★ Randomized Algorithms:

① Application by Verifying Polynomial Identities.

$$f(x) = \prod_{i=1}^{n} (x - a_i)$$

$$g(x) = \sum_{i=0}^{n} c_i x^i$$

$$f(x) \stackrel{?}{\equiv} g(x)$$

→ Algo chooses integer r uniformly at random in range $\{1, \ldots, 100n\} : S$

∴ $f(r)$ in $O(n)$ (Multiply n numbers)

$g(r)$ in $O(n)$ (Horner's Rule)

If $f(x) \equiv g(x)$ & $f(r) = g(r)$ } Correct ans

$f(x) \not\equiv g(x)$ & $f(r) \neq g(r)$ }

If $f(x) \not\equiv g(x)$ & $f(r) = g(r)$ Incorrect ans

⇒ $f(x) - g(x) = 0$ has a root r

where deg is atmost n

∴ ∃ atmost n roots in S

∴ Error $\leq 1/100$

Run Algo for K times. Error $\leq \left(\frac{1}{100}\right)^k$

→ Monte Carlo Algo.

② Verifying Matrix Multiplication (All operations done modulo 2)

→ $A_{n\times n}$ $B_{n\times n}$ $C_{n\times n}$ (all entries are either 0 or 1)

$\bar{r} = (r_1, r_2, \ldots, r_n) \in \{0,1\}^n$

∴ Prob to pick 1 random vector $1/2^n$

$(A(Br)) \overset{?}{=} Cr \quad \rightarrow \Theta(n^2)$

$X \quad Y$

If $AB = C$ and $ABr = Cr$ } Correct

$AB \neq C$ and $ABr \neq Cr$ }

If $AB \neq C$ and $ABr = Cr$ then ans is wrong.

∴ $D = AB - C \neq 0$

∴ ∃ atleast one component of D not equal to zero

WLOG assume $d_{11} \neq 0$

$X = Y$

$\Rightarrow (AB)r = Cr$

$\Rightarrow (AB - C)r = 0$

$\Rightarrow Dr = 0$

$\therefore d_{11}r_1 + d_{12}r_2 + \ldots + d_{1n}r_n = 0$

$\Rightarrow r_1 = -\dfrac{(d_{12}r_2 + \ldots + d_{1n}r_n)}{d_{11}}$

* $(r_2, \ldots, r_n)$ is determined already then we have only one choice of r_1 for which the equality holds out of the 2 choices of r_1 i.e. 0 & 1.

$\therefore$ error $\leq 1/2$

after k runs error $\leq (1/2)^k$: Monte Carlo

→ Fingerprinting

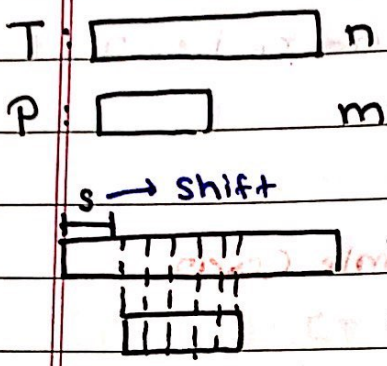
Try only at one sample point rather than at every point.

* Abundance of Witness
* Above Idea (Principle of deffered decisions)

If ∃ several random variables such as r_i of F some are set at one point in algo & other left random or deffered until some further point in analysis.

* LAS Vegas: Randomized algo that always returns a correct result. But running time may vary between executions.
* Monte Carlo: A randomized algo that terminates in polynomial time, but might produce erroneous result.

③ Pattern Matching (Only Binary strings considered)



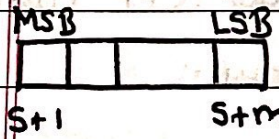
: First occurrance of P in T

★ Worst case T.C. : $O((n-m+1)m)$

→ Valid shift

Invalid Shift

★ fingerprint function



$$f(T[s+1 \ldots s+m]) = T[s+m] \times 2^0 + T[s+m-1] \times 2^1 + \vdots + T[s+1] \times 2^{m-1}$$

★ Horner's Rule:

$T[s+m] + 2(\ldots 2(T[s+2] + 2T[s+1]))$

∴ it takes $O(m)$ time to compute the fingerprint

∴ $f(T[s+1 \ldots s+m]) = t_s$

$f(P[1 \ldots m]) = p$

$f(T[s+2 \ldots s+m+1]) = t_{s+1}$

$\therefore t_{s+1} = 2(t_s - 2^{m-1} T[s+1]) + T[s+m+1]$

∴ Given t_s we can compute t_{s+1} in O(1) time

* Precompute powers of 2 till m

→ s is a valid shift ⇔ $t_s = p$

Hence to compute:

p : O(m)

$t_0, t_1 \ldots t_{n-m}$: O(n-m+1)

∴ Compute p : O(m) & t_0 : O(m)

& $t_1 \ldots t_{n-m}$: O(n-m)

∴ T.C. O(2m + (n-m)) = O(n+m)

* ∵ t_s & p can be ↑ calculate them w.r.t modulo q ∈ Prime

* Whenever string match we compare the substring with P

Fingerprints

∴ Worst case T.C. = O(2m + (n-m) + (n-m+1)m)

≈ O(nm)

Max possible no of checks

* Avg case:

f(P) = k where $k \in [0, 1, 2, \ldots, q-1]$

* Supurious hits:

Finger prints match but strings donot.

