1.3 Derivatives of Trigonometric Functions

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 1.4 with the *angle sum identity* for the sine:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Sine angle sum identity} \\
&= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 5a and Theorem 7, Section 1.4} \\
&= \cos x.
\end{aligned}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 Derivatives Involving the Sine

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x.$$

(b) $y = e^x \sin x$:

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x)\sin x \qquad \text{Product Rule}$$
$$= e^x \cos x + e^x \sin x$$
$$= e^x(\cos x + \sin x).$$

(c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x\cos x - \sin x}{x^2}.$$

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Derivative of the Cosine Function

With the help of the angle sum formula for the cosine,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we have

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} \qquad \text{Derivative definition}$$
$$= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \qquad \text{Cosine angle sum identity}$$
$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x.$$

Example 5a and Theorem 7, Section 1.4

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE 2 Derivatives Involving the Cosine

(a) $y = 5e^x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x)$$ Sum Rule

$$= 5e^x - \sin x.$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$ Product Rule

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x.$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$ Quotient Rule

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$ $\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{1 - \sin x}.$$

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE 4:

Find $d(\tan x)/dx$.

Solution

$$\frac{d}{dx}\left(\tan x\right) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}\left(\sin x\right) - \sin x \frac{d}{dx}\left(\cos x\right)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Solution

$$y = \sec x$$

$$y' = \sec x \tan x$$

EXAMPLE 5:

Find y'' if $y = \sec x$.

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}\left(\tan x\right) + \tan x \frac{d}{dx}\left(\sec x\right) \qquad \text{Product Rule}$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x \qquad \blacksquare$$

Excerxise 1.3

Find $\frac{dy}{dx}$ in the following problems:

1. $y = \frac{3}{x} + 5 \sin x$
2. $y = (\sec x + \tan x)(\sec x - \tan x)$
3. $y = \frac{\cos x}{1 + \sin x}$
4. $y = x^2 \sin x + 2x \cos x - 2 \sin x$

5. Find y'' if

   a. $y = \csc x$. **b.** $y = \sec x$.

1.4 The Chain Rule and Parametric Equations

Chain Rule:

if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE 1: Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \qquad \frac{dx}{du} \text{ evaluated at } u \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

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EXAMPLE 2: Differentiating from the Outside In

Differentiate $\sin(x^2 + e^x)$ with respect to x.

Solution

$$\frac{d}{dx}\sin(\underbrace{x^2 + e^x}_{\text{inside}}) = \cos(\underbrace{x^2 + e^x}_{\text{inside}})\cdot\underbrace{(2x + e^x)}_{\text{derivative of}}$$

EXAMPLE 3: Applying the Chain Rule to the Exponential Function

Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x. \quad \blacksquare$$

Slopes of Parametrized Curves

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{2}$$

EXAMPLE 4: Moving Along the Ellipse $x^2/a^2 + y^2/b^2 = 1$

Describe the motion of a particle whose position $P(x, y)$ at time t is given by

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

Find the line tangent to the curve at the point $(a/\sqrt{2}, b/\sqrt{2})$, where $t = \pi/4$. (The constants a and b are both positive.)

Solution We find a Cartesian equation for the particle's coordinates by eliminating t between the equations

$$\cos t = \frac{x}{a}, \qquad \sin t = \frac{y}{b}.$$

The identity $\cos^2 t + \sin^2 t = 1$ yields

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The particle's coordinates (x, y) satisfy the equation $(x^2/a^2) + (y^2/b^2) = 1$, so the particle moves along this ellipse. When $t = 0$, the particle's coordinates are

$$x = a\cos(0) = a, \qquad y = b\sin(0) = 0,$$

so the motion starts at $(a, 0)$. As t increases, the particle rises and moves toward the left, moving counterclockwise. It traverses the ellipse once, returning to its starting position $(a, 0)$ at $t = 2\pi$.

The slope of the tangent line to the ellipse when $t = \pi/4$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/4} \qquad \text{Eq. (2)}$$

$$= \left.\frac{b\cos t}{-a\sin t}\right|_{t=\pi/4}$$

$$= \frac{b/\sqrt{2}}{-a/\sqrt{2}} = -\frac{b}{a}.$$

The tangent line is

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$

$$y = \frac{b}{\sqrt{2}} - \frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$

or

$$y = -\frac{b}{a}x + \sqrt{2}b.$$ ■

Excerxise 1.4

Find $\frac{dy}{dx}$ **in the following problems:**

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = \tan u, \quad u = 10x - 5$
3. $y = \sin u, \quad u = x - \cos x$
4. $y = 2u^3, \quad u = 8x - 1$

Write the function in the form $y = f(u)$ **and** $u = g(x)$ **then use chain rule to find** $\frac{dy}{dx}$

5. $y = (2x + 1)^5$
6. $y = \sec(\tan x)$
7. $y = e^{2x/3}$

Find $\frac{dy}{dx}$ **for the following functions using parametric differentiation**

8. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \leq t \leq \pi$
9. $x = 4 \sin t, \quad y = 5 \cos t, \quad 0 \leq t \leq 2\pi$
10. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$