Heat Transfer: Conduction

Solution of the differential equation for fins

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Learning goals

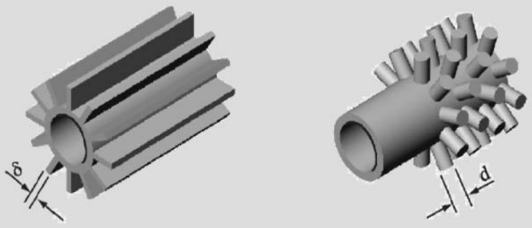
Fins differential equation:

- Homogenization of the fin differential equation
- General solution of the differential equation

$$\frac{\partial^2 \theta}{\partial x^2} - m^2\,\theta(x) = 0$$

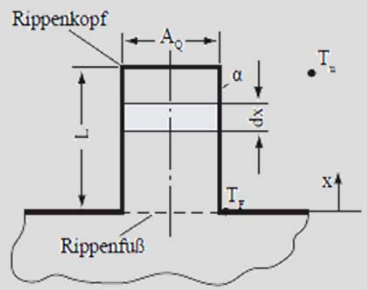
Definition of the fin parameter m :

- Interpretation of the fin parameter m for different fin geometries

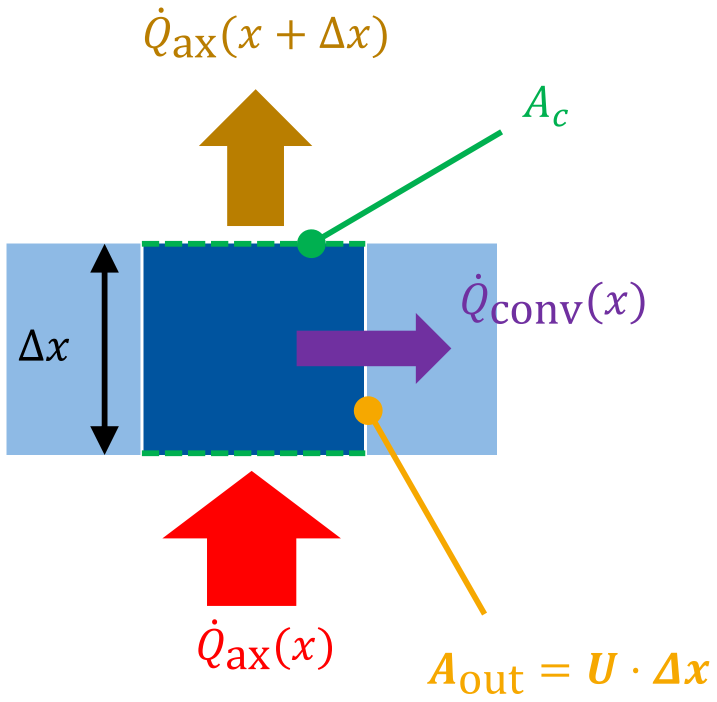


Boundary conditions:

- Recognition and implementation of different boundary conditions for the fin problem



Review: Differential equation for fins



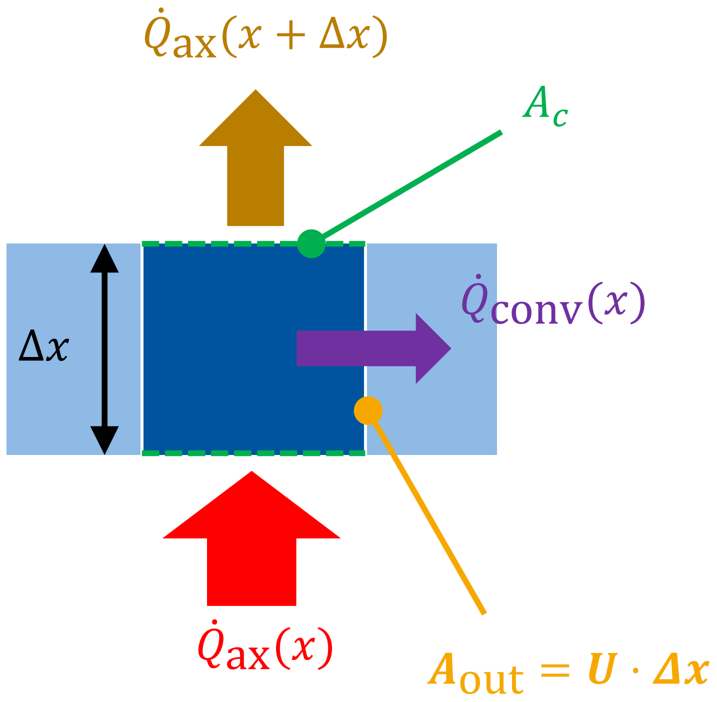
Inhomogeneous differential equation 2nd order:

$$-\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} = \alpha \cdot U \left(T(x) - T_A\right)$$

