Heat Transfer: Conduction

Semi-infinite body

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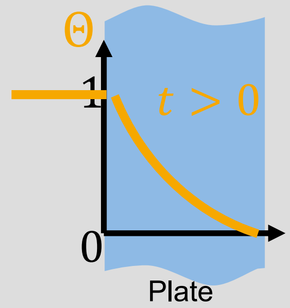
Learning goals

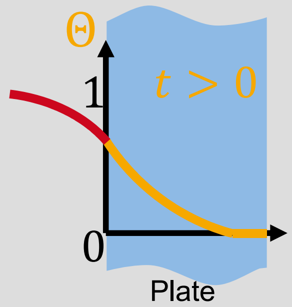
With imposed wall temperature:

- Understanding the applied boundary conditions of semi-infinite body with imposed wall temperature
- Solution of the problem with the Error Function Table

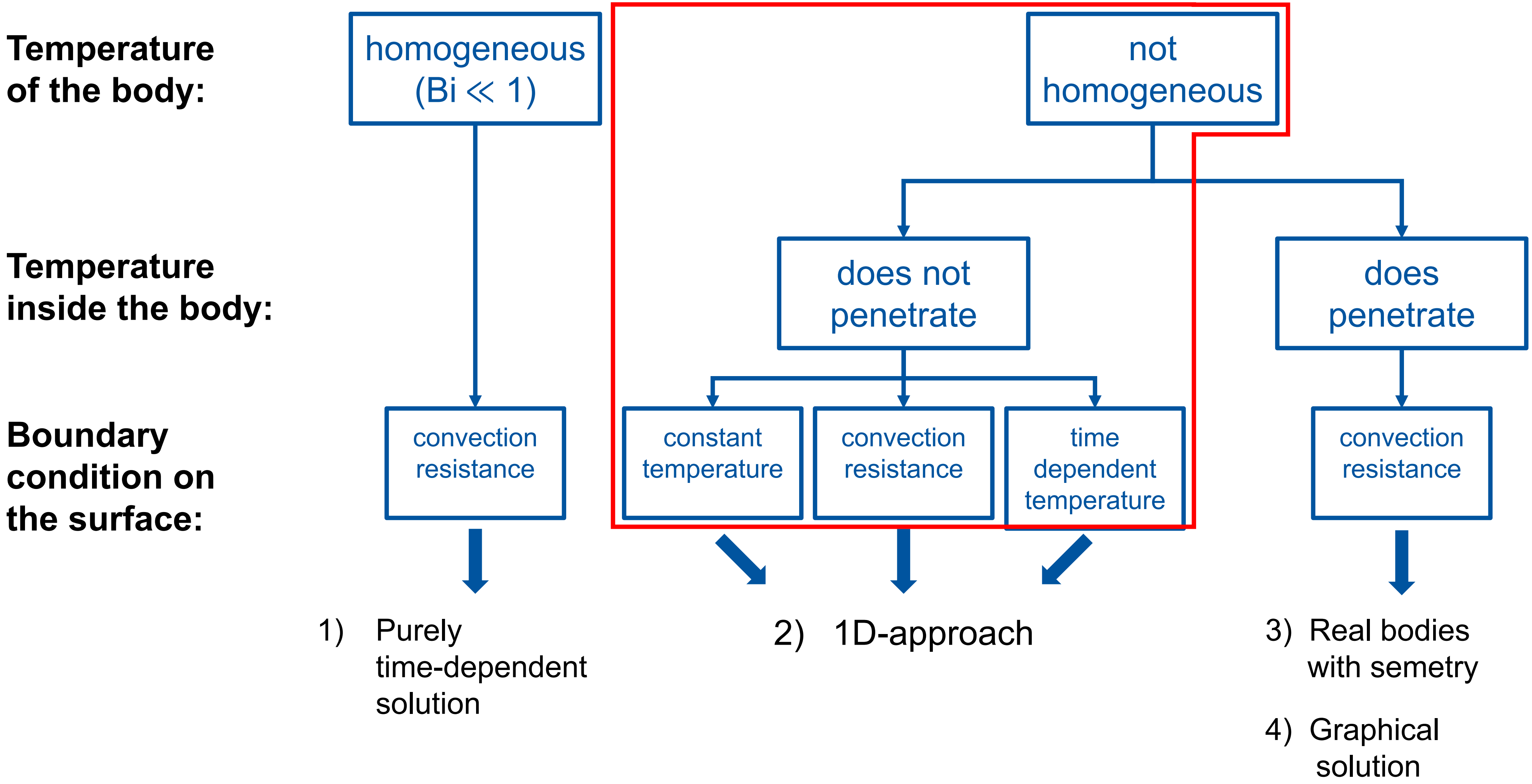


With non-negligible heat transfer resistance:

- Understanding the applied boundary conditions of semi-infinite body with non-negligible heat transfer resistance



How to simplify the problem?



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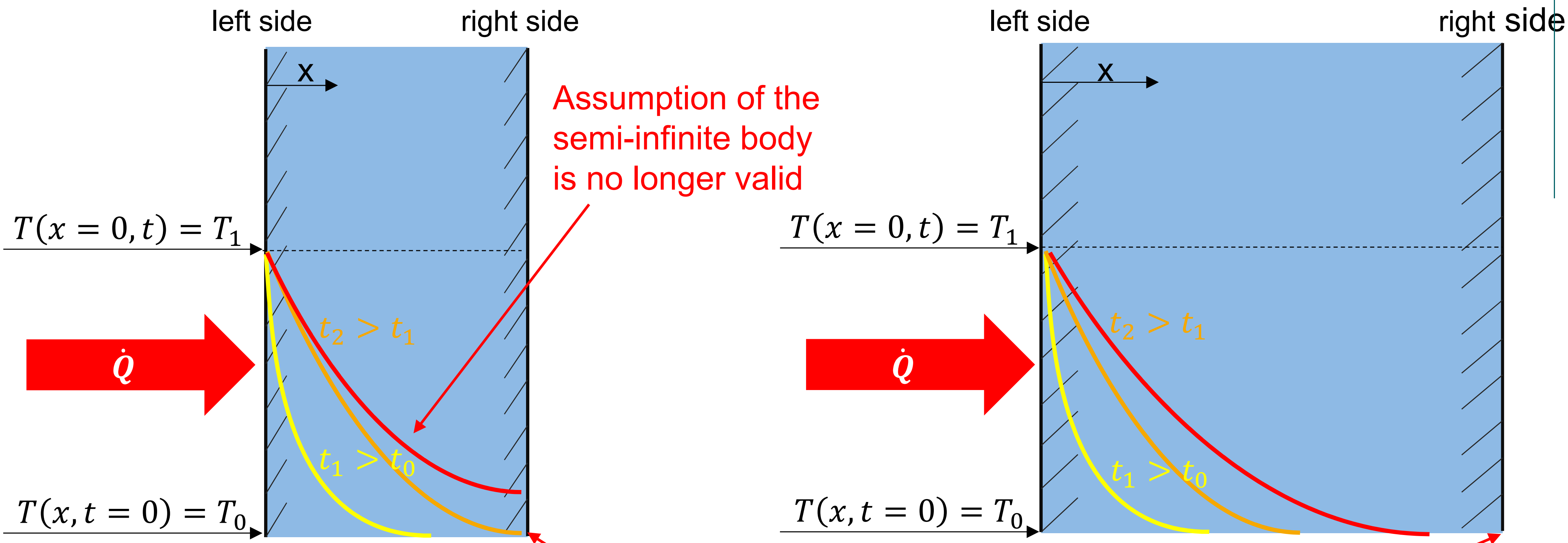
Definition of the semi-infinite body

Temperature variation in a body depends on location and time T = T(x,y,z,t). The temperature change on the right side is negligible, so that the plate thickness is not a parameter that influences the problem.

