Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 23 (October 17, 2019)

Multinomial Distribution

Def: Consider n independent trails, each of which results in one of the outcomes 1, 2, ..., r, with respective probabilities $p_1, p_2, \ldots, p_r$, where $\sum_{i=1}^{r} p_i = 1$. Let N_i be the number of trails that result in outcome i. Then $(N_1, \ldots, N_r)$ is said to have a multinomial distribution.

Theorem: The joint PMF of $(N_1, N_2, \ldots, N_r)$ is given by

$$f(n_1, n_2, \ldots, n_r) = \begin{cases} \binom{n}{n_1, n_2, \ldots, n_r} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \\ \qquad \text{for } n_1 \geq 0, \ldots, n_r \geq 0, \sum_{i=1}^{r} n_i = n \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$.

Remark: Notation: $Mult(n, p_1, p_2, \ldots, p_r)$.

Theorem: $N_i \sim Bin(n, p_i)$ for all $i = 1, 2, \ldots, r$.

Theorem: Let $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, r\}$. Then the JPMF of $(N_{i_1}, \ldots, N_{i_k})$ is given by

$$f(n_{i_1}, \ldots, n_{i_k}) = \begin{cases} \frac{n!}{w! n_{i_1}! \ldots n_{i_k}!} (1 - \sum_{s=1}^{k} p_{i_s})^w p_{i_1}^{n_{i_1}} \ldots p_{i_k}^{n_{i_k}} \\ \qquad \text{if } n_{i_1} \geq 0, \ldots, n_{i_k} \geq 0, \sum_{s=1}^{k} n_{i_s} \leq n \\ 0 \quad \text{otherwise,} \end{cases}$$

where $w = n - \sum_{s=1}^{k} n_{i_s}$.

Theorem: Let k and l be natural numbers such that $k + l = r$. Let $A = \{i_1, \ldots, i_k\}$ and $B = \{j_1, \ldots, j_l\}$ be a partition of $\{1, 2, \ldots, r\}$. Then the conditional distribution of $(N_{i_1}, \ldots, N_{i_k})$ given $N_{j_1} = n_1, \ldots, N_{j_l} = n_l$ is

$$Mult\left(n - \sum_{j=1}^{l} n_j, \frac{p_{i_1}}{1 - \sum_{s=1}^{l} p_{j_s}}, \ldots, \frac{p_{i_k}}{1 - \sum_{s=1}^{l} p_{j_s}}\right).$$

Theorem: $Cov(N_i, N_j) = -np_ip_j$.

Example 1: Suppose that the lifetime of electric bulbs manufactured by a manufacturer follows exponential distribution with mean of 50 hours. Eight such bulbs are chosen at random. Find the expected number of bulbs in the lot of 8 chosen bulbs with lifetime between 60 and 80 hours, given that the number of bulbs in the lot with lifetime anywhere between 40 and 60 hours is 2.