Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 20 (September 26, 2019)

Limit Theorems

Theorem: (Strong Law of Large Numbers) Let $\{X_n\}$ be a sequence of i.i.d. RVs with finite mean μ. Define $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $\{\overline{X}_n\}$ converges to μ almost suerly.

Example 1: Bernoulli proportion converges to success probability.

Example 2: Monte Carlo Integration.

Theorem: (Cental Limit Theorem) Let $\{X_n\}$ be a sequence of i.i.d. RVs with mean μ and varance $\sigma^2 < \infty$. Then, as $n \to \infty$,

$$P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \leq a\right) \to \Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

Example 3: $X_n \sim Bin(n, p)$. Then

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq a\right) \to \Phi(a).$$

Example 4: The lifetimes of a special type of battery is a RV with mean 40 hours and standard deviation 20 hours. A battery is used untill it fails, at which point it is replaced by a new one. Assume a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the proability that over 1100 hours of use can be obtained. $[\Phi(1) = 0.8413]$