

DATA STRUCTURES I, II, III, AND IV

- I. Amortized Analysis
- II. Binary and Binomial Heaps
- III. Fibonacci Heaps
- IV. Union-Find

Lecture slides by Kevin Wayne

<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

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Data structures

Static problems. Given an input, produce an output.

Ex. Sorting, FFT, edit distance, shortest paths, MST, max-flow, ...

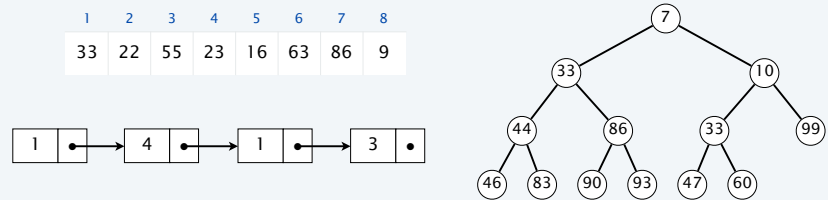
Dynamic problems. Given a sequence of operations (given one at a time), produce a sequence of outputs.

Ex. Stack, queue, priority queue, symbol table, union-find,

Algorithm. Step-by-step procedure to solve a problem.

Data structure. Way to store and organize data.

Ex. Array, linked list, binary heap, binary search tree, hash table, ...



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Appetizer

Goal. Design a data structure to support all operations in $O(1)$ time.

- $\text{INIT}(n)$: create and return an **initialized** array (all zero) of length n .
- $\text{READ}(A, i)$: return element i in array.
- $\text{WRITE}(A, i, \text{value})$: set element i in array to *value*.

Assumptions.

- Can MALLOC an uninitialized array of length n in $O(1)$ time.
- Given an array, can read or write element i in $O(1)$ time.

true in C or C++, but not Java

Remark. An array does INIT in $\Theta(n)$ time and READ and WRITE in $\Theta(1)$ time.

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Appetizer

Data structure. Three arrays $A[1..n]$, $B[1..n]$, and $C[1..n]$, and an integer k .

- $A[i]$ stores the current value for READ (if initialized).
- k = number of initialized entries.
- $C[j]$ = index of j^{th} initialized element for $j = 1, \dots, k$.
- If $C[j] = i$, then $B[i] = j$ for $j = 1, \dots, k$.

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. Ahead.

	1	2	3	4	5	6	7	8
A[]	?	22	55	99	?	33	?	?
B[]	?	3	4	1	?	2	?	?
C[]	4	6	2	3	?	?	?	?

$k = 4$

$A[4]=99, A[6]=33, A[2]=22$, and $A[3]=55$ initialized in that order

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Appetizer

INIT (A, n)

```

 $k \leftarrow 0$ .
 $A \leftarrow \text{MALLOC}(n)$ .
 $B \leftarrow \text{MALLOC}(n)$ .
 $C \leftarrow \text{MALLOC}(n)$ .
    
```

READ (A, i)

```

IF (IS-INITIALIZED ( $A[i]$ ))
    RETURN  $A[i]$ .
ELSE
    RETURN 0.
    
```

WRITE (A, i, value)

```

IF (IS-INITIALIZED ( $A[i]$ ))
     $A[i] \leftarrow \text{value}$ .
ELSE
     $k \leftarrow k + 1$ .
     $A[i] \leftarrow \text{value}$ .
     $B[i] \leftarrow k$ .
     $C[k] \leftarrow i$ .
    
```

IS-INITIALIZED (A, i)

```

IF ( $1 \leq B[i] \leq k$ ) and ( $C[B[i]] = i$ )
    RETURN true.
ELSE
    RETURN false.
    
```

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Appetizer

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. \Rightarrow

- Suppose $A[i]$ is the j^{th} entry to be initialized.
- Then $C[j] = i$ and $B[i] = j$.
- Thus, $C[B[i]] = i$.

	1	2	3	4	5	6	7	8
A[]	?	22	55	99	?	33	?	?
B[]	?	3	4	1	?	2	?	?
C[]	4	6	2	3	?	?	?	?

