Heat transfer: Conduction

Lecture notes

August 2021

Contents

Chapter 1

Fouriers law and the energy conservation equations

This section deals with the simplest heat transport mechanism, heat conduction, excluding the discussion on the additional combined influence of convection and radiation. The thermal conductivity λ $[\frac{\mathrm{W}}{\mathrm{mK}}]$, already known from previous chapters, defines the material properties, so that knowing the molecular structure of the body or fluid transporting heat through conduction is not necessary. It is sufficient to assume the medium to be homogenous in structure.

The following experiment will be examined: if the surface temperatures of a flat plate with thickness δ were adjust to T_1 and T_2 by heating or cooling, then a linear relationship between the rate of heat flow per unit area $\frac{\dot{Q}}{A}$ and the temperature difference $(T_1 - T_2)$ per plate thickness δ can be observed. The temperature shape in the plate is thus linear progressive and the heat flow rate per unit area can be derived

$$\frac{\dot{Q}}{A} \equiv \dot{q}^{''} = \lambda \frac{T_1 - T_2}{\delta} \quad \left[\frac{W}{m^2}\right] \tag{1.1}$$

The experiment above shows that a general relationship, determined by Fourier in 1822, can be also expressed in the form

$$\dot{q}^{''} = -\lambda \frac{\partial T}{\partial n} \tag{1.2}$$

(Fourier's equation)

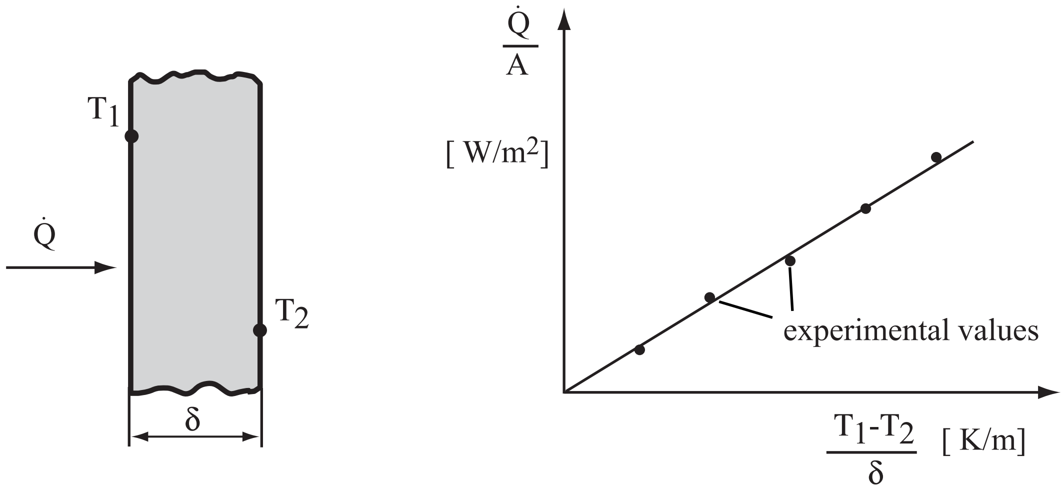


Figure 1.1: Heat flow rate per unit area as a function of the temperature difference

where $\frac{\partial T}{\partial n}$ is the temperature gradient perpendicular to a surface of constant temperature.

The negative sign, as implied by the second law of thermodynamics, indicates that heat can only flow from regions of higher to regions of lower temperature.

The temperature field of a body, which can include heat sources, such as Joule's heat or reaction heat, is described by equations of energy conservation on a control volume.

Theorem: The increase of inner energy in time of a control volume is the difference between the incoming and outflowing energy fluxes and the heat produced within the body.

The incoming energy flow at the surface area $dydz$ is solely a conductive heat flux, since we exclude convection. Using Fourier's equation (1.2) the incoming conductive heat flow rate in x-direction $d\dot{Q}_x$ is:

$$d\dot{Q}_x = -\lambda \frac{\partial T}{\partial x} dydz \tag{1.3}$$

and for the outflowing rate of conductive heat flow in x-direction

$$\begin{aligned} d\dot{Q}_{x+dx} &= d\dot{Q}_x + \tfrac{\partial}{\partial x}\left(d\dot{Q}_x\right) dx \\ &= d\dot{Q}_x + \tfrac{\partial}{\partial x}\left(-\lambda \tfrac{\partial T}{\partial x}\right) dxdydz \end{aligned} \tag{1.4}$$

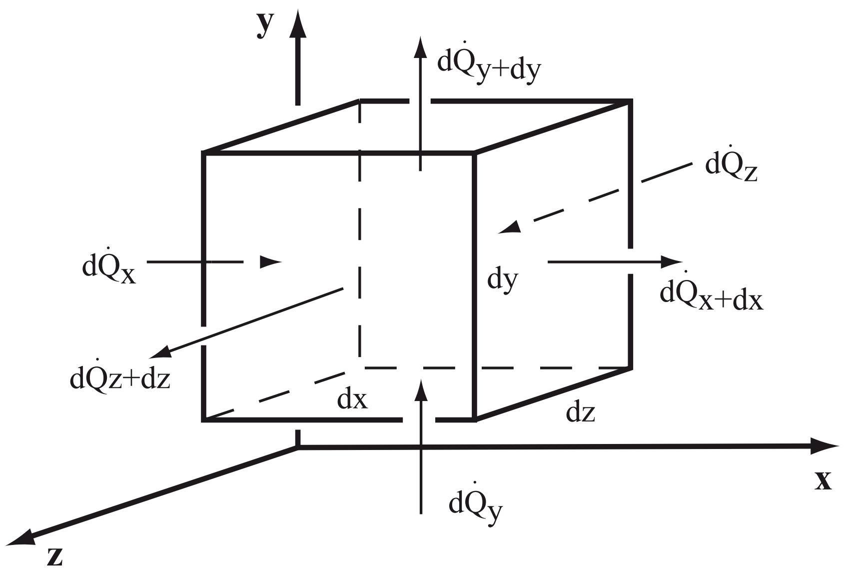


Figure 1.2: Energy balance of a static control volume

Similar expressions are obtained for the other two axes, respectively. The heat produced in the volume element is

$$\dot{\Phi}''' dx\ dy\ dz \tag{1.5}$$

with the source strength

$$\dot{\Phi}''' \left[\frac{\mathrm{W}}{\mathrm{m}^3}\right] \tag{1.6}$$

The rate of change of inner energy of the element, assuming negligible changes of mass in the volume element, can be written as

$$\frac{dU}{dt} = \frac{d(mu)}{dt} = \rho\ c_{\mathrm{v}} dxdydz \frac{\partial T}{\partial t} \tag{1.7}$$

where ρ [$^{\mathrm{kg}}/_{\mathrm{m}^3}$] is the density and c_{v} [$^{\mathrm{J}}/_{\mathrm{kgK}}$] the specific heat capacity at constant volume (the index v can be neglected for solids).

The law of conservation of energy gives the differential equation for the temperature field in cartesian coordinates. Similar differential equations can be derived for cylindrical and spherical coordinates.

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \dot{\Phi}''' \tag{1.8}$$

(Cartesian coordinates (x,y,z, t))

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \dot{\Phi}''' \tag{1.9}$$

(Cylindrical coordinates (r,θ,z,t))

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \theta}\left(\lambda \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(\lambda \frac{\partial T}{\partial \phi}\right) + \dot{\Phi}''' \tag{1.10}$$

(Spherical coordinates (r,θ,ϕ,t))

In section **??** “Convection” it will be shown in detail that the equations (1.8) - (1.10) are also valid in cases where mass changes are considered, i.e. density changes, if the value of the specific heat c will be substituted with that at constant pressure c_p.

In most cases, the thermal conductivity λ is regarded as constant, which simplifies equations (1.8) - (1.10). Hence, (1.8) can be written as follows

$$\frac{\rho c}{\lambda}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{\Phi}'''}{\lambda}$$

or

$$\frac{1}{a}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{\Phi}'''}{\lambda} \tag{1.11}$$

where a is the so-called *thermal diffusivity* $a \equiv \frac{\lambda}{\rho c}$, and has the unit $[\mathrm{m}^2/\mathrm{s}]$. The solution of Fourier’s differential equation (1.11) including the boundary conditions gives the temperature field of the body. It is clear from equation (1.11) that the ability of a material to ‘let heat pass through it’ increases with increasing thermal diffusivity. This can be due to a high thermal conductivity λ or a low heat capacity ρc of the material.

The following layers discuss solutions of the differential equation (1.8) - (1.10) for specific applications. The temperature field determined for a specific problem under consideration of its boundary conditions is used in equation (1.2) to determine the heat flux through a surface.

Fourier's differential equation (1.11) can also be found in the literature termed as

a) Poisson's equation when the unsteady state term $\frac{1}{a}\frac{\partial T}{\partial t}$ is not considered

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{\Phi}'''}{\lambda} = 0$$

b) Laplace equation, when additional heat sources are not present $\frac{\dot{\Phi}'''}{\lambda}$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1.12}$$

Chapter 2

Multi-layer walls

2.1 Plane walls and thermal resistances

For steady state, one-dimensional problems, without heat sources and constant heat conductivity λ, equation (1.8) simplifies to

$$\frac{d^2T}{dx^2} = 0 \tag{2.1}$$

Integrating yields $T = Ax+B$, and hence, the already mentioned linear temperature

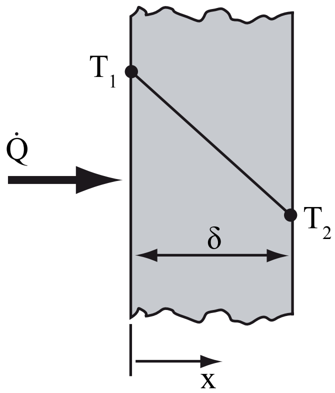


Figure 2.1: Conduction through a flat wall

profile. The constants A and B are determined from the boundary conditions

$$\begin{array}{rrcrl} & x = & 0 & : & T = & T_1 \\ \text{and} & x = & \delta & : & T = & T_2 \end{array}$$

The temperature profile in the plate is thus

$$T = T_1 + \frac{T_2 - T_1}{\delta} x \tag{2.2}$$

and using equation (1.2) the heat flux is

$$\dot{Q} = -\lambda A \frac{dT}{dx} = \lambda A \frac{T_1 - T_2}{\delta} \quad [\mathrm{W}] \tag{2.3}$$

In case of a composite wall, i.e. consisting of many layers with different thicknesses and materials, equation (2.3) must be calculated for each section in turn. The heat entering section 1, flows unchanged out of section 3 for steady state, one-dimensional cases.

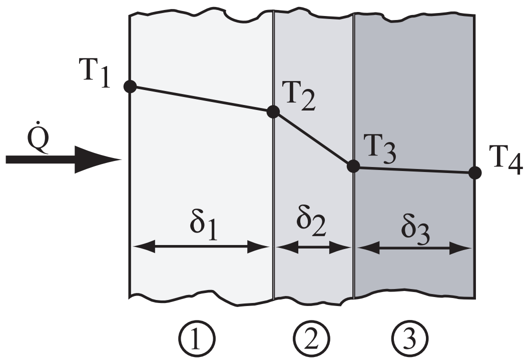


Figure 2.2: Conduction through a composite wall

$$\dot{Q} = \lambda_1 \frac{A}{\delta_1} (T_1 - T_2) = \lambda_2 \frac{A}{\delta_2} (T_2 - T_3) = \lambda_3 \frac{A}{\delta_3} (T_3 - T_4)$$

Hence,

$$\dot{Q} = \frac{A}{\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \frac{\delta_3}{\lambda_3}} (T_1 - T_4)$$

or in general for n layers,

$$\dot{Q} = \frac{1}{\sum\limits_{i=1}^{n} \frac{\delta_i}{A\lambda_i}} (T_1 - T_{n+1}) = \frac{1}{R_{\text{cond.}}} (T_1 - T_{n+1}) \tag{2.4}$$

The heat flow is equal to a temperature potential divided by the sum of the thermal resistances $R_{\text{cond.},i}$ of each layer, where

$$R_{\text{cond.}} = \sum_{i=1}^{n} R_{\text{cond.},i} = \sum_{i=1}^{n} \left(\frac{\delta_i}{A\lambda_i} \right) \quad \left[\frac{\text{K}}{\text{W}} \right]$$

Equation (2.4) is thus in a form matching that of Ohm's law for electric conductors.

