CSE 548: (*Design and*) Analysis of Algorithms

Randomized Algorithms

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Example 1: Routing

- What is the best way to route a packet from X to Y, esp. in high speed, high volume networks
  - A: Pick the shortest path from X to Y
  - B: Send the packet to a random node Z, and let Z route it to Y (possibly using a shortest path from Z to Y)
- Valiant showed in 1981 that surprisingly, B works better!
  - Turing award recipient in 2010

Example 2: Transmitting on shared network

- What is the best way for n hosts to share a common a network?
  - A: Give each host a turn to transmit
  - B: Maintain a queue of hosts that have something to transmit, and use a FIFO algorithm to grant access
  - C: Let every one try to transmit. If there is contention, use random choice to resove it.
- Which choice is better?

Topics

Simplify, Decentralize, Ensure Fairness

- Randomization can often:
  - Enable the use of a simpler algorithm
  - Cut down the amount of book-keeping
  - Support decentralized decision-making
  - Ensure fairness
- *Examples:*

  Media access protocol: Avoids need for coordination — important here, because coordination needs connectivity!

  Load balancing: Instead of maintaining centralized information about processor loads, dispatch jobs randomly.

  Congestion avoidance: Similar to load balancing

A Randomized Protocol for Medium Access

- Suppose n hosts want to access a shared medium
  - If mutiple hosts try at the same time, there is contention, and the "slot" is wasted.
  - A slot is wasted if no one tries.
  - How can we maximize the likelihood of every slot being utilized?
- Suppose that a randomized protocol is used.
  - Each host transmits with a probability p
  - What should be the value of p?
- We want the likelihood that one host will attempt access (probability p), while others don't try (probability $(1-p)^{n-1}$)
  - Find p that maximizes $p(1-p)^{n-1}$
  - Using differentiation to find maxima, we get $p = 1/n$

A Randomized Protocol for Medium Access

- Maximum probability (when $p = 1/n$)
$$\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$$
- Note $\left(1-\frac{1}{n}\right)^{n-1}$ converges to $1/e$ for reasonably large n
  - About 5% off e at $n = 10$.
  - So, let us simplify the expression to $1/ne$ for future calculations
- What is the *efficiency* of the protocol?
  - The probability that *some* host gets to transmit is $n \cdot 1/ne = 1/e$
- Is this protocol a reasonable choice?
  - Wasting almost 2/3rd of the slots is rarely acceptable

A Randomized Protocol for Medium Access

- How long before a host i can expect to transmit successfully?
  - The probability it fails the first time is $(1-1/ne)$
  - Probability i fails in k attempts: $(1-1/ne)^k$
  - This quantity gets to be reasonably small (specifically, $1/e$) when $k = ne$
  - For larger k, say $k = ne \cdot c \ln n$, the expression becomes
$$(\underline{(1-1/ne)^{ne}})^{c\ln n} = (1/e)^{c\ln n} = (\underline{e^{\ln n}})^{-c} = n^{-c}$$
- So, a host has a reasonable success chance in $O(n)$ attempts
  - This becomes a virtual certainty in $O(n \ln n)$ attempts

A Randomized Protocol for Medium Access

- What is the expected wait time?
  - "Average" time a host can expect to try before succeeding.

$$E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X = j]$$

- For our protocol, expected wait time is given by

$$1 \cdot p + 2 \cdot (1-p)p + 3 \cdot (1-p)^2 p \cdots = p \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}$$

- How do we sum the series $\sum ix^{i-1}$?
- Note that $\sum_{i=1}^{\infty} x^i = \frac{1}{(1-x)}$. Now, differentiate both sides:

$$\sum_{i=1}^{\infty} ix^{i-1} = -\frac{1}{(1-x)^2}$$

Intro Decentralize Taming distribution Probabilistic Algorithms Medium Access Coupon Collection Birthday Balls and Bins

A Randomized Protocol for Medium Access

- Expected wait time is

$$p \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} = \frac{p}{p^2} = 1/p$$

- We get an intuitive result — a host will need to wait $1/p = ne$ slots on the average
- *Note:* The derivation is a general one, applies to any event with probability p; it is not particular to this access protocol

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A Randomized Protocol for Medium Access

- How long will it be before every host would have a high probability of succeeding?
- We are interested in the probability of

$$S(k) = \bigcup_{i=1}^{n} S(i,k)$$

- Note that failures are not independent, so we cannot say that

$$Pr[S(k)] = \sum_{i=1}^{n} Pr[S(i,k)]$$

but certainly, the rhs is an upper bound on $Pr[F(k)]$.
  - We use this approximate *union bound* for our asymptotic analysis

