Solution of Problem Set 10

$\boxed{Q1}$ ~~Ques~~ Let $i, j \in S$ and $n > 0$ such that $p_{ij}^{(n)} > 0$. That means there exist a path from i to j in n step. Let the intermediate states are $i \to i_1 \to i_2 \to \cdots \to i_{n-1} \to j$.

As $Q = \frac{1}{2}(I+P) \Rightarrow q_{ij} > 0$ if $p_{ij} > 0$.

As $q_{ij}^{(n)} > q_{ii_1} q_{i_1 i_2} \cdots q_{i_{n-1} j} > 0$. $\square$

Q is aperiodic as $q_{ii} > 0 \;\; \forall i$. $\square$

If π is a stationary dist^{n} of P

$\Leftrightarrow \pi P = \pi$

$\Leftrightarrow \pi Q = \frac{1}{2}\pi(I+P) = \pi$

$\Leftrightarrow \pi$ is a stationary dist^{n} of Q. $\square$

$\boxed{Q2}$ For any two distinct states i, j, there is a path that takes the chain from i to j. Cut out any loops from this path and you still have a path that takes you from i to j. But this path has ~~dis~~ distinct states and distinct ~~arr~~ arrows. There are at most $(N-1)$ such arrows. $\square$

$\boxed{Q3}$ The transition probability matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

The characteristic polynomial is

$$|xI - P| = x^3 - \frac{1}{2}x^2 - \frac{1}{2}x,$$

and the characteristic roots are $x_1 = 0$, $x_2 = 1$, $x_3 = -\frac{1}{2}$.

As the roots are distinct, P is digonalizable. Therefore

$$P = Q \Lambda Q^{-1}, \quad \text{where } \Lambda = \text{diag}(0, 1, -\tfrac{1}{2}).$$

We need to find $P^n = Q \Lambda^n Q^{-1}$

Therefore
$$p_{11}^{(n)} = c_1 x_1^n + c_2 x_2^n + c_3 x_3^n$$
$$= c_2 + (-\tfrac{1}{2})^n c_3.$$

Now
$$\left.\begin{array}{l} p_{11}^{(0)} = 1 \Rightarrow c_2 + c_3 = 1. \\ p_{11}^{(1)} = 0 \Rightarrow c_2 - \frac{1}{2}c_3 = 0 \end{array}\right\} \Rightarrow c_2 = \frac{1}{3}, \quad c_3 = 2/3.$$

Hence $p_{11}^{(n)} = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$. $\square$.

$\boxed{Q4}$ Consider a MC with state space $S = \{0, 1, 2, 3, 4\}$, where state i represents the number of umbrellas in the place (office or home) where I am currently at.

The transition probability matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q & p \\ 0 & 0 & q & p & 0 \\ 0 & q & p & 0 & 0 \\ \cancel{p}q & p & 0 & 0 & 0 \end{pmatrix} \end{array}$$

If $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4)$ be ~~th~~ ~~the~~ the stationary dist^n (it will exists a it is a irreducible ~~an~~ finite MC), then

$$\left.\begin{aligned} q\pi_4 &= \pi_0, \\ q\pi_3 + p\pi_4 &= \pi_1 \\ q\pi_2 + p\pi_3 &= \pi_2 \\ q\pi_1 + p\pi_2 &= \pi_3 \\ \pi_0 + p\pi_1 &= \pi_4 \end{aligned}\right\} \Rightarrow \begin{aligned} \pi_0 &= q\pi_4 \\ \pi_1 &= \pi_4 \\ \pi_3 &= \pi_4 \\ \pi_2 &= \pi_4 \end{aligned}$$

As $\quad \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \Rightarrow q\pi_4 + \pi_4 + \pi_4 + \pi_4 + \pi_4 = 1$

$$\Rightarrow \pi_4 = \frac{1}{4+q}$$

$$\Rightarrow \pi_0 = \frac{q}{4+q}.$$

I get wet every time I happen to be in state 0 and it rains. The chance I am in state 0 is $\pi(0)$. Hence the required prob. of the part (a) is $\dfrac{pq}{q+4}$. □

$\boxed{\text{Q4(b)}}$ Let I need N umbrellas. Set up the MC as above. It is clear that

$$\pi_N = \pi_{N-1} = \cdots = \pi_1 \quad \text{and} \quad \pi_0 = q\pi_N.$$

Now $\sum_{i=0}^{N} \pi_i = 1 \Rightarrow \pi_0 = \dfrac{q}{q+N} \Rightarrow P(\text{Wet}) = \dfrac{pq}{q+N}$.