Explanation:

$\dot{Q}_{\mathrm{ax}}$:	Heat conduction in axial direction
$\dot{Q}_{\mathrm{conv}}$:	Convective heat dissipation to environment
Δx:	Length of the finite element
A_c:	Cross-sectional area of the fin
A_{out}:	Outer surface area (shell area) of the finite element
U:	Circumference (perimeter) of the fin
$\mathrm{T_A}$:	Ambient temperature

Solution of the differential equation for fins



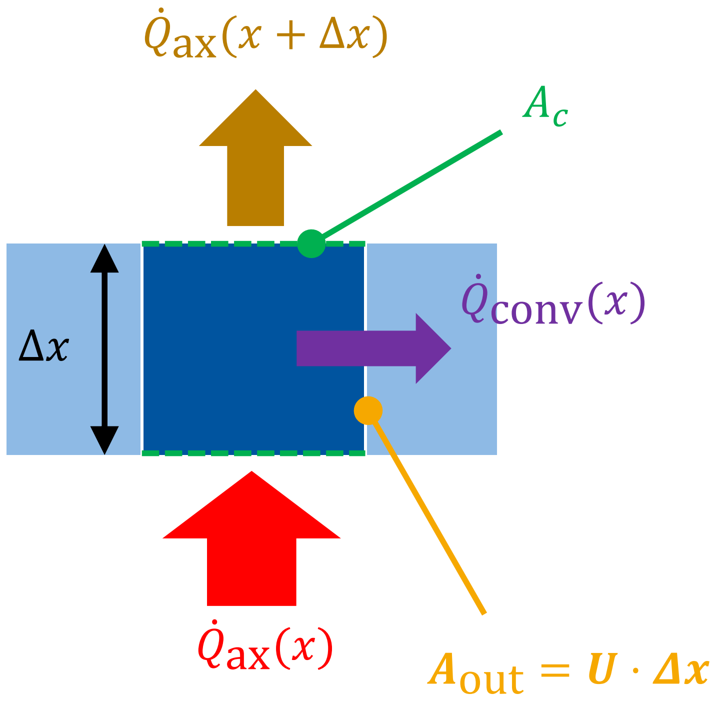
Inhomogeneous differential equation 2nd order:

$$-\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} = \alpha \cdot U \left(T(x) - T_A\right)$$

Due to T_A as a constant ambient temperature in the differential equation, the equation is inhomogeneous.

For the solution of the differential equation the method of homogenization is suitable.

Homogenization of the differential equation



Inhomogeneous differential equation 2nd order:

$$-\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} - \alpha \cdot U\ (T\ (x) - T_A) = 0$$

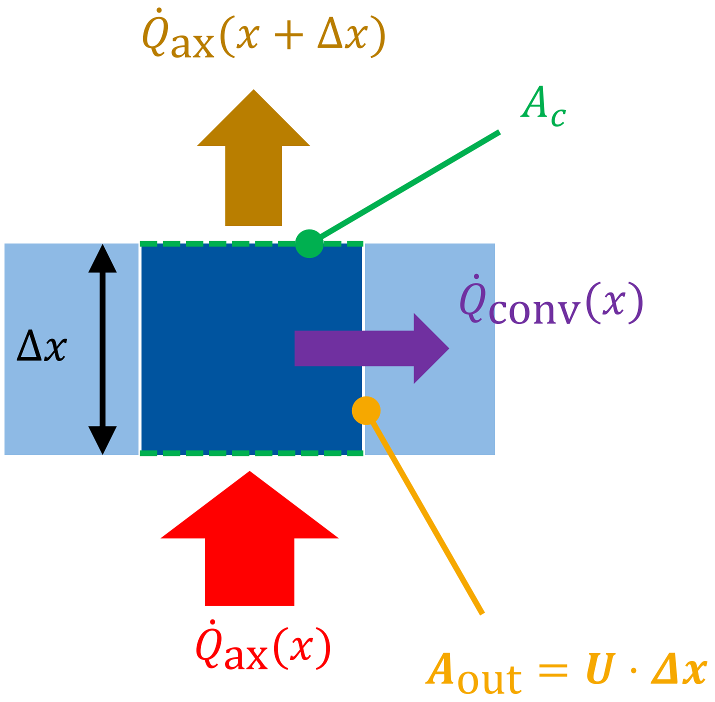
Homogenization of the equation by introducing the parameter θ (temperature difference):

