Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 29 (November 04, 2019)

Def: For $i \in S$, $N_n(i) = \#\{t : 0 \leq t \leq n, X_t = i\}$ is the number of visitis to state i during $\{0, 1, \ldots, n\}$.

Def: For $i \in S$, define $L_n(i) = \frac{N_n(i)}{n+1}$. Then $\{L_n(i) : i \in S\}$ is called emprical distribution at time n.

Theorem: Fix the state $i \in S$. Then

1. i is transient iff $\sum_{k=0}^{\infty} p_{ii}^{(k)} < \infty$.
2. i is null recurrent iff $\sum_{k=0}^{\infty} p_{ii}^{(k)} = \infty$ and
$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ii}^{(k)} = 0.$$
3. i is positive recurrent iff $\lim\limits_{n\to\infty} \dfrac{1}{n+1} \sum\limits_{k=0}^{n} p_{ii}^{(k)} > 0$.

Example 1: For a simple symetric random walk, the state 0 is null recurrent.

Stationary Distribution

Def: A vector $\{\pi_i\}_{i\in S}$ is called a stationary distribution for a MC with transition probability matrix $P = ((p_{ij}))$ if $\pi_i \geq 0$ for all $i \in S$, $\sum_{i\in S} \pi_i = 1$ and $\sum_{j\in S} \pi_j p_{ji} = \pi_i$ for all $i \in S$.

Remark: $\sum_{j\in S} \pi_j p_{ji} = \pi_i$ for all $i \in S \implies \underline{\pi} P = \underline{\pi}$. Thus a stationary distribution is a left eigen vector corrosponding to eigen value 1 and $\underline{\pi}\,\underline{1} = 1$.

Remark: $P(X_n = i) = \pi_i$ if $P(X_0 = i) = \pi_i$ for all $i \in S$.

Remark: If S is finite, then stationary distribution exists.

Theorem: Let $\{X_n\}$ be a MC having stationary distribution π, then $\pi(i) > 0 \implies i$ is positive recurrent.

Theorem: Let $\{X_n\}$ be a MC, then a stationary distribution exists iff there is at least one positive recurrent state.

Corollary: Simple random walk does not admit a stationary distribution.

Corollary: A finite state MC has atleast one positive recurrent state.
Remark: In general, stationary distribution may not exist. If it exist, it may not be unique.