1. $|\rho(X,Y)|=1$, precisely when there is equality in Cauchy-Schwartz Inequality.

Thus $|\rho(X,Y)|=1$

$\Rightarrow a'(X-EX) = Y-EY$ for some $a' \in \mathbb{R}$.

$\Rightarrow Y = -a'EX + EY + a'X$

$\Rightarrow Y = a+bX$ for some $a, b \in \mathbb{R}$.

2. a) $M_{Y,Z}(t_1,t_2) = E\left[e^{t_1 Y + t_2 Z}\right] = E\left[e^{t_1(X_1+X_2) + t_2(X_1^2+X_2^2)}\right]$

$= E\left[e^{t_1 X_1 + t_2 X_1^2}\right] E\left[e^{t_1 X_2 + t_2 X_2^2}\right]$

$= \left(E\left[e^{t_1 X + t_2 X^2}\right]\right)^2$ where $X \sim N(0,1)$.

$E\left[e^{t_1 X + t_2 X^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t_1 u + t_2 u^2} e^{-\frac{u^2}{2}} du$ (Provided $t_2 < \frac{1}{2}$)

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(1-2t_2)}{2}\left\{u^2 - 2u\frac{t_1}{1-2t_2} + \frac{t_1^2}{(1-2t_2)^2} - \frac{t_1^2}{(1-2t_2)^2}\right\}} du$

$= \frac{1}{(1-2t_2)^{-1/2}} e^{\frac{t_1^2}{2(1-2t_2)}}$.

Thus $M_{Y,Z}(t_1,t_2) = \frac{1}{(1-2t_2)^{-1}} e^{\frac{t_1^2}{1-2t_2}}$ for $t_2 < \frac{1}{2}$.

b) $EY = \dfrac{\partial}{\partial t_1} M_{Y,Z}(t_1, t_2)\Big|_{t_1=0,\, t_2=0}$

$EZ = \dfrac{\partial}{\partial t_2} M_{Y,Z}(t_1, t_2)\Big|_{t_1=0,\, t_2=0}$

$EYZ = \dfrac{\partial^2}{\partial t_1 \partial t_2} M_{Y,Z}(t_1, t_2)\Big|_{t_1=0,\, t_2=0}$

$EY^2 = \dfrac{\partial^2}{\partial t_1^2} M_{Y,Z}(t_1, t_2)\Big|_{t_1=0,\, t_2=0}$

$EZ^2 = \dfrac{\partial^2}{\partial t_2^2} M_{Y,Z}(t_1, t_2)\Big|_{t_1=0,\, t_2=0}$

Do the calculations.

3. The given MGF is the MGF of a $\text{Bin}(2, \frac{1}{4})$.

(a) $P(X_1 = k) = \binom{2}{k}\left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{2-k}$ for $k = 0, 1, 2$

$= 0$ otherwise

(b) $Y \sim \text{Bin}(6, \frac{1}{4})$.

Thus $P(Y = k) = \binom{6}{k}\left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{6-k}$ for $k = 0, 1, 2, 3, 4, 5, 6$

$= 0$ otherwise.

~~4.~~ $\cancel{P(\{X \le x\} \mid \{4.999 < Y \le 5.001\})}$

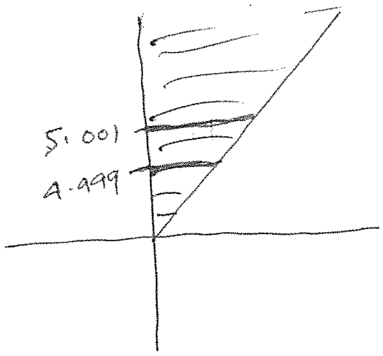
$\cancel{= \dfrac{15 \int_{4.999}^{5.001} \int_0^{y} e^{-(2x+3y)}\, dx\, dy}{\int}}$

4. $P(\{X \le x_1\} \mid \{4.999 < Y \le 5.001\})$

$$= \frac{P(\{X \le x_1,\ 4.999 < Y \le 5.001\})}{P(4.999 < Y \le 5.001)}$$

$$P(4.999 < Y \le 5.001) = 15\int_{4.999}^{5.001}\int_{0}^{y} e^{-(2x+3y)}\,dx\,dy$$

$$= \frac{15}{2}\int_{4.999}^{5.001}\left(e^{-3y} - e^{-5y}\right)dy$$



$$P(\{X \le x_1,\ 4.999 < Y \le 5.001\}) = 0 \text{ if } x_1 \le 0$$

$$= 15\int_{0}^{x_1}\int_{4.999}^{5.001} e^{-(2x+3y)}\,dy\,dx \text{ for } 0 < x_1 \le 4.999$$