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A Randomized Protocol for Medium Access

- If we use $k = ne$, then

$$\sum_{i=1}^{n} Pr[S(i,k)] = \sum_{i=1}^{n} \frac{1}{e} = n/e$$

which suggests that the likelihood some hosts failed within ne attempts is rather high.
- If we use $k = cn \ln n$ then we get a bound:

$$\sum_{i=1}^{n} Pr[S(i,k)] = \sum_{i=1}^{n} n^{-c/e} = n^{(e-c)/e}$$

which is relatively small — $O(n^{-1})$ for $c = 2e$.
- Thus, it is highly likely that all hosts will have succeeded in $O(n \ln n)$ attempts.

A Randomized Protocol: Conclusions

- High school probability background is sufficient to analyze simple randomized algorithms
- Carefully work out each step
  - Intuition often fails us on probabilities
- If every host wants to transmit in every slot, this randomized protocol is a bad choice.
  - 63% wasted slots is unacceptable in most cases.
  - Better off with a round-robin or queuing based algorithm.
- How about protocols used in Ethernet or WiFi?
  - Optimistic: whoever needs to transmit will try in the next slot
  - Exponential backoff when collisions occur
    - Each collision halves p

Coupon Collector Problem

- Suppose that your favorite cereal has a coupon inside. There are n types of coupons, but only one of them in each box. How many boxes will you have to buy before you can expect to have all of the n types?
- What is your guess?
- Let us work out the expectation. Let us say that you have so far $j-1$ types of coupons, and are now looking to get to the jth type. Let X_j denote the number of boxes you need to purchase before you get the $j+1$th type.

Coupon Collector Problem

- Note $E[X_j] = 1/p_j$, where p_j is the probability of getting the *jth* coupon.
- Note $p_j = (n-j)/n$, so, $E[X_j] = n/(n-j)$
- We have all n types when we finish the X_{n-1} phase:

$$E[X] = \sum_{i=0}^{n-1} E[X_j] = \sum_{i=0}^{n-1} n/(n-j) = nH(n)$$

  - Note $H(n)$ is the harmonic sum, and is bounded by $\ln n$
- Perhaps unintuitively, you need to buy $\ln n$ cereal boxes to obtain one useful coupon.
- *Abstracts the media access protocol just discussed!*

Birthday Paradox

- What is the smallest size group where there are at least two people with the same birthday?
  - 365
  - 183
  - 61
  - 25

Birthday Paradox

- The probability that the $i+1$th person's birthday is distinct from previous i is approx.[1]

$$p_i = \frac{N-i}{N}$$

- Let X_i be the number of *duplicate* birthdays added by i:

$$E[X_i] = 0 \cdot p_i + 1 \cdot (1-p_i) = 1 - p_i = \frac{i}{N}$$

- Sum up E_i's to find the # of distinct birthdays among n:

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{i}{N} = \frac{n(n-1)}{2N}$$

Thus, when $n \approx 27$, we have one duplicate birthday[2]

[1] We are assuming that $i-1$ birthdays are distinct: reasonable if $n \ll N$

[2] More accurate calculation will yield $n = 24.6$

Birthday Paradox Vs Coupon Collection

- Two sides of the same problem

  Coupon Collection: What is the minumum number of samples needed to cover every one of N values

  Birthday problem: What is the maximum number of samples that can avoid covering any value more than once?

- So, if we want enough people to ensure that every day of the year is covered as a birthday, we will need $365 \ln 365 \approx 2153$ people!
  - Almost 100 times as many as needed for one duplicate birthday!

Balls and Bins

If m balls are thrown at random into n bins:

- What should m be to have more than one ball in some bin?
  - Birthday problem
- What should m be to have at least one ball per bin?
  - Coupon collection, media access protocol example
- What is the maximum number of balls in any bin?
  - Such problems arise in load-balancing, hashing, etc.

