Model Solution

Problem Set 02

(MA225)

$\boxed{Q1(a)}$

We need to check the following conditions:

(i) If $F_1(\cdot)$ is non-decreasing.

(ii) If $F_1(\cdot)$ is right continuous.

(iii) If $\lim_{x\to+\infty} F_1(x) = 1$ & $\lim_{x\to-\infty} F_1(x) = 0$.

F_1 is not right continuous at $x = 0.5$. Hence it is not a CDF. $\square$

$\boxed{Q1(b)}$

(i) As $\tan^{-1}(x)$ is increasing, F_2 is increasing.

(ii) $\tan^{-1}(x)$ is continuous $\Rightarrow$ F_2 is continuous $\Rightarrow F_2$ is right continuous

(iii) $\lim_{x\to\infty} F_2(x) = 1$ & $\lim_{x\to-\infty} F_2(x) = 0$. $\square$.

Hence, $F_2(\cdot)$ is a CDF. $\square$

$\boxed{Q1(c)}$ Yes it is a CDF.

$\boxed{Q2}$

$\boxed{Q2(a)}$ As F is a CDF, it must be right-continuous.

Consider the point $x = 3$.

$$\lim_{x\to 3+} F(x) = F(3) \Rightarrow 4c^2-9c+6 = 4 \Rightarrow 4c^2-9c+2=0$$

$$\Rightarrow (c-2)(4c-1) = 0 \Rightarrow c = 2 \text{ or } c = \frac{1}{4}.$$

For $c=2$, $\cancel{F(3)}$ $F(1) = -\frac{5}{6} < 0$. Hence $c \neq 2$.

For $c = \frac{1}{4}$, $F(x) = \frac{7-3/2}{6} = \frac{11}{12} > 0$. for $x \in [1,2)$

$$F(x) = \frac{4\times\frac{1}{4^2}+6-9\times\frac{1}{4}}{4} = \frac{16}{16} = 1 \quad \text{for } x \in [2,\infty)$$

$\square$

$\boxed{Q2(b)}$

$P(1 < X < 2) = F(2-) - F(1) = \frac{11}{12} - \frac{11}{12} = 0.$

$P(2 \le X < 3) = F(3-) - F(2-) = 1 - \frac{11}{12} = \frac{1}{12}.$

$P(0 < X \le 1) = F(1) - F(0) = \frac{11}{12} - \frac{2}{3} = \frac{1}{4}$

$P(1 \le X \le 2) = F(2) - F(1-) = 1 - \frac{2}{3} = \frac{1}{3}.$

$P(X \ge 3) = 1 - F(3-) = 0$

$P(X = 2.5) = F(2.5) - F(2.5-) = 0.$ $\square$

$\boxed{Q2(c)}$

$$P(X=1 \mid 1 \le X \le 2) = \frac{P(X=1)}{F(2)-F(1-)} = \frac{F(1)-F(1-)}{F(2)-F(1-)} = \frac{\frac{11}{12}-\frac{2}{3}}{1-2/3} = \frac{3}{4} \quad \square$$

$$P(1 \le X < 2 \mid X > 1) = \frac{F(2-)-F(1)}{1-F(1)} = \frac{\frac{11}{12}-\frac{11}{12}}{1-\frac{11}{12}} = 0 \quad \square$$

$$P(1 \le X \le 2 \mid X=1) = \frac{F(1)-F(1-)}{F(1)-F(1-)} = 1. \quad \square.$$

$\boxed{Q2(d)}$ Take $D = \{0, 1, 2\}$.

$P(X=0) = F(0) - F(0-) = \frac{2}{3}$, $P(X=1) = F(1) - F(1-) = \frac{1}{4}$, $P(X=2) = \frac{1}{12}$.

As $P(X=0) + P(X=1) + P(X=2) = 1$, X is a DRV with PMF

$$f_X(x) = \begin{cases} \frac{2}{3} & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x = 1 \\ \frac{1}{12} & \text{if } x = 2 \\ 0 & \text{o.w.} \end{cases} \quad \square$$

$\boxed{Q3}$ The set of points, where $F(\cdot)$ has jumps, is $\{1, 2, \dots\} = \mathbb{N}$.

Now, for $x \in \mathbb{N}$,

$$P(X = x) = F(x) - F(x-)$$
$$= 1 - (1-p)^{x} - 1 + (1-p)^{x-1}.$$
$$= (1-p)^{x-1} p.$$

As $\sum_{x=1}^{\infty} (1-p)^{x-1} p = 1$, X is a DRV with PMF

$$f_X(x) = \begin{cases} (1-p)^{x-1} p & \text{if } x = 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$ □

$\boxed{Q4}$ If a ball goes to B_1, B_2 or B_3, label it as a success. Then probability of success is $\frac{1}{2}$. We need to find the prob. of 6 successes out of 20 trails where trails are independent and prob. of success in each trail remains fixed.

