# CSE 548: (*Design and*) Analysis of Algorithms

## Graphs

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# Overview

- Graphs provide a concise representation of a range problems

  Map coloring – more generally, resource contention problems

  Networks — communication, traffic, social, biological, ...


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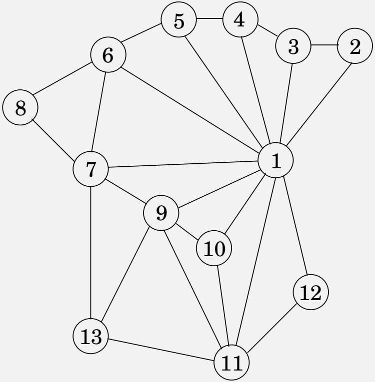


# Definition and Representations

A *graph* $G = (V, E)$, where $V$ is a set of vertices, and $E$ a set of edges. An edge $e$ of the form $(v_1, v_2)$ is said to span vertices $v_1$ and $v_2$. The edges in a *directed graph* are directed.
A $G' = (V', E')$ is called a *subgraph* of $G$ if $V' \subseteq V$ and $E'$ includes every edge in $E$ between vertices in $V'$.

**Adjacency matrix**

A graph $(V = \{v_1, \ldots, v_n\}, E)$ can be represented by an $n \times n$ matrix $a$, where $a_{ij} = 1$ iff $(v_i, v_j) \in E$

**Adjacency list**

Each vertex $v$ is associated with a linked list consisting of all vertices $u$ such that $(v, u) \in E$.

Note that adjacency matrix uses $O(n^2)$ storage, while adjacency list uses $O(|V| + |E|)$ storage. Both can represent directed as well as undirected graphs.

# Depth-First Search (DFS)

- A technique for traversing all vertices in the graph
- Very versatile, forms the linchpin of many graph algorithms

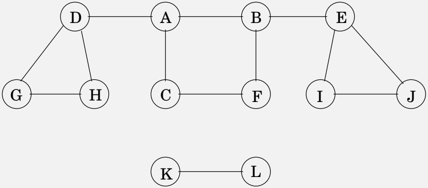
*dfs*$(V, E)$

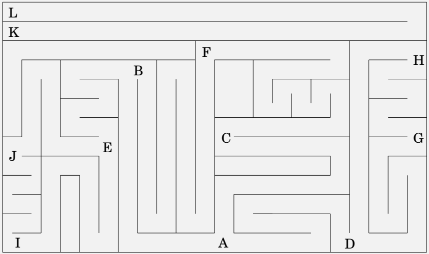
**foreach** $v \in V$ **do** $visited[v] = false$
**foreach** $v \in V$ **do**
  **if not** $visited[v]$ **then** $explore(V, E, v)$

*explore*$(V, E, v)$

$visited[v] = true$
$previsit(v)$ /*A placeholder for now*/
**foreach** $(v, u) \in E$ **do**
  **if not** $visited[u]$ **then** $explore(G, V, u)$
$postvisit(v)$ /*Another placeholder*/

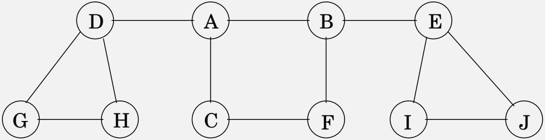
# Graphs, Mazes and DFS

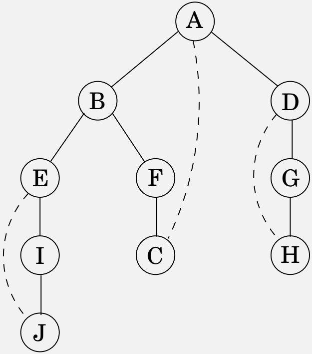



If a maze is represented as a graph, then DFS of the graph amounts to an exploration and mapping of the maze.

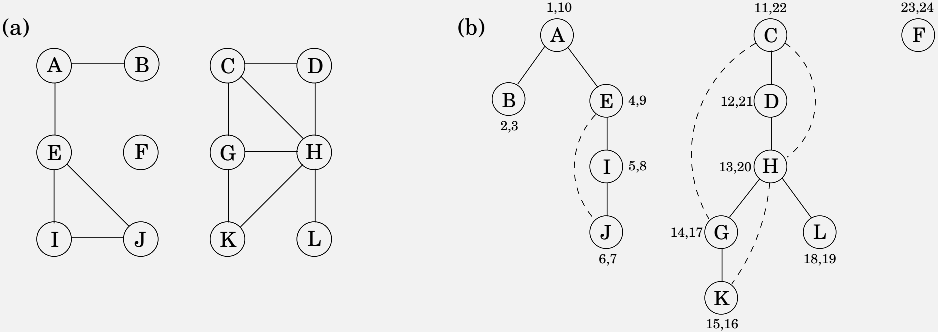
# A graph and its DFS tree




DFS uses $O(|V|)$ space and $O(|E| + |V|)$ time.