Definition: $\theta(x) = T(x) - T_A$

1. Derivation: $\frac{\partial \theta(x)}{\partial x} = \frac{\partial T(x)}{\partial x}$

2. Derivation: $\frac{\partial^2 \theta(x)}{\partial x^2} = \frac{\partial^2 T(x)}{\partial x^2}$

Introduction of the fin parameter m



Inhomogeneous differential equation 2nd order:

$$-\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} - \alpha \cdot U \left(T(x) - T_A\right) = 0$$

$$\theta(x) = T(x) - T_A$$

Substituting $\theta(x)$ into the equation:

$$\frac{\partial^2 \theta(x)}{\partial x^2} - \underbrace{\frac{\alpha \cdot U}{\lambda \cdot A_c}}_{= m^2 \text{ „Fin parameter“}} \theta(x) = 0$$

$$\frac{\partial^2 \theta(x)}{\partial x^2} - m^2\, \theta(x) = 0$$

Fin parameter m

Fin parameter is dependent on:

- Thermal conductivity of the fin
- Geometry of the fin
- Heat transfer coefficient to the surrounding medium

$$\boxed{m^2 = \frac{\alpha \cdot U}{\lambda \cdot A_c}}$$

Example of geometry:

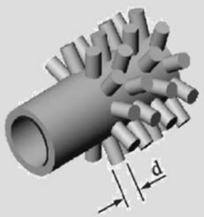
Bar fins:

Perimeter: $U = \pi\, d$

Cross section area: $A_c = \frac{\pi d^2}{4}$

$$m^2 = \frac{\pi\, d}{\pi d^2/4}$$

$$\boxed{m^2 = \frac{4\,\alpha}{\lambda\, d}}$$



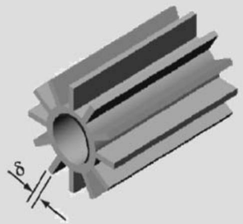
Plane fins:

Perimeter: $U = 2\,(\,\delta + T)$

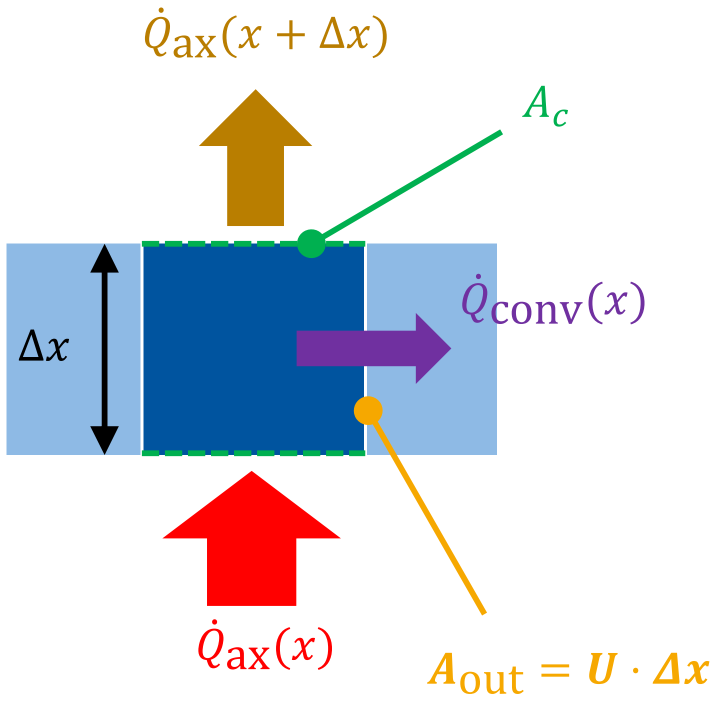
Cross section area: $A_c = \delta \cdot T$

$$m^2 = \frac{2\,(\,\delta + T)}{\delta \cdot T}$$

For $\delta \ll T$: $\boxed{m^2 \approx \frac{2\,\alpha}{\lambda \delta}}$



General solution of the fin equation



Inhomogeneous differential equation 2nd order:

$$\frac{\partial^2 \theta(x)}{\partial x^2} - m^2\,\theta(x) = 0$$

General solution of the fin differential equation:

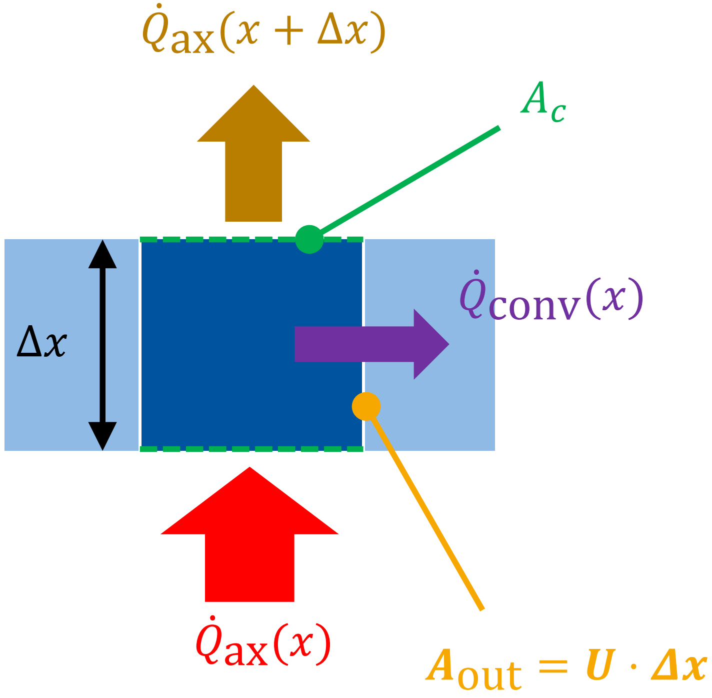
$$\theta(x) = A \cdot \sinh(m \cdot x) + B\cosh(m \cdot x)$$

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

Diff. eq. 2nd order $\Rightarrow$ **Two boundary conditions required!**

A , B and C, D are the unknown constants which are to be found with the help of boundary conditions.

Mathematical relations



General solution of the fin differential equation:

$$\theta(x) = A \cdot \sinh(m \cdot x) + B \cosh(m \cdot x)$$

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

Mathematical transformation:

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\Rightarrow \quad C = \frac{A + B}{2}, \qquad D = \frac{A - B}{2}$$

Boundary conditions

Boundary conditions:

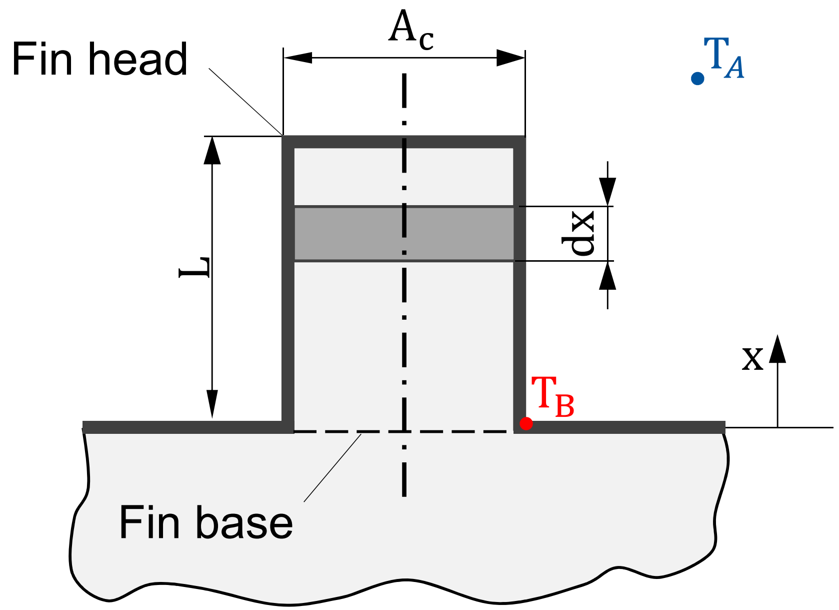
- Usually boundary conditions are defined at the base and head of the fins.
- The base of the fin is where the fin starts to dissipate heat to the environment by convection.

Boundary condition at the fin base $(x = 0)$:

Known temperature at the fin base:

$$T(x = 0) = T_B$$

$$\theta(x = 0) = T_B - T_A$$



Boundary conditions

Boundary conditions:

- Usually, boundary conditions are defined at the base and head of the fins.
- The base of the fin is where the fin starts to dissipate heat to the environment by convection.