2.2 Cylindrical wall

Within the control volume, $2\pi r dr L$, the heat flowing through the surface $2\pi r L$ is

$$\dot{Q}_r = -\lambda\, 2\pi r L \frac{dT}{dr}$$

and the leaving heat flow through the area $2\pi(r + dr)L$ is

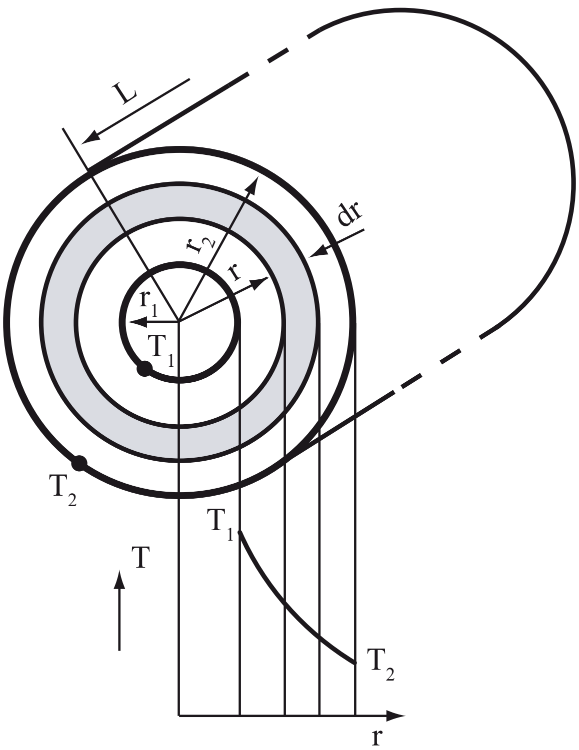


Figure 2.3: Conduction through a tube wall

$$\begin{aligned}\dot{Q}_{r+dr} &= \dot{Q}_r + \tfrac{d\dot{Q}_r}{dr}dr \\ &= \dot{Q}_r - 2\pi L\lambda \tfrac{d}{dr}\left(r\tfrac{dT}{dr}\right)dr\end{aligned}$$

The heat flow remains constant for steady state, one-dimensional case

$$\dot{Q}_r - \dot{Q}_{r+dr} = 0 \tag{2.5}$$

Which yields

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{2.6}$$

or

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \tag{2.7}$$

The relationship (2.6) can also be obtained directly from equation (1.9).

The solution of Laplace's equation (2.6),

$$T = A\ \ln\left(\frac{r}{r_0}\right) + B \tag{2.8}$$

where r_0 is the reference radius.

Using boundary conditions $r = r_1$: $T = T_1$ und $r = r_2$: $T = T_2$ the constants A and B from equation (2.8) can be determined

$$A = \frac{T_2 - T_1}{\ln}\left(\frac{r_2}{r_1}\right) \quad ; \quad B = T_1 - \frac{\ln\left(\frac{r_1}{r_0}\right)(T_2 - T_1)}{\ln\left(\frac{r_2}{r_1}\right)}$$

The temperature in the wall of the tube proves to have logarithmic character, which is given by

$$T = T_1 + \ln\left(\frac{r}{r_1}\right)\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \tag{2.9}$$

or

$$T = T_2 + \ln\left(\frac{r}{r_2}\right)\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \tag{2.10}$$

By using Fourier's equation, (1.2), the heat flux through the walls can be described

$$\dot{Q} = -\lambda A\frac{dT}{dr} = -\lambda 2\pi r L\frac{dT}{dr}$$

and with equation (2.9)

$$\dot{Q} = +2\pi\lambda L \frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)} \tag{2.11}$$

Introducing again, as stated in equation (2.4), the thermal resistance, then equation (2.11) tranforms into

$$\dot{Q} \equiv \frac{T_1 - T_2}{R_{\text{cond.}}} \quad \text{with} \quad R_{\text{cond.}} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda L} \tag{2.12}$$

In analogy to the composite wall multiple layer tubes or shells can also be considered. The heat flow can be written as:

$$\dot{Q} = \frac{T_1 - T_{n+1}}{\sum\limits_{i=1}^{n} R_{\text{cond},i}} = \frac{T_1 - T_{n+1}}{\frac{1}{2\pi L}\sum\limits_{i=1}^{n} \frac{1}{\lambda_i} \ln\frac{r_{i+1}}{r_i}} \tag{2.13}$$

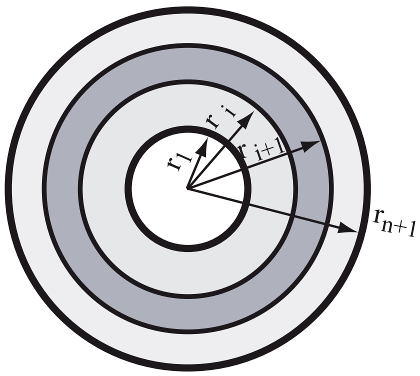


Figure 2.4: Conduction through a spherical shell

2.3 Plane walls with convective heat transfer

Although chapter 3 discusses only heat transfer through conduction, the convective heat transfer will be introduced in this section, since the latter is of importance and related by its boundary conditions to conduction. It has already been mentioned in the introduction that the energy transport from a flowing fluid to the neighbouring wall is not solely taken over by conduction. Only in the direct proximity of the wall, where due to the viscosity of the fluid the velocity can be neglected, it is able to calculate the heat flow as usual according to equation (1.2)

$$\dot{Q} = -A\left(\lambda \frac{dT}{dx}\right)_{\text{fluid,wall}} \tag{2.14}$$

using the temperature gradient in the fluid and the thermal conductivity λ_{fl} as parameters. The temperature gradient on the wall depends, in addition, on the flow velocity of the fluid, so that a theoretical solution is possible only in a limited number of situations, described in chapter 4.

Therefore, instead of equation (2.14), we normally use an empirical assumption for the convective heat transfer, e.g. on the side of the fluid A,

$$\dot{Q} = A\alpha_{\text{A}}\left(T_{\text{A}} - T_1\right) \tag{2.15}$$

where the heat transfer coefficient α $\left[\frac{\text{W}}{\text{m}^2\text{K}}\right]$ is assumed to be known.
For the above case of a fluid flowing over a wall, we get the heat flux

- from fluid A to the wall by convection

$$\dot{Q}_{\text{conv.A}} = A\alpha_{\text{A}}\left(T_{\text{A}} - T_1\right)$$

- through the wall by conduction

$$\dot{Q}_{\text{cond.}} = A\frac{\lambda}{\delta}\left(T_1 - T_2\right)$$

- from the wall to fluid B by convection

$$\dot{Q}_{\text{conv.B}} = A\alpha_{\text{B}}\left(T_2 - T_{\text{B}}\right)$$

In steady state it is,

$$\dot{Q}_{\text{conv.A}} = \dot{Q}_{\text{cond.}} = \dot{Q}_{\text{conv.B}} = \dot{Q}$$

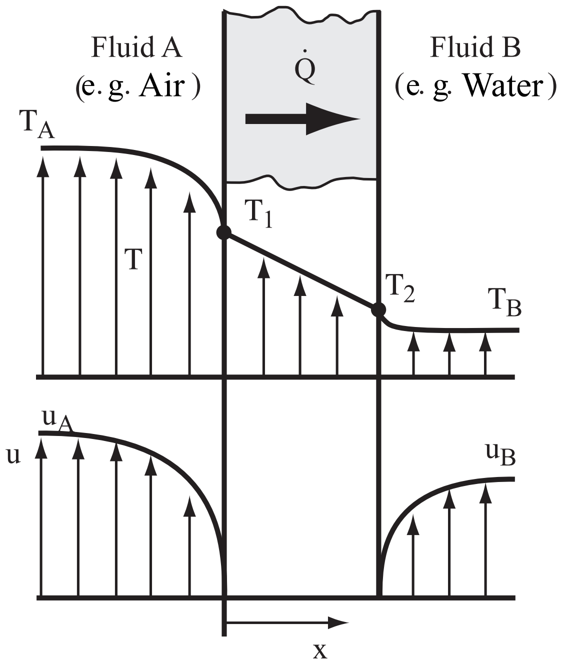


Figure 2.5: Heat transfer through a wall at convective boundary conditions

and thus

$$\dot{Q} = \frac{A}{\frac{1}{\alpha_\mathrm{A}} + \sum\limits_{i=1}^{n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_\mathrm{B}}} \left(T_\mathrm{A} - T_\mathrm{B}\right) \tag{2.16}$$

The denominator in equation (2.16) may be substituted by defining the overall heat transfer coefficient $k \left[\frac{\mathrm{W}}{\mathrm{m^2K}}\right]$

$$\frac{1}{k} = \frac{1}{\alpha_\mathrm{A}} + \sum_{i=1}^{n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_\mathrm{B}} \tag{2.17}$$

We get

$$\dot{Q} = kA \left(T_\mathrm{A} - T_\mathrm{B}\right) \tag{2.18}$$

or also by introducing the 'heat transfer resistance' for convection

$$R_\mathrm{conv.} \equiv \frac{1}{A\alpha} \quad \left[\frac{\mathrm{K}}{\mathrm{W}}\right]$$

$$\dot{Q} = \frac{1}{R_\mathrm{conv.A} + R_\mathrm{cond.} + R_\mathrm{conv.B}} \left(T_\mathrm{A} - T_\mathrm{B}\right) = \frac{1}{\sum W_i} \left(T_\mathrm{A} - T_\mathrm{B}\right) \tag{2.19}$$

By adding additional thermal resistances of conduction $W_{\text{cond}} = W_{\text{L}_i}$, equation 2.19 can be expanded to include cases of composite walls.

2.4 Tube wall with convective heat transfer

The derived relationships from the previous section can be used to solve problems where single or composite tube wall are used. As a practical example the calculation of the heat loss of an insulated hot water pipe is given.

The thermal resistances W_{L_i} from equation (2.13) have to be modified in order to include the heat transfer resistances on the tube's inner and outer side $W_{\text{conv}_\text{A}} = \frac{1}{\alpha_\text{A} A_\text{A}}$ and $W_{\text{conv}_\text{B}} = \frac{1}{\alpha_\text{B} A_\text{B}}$, respectively, where $A_\text{A} = 2\pi r_1 L$ and $A_\text{B} = 2\pi r_{n+1} L$. Hence, we get for the heat flux

$$\dot{Q} = \frac{2\pi L}{\frac{1}{\alpha_\text{A} r_1} + \sum\limits_{i=1}^{n} \frac{1}{\lambda_i} \ln \frac{r_{i+1}}{r_i} + \frac{1}{\alpha_\text{B} r_{n+1}}} \left(T_\text{A} - T_\text{B}\right) \tag{2.20}$$

or instead of using the radii, using the diameters

$$\dot{Q} = \frac{\pi L}{\frac{1}{\alpha_\text{A} d_1} + \frac{1}{2} \sum\limits_{i=1}^{n} \frac{1}{\lambda_i} \ln \frac{d_{i+1}}{d_i} + \frac{1}{\alpha_\text{B} d_{n+1}}} \left(T_\text{A} - T_\text{B}\right)$$

Comparing the equation above to the form where the overall heat transfer coefficient k is used

$$\dot{Q} = kA\left(T_\text{A} - T_\text{B}\right) = k\pi dL\left(T_\text{A} - T_\text{B}\right) \tag{2.21}$$

we get

$$\frac{1}{k} = \frac{1}{\alpha_\text{A}} \frac{d}{d_1} + \frac{d}{2} \sum_{i=1}^{n} \frac{1}{\lambda_i} \ln \frac{d_{i+1}}{d_i} + \frac{1}{\alpha_\text{B}} \frac{d}{d_{n+1}} \tag{2.22}$$

Where any surface area A or any diameter d may be used in equation (2.21) and equation (2.22), respectively. Normally, in practice, the outer diameter d_{n+1} is used as reference. For thin-walled tubes the relationship for flat plates (2.16) may be used, if the area will be substituted with $A = \pi d_m L$, where d_m describes the arithmetic mean diameter. Thus, for problems with heat transfer coefficients of equal order of magnitude on both sides of the fluid, it is good practice to use the mean value of the inner and outer diameter for calculations. If the heat transfer coefficients greatly deviate from each other, then it is recommended to select the diameter with the lower heat transfer coefficient

Chapter 3

Fins

The above stated relationships are of special technical importance to predict the temperature distribution and heat flux in fins. Fins are usually placed in the form of cylindrical rod fins, plane fins, circular fins, etc. on the surface of a heat emitting (or heat absorbing) body in areas where the greatest heat transfer resistances are located. The efficiency of a fin is given by the fin efficiency η_F, which is defined as

$$\eta_\mathrm{F} = \frac{\text{transferred heat}}{\text{maximum transferable heat}} \tag{3.1}$$

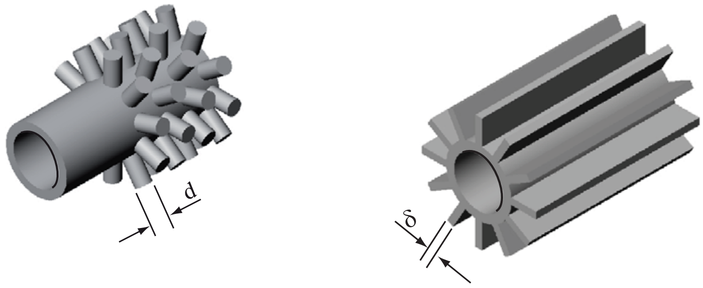
where the 'maximal transferable heat' is the heat that can be transferred by the fin if its total surface area reached the temperature of the fin base. In the following we will discuss those fin geometries, which approximately can be regarded one-dimensional and thus allow an analytical approach. Two-dimensional and three-dimensional fins can only be described by using numerical methods.