# DFS and Connected Components




A *connected component* of a graph is a maximal subgraph where there is path between any two vertices in the subgraph, i.e., it is a maximal *connected subgraph*.

# DFS Numbering

Associate post and pre numbers with each visited node by defining *previsit* and *postvisit*

**$previsit(v)$**

$pre[v] = clock$

$clock{+}{+}$

**$postvisit(v)$**

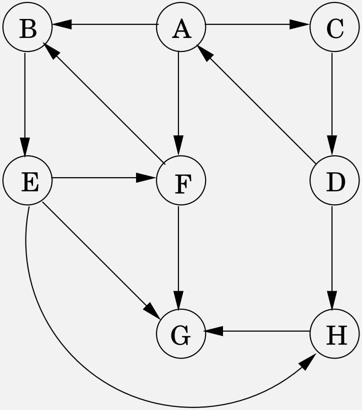
$post[v] = clock$

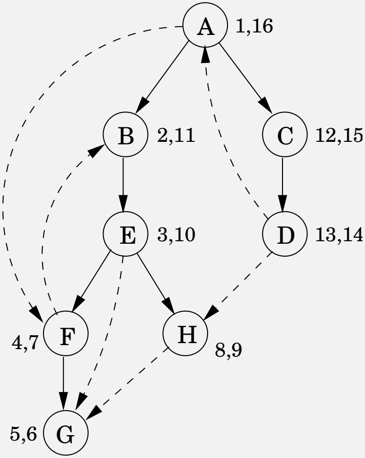
$clock{+}{+}$

**Property**

*For any two vertices $u$ and $v$, the intervals $[pre[u], post[u]]$ and $[pre[v], post[v]]$ are either disjoint, or one is contained entirely within another.*

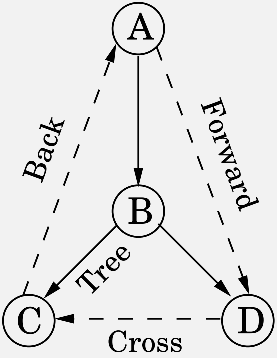
# DFS of Directed Graph



# DFS and Edge Types

## DFS tree




| `pre/post` *ordering for* $(u, v)$ | *Edge type* |
|---|---|
| $[_u\ \ [_v\ \ ]_v\ \ ]_u$ | Tree/forward |
| $[_v\ \ [_u\ \ ]_u\ \ ]_v$ | Back |
| $[_v\ \ ]_v\ \ [_u\ \ ]_u$ | Cross |

*No cross edges in undirected graphs!*

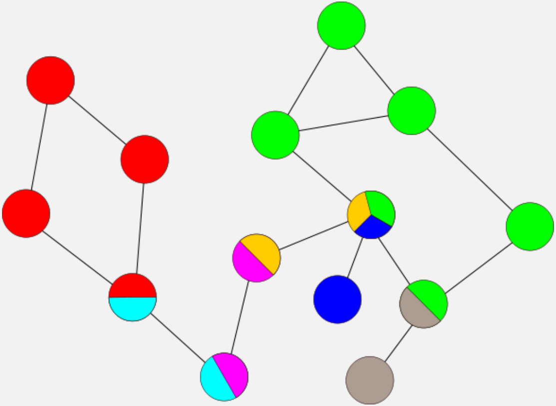
Back and forward edges merge

# Biconnectivity in Undirected Graphs

## Definition (Biconnected Components)

- *An articulation point in a graph is any vertex $a$ that lies on every path from between vertices $v, w$.*
- *A biconnected graph contains no articulation points.*
- *A biconnected component of a graph is a maximal subgraph that is biconnected.*

# Illustration of Biconnected Components



- Each biconnected component given a different color
- Articulation points have multiple colors

# Biconnected Graphs

- Biconnected components are *not* disjoint.
- Articulation points are duplicated across them.

- *Articulation points* $\equiv$ *single-points of failure*
  Graph is disconnected if any of them go down.
- Between any two $u, v \in V$ in a biconnected graph there are two node-disjoint paths (and hence a cycle).

# Finding articulation points during DFS

During DFS visit of vertex $v$, we want to decide if it is an articulation point or not. Divide $V$ into to disjoint sets:

- $V_i$ includes nodes *inside* the DFS tree rooted at $v$, and
- $V_o = V - V_i$ of nodes *outside* the DFS tree rooted at $v$.

