Chapter 3:

INTEGRATION

Indefinite Integrals and Antiderivative

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative (from its known rate of change).

to find a function F from its derivative f. If such a function F exists, it is called an *antiderivative* of f.

Finding Antiderivatives

DEFINITION Antiderivative
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Finding Antiderivatives

Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(c) $h(x) = \dfrac{1}{x} + 2e^{2x}$

Solution

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

(c) $H(x) = \ln|x| + e^{2x}$

> If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is
>
> $$F(x) + C$$
>
> where C is an arbitrary constant.

EXAMPLE 2 Finding a Particular Antiderivative

Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

Solution Since the derivative of $-\cos x$ is $\sin x$, the general antiderivative

$$F(x) = -\cos x + C$$

gives all the antiderivatives of $f(x)$. The condition $F(0) = 3$ determines a specific value for C. Substituting $x = 0$ into $F(x) = -\cos x + C$ gives

$$F(0) = -\cos 0 + C = -1 + C.$$

Since $F(0) = 3$, solving for C gives $C = 4$. So

$$F(x) = -\cos x + 4$$

The Power Rule in Integral Form

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n\, du,$$

obtained by treating the dx's as differentials that cancel. We are thus led to the following rule.

> If u is any differentiable function, then
>
> $$\int u^n\, du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1,\ n \text{ any number}). \qquad (1)$$

TABLE 3.2 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative		Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C,\quad n \neq -1$	**8.**	e^{kx}	$\frac{1}{k}e^{kx} + C$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.**	$\frac{1}{x}$	$\ln\lvert x\rvert + C,\quad x \neq 0$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$	**10.**	$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.**	$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.**	$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C,\quad kx > 1$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.**	a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C,\quad a > 0,\ a \neq 1$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$			

Example 3:

Find the following integration:

$$\int x^5 + x^3 + \sqrt{x} \; dx$$

Solution: $f(x) = x^5 + x^3 + \sqrt{x}$

$$F(x) = \frac{x^6}{6} + \frac{x^4}{4} + \frac{x^{3/2}}{3/2}$$

3.2 Substitution:

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

EXAMPLE 1 Using the Power Rule

$$\begin{aligned}
\int \sqrt{1+y^2}\cdot 2y\,dy &= \int \sqrt{u}\cdot\left(\frac{du}{dy}\right)dy && \text{Let } u = 1 + y^2,\ du/dy = 2y \\
&= \int u^{1/2}\,du \\
&= \frac{u^{(1/2)+1}}{(1/2)+1} + C && \text{Integrate, using Eq. (1) with } n = 1/2. \\
&= \frac{2}{3}u^{3/2} + C && \text{Simpler form} \\
&= \frac{2}{3}(1+y^2)^{3/2} + C && \text{Replace } u \text{ by } 1 + y^2.
\end{aligned}$$

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

EXAMPLE 2 Adjusting the Integrand by a Constant

$$\begin{aligned}
\int \sqrt{4t-1}\,dt &= \int \frac{1}{4}\cdot\sqrt{4t-1}\cdot 4\,dt \\
&= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right)dt && \text{Let } u = 4t-1,\ du/dt = 4. \\
&= \frac{1}{4}\int u^{1/2}\,du && \text{With the } 1/4 \text{ out front, the integral is now in standard form.} \\
&= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C && \text{Integrate, using Eq. (1) with } n = 1/2. \\
&= \frac{1}{6}u^{3/2} + C && \text{Simpler form} \\
&= \frac{1}{6}(4t-1)^{3/2} + C && \text{Replace } u \text{ by } 4t-1.
\end{aligned}$$

1. Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral

$$\int f(u)\,du.$$

2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

EXAMPLE 3 Using Substitution

$$\int \cos(7\theta + 5)\, d\theta = \int \cos u \cdot \frac{1}{7}\, du$$

Let $u = 7\theta + 5$, $du = 7\, d\theta$, $(1/7)\, du = d\theta$.

$$= \frac{1}{7}\int \cos u\, du$$

With the (1/7) out front, the integral is now in standard form.

$$= \frac{1}{7}\sin u + C$$

Integrate with respect to u, Table 4.2.

$$= \frac{1}{7}\sin(7\theta + 5) + C$$

Replace u by $7\theta + 5$.

EXAMPLE 4 Using Substitution

$$\int x^2 e^{x^3}\, dx = \int e^{x^3} \cdot x^2\, dx$$

$$= \int e^u \cdot \frac{1}{3}\, du$$

Let $u = x^3$, $du = 3x^2\, dx$, $(1/3)\, du = x^2\, dx$.

$$= \frac{1}{3}\int e^u\, du$$

$$= \frac{1}{3} e^u + C$$

Integrate with respect to u.

$$= \frac{1}{3} e^{x^3} + C$$

Replace u by x^3.

EXAMPLE 5 Multiplying by a Form of 1

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\, dx}{e^{2x} + 1}$$

Multiply by $(e^x/e^x) = 1$.

$$= \int \frac{du}{u^2 + 1}$$

Let $u = e^x$, $u^2 = e^{2x}$, $du = e^x\, dx$.

$$= \tan^{-1} u + C$$

Integrate with respect to u.

$$= \tan^{-1}(e^x) + C$$

Replace u by e^x.

EXAMPLE 6 Simplifying the Integrand

$$\begin{aligned}
\int \frac{\ln x^2}{x}\,dx &= \int \frac{2\ln x}{x}\,dx && \text{Power Rule for logarithms} \\
&= \int 2\ln x \cdot \frac{1}{x}\,dx \\
&= \int 2u\,du && \text{Let } u = \ln x,\ du = (1/x)\,dx. \\
&= u^2 + C && \text{Integrate with respect to } u. \\
&= (\ln x)^2 + C && \text{Replace } u \text{ by } \ln x.
\end{aligned}$$

EXAMPLE 7 Using Identities and Substitution

$$\begin{aligned}
\int \frac{1}{\cos^2 2x}\,dx &= \int \sec^2 2x\,dx && \frac{1}{\cos 2x} = \sec 2x \\
&= \int \sec^2 u \cdot \frac{1}{2}\,du && u = 2x,\ du = 2\,dx,\ dx = (1/2)\,du \\
&= \frac{1}{2}\int \sec^2 u\,du \\
&= \frac{1}{2}\tan u + C && \frac{d}{du}\tan u = \sec^2 u \\
&= \frac{1}{2}\tan 2x + C && u = 2x
\end{aligned}$$

The Integrals of $\sin^2 x$ and $\cos^2 x$

Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can using the substitution rule. Here is an example giving the integral formulas for $\sin^2 x$ and $\cos^2 x$ which arise frequently in applications.

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EXAMPLE 9

(a) $$\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$

As in part (a), but with a sign change

3.3 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\, dx.$$

Important note:

$$\int f(x)g(x)\, dx \text{ is not equal to } \int f(x)\, dx \cdot \int g(x)\, dx.$$

If f and g are differentiable functions of x, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\, dx = \int [f'(x)g(x) + f(x)g'(x)]\, dx$$

$$\boxed{\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx \qquad (1)}$$