Rod fins and plane fins Allowing the simplified assumptions that

- the temperature shape in a fin is one-dimensional, i.e. the temperature changes only as a function of the length of the fin, but not it's radius
- the heat transfer coefficient is known,

we get from the heat balance on the fin element with cross section area A_C, that the difference between the incoming and outgoing heat flux must be dissipated by convection to the surroundings.

$$\dot{Q}_x - \dot{Q}_{x+dx} = d\dot{Q}_\mathrm{conv}$$



(a) Rod fins (b) Plane fins

Figure 3.1: Tubes with finned surfaces

Using equation (1.2) the heat fluxes are

$$\dot{Q}_x = -\lambda A_\mathrm{c} \frac{dT}{dx}$$

$$\dot{Q}_{x+dx} = \dot{Q}_x + \frac{d\dot{Q}_x}{dx} dx = -\lambda A_\mathrm{c} \left(\frac{dT}{dx} + \frac{d^2T}{dx^2} dx \right)$$

and with equation (2.15) the convection flux from the element with circumference U and length dx to the surrounding

$$d\dot{Q}_\mathrm{conv} = \alpha U \left(T - T_\mathrm{u} \right) dx$$

The balance yields, after introducing a temperature difference $\theta \equiv T - T_\mathrm{a}$ the differential equation for the temperature of the fin

$$\frac{d^2\theta}{dx^2} - \underbrace{\frac{\alpha U}{\lambda A_\mathrm{c}}}_{\mathrm{m}^2} \theta = 0 \tag{3.2}$$

Which has the general solution with

$$\theta = A\, e^{\mathrm{mx}} + B\, e^{-\mathrm{mx}} = A^* \,\sinh(\mathrm{mx}) + B^* \cosh(\mathrm{mx}) \tag{3.3}$$

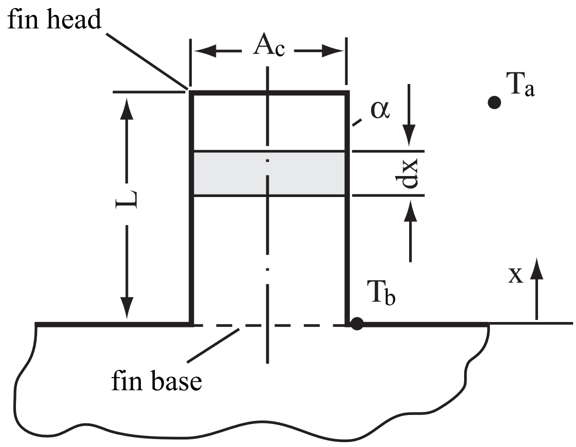


Figure 3.2: Energy balance of a fin

with

$$m \equiv \left(\frac{\alpha U}{\lambda A_c}\right)^{\frac{1}{2}} \quad \left[\frac{1}{\mathrm{m}}\right],$$

For the rod fin with circular cross section $m = \left(\frac{4\alpha}{\lambda d}\right)^{\frac{1}{2}}$ and for the plane fin $m = \left(\frac{2\alpha}{\lambda d}\right)^{\frac{1}{2}}$

Boundary conditions

1. $x = 0$: $\theta = \theta_b$
2. $x = L$:

   At the top of the fin many combinations of boundary conditions are possible

   a) The heat flow out of the top of the fin is negligible in comparison to the total dissipated heat, hence the top of the fin (which also can be called 'head') can be regarded as adiabatic
   $\left(\frac{d\theta}{dx}\right)_{x=\mathrm{L}} = 0$

   b) The fin is very long, thus the temperature of the top is approximately equal to the surrounding temperature
   $(\theta)_{x=\mathrm{L}} = 0$

   c) The fin has a finite height and transfers heat from the head as well
   $(Q)_{x=\mathrm{L}} = \alpha A_c \theta_{\text{head}}$

In the following the boundary condition 2a will be considered.

Then, from equation (3.3) the temperature over the height of the fin is

$$\theta = \theta_{\mathrm{F}} \frac{e^{\mathrm{m(L-x)}} + e^{-\mathrm{m(L-x)}}}{e^{\mathrm{mL}} + e^{-\mathrm{mL}}} \tag{3.4}$$

or

$$\theta = \theta_{\mathrm{F}} \frac{\cosh\left[\mathrm{m}\left(\mathrm{L}-\mathrm{x}\right)\right]}{\cosh\left(\mathrm{mL}\right)} \tag{3.5}$$

The following diagram shows the dimensionless temperature shape of a rod fin with circular cross section, for different fin materials with $\mathrm{m} = \left(\frac{4\alpha}{\lambda d}\right)^{\frac{1}{2}}$.

The fin's diameter is $d = 8$mm, its length $L = 40$mm and a heat transfer coefficient against surrounding air $\alpha = 10\ \frac{\mathrm{W}}{\mathrm{m}^2 K}$.

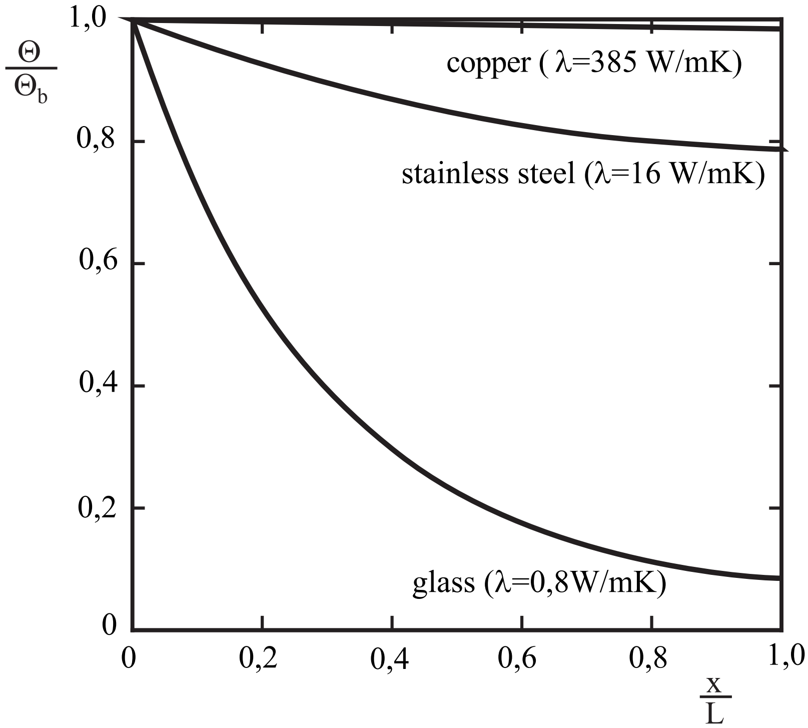


Figure 3.3: Change of temperature over the length of a rod fin

From the temperature curve above the total heat transferred to the surroundings by the fin can be determined. Energy balance along the fin shows that in steady state the heat, entering the fin base by conduction, is equal to the heat transferred to the surroundings by convection

$$\dot{Q} = \dot{Q}_{\mathrm{b}} = -\lambda A_{\mathrm{c}} \left(\frac{d\theta}{dx}\right)_{\mathrm{x=0}}$$

Using equation (3.4)

$$\dot{Q} = \lambda A_{\mathrm{c}} \mathrm{m} \theta_{\mathrm{b}} \left(\frac{e^{\mathrm{mL}} - e^{-\mathrm{mL0}}}{e^{\mathrm{mL}} + e^{-\mathrm{mL}}}\right) \tag{3.6}$$

or

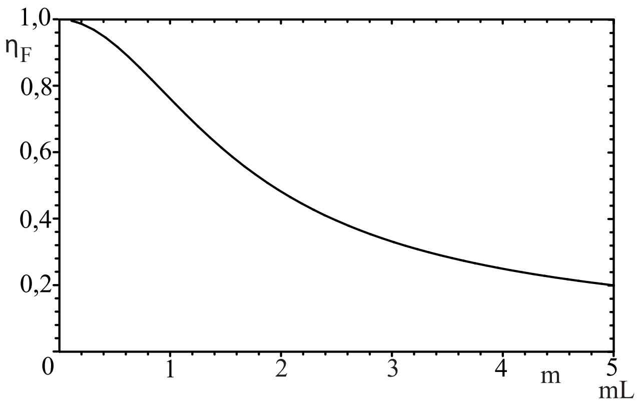
$$\dot{Q} = \lambda A_{\mathrm{b}} \mathrm{m} \theta_{\mathrm{b}} \tanh\left(\mathrm{mL}\right) \tag{3.7}$$

To determine the efficiency of the fin, the calculated heat flux from equation (3.7) is rated over the maximum possible heat flux of that particular fin geometry, see equation (3.1),

$$\eta_{\mathrm{F}} \equiv \frac{\dot{Q}}{\dot{Q}_{\mathrm{max}}} = \frac{\lambda A_{\mathrm{c}} \mathrm{m} \theta_{\mathrm{b}} \tanh(mL)}{\alpha\ U\ L\ \theta_{\mathrm{b}}} \tag{3.8}$$

$$\eta_{\mathrm{F}} = \frac{\tanh(mL)}{mL}$$

This function is shown in fig. 3.4



Rod fin with Diameter d	$\mathrm{m} = \left(\frac{4\,\alpha}{\lambda\,\mathrm{d}}\right)^{1/2}$
Plane fin	$\mathrm{m} = \left(\frac{2\,\alpha}{\lambda\,\delta}\right)^{1/2}$

Figure 3.4: Efficiency coefficient of a rod fin

As can be seen from the curve, the efficiency cannot be optimised solely over the length of the fin, since factors like mass of the fin or volume must also be considered.

Equations (3.4) - (3.8) have been derived for the boundary condition 2a). For cases 2b) and 2c) the solutions can be derived in the same manner.

Chapter 4

Heat sources and sinks

In many practical situations heat transfer processes for systems with inner heat production are to be estimated. Examples include fuel rods of nuclear plants, electric resistances or tanks in which chemical reactions take place, just to name a few. Often a one-dimensional analysis suffices as a first approximation.

As an example we use the cylindrical fuel rod to derive the differential equation of the temperature field.

Energy balance at the volume element $dV = 2\pi r dr L$ yields

$$+\dot{Q}_r - \dot{Q}_{r+dr} + \dot{\Phi}''' dV = 0$$

and thus

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{\Phi}'''}{\lambda} = 0 \tag{4.1}$$

or

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{\dot{\Phi}'''}{\lambda} = 0 \tag{4.2}$$

This solution or the equivalent for the plane or spherical case can be taken from equation (1.8). The boundary conditions are

1.
$$r = 0 \quad : \quad \left(\frac{dT}{dr}\right)_{r=0} = 0 \tag{4.3}$$

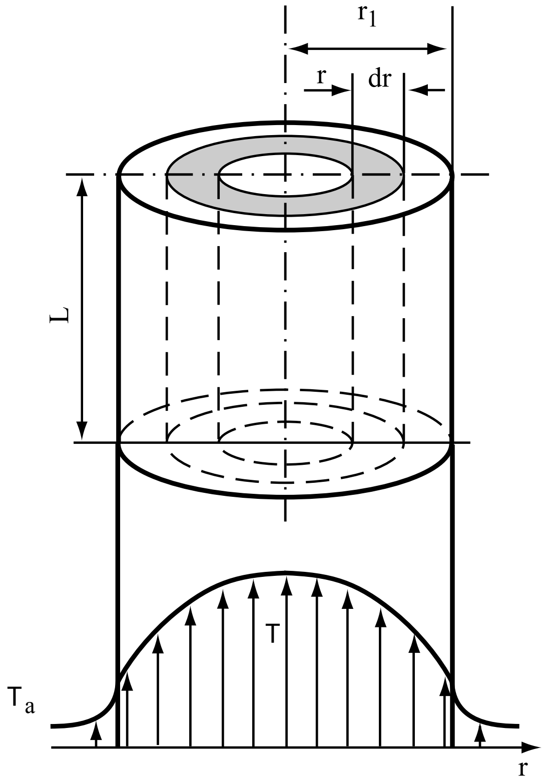


Figure 4.1: Heat conduction in a cylinder with heat sources

2.

