CSE 548: (*Design and*) Analysis of Algorithms

NP and Complexity Classes

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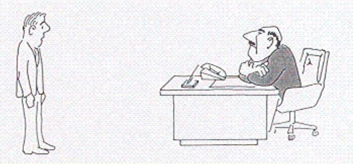
Search and Optimization Problems

- Many problems of our interest are search problems with exponentially (or even infinitely) many solutions
  - Shortest of the paths between two vertices
  - Spanning tree with minimal cost
  - Combination of variable values that minimize an objective
- We should be surprised we find efficient (i.e., polynomial-time) solutions to these problems
  - It seems like these should be the exceptions rather than the norm!
- What do we do when we hit upon other search problems?

Hard Problems: Where you find yourself ...



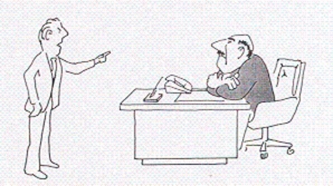
I can't find an efficient algorithm, I guess I'm just too dumb.

Images from "Computers and Intractability" by Garey and Johnson

Search and Optimization Problems

- What do we do when we hit upon hard search problems?
  - Can we prove they can't be solved efficiently?

Hard Problems: Where you would like to be ...



I can't find an efficient algorithm, because no such algorithm is possible.

Images from "Computers and Intractability" by Garey and Johnson

Search and Optimization Problems

- Unfortunately, it is very hard to prove that efficient algorithms are impossible
- Second best alternative:
  - Show that the problem is as hard as many other problems that have been worked on by a host of brilliant scientists over a very long time
- Much of complexity theory is concerned with categorizing hard problems into such *equivalence classes*

$\boxed{P\text{, } NP\text{, } Co\text{-}NP\text{, } NP\text{-hard and } NP\text{-complete}}$

Nondeterminism and Search Problems

- Nondeterminism is an oft-used abstraction in language theory
  - Non-deterministic FSA
  - Non-deterministic PDA
- So, why not non-deterministic Turing machines?
  - Acceptance criteria is analogous to NFA and NPDA
    - if there is a sequence of transitions to an accepting state, an NDTM will take that path.
- What does nondeterminism, a theoretical construct, mean in practice?
  - You can think of it as a boundless potential to search for and identify the correct path that leads to a solution
  - So, it does not change the class of problems that can be solved, just the time/space needed to solve.

Class *NP*: Non-deterministic Polynomial Time

How they operate:

- Guess a solution
- verify correctness in polynomial time

Polynomial time verifiability is the key property of *NP*.

- This is how you build a path from *P* to *NP*.
- Ideal formulation for search problems, where correct solutions are hard to find but easy to recognize.

Example: Boolean formula satisfiability (*SAT*)

- Given a boolean formula in CNF, find an assignment of {`true`, `false`} to variables that makes it true.
  - Why not DNF?

What are the bounds of *NP*?

- *Only* **Decision problems:**
  - Problems with an "yes" or "no" answer
  - Optimization problems are generally not in *NP*
    - But we can often find optimal solutions using "binary search"
- *"No" answers are usually not verifiable in P-time*
  - So, complement of *NP* problems are often not *NP*.
  - *UNSAT* — show that a CNF formula is false for all truth assignments[1]
- **Key point:** You cannot negate nondeterministic automata.
  - So, we are unable to convert an NDTM for *SAT* to solve *UNSAT* in *NP*-time.

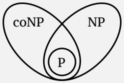
[1] Whether *UNSAT* $\in$ *NP* is unknown!

What are the bounds of *NP*?

- *Existentially quantified vs Universally quantified formulas*
  - *NP* is good for $\exists \overline{x}\, P(\overline{x})$: guess a value for $\overline{x}$ and check if $P(\overline{x})$ holds.
  - *NP* is not good for $\forall \overline{x}\, P(\overline{x})$:
    - Guessing does not seem to help if you need to check all values of $\overline{x}$.
- Negation of existential formula yields a universal formula.
  - No surprise that complement of *NP* problems are typically not in *NP*.
  - *UNSAT*: $\forall \overline{x} \neg P(\overline{x})$ where P is in CNF
  - *VALID*: $\forall \overline{x} P(\overline{x})$, where P is in DNF
- *NP seems to be a good way to separate hard problems from even harder ones!*

Co-NP: Problems whose complement is in NP

- Decision problems that have a polynomially checkable proof when the answer is "no"



What we *think* the world looks like.

