Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 24 (October 21, 2019)

Stochastic Processes

Def: Let T be a countable set of time index. A function $X : \mathcal{S} \times T \to \mathbb{R}$ is called a stochastic process.

Remark: As T is countable, we can take $T = \{0, 1, \ldots\}$, and we generally denote a stochastic process by $\{X_n : n \geq 0\}$.

Example 1: No. of students obtaining AA grade in MA225 in each year.
Example 2: Price of gold.
Example 3: Minimum temperature of each day.

Def: The set of all possible values taken by a stochastic process $\{X_n : n \geq 0\}$ is called state space.

Remark: We will only consider atmost countable state spaces and it will be denoted by $\{0, 1, \ldots\}$.
Remark: If $X_n = i$, then the process in said to be in state i at time n.

Markov Chain

Def: (Markov chain) A stochastic process $\{X_n : n \geq 0\}$ is said to be a Markov chain (MC) if

$$P\left(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right) = P\left(X_{n+1} = j | X_n = i\right)$$

for all states $i_0, i_1, \ldots, i_{n-1}, i$ and j and all $n \geq 0$.

Remark: For a MC, the conditional distribution of X_{n+1} (future) given the past $X_0, X_1, \ldots, X_{n-1}$ and present X_n depends only on the present X_n and not on past $X_0, X_1, \ldots, X_{n-1}$.

Time-homogeneous Markov Chain

Def: (Time-homogeneous MC) A MC $\{X_n : n \geq 0\}$ is said to be a time-homogeneous MC if

$$P\left(X_{n+1} = j | X_n = i\right) = P\left(X_n = j | X_{n-1} = i\right) = p_{ij}$$

for all $n \geq 1$.

Remark: We will only consider time-homogeneous MC in this course.
Remark: $p_{ij} \geq 0$ for all i and j.
Remark: $\sum_{j=0}^{\infty} p_{ij} = 1$.
Remark: p_{ij} is called one-step transition probability from state i to state j.
Remark: The matrix $P = ((p_{ij}))_{i,j \geq 0}$ is called one-step transition probability matrix.

Example 4: Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather condition. Suppose that if it is raining today, then it will rain tomorrow with probability α. If it is not raining today, then it will rain tomorrow with probability β.

Example 5: Suppose that the chance of rain tomorrow depends on previous weather conditions through last two days. Assume

$$P\left(\text{Rain tomorrow}|\text{rain for past two days}\right) = 0.7$$

$$P\left(\text{Rain tomorrow}|\text{rain today, no rain yesterday}\right) = 0.5$$

$$P\left(\text{Rain tomorrow}|\text{rain yesterday, no rain today}\right) = 0.4$$

$$P\left(\text{Rain tomorrow}|\text{no rain for past two days}\right) = 0.2$$

Example 6: Consider a communication system that transmit the digits 0 and 1. Each digit, transmitted must pass several stages, at each stage of which there is a probability p that the digit entered will remain unchanged when it leaves.

Example 7: A MC whose state space is given by set of integers is said to be simple random walk(SRW) if for some $0 < p < 1$, $p_{i,i+1} = p = 1 - p_{i,i-1}$. It is said to be simple symmetric random walk(SSRW) if $p = 1/2$.

Example 8: [A Gambling Model] Consider a gambler who at each play of the game either wins Re. 1 with probability p or losses Re. 1 with probability $1 - p$. Suppose that the gambler quits plays either when he goes broke or he attains a fortune of Rs. N.

Chapman-Kolmogorov Equations

Theorem: Consider a MC having state space $\{0, 1, \ldots\}$ and one-step transition probabilities p_{ij} for $i, j = 0, 1, \ldots$. Let us define

$$p_{ij}^{(n)} = P\left(X_n = j | X_0 = i\right) = P\left(X_{n+k} = j | X_k = i\right).$$

The Chapman-Kolmogorov equations are given by

$$p_{ij}^{(m+n)} = \sum_{k=0}^{\infty} p_{ik}^{(m)} p_{kj}^{(n)}$$

for all $m, n \geq 0$ and all $i, j = 0, 1, \ldots$.

Remark: If we denote n-step transition probability matrix by $P^{(n)}$, then

$$P^{(n+m)} = P^{(n)} P^{(m)} \implies P^{(n)} = P^n.$$