∴ Probability for ts to match with p modulo q is $1/q$

∴ Avg case: $O(2m + n - m + (\frac{n-m+1}{q})m)$

if $q > m$: $O(n+m)$

$Pr(f(T[s+1..s+m]) = f(P) \mid T[s+1..s+m] \neq P)$

$= Pr(ts - p \text{ is divisible by } q \mid T[s+1..s+m] \neq P)$

$T[s+1] \ldots T[s+m]$: max value $2^m - 1$ (# CP)

∴ $ts - p < 2^m$

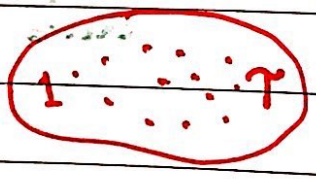
⇒ $ts - p$ has almost m prime divisor

If > m prime divisors:

$ts - p > 2^{>m} > 2^m$ (#)

* Choose q uniformly from $[1, T]$

∴ no. of prime nos $\frac{T}{\ln T}$



Only almost m of these can divide $ts - p$

∴ $Pr(ts \equiv p \pmod q \mid T[s+1..s+m] \neq P) = Pr(Z)$

$$\therefore Pr(Z) \leq \frac{m}{T/\ln T} = \frac{m \ln T}{T}$$

$$\Rightarrow T = n^2 m \lg(n^2 m)$$

$\ln T \approx \log T$ (# only a const factor)

$$\therefore Pr(Z) \leq \frac{m \lg(n^2 m \lg(n^2 m))}{n^2 m \lg(n^2 m)}$$

$$\leq \frac{\lg(\alpha \lg(\alpha))}{n^2 \lg(\alpha)} \quad \text{where } \alpha = n^2 m$$

$$\lim_{\alpha \to \infty} \frac{\lg(\alpha \lg(\alpha))}{\lg(\alpha)} = \frac{\frac{1}{\alpha \lg(\alpha)}(1 + \lg \alpha)}{1/\alpha} = 1 + \frac{1}{\lg(\alpha)} \approx 1$$

$$\therefore Pr(Z) \leq 1/n^2$$

∴ Monte Carlo : $O(n+m)$

Error $\leq 1/n^2$

→ LAS VEGAS:

Whenever two fingerprint match explict check by naive

$$O(2m + (n-m))\left(1 - 1/n^2\right) + O(nm)\left(\frac{1}{n^2}\right) : \text{Expected time}$$

$$= O\left(n+m\left(\frac{n^2-1}{n^2}\right) + \frac{m}{n}\right)$$

$$\approx O(n + m + m/n)$$

✱	TC
Naive	$O((n-m+1)m)$
W.C.	$O(nm)$
Avg Case	$O(n+m)$
Monte Carlo	$O(n+m)$ Error $\leq 1/n^2$
Las Vegas	$O(n+m+m/n)$: Expected time

→ Fingerprinting Types:

① Matrix Mul. } fix field then choose point of eval for fingerprint.
Poly Check }

② Pattern matching } After fixing point of evaluation we fix 1 and 7 (i.e. field)

⑤ Determine whether a odd number is prime or not.

Naive: $O(\sqrt{n}\ div(n)) = O(2^{\lg n/2}\ div(n))$

↓ large n

↓ Psuedo polynomial time algo

★ Fermat's little theorem:

→ If n is prime then $\forall\ 1 \le a < n$

$a^{n-1} \equiv 1 \pmod n$

→ Contrapositive: $\exists\ a \in 1 \le a < n \ni a^{n-1} \not\equiv 1 \pmod n$

then n is not a prime

→ Possibility of error one-sided → If no. of maybe k then error k-sided

i.e. said prime but is composite (maybe)

said composite must be composite

★ Psuedoprime to base a:

A composite integer n is psuedoprime to base a whenever $a^{n-1} \equiv 1 \pmod n$ ○ $a > 1$

★ Absolute psuedoprime (Carmichael number)

A composite integer n is called absolute psuedoprime

then ○ ~~~~ $a^{n-1} \equiv 1 \pmod n$

$\forall\ a \in [1, n) \ni \gcd(a, n) = 1$

* If composite $n \in \mathbb{Z}$ is not an absolute pseudoprime then $\exists\ b \in [1,n)$ s.t. $\gcd(b,n)=1$ & $b^{n-1} \not\equiv 1 \pmod n$

* Note ∃ only 646 carmichael numbers below 10^9. so we can safely assume n is not a carmichael number

∴ $\exists\ b \in [1,n)$ s.t. $\gcd(b,n)=1$ & $b^{n-1} \not\equiv 1 \pmod n$

n is composite & not a carmichael number

then

If $a \in [1,n)$ & $a^{n-1} \equiv 1 \pmod n$

i.e. a passes fermat's test

then $\exists\ x \in [1,n)$ s.t. $x^{n-1} \not\equiv 1 \pmod n$

(Here $x \equiv ab \pmod n$)

∴ no. of numbers fail the test ≥ no. of no.s pass the test

∴ Pr(Passing the test | n is composite & not carmichael)

$\leq 1/2$

→ Abundance of Witness paradigm applied to reduce the error probability to $(1/2)^k$

∴ T.C $O(k \log(n)\ mult(n)) \Rightarrow$ weakly polynomial.

→ Miller Rabin error $(1/4)^k$

→ AKS algo T.C. $O((\log n)^5)$

→ Pattern matching revisited

q choosen from [1, T)

No. of prime numbers ≈ $T/\ln T$

∴ probability to choose a prime $\frac{T/\ln T}{T} = \frac{1}{\ln T}$

∴ we need $\ln T$ iterations to find a prime

(i.e. choose a no. put it back if not prime)

∴ T.C. Monte Carlo : $O(\ln T + n + m)$: Expected Time

→ Monte Carlo algo usually donot have Expected Times but in this case it does.