Boundary condition at the fin head $(x = \mathrm{L})$:

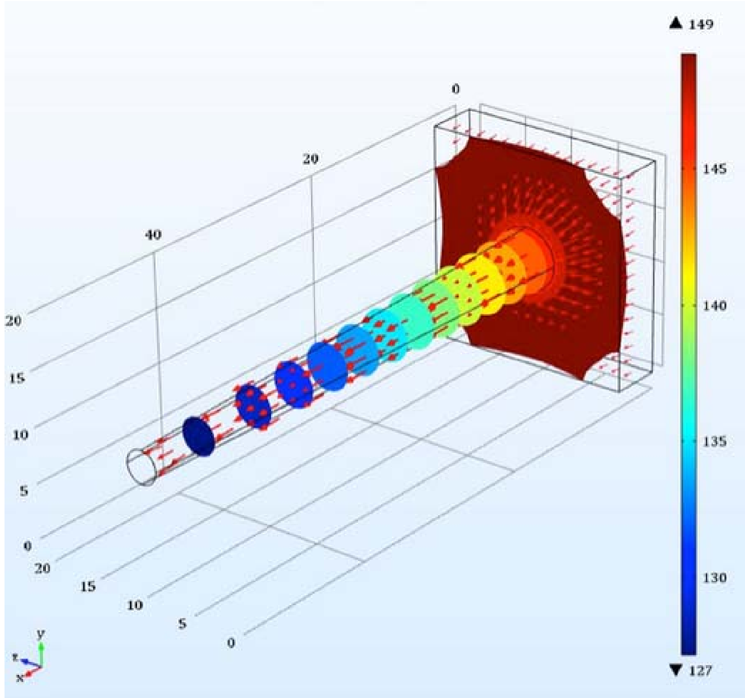
I. Sufficiently long fin:

$$\dot{Q}_{\text{head}} = 0 \quad \Longrightarrow \quad \left.\frac{dT}{dx}\right|_{x=L} = 0$$

identical

II. $A_{\text{Head}} \ll A_{\text{Surface}}$:

$$\dot{Q}_{\text{head}} = 0 \quad \Longrightarrow \quad \left.\frac{dT}{dx}\right|_{x=L} = 0$$



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Boundary conditions

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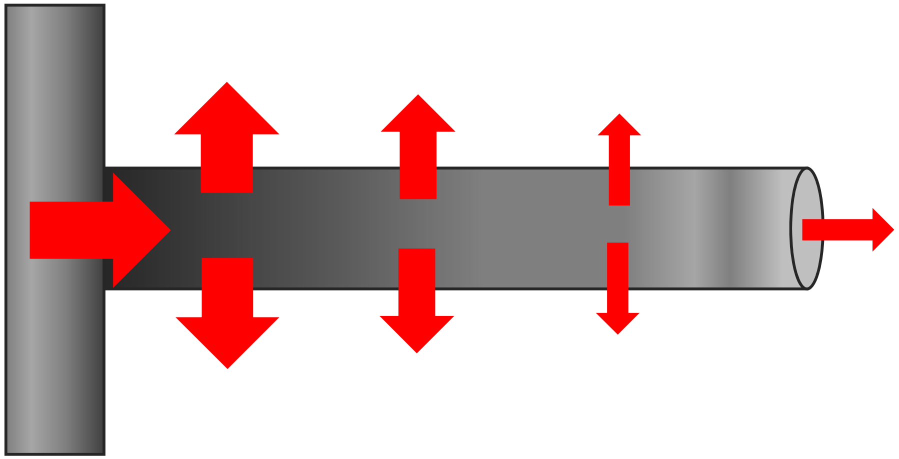
Boundary condition at the fin head:

III. If heat flow at the head is not negligible:

$$\dot{Q}_{\text{head}} \neq 0$$

$$\dot{Q}_{\text{head}} = \dot{Q}_L = \alpha\, A_c\, \theta_{\text{head}}$$

$$\Rightarrow \quad \dot{Q}_{\text{head}} = \dot{Q}_L = \alpha\, A_c\, (T_{\text{H}} - T_{\text{A}})$$