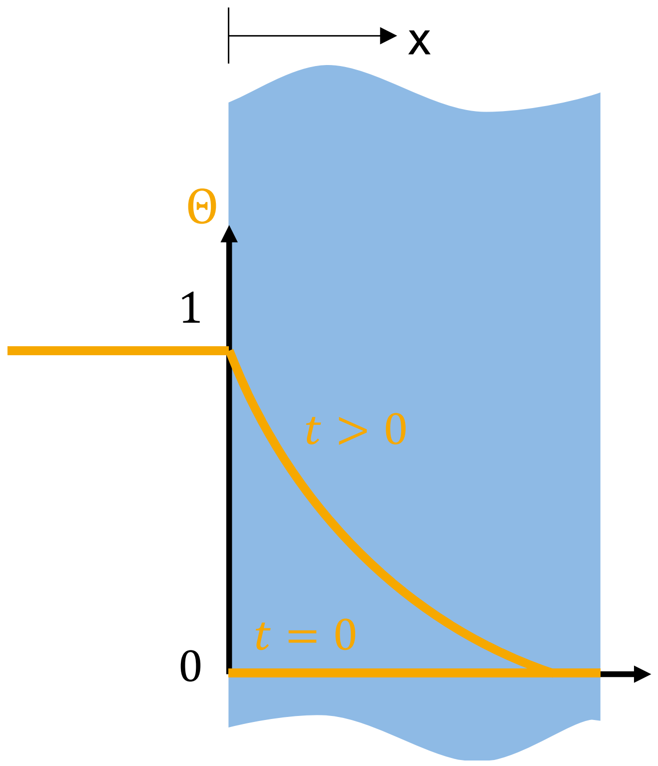


BC for right side: $\mathrm{T}(\mathrm{t} > 0, \mathrm{x} \approx \delta) = T_0$
Equivalent to: $\mathrm{T}(\mathrm{t} > 0, \mathrm{x} \rightarrow \infty) = T_0$

Differential equation and initial and boundary bonditions

Differential equation:

1st order in time → 1 IC
2nd order in space → 2 BC



Differential equation:

$$\frac{\partial \Theta^*}{\partial t} = a \frac{\partial^2 \Theta^*}{\partial x^2} \qquad \text{with} \qquad a = \frac{\lambda}{\rho c_p}$$

Dimensionless temperature difference:

$$\Theta^* = \frac{T - T_0}{T_A - T_0}$$

Initial and boundary conditions:

Initial condition:

$$\left.\begin{aligned} t &= 0 \\ 0 \le x &\le \infty \end{aligned}\right\} T = T_0 \quad |\Theta^* = 0$$

Boundary condition:

① $$\left.\begin{aligned} t &> 0 \\ x &= 0 \end{aligned}\right\} T = T_A \quad |\Theta^* = 1$$

② $$\left.\begin{aligned} t &> 0 \\ x &\to \infty \end{aligned}\right\} T = T_0 \quad |\Theta^* = 0$$

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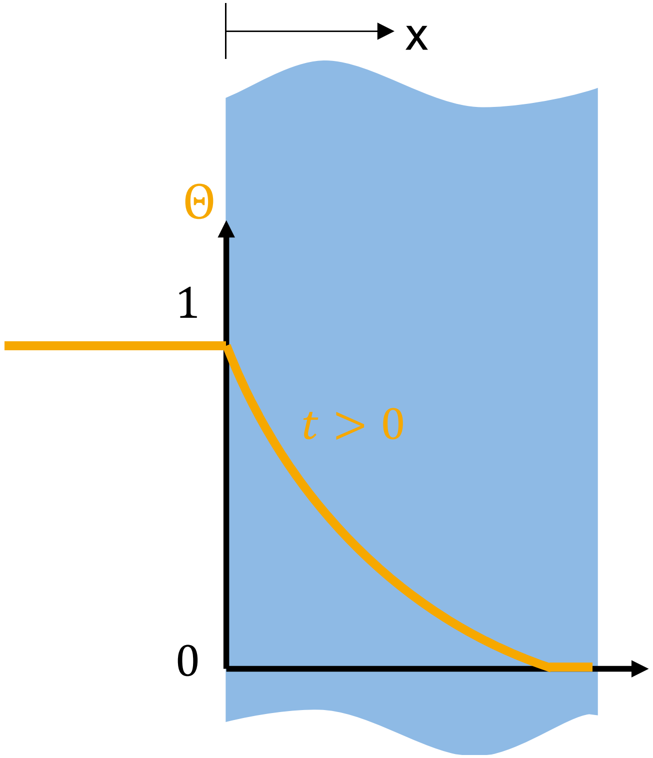
Connecting time and space coordinates by substitution



Plate

Differential equation:

$$\frac{\partial\Theta^*}{\partial t} = a\,\frac{\partial^2\Theta^*}{\partial x^2}$$ introduction of $\eta = \frac{x}{\sqrt{4at}}$

$$\frac{\partial\Theta^*}{\partial t} = \frac{\partial\Theta^*}{\partial\eta}\cdot\frac{\partial\eta}{\partial t} = -\frac{x}{\sqrt{4at^3}}\frac{\partial\Theta^*}{\partial\eta}$$

$$\frac{\partial^2\Theta^*}{\partial x^2} = \frac{\partial^2\Theta^*}{\partial\eta^2}\overset{\frac{1}{2at}}{\left(\cancel{\frac{\partial\eta}{\partial x}}\right)} + \frac{\partial\Theta^*}{\partial\eta}\cdot\cancel{\frac{\partial^2\eta}{\partial x^2}}$$

Derivation at the end of the video

Diff. equation as a function of $\eta(t,x)$:

$$-\frac{x}{\sqrt{4at^3}}\frac{\partial\Theta^*}{\partial\eta} = a\,\frac{1}{2at}\frac{\partial^2\Theta^*}{\partial\eta^2}$$

$$\frac{\partial^2\Theta^*}{\partial\eta^2} + 2\eta\,\frac{\partial\Theta^*}{\partial\eta} = 0$$

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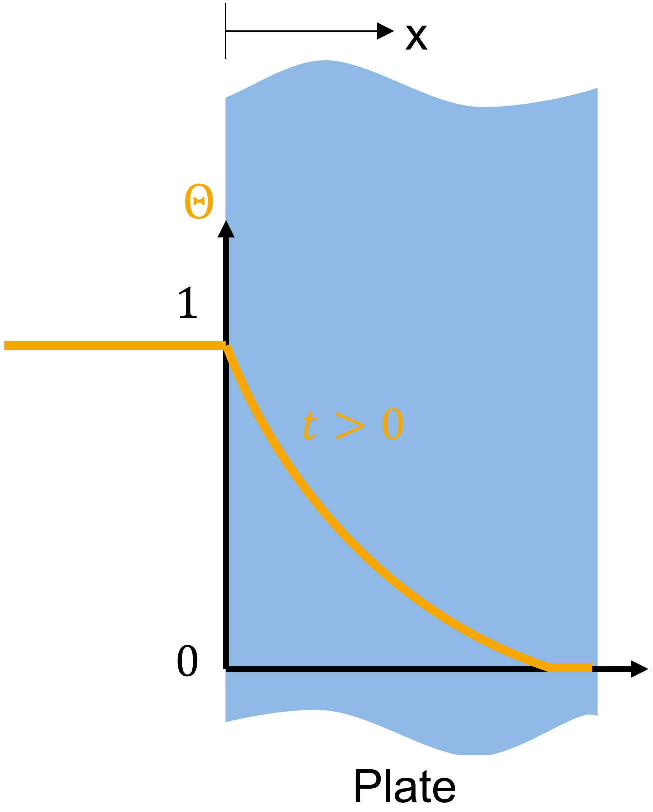
Solution of the substituted differential equation



2. substitution to solve the diff. eq.:

$$\frac{\partial \Theta^*}{\partial \eta} = Z \qquad \frac{\partial Z}{\partial \eta} = \frac{\partial^2 \Theta^*}{\partial \eta^2}$$

$$\frac{\partial Z}{\partial \eta} + 2\eta Z = 0$$

$$\frac{\partial Z}{Z} = -2\eta \partial \eta$$

Diff. equation as a function of $\eta(t, x)$:

$$\frac{\partial^2 \Theta^*}{\partial \eta^2} + 2\eta \frac{\partial \Theta^*}{\partial \eta} = 0$$