Balls and Bins: Max Occupancy

- Probability $p_{1,k}$ that the first bin receives at least k balls:
  - Choose k balls in $\binom{m}{k}$ ways
  - These k balls should fall into the first bin: prob. is $(1/n)^k$
  - Other balls may fall anywhere, i.e., probability 1:[3]

$$\binom{m}{k}\left(\frac{1}{n}\right)^k = \frac{m \cdot (m-1) \cdots (m-k+1)}{k! n^k} \leq \frac{m^k}{k! n^k}$$

- Let $m = n$, and use Sterling's approx. $k! \approx \sqrt{2\pi k}(k/e)^k$:

$$P_k = \sum_{i=1}^{n} p_{i,k} \leq n \cdot \frac{1}{k!} \leq n \cdot \left(\frac{e}{k}\right)^k$$

- Some arithmetic simplification will show that $P_k < 1/n$ when

$$k = \frac{3 \ln n}{\ln \ln n}$$

[3] This is actually an upper bound, as there can be some double counting.

Balls and Bins: Summary of Results

m balls are thrown at random into n bins:

- Min. one bin with expectation of 2 balls: $m = \sqrt{2n}$
- No bin expected to be empty: $m = n \ln n$
- Expected number of empty bins: $ne^{-m/n}$
- Max. balls in any bin when $m = n$:

$$\Theta(\ln n / \ln \ln n)$$

  - This is a probabilistic bound: chance of finding any bin with higher occupancy is $1/n$ or less.
  - Note that the absolute maximum is n.

Randomized Quicksort

- Picks a pivot at random. What is its complexity?
- If pivot index is picked uniformly at random over the interval $[l, h]$, then:
  - every array element is equally likely to be selected as the pivot
  - every partition is equally likely
  - thus, *expected* complexity of *randomized* quicksort is given by:

$$T(n) = n + \frac{1}{n}\sum_{i=1}^{n-1}(T(i) + T(n-i))$$

  Summary: Input need not be random
  - Expected $O(n \log n)$ performance comes from *externally forced* randomness in picking the pivot

Cache or Page Eviction

- Caching algorithms have to evict entries when there is a miss
  - As do virtual memory systems on a page fault
- Optimally, we should evict the “farthest in future” entry
  - But we can’t predict the future!
- Result: many candidates for eviction. How can be avoid making bad (worst-case) choices repeatedly, even if input behaves badly?
- Approach: pick one of the candidates at random!

Closest pair

- We studied a deterministic divide-an-conquer algorithm for this problem.
  - Quite complex, required multiple sort operations at each stage.
  - Even then, the number of cross-division pairs to be considered seemed significant
  - Result: deterministic algorithm difficult to implement, and likely slow in practice.
- Can a randomized algorithm be simpler and faster?

Randomized Closest Pair: Key Ideas

- Divide the plane into small squares, hash points into them
  - Pairwise comparisons can be limited to points within the squares very closeby
- Process the points in some random order
  - Maintain min. distance δ among points processed so far.
  - Update δ as more points are processed
- At any point, the "small squares" have a size of $\delta/2$
  - At most one point per square (or else points are closer than δ)
  - Points closer than δ will at most be two squares from each other
    - Only constant number of points to consider
  - Requires rehashing all processed points when δ is updated.

Randomized Closest Pair: Analysis

- Correctness is relatively clear, so we focus on performance
- Two main concerns

  Storage: # of squares is $1/\delta^2$, which can be very large
  - Use a dictionary (hash table) that stores up to n points, and maps $(2x_i/\delta, 2y_i/\delta)$ to $\{1, \ldots, n\}$
  - To process a point (x_j, y_j)
    - look up the dictionary at $(x_j/\delta \pm 2, y_j/\delta \pm 2)$
    - insert if it is not closer than δ

  Rehashing points: If closer than δ — very expensive.
- Total runtime can all be "charged" to insert operations,
  - incl. those performed during rehashing

  so we will focus on estimating inserts.

Randomized Closest Pair: # of Inserts

Theorem

If random variable X_i denotes the likelihood of needing to rehash after processing k points, then

$$X_i \leq \frac{2}{i}$$

- Let $p_1, p_2, \ldots, p_i$ be the points processed so far, and p and q be the closest among these
- Rehashing is needed while processing p_i if $p_i = p$ or $p_i = q$
- Since points are processed in random order, there is a $2/i$ probability that p_i is one of p or q

Randomized Closest Pair: # of Inserts

Theorem

The expected number of inserts is $3n$.

- Processing of p_i involves
  - i inserts if rehashing takes place, and 1 insert otherwise
- So, expected inserts for processing p_i is

$$i \cdot X_i + 1 \cdot (1 - X_i) = 1 + (i-1) \cdot X_i = 1 + \frac{2(i-1)}{i} \leq 3$$

- Upper bound on expected inserts is thus $3n$

Look Ma! I have a linear-time randomized closest pair algorithm—And it is not even probabilistic!

