CSE 548: (*Design and*) Analysis of Algorithms

Graphs

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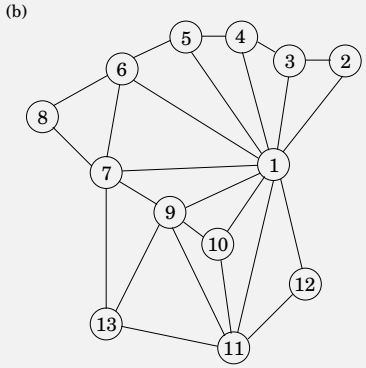
Overview

- Graphs provide a concise representation of a range problems
  Map coloring – more generally, resource contention problems
  Networks — communication, traffic, social, biological, ...



Definition and Representations

A *graph* $G = (V, E)$, where V is a set of vertices, and E a set of edges. An edge e of the form (v_1, v_2) is said to span vertices v_1 and v_2. The edges in a *directed graph* are directed.
A $G' = (V', E')$ is called a *subgraph* of G if $V' \subseteq V$ and E' includes every edge in E between vertices in V'.

Adjacency matrix

A graph $(V = \{v_1, \ldots, v_n\}, E)$ can be represented by an $n \times n$ matrix a, where $a_{ij} = 1$ iff $(v_i, v_j) \in E$

Adjacency list

Each vertex v is associated with a linked list consisting of all vertices u such that $(v, u) \in E$.

Note that adjacency matrix uses $O(n^2)$ storage, while adjacency list uses $O(|V| + |E|)$ storage. Both can represent directed as well as undirected graphs.

Depth-First Search (DFS)

- A technique for traversing all vertices in the graph
- Very versatile, forms the linchpin of many graph algorithms

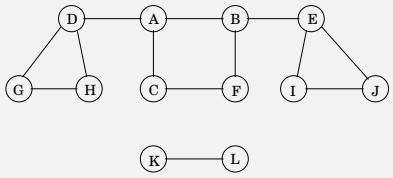
dfs(V, E)

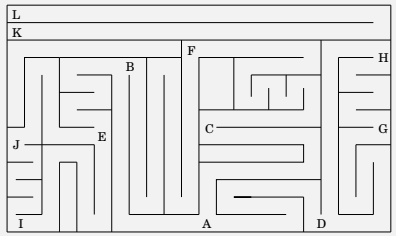
foreach $v \in V$ **do** *visited*$[v] =$ *false*
foreach $v \in V$ **do**
  if not *visited*$[v]$ **then** *explore*(V, E, v)

explore(V, E, v)

visited$[v] =$ *true*
previsit(v) /*A placeholder for now*/
foreach $(v, u) \in E$ **do**
  if not *visited*$[u]$ **then** *explore*(G, V, u)
postvisit(v) /*Another placeholder*/

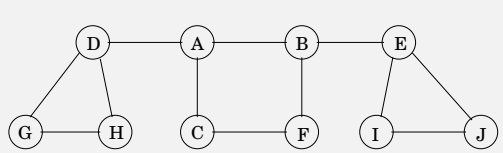
Graphs, Mazes and DFS

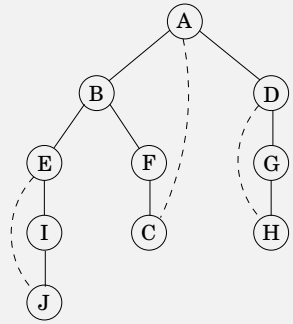




If a maze is represented as a graph, then DFS of the graph amounts to an exploration and mapping of the maze.

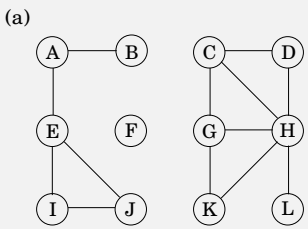
A graph and its DFS tree

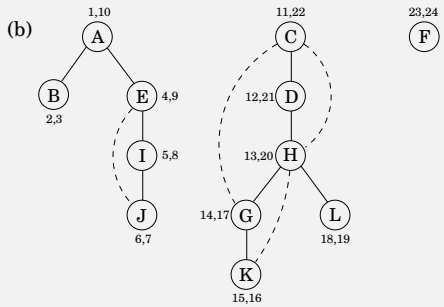




DFS uses $O(|V|)$ space and $O(|E| + |V|)$ time.