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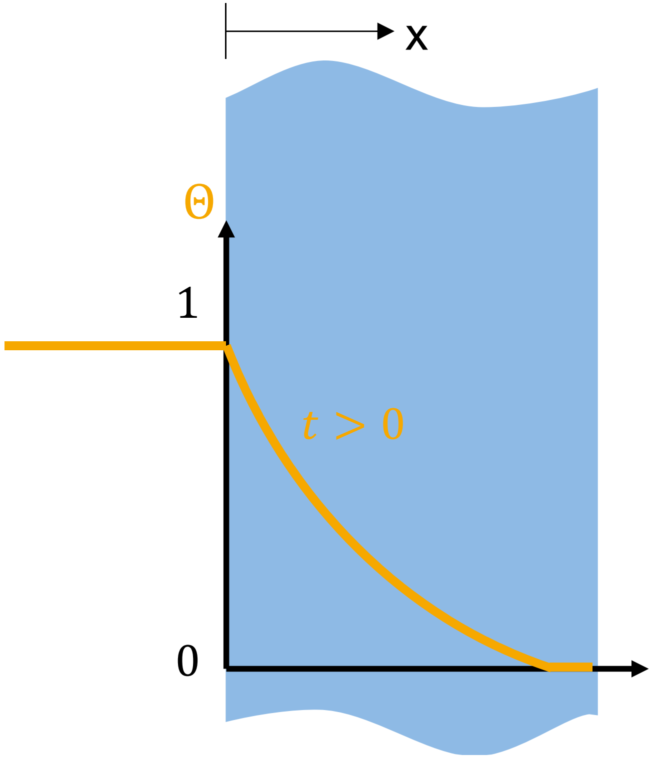
Solution of the substituted differential equation



Plate

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$$\frac{\partial \Theta^*}{\partial \eta} = Z \qquad \frac{\partial Z}{\partial \eta} = \frac{\partial^2 \Theta^*}{\partial \eta^2}$$

$$\frac{\partial Z}{\partial \eta} + 2\eta Z = 0$$

$$\frac{\partial Z}{Z} = -2\eta \partial \eta$$

Integration:

$$\ln Z = -\eta^2 + C_1$$

$$Z = \frac{\partial \Theta^*}{\partial \eta} = e^{-\eta^2 + C_1} = C_2 e^{-\eta^2}$$

$$\int_{\Theta_1^*}^{\Theta_2^*} d\Theta^* = C_2 \int_{\eta_1}^{\eta_2} e^{-\eta^2} \, d\eta$$

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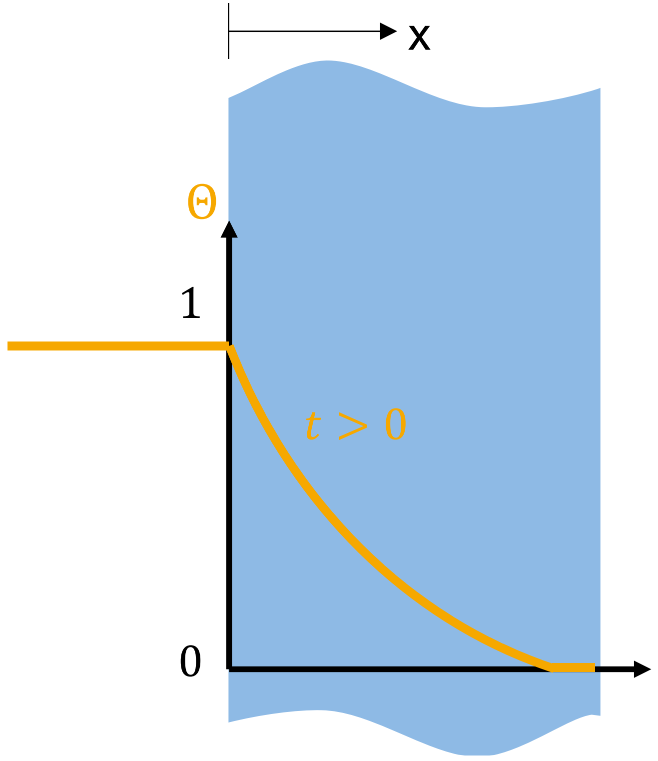
Solution of the substituted differential equation



Plate

Boundary conditions:

① $\left.\begin{matrix} t > 0 \\ x = 0 \end{matrix}\right\} \eta = 0 \;|\; \Theta^* = 1$

② $\left.\begin{matrix} t > 0 \\ x \to \infty \end{matrix}\right\} \eta = \infty \;|\; \Theta^* = 0$

$$\eta = \frac{x}{\sqrt{4at}}$$

$$\Theta^*_{\eta=\infty} - \Theta^*_{\eta=0} = -1 = C_2 \int_0^\infty \boxed{e^{-\eta^2}\, d\eta} = \frac{\sqrt{\pi}}{2}$$

WolframAlpha

Integration:

$$\ln Z = -\eta^2 + C_1$$

$$Z = \frac{\partial \Theta^*}{\partial \eta} = e^{-\eta^2 + C_1} = C_2 e^{-\eta^2}$$

$$\int_{\Theta^*_1}^{\Theta^*_2} d\Theta^* = C_2 \int_{\eta_1}^{\eta_2} e^{-\eta^2}\, d\eta$$