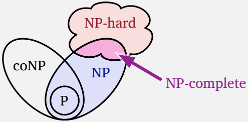
- Biggest open problem: Is $P = NP$?
  - Will also imply $co\text{-}NP = P$

The class $Co\text{-}NP \cap NP$

- Often, problems that are in $NP \cap co\text{-}NP$ are in P
- It requires considerable insight and/or structure in the problem to show that something is both NP and $co\text{-}NP$
  - This can often be turned into a P-time algorithm
- Examples
  - Linear programming [1979]
    - Obviously in NP. To see why it is in $co\text{-}NP$, we can derive a lower bound by multiplying the constraints by a suitable (guessed) number and adding.
  - Primality testing [2002]
    - Obviously in co−NP; See "primality certificate" for proof it is NP
  - Integer factorization?

NP-hard and NP-complete

- A problem Π is NP-hard if the availability of a polynomial solution to Π will allow NP-problems to be solved in polynomial time.
  - Π is NP-hard $\Leftrightarrow$ if Π can be solved in P-time, $P = NP$
- NP-complete = NP-hard $\cap$ NP



More of what we *think* the world looks like.

Polynomial-time Reducibility

- Show that a problem A could be transformed into problem B in polynomial time
  - Called a polynomial-time reduction of A to B
  - The crux of proofs involving NP-completeness
- *Implication:* if B can be solved in P-time, we can solve A in P-time
- *An NP-complete problem* is one to which any problem in NP can be reduced to.
- **Never forget the direction:** To prove a problem Π is NP-complete, need to show how all other NP problems can be solved using Π, not vice-versa!

Wait! How can I reduce *every NP* to my problem?

- If a particular NP-problem A is given to you, then you can think of a way to reduce it to your problem B
- But how do you go about proving that *every NP* problem X can be reduced to B
  - You don't even know X — indeed, the class NP is infinite!
- *If you already knew an NP-complete problem, your task is easy!*
  - Simply reduce this NP-complete problem to B, and by transitivity, you have a reduction of every $X \in NP$ to B
- So, who will bell the cat?
  - Stephen Cook [1970] and Leonid Levin [1973] managed to do this!
  - Cook was denied reappointment/tenure in 1970 at Berkeley, but won the Turing award in 1982!

The first *NP*-complete problem: *SAT*

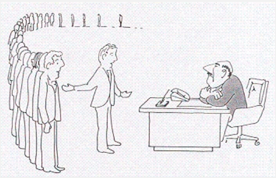
How do you show reducibility of arbitrary *NP*-problems to *SAT*? You start from the definition, of course!

- The class *NP* is defined in terms of an NDTM
  - X is in *NP* if there is an NDTM T_X that solves X in polynomial time
- Use this NDTM as the basis of proof.

Specifically, show that acceptance by an NDTM can be encoded in terms of a boolean formula

- Model T_X tape contents, tape heads, and finite state at each step as a vector of boolean variables
  - Need $(p(n))^2$ variables, where $p(n)$ is the (polynomial) runtime of T_X
- Model each transition as a boolean formula

Thanks to Cook-Levin, you can say ...

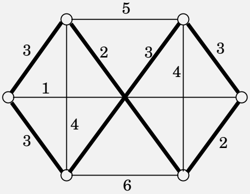


I can't find an efficient algorithm, but neither can all these famous people.

Thanks to *NP*-completeness results, you can say this even if you have been working on an obscure problem that no one ever looked at!

Some Hard Decision Problems

Traveling Salesman Problem



Given n vertices and $n(n-1)/2$ distances between them, is there a *tour* (i.e., cycle) of length b or less that passes through all vertices?