DFS and Connected Components





A *connected component* of a graph is a maximal subgraph where there is path between any two vertices in the subgraph, i.e., it is a maximal *connected subgraph*.

DFS Numbering

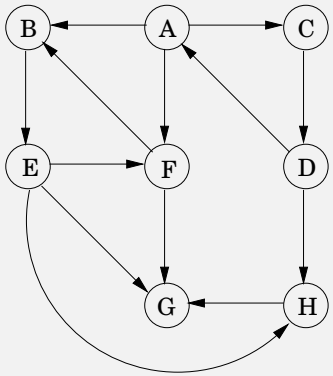
Associate post and pre numbers with each visited node by defining *previsit* and *postvisit*

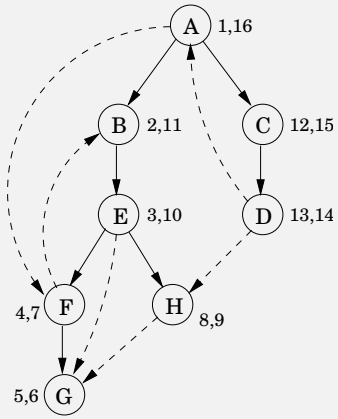
previsit(v)	*postvisit*(v)
$pre[v] = clock$	$post[v] = clock$
$clock{+}{+}$	$clock{+}{+}$

Property

For any two vertices u and v, the intervals $[pre[u], post[u]]$ *and* $[pre[v], post[v]]$ *are either disjoint, or one is contained entirely within another.*

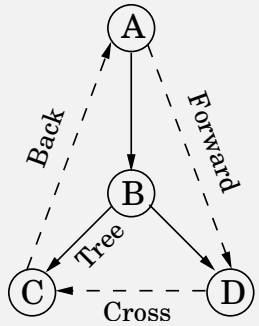
DFS of Directed Graph





DFS and Edge Types

DFS tree



`pre/post` *ordering for* (u, v)	*Edge type*
$[_u \; [_v \;]_v \;]_u$	Tree/forward
$[_v \; [_u \;]_u \;]_v$	Back
$[_v \;]_v \; [_u \;]_u$	Cross

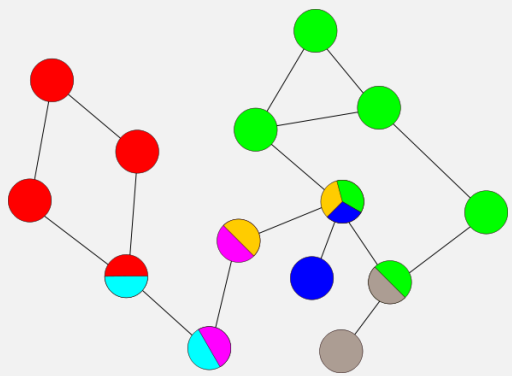
No cross edges in undirected graphs!
Back and forward edges merge

Biconnectivity in Undirected Graphs

Definition (Biconnected Components)

- *An articulation point in a graph is any vertex a that lies on every path from between vertices v, w.*
- *A biconnected graph contains no articulation points.*
- *A biconnected component of a graph is a maximal subgraph that is biconnected.*

Illustration of Biconnected Components



- Each biconnected component given a different color
- Articulation points have multiple colors

Biconnected Graphs

- Biconnected components are *not* disjoint.
- Articulation points are duplicated across them.

- *Articulation points ≡ single-points of failure*
  Graph is disconnected if any of them go down.
- Between any two $u, v \in V$ in a biconnected graph there are two node-disjoint paths (and hence a cycle).

Finding articulation points during DFS

During DFS visit of vertex v, we want to decide if it is an articulation point or not. Divide V into to disjoint sets:

- V_i includes nodes *inside* the DFS tree rooted at v, and
- $V_o = V - V_i$ of nodes *outside* the DFS tree rooted at v.