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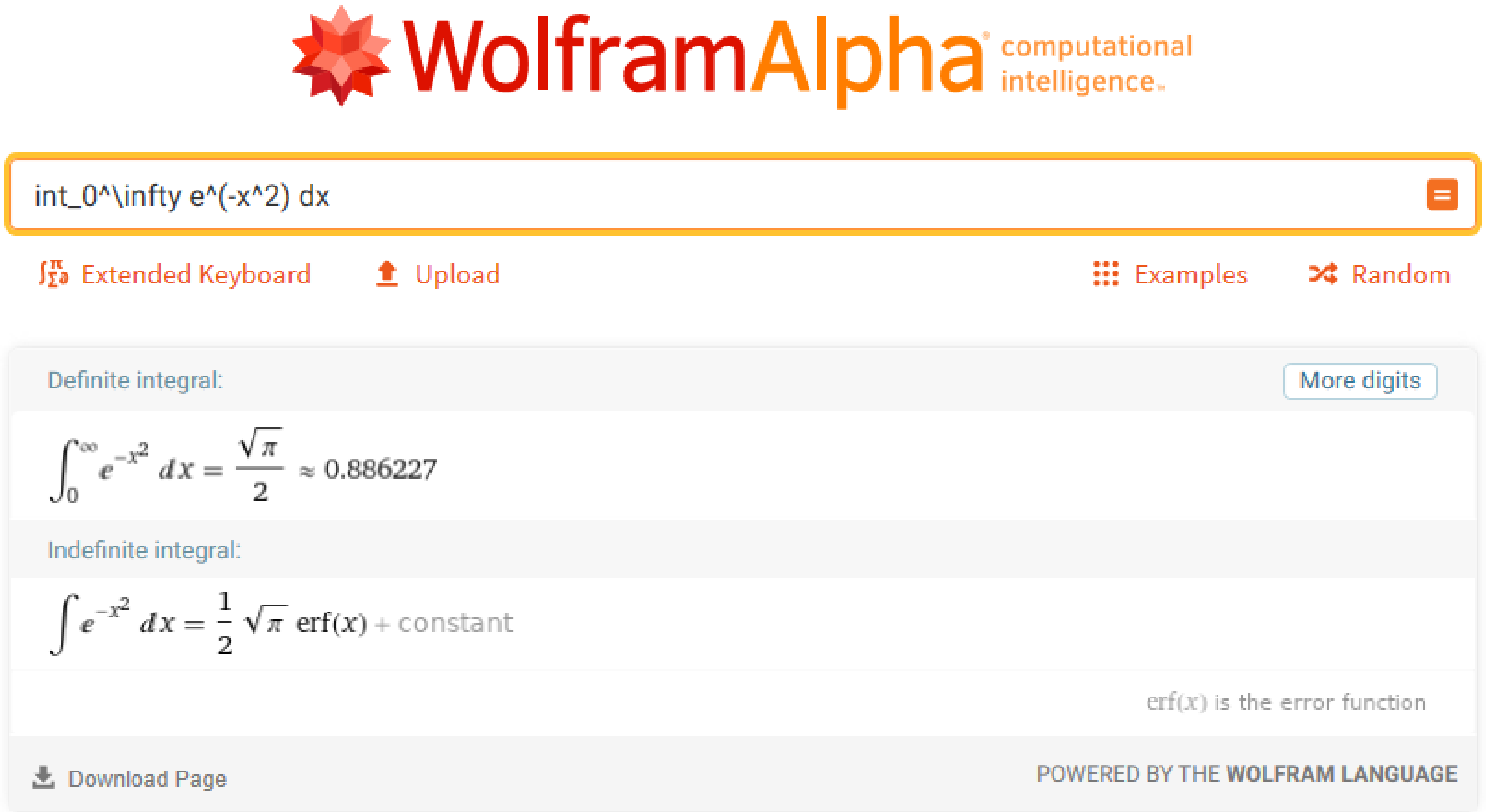
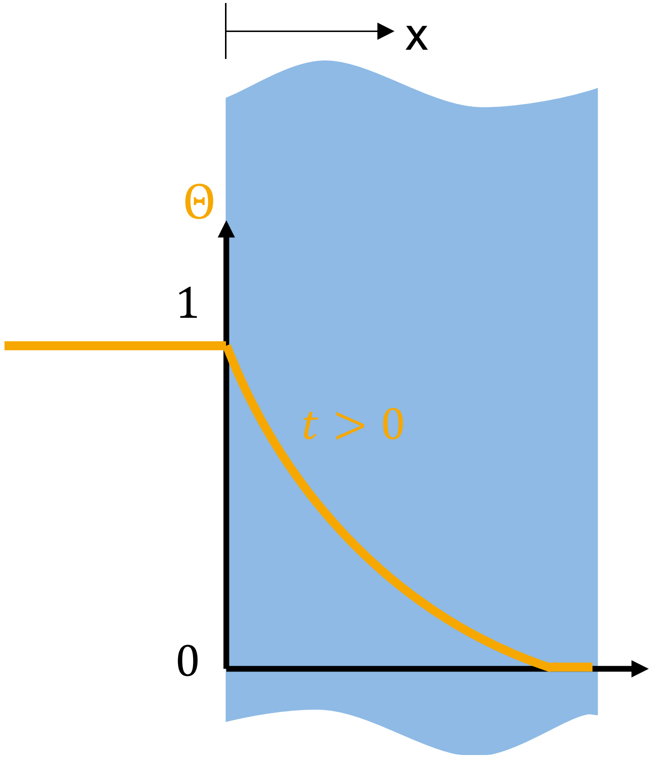
Wolfram Alpha



Solution of the substituted differential equation



Boundary conditions:

① $\eta = 0 \mid \Theta^* = 1$

② $\eta = \infty \mid \Theta^* = 0$

$$\eta = \frac{x}{\sqrt{4at}}$$

$$\Theta^*_{\eta=\infty} - \Theta^*_{\eta=0} = -1 = C_2 \boxed{\int_0^\infty e^{-\eta^2}\, d\eta} = \frac{\sqrt{\pi}}{2}$$

WolframAlpha

Solution:

$$C_2 = -\frac{2}{\sqrt{\pi}}$$

$$\Theta^*(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2}\, d\xi$$

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Description of the error function

Formulary Appendix B

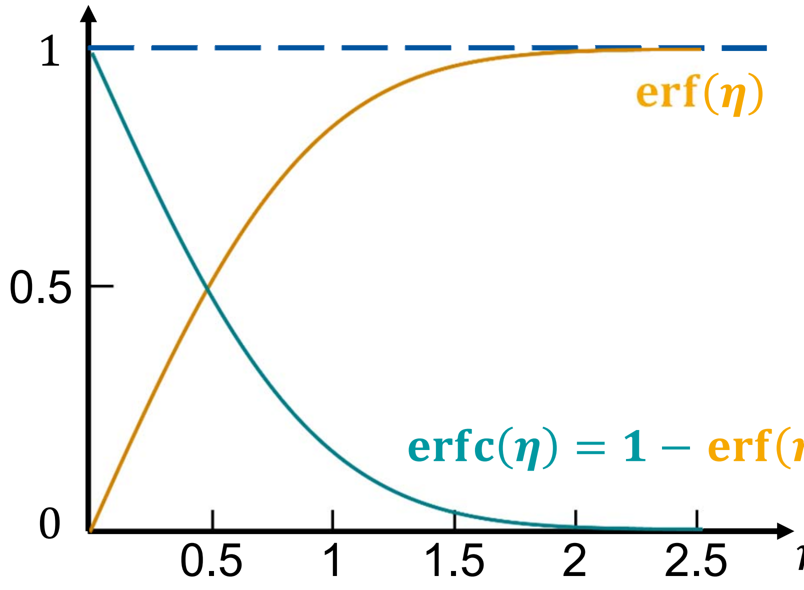
Tabelle 8: Auswertung der Fehlerfunktion

η	$\mathrm{erf}(\eta)$	$\mathrm{erfc}(\eta)$	$2/\sqrt{\pi}\,\exp(-\eta^2)$
0	0	1	1,128
0,05	0,056	0,944	1,126
0,1	0,112	0,888	1,117
0.15	0,168	0,832	1,103
0,2	0,223	0,777	1,084
0,25	0,276	0,724	1,060
0,3	0,329	0,671	1,031
0,35	0,379	0,621	0,998
0,4	0,428	0,572	0,962
0,45	0,475	0,525	0,922
0,5	0,520	0,480	0,879
0,55	0,563	0,437	0,834
0,6	0,604	0,396	0,787
0,65	0,642	0,378	0,740
0,7	0,678	0,322	0,691
0,75	0,711	0,289	0,643
0,8	0,742	0,258	0,595
0,85	0,771	0,229	0,548
0,9	0,797	0,203	0,502
0,95	0,821	0,179	0,458
1	0,843	0,157	0,415
1,1	0,880	0,120	0,337
1,2	0,910	0,090	0,267
1,3	0,934	0,066	0,208
1,4	0,952	0,048	0,159
1,5	0,966	0,034	0,119
1,6	0,976	0,024	0,087
1,7	0,984	0,016	0,063
1,8	0,989	0,011	0,044
1,9	0,993	0,007	0,030
2	0,995	0,005	0,021

Error function $\mathrm{erf}(\eta)$:

$$\Theta^*(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2}\, d\xi$$

or: $$\Theta^*(\eta) = 1 - \mathrm{erf}(\eta)$$



$$\eta = \frac{x}{\sqrt{4at}}$$

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Example: Comparison of the thermal penetration depth of different materials

At which position x is $\Theta^*(\eta) = 0.01$ reached after $t = 10s$?

$$\Theta^*(\eta) = 0.01 = 1 - \text{erf}(\eta) \quad \rightarrow \text{erf}(\eta) = 1 - 0.01 = 0.99 \quad \boxed{\rightarrow \eta = 1.8}$$

$$\eta = \frac{x}{\sqrt{4at}} \qquad \boxed{\rightarrow \; x = 2 \cdot \eta \cdot \sqrt{at}}$$

a) Copper with $a = 117 \cdot 10^{-6} \frac{m^2}{s}$:

$$x = 2 \cdot 1.8 \cdot \sqrt{117 \cdot 10^{-6} \frac{m^2}{s} \cdot 10s}$$

$$x_{Cu} = 0.123\, m$$

b) Paper with $a = 0.14 \cdot 10^{-6} \frac{m^2}{s}$:

$$x = 2 \cdot 1.8 \cdot \sqrt{0.14 \cdot 10^{-6} \frac{m^2}{s} \cdot 10s}$$

$$x_P = 0.0043\, m$$

Comparison of both penetration depths:

$$28 \cdot x_{Cu} \approx x_P$$

The thermal diffusivity a determines the speed with which a temperature information propagates in a body!