**Observation (I)**

*Suppose that every inside node $v_i$ (i.e., $v_i \in V_i - \{v\}$) can reach some outside node $v_o$ (i.e., $v_o \in V_o$) without going through $v$. Then $v$ is* ***not*** *an articulation point.*

# Finding articulation points during DFS (2)

**Proof:**

Paths within $V_o$: By properties of DFS tree, any two vertices in $V_o$ can reach each other without going through $v$

Paths between $v_i$ and $V_o$: Once $v_i$ can reach any node in $V_o$, it can reach all other nodes in $V_o$ without having to go through $v$.

Paths within $V_i$: Consider nodes $v_i$ and $v_i'$ in $V_i - \{v\}$. By our assumption there is a path from $v_i$ to some $v_o \in V_o$, and another path from $v_i'$ to some $v_o' \in V_o$, neither involving $v$. Consider the path $v_i$ to $v_o$ to $v_o'$ to $v_i'$, which does not pass through $v$.

Thus, for every pair of vertices, there is a path between them that avoids $v$, and hence $v$ is *not* an articulation point.

# Finding articulation points during DFS (3)

**Observation (2)**

*If Observation (1) does not hold then $v$ is an articulation point.*

This means that there exists some $v_i$ that cannot reach an outside node $v_o$ without going through $v$. By definition, this means that $v$ is an articulation point.

# Finding articulation points during DFS (3)

We combine and slightly strengthen Observations (1), (2), while omitting the proof, as it is essentially the same as before.

**Observation (3)**

*Let $Y$ denote the set of ancestors of $v$ and $X$ its immediate children in the DFS tree. Now, $v$ is not an articulation point iff each $x \in X$ has a path to some $y \in Y$ without going through $v$.*

- By focusing on just the children and ancestors of $v$, this makes it easier to decide on articulation points during DFS.
- *Note than any such path from $x$ to $y$ will follow zero or more tree edges down, followed by a back edge.*

# Finding articulation points during DFS (4)

*Note than any such path from $x$ to $y$ will follow zero or more tree edges down, followed by a back edge.*

- If this path bypasses $v$, this is going to occur due to a single back-edge that goes to an ancestor of $v$. So there is no need to consider paths with multiple back-edges.
- Note: Pre-number increases while following tree edges down, while it decreases when a back edge is followed.

*Key Idea:* During DFS, for each vertex $x$, maintain the highest ancestor that can be reached from $x$ by following tree edges down and then a back edge.

# Finding articulation points during DFS (5)

*Key Idea:* During DFS, maintain the highest ancestor reachable from $x$ by following tree edges down and then a back edge.

- This info is maintained in the array *low*.
- $low[x] \leq low[x']$ for every child $x'$ of $x$ in DFS tree.
  - This captures the fact that we can follow the tree edge down from $x$ to $x'$, then go to whichever ancestor is reachable from $x'$.
  - Algorithmically, let $low[x] = min(low[x], low[x'])$ when $explore(x')$ returns
- $low[x] \leq pre[x'']$ for $x''$ adjacent to $x$ but not a parent or child in the DFS tree.
  - Algorithmically, set $low[x] = min(low[x], pre[x''])$
  - By properties of DFS, $x''$ is either a descendant or ancestor of $x$.
    - As we are taking *min*, statement effective only if $x''$ is an ancestor.

# Finding articulation points during DFS (6)

*Key Idea:* Maintain *low* during DFS

- when visiting $x$, initialize $low[x]$ to $pre[x]$.
- when a DFS call on a child $x'$ of $x$ returns, check if $low[x'] \geq pre[x]$.
  - If so, the highest ancestor $x'$ can reach is not higher than $x$, i.e., $x'$ cannot go to ancestors of $x$ without going through $x$.
  - So, mark $x$ as articulation point
- If an adjacent vertex $x''$ is already visited when $x$ considers it, set $low[x] = min(low[x], pre[x''])$ unless $x''$ is parent of $x$

All these can be done during a DFS, while increasing the cost of each step by only a constant amount — so, overall complexity is $O(|E| + |V|)$

# Directed Acyclic Graphs (DAGs)

A directed graph that contains no cycles.

Often used to represent (acyclic) dependencies, partial orders,...

