Lect 04
Number System, Gates, Boolean Algebra

CS221: Digital Design

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Outline

- Gates in Digital System
  - Basic Gates (AND, OR & NOT)
  - Universal Gates (NAND & NOR)
  - Others : XOR, XNOR
- Boolean Algebra
  - Axioms
- Boolean Functions

Boolean Algebra

How to prove 2+2=5?

We know 2+2=4

$$2+2=4-\frac{9}{2}+\frac{9}{2}=\boxed{\sqrt{(4-\frac{9}{2})^2}}+\frac{9}{2}$$

$$=\sqrt{16-2.4.\frac{9}{2}+(\frac{9}{2})^2}+\frac{9}{2}$$

$$=\sqrt{-20+(\frac{9}{2})^2}+\frac{9}{2}=\sqrt{25-45+(\frac{9}{2})^2}+\frac{9}{2}$$

$$=\sqrt{5^2-2.4.\frac{9}{2}+(\frac{9}{2})^2}+\frac{9}{2}=\sqrt{(5-\frac{9}{2})^2}+\frac{9}{2}$$

$$=5-\frac{9}{2}+\frac{9}{2}=5$$

Where is the mistake?

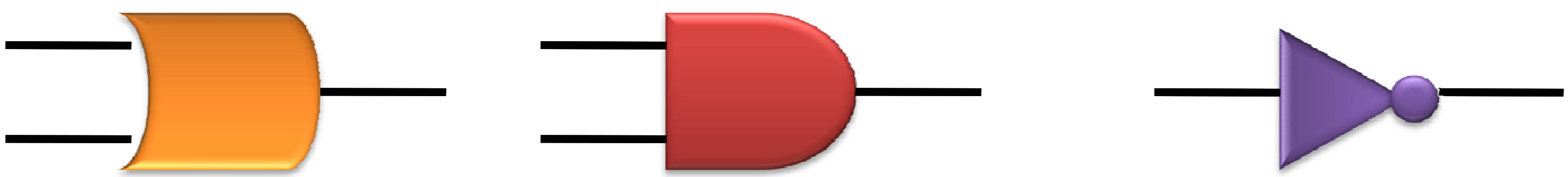
$\sqrt{x^2}=x$ is true only when $x\geq0$

Boolean Algebra

- Computer hardware using binary circuit greatly simply design
- George Boole (1813-1864): developed a mathematical structure in **1847**
  - To deal with binary operations with just two values
- Binary circuits: To have a conceptual framework to manipulate the circuits algebraically
  - **Claude Shannon : 1937, Master Thesis**

Basic Gates in Binary Circuit

- Element 0 : “FALSE”. Element 1 : “TRUE”.
- ‘+’ operation “OR”,‘*’ operation “AND” and ’ operation “NOT”.



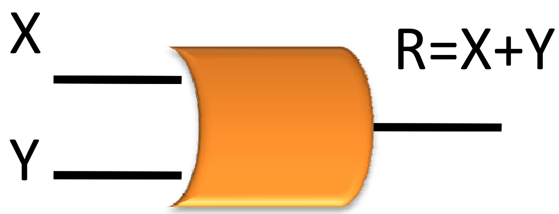
OR	0	1
0	0	1
1	1	1

AND	0	1
0	0	0
1	0	1

NOT	
0	1
1	0

OR Gate

- '+' operation "OR"



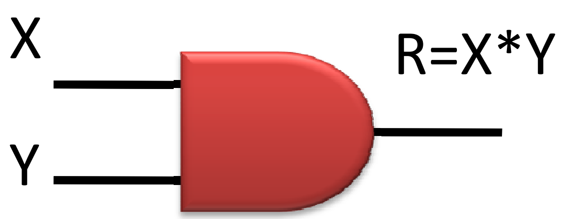
OR	*0*	*1*
0	*0*	*1*
1	*1*	*1*

X	Y	R=X OR Y R= X + Y
0	0	0
0	1	1
1	**0**	**1**
1	**1**	**1**

1 + Y = 1

AND Gate

- ‘*’ operation “AND”



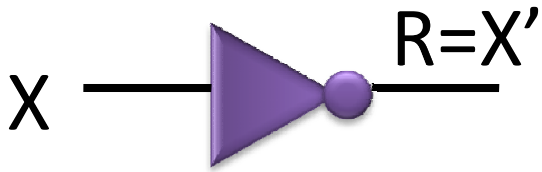
AND	0	1
0	0	0
1	0	1

X	Y	R=X AND Y R= X * Y
0	**0**	**0**
0	**1**	**0**
1	0	0
1	1	1

0 * Y = 0

NOT Gate

- ‘ operation “NOT” or use BAR
- R= $\overline{X}$



X	R=X’ R= NOT X
0	1
1	0

Boolean Algebra Defined

- Boolean Algebra B : 5-tuple

{B, +, *, ’, 0, 1}

- **+** and ***** are *binary* operators,
- **‘** is a *unary* operator.