$P(\text{Wet}) \leq 0.01 \Rightarrow N \geq 23.6.$

Hence I need 24 umbrellas to make $P(\text{wet}) \leq 0.01$. ∎

Q5 (a) Trivial , (b) Trivial.

(c) For $p = \frac{1}{2}$, the transition probability matrix is doubly stochastic. Hence the stationary dist^n is $\pi_i = \frac{1}{N+1}$. □

for $p \neq \frac{1}{2}$, $\underline{\pi} = (\pi_0, \ldots, \pi_N)$ satisfies

$q(\pi_0 + \pi_1) = \pi_0$, $p\pi_0 + q\pi_2 = \pi_1, \ldots,$ $p\pi_{N-2} + q\pi_N = \pi_{N-1}$, $p(\pi_{N-2} + \pi_{N-1}) = \pi_{N-1}$

Solve it to obtain the answer.

Q6 Consider a MC with state space $S = \{1, 2, 3, 4\}$, where

1 means that the particle is at the point A.

2 " " " " " " " " B.

3 " " " " " " " " C.

4 " " " " " " " " D.

The transition probability matrix is

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

P is ~~aperiodic~~, irriducible, finite and doubly stochastic. Hence ~~the~~ unique stationary dist^n exists and is given by

$\pi_i = \frac{1}{4}$ for all $i = 1, 2, 3, 4$.

As $\pi_i = \dfrac{1}{E(T_i \mid X_0 = i)} = \frac{1}{4} \Rightarrow E(T_i \mid X_0 = i) = 4$.

□.

In particular $E(T_1 \mid X_0 = 1) = 4$.

$\boxed{Q7}$ The required probability is

$P($All the states have been visited by time $T \mid X_0 = 0)$.

$= P($All the states have been visited by time $T \mid X_1 = 1, X_0 = 0)\, P(X_1 = 1 \mid X_0 = 0)$
$+ P($ ______________________ $\mid X_1 = n, X_0 = 0)\, P(X_1 = n \mid X_0 = 0)$.

Now, $P($All the states have been visited by time $T \mid X_1 = 1, X_0 = 0)$

$= P($Starting from 1, the chain will hit N before hitting $n)$

$= \frac{1}{n}$ using Gambler's Ruin problem.

Similarly, $P($All the states have been visited by time $T \mid X_0 = 0, X_1 = n)$

$= \frac{1}{n}$.

Hence the required probability is $\frac{1}{n}$. $\square$

$\boxed{Q8}$

(a) Trivial.

(b) As P is irreducible and finite, it has unique stationary dist^n. The stationary dist is $\pi = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.
As P is aperiodic

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$\boxed{Q9}$ Check that any π of the form $(p, 0, 1-p)$ with $0 \le p \le 1$, is a stationary dist^n. $\square$

$$P^n = \begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} & p_{13}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & p_{23}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & p_{33}^{(n)} \end{pmatrix} \cancel{=} \cancel{\begin{pmatrix} 1 & 0 & 0 \\ 1-\left(\frac{3}{4}\right)^n & \left(\frac{1}{2}\right)^n & 1-\left(\frac{3}{4}\right)^n \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1-0.5^n}{2} & \left(\frac{1}{2}\right)^n & \frac{1-0.5^n}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Here $p_{21}^{(n)} = \frac{1}{4}\left[1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{n-1}\right] = \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right).$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$\blacksquare$

$\boxed{110.}$ Let T be the first time the answer is passed incorrectly. The president said yes. So incorrect means the first time someone passed no. Thus

$$P(T = k) = \left(\frac{1}{2}\right)^k$$

Thus $ET = \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k} = 2.$

The transition probability matrix is

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{pmatrix}.$$

The stationary distribution is $(0.6, 0.4)$.