**Property (DAGs and DFS)**

- *A directed graph has a cycle iff its DFS reveals a back edge.*
- *In a dag, every edge leads to a vertex with lower post number.*
- *Every dag has at least one source and one sink.*

# Strongly Connected Components (SCC)

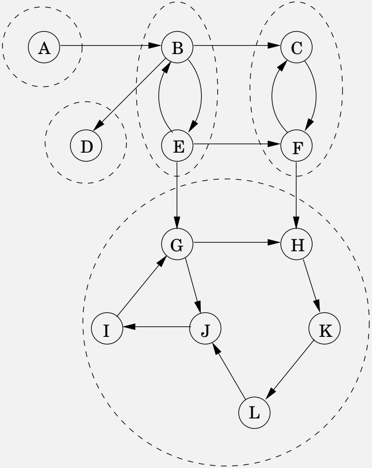
Analog of connected components for undirected graphs

**Definition (SCC)**

- *Two vertices $u$ and $v$ in a directed graph are* connected *if there is a path from $u$ to $v$ and vice-versa.*
- *A directed graph is* strongly connected *if any pair of vertices in the graph are connected.*
- *A subgraph of a directed graph is said to be an SCC if it is a maximal subgraph that is strongly connected.*

SCCs are also similar to biconnected components!

# SCC Example

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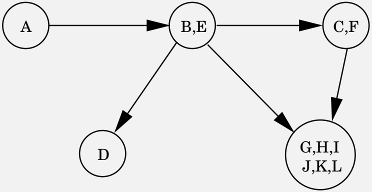



The textbook describes an algorithm for computing SCC in linear-time using DFS.

# Breadth-first Search (BFS)

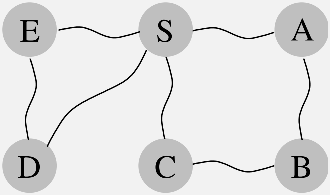
- Traverse the graph by “levels”
  - $BFS(v)$ visits $v$ first
  - Then it visits all immediate children of $v$
  - then it visits children of children of $v$, and so on.
- As compared to DFS, BFS uses a queue (rather than a stack) to remember vertices that still need to be explored

# BFS Algorithm

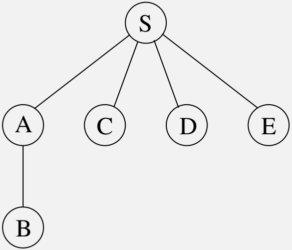
*bfs*$(V, E, s)$

**foreach** $u \in V$ **do** $visited[u] = false$
$q = \{s\}$; $visited[s] = true$
**while** $q$ is nonempty **do**
  $u = deque(q)$
  **foreach** edge $(u, v) \in E$ **do**
    **if not** $visited[v]$ **then**
      $queue(q, v)$; $visited[v] = true$

# BFS Algorithm Illustration




| Order of visitation | Queue contents after processing node |
|---|---|
| | $[S]$ |
| $S$ | $[A\ C\ D\ E]$ |
| $A$ | $[C\ D\ E\ B]$ |
| $C$ | $[D\ E\ B]$ |
| $D$ | $[E\ B]$ |
| $E$ | $[B]$ |
| $B$ | $[\,]$ |



# Shortest Paths and BFS

BFS automatically computes shortest paths!

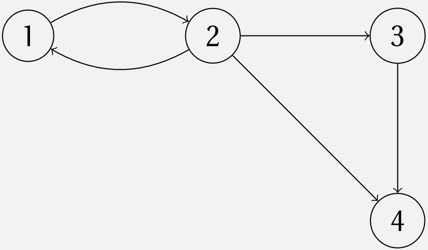
$bfs(V, E, s)$

**foreach** $u \in V$ **do** $dist[u] = \infty$
$q = \{s\}; dist[s] = 0$
**while** $q$ is nonempty **do**
  $u = deque(q)$
  **foreach** edge $(u, v) \in E$ **do**
    **if** $dist[v] = \infty$ **then**
      $queue(q, v); dist[v] = dist[u] + 1$

But not all paths are created equal! We would like to compute shortest weighted path — a topic of future lecture.

# Graph paths and Boolean Matrices

A graph and its boolean matrix representation




$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Graph paths and Boolean Matrices

- Let $A$ be the adjacency matrix for a graph $G$, and $B = A \times A$. Now, $B_{ij} = 1$ iff there is path in the graph of length 2 from $v_i$ to $v_j$
- Let $C = A + B$. Then $C_{ij} = 1$ iff there is path of length $\leq 2$ between $v_i$ and $v_j$
- Define $A^* = A^0 + A^1 + A^2 + \cdots$. If $D = A^*$ then $D_{ij} = 1$ iff $v_j$ is reachable from $v_i$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Shortest paths and Matrix Operations

- Redefine operations on matrix elements so that $+$ becomes *min*, and $*$ becomes integer addition.
- $D = A^*$ then $D_{ij} = k$ iff the shortest path from $v_j$ to $v_i$ is of length $k$