Chapter 6
Sampling Distributions

Sampling Distributions

Q: A company manufactures 100 Ohms resistors. A sample of 40 resistors from the assembly line ~~[illegible]~~ is found to have a mean of 105 Ohms. How likely is the population mean (the mean of the probability density function) to be 100 Ohms?

A: In questions like this, we need to make inferences about the population mean based on the sample mean. To do this, we need to know the probability distribution of the sample mean!!!

Defn: The probability distribution of a statistic is called a sampling distribution.

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Sampling Distribution of Means

Given a sample with n observations: $X_1, X_2, \ldots, X_n$

Sample mean $\bar{X} = \frac{1}{n}\left(X_1 + X_2 + \cdots + X_n\right)$

$\bar{X}$ itself is a random variable, in fact, it is a linear combination of random variables $X_1 \ldots X_n$.

Now assume the observations were taken from a population with mean μ and standard deviation σ.

What is the mean of $\bar{X}$?

$$\mu_{\bar{X}} = \frac{1}{n}\underbrace{\left(\mu + \mu + \cdots + \mu\right)}_{n \text{ terms}} = \mu$$

What is the ~~[illegible]~~ variance of $\bar{X}$?

$$\sigma_{\bar{X}}^2 = \frac{1}{n^2}\left(\sigma^2 + \sigma^2 + \cdots + \sigma^2\right) = \frac{\sigma^2}{n}$$

Note: observations are independent of each other which allows us to use the simplified formulas from ~~[illegible]~~ Chapter 4.

What do these results mean?

(1) If I take <u>many</u> samples from the population, each with n observations, <u>the mean of the sample means</u> will equal the population mean.

(2) If I take <u>many</u> samples from the population, each with n observations, <u>the standard deviation of the sample means</u> will equal $\sqrt{\frac{\sigma^2}{n}}$ (equivalently the variance of the sample means will be $\frac{\sigma^2}{n}$) where σ is the population ~~mean~~ standard deviation.

<u>Central Limit Theorem</u>: If $\bar{X}$ is the mean of a random sample of size n taken from a population with mean μ and variance σ^2, then the limiting form of the probability distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

as $n \to \infty$ is the standard normal distribution $n(z; 0, 1)$

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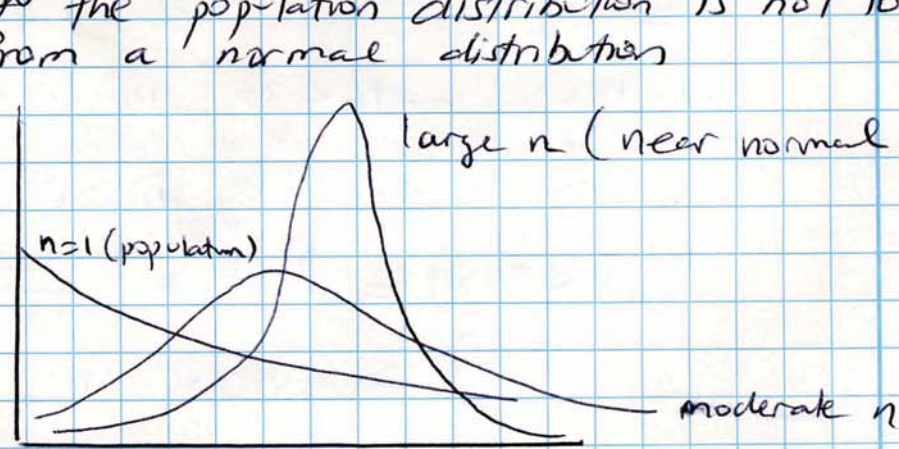
Notice that regardless of the population distribution $f(x)$, the central limit theorem states that the distribution of the sample mean is a normal distribution!!!

The normal distribution result of the central limit theorem is good if $n \geq 30$. For $n < 30$ it is only good if the population distribution is not too different from a normal distribution

Distribution of $\bar{X}$



Example: Manufacture resistors. Population mean 100 Ohms population standard deviation 20 Ohms. Find the probability that a random sample of 50 resistors will have a mean resistance of 101 Ohms or larger.

