Heat Transfer: Conduction

Derivation of the steady state energy conservation equations

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Video overview

Steady state energy conservation equation without sources:

- Steady state 1-D heat conduction without sources
- Steady state ***2-D*** heat conduction without sources

Steady state energy conservation equation with sources:

- Steady state 2-D heat conduction ***with sources***

Transient energy conservation equation:

- ***Transient*** 2-D heat conduction with sources

Transient 3-D energy conservation equation with sources:

- ***3-D*** conservation equation without advection

Learning goals

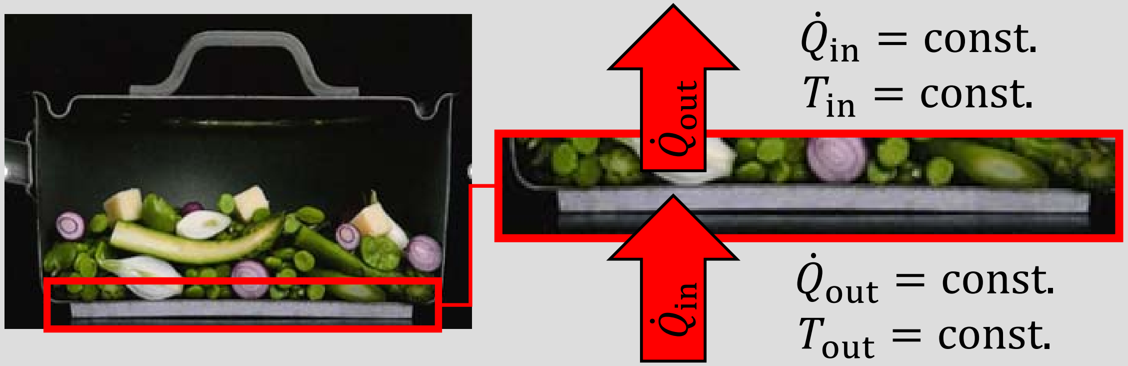
Energy balances:

- Setting up energy balances for different cases
- Development of a differential equation from the energy balance using Taylor series expansion
- Establish necessary boundary conditions
- Solving the differential equation for simple cases

Examples from our everyday life

Steady state:

Heat is conducted through the bottom of the pot. Both the temperature difference and the heat flow are constant in time.



Steady state energy conservation equation with sources:

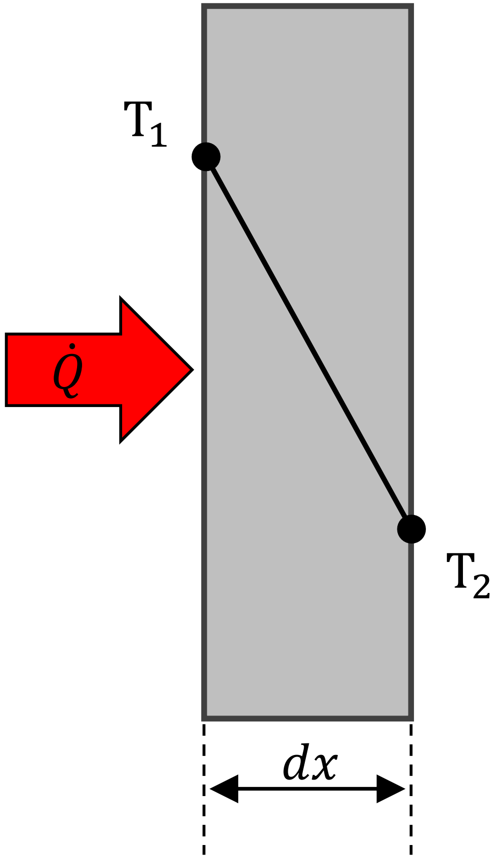
The amount of heat/temperature of an object changes over time.

- Example: The coffee cools down.

Transient energy conservation equation:

Thermal energy within a body is generated or absorbed by the conversion of other types of energy into heat.