$k = 4$

$A[4]=99$, $A[6]=33$, $A[2]=22$, and $A[3]=55$ initialized in that order

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Appetizer

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. \Leftarrow

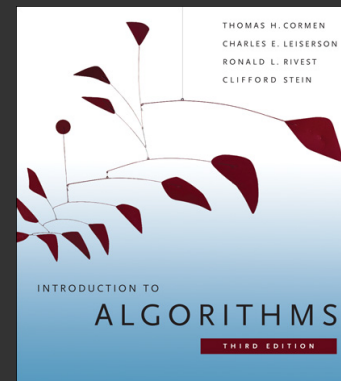
- Suppose $A[i]$ is uninitialized.
- If $B[i] < 1$ or $B[i] > k$, then $A[i]$ clearly uninitialized.
- If $1 \leq B[i] \leq k$ by coincidence, then we still can't have $C[B[i]] = i$ because none of the entries $C[1..k]$ can equal i . ■

	1	2	3	4	5	6	7	8
A[]	?	22	55	99	?	33	?	?
B[]	?	3	4	1	?	2	?	?
C[]	4	6	2	3	?	?	?	?

$k = 4$

$A[4]=99$, $A[6]=33$, $A[2]=22$, and $A[3]=55$ initialized in that order

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AMORTIZED ANALYSIS

- binary counter
- multi-pop stack
- dynamic table

Amortized analysis

Worst-case analysis. Determine worst-case running time of a data structure operation as function of the input size n .

can be too pessimistic if the only way to encounter an expensive operation is when there were lots of previous cheap operations

Amortized analysis. Determine worst-case running time of a **sequence** of n data structure operations.

Ex. Starting from an empty stack implemented with a dynamic table, any sequence of n push and pop operations takes $O(n)$ time in the worst case.

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Amortized analysis: applications

- Splay trees.
- Dynamic table.
- Fibonacci heaps.
- Garbage collection.
- Move-to-front list updating.
- Push-relabel algorithm for max flow.
- Path compression for disjoint-set union.
- Structural modifications to red-black trees.
- Security, databases, distributed computing, ...

SIAM J. ALG. DISC. METH.
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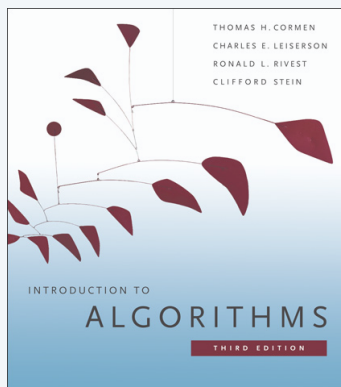
AMORTIZED COMPUTATIONAL COMPLEXITY*

ROBERT ENDRE TARJAN†

Abstract. A powerful technique in the complexity analysis of data structures is *amortization*, or averaging over time. Amortized running time is a realistic but robust complexity measure for which we can obtain surprisingly tight upper and lower bounds on a variety of algorithms. By following the principle of designing algorithms whose amortized complexity is low, we obtain “self-adjusting” data structures that are simple, flexible and efficient. This paper surveys recent work by several researchers on amortized complexity.

ASM(MOS) subject classifications. 68C25, 68E05

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CHAPTER 17

AMORTIZED ANALYSIS

- ▶ *binary counter*
- ▶ *multi-pop stack*
- ▶ *dynamic table*

Binary counter

Goal. Increment a k -bit binary counter (mod 2^k).

Representation. $A[j] = j^{\text{th}}$ least significant bit of counter.

Counter value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
14	0	0	0	0	1	1	1	0
15	0	0	0	0	1	1	1	1
16	0	0	0	1	0	0	0	0

Cost model. Number of bits flipped.

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Binary counter

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Representation. $A[j] = j^{\text{th}}$ least significant bit of counter.

Counter value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
14	0	0	0	0	1	1	1	0
15	0	0	0	0	1	1	1	1
16	0	0	0	1	0	0	0	0

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(nk)$ bits. ← overly pessimistic upper bound

Pf. At most k bits flipped per increment. ■

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Aggregate method (brute force)

Aggregate method. Analyze cost of a **sequence** of operations.

Counter value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

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Binary counter: aggregate method

Starting from the zero counter, in a sequence of n INCREMENT operations:

- Bit 0 flips n times.
- Bit 1 flips $\lfloor n/2 \rfloor$ times.
- Bit 2 flips $\lfloor n/4 \rfloor$ times.
- ...

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

- Bit j flips $\lfloor n/2^j \rfloor$ times.
- The total number of bits flipped is $\sum_{j=0}^{k-1} \left\lfloor \frac{n}{2^j} \right\rfloor < n \sum_{j=0}^{\infty} \frac{1}{2^j} = 2n$. ■

Remark. Theorem may be false if initial counter is not zero.

