2.2 The Mean Value Theorem

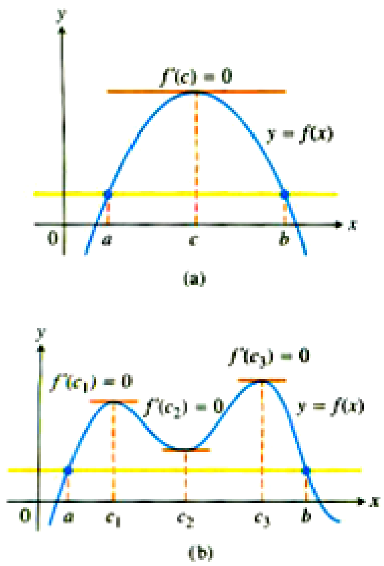


FIGURE 2.4 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

We know that constant functions have zero derivatives, but could there be a complicated function, with many terms, the derivatives of which all cancel to give zero? What is the relationship between two functions that have identical derivatives over an interval? What we are really asking here is what functions can have a particular *kind* of derivative. These and many other questions we study in this chapter are answered by applying the Mean Value Theorem. To arrive at this theorem we first need Rolle's Theorem.

Rolle's Theorem

Drawing the graph of a function gives strong geometric evidence that between any two points where a differentiable function crosses a horizontal line there is at least one point on the curve where the tangent is horizontal (Figure 2.4). More precisely, we have the following theorem.

THEOREM 3 Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

THEOREM 4 The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

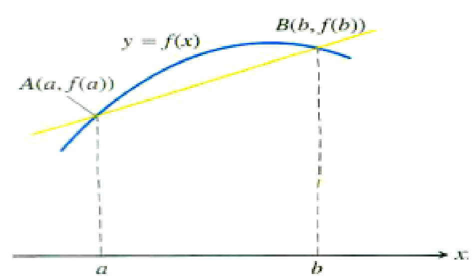


FIGURE 2.5 The graph of f and the chord AB over the interval $[a, b]$.

EXAMPLE 3

The function $f(x) = x^2$ (Figure 2.6) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this (exceptional) case we can identify c by solving the equation $2c = 2$ to get $c = 1$. ■

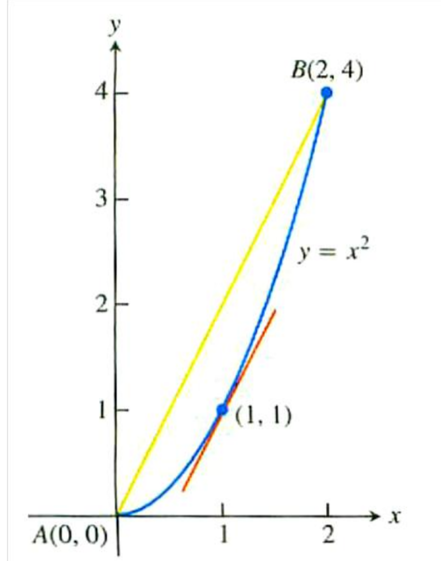
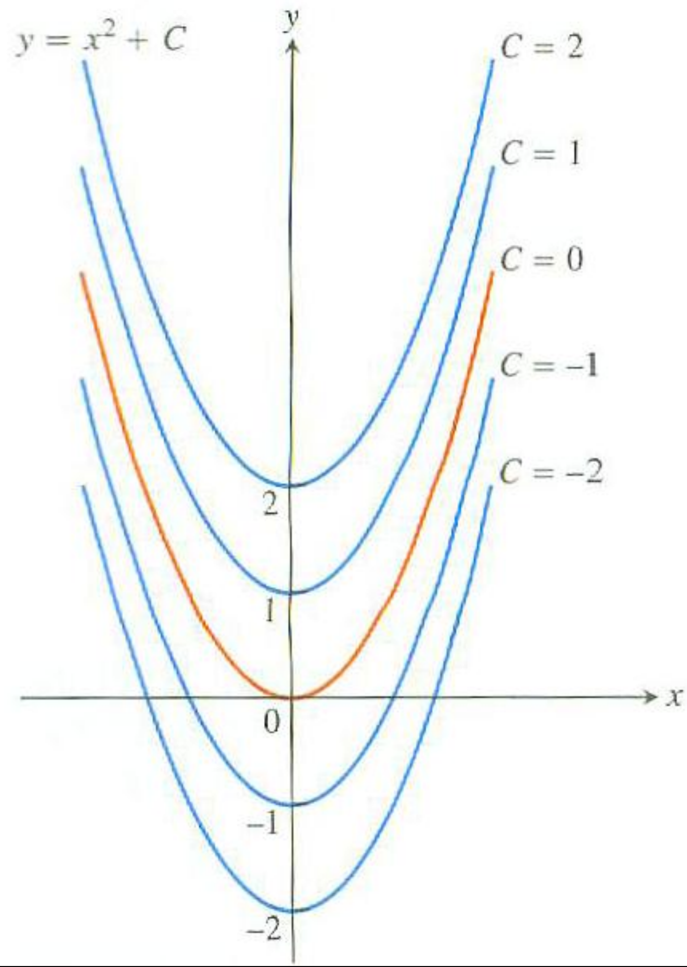


FIGURE 2.6 As we find in Example 3, $c = 1$ is where the tangent is parallel to the chord.

COROLLARY 1 Functions with Zero Derivatives Are Constant
If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 Functions with the Same Derivative Differ by a Constant
If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant on (a, b).

FIGURE 2.6 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.



EXAMPLE 5

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$ for some constant C. The value of C can be determined from the condition that $f(0) = 2$ (the graph of f passes through $(0, 2)$):

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$. ■

Exercise 2.2

- **Show that the function $f(x)$ satisfies Rolle's theorem on the given interval (find the values of *c* satisfying the theorem) :**

    1. $f(x) = x^2 - 8\,x + 15$ **, on [3, 5]**
    2. $f(x) = \cos x$ **, on** $[\frac{\pi}{2}, \frac{3\pi}{2}]$

- **Show that the function $f(x)$ satisfies the Mean value theorem on the given interval (find the values of *c* satisfying the theorem):**

    1. $f(x) = x^2 - x$ **, on** $[-3, 5]$
    2. $f(x) = x^3 + x - 4$ **, on** $[-1, 2]$