$$r = r_1 \quad :$$

As already mentioned in section 2, on the cylinder surface either a) the temperature or b) a heat transfer coefficient can be given

a)

$$T = T_1 \tag{4.4}$$

b)

$$\dot{Q} = 2\pi r_1 L \alpha \left(T_1 - T_\mathrm{a}\right) \tag{4.5}$$

since the dissipated heat for steady state must be produced by the heat sources within the cylinder, we get

$$\dot{Q} = \pi r_1^2 L \dot{\Phi}'''$$

and hence

$$T_1 = T_{\mathrm{a}} + \frac{r_1 \dot{\Phi}'''}{2\alpha}$$

Integrating the differential equation (4.1) using equation (4.3) and (4.5) gives the temperature shape in the cylinder

$$T = T_{\mathrm{a}} + \frac{\dot{\Phi}''' r_1^2}{4\lambda} \left(1 + \frac{2\lambda}{\alpha r_1} - \left(\frac{r}{r_1}\right)^2\right) \tag{4.6}$$

This equation can be given in a general form valid both for the plane and spherical geometries

$$T = T_{\mathrm{a}} + \frac{\dot{\Phi}''' s^2}{2\,(n+1)\,\lambda} \left(1 + \frac{2\lambda}{\alpha s} - \left(\frac{\xi}{s}\right)^2\right) \tag{4.7}$$

the parameter ξ, the characteristic length s and the control parameter n are to be included using the following rule:

Table 4.1: Definitions of the parameters in equation (4.7)

	Plate*)	Cylinder	Sphere
ξ	x	r	r
s	δ	r_1	r_1
n	0	1	2

*)For plates x should be with reference to the plane of symmetry and δ is half the plate thickness

Using (4.7) the maximum temperature in the body ($\xi = 0$)

$$T_{\max} = T_{\mathrm{a}} + \frac{\dot{\Phi}''' s^2}{2\,(n+1)\,\lambda} \left(1 + \frac{2\lambda}{\alpha s}\right) \tag{4.8}$$

and the surface temperature ($\xi = s$) can be calculated

$$T_{\mathrm{s}} = T_{\mathrm{a}} + \frac{\dot{\Phi}''' s}{(n+1)\,\alpha} \tag{4.9}$$

Chapter 5

Unsteady heat conduction

Previous chapters have dealt with conduction processes, for which a balance state has been reached. The obtained temperature field was not a function of time.

Here, the unsteady states of heating and cooling processes until an balance temperature is reached will be discussed. These processes are described by the differential equations (1.8), (1.9) or (1.10) from section 1 satisfying the initial and boundary conditions of the problem in question.

Since a general solution of these equations is not possible, we can only describe typical examples used in practice.

5.1 Lumped capacity model for a body of homogeneous temperature

If a body has a high thermal conductivity, so that the thermal resistance of the body is small compared to the heat transfer resistance between the body and the surrounding fluid, then a nearly homogeneous temperature will be established at each point in time during heating or cooling.

Hence, it is not necessary to implement the differential equation for the temperature field, equation (1.8) - (1.10). It is much easier to formulate the energy balance for the entire body, which states that the inner energy of the body, which is a function of the time, will change due to heat transfer by convection from the surface to the surroundings.

Equating the energy:

$$\frac{dU}{dt} = -\alpha A \left(T - T_{\mathrm{a}}\right)$$

or

$$\rho c V \frac{dT}{dt} + \alpha A \left(T - T_{\mathrm{a}}\right) = 0 \tag{5.1}$$

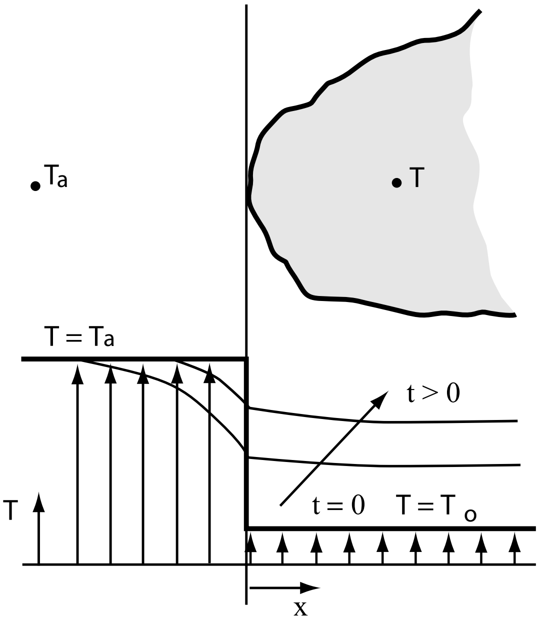


Figure 5.1: Heating of a body

Defining the dimensionless temperature $\theta^* \equiv \frac{(T-T_0)}{(T_{\mathrm{a}}-T_0)}$, where T is the temperature as a function of the time and T_0 is the initial temperature of the body, it is:

$$\frac{d\theta^*}{dt} + \frac{\alpha}{\rho c}\frac{A}{V}\left(\theta^* - 1\right) = 0$$

and thus

$$\frac{d\theta^*}{(\theta^* - 1)} = -\frac{\alpha}{\rho c}\frac{A}{V}dt$$

This differential equation can be integrated using the boundary condition

$t = 0$: $\quad T = T_0 \quad$ i.e. $\quad \theta^* = 0$

for $0 \leq \theta^* \leq 1$ which finally yields

$$\frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \exp\left[-\frac{\alpha}{\rho c}\frac{A}{V}t\right] \tag{5.2}$$

Introducing two dimensionless numbers, namely, the Biot number

$$\mathrm{Bi} \equiv \frac{\alpha L}{\lambda}, \tag{5.3}$$

and the Fourier number

$$\mathrm{Fo} \equiv \frac{\lambda t}{\rho c L^2} = \frac{at}{L^2}, \tag{5.4}$$

where L is the characteristic length, resulting from the ratio $\frac{V}{A}$, equation (5.2) can be expressed in the following form:

$$\frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \exp\left[-\mathrm{Bi}\cdot\mathrm{Fo}\right] \tag{5.5}$$

In other words, the dimensionless temporal progress of the temperature equilibrium

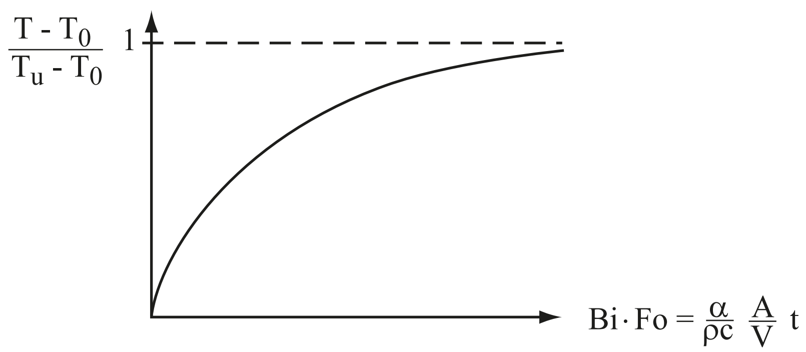


Figure 5.2: The temperature of a body as a function of the time

between a body with high thermal conductivity and its surrounding (written in the form of equation (5.5)) is in general the same for all problems, where the boundary conditions are described by equal Biot numbers. Moreover, this is valid for all problems where the Biot number is very small, Bi $\ll$ 1. Equation (5.5) may still be used as a first approximation of the heat transfer behaviour of a body, in cases where the above conditions do not apply.

5.2 One-dimensional semi-infinite plate solution

In cases where the thermal resistance within a body cannot be neglected, a general, analytical solution is possible only after complex calculations and only for a limited number of geometries and boundary conditions.

Since often in practice these solutions, usually shown in dimensionless diagrams, suffice for the preliminary approximation of the temperature/time behaviour, we will discuss next the basic mathematical procedures for selected examples.

Semi-infinite plate with given surface temperature

A very useful case for approximations of the temperature/time behaviour of bodies with high Biot numbers is the semi-infinite body, i.e. a model of a body for which the given temperature change has not yet penetrated deep under the surface and which can by analysed one-dimensionally.

Under these conditions, equation (1.8) simplifies to

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c}\frac{\partial^2 T}{\partial x^2} \tag{5.6}$$

with the initial and boundary conditions

IC 1:

$$\left.\begin{matrix} t = 0 \\ 0 < x < \infty \end{matrix}\right\} T = T_0$$

BC 2: since $\mathrm{Bi} = \frac{\alpha L}{\lambda} \gg 1$

$$\left.\begin{matrix} t > 0 \\ x = 0 \end{matrix}\right\} T = T_\mathrm{a}$$

BC 3:

$$\left.\begin{matrix} t > 0 \\ x \to \infty \end{matrix}\right\} T = T_0$$

Introducing the dimensionless temperature difference $\theta^* \equiv \frac{T-T_\mathrm{o}}{T_\mathrm{a}-T_\mathrm{o}}$ and the thermal

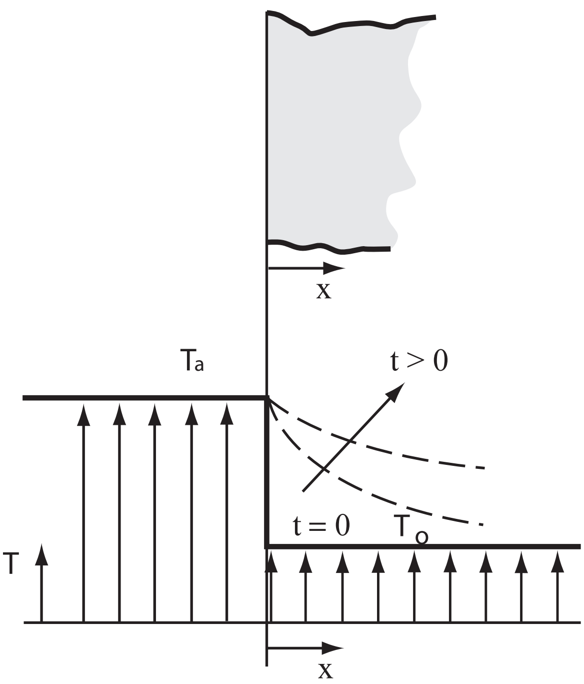


Figure 5.3: Initial and boundary conditions

diffusivity $a \equiv \frac{\lambda}{\rho\, c}$, it can also be written

$$\frac{\partial \theta^*}{\partial t} = a \left(\frac{\partial^2 \theta^*}{\partial x^2} \right) \tag{5.7}$$

and

IC 1:

$$\left.\begin{array}{c} t = 0 \\ 0 < x < \infty \end{array}\right\} \theta^* = 0$$

BC 2:

$$\left.\begin{array}{c} t > 0 \\ x = 0 \end{array}\right\} \theta^* = 1$$

BC 3:

$$\left.\begin{array}{c} t > 0 \\ x \to \infty \end{array}\right\} \theta^* = 0$$

This differential equation can be solved using the method of variable separation.

However, the present case can be solved by an easier way. Instead of the two independent variables x and t, only one independent variable $\eta(x,t)$ will be determined, for which the partial differential equation reduces to an ordinary differential equation, so that the following is valid

$$\theta^*(\mathrm{x,t}) = \theta^*\left[\eta(\mathrm{x,t})\right]$$

For the independent variable, η, the following formulation will be used

$$\eta = b\, x^{\mathrm{c}} t^{\mathrm{d}}$$

Additionally the differentials of equation (5.7) will be rewritten, so that they depend on η.

$$\frac{\partial\theta^*}{\partial t} = \left(\frac{d\theta^*}{d\eta}\right)\left(\frac{\partial\eta}{\partial t}\right)$$

and

$$\frac{\partial^2\theta^*}{\partial x^2} = \left(\frac{d^2\theta^*}{d\eta^2}\right)\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{d\theta^*}{d\eta}\right)\left(\frac{\partial^2\eta}{\partial x^2}\right)$$

taking account of the mentioned formulation $\eta = \mathrm{b}\ x^{\mathrm{c}} t^{\mathrm{d}}$ the differential equation (5.7) is

$$\left(\frac{d\theta^*}{d\eta}\right) b\, x^{\mathrm{c}} d t^{\mathrm{d}-1} = a\left(\frac{d^2\theta^*}{d\eta^2}\right)\left(\mathrm{bc}\ x^{\mathrm{c}-1} t^{\mathrm{d}}\right)^2 + a\left(\frac{d\theta^*}{d\eta}\right)\left(\mathrm{bc}(\mathrm{c}-1) x^{\mathrm{c}-2} t^{\mathrm{d}}\right)$$

and by rearranging it gives

$$\frac{d^2\theta^*}{d\eta^2} + \frac{\left(\frac{\mathrm{ac}(\mathrm{c}-1)t}{dx^2} - 1\right)}{\left(\frac{\mathrm{abc}^2 t^{\mathrm{d}+1}}{dx^{2-\mathrm{c}}}\right)}\frac{d\theta^*}{d\eta} = 0$$

Thus, if this were to be an ordinary differential equation dependent only on η then the expressions in the brackets must also depend only on η.