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Accounting method (banker's method)

Assign (potentially) different charges to each operation.

- D_i = data structure after i^{th} operation.
- c_i = actual cost of i^{th} operation.
- \hat{c}_i = amortized cost of i^{th} operation = amount we charge operation i .
- When $\hat{c}_i > c_i$, we store credits in data structure D_i to pay for future ops; when $\hat{c}_i < c_i$, we consume credits in data structure D_i .
- Initial data structure D_0 starts with 0 credits.

can be more or less than actual cost

Credit invariant. The total number of credits in the data structure ≥ 0 .

$$\sum_{i=1} \hat{c}_i - \sum_{i=1} c_i \geq 0 \quad \leftarrow \text{our job is to choose suitable amortized costs so that this invariant holds}$$



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- Initial data structure D_0 starts with 0 credits.

can be more or less than actual cost

Credit invariant. The total number of credits in the data structure ≥ 0 .

$$\sum_{i=1}^n \hat{c}_i - \sum_{i=1}^n c_i \geq 0$$

Theorem. Starting from the initial data structure D_0 , the total actual cost of any sequence of n operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of n operations is: $\sum_{i=1}^n \hat{c}_i \geq \sum_{i=1}^n c_i$. credit invariant

Intuition. Measure running time in terms of credits (time = money).

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Binary counter: accounting method

Credits. One credit pays for a bit flip.

Invariant. Each 1 bit has one credit; each 0 bit has zero credits.

Accounting.

- Flip bit j from 0 to 1: charge 2 credits (use one and save one in bit j).

increment



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Binary counter: accounting method

Credits. One credit pays for a bit flip.

Invariant. Each 1 bit has one credit; each 0 bit has zero credits.

Accounting.

- Flip bit j from 0 to 1: charge 2 credits (use one and save one in bit j).
- Flip bit j from 1 to 0: pay for it with the 1 credit saved in bit j .

increment



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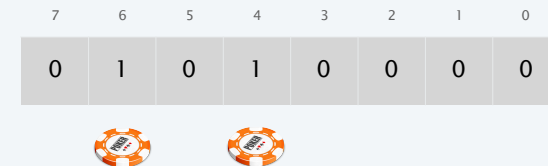
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Binary counter: accounting method

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Invariant. Each 1 bit has one credit; each 0 bit has zero credits.

Accounting.

- Flip bit j from 0 to 1: charge 2 credits (use one and save one in bit j).
- Flip bit j from 1 to 0: pay for it with the 1 credit saved in bit j .

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

- Each INCREMENT operation flips at most one 0 bit to a 1 bit, so the amortized cost per INCREMENT ≤ 2 .
- Invariant \Rightarrow number of credits in data structure ≥ 0 .
- Total actual cost of n operations \leq sum of amortized costs $\leq 2n$. ■

accounting method theorem

the rightmost 0 bit
(unless counter overflows)

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Potential method (physicist's method)

Potential function. $\Phi(D_i)$ maps each data structure D_i to a real number s.t.:

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each data structure D_i .

Actual and amortized costs.

- c_i = actual cost of i^{th} operation.
- $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$ = amortized cost of i^{th} operation.

our job is to choose
a potential function
so that the amortized cost
of each operation is low

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Potential method (physicist's method)

Potential function. $\Phi(D_i)$ maps each data structure D_i to a real number s.t.:

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each data structure D_i .

Actual and amortized costs.

- c_i = actual cost of i^{th} operation.
- $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$ = amortized cost of i^{th} operation.

Theorem. Starting from the initial data structure D_0 , the total actual cost of any sequence of n operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of operations is:

$$\begin{aligned} \sum_{i=1}^n \hat{c}_i &= \sum_{i=1}^n (c_i + \Phi(D_i) - \Phi(D_{i-1})) \\ &= \sum_{i=1}^n c_i + \Phi(D_n) - \Phi(D_0) \\ &\geq \sum_{i=1}^n c_i \quad \blacksquare \end{aligned}$$

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Binary counter: potential method

Potential function. Let $\Phi(D)$ = number of 1 bits in the binary counter D .

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

increment

7	6	5	4	3	2	1	0
0	1	0	0	1	1	1	1



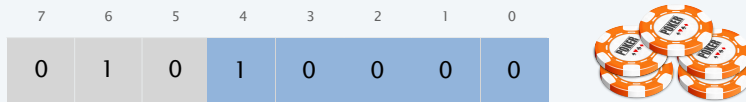
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Binary counter: potential method

Potential function. Let $\Phi(D)$ = number of 1 bits in the binary counter D .