$$= 15\left[\int_{0}^{4.999}\int_{4.999}^{5.001} e^{-2x+3y}\,dy\,dx + \int_{4.999}^{x_1}\int_{x}^{5.001} e^{-2x+3y}\,dy\,dx\right]$$

for $4.999 < x_1 \le 5.001$

$$= P(4.999 < Y \le 5.001) \text{ for } x > 5.001.$$

5. $P(X=5) = 0.04 + 0.12 + 0.21 + 0.05$

$= 0.42$

$P(Y=1 | X=5) = \dfrac{0.04}{0.42} = \dfrac{2}{21}$

$P(Y=2 | X=5) = \dfrac{0.12}{0.42} = \dfrac{2}{7}$

$P(Y=3 | X=5) = \dfrac{0.21}{0.42} = \dfrac{1}{2}$

$P(Y=4 | X=5) = \dfrac{0.05}{0.42} = \dfrac{5}{42}$

6. $P(Y=k) = \sum_{j=0}^{\infty} P(Y=k | X=j)\, P(X=j)$

$= \sum_{j=k}^{\infty} P(Y=k | X=j)\, P(X=j)$

$= \sum_{j=k}^{\infty} \binom{j}{k} p^k (1-p)^{j-k} \dfrac{e^{-\lambda} \lambda^j}{j!}$

$= \dfrac{(\lambda p)^k}{k!} \sum_{j=k}^{\infty} \dfrac{e^{-\lambda} \{\lambda(1-p)\}^{j-k}}{(j-k)!} = e^{-\lambda} e^{\lambda(1-p)} \dfrac{(\lambda p)^k}{k!}$

$= e^{-\lambda p} \dfrac{(\lambda p)^k}{k!}.$

$P(X=k | Y=y) = \dfrac{P(X=k, Y=y)}{P(Y=y)} = \dfrac{P(Y=y | X=k)\, P(X=k)}{e^{-\lambda p} \dfrac{(\lambda p)^y}{y!}}$

for $k = y, y+1, \ldots$

$= 0$ otherwise

$$= \frac{\binom{k}{y} p^y (1-p)^{k-y} \times e^{-\lambda} \frac{\lambda^k}{k!}}{e^{-\lambda p} \frac{(\lambda p)^y}{y!}} \quad \text{for } k = y, y+1, \dots$$

$$= 0 \quad \text{otherwise.}$$

$$= \frac{(1-p)^{k-y} \, \lambda^{k-y} \, e^{-\lambda(1-p)}}{(k-y)!} \quad \text{for } k = y, y+1, \dots$$

$$= 0 \quad \text{otherwise.}$$

Then conditional distribution of X given $Y = y$ is $y + P((1-\lambda)p)$.

7. Problem Set 4, Q. 5 we have seen that

$$X \sim NB\left(k, 1 - \frac{\theta_1}{1-\theta_2}\right) \text{ and}$$

$$Y \sim NB\left(k, 1 - \frac{\theta_2}{1-\theta_2}\right).$$

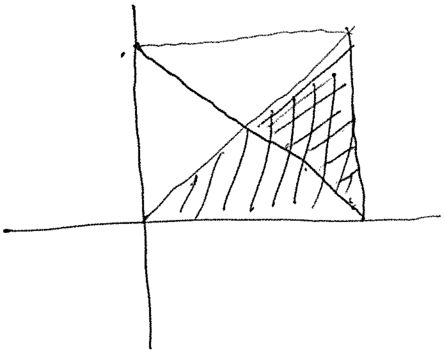
Rest is trivial.

8. Conditional distribution of X_2 given $X_1 = x$ is $U(0, x)$.

$$f_{X_1, X_2}(x_1, x_2) = f_{X_2|X_1}(x_2|x_1) \, f_{X_1}(x_1)$$

$$= \frac{1}{x_1} \quad \text{for } 0 < x_2 < x_1 < 1.$$

$$P\left(X_1 + X_2 \geq 1\right) = \int_{1/2}^{1} \int_{1-x_1}^{x_1} \frac{1}{x_1}\, dx_2\, dx_1$$



$$= \int_{1/2}^{1} \frac{2x_1 - 1}{x_1}\, dx_1$$

$$= 2\left(1 - \frac{1}{2}\right) - \log x_1 \Big|_{\frac{1}{2}}^{1}$$

$$= 1 - \log 2 .$$

$$f_{X_2}(x_2) = \int_{x_2}^{1} \frac{1}{x_1}\, dx_1 = -\log x_2 .$$

$$E\left[X_1 \mid X_2 = x_2\right] = \frac{\int_{x_2}^{1} x_1 \cdot \frac{1}{x_1}\, dx_1}{-\log x_2} = \frac{1 - x_2}{-\log x_2}$$

$$= \frac{x_2 - 1}{\log x_2} .$$

[Q9] The marginals are found in Problem Set 04, Question no. 9.