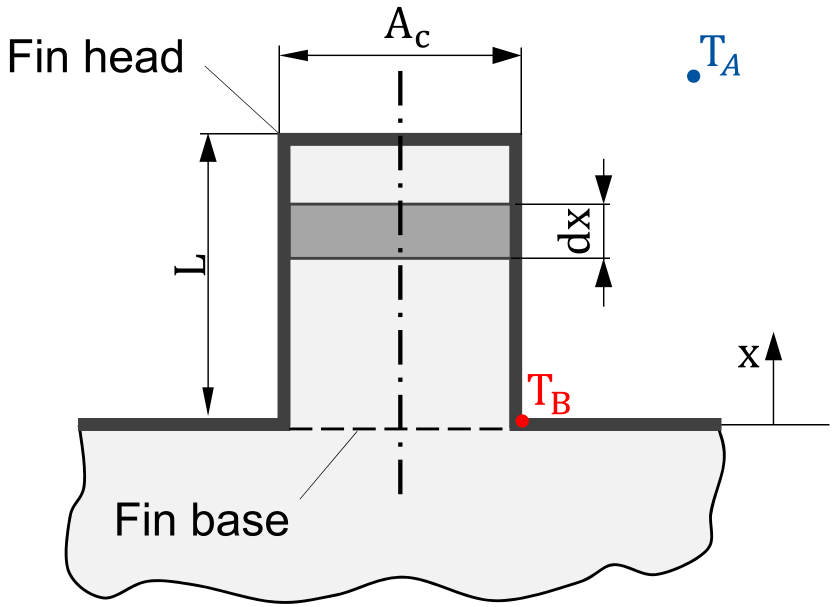


Replacing boundary conditions and solving the differential equation

General solution of the diff. equation:

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

$$\frac{\partial \theta(x)}{\partial x} = m \cdot C \cdot e^{m\,x} - m \cdot D \cdot e^{-m\,x}$$



Replacing boundary conditions:

BC1: Given base temperature at $x = 0$:

$$\Longrightarrow \quad \theta(x) = \theta_B$$

$$\theta_B = C \cdot e^0 + De^0$$

$$\theta_B = C + D$$

$$C = \theta_B - D$$

BC2: No heat flow at x = L :

$$\dot{Q}_{Head} = 0 \quad \Longrightarrow \quad \left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

$$m \cdot C \cdot e^{m\,L} - m \cdot D \cdot e^{-m\,L} = 0$$

$$(\theta_B - D) \cdot e^{m\,L} - D \cdot e^{-m\,L} = 0$$

$$\theta_B \cdot e^{m\,L} = D \cdot (e^{m\,L} + e^{-m\,L})$$

$$\Longrightarrow \quad \boxed{\begin{aligned} D &= \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}} \\ C &= \theta_B - \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}} \end{aligned}}$$

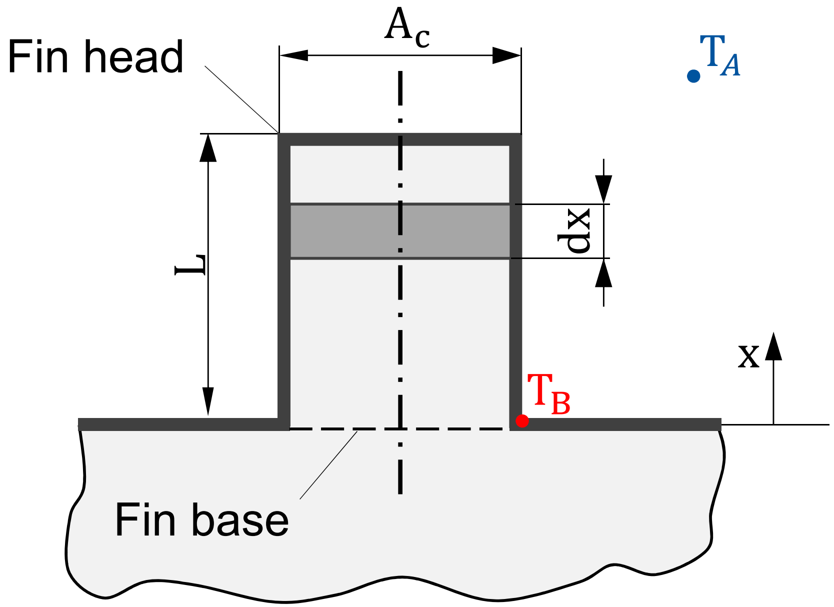
Replacing boundary bonditions and solving the differential equation

General solution of the diff. equation:

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

$$\boxed{C = \theta_B - \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}}}$$

$$\boxed{D = \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}}}$$



Replace C and D in diff. equation:

$$\theta(x) = \left(\theta_B - \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}}\right) \cdot e^{m\,x} + \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}} \cdot e^{-m\,x}$$

Mathematically reformulation and simplification:

$$\boxed{\theta(x) = \theta_B \cdot \left(\frac{e^{m\,(L-x)} + e^{-m\,(L-x)}}{e^{m\,L} + e^{-m\,L}}\right)}$$

Alternatively:

$$\boxed{\theta(x) = \theta_B \cdot \left(\frac{\cosh(m(L-x))}{\cosh(mL)}\right)}$$

