PROBABILITY THEORY AND RANDOM PROCESSES (MA225)

LECTURE SLIDES

Lecture 18 (September 23, 2019)

Modes of Convergence

Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Let X be a random variable defined on the same probability space $(\mathcal{S}, \mathcal{F}, P)$.

Def: (**Almost sure convergence**) We say that X_n converges almost surely or with probability 1 to a random variable X if

$$P(\omega : X_n(\omega) \to X(\omega)) = 1\,.$$

Example 1: Let $\mathcal{S} = [0,1], \mathcal{F} = \mathcal{B}([0,1])$ and P be the uniform measure. Define $X_n = 1_{[0,\frac{1}{n}]}$. Then X_n converges almost surely (w. p. 1) to the zero random variable.

Theorem: Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Suppose $X_n \to X$ w. p. 1 and $Y_n \to Y$ w. p. 1. Then

- $X_n + Y_n \to X + Y$ w. p. 1.
- $X_n Y_n \to XY$ w. p. 1.
- $f(X_n) \to f(X)$ w. p. 1, for any f continuous.

Def: (**Convergence in r^{th} mean**) We say that X_n converges in r^{th} mean to a random variable X if

$$E|X_n - X|^r \to 0\,.$$

Example 2: Let $\mathcal{S} = [0,1], \mathcal{F} = \mathcal{B}([0,1])$ and P be the uniform measure. Define $X_n = 1_{[0,\frac{1}{n}]}$. Then X_n converges in r^{th} mean to the zero random variable.

Theorem: Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$.

- If $X_n \to X$ in r^{th} mean and $Y_n \to Y$ in r^{th} mean, then $X_n + Y_n \to X + Y$ in r^{th} mean.
- If $X_n \to X$ in r^{th} mean then $f(X_n) \to f(X)$ in r^{th} mean, for any f bounded continuous.

Def: (**Convergence in probability**) We say that X_n converges in probability to a random variable X if for any $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0\,.$$

Example 3: Let $\mathcal{S} = [0,1], \mathcal{F} = \mathcal{B}([0,1])$ and P be the uniform measure. Define $X_n = n1_{[0,\frac{1}{n}]}$. Then X_n converges in probability to the zero random variable.

Theorem: Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Suppose $X_n \to X$ in probability and $Y_n \to Y$ in probability. Then

- $X_n + Y_n \to X + Y$ in probability.
- $X_n Y_n \to XY$ in probability.
- $f(X_n) \to f(X)$ in probability, for any f continuous.

Def: (**Convergence in distribution**) We say that X_n converges in distribution to a random variable X if

$$F_n(x) \to F(x)$$

for all x where F is continuous and where F_ns are the distribution functions of X_ns and F is the distribution function of X.

Remark: Unlike the first three modes of convergence, here X_ns can be defined on different probability spaces. We are only interested in the distribution functions. This flexibility makes this mode of convergence very useful.

Example 4: Suppose X_ns are random variables such that $P(X_n = 1/n) = 1$. Then X_n converges in distribution to the zero random variable.

Theorem: Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Suppose $X_n \to X$ in distribution and $Y_n \to c$ in probability for some constant c. Then

- $X_n + Y_n \to X + c$ in distribution.
- $X_n Y_n \to cX$ in distribution.
- $f(X_n) \to f(X)$ in distribution, for any f continuous.

Important: If X_n converges to X in distribution and Y_n converges to Y in distribution then $X_n + Y_n$ may not converge to $X + Y$ in distribution. Same for product.