Comparing the exponents of x and t we get

$$\mathrm{c} = 1 \quad \mathrm{d} = -\tfrac{1}{2} \quad \text{and } \mathrm{b} = \tfrac{1}{\sqrt{4a}}$$

and thus the ordinary differential equation

$$\frac{d^2\theta^*}{d\eta^2} + 2\eta\frac{d\theta^*}{d\eta} = 0 \tag{5.8}$$

with $\eta \equiv \frac{x}{\sqrt{4at}}$. The new variable η can be rewritten using the Fourier number Fo, where the distance from the plate surface will be used as the characteristic length,

$$\eta \equiv \frac{x}{\sqrt{4at}} = \frac{1}{\sqrt{4\mathrm{Fo}}}$$

The second-order differential equation (5.8) requires two boundary conditions in the new coordinates η and θ^*. Transforming the boundary conditions yields

$$\begin{aligned} &\text{BC1:} \quad \eta = \infty && : \quad \theta^* = 0 \\ &\text{BC2:} \quad \eta \to 0 && : \quad \theta^* = 1 \\ &\text{BC3:} \quad \eta \to \infty && : \quad \theta^* = 0 \end{aligned}$$

Since BC1 equals BC3 after transformation, and they do not contradict each other, a similar solution is possible using the variable η.

The solution of this second-order differential equation is carried out with the usual methods of substitution,

$$z \equiv \frac{d\theta^*}{d\eta}\,, \qquad \frac{dz}{d\eta} = \frac{d^2\theta^*}{d\eta^2}$$

which yields

$$\frac{dz}{d\eta} + 2\eta z = 0$$

and integrating

$$\ln z = -\eta^2 + C_1 \quad \text{or} \quad \frac{d\theta^*}{d\eta} = C_2 e^{-\eta^2}$$

The second integration yields, using integration variable ξ and BC 2

$$\theta^*(\eta) - 1 = C_2 \int\limits_{\xi=0}^{\xi=\eta} e^{-\xi^2} d\xi$$

C_2 can be determined from boundary condition

$$C_2 = \frac{-1}{\int\limits_0^\infty e^{-\xi^2} d\xi} = -\frac{2}{\sqrt{\pi}}$$

Which yields the dimensionless temperature field

$$\theta^* = 1 - \frac{2}{\sqrt{\pi}} \int\limits_{\xi=0}^{\xi=\eta} e^{-\xi^2} d\xi \tag{5.9}$$

The second term on the right side of equation (5.9) is called the error function $\mathrm{erf}(\eta)$ and can be found listed on the Appendix. So that,

$$\begin{aligned} \theta^* &= 1 - \mathrm{erf}\left[\eta\right] \qquad \text{or} \\ \theta^* &= 1 - \mathrm{erf}\left[\tfrac{1}{\sqrt{4\mathrm{Fo}}}\right] \end{aligned} \tag{5.10}$$

This function is shown in the following diagram 5.4. The diagram shows that the

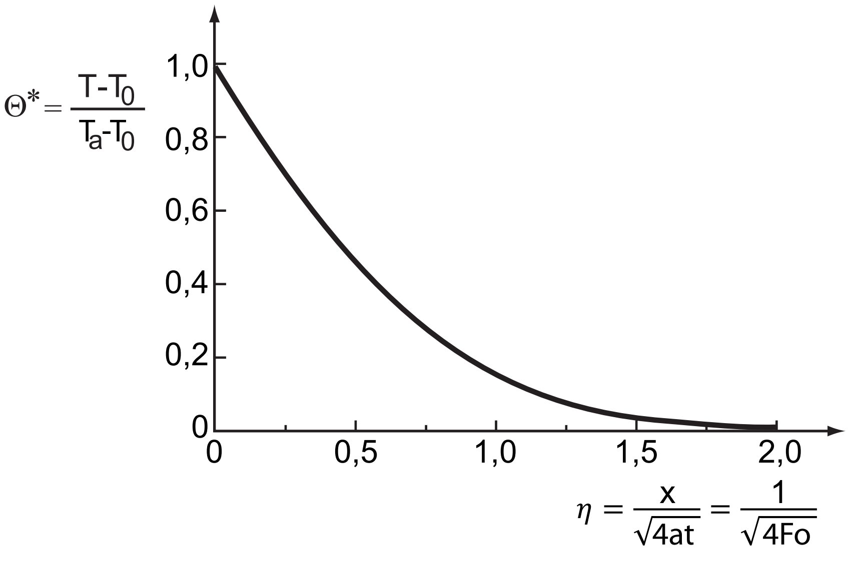


Figure 5.4: Temperature characteristics in a semi-infinite body - constant surface temperature

initial temperature difference is reduced by 1% at a value of $\eta = 1{,}8$. This value is often used to define the penetration depth or thickness of the temperature boundary layer

$$\delta(t) = 3{,}6\sqrt{at} \tag{5.11}$$

From the temperature field (equation (5.9)) the heat transfer rate at the surface can be determined

$$\dot{q}''\Big|_{x=0} = -\,\lambda \frac{\partial T}{\partial x}\Big|_{x=0}$$

Differentiation yields:

$$\begin{aligned} \dot{q}''\Big|_{x=0} &= \tfrac{\lambda}{\sqrt{\pi a t}}\,(T_\mathrm{a} - T_\mathrm{o}) \\ &= \sqrt{\tfrac{\lambda c\rho}{\pi t}}\,(T_\mathrm{a} - T_\mathrm{o}) \end{aligned} \tag{5.12}$$

In other words, the heat drops steadily with time, so that at time t the heat is reduced to

$$\int\limits_{t=0}^{t} \dot{q}''\Big|_{x=0} dt = 2\sqrt{\frac{\lambda c\rho}{\pi} t}\,(T_\mathrm{a} - T_\mathrm{o}) \tag{5.13}$$

Semi-infinite plate with non negligible heat transfer resistance

In the previous section it is assumed that the temperature at the body surface equals the surrounding temperature for all $t > 0$. This is valid only if the heat transfer resistance is small compared to the thermal resistance within the body, or if $\quad \mathrm{Bi} = \frac{\alpha L}{\lambda} \gg 1$.

If the heat transfer resistance cannot be neglected, then a surface temperature will be observed which is between the surrounding temperature and the initial temperature of the body. Instead of BC2, a new boundary condition is introduced
BC 2a:

$$\alpha\,(T_\mathrm{a} - T_{x=0}) = -\lambda \left(\frac{\partial T}{\partial x}\right)_{x=0}$$

or

$$\left(\frac{\partial T}{\partial x}\right)_{x=0} = \frac{\alpha}{\lambda}\,(T_{x=0} - T_\mathrm{a}) \tag{5.14}$$

This relationship can be presented illustratively, as shown in the following diagram. The extrarpolations of all gradient lines intersect at point P, which is given by the coordinates T_a and $-\frac{\lambda}{\alpha}$.

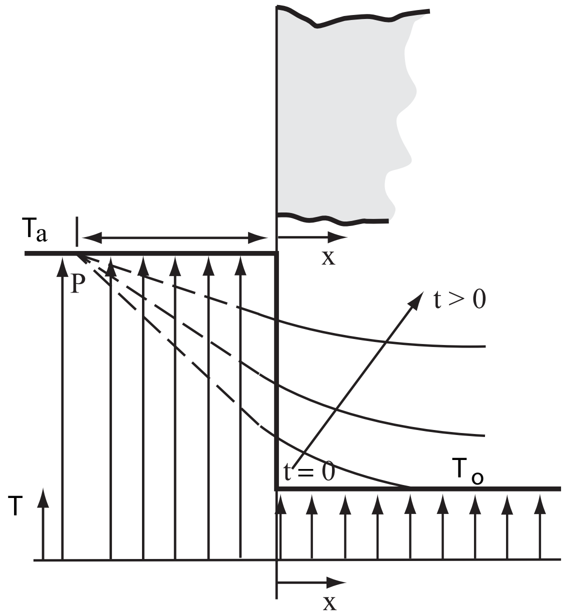


Figure 5.5: Boundary conditions

With these boundary conditions the computation load for solving equation (5.6) is much greater than before. Detailed discussion can be found in **?**. The following equation describes the temperature field

$$\theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\mathrm{Fo}}}\right) - \left[\exp\left(\mathrm{Bi} + \mathrm{Fo}\ \mathrm{Bi}^2\right)\right]\left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\mathrm{Fo}}} + \sqrt{\mathrm{Fo}} \cdot \mathrm{Bi}\right)\right] \tag{5.15}$$

with $\mathrm{Fo} = at/x^2$ and $\mathrm{Bi} = \alpha x/\lambda$.

The solution of this equation is shown in figure 5.6. The previously discussed case of negligible heat transfer resistances is thus a special case, $\sqrt{\mathrm{Fo}}\mathrm{Bi} \to \infty$, as shown in the diagram.

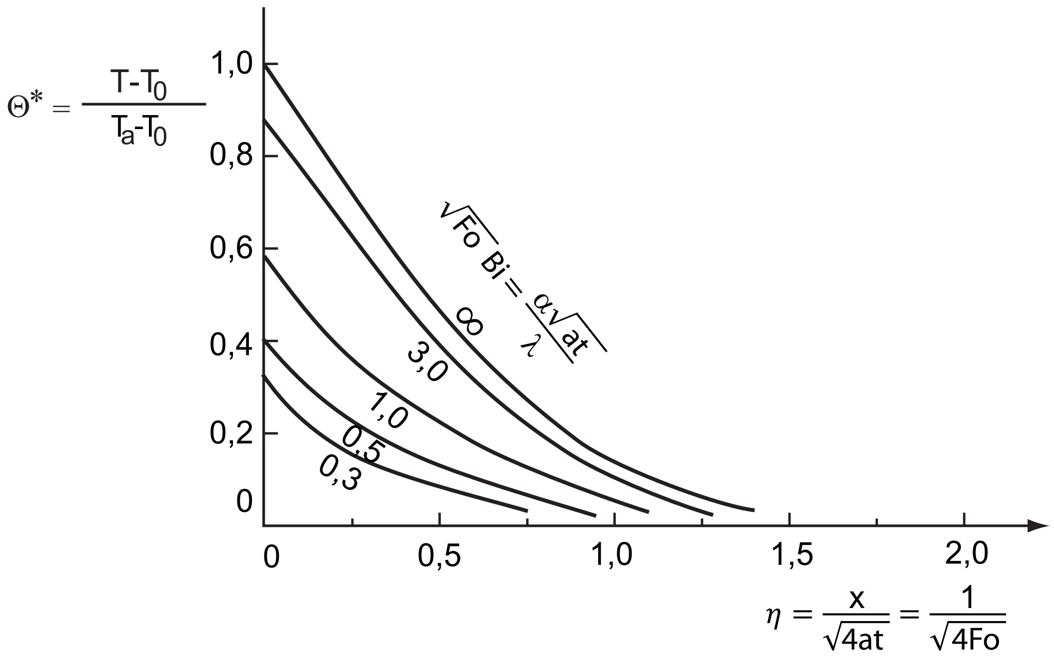


Figure 5.6: The temperature as a function of time of a semi-infinite body with finite convective heat transfer resistance.

Semi-infinite plate with time dependent surface temperatures

Many temperature processes have boundary conditions that change periodically, e.g. the daily or annual temperature variations, which have consequences on the temperature distribution in the walls of buildings or floors, or the cyclic heat load of the cylinder wall of a combustion engine.

In general, such processes with boundary conditions that change periodically cannot be described by analytical solutions of the differential equation for the temperature field (eq.(1.8)).

Yet, the basic relationships can still be discussed using the model of the semi-infinite body. Equation (1.8) thus simplifies to

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} \tag{5.16}$$

BC1: $t = 0:$ $T = T_o(x)$

BC1: $x = 0:$ $T_{x=0} = f(t)$

BC2: $x \to \infty:$ $T_{x\to\infty} = T_m$

If, for the simplest case, we assume that the surface temperature changes periodically according to a step function between T_{max} and T_{min}, then we can get results for the temperature profile within the body as shown in the diagram below.