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

increment

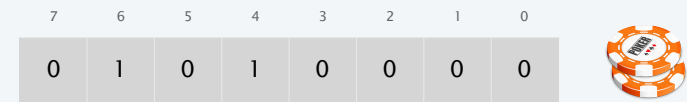


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Binary counter: potential method

Potential function. Let $\Phi(D)$ = number of 1 bits in the binary counter D .

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .



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Binary counter: potential method

Potential function. Let $\Phi(D)$ = number of 1 bits in the binary counter D .

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

- Suppose that the i^{th} INCREMENT operation flips t_i bits from 1 to 0.
- The actual cost $c_i \leq t_i + 1$. ← operation flips at most one bit from 0 to 1 (no bits flipped to 1 when counter overflows)
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

$$\leq c_i + 1 - t_i$$
← potential decreases by 1 for t_i bits flipped from 1 to 0 and increases by 1 for bit flipped from 0 to 1

$$\leq 2.$$
- Total actual cost of n operations \leq sum of amortized costs $\leq 2n$. ■

↑
potential method theorem

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Famous potential functions

Fibonacci heaps. $\Phi(H) = 2 \text{ trees}(H) + 2 \text{ marks}(H)$

Splay trees. $\Phi(T) = \sum_{x \in T} \lfloor \log_2 \text{size}(x) \rfloor$

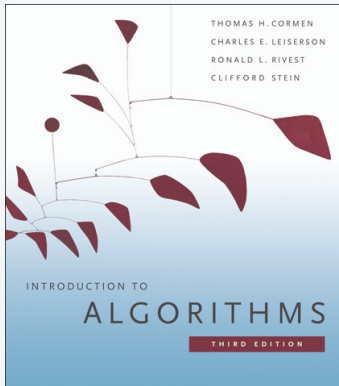
Move-to-front. $\Phi(L) = 2 \text{ inversions}(L, L^*)$

Preflow-push. $\Phi(f) = \sum_{v: \text{excess}(v) > 0} \text{height}(v)$

Red-black trees. $\Phi(T) = \sum_{x \in T} w(x)$

$$w(x) = \begin{cases} 0 & \text{if } x \text{ is red} \\ 1 & \text{if } x \text{ is black and has no red children} \\ 0 & \text{if } x \text{ is black and has one red child} \\ 2 & \text{if } x \text{ is black and has two red children} \end{cases}$$

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SECTION 17.4

AMORTIZED ANALYSIS

- ▶ *binary counter*
- ▶ *multi-pop stack*
- ▶ *dynamic table*

Multipop stack

Goal. Support operations on a set of elements:

- $\text{PUSH}(S, x)$: add element x to stack S .
- $\text{POP}(S)$: remove and return the most-recently added element.
- $\text{MULTI-POP}(S, k)$: remove the most-recently added k elements.

```
MULTI-POP( $S, k$ )
FOR  $i = 1$  TO  $k$ 
    POP( $S$ ).
```

Exceptions. We assume POP throws an exception if stack is empty.

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Multipop stack

Goal. Support operations on a set of elements:

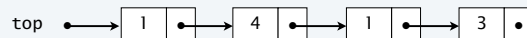
- $\text{PUSH}(S, x)$: add element x to stack S .
- $\text{POP}(S)$: remove and return the most-recently added element.
- $\text{MULTI-POP}(S, k)$: remove the most-recently added k elements.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n^2)$ time.

Pf.

- Use a singly linked list.
- POP and PUSH take $O(1)$ time each.
- MULTI-POP takes $O(n)$ time. ■

← overly pessimistic upper bound



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Multipop stack: aggregate method

Goal. Support operations on a set of elements:

- $\text{PUSH}(S, x)$: add element x to stack S .
- $\text{POP}(S)$: remove and return the most-recently added element.
- $\text{MULTI-POP}(S, k)$: remove the most-recently added k elements.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

- An element is popped at most once for each time that it is pushed.
- There are $\leq n$ PUSH operations.
- Thus, there are $\leq n$ POP operations (including those made within MULTI-POP). ■

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Multipop stack: accounting method

Credits. 1 credit pays for either a PUSH or POP.

Invariant. Every element on the stack has 1 credit.

Accounting.

