Heat Transfer: Conduction

Heat conduction in a cylindrical coordinate system

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Video overview

Steady state 1-D Heat conduction in pipe wall:

- Schematic curves for temperature, cross section and heat flow
- Derivation of the differential equation via energy balances
- Mathematical solution of the differential equation

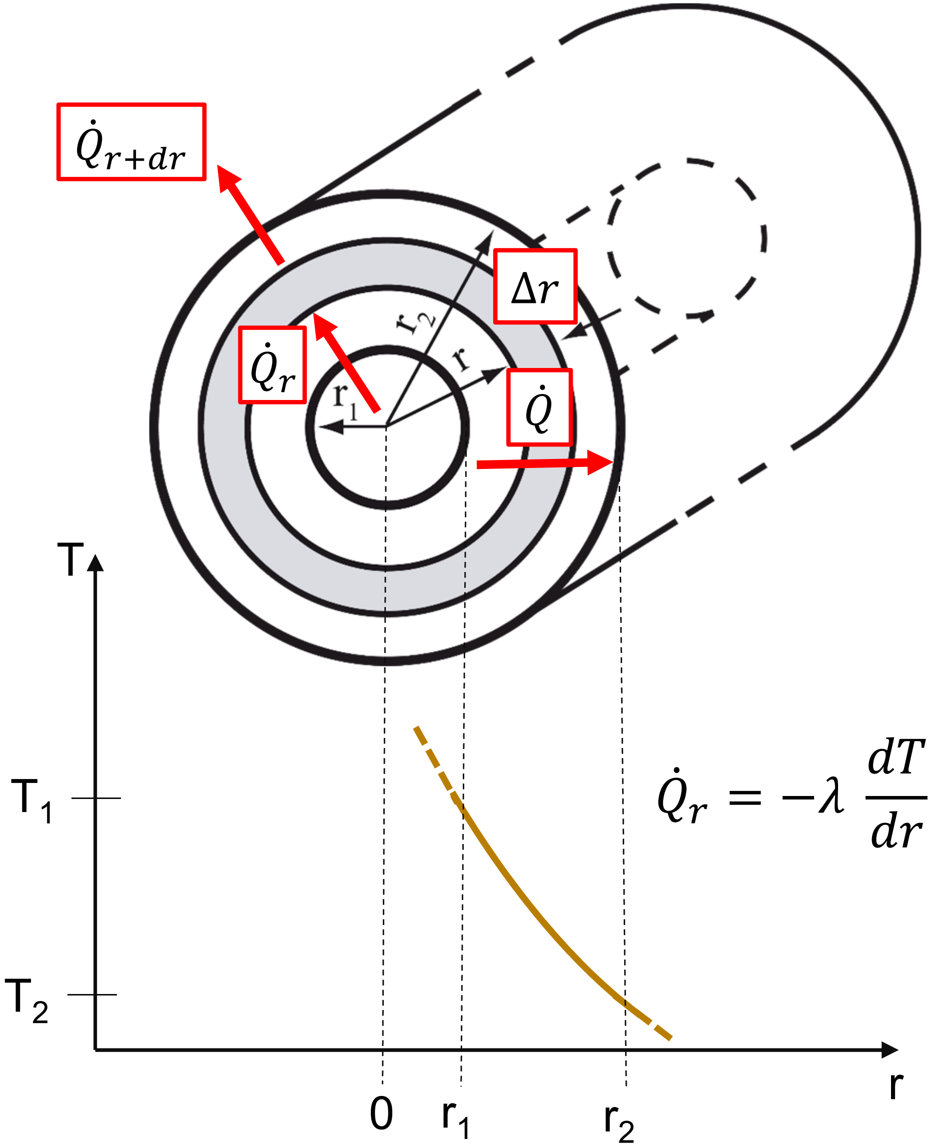
Steady state 1-D Heat conduction for multilayer pipe walls:

- Expand the equation to several resistors

Special case of steady state 1-D heat conduction for very thin pipe walls $\delta \ll r_1$:

- Simplification of the problem (engineering approach)