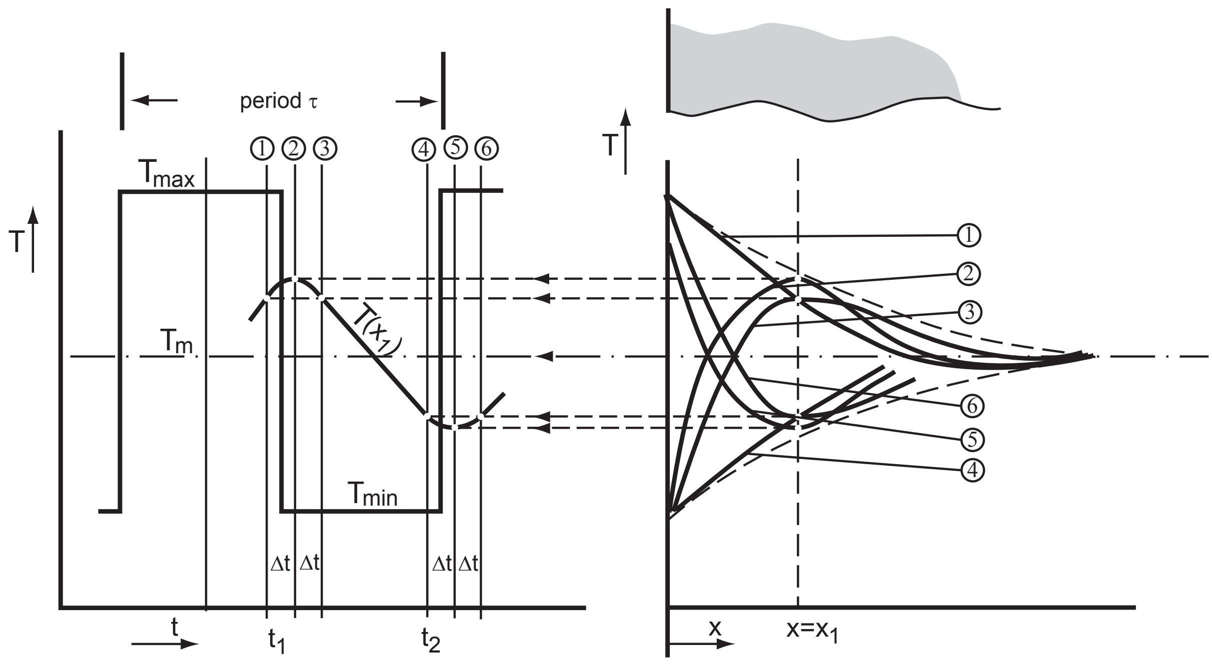


Figure 5.7: Penetration of a periodically changing surrounding temperature in a semi-infinite body

At time t_1 shortly before the temperature T_{max} changes to T_{min} we would expect a temperature distribution according to curve 1. Only in the deeper layers this curve differs from the symptotic case of long-period oscillations. Shortly after the change Δt to temperature T_{min} a new temperature profile, curve 2, exists for which, although the temperature maximum in the deep layers flattens through heat conduction. In the areas close to the surface the reduced surface temperature is noticeable. The process of heat conduction e balance leads shortly before changing again from T_{min} to T_{max} at t_2 to a new temperature profile, curve 4, which is the mirror image of curve 1. After the shift, the same process begins again. From these profiles on the right side, which are qualitatively derived, we can get the temperature profiles as a function of the time e.g. for a layer at depth $x = x_1$ sketched on the left side. How this is done is shown in the diagram.

It can be concluded that

- the amplitude of the temperature oscillation diminishes at greater depths,
- the temperature maximum in the inner part of the body are phase shifted

If the surface temperature is described by a harmonic oscillation,

$$T_{x=0} - T_{\mathrm{m}} = (T_{\max} - T_{\mathrm{m}}) \cos\left(\frac{2\pi}{\tau} t\right)$$

and

$$\theta_{x=0} = \frac{T_{x=0} - T_{\mathrm{m}}}{T_{\max} - T_{\mathrm{m}}} \cos\left(\frac{2\pi}{\tau} t\right)$$

with the period τ, then we get an analytical solution for the temperature field in the form of

$$\theta = \frac{T - T_{\mathrm{m}}}{T_{\max} - T_{\mathrm{m}}} = \underbrace{\exp\left(-\sqrt{\frac{\pi x^2}{a\tau}}\right)}_{1} \cos\left(\underbrace{2\pi \frac{t}{\tau}}_{2} - \underbrace{\sqrt{\frac{\pi x^2}{a\tau}}}_{3}\right) \tag{5.17}$$

with the amplitude (1):

$$\exp\left(-\sqrt{\frac{\pi x^2}{a\tau}}\right)$$

angular frequency (2):

$$\frac{2\pi}{\tau} \tag{5.18}$$

and phase shift (3):

$$\sqrt{\frac{\pi x^2}{a\tau}}$$

The above presented qualitative results of how the amplitudes of the temperature oscillations and phase shifts subside can be evaluated quantitatively using equation (5.17).

5.3 Dimensionless numbers and Heisler diagrams

In the previous layers, a number of examples were given, which had simple, analytical solutions of the Fourier equation (1.8),

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

However, if in cases of systems with complicated geometries or boundary and initial conditions, analytical solutions do not suffice, then numerical solutions using the method of finite differences are nowadays available.

One of the major disadvantages of such methods is the fact that, in general, the derived temperature field is dependent on a large number of parameters

$$T = T\,(\mathrm{x,y,z,t,}\rho\mathrm{,c,}\lambda, \text{initial and boundary conditions})$$

and that if one of the parameters changes, then the entire calculation has to be redone.

Next, using a simple example of unsteady state heat conduction in a plate, it will be shown that the number of dependent parameters can be considerably reduced by introducing dimensionless numbers, some of which have already been discussed in previous examples.

First two plates will be considered, with lengths much greater than their thickness and which are brought at a given time in another environment with different temperature. The thickness of the plate, its initial temperature, the new temperature of the surroundings, the material properties and the boundary conditions are labeled A for the first and B for the second plate.

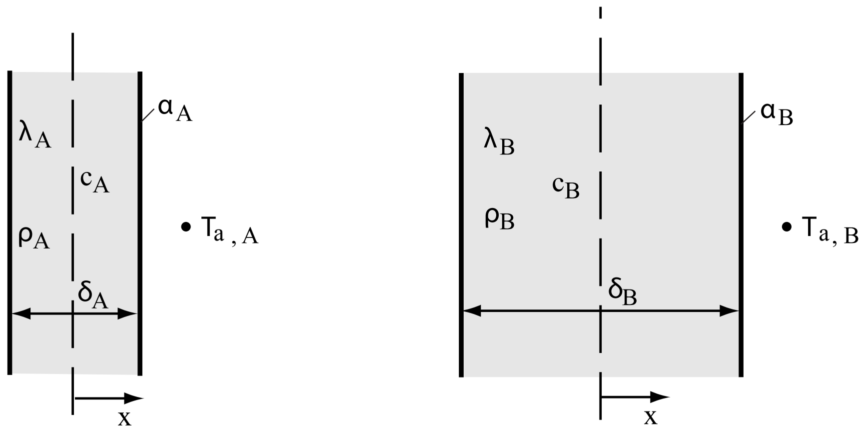


Figure 5.8: Unsteady heat transfer for a two different plate configurations

The temperatures in the plates are distributed according to the differential equations

System A

$$\frac{\partial T_A}{\partial t} = \left(\frac{\lambda}{\rho c}\right)_A \frac{\partial^2 T_A}{\partial x^2}$$

System B

$$\frac{\partial T_B}{\partial t} = \left(\frac{\lambda}{\rho c}\right)_B \frac{\partial^2 T_B}{\partial x^2}$$

with the initial and boundary conditions

$$\begin{array}{lll} t = 0 & -\frac{\delta}{2} < x < +\frac{\delta}{2} & T = T_0 \\ t \to \infty & -\frac{\delta}{2} < x < +\frac{\delta}{2} & T = T_a \\ t > 0 & x = -\frac{\delta}{2} \text{ and } x = +\frac{\delta}{2} & \dot{q}''_{\text{conduction}} = \dot{q}''_{\text{convection}} \end{array}$$

The differential equations can be written in a dimensionless form. To this purpose, reference values will be selected that characterize the system. The geometry of both plates is described by the plate thickness δ. If the heat conduction process is periodic in time the process is similar if only the period of oscillations τ is different for both plates. The temperature, or the temperature difference $\theta \equiv T - T_a$, is referenced to the characteristic temperature difference $\theta_0 \equiv T_0 - T_a$ of the system.

Hence, the dimensionless variables are

$$\begin{aligned} \text{x}^* &\equiv \frac{x_A}{\delta_A} = \frac{x_B}{\delta_B}, \\ \text{t}^* &\equiv \frac{t_A}{\tau_A} = \frac{t_B}{\tau_B} \\ \text{and } \theta^* &\equiv \frac{\theta_A}{\theta_{A_0}} = \frac{\theta_B}{\theta_{B_0}} \end{aligned} \tag{5.19}$$

by substituting them into the differential equations for the temperature field they read as follows

System A

$$\frac{\partial \theta^*}{\partial t^*} = \left(\frac{a_\mathrm{A}\tau_\mathrm{A}}{\delta_\mathrm{A}^2}\right)\frac{\partial^2\theta^*}{\partial x^{*2}}$$

System B

$$\frac{\partial \theta^*}{\partial t^*} = \left(\frac{a_\mathrm{B}\tau_\mathrm{B}}{\delta_\mathrm{B}^2}\right)\frac{\partial^2\theta^*}{\partial x^{*2}}$$

Thus, the differential equations are identical, if the Fourier numbers

$$\mathrm{Fo} \equiv \frac{a\tau}{\delta^2} = \frac{\lambda}{\rho c}\frac{\tau}{\delta^2}$$

of both systems are equal. The equivalence of the parameters which are included in the Fourier number is not necessary. The dimensionless temperature fields are only then equal, if the boundary conditions of both systems are equal, too. At all times t, the energy balance at the wall must be obeyed. In other words, the conductive heat flow rate to the surface of the body must equal the convective heat flow from to surface to the surroundings.

$$\dot{q}''_\mathrm{wall} = -\left(\lambda\frac{\partial T}{\partial x}\right)_\mathrm{wall} = \alpha\left(T_\mathrm{wall} - T_\mathrm{a}\right) \tag{5.20}$$

and hence

$$\left(\frac{\partial T}{\partial x}\right)_\mathrm{wall} = -\frac{\alpha}{\lambda}\left(T_\mathrm{wall} - T_\mathrm{a}\right)$$

In the dimensionless form, the boundary conditions are

System A

$$\left(\frac{\partial \theta^*}{\partial x^*}\right)_\mathrm{Wand} = -\left(\frac{\alpha_\mathrm{A}\delta_\mathrm{A}}{\lambda_\mathrm{A}}\right)\theta^*_\mathrm{Wand}$$

System B

$$\left(\frac{\partial \theta^*}{\partial x^*}\right)_\mathrm{Wand} = -\left(\frac{\alpha_\mathrm{B}\delta_\mathrm{B}}{\lambda_\mathrm{B}}\right)\theta^*_\mathrm{Wand}$$

Hence, it follows that the boundary conditions are identical, if the Biot numbers

$$\mathrm{Bi} \equiv \frac{\alpha\delta}{\lambda}$$

of both systems are equal. If the system had to be extended to consider three-dimensional bodies, the dimensionless temperature field has to be described by dimensionless parameters

$$\begin{aligned}\frac{T - T_\mathrm{a}}{T_0 - T_\mathrm{a}} &= \frac{T - T_\mathrm{a}}{T_0 - T_\mathrm{a}}\left(\frac{\mathrm{x}}{\delta_1},\frac{\mathrm{y}}{\delta_2},\frac{\mathrm{z}}{\delta_3},\frac{t}{\tau},\left(\frac{a\tau}{\delta^2}\right)_{1,2,3},\left(\frac{\alpha\delta}{\lambda}\right)_{1,2,3}\right)\\ &= \frac{T - T_\mathrm{a}}{T_0 - T_\mathrm{a}}\left(\mathrm{x}^*,\mathrm{y}^*,\mathrm{z}^*,\mathrm{t}^*,\mathrm{Fo}_{1,2,3},\mathrm{Bi}_{1,2,3}\right)\end{aligned} \tag{5.21}$$

The previously mentioned examples can be derived using theses parameters, respectively. The analytical or numerical solutions of the differential equations are

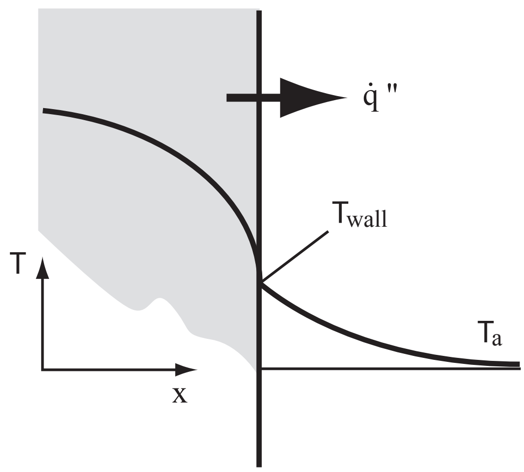


Figure 5.9: Boundary conditions at the surface

often presented in diagrams using these dimensionless parameters. In the following three examples will be shown which are useful to approximate many practical problems. These examples will show the temperature profile and the transferred heat during cooling (or heating) of extended plates, long cylinders and spheres, whose surrounding temperatures have been abruptly changed at a given time. Although these one-dimensional problems can be solved analytically, because of their complicated computations it is recommended to use the diagrams given by **?**. The following diagrams show the mid-plane temperature as a function of the time for a plate, cylinder and sphere, together with additional diagrams used for the determination of the temperatures at other points of the body. The diagrams will be interpolated appropriately in order to get the heat as a function of the time at the surface of the body.