Tabelle 8: Auswertung der Fehlerfunktion

η	erf(η)	erfc(η)	$2/\sqrt{\pi}\exp(-\eta^2)$
0	0	1	1,128
0,05	0,056	0,944	1,126
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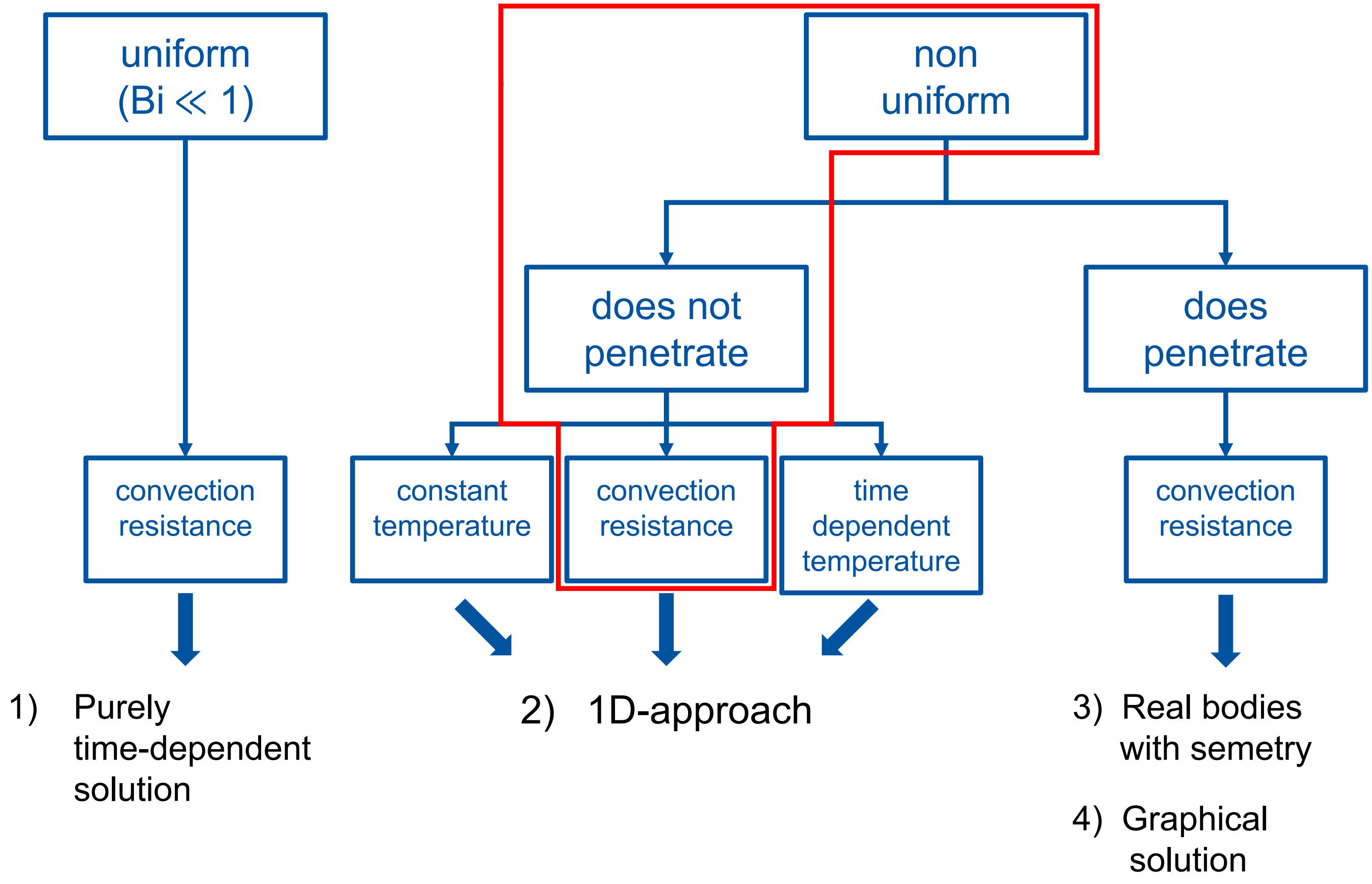
How can the problem be simplified?

Temperature of the body:

Temperature inside the body:

Boundary condition on the surface:



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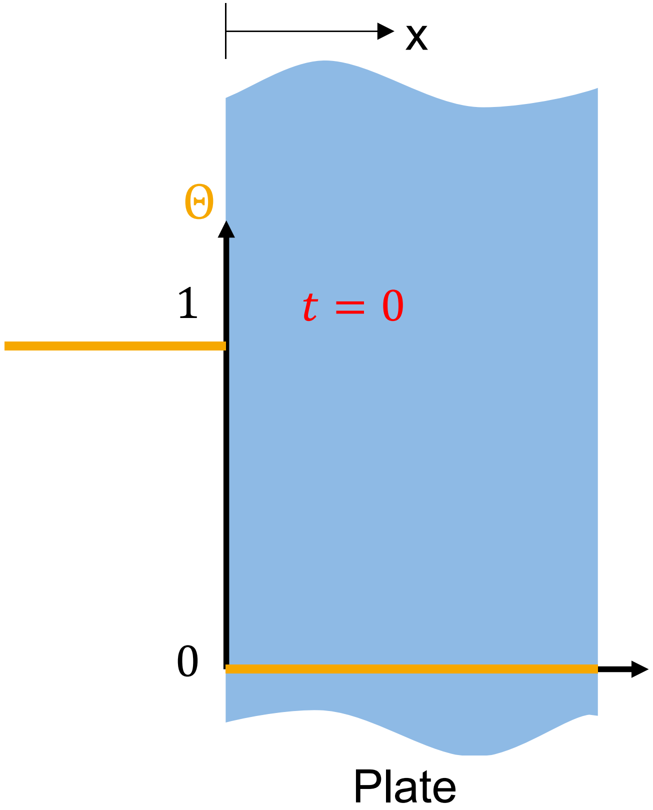
With non-negligible heat transfer resistance

External heat transfer resistance is **not** negligible.



Differential equation:

$$\frac{\partial \Theta^*}{\partial t} = a \frac{\partial^2 \Theta^*}{\partial x^2} \qquad \text{with} \qquad a = \frac{\lambda}{\rho c_p}$$

Dimensionless temperature difference:

$$\Theta^* = \frac{T - T_0}{T_A - T_0}$$

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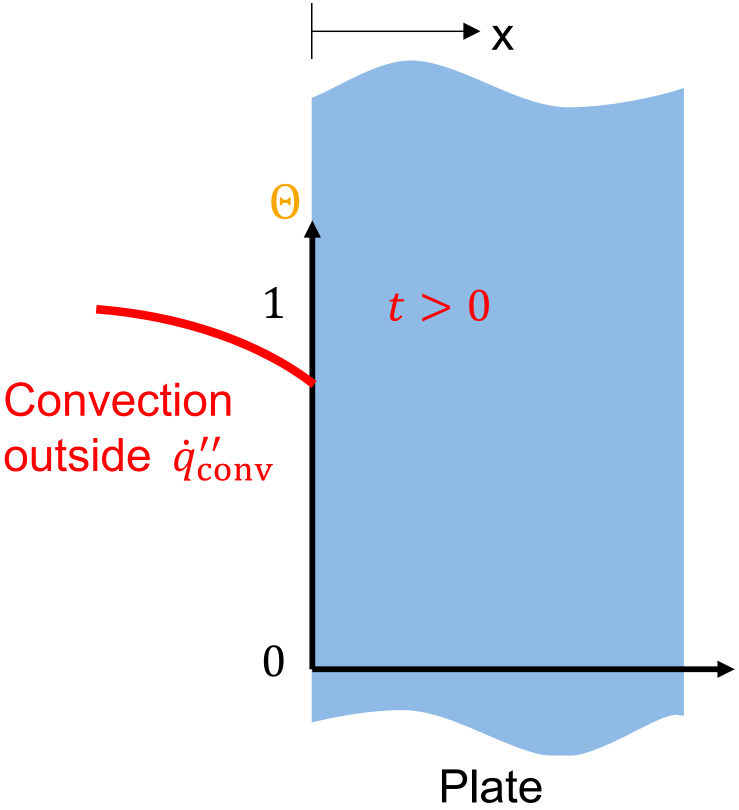