Differential equation derivation: Steady state 1-D heat conduction in a cylindrical body without sources



$$\dot{Q}_r = -\lambda \frac{dT}{dr} A(r)$$

Fourier's law for a cylindrical body:

$$\dot{Q}_{\mathrm{r}} = -\lambda\, A \frac{dT}{dr}$$

Change of area:

$$\mathrm{A(r)} \uparrow \quad \text{with} \quad \mathrm{r} \uparrow$$

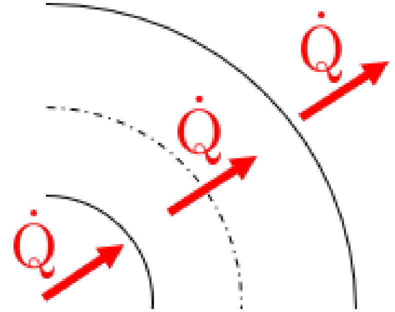
Resulting temperature gradient:

$$\dot{Q}_{\mathrm{r}} = \text{const.} \quad \Rightarrow \frac{dT}{dr} \downarrow$$

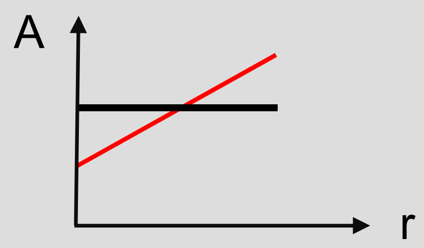
Energy balance around ring element:

$$0 = \dot{Q}_{\mathrm{r}} - \dot{Q}_{\mathrm{r+dr}}$$

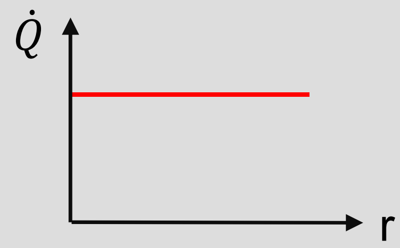
Differential equation derivation:
Steady state 1-D heat conduction in a cylindrical body without sources



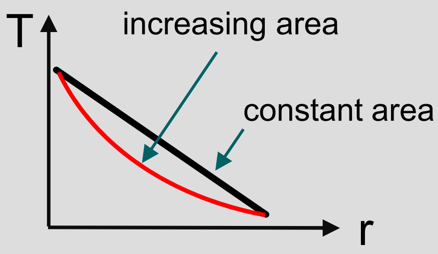
Area A :



Heat flow $\dot{Q}$:



Temperature T :



Fourier's law for a cylindrical body:

$$\dot{Q}_\mathrm{r} = -\lambda \, \frac{dT}{dr} \, A(r)$$

Local temperature gradient for a cyl. body:

$$\frac{dT}{dr} = -\frac{\dot{Q}_\mathrm{r}}{\lambda \, A(r)}$$

Separation of the temperature gradient in constant and variable terms:

$$dT = -\frac{\dot{Q}_\mathrm{r}}{\lambda \, 2 \, \pi \, L} \cdot \frac{dr}{r}$$

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Differential equation derivation:
Steady state 1-D heat conduction in a cylindrical body without sources

Integration of Fourier's law (this approach is less common):

$$\int_{\mathrm{T_0}}^{\mathrm{T_r}} dT = -\mathrm{const.} \int_{\mathrm{r_0}}^{\mathrm{r}} \frac{dr}{r} \qquad \Longrightarrow T_0 - T_\mathrm{r} = \cdots$$

Differential equation derivation:
Steady state 1-D heat conduction in a cylindrical body without sources

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Proven Method: Temperature profile via an energy balance around an infinitesimal element:

$$\dot{Q}_\mathrm{in} = -\lambda\, A(r) \frac{dT}{dr} = -\lambda\, 2\,\pi\, r\, L \frac{dT}{dr}$$

$$\dot{Q}_\mathrm{out} = \dot{Q}_\mathrm{in} + \frac{d\dot{Q}_\mathrm{in}}{dr} \Delta r$$

$$0 = -\frac{d\dot{Q}_\mathrm{in}}{dr} =$$

Differential equation derivation: Steady state 1-D heat conduction in a cylindrical body without sources

