1.6 L'Hôpital's Rule

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

EXAMPLE 1 Using L'Hôpital's Rule

(a) $\displaystyle \lim_{x \to 0} \frac{3x - \sin x}{x} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$

(b) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \left.\frac{\frac{1}{2\sqrt{1+x}}}{1}\right|_{x=0} = \frac{1}{2}$

EXAMPLE 5 Working with the Indeterminate Form ∞/∞

Find

(a) $\displaystyle \lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\displaystyle \lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\displaystyle \lim_{x \to \infty} \frac{e^x}{x^2}.$

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

(c) $\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$ ■

EXAMPLE 6 Working with the Indeterminate Form $\infty \cdot 0$

Find

(a) $\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+} \sqrt{x} \ln x$

Solution

(a) $\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right)$ $\infty \cdot 0$

$= \lim_{h\to 0^+}\left(\frac{1}{h} \sin h\right) = 1$ Let $h = 1/x$.

(b) $\lim_{x\to 0^+} \sqrt{x} \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}}$ ∞/∞

$= \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}} = \lim_{x\to 0^+}\left(-2\sqrt{x}\right) = 0$

EXAMPLE 8 Working with the Indeterminate Form 1^∞

Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+} (1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x} \ln (1 + x),$$

l' Hôpital's Rule now applies to give

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$= \frac{1}{1} = 1.$$

Therefore,

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$$

EXAMPLE 9 An Indeterminant Form ∞^0

Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1/x}{1}$$

$$= \frac{0}{1} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$$

Excecise 1.6

use l'Hôpital's Rule to evaluate the limit.

1. $\lim_{x \to 2} \frac{x-2}{x^2-4}$

2. $\lim_{x \to 0} \frac{\sin 5x}{x}$

3. $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

4. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$

5. $\lim_{t \to 0} \frac{t \sin t}{1-\cos t}$

6. $\lim_{x \to 0^+} \left(1+\frac{1}{x}\right)^x$

7.	$\lim_{x \to 0^+} x^x$
8.	$\lim_{x \to 0} \frac{3^x-1}{2^x-1}$