Hamiltonian Cycle

- Simpler than TSP
  - Is there a cycle that passes through every vertex in the graph?
- Earliest reference, posed in the context of chess boards and knights (“Rudrata cycle”)
- *Longest path* is another version of the same problem
  - When posed as a decision problem, becomes the same as Hamiltonian path problem

Balanced Cuts

Does there exist a way to partition vertices V in a graph into two sets S and T such that

- there are at most b edges between S and T, and
- $|S| \geq |T| \geq |V|/3$

Integer Linear Programming (ILP) and Zero-One Equations (ZOE)

ILP: Linear programing, but solutions are limited to integers

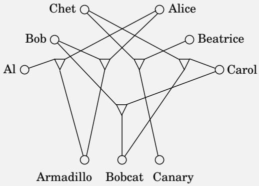
- Many problems are easy to solve over real numbers but much harder for integers.
- Examples:
  - Knapsack
  - solutions to equations such as $x^n + y^n = z^n$

ZOE: A special case of ILP, where the values are just 0 or 1.

- Find $\mathbf{x}$ such that $\mathbf{A}x = \mathbf{1}$ where $\mathbf{1}$ is a column matrix consisting of 1’s.

3d-Matching

- Given triples of compatibilities between men, women and pets, find perfect, 3-way matches.

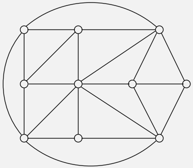


Independent set, vertex cover, and clique

Independent set: Does this graph contain a set of at least k vertices with no edge between them?

Vertex cover: Does this graph contain a set of at least k vertices that cover all edges?

Clique: Does this graph contain at least k vertices that are fully connected among themselves?

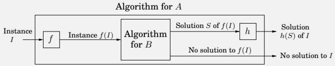


Easy Vs Hard Problems

Hard	Easy
3SAT	2SAT, HORN SAT
TSP	MST
Longest path	Shortest path
3d-matching	bipartite match
Independent set	Indep. set on trees
ILP	Linear programming
Hamiltonian cycle	Euler path,
	Knights tour
Balanced cut	Min-cut

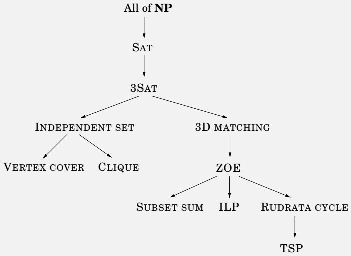
NP-completeness: Polynomial-time Reductions

- Show that a known *NP*-complete problem A could be transformed into problem B in polynomial time



- *Implication:* if B can be solved in P-time, we can solve A in P-time
- **Never forget the direction:**
  - We are proving that B is *NP*-complete here.

NP-completeness Reductions



Reducing all of *NP* to *SAT*

- We already discussed this
  - Show how to reduce acceptance by an NDTM to the *SAT* problem.
- *Exercise:* Show how to transform acceptance by an FSA into an instance of *SAT*

Reducing *SAT* to 3*SAT*

- 3*SAT*: A special case of *SAT* where each clause has ≤ 3 literals
- Reduction involves transforming a disjunction with many literals into a CNF of disjunctions with ≤ 3 literals per term
- The transformation below at most doubles the problem size.
- **Key Idea:** Introduce additional variables:
  - *Example:* $l_1 \vee l_2 \vee l_3 \vee l_4$ can be transformed into:

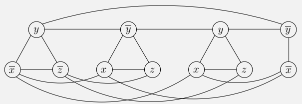
    $$(l_1 \vee l_2 \vee y_1) \wedge (\overline{y_1} \vee l_3 \vee l_4)$$

    For this conjunction to be true, one of $\{l_1, \ldots, l_4\}$ must be true:
    - So a solution to the transformed problem is a solution to the original — simply discard assignments for the new variables y_i.

Reducing 3*SAT* to Independent set

- Nontrivial reduction, as the problems are quite different in nature
- **Idea:** Model each of k clauses of 3*SAT* by a "triangle" in a graph

The graph corresponding to $(\overline{x} \vee y \vee \overline{z})\ (x \vee \overline{y} \vee z)\ (x \vee y \vee z)\ (\overline{x} \vee \overline{y})$.