Integrand:

$$\frac{d\left(r\,\frac{dT}{dr}\right)}{dr}$$

1st integration yields:

$$r\,\frac{dT}{dr} = const. = C_1$$

2nd integration yields:

$$T(r) = C_1 \ln(r) + C_2$$

Boundary conditions:

$$r = r_1 : T = T_1$$

$$r = r_2 : T = T_2$$

Results of 2nd integration:

$$T_1 = C_1 \ln(r_1) + C_2$$

$$T_2 = C_1 \ln(r_2) + C_2$$

Subtraction of temperature eq.:

$$T_1 - T_2 = C_1\left(\ln(r_1) - \ln(r_2)\right)$$

1st integration constant:

$$C_1 = \frac{T_1 - T_2}{\ln\left(\frac{r_1}{r_2}\right)} = \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}$$

Insert in temperature equation:

$$C_2 = T_1 - \frac{T_1 - T_2}{\ln\left(\frac{r_1}{r_2}\right)} \ln(r_1)$$

Differential equation derivation: Steady state 1-D heat conduction in a cylindrical body without sources

1st integration constant:

$$C_1 = \frac{T_2 - T_1}{\ln \frac{r_2}{r_1}}$$

2nd integration constant:

$$C_2 = T_1 - \frac{T_2 - T_1}{\ln \frac{r_2}{r_1}} \cdot \ln r_1$$

Final temperature profile:

$$T(r) = C_1 \cdot \ln \mathrm{r} + \mathrm{C}_2$$

$$T(r) = \frac{T_2 - T_1}{ln \frac{r_2}{r_1}} \cdot \ln r + T_1 - \frac{T_2 - T_1}{ln \frac{r_2}{r_1}} \cdot \ln r_1$$

$$\boxed{T(r) = T_1 + \frac{T_2 - T_1}{\ln \frac{r_2}{r_1}} \cdot \ln \frac{r}{r_1}}$$

Differential equation derivation:
Steady state 1-D heat conduction in a cylindrical body without sources

Temperature profile:

$$T(r) = T_1 + \frac{T_2 - T_1}{\ln \frac{r_2}{r_1}} \cdot \ln \frac{r}{r_1}$$

Heat flow:

$$\dot{Q} = -\lambda \cdot 2\pi \cdot r \cdot L \cdot \frac{dT(r)}{dr}$$

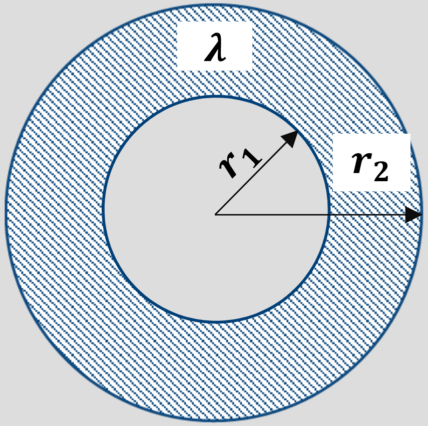
$$\Rightarrow \boxed{\frac{dT(r)}{dr} = \frac{T_2 - T_1}{\ln \frac{r_2}{r_1}} \cdot \frac{1}{r}}$$

Thermal resistances of a multilayer pipe wall
Steady state 1-D heat conduction multilayer cylindrical wall without sources

Thermal resistance R_c for single layer pipe wall:

$$\dot{Q} = \frac{T_1 - T_2}{\underbrace{\frac{\ln\frac{r_2}{r_1}}{\lambda \cdot 2 \cdot \pi \cdot L}}_{R_c}} = \frac{T_1 - T_2}{R_c}$$

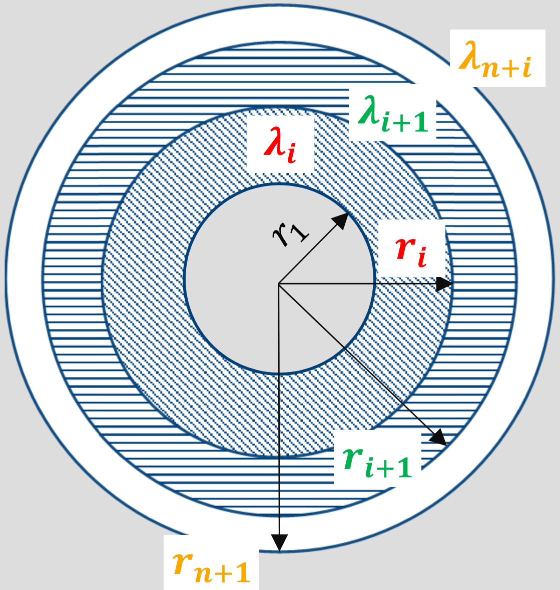
$$\boxed{R_c = \frac{\ln\frac{r_2}{r_1}}{\lambda \cdot 2 \cdot \pi \cdot L}}$$