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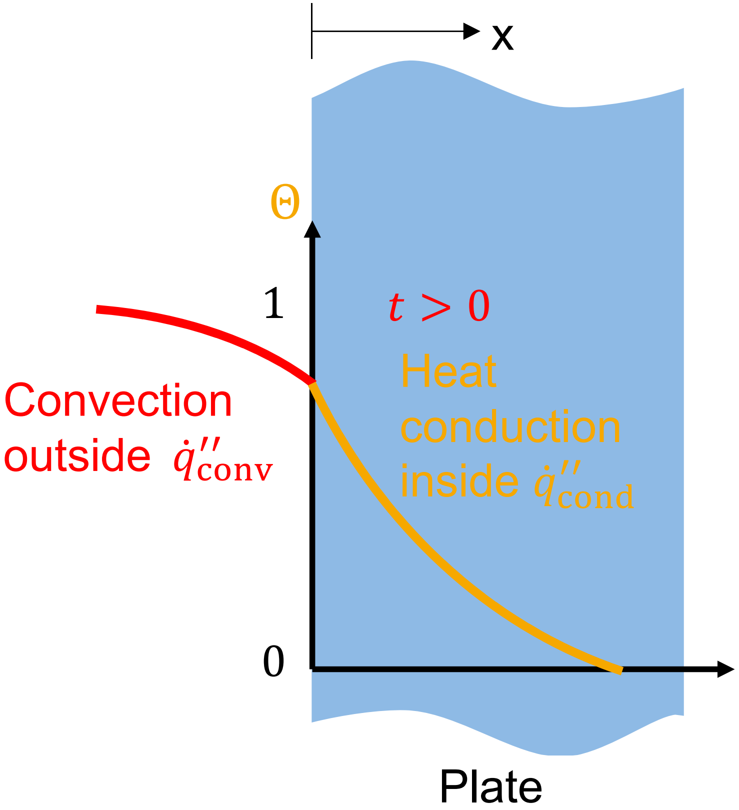

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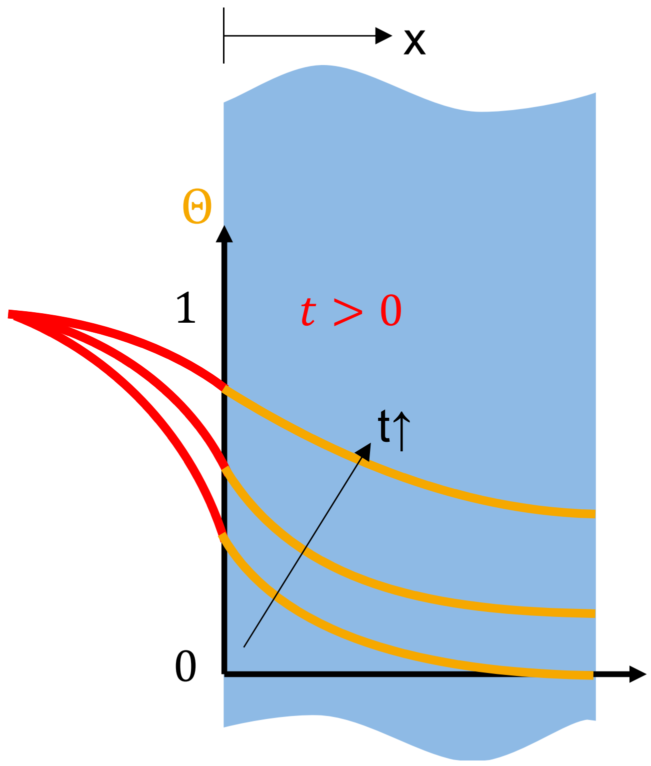

Boundary condition:

Boundary condition:

1 $\left.\begin{array}{l} t > 0 \\ x = 0 \end{array}\right\}$

$$\underbrace{\alpha(T_A - T_{x=0})}_{\dot{q}''_{\text{conv}}} = \underbrace{-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0}}_{\dot{q}''_{\text{cond}}} \quad \rightarrow \quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{\alpha}{\lambda}(T_{x=0} - T_A)$$

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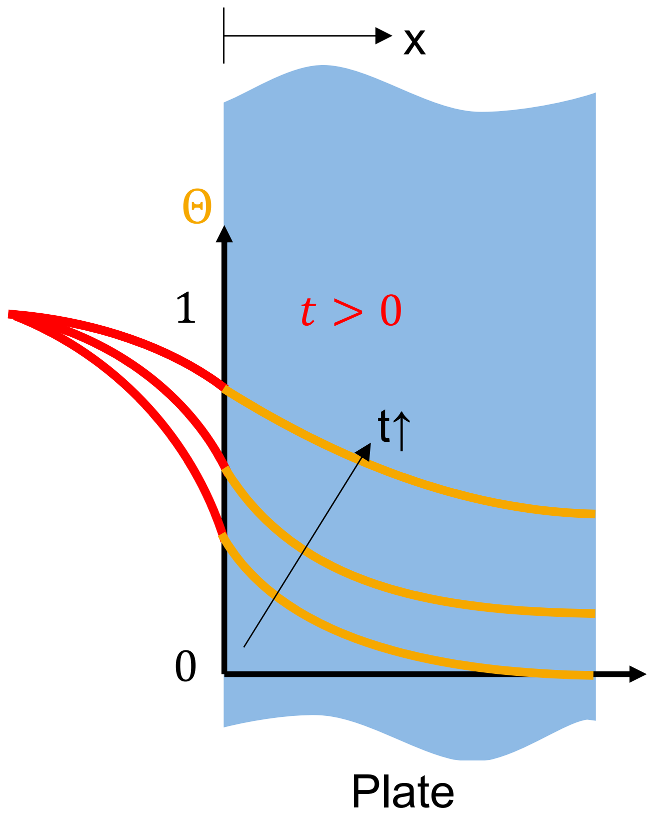


Boundary condition:

Boundary condition:

1 $$\left.\begin{matrix} t > 0 \\ x = 0 \end{matrix}\right\} \left.\frac{\partial \Theta^*}{\partial x}\right|_{x=0} = -\frac{\alpha}{\lambda}$$