Hash Tables

- A data structure for implementing:
  - Dictionaries: Fast look up of a record based on a key.
  - Sets: Fast membership check.
- Support expected $O(1)$ time *lookup*, *insert*, and *delete*
- Hash table entries may be:
  - fat: store a pair $(key, object)$
  - lean: store pointer to object containing key
- Two main questions:
  - *How to avoid $O(n)$ worst case behavior?*
  - How to ensure *average case performance* can be realized *for arbitrary distribution of keys?*

Hash Table Implementation

Direct access: A fancy name for arrays. Not applicable in most cases where the universe $\mathcal{U}$ of keys is very large.

Index based on hash: Given a hash function h (fixed for the entire table) and a key x, use $h(x)$ to index into an array A.

- Use $A[h(x) \mod s]$, where s is the size of array
  - Sometimes, we fold the mod operation into h.
- Array elements typically called *buckets*
- *Collisions bound to occur* since $s \ll |\mathcal{U}|$
  - Either $h(x) = h(y)$, or
  - $h(x) \neq h(y)$ but $h(x) \equiv h(y) \pmod{s}$

Collisions in Hash tables

- *Load factor α:* Ratio of number of keys to number of buckets
- *If* keys were random:
  - What is the max α if we want ≤ 1 collisions in the table?
  - If $\alpha = 1$, what is the maximum number of collisions to expect?
- Both questions can be answered from balls-and-bins results: $1/\sqrt{n}$, and $O(\ln n/\ln\ln n)$
- **Real world keys are not random.** Your hash table implementation needs to achieve its performance goals independent of this distribution.

Chained Hash Table

- Each bucket is a linked list.
- Any key that hashes to a bucket is inserted into that bucket.
- What is the *average* search time, as a function of α?
  - It is $1 + \alpha$ if:
    - you assume that the distribution of lookups is independent of the table entries, OR,
    - the chains are not too long (i.e., α is small)

Open addressing

- If there is a collision, probe other empty slots
  - Linear probing: If $h(x)$ is occupied, try $h(x) + i$ for $i = 1, 2, ...$
  - Binary probing: Try $h(x) \oplus i$, where $\oplus$ stands for exor.
  - Quadratic probing: For ith probe, use $h(x) + c_1 i + c_2 i^2$
- Criteria for secondary probes
  - Completeness: Should cycle through all possible slots in table
  - Clustering: Probe sequences shouldn't coalesce to long chains
  - Locality: Preserve locality; typically conflicts with clustering.
- Average search time can be $O(1/(1-\alpha)^2)$ for linear probing, and $O(1/(1-\alpha))$ for quadratic probing.

Chaining Vs Open Addressing

- Chaining leads to fewer collisions
  - Clustering causes more collisions w/ open addressing for same α
  - However, for lean tables, open addressing uses half the space of chaining, so you can use a much lower α for same space usage.
- Chaining is more tolerant of "lumpy" hash functions
  - For instance, if $h(x)$ and $h(x+1)$ are often very close, open hashing can experience longer chains when inputs are closely spaced.
  - Hash functions for open-hashing having to be selected very carefully
- Linked lists are not cache-friendly
  - Can be mitigated w/ arrays for buckets instead of linked lists
- Not all quadratic probes cover all slots (but some can)

Resizing

- Hard to predict the right size for hash table in advance
  - Ideally, $0.5 \le \alpha \le 1$, so we need an accurate estimate
- *It is stupid to ask programmers to guess the size*
  - Without a good basis, only terrible guesses are possible
- *Right solution:* Resize tables automatically.
  - When α becomes too large (or small), rehash into a bigger (or smaller) table
  - Rehashing is $O(n)$, but if you increase size by a factor, then amortized cost is still $O(1)$
  - Exercise: How to ensure amortized $O(1)$ cost when you resize up as well as down?

Average Vs Worst Case

- Worst case search time is $O(n)$ for a table of size n
- *With hash tables, it is all about avoiding the worst case, and achieving the average case*
- Two main chalenges:
  - *Input is not random,* e.g., names or IP addresses.
  - Even when input is random, h may cause "lumping," or non-uniform dispersal of $\mathcal{U}$ to the set $\{1, \ldots, n\}$
- Two main techniques
  - Universal hashing
  - Perfect hashing

Universal Hashing

- No single hash function can be good on all inputs
  - Any function $\mathcal{U} \to \{1, \ldots, n\}$ must map $|\mathcal{U}|/n$ inputs to same value!