Solution: $\mu = 100 \quad \sigma = 20$

$$\mu_{\bar{X}} = 100 \qquad \sigma_{\bar{X}} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{50}}$$

$$= 2.83$$

since $n = 50 > 30$ we can use the Central limit theorem (even though we don't know the population distribution) to say that the sample mean $\bar{X}$ has a normal distribution with $\mu_{\bar{X}} = 100$ and $\sigma_{\bar{X}}^2 = 2.83$

Then $P(\bar{X} > 101) = P\left(Z > \frac{101 - 100}{2.83}\right)$

$= P(Z > 0.35)$

$= 1 - P(Z < 0.35)$

$= 1 - 0.6368$ From table A.3

$= 0.3632$

Example: Manufacture light bulbs. Length of life approximately normally distributed with mean 800 hours and a standard deviation 40 hours. Find the probability that a random sample of 16 bulbs will have an average life of less than 775 hours.

Solution: $\mu = 800 \quad \sigma = 40, \quad \mu_{\bar{X}} = 800 \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{40}{\sqrt{16}} = 10$

Even though $n = 16 < 30$, the Central limit theorem can be used because it is stated that the population distribution is approximately normal.

$$P(\bar{X} < 775) = P\left(Z < \frac{775 - 800}{10}\right) = P(Z < -2.5)$$

$= 0.0062$ from Table A.3.

Inference on the population mean: Given the probability we calculated in the previous example, we could ask the question how likely is the population mean to be really 800 hours? Not very likely since the prob value was so small. We will learn to answer this question in a formal manner when we discuss hypothesis testing.

Sampling Distribution of Difference of Two Means

Sometimes we are interested in comparing two populations, i.e is one manufacturing process better than the other (according to longer life expectancy or similar critrion)

Population 1	Population 2
μ_1, σ_1	μ_2, σ_2
$\downarrow$	$\downarrow$
Sample 1 with n_1 observations	Sample 2 with n_2 observations
$\bar{X}_1 : \mu_{\bar{X}_1} = \mu_1$	$\bar{X}_2 : \mu_{\bar{X}_2} = \mu_2$
$\sigma_{\bar{X}_1} = \sigma_1/\sqrt{n_1}$	$\sigma_{\bar{X}_2} = \sigma_2/\sqrt{n_2}$

What can we say about the sampling distribution of $\bar{X}_1 - \bar{X}_2$?

From what we learned in Chapter 4: $\mu_{\bar{X}_1 - \bar{X}_2} = \mu_{\bar{X}_1} - \mu_{\bar{X}_2}$

$$= \mu_1 - \mu_2$$

$$\sigma^2_{\bar{X}_1 - \bar{X}_2} = \sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2}$$

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Generalization of central limit theorem

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$ is a

standard normal variable. Good when both n_1 and $n_2 \geq 30$ or when the population dists. are approximately normal.

Example (8.9 in textbook) lifetime of product

Population 1: $\mu_1 = 6.5$ $\sigma_1 = 0.9$

Population 2: $\mu_2 = 6.0$ $\sigma_2 = 0.8$

Sample with $n_1 = 36$ observations from Population 1

" " $n_2 = 49$ observations " " 2

What is the probability that a random sample of 36 TVs from population 1 will have a mean life that is at least 1 year longer than the sample of 49 TVs from population 2?

$$P(\bar{X}_1 - \bar{X}_2 \geq 1.0) = ?$$

Soln: Since both n_1 and $n_2 \geq 30$ the sampling distribution of $\bar{X}_1 - \bar{X}_2$ will be approximately normal with

$$\mu_{\bar{X}_1 - \bar{X}_2} = \mu_{\bar{X}_1} - \mu_{\bar{X}_2} = \mu_1 - \mu_2 = 6.5 - 6.0 = 0.5$$