Hence we can use Binomial dist. Let X be the random variable that denotes the no. of success out of $n = 20$ trail. The required probability is

$$P(X = 6) = \binom{20}{6} \left(\frac{1}{2}\right)^{20}.$$ □.

$\boxed{Q5}$ Let X denote the number of boys in a family that has n children. Then $X \sim Bin(n, \frac{1}{2})$. We need to find smallest n, such that

$$P(1 \le X \le n-1) \ge 0.90.$$

$$\Rightarrow \quad 1 - P(X=0) - P(X=n) \geq 0.90$$

$$\Rightarrow \quad \left(\tfrac{1}{2}\right)^n + \left(\tfrac{1}{2}\right)^n \leq 0.10$$

$$\Rightarrow \quad (n-1)\ln 2 \geq \ln 10$$

$$\Rightarrow \quad n \geq \frac{\ln 10}{\ln 2} + 1 = 4.322$$

Hence the required minimum n is 5.

$\boxed{Q6}$ The PMF of X is

$$f_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, 2, \ldots, n \\ 0 & \text{o.w.} \end{cases}$$

Consider the ratio $\dfrac{f_X(k+1)}{f_X(k)} = \dfrac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}}$

$$= \frac{\dfrac{\cancel{n!}}{(k+1)!\,(n-k-1)!}\; p}{\dfrac{\cancel{n!}}{k!\,(n-k)!}\;(1-p)}$$

$$= \frac{n-k}{k+1} \times \frac{p}{1-p}.$$

Now $f_X(k) < f_X(k+1) \iff \dfrac{f_X(k+1)}{f_X(k)} > 1$

$$\iff \frac{n-k}{k+1} \times \frac{p}{1-p} > 1$$

$$\iff np - kp > k - pk + 1 - p$$

$$\iff k < np + p - 1 = (n+1)p - 1.$$

~~Hence for $k = 0, 1, 2, \ldots, [(n+1)p] - 1$~~

~~Similarly $f_X(k) = f_X(k+1) \iff k = (n+1)p - 1$~~

Similarly $f_X(k) > f_X(k+1) \iff k > (n+1)p - 1.$

Case - I: $(n+1)p$ is not an integer

Then $f_x(0) < f_x(2) < \cdots < f_x([(n+1)p]-1) < f_x([(n+1)p])$

$> f_x([(n+1)p]+1) > \cdots > f_x(n)$.

Hence we have mode at $k = [(n+1)p]$

Case - II: $(n+1)p$ is an integer.

Then $f_x(k) = f_x(k+1) \iff k = (n+1)p - 1$.

Hence $f_x(0) < f_x(1) < \cdots < f_x((n+1)p-1) = f_x((n+1)p) > f_x((n+1)p+1)$

$\cdots > f_x(n)$.

~~Hence in~~

Thus in this case we have two modes at $k = (n+1)p-1$ and $k = (n+1)p$. □

$\boxed{Q7}$ Consider the following indefinite inegral,

$$I_k = k\binom{n}{k}\int t^{k-1}(1-t)^{n-k}\,dt$$

$$= \binom{n}{k} t^k (1-t)^{n-k} + I_{k+1}$$

$$\vdots$$

$$= \binom{n}{k} t^k (1-t)^{n-k} + \cdots + \binom{n}{n-1} t^{n-1}(1-t) + I_n$$

$$= \binom{n}{k} t^k (1-t)^{n-k} + \cdots + \binom{n}{n-1} t^{n}(1-t) + t^n.$$

Hence $P(X \geq k) = k\binom{n}{k}\int_0^p t^{k-1}(1-t)^{n-k}\,dt.$ □

Q8

$$\sum_{j=r}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

$= P(\text{At least } r \text{ success in } n \text{ trails})$

$= P(\text{At least } r \text{ success in first } (n-1) \text{ trails})$
$\times P(\text{Any result in last trail})$
$+ P(\text{Exactly } (r-1) \text{ success in first } (n-1) \text{ trails})$
$\times P(\text{Success in last trail}).$

$$= \left\{ \sum_{j=r}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \right\} \times 1 + \left\{ \binom{n-1}{r-1} p^{r-1} (1-p)^{n-1-r+1} \right\} \times p$$

$$= \sum_{j=r}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} + \binom{n-1}{r-1} p^r (1-p)^{n-r}. \quad \square$$