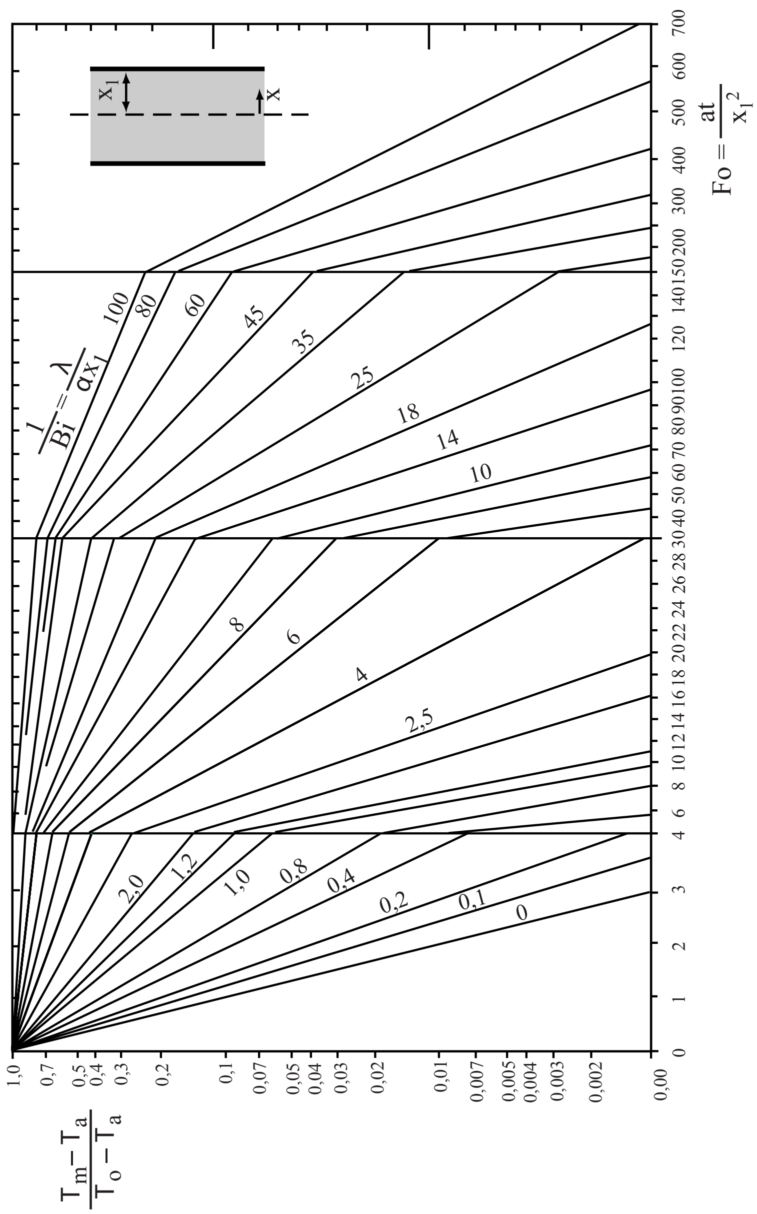


Figure 5.10: Mid-plane temperature of a plate with thickness $2x_1$

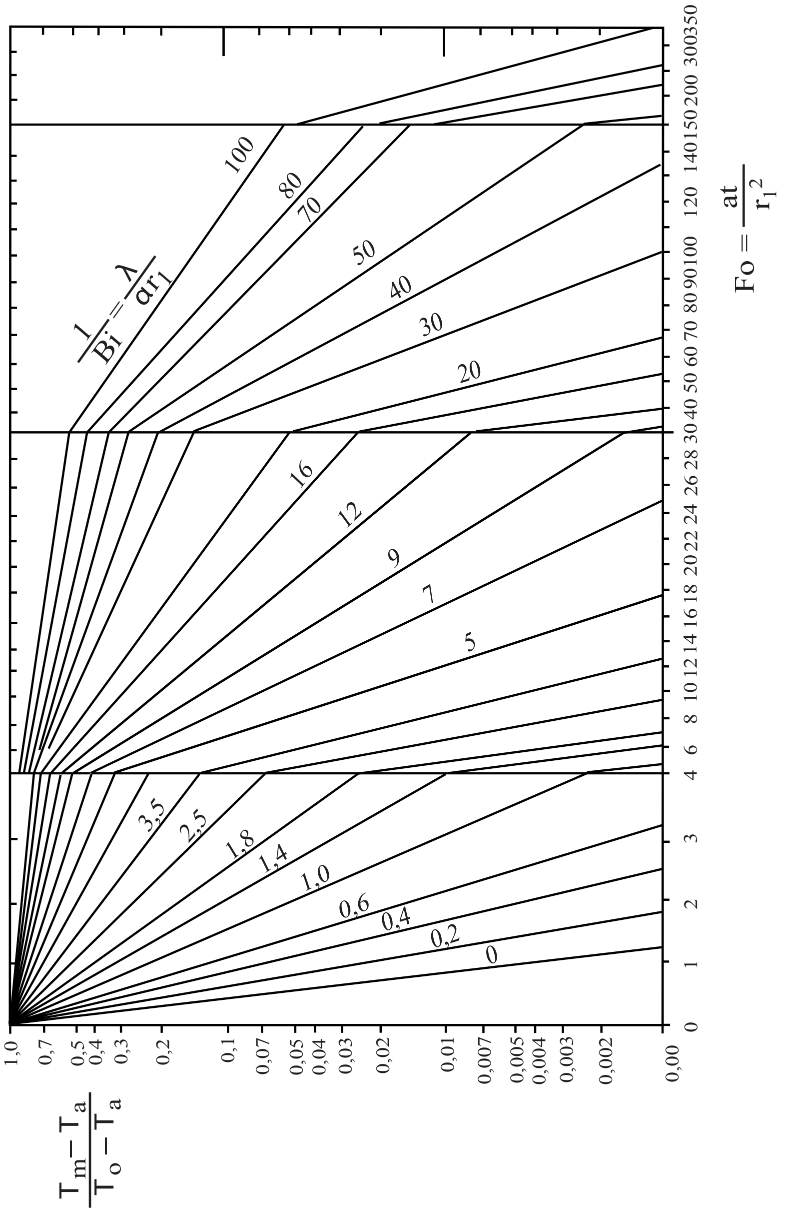


Figure 5.11: Temperature along the axis of a cylinder with radius r_1

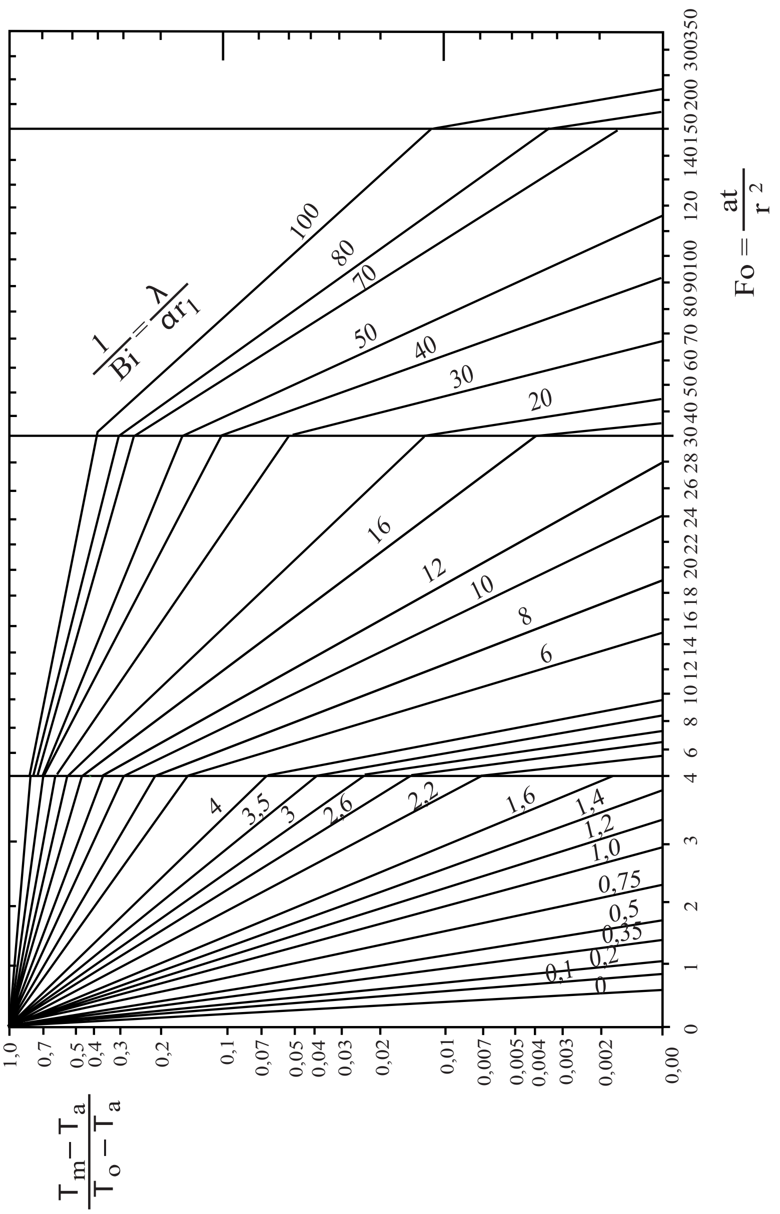


Figure 5.12: Temperature in the centre of a sphere with radius r_1

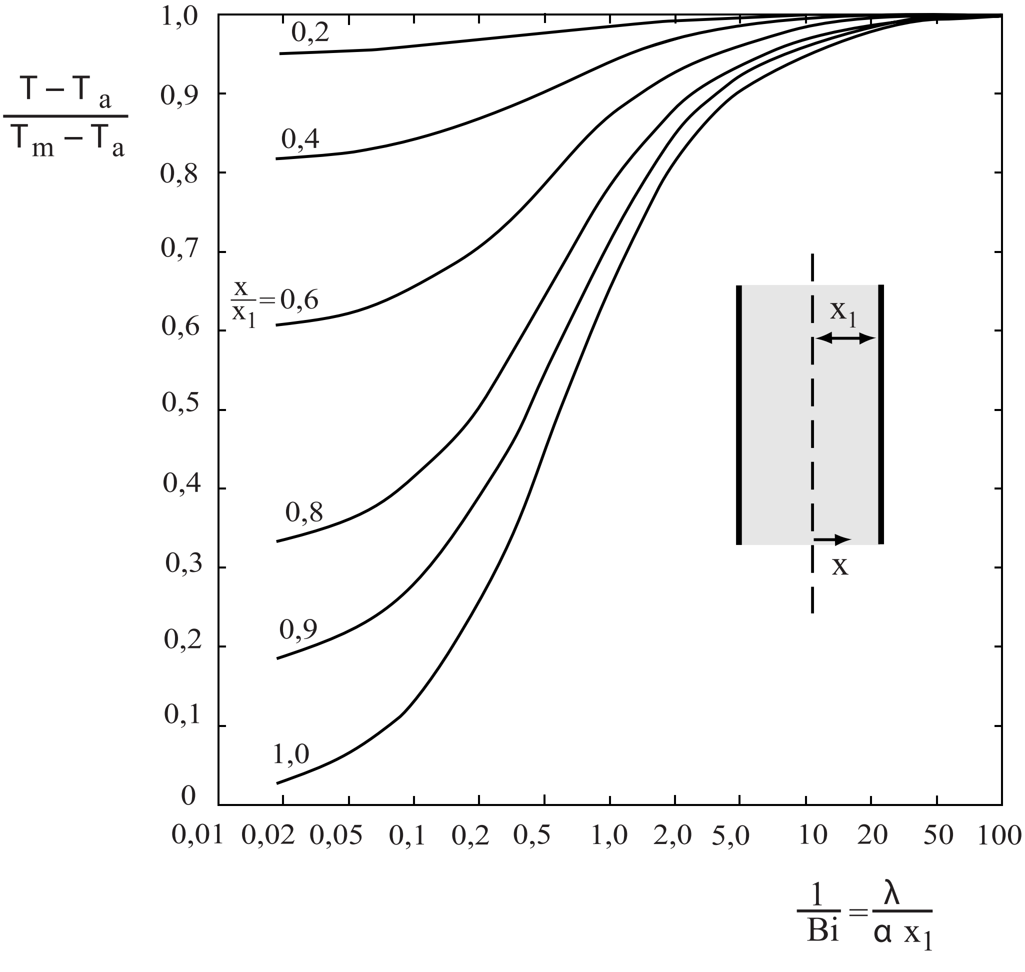


Figure 5.13: Temperature distribution in a plate (valid for Fo > 0,2)