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\sigma_{\bar{X}_1}^2 + \sigma_{\bar{X}_2}^2}$$

$$= \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} = \sqrt{\frac{0.81}{36} + \frac{0.64}{49}}$$

$$= 0.189$$

Therefore

$$P(\bar{X}_1 - \bar{X}_2 \geq 1.0) = P\left(Z \geq \frac{1.0 - 0.5}{0.189}\right)$$

$$= 1 - P(Z < 2.65)$$

↓ Table A.3

$$= 1 - 0.996 = 0.004$$

Example (similar to example 8.8 textbook)

An electrical company is evaluating two different production methods for long lasting light bulbs. Call these methods A and B. The population distribution and the means for A and B are unknown, but we know that the standard deviation for both is 50 hours. Assuming that the mean life-time for both methods is the same, find $P(\bar{X}_A - \bar{X}_B \geq 15)$ when a random sample of $n_A = 100$ is taken from population A and a random sample of $n_B = 100$ is taken " " B.

Soln: Since both n_A and $n_B \geq \overset{30}{\cancel{100}}$, the sampling distribution for $\bar{X}_A - \bar{X}_B$ will be approximately normal with

$$\mu_{\bar{X}_A} - \mu_{\bar{X}_B} = \mu_A - \mu_B = 0 \quad \left[\mu_A, \mu_B \text{ unknown but assumed to be equal} \right]$$

$$\sigma_{\bar{X}_A - \bar{X}_B} = \sqrt{\frac{50^2}{100} + \frac{50^2}{100}} = \sqrt{50} = 7.07$$

Therefore $P(\bar{X}_1 - \bar{X}_2 \geq 15) = P\left(Z \geq \frac{15 - 0}{7.07}\right)$

$$= 1 - P(Z < 2.12) = 1 - 0.983 \text{ (Table A.3)}$$

$$= 0.017.$$

This low probability suggests that the mean lifetime of the two populations likely are not the same based on these observations. Most likely A has a longer lifetime. Later we will use hypothesis testing to determine this.

Exercise 8.18 text-book

$$f(x) = \begin{cases} 1/3, & x = 2, 4, 6 \\ 0, & \text{elsewhere} \end{cases}$$

Random sample size $n = 54$

$$P(4.15 < \bar{X} < 4.35) = ?$$

First find population mean and standard deviation.

$$\mu = \sum_x x f(x) = 2 \times \frac{1}{3} + 4 \times \frac{1}{3} + 6 \times \frac{1}{3} = 4$$

$$\sigma^2 = E[X^2] - \mu^2 = \left(\sum_x x^2 f(x)\right) - 16$$

$$= 4 \times \frac{1}{3} + 16 \times \frac{1}{3} + 36 \times \frac{1}{3} - 16$$

$$= \frac{56}{3} - 16 = \frac{56 - 48}{3} = 8/3$$

Now the sample mean and variance

$$\mu_{\bar{X}} = \mu = 4 \quad \sigma_{\bar{X}}^2 = \frac{\sigma^2}{n} = \frac{8/3}{54} = \frac{4}{81}$$

$$\sigma_{\bar{X}} = 2/9$$

Finally $P(4.15 < \bar{X} < 4.35) = P\left(\frac{4.15 - 4}{2/9} < Z < \frac{4.35 - 4}{2/9}\right)$

$$= P(0.68 < Z < 1.58) = 0.9429 - 0.7517 \quad \left[\text{Table A.3}\right]$$

$$= 0.1912$$

Sampling Distribution of S^2

Remember S^2 is sample variance. This completly different than $\sigma_{\bar{X}}^2$ so don't get confused.

Given a sample: 5, 11, 9, 5, 10, 15, 6, 10, 5, 10

$$\bar{X} = \frac{1}{10}(5+11+9+5+10+15+6+10+5+10) = 8.4$$

and
$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{9}\left[(5-8.6)^2 + (11-8.6)^2 + \cdots + (10-8.6)^2\right]$$
$$= 10.933$$

Now we will talk about the probability distribution for S^2.
Before we were talking about " " " for $\bar{X}$.

Sampling distribution of S^2 is used in studying variability.

