Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 31 (November 08, 2019)

Exponential Distribution

Theorem: Suppose X has exponential distribution with parameter λ. Then for any $t, s \geq 0$ $P(X > t + s | X > t) = P(X > s)$. This is called the memoryless property. Further exponential distribution is the only continuous distribution with this property.

Def: For a continuous RV X having distribution function F_X and probability density function f_X, the failure rate or hazart rate function is defined by

$$r(t) = \frac{f_X(t)}{1 - F_X(t)} .$$

Remark: Interpretation of $r(t)$ is that given that there has been no failure upto time t, what is the probability that there will be a failure in time $t + dt$.

Remark: If X is a RV having hazard rate function $r(\cdot)$ then its distribution function is given by

$$F(x) = 1 - e^{-\int_0^x r(t)dt}.$$

Remark: $X \sim Exp(\lambda)$ iff $r(t) \equiv \lambda$.

Properties of exponential distribution

1. If $X_1, X_2 \ldots, X_n$ are i.i.d. $\text{Exp}(\lambda)$ then $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$.
2. If $X_1, X_2 \ldots, X_n$ are independent $\text{Exp}(\lambda_i)$ then $\min_i X_i \sim \text{Exp}(\sum_i \lambda_i)$.
3. If $X_1, X_2 \ldots, X_n$ are independent $\text{Exp}(\lambda_i)$ then $P(X_i = \min_j X_j) = \dfrac{\lambda_i}{\sum_j \lambda_j}$.
4. If $X_1, X_2 \ldots, X_n$ are independent $\text{Exp}(\lambda_i)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$ then

$$f_{X_1+X_2+\ldots+X_n}(x) = \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i x},$$

where $C_{i,n} = \prod_{j \neq i} \dfrac{\lambda_j}{\lambda_j - \lambda_i}$.

Some Problems

Example 1: Consider a post office that is run by two clerks. Suppose that when Mr. Amar enters the system he discovers that Mr. Akbar is being served by one of the clerks and Mr. Antony by the other clerk. Suppose also that Mr. Amar is told that his service will begin as soon as either Mr. Akbar or Mr. Antony leaves. If the amount of time that a clerk spends with a customer is exponential with mean $1/\lambda$, what is the probability that of the three customers Mr. Amar is the last to leave the post office?

Some Problems

Example 2: Suppose you arrive at a post office having two clerks, at a moment when both are busy but no one else is waiting. You will enter service when either clerk becomes free. If service times for clerk i is exponential with mean $1/\lambda_i$, $i = 1, 2$. Find $E(T)$, where T is the amonunt of time that you sepnd at the post office.

Order Statistics

Theorem: Let $X_1, X_2, \ldots, X_n$ be i.i.d. CRVs with PDF $f(\cdot)$. If we let $X_{(i)}$ denote the ith smallest of these RVs, then $X_{(1)}, \ldots, X_{(n)}$ are called the order statistics. The JPDF of $(X_{(1)}, \ldots, X_{(n)})$ is given by

$$f_{(X_{(1)},\ldots,X_{(n)})}(x_1, \ldots, x_n) = \begin{cases} n! \prod_{i=1}^{n} f(x_i) & \text{for} \quad x_1 < x_2 \ldots < x_n \\ 0 & \text{otherwise}\,. \end{cases}$$

Corollary: Let $X_1, X_2, \ldots, X_n$ be i.i.d. CRVs with PDF $f(\cdot)$ and CDF $F(\cdot)$. Then

$$f_{X_{(i)}}(x) = \frac{n!}{(n-i)!(i-1)!} f(x)(F(x))^{i-1}(1 - F(x))^{n-i}\,.$$