Review Fourier law:
Steady state 1-D heat conduction in a plane wall without source



Fourier's law:

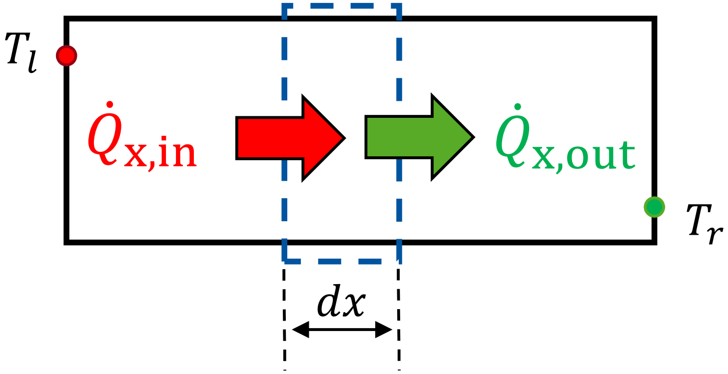
$$\dot{Q}_x = -A\,\lambda\,\frac{\Delta T}{\Delta x}$$

Heat flow through the wall:

$$\dot{Q} = \dot{q}'' \cdot A \;= -\lambda \cdot A \cdot \frac{T_2 - T_1}{dx}\;[W]$$

Differential equation derivation:
Steady state 1-D heat conduction without sources

Stationary: thermal energy does not change over time!



Energy balance of the element dx:

$$0 = \dot{Q}_{\mathrm{x,in}} - \dot{Q}_{\mathrm{x,out}}$$

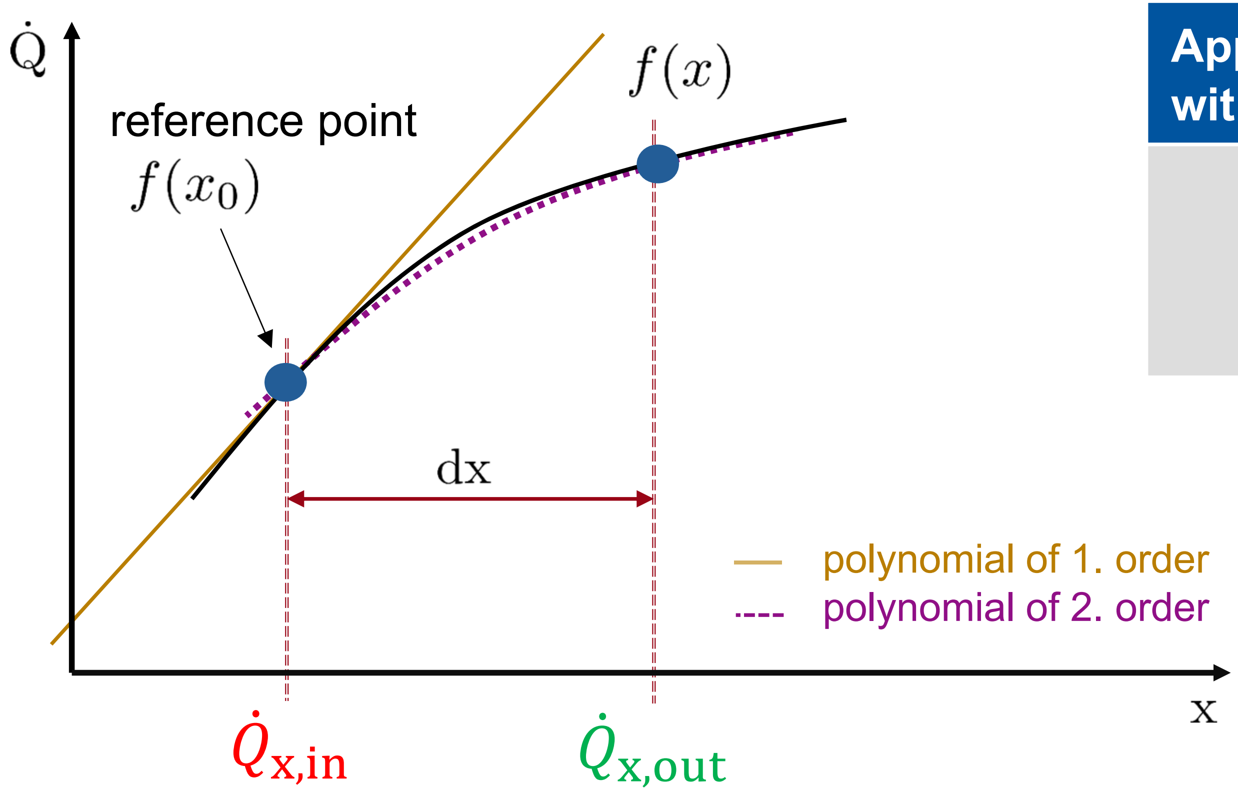
Definition of $\dot{Q}_{\mathrm{x,out}}$ by *Taylor series expansion*:

$$\dot{Q}_{\mathrm{x,out}} = \dot{Q}_{\mathrm{x,in}} + \frac{\partial \dot{Q}_{\mathrm{x,in}}}{\partial x} dx$$

Mathematical Add-on: Taylor series expansion

Approx. of a function $f(x)$ at the point $x = x_0$ with *Taylor series expansion*:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n$$

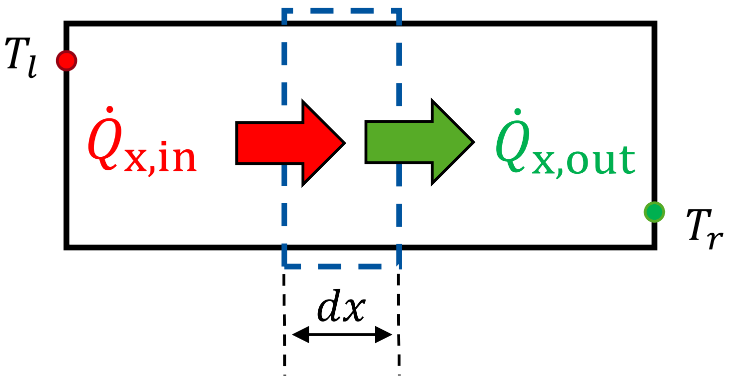


Approximation of the heat flow $\dot{Q}_{\mathrm{x,out}}$ with the polynomial of 1. order

$$\dot{Q}_{\mathrm{x,out}} = \dot{Q}_{\mathrm{x,in}} + \frac{\partial \dot{Q}_{\mathrm{x,in}}}{\partial x} dx$$

Differential equation derivation:
Steady state 1-D heat conduction without sources

Steady state: thermal energy does not change over time!



Energy balance around a infinitesimal element:

$$0 = \dot{Q}_{\mathrm{x,in}} - \dot{Q}_{\mathrm{x,out}}$$

Taylor series expansion:

$$\dot{Q}_{\mathrm{x,out}} = \dot{Q}_{\mathrm{x,in}} + \frac{\partial \dot{Q}_{\mathrm{x,in}}}{\partial x} dx$$

Insert Taylor series expansion in energy balance:

$$0 = \cancel{\dot{Q}_{\mathrm{x,in}}} - \cancel{\dot{Q}_{\mathrm{x,in}}} - \frac{\partial \dot{Q}_{\mathrm{x,in}}}{\partial x} \cancel{dx}$$

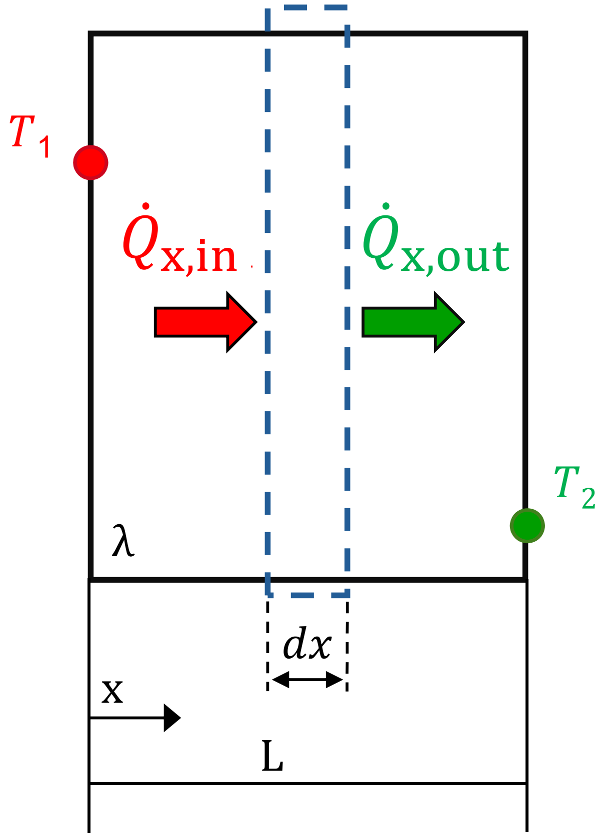
Apply Fourier's law:

$$\dot{Q} = \dot{q}'' \cdot A = -\lambda \cdot A \cdot \frac{\partial T}{\partial x}$$

Resulting differential equation:

$$0 = -\frac{\partial \dot{Q}_{\mathrm{x,in}}}{\partial x} = \lambda \cdot A \cdot \frac{\partial^2 T}{\partial x^2}$$

Temperature profile:
Steady state 1-D heat conduction without sources



Diff. eq. 2. order ➡ 2 b.c. needed:

$$0 = -\frac{\partial \dot{Q}_{x,in}}{\partial x} = \lambda \cdot A \cdot \frac{\partial^2 T}{\partial x^2}$$

Boundary conditions:

$$\mathrm{x} = 0 \quad , \quad T = T_1$$
$$\mathrm{x} = \mathrm{L} \quad , \quad T = T_2$$

2-times integration yields temperature:

$$T = T_1 + \frac{T_2 - T_1}{L} x$$

Heat flow:

$$\dot{Q}_x = -\lambda A \frac{\partial T}{\partial x} = -\lambda A \frac{\partial \left(T_1 + \frac{T_2 - T_1}{L} x \right)}{\partial x} = -\lambda A \frac{T_2 - T_1}{L} \,[\mathrm{W}]$$

Steady state 2-D heat conduction without sources

Energy balance:

$$0 = \left(\dot{Q}_{\mathrm{x,in}} - \dot{Q}_{\mathrm{x,out}}\right) + \left(\dot{Q}_{\mathrm{y,in}} - \dot{Q}_{\mathrm{y,out}}\right)$$

$$0 = \left(\dot{q}''_{\mathrm{x,in}} - \dot{q}''_{\mathrm{x,out}}\right) \cdot dy \cdot 1 + \left(\dot{q}''_{\mathrm{y,in}} - \dot{q}''_{\mathrm{y,out}}\right) \cdot dx \cdot 1$$

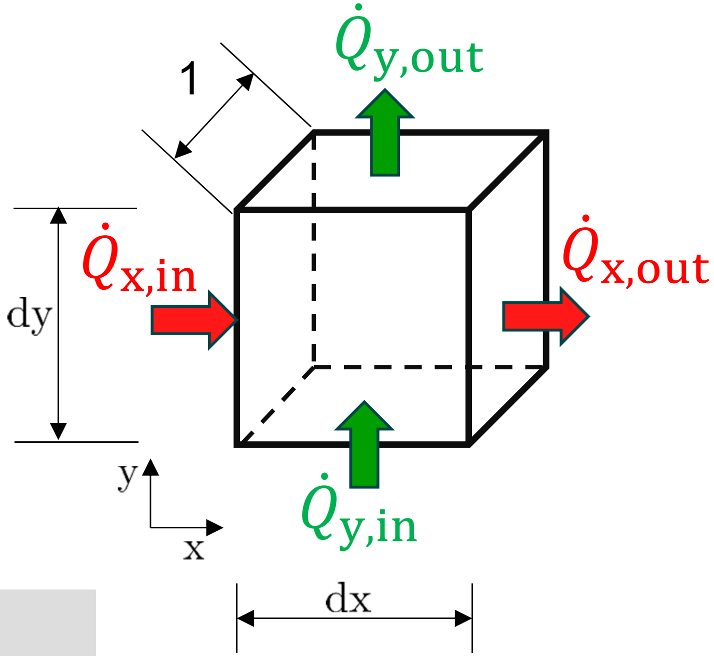
$\dot{q}''_{x,out}$ and $\dot{q}''_{y,out}$ with Taylor series expansion :