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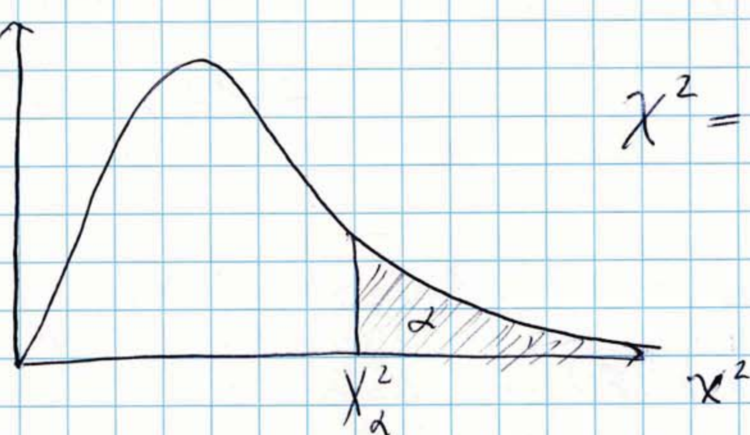
* A random variable of the form $Y = \sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2}$ has a χ^2 (chi-squared) distribution with n degrees of freedom if the X_i are normally distributed.

* The mathematical expression for the χ^2 distribution is complicated (Section 6.8 textbook), the shape looks like



$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

χ_α^2 represents the χ^2 value above which we find an area of α. (Table A.5)

Sample = $X_1, ---, X_n$ population mean μ, standard dev σ

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}\left((X_i-\bar{X})+(\bar{X}-\mu)\right)^2$$

$$= \sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{i=1}^{n}(\bar{X}-\mu)^2 + \sum_{i=1}^{n} 2(\bar{X}-\mu)(X_i-\bar{X})$$

$$= \underbrace{\sum_{i=1}^{n}(X_i-\bar{X})^2}_{\text{this is } (n-1)S^2} + n(\bar{X}-\mu)^2 + \cancel{2(\bar{X}-\mu)\underbrace{\sum_{i=1}^{n}(X_i-\bar{X})}_{\emptyset \text{ since } \sum_{i=1}^{n} X_i = \bar{X}n}}$$

Divide both sides by σ^2

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{(n-1)S^2}{\sigma^2} + \frac{(\bar{X}-\mu)^2}{\sigma^2/n}$$

$$\underbrace{\frac{(n-1)S^2}{\sigma^2}}_{\substack{\chi^2 \text{ with} \\ n-1 \text{ degrees} \\ \text{of freedom.}}} = \underbrace{\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2}_{\substack{\chi^2 \text{ with } n \\ \text{degrees of} \\ \text{freedom}}} - \underbrace{\frac{1}{\sigma^2/n}(\bar{X}-\mu)^2}_{\substack{\chi^2 \text{ with} \\ 1 \text{ degrees of} \\ \text{freedom}}}$$

<u>Theorem</u>. If S^2 is the variance of a random sample of size n taken from a normal population having the variance σ^2, then the statistic

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{\sigma^2}$$

has a χ^2 distribution with $v = n-1$ degrees of freedom.

Example: (Example 8-10 textbook)

Car batteries supposed to last 3 years on average with standard deviation of 1 year. If a sample of 5 batteries are found to have life-times: 1.9, 2.4, 3.0, 3.5 and 4.2 years, is the claim that the population standard deviation is 1 year valid? (Assume the battery lifetimes are normally distributed)

Solution

first find S^2 value.

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \quad \text{where} \quad \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$\bar{X} = \frac{1}{5}(1.9 + 2.4 + 3.0 + 3.5 + 4.2) = 3.0$$

$$S^2 = \frac{1}{4}\left[(1.9-3)^2 + \dots + (4.2-3)^2\right] = 0.815$$

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{4 \times 0.815}{1} = 3.26$$