Boolean Algebra Defined

- ***Axiom #1: Closure***

  If a and b are Boolean

  (a + b) and (a * b) are Boolean.

- ***Axiom #2: Cardinality/Inverse***

  if a is Boolean then a’ is Boolean

- ***Axiom #3: Commutative***

  (a + b) = (b + a)

  (a * b) = (b * a)

Boolean Algebra Defined

- ***Axiom #4: Associative :*** **If a and b are Boolean**

$$(a + b) + c = a + (b + c)$$
$$(a * b) * c = a * (b * c)$$

- ***Axiom #6: Distributive***

$$a * (b + c) = (a * b) + (a * c)$$
$$a + (b * c) = (a + b) * (a + c)$$

2nd one is Not True for Decimal numbers System
$5+(2*3) \neq (5+2)*(5+3)$
$11 \neq 56$

Boolean Algebra Defined

- ***Axiom #5: Identity Element :***
  - **B has identity to + and ***

0 is identity element for + : $a + 0 = a$

1 is identity element for * : $a * 1 = a$

- ***Axiom #7: Complement Element***

$$a + a' = 1$$

$$a * a' = 0$$

Terminology

- **Juxtaposition implies * operation:**

$$ab = a * b$$

- **Operator order of precedence is:**

$$(\,) > ' > * > +$$

$$a+bc = a+(b*c) \neq (a+b)*c$$

$$ab' = a(b') \neq (a*b)'$$

Named Theorems

Idempotent	a + a = a	a * a = a
Boundedness	a + 1 = 1	a * 0 = 0
Absorption	**a + (a*b) = a**	**a*(a+b) = a**
Associative	(a+b)+c= a+(b+c)	(a*b)*c= a*(b*c)

Involution	(a')' = a	
DeMorgan's	(a+b)' = a' * b'	(a*b)'=a' + b'

Simplification Theorem

- Uniting :

$XY + XY' = X$ $X(Y+Y')=X.1=X$

$(X + Y)(X + Y') = X$ $XX+XY'+YX+YY'=X+X(\mathbf{Y+Y'})+0=X$

- Absorption:

$X + XY = X$ $X(1+Y)=X.1=X$

$X(X + Y) = X$ $XX+XY=X+XY=X$

- Adsorption

$(X + Y')Y = XY, \quad XY' + Y = X + Y$ $XY+YY'=XY+0=XY$

Principle of Duality

- Dual of a statement S is obtained
  - By interchanging * and +
  - By interchanging 0 and 1
- Dual of $(a*1)*(0+a') = 0$ is $(a+0)+(1*a') = 1$

Duality examples

- $x + 0 = x$ $x.1=x$
- $z + x'=1$ $x.x'=0$
- $A+B'C$ $A.(B'+C)$
- $A'B'+AB$ $(A'+B').(A+B)$

Consensus Theorem

- $XY + X'Z + YZ = XY + X'Z$

$$
\begin{aligned}
& XY + X'Z + YZ \\
&= xy + x'z + (x + x')yz \\
&= xy + x'z + xyz + x'yz \\
&= xy + xyz + x'z + x'yz \\
&= xy(1 + z) + x'z(1 + y) \\
&= xy + x'z
\end{aligned}
$$

Duality

Consensus (collective opinion) of X.Y and X'.Z is Y.Z

- $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$

Shannon Expansion

- **$F(A,B) = A' \cdot F(0,B) + A \cdot F(1,B)$**

Example:

$F(A,B) = A'.B+A.B'$

$=A'.(1.B+0.B')+A.(0.B+1B')$

$=A'B+AB'$

Shannon Expansion

- **$F(X, Y, Z) = X \cdot F(1,Y,Z) + X' \cdot F(0, Y, Z)$**

Example:

$XY+X'Z+YZ$

$=X.(1.Y+0.Z+YZ) + X'(0.Y+1.Z+YZ)$

$=X.(Y+YZ)+X'(Z+YZ)$

$=XY+XYZ+X'Z+X'YZ$

$=XY+XYZ+X'Z+X'YZ+\mathbf{XYZ}$

$= XY(1+Z)+X'Z+YZ(X'+X)$

$=XY+X'Z+YZ$