  - Independent set of size k must contain one literal from each clause
    - By setting that literal to *true*, we obtain a solution for 3*SAT*
  - *Key point:* Avoid conflicts, e.g., assigning *true* to both x and $\overline{x}$
    - ensure using edges between every variable and its complement

Reducing Independent set to Vertex Cover

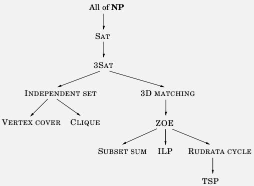
- If S is an independent set then $V - S$ is a vertex cover
  - Consider any edge e in the graph
  - *Case 1:* Both ends of e are in $V - S$
  - *Case 2:* At least one end of e is S. The other end of e cannot be in S or else S won't be independent.
  - Thus, in both cases, at least one side of e must go to $V - S$.
  - In other words $V - S$ is a vertex cover
- Thus, we have reduced independent set to vertex cover problem.

Reducing Independent set to Clique

- If S is an independent set then S is clique in $\overline{G} = (V, \overline{E})$
  - For any pair $v_1, v_2 \in S$ there is no edge in E
    - means that there is an edge between any such pair in G'
    - i.e, S is a clique in $\overline{G}$
- Thus, we have reduced independent set to the clique problem, while only using polynomial time and space.

NP-completeness Reductions

- We have discussed the left half of this picture
- We won't discuss the right half, since the proofs are similar in many ways, but are more involved.
  - You can find those reductions in the text book.



Beyond NP: PSPACE

- **PSPACE:** The class of problems that can be solved using only polynomial amount of space.
  - It is OK to take exponential (or super-exponential) time.
- *Key point:* Unlike time, space is reusable.
  - Result: many exponential algorithms are in PSPACE.
    - Consider universal formulas. We can check them in polynomial space by rerunning the same computation (say, $check(v)$) for each v.
    - The space used for *check* is recycled, but the time adds up for different v's.
- **Note:** *SAT* is in PSPACE
  - Try every possible truthe assignment for variables.
- Thus, all *NP*-complete problems are in PSPACE.

PSPACE-hard and PSPACE-complete

PSPACE-hard: A problem Π is PSPACE-hard if for any problem Π' in PSPACE there is a *P*-time reduction to Π.

PSPACE-complete: PSPACE-hard problems that are in PSPACE.

- *Examples:*

  QBF: Quantified boolean formulae

  NFA totality: Does this NFA accept all strings?

Is $NP \subsetneq$ PSPACE?

- We think so, but we can't even prove $P \subsetneq PSPACE$

Classes EXP, EXP-hard and EXP-complete

- The class EXP (aka EXPTIME) consists of the class of problems that can be solved in $O(2^{n^k})$ time for some k.
- PSPACE $\subseteq$ EXP.
  - Intuitively, you can't do more than EXP work using a PSPACE algorithm because you need polynomial amount of space even if the only thing you did is to count up to 2^n.
- As usual, EXP-hard and EXP-complete are defined using P-time reductions.
- Generalized versions of games such as chess and checkers are EXP-hard.
- We *think* $PSPACE \subsetneq EXP$, but can only prove $P \subsetneq EXP$.

Where do we stop?

- These classes can be extended for ever:

  NEXP: Nondeterministic exponential time

  EXPSPACE: Problems solvable with exponential space.

  EEXP: Problems solvable in double exp. time ($O(2^{2^{(n^k)}})$) for some k
- *Examples:*
  - Equivalence of regexpr with intersection is EXPSPACE-hard.
  - REs with negation can't be decided even in E^kEXPTIME for any k.
- $P \subseteq NP \subseteq$ PSPACE $\subseteq$ EXP $\subseteq$ NEXP $\subseteq$ EXPSPACE $\subseteq$ EEXP $\subseteq$ NEEXP $\subseteq$ EEXPSPACE $\subseteq \cdots$
- We *think* these classes are distinct, but have proofs only for classes that are 3 places apart, e.g., P and EXP.