Then $$\lim_{n\to\infty} P^n = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}.$$

$\boxed{Q11}$ (a) The transition probabilities are

$$p_{ij} = \begin{cases} 1-3\alpha & \text{if } j = i \\ \alpha & \text{if } j \neq i \end{cases}$$

By symmetry, $p_{ij}^{(n)} = \frac{1}{3}\left(1 - p_{ii}^{(n)}\right)$ for $i \neq j$.

Let us prove by induction that

$$p_{ij}^{(n)} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n} & \text{if } \cancel{i \neq j}\ i = j \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha)^{n} & \text{if } i \neq j \end{cases}$$

This is true for $n = 1$.

Now
$$p_{ii}^{(n+1)} = p_{ii}^{(n)} p_{ii} + \sum_{j\neq i} p_{ij}^{(n)} p_{ji}$$

$$= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n+1}$$

By symmetry, $p_{ij}^{(n+1)} = \frac{1}{4} - \frac{1}{4}(1-4\alpha)^{n+1}$ for $i \neq j$.

(b) As the transition probability matrix is doubly stochastic, irreducible and finite, the stationary dist^{n} is $\pi_i = \frac{1}{4}\ \forall i$.

$\boxed{Q12}$

Let x_n = No. of pairs of shoes at font door at ~~time n~~ n^{th} morning.

Then $\{X_n\}$ is a MC with transition ~~probabilites matrix~~ probabilities

$$p_{00} = \frac{3}{4} = 1 - p_{01}, \quad p_{k,k-1} = \frac{1}{4} = 1 - p_{k,k}$$

$$p_{i,i+1} = \frac{1}{4} = p_{i,i-1}, \quad p_{ii} = \frac{1}{2} \quad \text{for } i = 1, 2, \ldots, k-1.$$

Note that the transition probability matrix is doubly stochastic, irreducible, aperiodic and finite. Hence the stationary distn is

$$\pi_i = \frac{1}{k+1} \quad \text{for } i = 0, 1, 2, \ldots, k.$$

Hence the required probability is $\frac{1}{2}\pi_0 + \frac{1}{2}\pi_k = \frac{1}{k+1}$ □

$\boxed{Q13}$ let $\{X_n\}$ be the type of the n^{th} vehicle. Thus the TPM is

$$\begin{array}{c|c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

If $\Pi = (\Pi_c, \Pi_T)$ is the stationary dist. then

$$[\Pi_c \ \Pi_T] \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = [\Pi_c \ \Pi_T]$$

$$\frac{4}{5}\Pi_c + \frac{3}{4}\Pi_T = \Pi_c \Rightarrow \frac{3}{4}\Pi_T = \frac{1}{5}\Pi_c \Rightarrow \Pi_c = \frac{15}{4}\Pi_T$$

$$\frac{1}{5}\Pi_c + \frac{1}{4}\Pi_T = \Pi_T \Rightarrow \frac{1}{5}\Pi_c = \frac{3}{4}\Pi_T \Rightarrow \Pi_c = \frac{15}{4}\Pi_T$$

Also $\pi_C + \pi_T = 1$ $\Rightarrow \left(\frac{15}{4} + 1\right)\pi_T = 1$

$\cancel{\Rightarrow \frac{15}{4}(\pi_T + \pi_T) = 1}$ $\Rightarrow \pi_T = \frac{4}{19}, \quad \pi_C = \frac{15}{19}$

Thus $\frac{4}{19}$ fraction of vehicles are trucks.

(14) Let $\{X_n\}$ be the MC with states 0, 1, 2, where $X_n = i$ if i switches are on n^{th} day for $i = 0, 1, 2$. Thus the TPM. is

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{9}{16} & \frac{6}{16} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{16} & \frac{6}{16} & \frac{9}{16} \end{bmatrix} \end{array}.$$

The stationary dist. is $\pi_0 = \frac{2}{7}, \pi_1 = \frac{3}{7}, \pi_2 = \frac{2}{7}$

Thus both switches are on $\frac{2}{7}$ fraction of days

" " " off on $\frac{2}{7}$ " " "