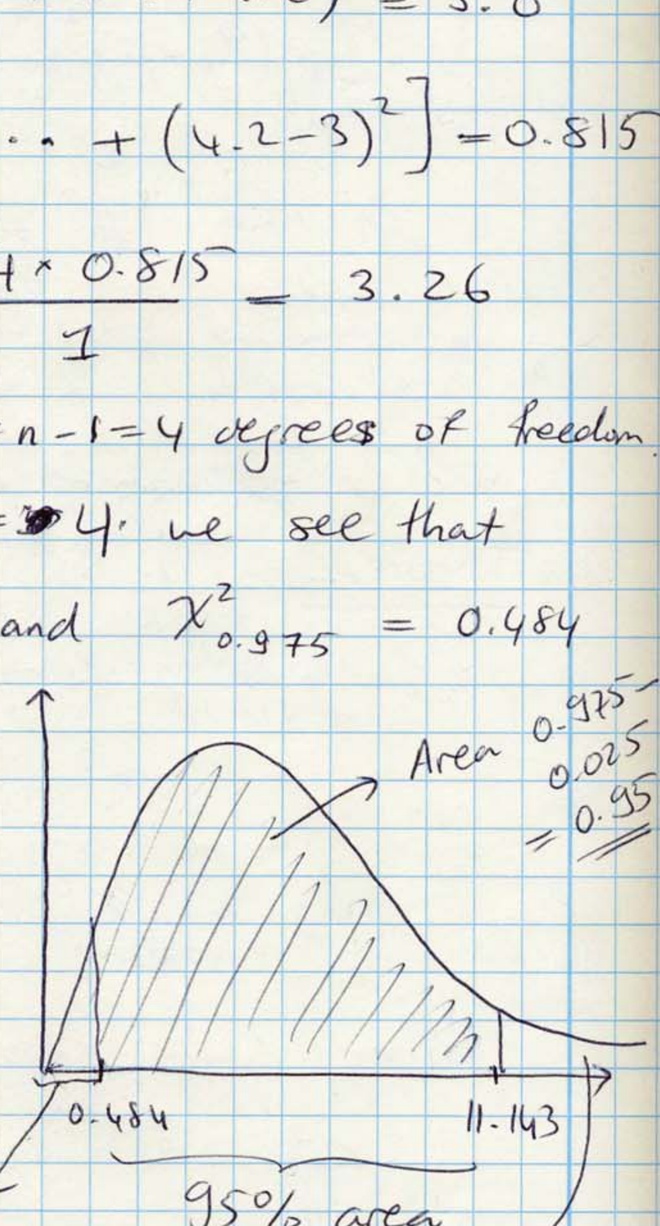
Since $n = 5$, χ^2 has $v = n-1 = 4$ degrees of freedom.

From Table A.5 row $v = 4$. we see that

$$\chi^2_{0.025} = 11.143 \quad \text{and} \quad \chi^2_{0.975} = 0.484$$

This means that 0.95 area under the χ^2 curve falls between 0.484 and 11.143.

Since our χ^2 value for this sample $\chi^2 = 3.26$ falls within this range it is reasonable.



If the computed χ^2 fell here it would be a sign that $\sigma = 1$ is likely wrong.

t - Distribution

We saw how to use the normal distribution to compute probabilities about the sample mean when we know (or can compute) the population mean and variance.
In some experiments we might know the population mean but not the variance. In these cases the sample variance can be substituted for the population variance and the resulting sampling distribution is called the t-distribution.

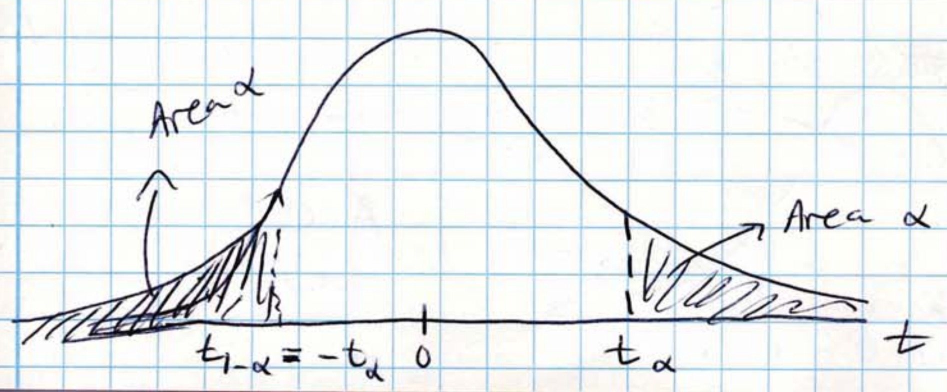
Remember $\underbrace{Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}}$	$T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$
σ known	σ unknown
Normal distribution $n \geq 30$ or any n but population distribution known to be close to normal	T-distribution. For $n \geq 30$ T-distribution close to normal distribution However, if n small S^2 can fluctuate significantly from sample to sample

