1.5 Implicit Differentiation

Implicitly Defined Functions

We begin with an example.

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.37). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?

The answer is yes. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

The Chain Rule gives $\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{2y}.$$

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$ ■

EXAMPLE 2 Slope of a Circle at a Point

Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x. Rather it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 1.36). The point $(3, -4)$ lies on the graph of y_2, so we can find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $\left.-\frac{x}{y}\right|_{(3,\,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

EXAMPLE 3 Differentiating Implicitly

Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 1.39).

Solution

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy) \qquad \text{Differentiate both sides with respect to } x \ldots$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy) \qquad \ldots \text{treating } y \text{ as a function of } x \text{ and using the Chain Rule.}$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right) \qquad \text{Treat } xy \text{ as a product.}$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y \qquad \text{Collect terms with } dy/dx \ldots$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy \qquad \ldots \text{and factor out } dy/dx.$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy} \qquad \text{Solve for } dy/dx \text{ by dividing.}$$

Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables x and y, not just the independent variable x. ■

Excerxise 1.5

Find $\frac{dy}{dx}$ by using implicit differentiation in the following problems:

1. $x^2y + xy^2 = 6$
2. $x = \tan y$
3. $x + \sin y = xy$
4. $x^3 - xy + y^3 = 1$

1.6 Derivatives of Inverse Functions and Logarithms

Derivative of the Natural Logarithm Function

$$
\begin{aligned}
y &= \ln x \\
e^y &= x && \text{Inverse function relationship} \\
\frac{d}{dx}(e^y) &= \frac{d}{dx}(x) && \text{Differentiate implicitly} \\
e^y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x} && e^y = x
\end{aligned}
$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

EXAMPLE 1: Derivatives of Natural Logarithms

(a) $\frac{d}{dx} \ln 2x = \frac{1}{2x} \frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$

(b) $\frac{d}{dx} \ln (x^2 + 3) = \frac{1}{x^2 + 3} \cdot \frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3} \cdot 2x = \frac{2x}{x^2 + 3}.$

The Derivative of a^u

We start with the equation $a^x = e^{\ln (a^x)} = e^{x \ln a}$:

$$
\begin{aligned}
\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} &= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) && \frac{d}{dx} e^u = e^u \frac{du}{dx} \\
&= a^x \ln a.
\end{aligned}
$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a.$$

The Derivative of $\log_a u$

$$\log_a x = \frac{\ln x}{\ln a}.$$

Taking derivatives, we have

$$\begin{aligned}
\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\
&= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x \qquad \text{Since } \ln a \text{ is a constant} \\
&= \frac{1}{\ln a}\cdot\frac{1}{x} \\
&= \frac{1}{x\ln a}.
\end{aligned}$$

Inverse Trigonometric Functions

1. **Deravitive of $sin^{-1}x$**

$$y = \sin^{-1} x$$

$$\begin{aligned}
\sin y &= x && y = \sin^{-1} x \Leftrightarrow \sin y = x \\
\frac{d}{dx}(\sin y) &= 1 && \text{Derivative of both sides with respect to } x \\
\cos y\,\frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\cos y} && \text{We can divide because } \cos y > 0 \text{ for } -\pi/2 < y < \pi/2. \\
&= \frac{1}{\sqrt{1-x^2}} && \cos y = \sqrt{1-\sin^2 y}
\end{aligned}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1+u^2}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1+u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

Excerxise 1.5:

Find the first derivative of y for the following problems:

1. $y = \ln x^3$
2. $y = \dfrac{\ln t}{t}$
3. $y = \ln(\ln x)$
4. $y = 2^x$
5. $y = \log_2 5\theta$
6. $y = \cos^{-1}(x^2)$
7. $y = x \sin^{-1} x$
8. $y = \sec^{-1} 5s$
9. $y = \csc^{-1} \dfrac{x}{2}$
10. $y = \cot^{-1} \sqrt{t}$
11. $y = \tan^{-1}(\ln x)$
12. $y = \ln(x^2 + 4) - x \tan^{-1}\left(\dfrac{x}{2}\right)$