Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 21 (September 27, 2019)

Bivariate normal

Def: A two dimensional random vector $\mathbf{X} = (X,\ Y)$ is said to have a bivariate normal distribution if $aX + bY$ is a univariate normal for all $(a,\ b) \in \mathbb{R}^2 \setminus (0, 0)$.

Theorem: If $\boldsymbol{\mu} = E(\mathbf{X})$ and Σ is the variance-covariance matrix of $\mathbf{X}$, then for any fixed $\mathbf{u} = (a,\ b) \in \mathbb{R}^2 \setminus (0, 0)$, $\mathbf{u}'\mathbf{X} \sim N(\mathbf{u}'\mu, \mathbf{u}'\Sigma\mathbf{u})$.

Theorem: Let $\mathbf{X}$ be a bivariate normal random vector, then $M_{\mathbf{X}}(\mathbf{t}) = e^{\mathbf{t}'\mu + \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}}$ for all $\mathbf{t} \in \mathbb{R}^2$.

Remark: Thus the bivariate normal distribution is completely specified by the mean vector $\boldsymbol{\mu}$ and the variance-covariance matrix Σ. We may therefore denote a bivariate normal distribution by $N_2(\boldsymbol{\mu},\ \Sigma)$.

Def: A two dimensional random vector $\mathbf{X}$ is said to have a bivariate normal distribution if it can be expressed in the form $\mathbf{X} = \boldsymbol{\mu} + A\mathbf{Y}$, where A is a 2×2 matrix of real numbers, $\mathbf{Y} = (Y_1, Y_2)$ and Y_1 and Y_2 are i.i.d $N(0, 1)$. In this case $E(\mathbf{X}) = \boldsymbol{\mu}$ and $\Sigma = AA'$.

Theorem: If $\mathbf{X} \sim N_2(\boldsymbol{\mu}, \Sigma)$, then $X \sim N(\mu_1, \sigma_{11})$ and $Y \sim N(\mu_2, \sigma_{22})$.

Remark: The converse of the above theorem is not true.

Remark: If $Cov(X, Y) = 0$, then X and Y are independent.

Theorem: Let $\mathbf{X} \sim N_2(\boldsymbol{\mu}, \Sigma)$ be such that Σ is invertible, then, for all $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{X}$ has a joint PDF given by

$$
\begin{aligned}
f(\mathbf{x}) &= \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{A(x,y,\mu_x,\mu_y,\sigma_x,\sigma_y,\rho)},
\end{aligned}
$$

where

$$
A = -\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}.
$$