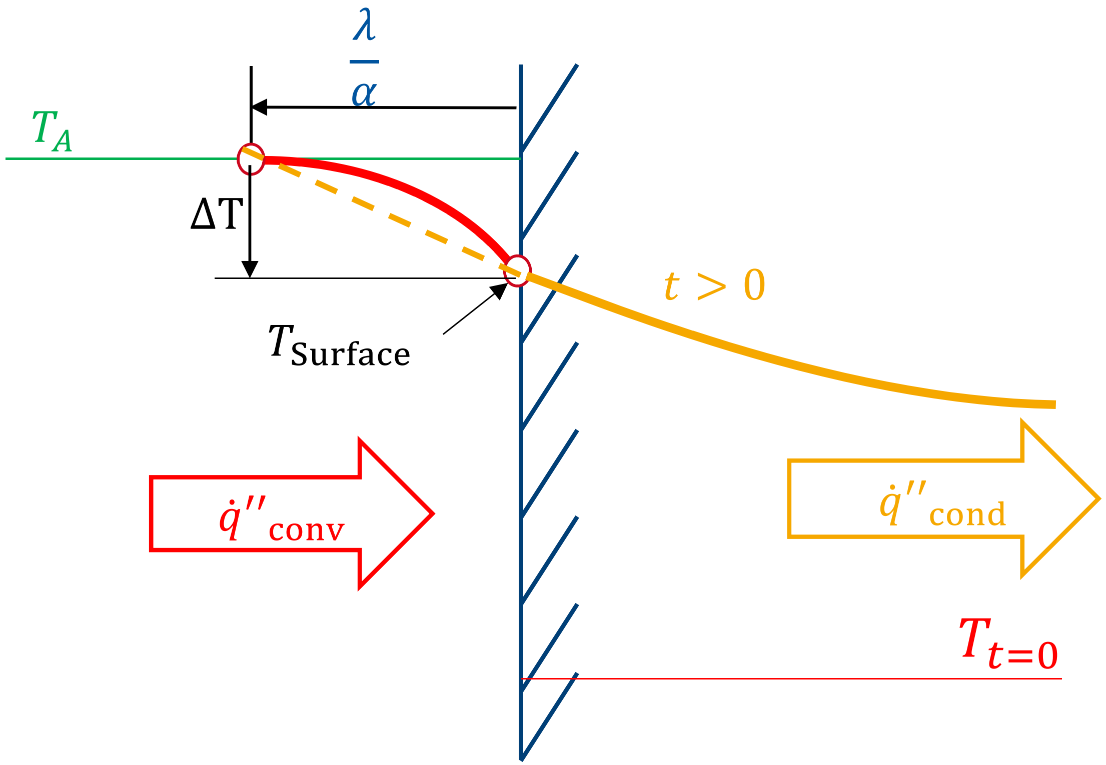
$$\underbrace{\alpha(T_A - T_{x=0})}_{\dot{q}''_{\mathrm{conv}}} = \underbrace{-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0}}_{\dot{q}''_{\mathrm{cond}}} \rightarrow \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{\alpha}{\lambda}(T_{x=0} - T_A)$$

2 $$\left.\begin{matrix} t > 0 \\ x \to \infty \end{matrix}\right\} T = T_0 \;|\Theta^* = 0$$

Initial condition:

$$\left.\begin{matrix} t > 0 \\ 0 \leq x \leq \infty \end{matrix}\right\} T = T_0 \;|\Theta^* = 0$$

Determination of the temperature gradient on the wall for BC $\dot{q}''_{\mathrm{conv}}$



Solution:

$$\underbrace{\alpha(T_A - T_{x=0})}_{\dot{q}''_{\mathrm{conv}}} = \underbrace{-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0}}_{\dot{q}''_{\mathrm{cond}}}$$

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = \underbrace{\boxed{\frac{\alpha}{\lambda}}}_{[1/\mathrm{m}]}(T_{x=0} - T_A)$$

$\frac{\lambda}{\alpha}$ is the spatial distance between T_A and T_{Surface}

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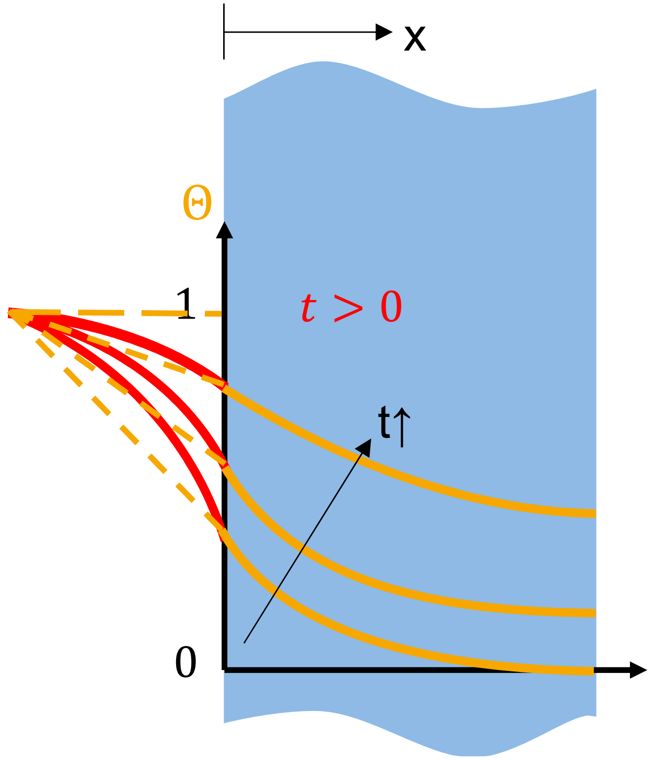

Boundary condition:

Boundary condition:

① $$\left.\begin{matrix} t > 0 \\ x = 0 \end{matrix}\right\} \left.\frac{\partial \Theta^*}{\partial x}\right|_{x=0} = -\frac{\alpha}{\lambda}$$

$$\underbrace{\alpha(T_A - T_{x=0})}_{\dot{q}''_{\mathrm{conv}}} = \underbrace{-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0}}_{\dot{q}''_{\mathrm{cond}}} \rightarrow \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{\alpha}{\lambda}(T_{x=0} - T_A)$$

② $$\left.\begin{matrix} t > 0 \\ x \rightarrow \infty \end{matrix}\right\} T = T_0 \; |\Theta^* = 0$$

Initial condition:

$$\left.\begin{matrix} t > 0 \\ 0 \leq x \leq \infty \end{matrix}\right\} T = T_0 \; |\Theta^* = 0$$

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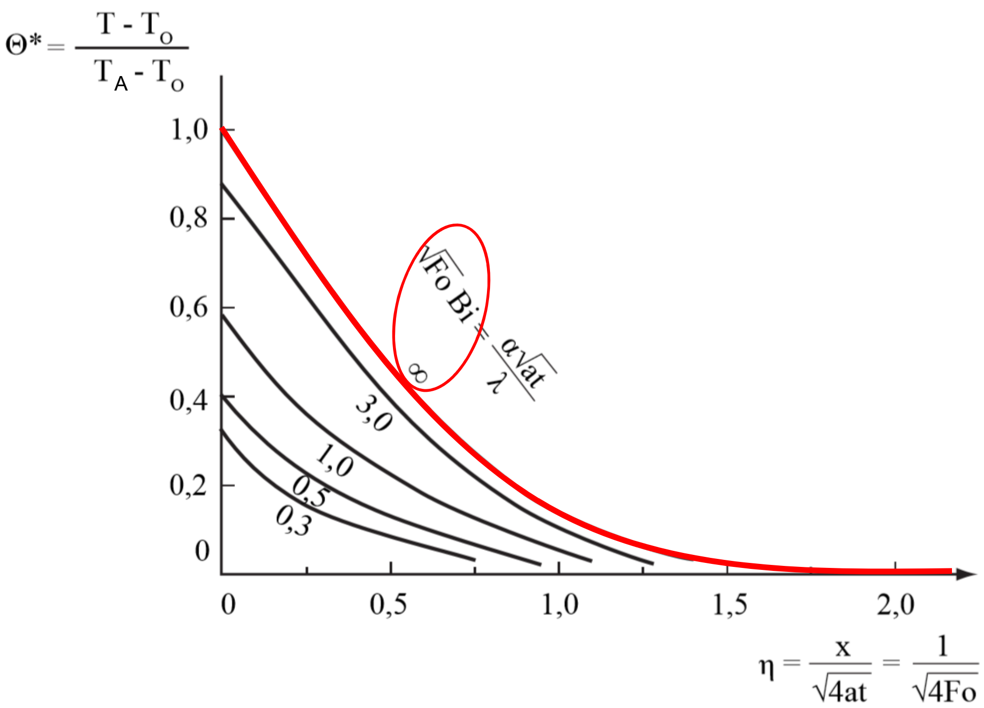

Review $\alpha = \infty$:

$$\Theta^*(\eta) = 1 - \mathrm{erf}(\eta) = 1 - \mathrm{erf}\left(\frac{x}{\sqrt{4at}}\right)$$

$$\Theta^*(\eta) = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4Fo}}\right)$$

