Probability Theory and Random Processes (MA225)

Lecture Slides

Lecture 10 (August 22, 2019)

Moment Inequality

Theorem: Let X be a RV and $g : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $E\left(g(|X|)\right)$ is finite. Then for any $c > 0$ with $g(c) > 0$, then

$$P(|X| \geq c) \leq \frac{E\left(g(|X|)\right)}{g(c)}.$$

Corollary: (Markov Inequality) Let X be a RV with $E\left(|X|^r\right) < \infty$ for some $r > 0$. Then for any $c > 0$,

$$P(|X| \geq c) \leq \frac{E\left(|X|^r\right)}{c^r}.$$

Corollary: (Chebyshev Inequality) Let X be a RV with $E(X^2) < \infty$. Let us denote $\mu = E(X)$ and $\sigma^2 = Var(X)$. Then for any $k > 0$,

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}.$$

Example 1: (Chebyshev inequality is tight) Let X be a DRV with PMF

$$f_X(x) = \begin{cases} \frac{1}{8} & \text{if } x = -1, 1 \\ \frac{3}{4} & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $E(X) = 0$ and $E(X^2) = \frac{1}{4}$.
Using Chebyshev inequality, $P(|X| \geq 1) \leq \frac{1}{4}$.
Using PMF, $P(|X| \geq 1) = \frac{1}{4}$.

Gamma Function

Def: For $\alpha > 0$, define

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Theorem:

1. The functional equation $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ holds for $\alpha > 0$.
2. $\Gamma(n + 1) = n!$ for $n = 1, 2, \ldots$.

Beta Function

Def: For $\alpha > 0$ and $\beta > 0$, define

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx.$$

Theorem: For $\alpha > 0$ and $\beta > 0$,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Corollary: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Gamma Distribution

A RV X is said to have a gamma distribution if the PDF of the RV is given by

$$f_X(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. We will use the notation $X \sim Gamma(\alpha, \beta)$.

Remark: $E(X) = \frac{\alpha}{\beta}$, $Var(X) = \frac{\alpha}{\beta^2}$.

Remark: $M_X(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha}$ for $t < \beta$.

Beta Distribution

A RV X is said to have a beta distribution if the PDF of the RV is given by

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. We will use the notation $X \sim Beta(\alpha, \beta)$.

Remark: $E(X) = \frac{\alpha}{\alpha+\beta}$, $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.