  Note: $|\mathcal{U}|$ can be much, much larger than n.

Definition

A family of hash functions $\mathcal{H}$ is universal if

$$Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{n} \quad \text{for all } x \neq y$$

Meaning: If we pick h at random from the family $\mathcal{H}$, then, probability of collisions is the same for any two elements.

Contrast with non-universal hash functions such as

$$h(x) = ax \bmod n, \qquad (a \text{ is chosen at random})$$

Note y and $y + kn$ collide with a probability of 1 *for every* a.

Universal Hashing Using Multiplication

Observation (Multiplication Modulo Prime)

If p is a prime and $0 < a < p$

- $\{1a, 2a, 3a, \ldots, (p-1)a\} = \{1, 2, \ldots, p-1\}$ *(mod p)*
- $\forall a\, \exists b\; ab \equiv 1 \pmod{p}$

Prime multiplicative hashing

Let the key $x \in \mathcal{U}$, $p > |\mathcal{U}|$ be prime, and $0 < r < p$ be random. Then

$$h(x) = (rx \bmod p) \bmod n$$

is universal.

Prove: $Pr[h(x) = h(y)] = \frac{1}{n}$, for $x \neq y$

Universality of prime multiplicative hashing

- Need to show $Pr[h(x) = h(y)] = \frac{1}{n}$, for $x \neq y$
- $h(x) = h(y)$ means $(rx \bmod p) \bmod n = (ry \bmod p) \bmod n$
- Note $a \bmod n = b \bmod n$ means $a = b + kn$ for some integer k. Using this, we eliminate $\bmod n$ from above equation to get:

$$\begin{aligned} rx \bmod p &= kn + ry \bmod p, \text{ where } k \leq \lfloor p/n \rfloor \\ rx &\equiv kn + ry \pmod{p} \\ r(x-y) &\equiv kn \pmod{p} \\ r &\equiv kn(x-y)^{-1} \pmod{p} \end{aligned}$$

- So, x, y collide if $r = n(x-y)^{-1}, 2n(x-y)^{-1}, \ldots, \lfloor p/n \rfloor n(x-y)^{-1}$
- In other words, x and y collide for p/n out of p possible values of r, i.e., collision probability is $1/n$

Binary multiplicative hashing

- Faster: avoids need for computing modulo prime
- When $|\mathcal{U}| < 2^w$, $n = 2^l$ and a an odd random number

$$h(x) = \left\lfloor \frac{ax \mod 2^w}{2^{w-l}} \right\rfloor$$

- Can be implemented efficiently if w is the wordsize:

```
(a*x) >> (WORDSIZE-HASHBITS)
```

- Scheme is near-universal: collision probability is $O(1)/2^l$

Prime Multiplicative Hash for Vectors

Let p be a prime number, and the key x be a vector $[x_1, \ldots, x_k]$ where $0 \le x_i < p$. Let

$$h(x) = \sum_{i=1}^{k} r_i x_i \pmod{p}$$

If $0 < r_i < p$ are chosen at random, then h is universal.

- Strings can also be handled like vectors, or alternatively, as a polynomial evaluated at a random point a, with p a prime:

$$h(x) = \sum_{i=0}^{l} x_i a^i \ \ mod\ p$$

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$
- When collision occurs, $\sum_{j=1}^{k} r_j x_j = \sum_{j=1}^{k} r_j y_j \ \ (mod\ p)$
- Rearranging, $\sum_{j \neq i} r_j(x_j - y_j) = r_i(y_i - x_i) \ \ (mod\ p)$
- The lhs evaluates to some c, and we need to estimate the probability that rhs evaluates to this c
- Using multiplicative inverse property, we see that $r_i = c(y_i - x_i)^{-1} \ \ (mod\ p)$.
- Since $y_i, x_i < p$, it is easy to see from this equation that the collision-causing value of r_i is distinct for distinct y_i.
- Viewed another way, exactly one of p choices of r_i would cause a collision between x_i and y_i, i.e., $Pr_h[h(x) = h(y)] = 1/p$

Perfect hashing

Static: Pick a hash function (or set of functions) that avoids collisions for a given set of keys

Dynamic: Keys need not be static.

Approach 1: Use $O(n^2)$ storage. Expected collision on n items is 0.
But too wasteful of storage.
Don't forget: more memory usually means less performance due to cache effects.

