_n)$ satisfying
$x_1 + \cdots + x_n = r$

19).
There are $n!$ ways to arrange n keys.
The right key will be found in kth trial if the right key is in the kth position.
There are $(n-1)!$ ways to arrange n keys such that right key is in the kth position.
So, the required prob. is $1/n$

$$X_i = \begin{cases} 1 & \text{if the } i\text{th key is right} \\ 0 & \text{o.w} \end{cases}$$

Since the trials are independent,
$P(X_i = 1) = 1/n$

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(b). $P(\phi) = \sum_{x \in \phi} p(1-p)^x = 0$

$P(A) \geq 0 \quad \because 0 < p < 1$

let $A_1, A_2, \cdots \qquad A_i \cap A_j = \phi$

To show, $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

(c) sin

If This is a probability, then

$1 = P(S) = P\left(\bigcup_{i=1}^{\infty} \{i\}\right) = \sum_{i=1}^{\infty} P(\{i\}) = 0$

contradiction

so, this P is not probability.

If possible let this is a probability, then

let $A = \{0, 2, 4, \cdots\}$ and $B = \{1, 3, 5, \cdots\}$

$P(A) = 1, \quad P(B) = 1$

$A \cup B = S, \quad A \cap B = \phi$

so, $P(A \cup B) = P(S) = 1$

but $P(A \cup B) = P(A) + P(B) \because A \cap B = \phi$

$= 1 + 1 = 2$

$\therefore 1 = 2$ contradiction.

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