Q9 For $K = 0, 1, 2, \ldots$

$P(X=K) = P(\text{There are } K \text{ failures in first } (K+r-1) \text{ trails and the last trail results in a success}).$

$= P(K \text{ failures in } (r+K-1) \text{ trails}) \times P(\text{success in last tra}$

$$= \binom{r+K-1}{K} p^{r-1} (1-p)^K \times p$$

$$= \binom{r+K-1}{K} p^r (1-p)^K$$

Hence the PMF is

$$f_X(K) = \begin{cases} \binom{r+k-1}{r-1} p^r (1-p)^K & K = 0, 1, 2, \ldots \\ 0 & \text{O.W.} \end{cases} \quad \square$$

$\boxed{Q10}$ Let $X \sim \text{Geo}(p)$. The PMF of X is

$$f_X(k) = \begin{cases} (1-p)^k p & k = 0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

Hence the CDF of X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0. \\ 1-(1-p)^{k+1} & \text{if } k \le x < k+1, \; k = 0,1,2,\ldots \end{cases}$$

Now for integer $m \ge 0$, $n \ge 0$,

$$P(X \ge m+n \mid X \ge m) = \frac{P(X \ge m+n)}{P(X \ge m)}$$

$$= \frac{1 - P(X < m+n)}{1 - P(X < m)}$$

$$= \frac{1 - 1 + (1-p)^{m+n}}{1-1+(1-p)^m}$$

$$= (1-p)^n$$

$\square$

$P(X \ge n) = 1 - P(X < n) = (1-p)^n$.

$\boxed{Q11}$ Let E denote the event that the mathematician finds one of the box is empty and other box has exactly k matches.

Let EL denote the event that the mathematician finds the ~~lef~~ match box in left pocket is empty and the match box in right pocket has k matches.

Let ER denote the event that the mathematician finds the match box in right pocket is empty and the match box in left pocket has k matches.

Clearly $P(E) = P(EL) + P(ER)$.

Now we will try to calculate $P(EL)$. ~~The event EL occurs~~
Suppose that we call the selection of left pocket as success and that of right pocket as failure.
Then the event EL occurs iff there are $(n-k)$ failures before $(n+1)$st success. Hence

$$P(EL) = \binom{n+1+n-k-1}{n} \left(\frac{1}{2}\right)^{n+1+n-k}$$

$$= \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k+1}$$

Similarly $P(ER) = \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k+1}$

Hence $P(E) = \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k}$.

$\boxed{\text{Q 12}}$

$\boxed{\text{Q 12 (a)}}$ ~~The PMF of X is~~

$$\cancel{f_X(k) = \binom{N}{k} \left(\frac{a}{N}\right) \left(1-\frac{a}{N}\right)^{N-k} \quad k = 0, 1, \ldots, N}$$

The PMF of X is

$$f_X(k) = \begin{cases} \binom{n}{k} \left(\frac{a}{N}\right)^k \left(1 - \frac{a}{N}\right)^{n-k} & k = 0, 1, \ldots, n \\ 0 & \text{o.w.} \end{cases}$$

$\boxed{\text{Q 12 (b)}}$ The PMF of X is

$$f_X(k) = \begin{cases} \dfrac{\binom{a}{k}\binom{N-a}{n-k}}{\binom{N}{n}} & \text{if } \max\{0, n+a-N\} \le k \le \min\{a, n\} \text{ and } k \text{ is an integer} \\ 0 & \text{o.w.} \end{cases}$$

Let us denote S_i = Success at i^{th} ~~trait~~ draw $\quad i = 1, 2, \ldots, n$

f_i = Failure at i^{th} draw.

If the draws are to be independent, the following condition need to be hold true

$$P(S_1 \cap S_2) = P(S_1)\, P(S_2). \quad \text{———} \quad (*)$$

Now, $P(S_1) = \dfrac{a}{N}$, $\quad P(S_2) = P(S_2|S_1)\, P(S_1) + P(S_2|F_1)\, P(F_1)$

$$= \frac{a}{N}.$$

$$P(S_1 \cap S_2) = P(S_2|S_1)\, P(S_1) = \frac{(a-1)a}{N(N-1)}.$$

Clearly $(*)$ is not true. Hence the draws are not indep. ∎

$\boxed{Q13}$ There are four possibility

	d	r
d	dd	~~rd~~ dr
r	rd	rr

Assuming that they are equally likely and calling $\{dd, dr, rd\}$ as success, the success prob. is $\frac{3}{4}$.

Hence the required probability is $\binom{4}{1}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) = \frac{27}{64}$. ∎.