~~(15) Let the outcomes be $a, b, c \ldots$ (m of them). Suppose that a is the desirable outcome. Define a MC with state $0, 1, \ldots, k$, where state i means we are currently on a run of i a's for $i < k$, and state k means the run has occured.~~

~~Thus $p_{k,0} = \frac{m-1}{m}$, $p_{k,k+1} = \frac{1}{m}$.~~

Let $\psi(i)$ denote the expected number of steps to reach a run of k a's starting from i a's. We want to find $\psi(0)$. $\psi(k)=0$

Thus

$$\psi(i) = 1 + \frac{m-1}{m}\psi(0) + \frac{1}{m}\psi(i+1) \quad \text{for } i<k.$$

$$\psi(k-1) = 1 + \left(\frac{m-1}{m}\right)\psi(0)$$

$$\psi(k-2) = 1 + \left(\frac{m-1}{m}\right)\psi(0) + \frac{1}{m}\psi(k-1)$$

$$= 1 + \left(\frac{m-1}{m}\right)\psi(0) + \frac{1}{m}\left[1 + \left(\frac{m-1}{m}\right)\psi(0)\right]$$

$$= \left(1+\frac{1}{m}\right) + \left(1+\frac{1}{m}\right)\left(\frac{m-1}{m}\right)\psi(0)$$

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15 Consider the MC $\{X_n\}$ with state space $\{1, \ldots, k\}$ where state i for $i<k$ means we are on a run of length i (of some outcome) and state k means a run of k of one of the outcomes has occured.

Thus $p_{i1} = \frac{m-1}{m}$, $p_{i,i+1} = \frac{1}{m}$, for $i<k$.

$p_{kk} = 1$.

Let $\psi(i)$ denote the expected number of steps to reach ~~a run of k~~ state k starting from state i. Thus we are interested in

$1 + \psi(1)$.

Now $\psi(K) = 0$ & $\psi(i) = 1 + \left(\frac{m-1}{m}\right)\psi(1) + \frac{1}{m}\psi(i+1)$.

for $i=1$, $\psi(1) = 1 + \frac{m-1}{m}\psi(1) + \frac{1}{m}\psi(2)$

$\Rightarrow \psi(1) = m + \psi(2)$ ——— (*)

for $i=2$, $\psi(2) = 1 + \frac{m-1}{m}\psi(1) + \frac{1}{m}\psi(3)$

(*) $\Rightarrow \psi(1) = m + 1 + \frac{m-1}{m}\psi(1) + \frac{1}{m}\psi(3)$

$\Rightarrow \psi(1) = m + m^2 + \psi(3)$

$\vdots$

$\Rightarrow \psi(1) = m + m^2 + \cdots + m^{k-1} + \psi(k)$

$= m + m^2 + \cdots + m^{k-1}$

Hence the required quantity is $1 + \psi(1) = \frac{m^k - 1}{m-1}$. $\square$.

Q16 The TPM is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & & & \\ q & 0 & p & 0 & \cdots & & & \\ 0 & p & 0 & q & \cdots & & & \\ 0 & 0 & q & 0 & p & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \end{pmatrix}$$

If possible let $\underline{\pi} = (\pi_0, \pi_1, \cdots\cdots)$ be a stationary dist^n.
Solve $\underline{\pi} P = \underline{\pi}$ to obtain $\pi_0 = qc$ and $\pi_i = c$ for all $i = 1, 2, \ldots$
Hence stationary dist^n does not exist (as $\sum \pi_i = 1$ cannot be satisfied). $\Rightarrow$ States are not (+ve) recurrent.

Q17 Define x_n = Remainder when S_n is divided by 7.
The state space is $S = \{0, 1, 2, \cdots, 6\}$

The TPM is

$$P = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \cdots & \frac{1}{6} & \frac{1}{6} \\ \vdots & \vdots & \vdots & & \vdots & \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \cdots & 0 & \frac{1}{6} \end{pmatrix}$$

which is a doubly stochastic. Hence the stationary dist^n is $(\frac{1}{7}, \cdots\cdots, \frac{1}{7})$.

~~Now D_n = ...~~

$$\Rightarrow \frac{D_n}{n} \longrightarrow \pi_0 = \frac{1}{7} \quad a.s.$$

$\square$