Let $X_1, X_2, \ldots, X_n$ be independent random variables that are normally distributed with mean μ. Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

then $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ has a t-distribution with

$v = n-1$ degrees of freedom.

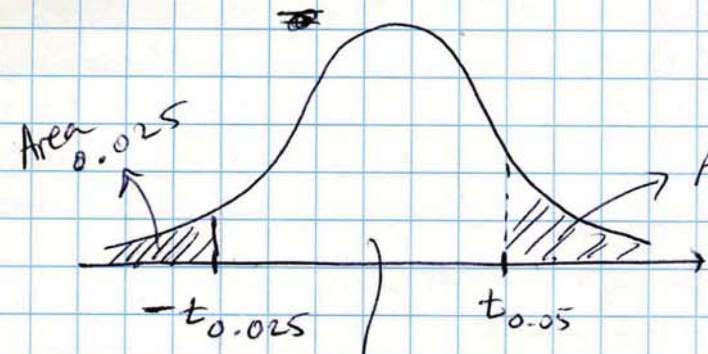


$$t_{0.95} = -t_{0.05}$$

$$t_{0.99} = -t_{0.01}$$

$$t_{1-\alpha} = -t_\alpha$$

<u>Example</u>: $P(-t_{0.025} < T < t_{0.05})$



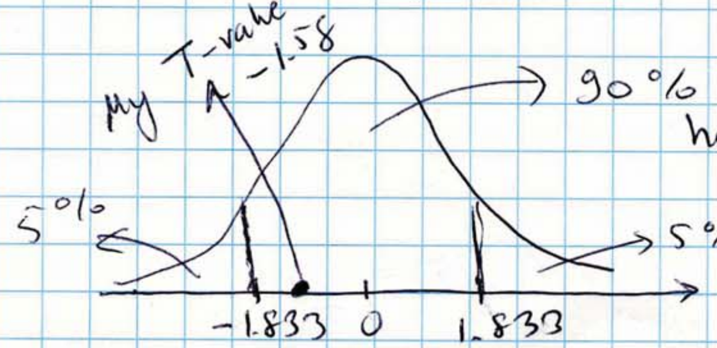
$$\text{Area} = 1 - (0.05 + 0.025) = 0.925$$

<u>Note</u>: t-distribution symmetric about mean of 0. In other words, the t-value leaving an area of $1-\alpha$ to the right and therefore an area of α to the left is equal to the negative t-value that leaves an area α to the right.

<u>Example</u>: An ISP claims that the mean connection speed provided to my house is 5 Mbps. To check this claim I measure the connection speed at 10 different occasions and find a mean connection speed of 4.5 Mbps and a sample standard deviation of 1.0 Mbps. I will be satisfied if I can show that the ISP's claim is true if the t-value for my sample falls between $-t_{0.05}$ and $t_{0.05}$. Should I accept their claim?

$$T = \frac{4.5 - 5}{1.0/\sqrt{10}} = -1.58$$

For $v = n-1 = 9$
$t_{0.05} = 1.833$
so
$-t_{0.05} = -1.833$



ACCEPT THE CLAIM.