- PUSH(S, x): charge 2 credits.
 - use 1 credit to pay for pushing x now
 - store 1 credit to pay for popping x at some point in the future
- POP(S): charge 0 credits.
- MULTIPOP(S, k): charge 0 credits.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

- Invariant \Rightarrow number of credits in data structure ≥ 0 .
- Amortized cost per operation ≤ 2 .
- Total actual cost of n operations \leq sum of amortized costs $\leq 2n$. ■

↑
accounting method theorem

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Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 1: push]

- Suppose that the i^{th} operation is a PUSH.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$.

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Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 2: pop]

- Suppose that the i^{th} operation is a POP.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$.

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Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 3: multi-pop]

- Suppose that the i^{th} operation is a MULTI-POP of k objects.
- The actual cost $c_i = k$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k - k = 0$. ■

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Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

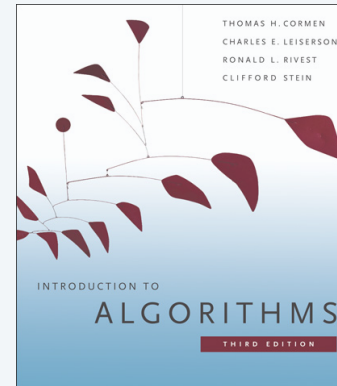
Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [putting everything together]

- Amortized cost $\hat{c}_i \leq 2$. ← 2 for push; 0 for pop and multi-pop
- Sum of amortized costs \hat{c}_i of the n operations $\leq 2n$.
- Total actual cost \leq sum of amortized cost $\leq 2n$. ▀

↑
potential method theorem

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SECTION 17.4

AMORTIZED ANALYSIS

- ▶ binary counter
- ▶ multi-pop stack
- ▶ dynamic table

Dynamic table

Goal. Store items in a table (e.g., for hash table, binary heap).

- Two operations: INSERT and DELETE.
 - too many items inserted \Rightarrow expand table.
 - too many items deleted \Rightarrow contract table.
- Requirement: if table contains m items, then space = $\Theta(m)$.

Theorem. Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n^2)$ time.

Pf. Each INSERT or DELETE takes $O(n)$ time. ▀

←
overly pessimistic
upper bound

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Dynamic table: insert only

- When inserting into an empty table, allocate a table of capacity 1.
- When inserting into a full table, allocate a new table of twice the capacity and copy all items.
- Insert item into table.

insert	old capacity	new capacity	insert cost	copy cost
1	0	1	1	–
2	1	2	1	1
3	2	4	1	2
4	4	4	1	–
5	4	8	1	4
6	8	8	1	–
7	8	8	1	–
8	8	8	1	–
9	8	16	1	8
⋮	⋮	⋮	⋮	⋮

Cost model. Number of items written (due to insertion or copy).

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Dynamic table: insert only (aggregate method)

Theorem. [via aggregate method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let c_i denote the cost of the i^{th} insertion.

$$c_i = \begin{cases} i & \text{if } i - 1 \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Starting from empty table, the cost of a sequence of n INSERT operations is:

$$\begin{aligned} \sum_{i=1}^n c_i &\leq n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j \\ &< n + 2n \\ &= 3n \quad \blacksquare \end{aligned}$$

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Dynamic table demo: insert only (accounting method)



Insert. Charge 3 credits (use 1 credit to insert; save 2 with new item).

Invariant. 2 credits with each item in right half of table; none in left half.

insert N

capacity = 16



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Dynamic table: insert only (accounting method)

Insert. Charge 3 credits (use 1 credit to insert; save 2 with new item).

Invariant. 2 credits with each item in right half of table; none in left half.

Pf. [induction]

- Each newly inserted item gets 2 credits. ↑
slight cheat if table capacity = 1
(can charge only 2 credits for first insert)
- When table doubles from k to $2k$, $k/2$ items in the table have 2 credits.
 - these k credits pay for the work needed to copy the k items
 - now, all k items are in left half of table (and have 0 credits)

Theorem. [via accounting method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf.

- Invariant \Rightarrow number of credits in data structure ≥ 0 .
- Amortized cost per INSERT = 3.
- Total actual cost of n operations \leq sum of amortized cost $\leq 3n$. ↑
accounting method theorem \blacksquare

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Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \text{ size}(D_i) - \text{capacity}(D_i)$.