Approach 2: Use a secondary hash table for each bucket of size n_i^2, where n_i is the number of elements in the bucket.
Uses only $O(n)$ storage, *if h is universal*

Hashing Summary

- Excellent average case performance
  - Pointer chasing is expensive on modern hardware, so improvement from $O(\log n)$ of binary trees to expected $O(1)$ for hash tables is significant.
- But all benefits will be reversed if collisions occur too often
  - Universal hashing is a way to ensure expected average case *even when input is not random.*
- Perfect hashing can provide efficient performance even in the worst case, but the benefits are likely small in practice.

Probabilistic Algorithms

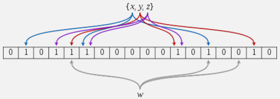
- Algorithms that produce the correct answer with some probability
- By re-running the algorithm many times, we can increase the probability to be arbitrarily close to 1.0.

Bloom Filters

- To resolve collisions, hash tables have to store keys: $O(mw)$ bits, where w is the number of bits in the key
- What if you want to store very large keys?
- *Radical idea:* Don't store the key in the table!
  - Potentially w-fold space reduction

Bloom Filters

- To reduce collisions, use multiple hash functions $h_1, \ldots, h_k$
- Hash table is simply a bitvector $B[1..m]$
- To insert key x, set $B[h_1(x)], B[h_2(x)], \ldots, B[h_k(x)]$



Images from Wikipedia Commons

- Membership check for y: all $B[h_i(y)]$ should be set
  - No false negatives, but false positives possible
- No deletions possible in the current algorithm.

Bloom Filters: False positives

- Prob. that a bit is *not* set by h_1 on inserting a key is $(1 - 1/m)$
  - The probability it is not set by any h_i is $(1 - 1/m)^k$
  - The probability it is not set after r key inserts is $(1 - 1/m)^{kr} \approx e^{-kr/m}$
- Complementing, the prob. p that a certain bit is set is $1 - e^{-kr/m}$
- For a false positive on a key y, all the bits that it hashes to should be a 1. This happens with probability

$$\left(1 - e^{-kr/m}\right)^k = (1 - p)^k$$

Bloom Filters

- Consider
$$\left(1 - e^{-kr/m}\right)^k$$
- Note that the table can potentially store very large number of entries with very low false positives
  - For instance, with $k = 20$, $m = 10^9$ bits (12M bytes), and a false positive rate of $2^{-10} = 10^{-3}$, can store $60M$ keys of arbitrary size!
- *Exercise:* What is the optimal value of k to minimize false positive rate for a given m and r?
  - But large k values introduce high overheads
- *Important:* Bloom filters can be used as a prefilter, e.g., if actual keys are in secondary storage (e.g., files or internet repositories)

Using arithmetic for substring matching

Problem: Given strings $T[1..n]$ and $P[1..m]$, find occurrences of P in T in $O(n+m)$ time.

Idea: To simplify presentation, assume P, T range over [0-9]

- Interpret $P[1..m]$ as digits of a number
$$p = 10^{m-1}P[1] + 10^{m-2}P[2] + \cdots 10^{m-m}P[m]$$
- Similarly, interpret $T[i..(i+m-1)]$ as the number t_i
- Note: P is a substring of T at i iff $p = t_i$
- To get t_{i+1}, shift $T[i]$ out of t_i, and shift in $T[i+m]$:
$$t_{i+1} = \left(t_i - 10^{m-1}T[i]\right) \cdot 10 + T[i+m]$$

We have an $O(n+m)$ algorithm. Almost: we still need to figure out how to operate on m-digit numbers in constant time!

Rabin-Karp Fingerprinting

Key Idea

- Instead of working with m-digit numbers,
- perform all arithmetic modulo a *random* prime number q,
- where $q > m^2$ fits within wordsize

- All observations made on previous slide still hold
  - Except that $p = t_i$ does not guarantee a match
  - Typically, we expect matches to be infrequent, so we can use $O(m)$ exact-matching algorithm to confirm probable matches.

Carter-Wegman-Rabin-Karp Algorithm

Difficulty with Rabin-Karp: Need to generate random primes, which is not an efficient task.

New Idea: Make the radix random, as opposed to the modulus

- We still compute modulo a prime q, but it is not random.

Alternative interpretation: We treat P as a polynomial
$$p(x) = \sum_{i=1}^{m} P[m-i] \cdot x^i$$
and evaluate this polynomial at a randomly chosen value of x

Like any probabilistic algorithm we can increase correctness probability by repeating the algorithm with different randoms.