Observation (1)

*Suppose that every inside node v_i (i.e., $v_i \in V_i - \{v\}$) can reach some outside node v_o (i.e., $v_o \in V_o$) without going through v. Then v is **not** an articulation point.*

Finding articulation points during DFS (2)

Proof:

Paths within V_o: By properties of DFS tree, any two vertices in V_o can reach each other without going through v

Paths between v_i and V_o: Once v_i can reach any node in V_o, it can reach all other nodes in V_o without having to go through v.

Paths within V_i: Consider nodes v_i and v_i' in $V_i - \{v\}$. By our assumption there is a path from v_i to some $v_o \in V_o$, and another path from v_i' to some $v_o' \in V_o$, neither involving v. Consider the path v_i to v_o to v_o' to v_i', which does not pass through v.

Thus, for every pair of vertices, there is a path between them that avoids v, and hence v is *not* an articulation point.

Finding articulation points during DFS (3)

Observation (2)

If Observation (1) does not hold then v is an articulation point.

This means that there exists some v_i that cannot reach an outside node v_o without going through v. By definition, this means that v is an articulation point.

Finding articulation points during DFS (3)

We combine and slightly strengthen Observations (1), (2), while omitting the proof, as it is essentially the same as before.

Observation (3)

Let Y denote the set of ancestors of v and X its immediate children in the DFS tree. Now, v is not an articulation point iff each $x \in X$ has a path to some $y \in Y$ without going through v.

- By focusing on just the children and ancestors of v, this makes it easier to decide on articulation points during DFS.
- *Note than any such path from x to y will follow zero or more tree edges down, followed by a back edge.*

Finding articulation points during DFS (4)

Note than any such path from x to y will follow zero or more tree edges down, followed by a back edge.

- If this path bypasses v, this is going to occur due to a single back-edge that goes to an ancestor of v. So there is no need to consider paths with multiple back-edges.
- Note: Pre-number increases while following tree edges down, while it decreases when a back edge is followed.

Key Idea: During DFS, for each vertex x, maintain the highest ancestor that can be reached from x by following tree edges down and then a back edge.

Finding articulation points during DFS (5)

Key Idea: During DFS, maintain the highest ancestor reachable from x by following tree edges down and then a back edge.

- This info is maintained in the array *low*.
- $low[x] \leq low[x']$ for every child x' of x in DFS tree.
  - This captures the fact that we can follow the tree edge down from x to x', then go to whichever ancestor is reachable from x'.
  - Algorithmically, let $low[x] = min(low[x], low[x'])$ when $explore(x')$ returns
- $low[x] \leq pre[x'']$ for x'' adjacent to x but not a parent or child in the DFS tree.
  - Algorithmically, set $low[x] = min(low[x], pre[x''])$
  - By properties of DFS, x'' is either a descendant or ancestor of x.
    - As we are taking *min*, statement effective only if x'' is an ancestor.

Finding articulation points during DFS (6)

Key Idea: Maintain *low* during DFS

- when visiting x, initialize $low[x]$ to $pre[x]$.
- when a DFS call on a child x' of x returns, check if $low[x'] \geq pre[x]$.
  - If so, the highest ancestor x' can reach is not higher than x, i.e., x' cannot go to ancestors of x without going through x.
  - So, mark x as articulation point
- If an adjacent vertex x'' is already visited when x considers it, set $low[x] = min(low[x], pre[x''])$ unless x'' is parent of x

All these can be done during a DFS, while increasing the cost of each step by only a constant amount — so, overall complexity is $O(|E| + |V|)$

Directed Acyclic Graphs (DAGs)

A directed graph that contains no cycles.
Often used to represent (acyclic) dependencies, partial orders,...