- $\Phi(D_0) = 0$. ↑
number of elements ↑
capacity of array
- $\Phi(D_i) \geq 0$ for each D_i . ← immediately after doubling
 $\text{capacity}(D_i) = 2 \text{ size}(D_i)$



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Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

\uparrow \uparrow
 number of capacity of
 elements array

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

Case 0. [first insertion]

- Actual cost $c_1 = 1$.
- $\Phi(D_1) - \Phi(D_0) = (2 \text{size}(D_1) - \text{capacity}(D_1)) - (2 \text{size}(D_0) - \text{capacity}(D_0))$
 $= 1$.
- Amortized cost $\hat{c}_1 = c_1 + (\Phi(D_1) - \Phi(D_0))$
 $= 1 + 1$
 $= 2$.

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Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

\uparrow \uparrow
 number of capacity of
 elements array

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

Case 1. [no array expansion] $\text{capacity}(D_i) = \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = (2 \text{size}(D_i) - \text{capacity}(D_i)) - (2 \text{size}(D_{i-1}) - \text{capacity}(D_{i-1}))$
 $= 2$.
- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
 $= 1 + 2$
 $= 3$.

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Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

\uparrow \uparrow
 number of capacity of
 elements array

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

Case 2. [array expansion] $\text{capacity}(D_i) = 2 \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1 + \text{capacity}(D_{i-1})$.
- $\Phi(D_i) - \Phi(D_{i-1}) = (2 \text{size}(D_i) - \text{capacity}(D_i)) - (2 \text{size}(D_{i-1}) - \text{capacity}(D_{i-1}))$
 $= 2 - \text{capacity}(D_i) + \text{capacity}(D_{i-1})$
 $= 2 - \text{capacity}(D_{i-1})$.
- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
 $= 1 + \text{capacity}(D_{i-1}) + (2 - \text{capacity}(D_{i-1}))$
 $= 3$.

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Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

\uparrow \uparrow
 number of capacity of
 elements array

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i .

[putting everything together]

- Amortized cost per operation $\hat{c}_i \leq 3$.
- Total actual cost of n operations \leq sum of amortized cost $\leq 3n$. ■

\uparrow
 potential method theorem

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Dynamic table: doubling and halving

Thrashing.

- INSERT: when inserting into a full table, double capacity.
- DELETE: when deleting from a table that is $\frac{1}{2}$ -full, halve capacity.

Efficient solution.

- When inserting into an empty table, initialize table size to 1; when deleting from a table of size 1, free the table.
- INSERT: when inserting into a full table, double capacity.
- DELETE: when deleting from a table that is $\frac{1}{4}$ -full, halve capacity.

Memory usage. A dynamic table uses $\Theta(n)$ memory to store n items.

Pf. Table is always between 25% and 100% full. ■

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Dynamic table demo: insert and delete (accounting method)



Insert. Charge 3 credits (1 to insert; save 2 with item if in right half).

Delete. Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

Invariant 1. 2 credits with each item in right half of table.

Invariant 2. 1 credit with each empty slot in left half of table.

delete M

capacity = 16



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Dynamic table: insert and delete (accounting method)

Insert. Charge 3 credits (1 to insert; save 2 with item if in right half).

Delete. Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

discard any existing or extra credits

Invariant 1. 2 credits with each item in right half of table. ← to pay for expansion

Invariant 2. 1 credit with each empty slot in left half of table. ← to pay for contraction

Theorem. [via accounting method] Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n)$ time.

Pf.

- Invariants \Rightarrow number of credits in data structure ≥ 0 .
- Amortized cost per operation ≤ 3 .
- Total actual cost of n operations \leq sum of amortized cost $\leq 3n$. ■

↑
accounting method theorem

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Dynamic table: insert and delete (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n)$ time.

Pf sketch.

- Let $\alpha(D_i) = \text{size}(D_i) / \text{capacity}(D_i)$.

- Define $\Phi(D_i) = \begin{cases} 2 \text{size}(D_i) - \text{capacity}(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\ \frac{1}{2} \text{capacity}(D_i) - \text{size}(D_i) & \text{if } \alpha(D_i) < 1/2 \end{cases}$

- $\Phi(D_0) = 0, \Phi(D_i) \geq 0$. [a potential function]
- When $\alpha(D_i) = 1/2, \Phi(D_i) = 0$. [zero potential after resizing]
- When $\alpha(D_i) = 1, \Phi(D_i) = \text{size}(D_i)$. [can pay for expansion]
- When $\alpha(D_i) = 1/4, \Phi(D_i) = \text{size}(D_i)$. [can pay for contraction]

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