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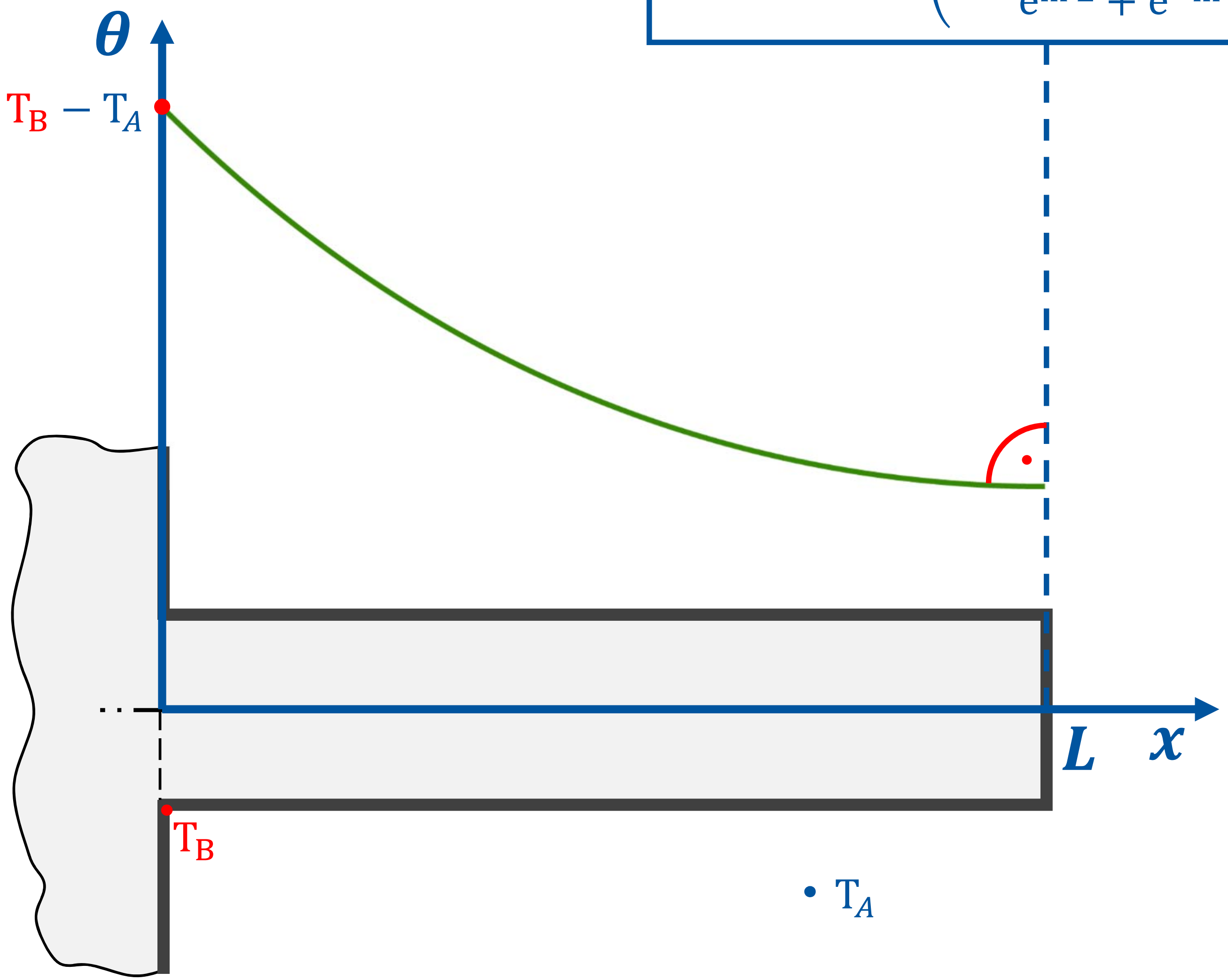
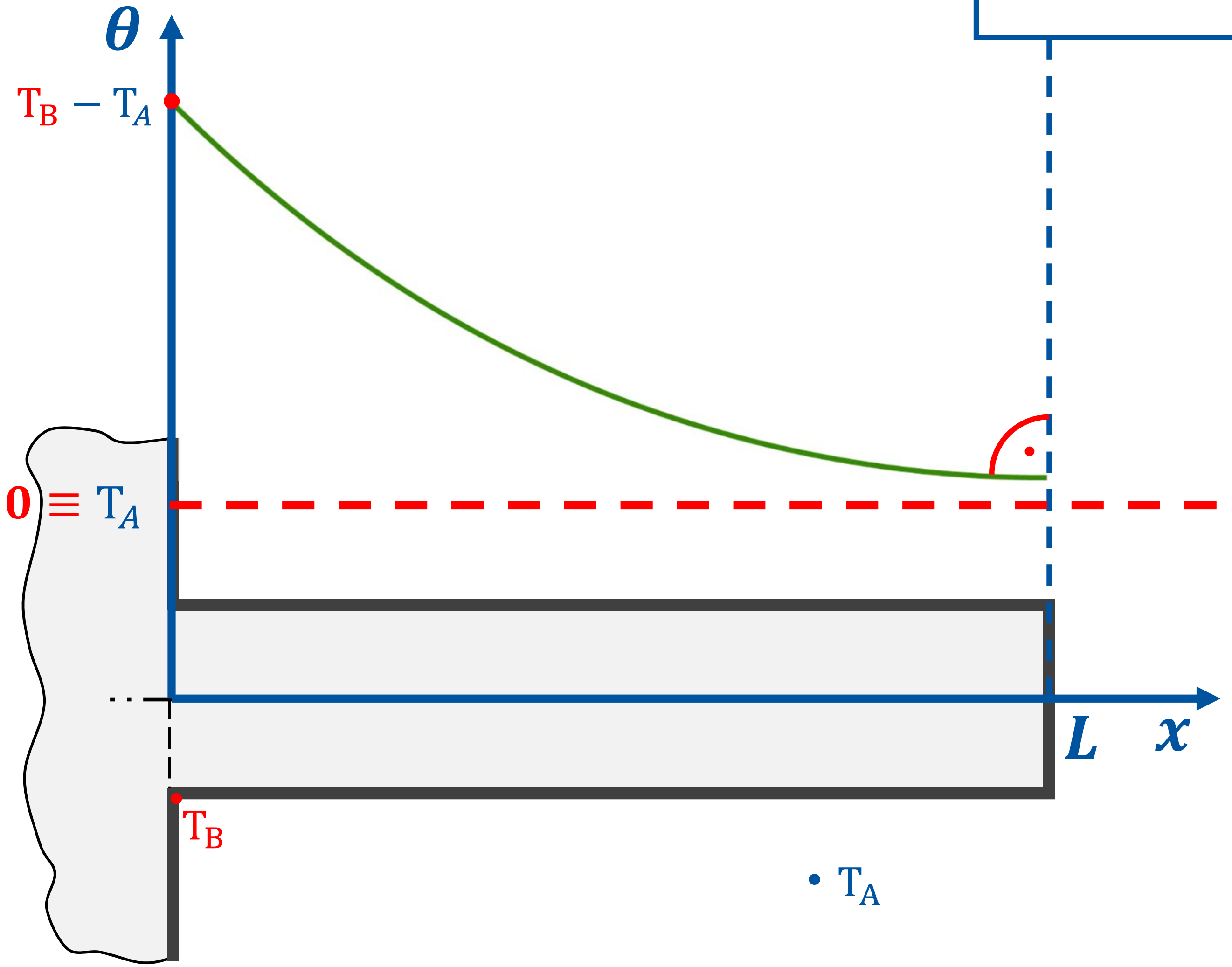
Temperature profile at the fin

$$\theta(x) = \theta_B \cdot \left(\frac{e^{m\,(L-x)} + e^{-m\,(L-x)}}{e^{m\,L} + e^{-m\,L}} \right)$$



Temperature profile at the fin



$$\theta(x) \quad = \theta_B \cdot \left(\frac{e^{m\,(L-x)} + e^{-m\,(L-x)}}{e^{m\,L} + e^{-m\,L}} \right)$$

Is the head temperature T_H equal to the ambient temperature when $\dot{Q}_{Head} = 0$?

For x = L :

$$\theta(L) \quad = \theta_B \cdot \left(\frac{e^{0} + e^{-0}}{e^{m\,L} + e^{-m\,L}} \right)$$

$$= \theta_B \cdot \left(\frac{2}{e^{m\,L} + e^{-m\,L}} \right)$$

Conclusion:
Even with $\dot{\boldsymbol{Q}}_{\text{head}} = \mathbf{0}$ the head temperature $\boldsymbol{T}_{\text{H}}$ is <u>always</u> above the ambient temperature and only approaches to it.

Comprehension questions

Which approach can be used to solve the inhomogeneous fin differential equation?

Which quantities influence the fin parameter m ?

Which common boundary conditions can be used to solve the fin temperature profile?