Property (DAGs and DFS)

- *A directed graph has a cycle iff its DFS reveals a back edge.*
- *In a dag, every edge leads to a vertex with lower post number.*
- *Every dag has at least one source and one sink.*

Strongly Connected Components (SCC)

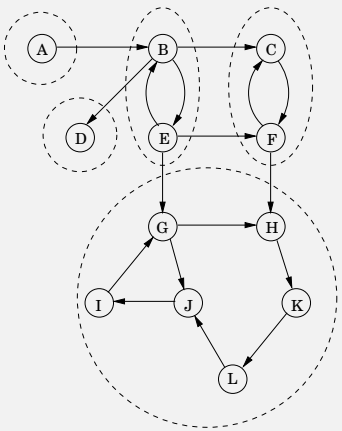
Analog of connected components for undirected graphs

Definition (SCC)

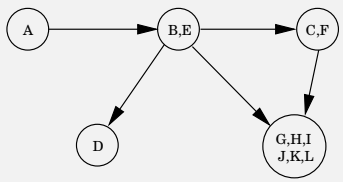
- *Two vertices u and v in a directed graph are* connected *if there is a path from u to v and vice-versa.*
- *A directed graph is* strongly connected *if any pair of vertices in the graph are connected.*
- *A subgraph of a directed graph is said to be an SCC if it is a maximal subgraph that is strongly connected.*

SCCs are also similar to biconnected components!

SCC Example



(b)



The textbook describes an algorithm for computing SCC in linear-time using DFS.

Breadth-first Search (BFS)

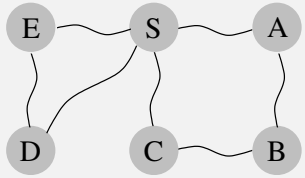
- Traverse the graph by "levels"
  - $BFS(v)$ visits v first
  - Then it visits all immediate children of v
  - then it visits children of children of v, and so on.
- As compared to DFS, BFS uses a queue (rather than a stack) to remember vertices that still need to be explored

BFS Algorithm

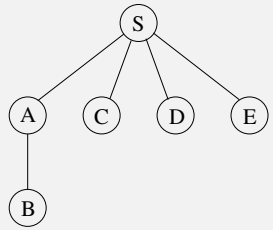
bfs(V, E, s)

foreach $u \in V$ **do** $visited[u] = false$
$q = \{s\}$; $visited[s] = true$
while q is nonempty **do**
  $u = deque(q)$
  foreach edge $(u, v) \in E$ **do**
    if not $visited[v]$ **then**
      $queue(q, v)$; $visited[v] = true$

BFS Algorithm Illustration



Order of visitation	Queue contents after processing node
	$[S]$
S	$[A\ C\ D\ E]$
A	$[C\ D\ E\ B]$
C	$[D\ E\ B]$
D	$[E\ B]$
E	$[B]$
B	$[\,]$



Shortest Paths and BFS

BFS automatically computes shortest paths!

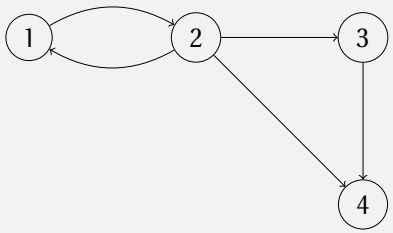
bfs(V, E, s)

foreach $u \in V$ **do** $dist[u] = \infty$
$q = \{s\}$; $dist[s] = 0$
while q is nonempty **do**
  $u = deque(q)$
  foreach edge $(u, v) \in E$ **do**
    if $dist[v] = \infty$ **then**
      $queue(q, v)$; $dist[v] = dist[u] + 1$

But not all paths are created equal! We would like to compute shortest weighted path — a topic of future lecture.

Graph paths and Boolean Matrices

A graph and its boolean matrix representation



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Graph paths and Boolean Matrices

- Let A be the adjacency matrix for a graph G, and $B = A \times A$. Now, $B_{ij} = 1$ iff there is path in the graph of length 2 from v_i to v_j
- Let $C = A + B$. Then $C_{ij} = 1$ iff there is path of length ≤ 2 between v_i and v_j
- Define $A^* = A^0 + A^1 + A^2 + \cdots$. If $D = A^*$ then $D_{ij} = 1$ iff v_j is reachable from v_i.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Shortest paths and Matrix Operations

- Redefine operations on matrix elements so that $+$ becomes *min*, and $*$ becomes integer addition.
- $D = A^*$ then $D_{ij} = k$ iff the shortest path from v_j to v_i is of length k