Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 32 (November 11, 2019)

Counting Process

Def: A stochastic process $\{N(t) : t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of "events" that occur by time t.

Remark: A counting process possesses the following properties.

1. $\{N(t) : t \geq 0\}$ is a continuous time stochastic process.
2. $N(t) \geq 0$ for all t.
3. $N(t)$ is integered values.
4. If $s < t$, then $N(s) \leq N(t)$.
5. For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

Examples

Example 1: $N(t) =$ The number of persons who enter a particular store upto time t — Counting process.

Example 2: $N(t) =$ Total number of people who were born upto time t — Counting process.

Example 3: $N(t) =$ the number of persons in a store at a time t — Not a counting process.

Independent and Stationary Increment

Def: A counting process $\{N(t) : t \geq 0\}$ is said to have independent increments if the number of events that occur in disjoint time intervals are independent, i.e., for any $t_1 < t_2 < t_3 < t_4$, the random variables $N(t_2) - N(t_1)$, $N(t_3) - N(t_2)$ and $N(t_4) - N(t_3)$ are independent.

Def: A counting process $\{N(t)\}$ is said to have stationary increment if the distribution of $N(t + s) - N(t)$ depends only on s for all $t \geq 0$.

Poisson Process

Def: A counting process $\{N(t) : t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if

1. $N(0) = 0$ with probability 1.
2. it has independent increments.
3. it has stationary increments .
4. $N(t)$ has $Poi(\lambda t)$ distribution.

Remark: The definition fixes all finite dimensional distributions of the stochastic process.

Remark: Fix any $T > 0$. Define a new process $N_T(\cdot)$ by $N_T(t) = N(T + t) - N(T)$. Then N_T is again a Poisson process with rate λ. Thus a Poisson process probabilistically restarts itself at any point of time (Markov property).

Interarrival times

Def: Let T_1 denote the time of occurrence of the first event. For $n \geq 2$, let T_n denote the time elapsed between $(n-1)st$ and nth event. Then $\{T_n\}_{n \geq 1}$ is called the sequence of interarrival times.

Theorem: T_ns are i.i.d. $Exp(\lambda)$ random variables.

Corollary: If S_n denotes the time of the nth event then S_n has $Gamma(n, \lambda)$ distribution.

Example

Example 4: Suppose that people immigrate into a territory according to Poisson process with rate $\lambda = 1$ per day.
a) What is the expected time until the 10th immigrant arrives?
b) What is the probability that the elapsed time between 10th and 11th arrival exceeds 2 days?