Multilayer pipe wall and heat flow:

$$\boxed{\sum_{i=1}^{i=n} R_{c,i} = \frac{1}{2\,\pi\,L}\sum_{i=1}^{n}\frac{1}{\lambda_i}\cdot\ln\frac{r_{i+1}}{r_i}}$$

$$\boxed{\dot{Q}_r = \frac{T_1 - T_{n+1}}{R_{c,i}} = \frac{T_1 - T_{n+1}}{\frac{1}{2\,\pi\,L}\cdot\sum_{i=1}^{n}\frac{1}{\lambda_i}\cdot\ln\frac{r_{i+1}}{r_i}}}$$



Special case of a thin pipe wall ($\delta \ll r$)

Introduction of the pipe thickness δ for the case: $\delta \ll r$

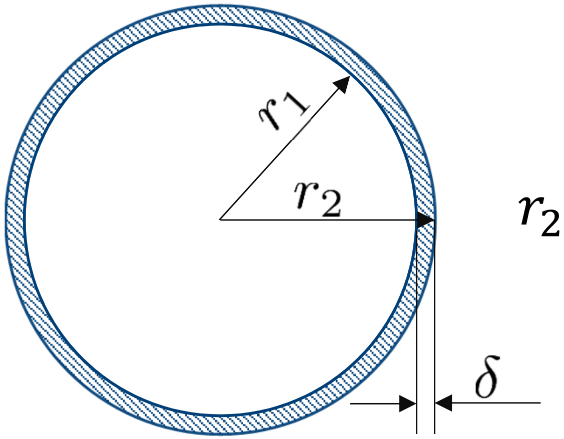
$$\ln \frac{r_2}{r_1} = \ln\left(\frac{r_1+\delta}{r_1}\right) \implies \boxed{\ln\left(1+\frac{\delta}{r_1}\right)}$$

Mathematical reformulation:

$\ln\left(1+\frac{\delta}{r_1}\right)$ is equivalent to $\ln(1+x)$

For $\delta \ll r$ the ln term simplifies to:

$\ln(1+x) \approx x$ for small x



$$\delta \ll r_1$$
$$r_2 = r_1 + \delta$$

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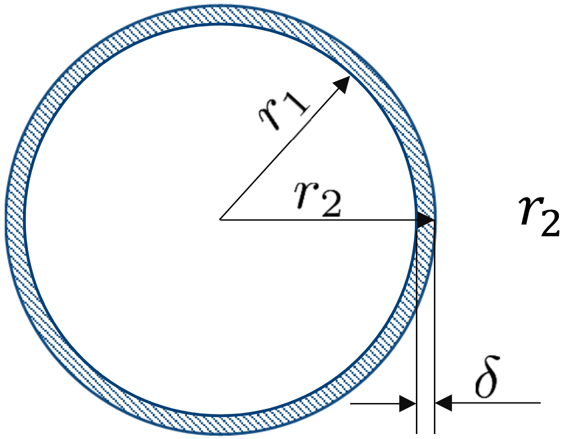

$$\delta \ll r_1$$
$$r_2 = r_1 + \delta$$

Simplified equation for the case: $\delta \ll r$

$$\dot{Q} = \frac{T_1 - T_2}{\frac{1}{2\pi L}\cdot\frac{1}{\lambda}\cdot\frac{\delta}{r_1}} = \underbrace{A\cdot\lambda\cdot\frac{\Delta T}{\delta}}_{\text{plane wall equation}}$$

Comprehension questions

What is the course of the temperature profile for cylindrical bodies?

How does the temperature profile of a cylindrical body differ from the temperature profile in a plane wall? What is the reason for this?

Under which conditions can the curvature of the cylinder and thus the change of the area inside the cylinder wall be neglected?