# Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 14 (September 05, 2019)

# Functions of RVs: Technique 2 for CRV

Example 1: Let $X_1$ and $X_2$ be *i.i.d.* $U(0, 1)$ random variables. Find the JPDF of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

Example 2: Let $X_1$ and $X_2$ be *i.i.d.* $N(0, 1)$ random variables. Find the PDF of $Y_1 = X_1/X_2$.

Remark: If $X$ and $Y$ are independent, then $g(X)$ and $h(Y)$ are also independent.

# Moment Generating Function

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a RV. The moment generating function (MGF) of $\boldsymbol{X}$ at $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ is defined by

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = E\left(\exp\left(\sum_{i=1}^{n} t_i X_i\right)\right),$$

provided the expectation exists.

Remark: $E(X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}) = \frac{\partial^{r_1+r_2+\ldots+r_n}}{\partial t_1^{r_1} \partial t_2^{r_2} \ldots \partial t_n^{r_n}} M_{\boldsymbol{X}}(\boldsymbol{t})\Big|_{\boldsymbol{t}=0}$.

Def: Two RVs $\boldsymbol{X}$ and $\boldsymbol{Y}$ are said to have the same distribution, denoted by $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y}$, if $F_{\boldsymbol{X}}(\cdot) = F_{\boldsymbol{Y}}(\cdot)$.

Theorem: Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two RVs. Let $M_{\boldsymbol{X}}(\boldsymbol{t}) = M_{\boldsymbol{Y}}(\boldsymbol{t})$ for all $\boldsymbol{t}$ in a neighborhood around 0, then $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y}$.

Example 3: Let $X_i$, $i = 1, 2, \ldots, k$ be independent $Bin(n_i, p)$ RVs. Then $\sum X_i \sim Bin(\sum n_i, p)$.

Example 4: Let $X_i$, $i = 1, 2, \ldots, k$ be iid $Exp(\lambda)$ RVs. Then $\sum X_i \sim Gamma(k, \lambda)$.

Example 5: Let $X_i$, $i = 1, 2, \ldots, k$ be independent $N(\mu_i, \sigma_i^2)$ RVs. Then $\sum X_i \sim N(\sum \mu_i, \sum \sigma_i^2)$.

Theorem: $X$ and $Y$ are independent iff $M_{X,Y}(t_1, t_2) = M_X(t_1) M_Y(t_2)$.

# Expectation and Variance of a Random Vector

Expectation of a random vector is given by

$$E(\boldsymbol{X}) = (EX_1, EX_2, \ldots, EX_n) = \boldsymbol{\mu}\,.$$

The variance-covariance matrix of a n-dimensional random vector, denoted by $\Sigma$, is defined by

$$\Sigma = [Cov(X_i, X_j)]_{i,j=1}^{n} = E(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^t\,.$$