The conditional PDF of ~~Y~~ X given Y is as follows:

For $y \in (0,1)$,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{\Gamma(\theta_1+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_3)} \dfrac{x^{\theta_1-1}(1-x-y)^{\theta_3-1}}{(1-y)^{\theta_1+\theta_3-1}} & \text{if } 0<x<1-y \\ 0 & \text{o.w.} \end{cases}$$

The conditional PDF of Y given X is as follows:

For $x \in (0,1)$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{\Gamma(\theta_2+\theta_3)}{\Gamma(\theta_2)\Gamma(\theta_3)} \times \dfrac{y^{\theta_2-1}(1-x-y)^{\theta_3-1}}{(1-x)^{\theta_2+\theta_3-1}} & \text{if } 0<y<1-x \\ 0 & \text{o.w.} \end{cases}$$

$\square$

[Q10] Let N denote the number of accidents in a week.

Let $X_1, X_2, \ldots, X_N$ denote the number of injured in accident $1, 2, \ldots, N$, respectively.

Here $E(N) = 4$, $E(X_i) = 2$ for all i.

The number of injured in a week is $\sum_{i=1}^{N} X_i$. We need to find

$$E\left(\sum_{i=1}^{N} X_i\right) = E\,E\left(\sum_{i=1}^{N} X_i \Big| N\right) \qquad (*)$$

Now $E\left(\sum_{i=1}^{N} X_i \Big| N=n\right) = E\left(\sum_{i=1}^{n} X_i \Big| N=n\right) = E\left(\sum_{i=1}^{n} X_i\right) = 2n$

$(*) \Rightarrow E\left(\sum_{i=1}^{N} X_i\right) = 2E(N) = 8.$

$\square$

Q11 Let us define $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ Person takes his/her own hat.} \\ 0 & \text{o.w.} \end{cases}$

Then total number of matches $X = \sum_{i=1}^{n} X_i$.

We ~~k~~ need to find $E(X \mid X_1 = 0)$.

We know that $E(X) = 1$.

Again $E(X) = E\,E(X \mid X_1)$

$$= E(X \mid X_1 = 0)\, P(X_1 = 0) + E(X \mid X_1 = 1)\, P(X_1 = 1). \quad (*)$$

Now $P(X_1 = 0) = \frac{n-1}{n}$ and $P(X_1 = 1) = \frac{1}{n}$.

$E(X \mid X_1 = \cancel{0}\,1)$

$= E\left(\sum_{i=1}^{n} X_i \mid X_1 = 1\right)$

$= E\left(1 + \sum_{i=2}^{n} X_i \mid X_1 = 1\right)$

$= 1 + E\left(\sum_{i=2}^{n} X_i \mid X_1 = 1\right)$

$= 1 + E\left(\sum_{i=2}^{n} X_i\right)$, As $X_1 = 1$, means that the 1^{st} person takes his/her own hat and then in the pool of hats, there are ~~only~~ hats of rest of the $(n-1)$ persons' hats. Hence the conditional expectation $E\left(\sum_{2}^{n} X_i \mid X = 1\right)$ is same as $E\left(\sum_{2}^{n} X_i\right)$

$= 2$

Now using ⊛

$$1 = E(X \mid X_1 = 0) \times \frac{n-1}{n} + \frac{2}{n}$$

$$\Rightarrow E(X \mid X_1 = 0) = \frac{n-2}{n-1}$$ □.

[Note : $E(X \mid X_1 = 0) \neq E(\sum_{i=2}^{n} X_i)$. WHY?].

$\boxed{Q12}$ Let us define

$$Y = \begin{cases} 1 & \text{if the } 1^{st} \text{ door is selected} \\ 2 & \text{if } '' \ 2^{nd} \ '' \ '' \ '' \\ 3 & \text{if } '' \ 3^{rd} \ '' \ '' \ '' \end{cases}$$

and X denote the length of time until the miner reaches safty.