$$\dot{q}''_{\mathrm{x,out}} = \dot{q}''_{\mathrm{x,in}} + \frac{\partial \dot{q}''_{\mathrm{x,in}}}{\partial x} dx + \ldots$$

$$\dot{q}''_{\mathrm{y,out}} = \dot{q}''_{\mathrm{y,in}} + \frac{\partial \dot{q}''_{\mathrm{y,in}}}{\partial y} dy + \ldots$$

Insert in the energy balance:

$$0 = -\frac{\partial \dot{q}''_{\mathrm{x,in}}}{\partial x} + -\frac{\partial \dot{q}''_{\mathrm{y,in}}}{\partial y}$$



With Fourier's law:

$$0 = \lambda \frac{\partial^2 T}{\partial x^2} + \lambda \frac{\partial^2 T}{\partial y^2}$$

Laplace equation:

$$0 = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Steady state 2-D heat conduction with sources

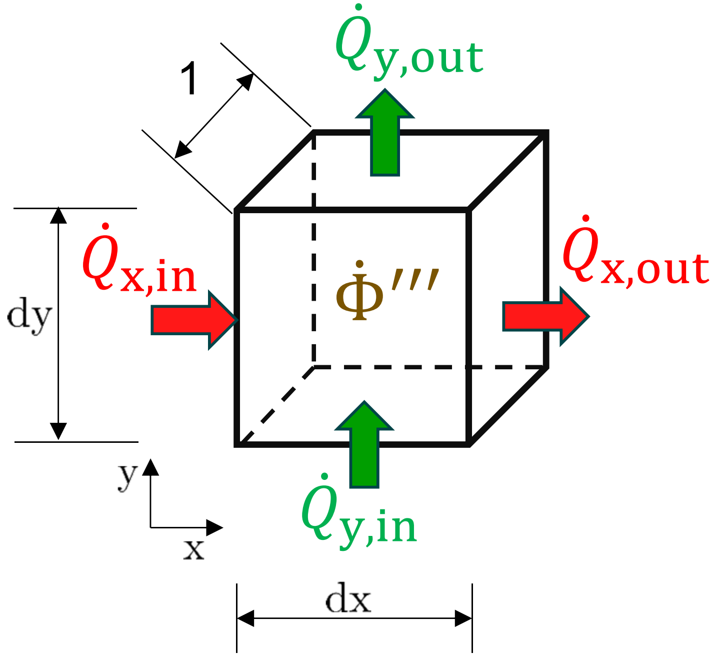
Energy balance:

$$0 = \left(\dot{Q}_{\mathrm{x,in}} - \dot{Q}_{\mathrm{x,out}}\right) + \left(\dot{Q}_{\mathrm{y,in}} - \dot{Q}_{\mathrm{y,out}}\right) + \dot{\Phi}''' \cdot V$$

$$0 = \left(\dot{q}''_{\mathrm{x,in}} - \dot{q}''_{\mathrm{x,out}}\right) \cdot dy \cdot 1 + \left(\dot{q}''_{\mathrm{y,in}} - \dot{q}''_{\mathrm{y,out}}\right) \cdot dx \cdot 1 + \dot{\Phi}''' \cdot dx \cdot dy \cdot 1$$

with

- Taylor series expansion
- Fourier's law



Poisson equation:

$$\lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \dot{\Phi}''' = 0$$

Volumetric source term :

$\dot{\Phi}''' \left[\frac{\mathrm{W}}{\mathrm{m}^3}\right]$

- can be positive (source)
- can be negative (sink)

Comprehension questions

What is the steady state temperature profile for a homogeneous, one-dimensional, flat wall without heat sources?

Under which conditions does Poisson's equation become Laplace's equation (heat conduction)?