Indian Institute of Technology Guwahati
Probability Theory and Random Processes (MA225)
Problem Set 11

1. If X_1 and X_2 are independent non-negative continuous random variables, show that

$$P\{X_1 < X_2 | min(X_1, X_2) = t\} = \frac{r_1(t)}{r_1(t) + r_2(t)}$$

where $r_i(t)$ is the hazard rate function of X_i.

2. Let X and Y be independent exponential random variables with respective rates λ and μ. Let $M = min(X, Y)$. Find (a) $E[MX|M = X]$, (b) $E[MX|M = Y]$.

3. In a certain system, a customer must first be served by server 1 and then by server 2. The service times at server i is exponential with rate μ_i, $i = 1, 2$. An arrival finding server 1 busy waits in the line for that server. Upon completion of service at sever 1, a customer either enters service with server 2 if that server is free or else remain with server 1 (blocking any other customer from entering service) until server 2 is free. Customers depart the system after being served by server 2. Suppose that when you arrive there is one customer in the system and that customer is being served by server 1. What is the expected total time you spend in the system?

4. Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Define $N(t) = N_1(t) + N_2(t)$. Prove that $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

5. Let $N(t)$ be a Poisson process with rate λ. Find the probability that there are two arrivals in $(0, 2]$ and three arrivals in $(1, 4]$.

6. Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Let N(t) be the merged process $N(t) = N_1(t) + N_2(t)$.
a) Find the probability that $N(1) = 2 \,\text{and}\, N(2) = 5$.
b) Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

7. Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Find the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$.

8. Customers arrive at a bank according to a Poisson process with rate λ. Suppose two customers have arrived during the first hour. What is the probability that
(a) both arrived during the first 20 minutes?
(b) at least one arrived during the first 20 minutes?

9. A machine works for an exponentially distributed time with rate μ and then fails. A repair crew checks the machine at times distributed according to a Poisson process with rate λ; if the machine is found to be failed then it is immediately replaced. Find the expected time of first replacement of the machine. [Hint: Note that if the machine fails at time t, the machine is replaced during the first inspection after the time t.]

10. Let $\{Y_i\}$ be an i.i.d. sequence of random variables, $N(t)$ a Poisson process (independent of the Y_is) with intensity λ and let

$$X(t) = \sum_{i=1}^{N(t)} Y_i \,.$$

be the associated compound Poisson process. Further, let T be an exponentially distributed random variable with parameter μ, independent of $N(t)$ and Y_i. Calculate $E\left[e^{-T} X(T)\right]$.