Review $\alpha \neq \infty$:

$$\Theta^*(\eta) = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4Fo}}\right)$$

$$-e^{Bi+FoBi^2}\left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{4Fo}} + \sqrt{Fo}\cdot Bi\right)\right]$$

with:

$$Fo = \frac{at}{L^2} \qquad Bi = \frac{\alpha x}{\lambda}$$

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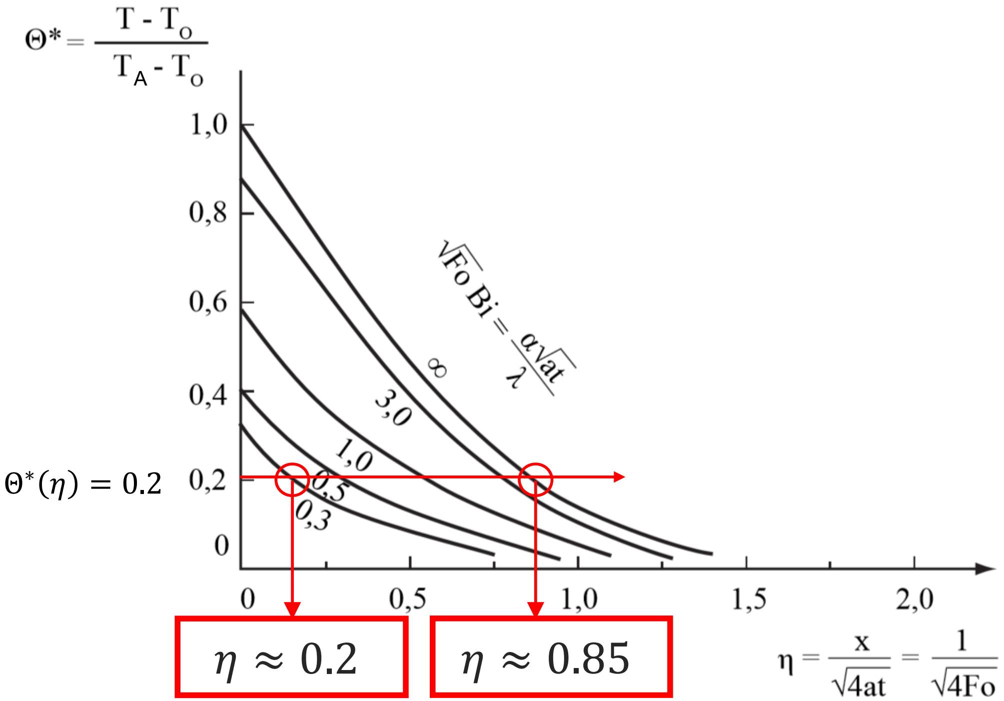
Example: Thermal penetration with convective resistance

At which position x is $\Theta^*(\eta) = 0.2$ reached after $t = 10\,s$? (for copper)

$a_K = 117 \cdot 10^{-6} \frac{m^2}{s}$; $\lambda_K = 401 \frac{W}{m \cdot K}$:

$$\sqrt{Fo} \cdot Bi = \frac{\alpha \sqrt{a \cdot t}}{\lambda} = \frac{\alpha \sqrt{117 \cdot 10^{-6} \frac{m^2}{s} \cdot 10s}}{401 \frac{W}{m \cdot K}} = \alpha \cdot 8.53 \cdot 10^{-5} \frac{m^2 \cdot K}{W}$$



Considering different cases:

A) $\alpha \to \infty$ - imposed wall temperature:

$$\sqrt{Fo} \cdot Bi = \frac{\alpha \sqrt{a \cdot t}}{\lambda} = \infty \to \eta \approx 0.85$$

$$x = \eta \sqrt{4at} = 0.85 \sqrt{4 \cdot 117 \cdot 10^{-6} \frac{m^2}{s} \cdot 10s} = \mathbf{0.058m}$$

B) Thermal resistance ($\sqrt{Fo} \cdot Bi = 0.3$)

$$\sqrt{Fo} \cdot Bi = \frac{\alpha \sqrt{a \cdot t}}{\lambda} = 0.3 \to \alpha \approx 3517 \frac{W}{m^2 K} \quad \to \eta \approx 0.2$$

$$x = \eta \sqrt{4at} = 0.2 \sqrt{4 \cdot 117 \cdot 10^{-6} \frac{m^2}{s} \cdot 10s} = \mathbf{0.0136m}$$

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Comparison – solution with error function and with diagram

At which position x is $\Theta^*(\eta) = 0.2$ reached after $t = 10s$? (for copper):

$$\Theta^*(\eta) = 0.2 = 1 - \mathrm{erf}(\eta) \quad \rightarrow \mathrm{erf}(\eta) = 1 - 0.2 = 0.8 \quad \boxed{\rightarrow \eta = 0.9}$$

$$\eta = \frac{x}{\sqrt{4at}} \quad \boxed{\rightarrow \; x = 2 \cdot \eta \cdot \sqrt{at}}$$

With table of error function:

$$x = 2 \cdot 0.9 \cdot \sqrt{117 \cdot 10^{-6} \frac{m^2}{s} \cdot 10s}$$

$$x_{\mathrm{Cu,e}} = 0.0615\, m$$

From graph:

$$x = 0.85 \sqrt{4 \cdot 117 \cdot 10^{-6} \frac{m^2}{s} \cdot 10s}$$

$$x_{\mathrm{Cu,d}} = 0.058\, m$$

Difference is due to reading inaccuracy,
since the equations for negligible thermal resistance are identical.

Tabelle 8: Auswertung der Fehlerfunktion

η	$\mathrm{erf}(\eta)$	$\mathrm{erfc}(\eta)$	$2/\sqrt{\pi}\exp(-\eta^2)$
0	0	1	1,128
0,05	0,056	0,944	1,126
0,1	0,112	0,888	1,117
0.15	0,168	0,832	1,103
0,2	0,223	0,777	1,084
0,25	0,276	0,724	1,060
0,3	0,329	0,671	1,031
0,35	0,379	0,621	0,998
0,4	0,428	0,572	0,962
0,45	0,475	0,525	0,922
0,5	0,520	0,480	0,879
0,55	0,563	0,437	0,834
0,6	0,604	0,396	0,787
0,65	0,642	0,378	0,740
0,7	0,678	0,322	0,691
0,75	0,711	0,289	0,643
0,8	0,742	0,258	0,595
0,85	0,771	0,229	0,548
0,9	0,797	0,203	0,502
0,95	0,821	0,179	0,458
1	0,843	0,157	0,415
1,1	0,880	0,120	0,337
1,2	0,910	0,090	0,267
1,3	0,934	0,066	0,208
1,4	0,952	0,048	0,159
1,5	0,966	0,034	0,119
1,6	0,976	0,024	0,087
1,7	0,984	0,016	0,063
1,8	0,989	0,011	0,044
1,9	0,993	0,007	0,030
2	0,995	0,005	0,021

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Comprehension questions

What is meant by a semi-infinite body and how is it defined?

Which two dimensionless numbers describe the transient temperature profile within a (semi-infinite) body with relevant convective resistance?

What is meant by the thermal penetration depth?

Mathematical explanation

Derivation:

$$\frac{\partial^2\Theta^*}{\partial x^2} = \frac{\partial^2\Theta^*}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial\Theta^*}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial x^2}$$

$$\frac{\partial^2\Theta^*}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\Theta^*}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\underset{u}{\frac{\partial\Theta^*}{\partial\eta}}\cdot\underset{v}{\frac{\partial\eta}{\partial x}}\right) \xrightarrow{\text{chain rule}} u'v + v'u$$

$$= \underset{v}{\frac{\partial\eta}{\partial x}}\cdot\underset{u'}{\left[\frac{\partial}{\partial\eta}\left(\frac{\partial\Theta^*}{\partial\eta}\right)\cdot\frac{\partial\eta}{\partial x}\right]} + \underset{u}{\frac{\partial\Theta^*}{\partial\eta}}\cdot\underset{v'}{\frac{\partial^2\eta}{\partial x^2}}$$

$$= \frac{\partial\Theta^*}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial\Theta^*}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial x^2}$$

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