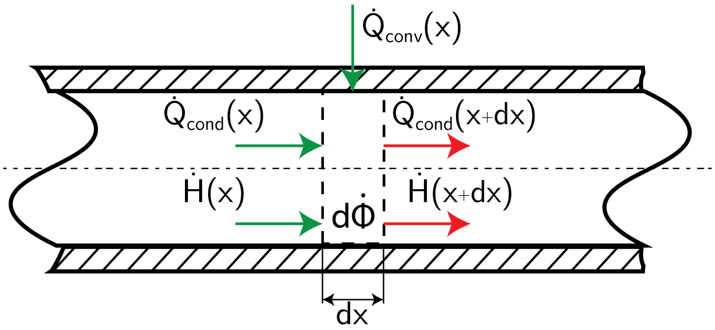
Fluid Flowing Through a Pipe

Derive the energy balance to describe the temperature profile in axial direction not neglecting axial diffusion.



1) Setting up an energy balance:

The temperature profile can be derived from the governing energy equation. The governing equation results from the energy balance around an infinitesimal element in the system:

$$\frac{dU}{dt} = \sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}}$$

Which for the steady-state case will be:

$$0 = \dot{Q}_{\text{cond}}(x) - \dot{Q}_{\text{cond}}(x+dx) + \dot{Q}_{\text{conv}}(x) + \dot{H}_{\text{cond}}(x) - \dot{H}_{\text{cond}}(x+dx) + d\dot{\Phi}$$

2) Defining the fluxes:

The ingoing conductive flux can be described by use of Fourier's law:

$$\dot{Q}_{\text{cond}}(x) = -\lambda A_c \frac{dT}{dx} = -\lambda \frac{\pi D^2}{4} \frac{dT}{dx}$$

The outgoing conductive flux for an infinitesimal element can be approximated by use of Taylor series:

$$\dot{Q}_{\text{cond}}(x+dx) = \dot{Q}_{\text{cond}}(x) + \frac{d\dot{Q}_{\text{cond}}(x)}{dx} \cdot dx$$

Furthermore, the gain due to convection for the infinitesimal element can be described by use of Newton's law of cooling (note that $T_{\text{w}} < T(x)$ will result in a outgoing flux):

$$\dot{Q}_{\text{conv}}(x) = \alpha A_{\text{s}} \left(T_{\text{w}} - T(x)\right) = \alpha \pi D dx \left(T_{\text{w}} - T(x)\right)$$

The ingoing convective flux can be described as:

$$\dot{H}(x) = \dot{m} c T(x) = \rho u \frac{\pi D^2}{4} c T(x)$$

The outgoing convective flux for an infinitesimal element can be approximated by use of Taylor series:

$$\dot{H}(x+dx) = \dot{H}(x) + \frac{d\dot{H}(x)}{dx} \cdot dx$$

Lastly, the heat generated inside the infinitesimal element can be expressed as:

$$d\dot{\Phi} = \dot{\Phi}''' \cdot dV = \dot{\Phi}''' \cdot \frac{\pi D^2}{4} dx$$

3) Inserting and rearranging:

Inserting the found fluxes yields:

$$\lambda\frac{\pi D^2}{4}\frac{d^2T}{dx^2}\cdot dx + \alpha\pi D dx\left(T_\mathrm{w} - T(x)\right) - \rho u\frac{\pi D^2}{4}c\frac{dT}{dx}\cdot dx + \dot{\Phi}'''\cdot\frac{\pi D^2}{4}dx$$

$$\boxed{\rightarrow \frac{d^2T}{dx^2} + \frac{4\alpha}{\lambda D}\left(T_\mathrm{w} - T(x)\right) - \frac{\rho uc}{\lambda}\frac{dT}{dx} + \frac{\dot{\Phi}'''}{\lambda}}$$