- Different prime numbers for Rabin-Karp
- Different values of x for CWRK

Carter-Wegman-Rabin-Karp Algorithm

$$p(x) = \sum_{i=1}^{m} P[m-i] \cdot x^i$$

Random choice does not imply high probability of being right.

- You need to explicitly establish correctness probability.

So, what is the likelihood of false matches?

- A false match occurs if $p_1(x) = p_2(x)$, i.e., $p_1(x) - p_2(x) = p_3(x) = 0$.
- Arithmetic modulo prime defines a *field,* so an mth degree polynomial has $m+1$ roots.
- Thus, $(m+1)/q$ of the q (recall q is the prime number used for performing modulo arithmetic) possible choices of x will result in a false match, i.e., probability of false positive = $(m+1)/q$

Primality Testing

Fermat's Theorem

$a^{p-1} \equiv 1 \pmod{p}$

- Recall $\{1a, 2a, 3a, \ldots, (p-1)a\} \equiv \{1, 2, \ldots, p-1\} \pmod{p}$
- Multiply all elements of both sides:

$$(p-1)!a^{p-1} \equiv (p-1)! \pmod{p}$$

- Canceling out $(p-1)!$ from both sides, we have the theorem!

Primality Testing

- Given a number N, we can use Fermat's theorem as a probabilistic test to see if it is prime:
  - if $a^{N-1} \not\equiv 1 \pmod{N}$ then N is not prime
  - Repeat with different values of a to gain more confidence
- *Question:* If N is *not* prime, what is the probability that the above procedure will fail?
  - For Carmichael's numbers, the probability is 1 — but ignore this for now, since these numbers are very rare.
  - For other numbers, we can show that the above procedure works with probability 0.5

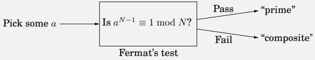
Primality Testing

Lemma

If $a^{N-1} \not\equiv 1 \pmod{N}$ for a relatively prime to N, then it holds for at least half the choices of $a < N$.

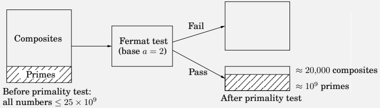
- If there is no b such that $b^{N-1} \equiv 1 \pmod{N}$, then we have nothing to prove.
- Otherwise, pick one such b, and consider $c \equiv ab$.
- Note $c^{N-1} \equiv a^{N-1}b^{N-1} \equiv a^{N-1} \not\equiv 1$
- Thus, for every b for which Fermat's test is satisfied, there exists a c that does not satisfy it.
  - Moreover, since a is relatively prime to N, $ab \not\equiv ab'$ unless $b \equiv b'$.
- Thus, at least half of the numbers $x < N$ that are relatively prime to N will fail Fermat's test.

Primality Testing



- When Fermat's test returns "prime" $Pr[N \text{ is not prime}] < 0.5$
- If Fermat's test is repeated for k choices of a, and returns "prime" in each case, $Pr[N \text{ is not prime}] < 0.5^k$
- In fact, 0.5 is an upper bound. Empirically, the probability has been much smaller.

Primality Testing



- Empirically, on numbers less than 25 billion, the probability of Fermat's test failing to detect non-primes (with $a = 2$) is more like 0.00002
- This probability decreases even more for larger numbers.

Prime number generation

Lagrange's Prime Number Theorem

For large N, primes occur approx. once every $\log N$ numbers.

Generating Primes

- Generate a random number
- Probabilistically test it is prime, and if so output it
- Otherwise, repeat the whole process

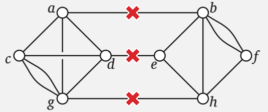
- What is the complexity of this procedure?
  - $O(\log^2 N)$ multiplications on $\log N$ bit numbers
- If N is not prime, should we try $N+1, N+2$, etc. instead of generating a new random number?
  - No, it is not easy to decide when to give up.

Rabin-Miller Test

- Works on Carmichael's numbers
- For prime number test, we consider only odd N, so $N - 1 = 2^t u$ for some odd u
- Compute
$$a^u, a^{2u}, a^{4u}, \ldots, a^{2^t u} = a^{N-1}$$
- If a^{N-1} is not 1 then we know N is composite.
- Otherwise, we do a follow-up test on a^u, a^{2u} etc.
  - Let $a^{2^r u}$ be the first term that is equivalent to 1.
  - If $r > 0$ and $a^{2^{r-1} u} \not\equiv -1$ then N is composite
- This combined test detects non-primes with a probability of at least 0.75 for all numbers.