Example : ISP claims mean connection speed of 5 MBPS. On 8 occasions I measure: 4.4, 5.5, 4.1, 5.1, 4.2, 2.6, 4.1, 3.8

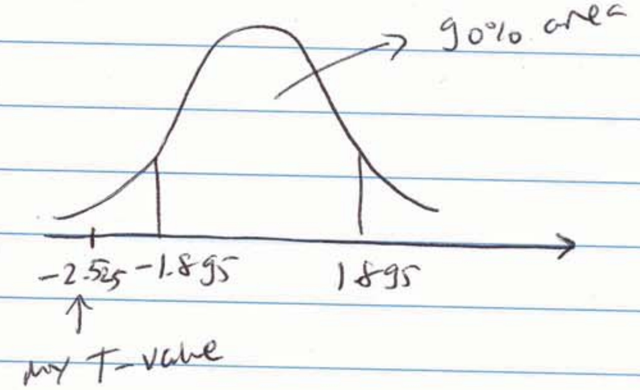
Does the t-value fall between $-t_{0.05}$ and $t_{0.05}$?

$\bar{X} = 4.225 \quad S = 0.8681$

$$T = \frac{4.225 - 5}{0.8681/\sqrt{8}} = -2.525$$

Since $n=8$, $V=7$ so we use that row in Table A.4.

$t_{0.05} = 1.895$ so $-t_{0.05} = -1.895$



Falls outside acceptable range. In fact, it even falls outside the range $-t_{0.025}$ to $t_{0.025}$ (95% area)

Note: in these two examples we assumed that the connection speeds were normally distributed which is necessary for using the t-distribution.

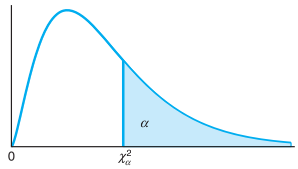


Table A.5 Critical Values of the Chi-Squared Distribution

	α									
v	**0.995**	**0.99**	**0.98**	**0.975**	**0.95**	**0.90**	**0.80**	**0.75**	**0.70**	**0.50**
1	0.0^4393	0.0^3157	0.0^3628	0.0^3982	0.00393	0.0158	0.0642	0.102	0.148	0.455
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	0.575	0.713	1.386
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	1.213	1.424	2.366
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	1.923	2.195	3.357
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	2.675	3.000	4.351
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
8	1.344	1.647	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.737	7.267	9.342
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
13	3.565	4.107	4.765	5.009	5.892	7.041	8.634	9.299	9.926	12.340
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.037	11.721	14.339
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	11.912	12.624	15.338
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	12.792	13.531	16.338
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	13.675	14.440	17.338
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	14.562	15.352	18.338
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	15.452	16.266	19.337
21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	16.344	17.182	20.337
22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	17.240	18.101	21.337
23	9.260	10.196	11.293	11.689	13.091	14.848	17.187	18.137	19.021	22.337
24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	19.037	19.943	23.337
25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	19.939	20.867	24.337
26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	20.843	21.792	25.336
27	11.808	12.878	14.125	14.573	16.151	18.114	20.703	21.749	22.719	26.336
28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	22.657	23.647	27.336
29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	23.567	24.577	28.336
30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	24.478	25.508	29.336
40	20.707	22.164	23.838	24.433	26.509	29.051	32.345	33.66	34.872	39.335
50	27.991	29.707	31.664	32.357	34.764	37.689	41.449	42.942	44.313	49.335
60	35.534	37.485	39.699	40.482	43.188	46.459	50.641	52.294	53.809	59.335

Table A.5 (continued) Critical Values of the Chi-Squared Distribution

	α									
v	**0.30**	**0.25**	**0.20**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.001**
1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.266
4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.466
5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.515
6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.321
8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.124
9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
11	12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
12	14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909
13	15.119	15.984	16.985	19.812	22.362	24.736	25.471	27.688	29.819	34.527
14	16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.124
15	17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.698
16	18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
17	19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.791
18	20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
19	21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.819
20	22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.314
21	23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.796
22	24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268
23	26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728
24	27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179
25	28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.619
26	29.246	30.435	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.051
27	30.319	31.528	32.912	36.741	40.113	43.195	44.140	46.963	49.645	55.475
28	31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.994	56.892
29	32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.335	58.301
30	33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.702
40	44.165	45.616	47.269	51.805	55.758	59.342	60.436	63.691	66.766	73.403
50	54.723	56.334	58.164	63.167	67.505	71.420	72.613	76.154	79.490	86.660
60	65.226	66.981	68.972	74.397	79.082	83.298	84.58	88.379	91.952	99.608