Indian Institute of Technology Guwahati
Probability Theory and Random Processes (MA225)
Problem Set 06

1. Let X and Y are RVs such that $|\rho(X, Y)| = 1$. Show that $Y = a + bX$ for some real constants a and b.

2. Let X_1 and X_2 be independent $N(0,1)$ random variables and let $Y = X_1 + X_2$, $Z = X_1^2 + X_2^2$.

   (a) Show that the joint MFG of (Y, Z) is $M_{Y,Z}(t_1, t_2) = (1 - 2t_2)^{-1} e^{\frac{t_1^2}{1-2t_2}}$ if $t_1 \in \mathbb{R}$ and $t_2 < \frac{1}{2}$.

   (b) Using (a), find $Corr(Y, Z)$.

3. Let X_1, X_2, X_3 be i.i.d. with common MGF $M(t) = ((3/4) + (1/4)e^t)^2$, for all $t \in \mathbb{R}$.

   (a) Determine the probabilities $P(X_1 = k)$ for $k \in \mathbb{R}$.

   (b) Find the MGF of $Y = X_1 + X_2 + X_3$, and then determine the probability $P(Y = k)$ for $k \in \mathbb{R}$.

4. Let (X, Y) be a random vector with PDF

$$f_{X,Y}(x, y) = \begin{cases} 15e^{-(2x+3y)} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

   Find $P(\{X \le x\}|\{4.999 < Y \le 5.001\})$.

5. Consider the following joint PMF of the random vector (X, Y). Find the conditional PMF of Y given $X = 5$.

x \ y	1	2	3	4
4	0.08	0.11	0.09	0.03
5	0.04	0.12	0.21	0.05
6	0.09	0.06	0.08	0.04

6. Suppose that the number, X, of eggs laid by a bird has the Poisson(λ) distribution with $\lambda > 0$, the probability that an egg would finally develop is $p \in (0, 1)$. Under the assumption of independence of development of eggs, show here that the number, Y, of eggs surviving has the Poisson(λp) distribution. Find the conditional distribution of X given $Y = y$.

7. For the bivariate negative binomial distribution, the PMF is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{(x+y+k-1)!}{x!y!(k-1)!}\theta_1^x\theta_2^y(1 - \theta_1 - \theta_2)^k & \text{if } x \in \{0, 1, 2, \ldots\}, y \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise,} \end{cases}$$

   k is a positive integer, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, and $0 < \theta_1 + \theta_2 < 1$. Find both the conditional distributions.

8. Let us choose at random a point from the interval (0,1) and let the random variable X_1 be equal to the number which corresponds to that point. Then choose a point at random from the interval $(0, X_1)$ and let X_2 be equal to the number which corresponds to this point. Compute $P(X_1 + X_2 \ge 1)$ and find the conditional mean $E(X_1|X_2 = x_2)$ for $x_2 \in (0, 1)$. Ans: $1 - \ln 2$, $(x_2 - 1)/\ln x_2$.

9. For the bivariate beta random vector (X, Y) having PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}x^{\theta_1-1}y^{\theta_2-1}(1 - x - y)^{\theta_3-1} & \text{if } x > 0, y > 0, x + y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

   where $\theta_i > 0$, $i = 1, 2, 3$. Find both the conditional PDFs.

10. Suppose that the expected number of accidents per week at an industrial plant is four. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean two. Assume that the number of workers injured in each accident is independent of the number of accidents that occur. What is the expected number of injuries during a week? Hint: Use the conditional expectations. Ans: 8.

11. At a party n, (> 1) men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the conditional expected number of matches given that the first person did not have a match. Ans: $\frac{n-2}{n-1}$.

12. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to the mine after five hours of travel. Assuming that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until the miner reaches safety? Ans: 10

13. Independent trials, each of which is a success with probability p, are performed until there are k consecutive successes. What is the mean number of necessary trails?

14. An insurance company supposes that the number of accidents at each of it's policyholders will have in a year is Poisson distributed with the mean of the Poisson depending on the policyholder. If the Poisson mean of a randomly chosen policyholder has a gamma distribution with PDF

$$g(\lambda) = \begin{cases} \lambda e^{-\lambda} & \text{if } \lambda > 0 \\ 0 & \text{otherwise,} \end{cases}$$

what is the probability that a randomly chosen policyholder has exactly n accidents in the next year?

15. Suppose that the number of the people who visit a yoga studio each day is a Poisson random variable with mean λ. Suppose further that each person who visits is, independently, female with probability p and male with probability $1 - p$. Find the joint probability that exactly n women and m men visit the studio today.