We need to find

$$E(X) = E\,E(X \mid Y)$$

$$= E(X \mid Y=1)\,P(Y=1) + E(X \mid Y=2)\,P(Y=2) + E(X \mid Y=3)\,P(Y=3).$$

$$= \frac{1}{3}\left[2 + 3 + E(X) + 5 + E(X)\right]$$

$$\Rightarrow 3E(X) = 10 + 2E(X)$$

$$\Rightarrow E(X) = 10.$$

$E(X \mid Y=2) = 3 + E(X)$, as the 2^{nd} door leads to a tunnel that returns him to the mine after three hours of travel. ~~and~~ Once he returns to the mine ~~every~~ the problem is as before, and his expected additional time until safty is $E(X)$. □.

Q13 Let N_K denote the number of ~~a~~ trails to get K consecutive successes, and let $E(N_K) = M_K$.

We will find a recursion ~~real~~ relation ~~to~~ on M_K's and then solve for it.

Note that $N_K = N_{K-1} + A_{K-1,K}$,

where $A_{K-1,K}$ denotes the number of additional trails needed to go from $(K-1)$ consecutive successes to having K in a row. Taking expectation, we have

$$M_K = M_{K-1} + E(A_{K-1,K}). \quad \text{——— } (**)$$

Let us denote $X = \begin{cases} 1 & \text{if there is a success on } (N_{K-1}+1)^{st} \text{ trail} \\ 0 & \text{if there is a failure on } (N_{K-1}+1)^{st} \text{ trail} \end{cases}$

Now $E(A_{K-1,K}) = E(A_{K-1,K} \mid X=1)\,P(X=1) + E(A_{K-1,K} \mid X=0)\,P(X=0)$ $(*)$

$E(A_{K-1,K} \mid X=1) = 1$, as $X=1 \Rightarrow A_{K-1,K} = 1$.

$E(A_{K-1,K} \mid X=0) = 1 + M_K$, as if $X=0$, then at that point we are starting all over and the expeted additional trails from then on would be $E(N_K)$.

$P(X=1) = p$ and $P(X=0) = 1-p$

$(*) \Rightarrow E(A_{K-1,K}) = p + (1+M_K)(1-p) = 1 + (1-p)M_K.$

$(**) \Rightarrow M_K = M_{K-1} + 1 + (1-p)M_K$

$\Rightarrow M_K = \frac{1}{p} + \frac{M_{K-1}}{p}.$

Since $N_1 \sim Geo(p)$, $M_1 = E(N_1) = \frac{1}{p}$.

$$M_2 = \frac{1}{p} + \frac{1}{p^2}$$

$$M_3 = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}$$

$$\vdots$$

$$M_k = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^k} = \frac{1-p^k}{p^k(1-p)}. \qquad \square.$$

$\boxed{Q14}$ Let X denote the number of accidents that a randomly chosen policy-holder ~~has~~ will have in next year. ~~Let Y denote~~ ~~be a~~ ~~Poisson~~ ~~RV with~~ ~~mean~~ ~~PDF~~

Let Y denote the mean of ~~accidents~~ number of accidents that a randomly chosen policy-holder will have in next year. Y has PDF $g(\cdot)$.

Here $X \mid Y = y \sim P(y)$ and

Now
$$P(X=n) = \int_0^\infty P(X=n \mid Y=y)\, g(y)\, dy$$

$$= \int_0^\infty \frac{e^{-y} y^n}{n!} \times y e^{-y}\, dy$$

$$= \frac{1}{n!} \int_0^\infty y^{n+1} e^{-2y}\, dy$$

$$= \frac{1}{n!} \frac{\Gamma(n+2)}{2^{n+2}}$$

$$= \frac{n+1}{2^{n+2}}. \qquad \square.$$

Q15 Let X denote the number of persons visits the studio today.

Let N and M denote the number of female and male, respectively.

$$P(N=n, M=m)$$

$$= \sum_{K=0}^{\infty} P(N=n, M=m \mid X=K)\, P(X=K)$$

$$= P(N=n, M=m \mid X=n+m)\, P(X=n+m) \qquad \text{———} \; (*)$$

As $P(N=n, M=m \mid X=K) = 0$ for $K \neq n+m$.

Here $X \sim P(\lambda)$ and $N \mid X=K \sim Bin(K, p)$.

Hence $(*) \Rightarrow$

$$P(N=n, M=m) = \binom{m+n}{n} p^n (1-p)^m \times \frac{e^{-\lambda} \lambda^{n+m}}{(n+m)!}$$

$$= \frac{e^{-\lambda} \lambda^{n+m} p^n (1-p)^m}{n!\, m!}$$

$$= \frac{e^{-(\lambda p)} (\lambda p)^n}{n!} \times \frac{e^{-(\lambda(1-p))} (\lambda(1-p))^m}{m!}$$

Note: N and M are indep and $N \sim P(\lambda p)$ and $M \sim P(\lambda(1-p))$ $\square$.