Global Min-cut in Undirected Graphs

- Compute the minimum number of edges that need to be severed to disconnect a graph
- Yields the edge-connectivity of the graph



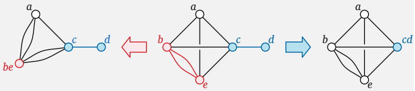
A multigraph whose minimum cut has three edges.

Deterministic Global Min-cut

- Replace each undirected edge by two (opposing) directed edges
- Pick a vertex s
- for each t in V compute the minimum $s-t$ cut
- The smallest among these is the global min-cut
- Repeating min-cut $O(|V|)$ times, so it is expensive *and* complex.

Randomized global min-cut

- Relies on repeated "collapsing" of edges, illustrated below
  - Pick a random edge (u, v), and delete it
  - Replace u and v by a single vertex uv
  - Replace each edge (x, u) by (x, uv)
  - Replace each edge (x, v) by (x, uv)
- Note: edges maintain their identity during this process



A graph G and two collapsed graphs $G/\{b, e\}$ and $G/\{c, d\}$.

Randomized global min-cut

GuessMinCut(V, E)
if $|V| = 2$ **then**
  return the only cut remaining
Pick an edge at random and collapse it to get V', E'
return *GuessMinCut*(V', E')

- Does this algorithm make sense? Why should it work?
- *Basic idea:* Only a small fraction of edges belong to the min-cut, reducing the likelihood of them being collapsed
- Still, when almost every edge is being collapsed, how likely is it that min-cut edges will remain?

GuessMinCut Correctness Probability

- If min-cut has k edges, then every node has min degree k
- So, there are $nk/2$ edges
- The likelihood of collapsing them in the first step is $2/n$
  - The likelihood of preserving min-cut edges is $(n-2)/n$
- We thus have the following recurrence for likelihood of preserving min-cut edges in the final solution:

$$P(n) \geq \frac{n-2}{n} \cdot P(n-1) \geq \frac{\cancel{n-2}}{n} \cdot \frac{\cancel{n-3}}{n-1} \cdot \frac{\cancel{n-4}}{\cancel{n-2}} \cdots \frac{2}{\cancel{4}} \cdot \frac{1}{\cancel{3}} = \frac{2}{n(n-1)}$$

So, the probability of being wrong is high

- by repeating it $O(n^2 \ln n)$ times, we reduce it to $1/n^c$.

Overall runtime is $O(n^4 \ln n)$, which is hardly impressive.

Power of Two Random Choices

If a single random choice yields unsatisfactory results, try making two choices and pick the better of two.

Example applications

Balls and bins: Maximum occupancy comes down from $O(\log n/\log\log n)$ to $O(\log\log n)$

Quicksort: Significantly increase odds of a balanced split if you pick three random elements and use their median as pivot

Load balancing: Random choice does not work well if different tasks take different time. Making two choices and picking the lighter loaded of the two can lead to much better outcomes

Power of Two Random Choices for Min-cut

- Divide random collapses into two phases
  - An initial "safe" phase that shrinks the graph to $1 + n/\sqrt{2}$ nodes
    - Probability of preserving min-cut is

$$\frac{(\cancel{n}/\sqrt{2})(n/\sqrt{2}+1)}{\cancel{n}(n-1)} \geq \frac{1}{2}$$

  - A second "unsafe" phase that is run twice, and the smaller min-cut is picked

Power of Two Random Choices for Min-cut

- A single run of unsafe phase is simply a recursive call
  - A kind-of-divide and conquer with power-of-two
    - Since input size decreases with each level of recursion, total time is reduced in spite of exponential increase in number of iterations
- We get the following recurrence for correctness probability:

$$P(n) \geq 1 - \left(1 - \frac{1}{2}P\left(\frac{n}{\sqrt{2}}+1\right)\right)^2$$

  which yields a result of $\Omega(1/\log n)$
  - Need $O(\log^2 n)$ repetitions to obtain low error rate
- For runtime, we have the recurrence

$$T(n) = O(n^2) + 2T(\frac{n}{\sqrt{2}}+1) = O(n^2 \log n)$$

  - Incl. $\log^2 n$ iterations, total runtime is $O(n^2 \log^3 n)$!