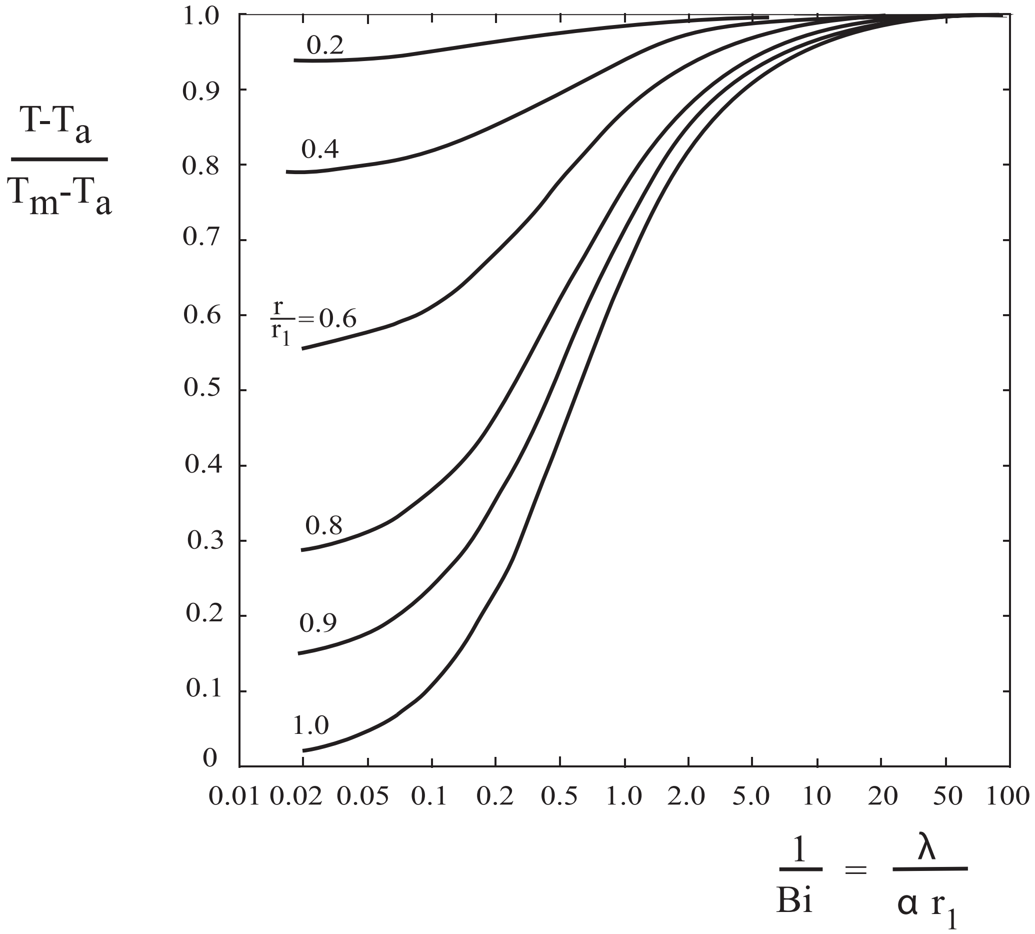


Figure 5.14: Temperature distribution in a cylinder (valid for Fo > 0,2)

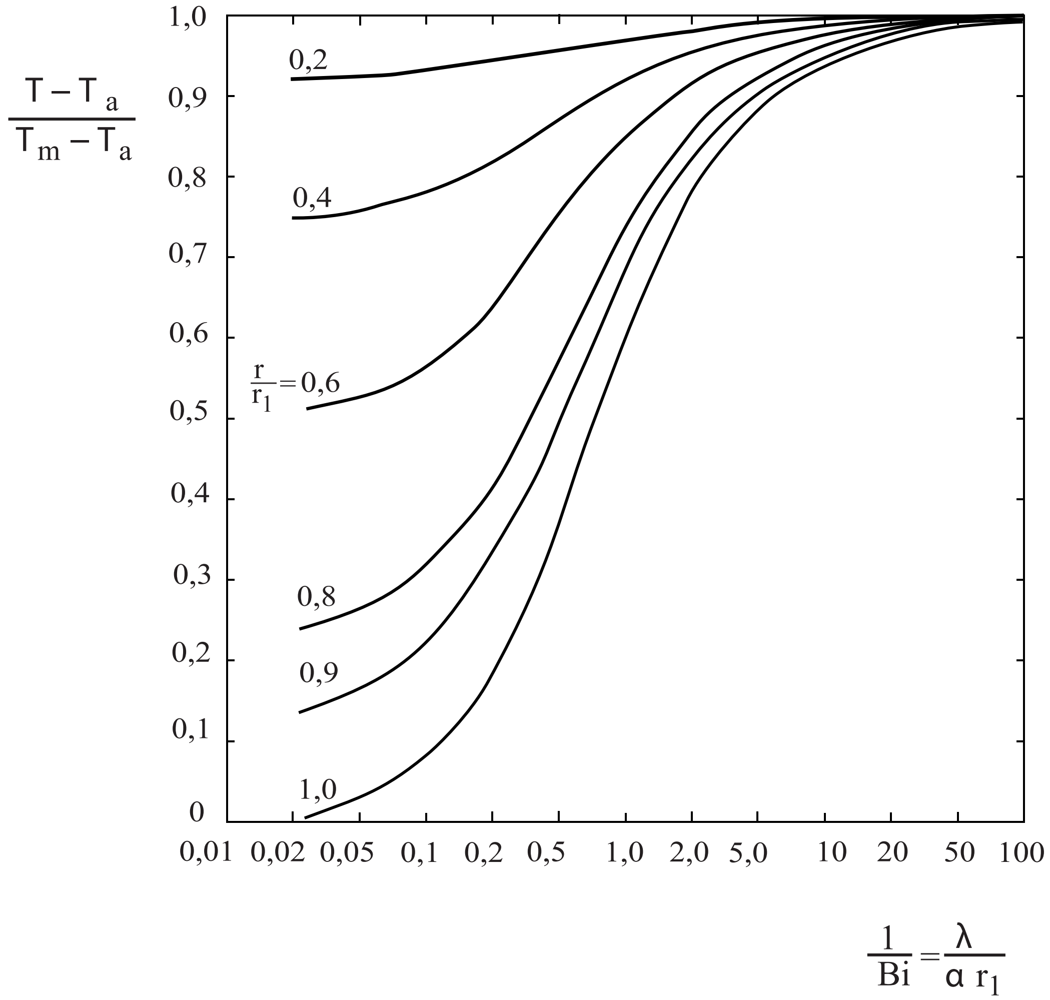


Figure 5.15: Temperature distribution in a sphere (valid for Fo > 0,2)

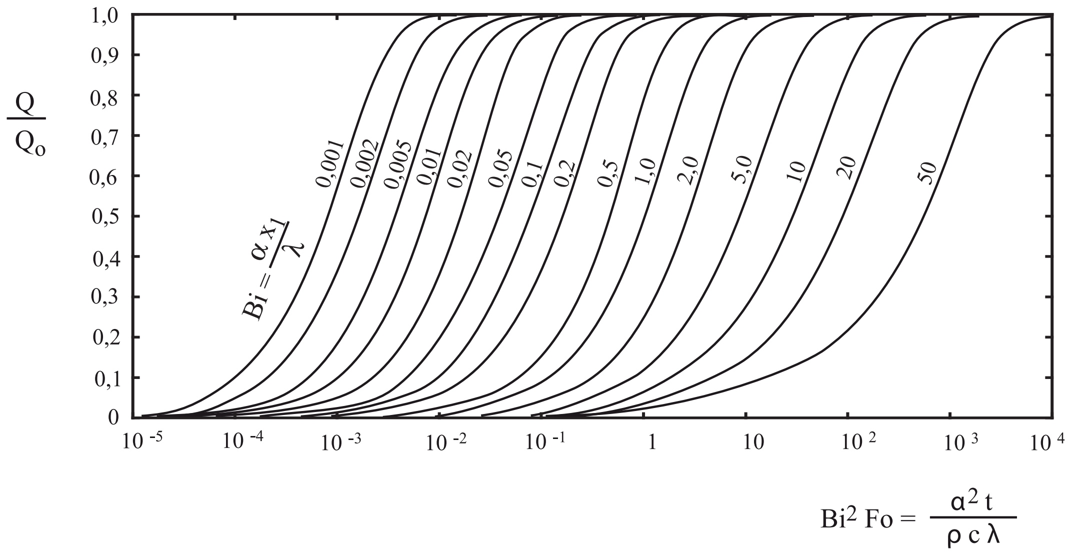


Figure 5.16: Heat loss of a plate

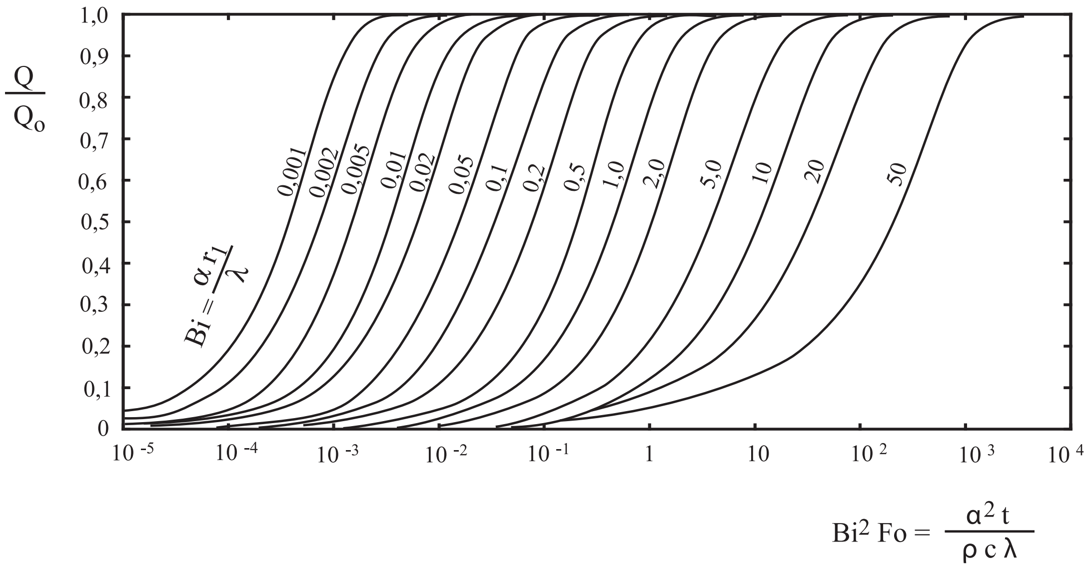


Figure 5.17: Heat loss of a cylinder

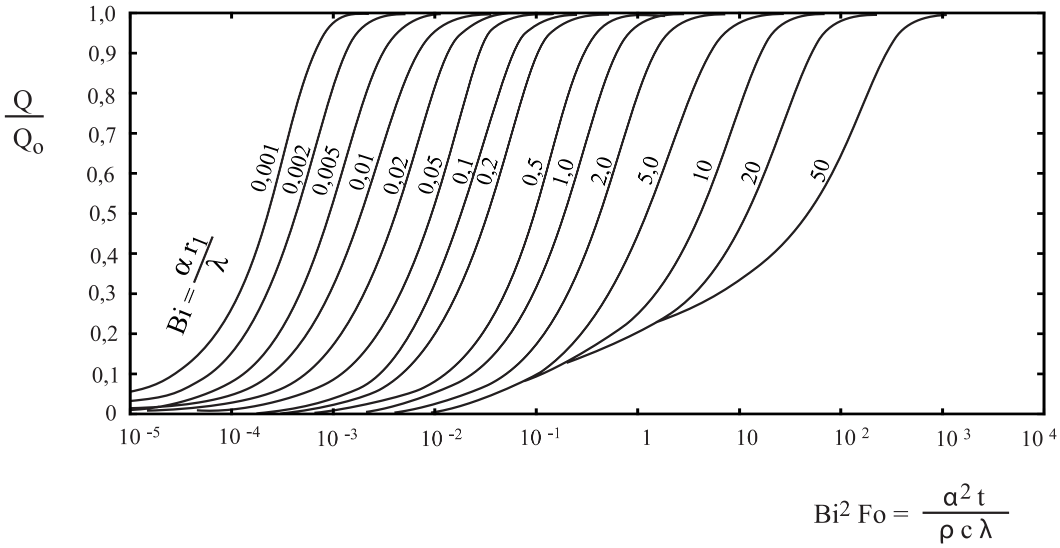


Figure 5.18: Heat loss of a sphere