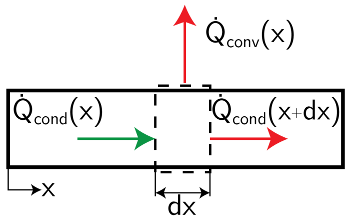
Temperature Profile in a Fin

Derive the energy balance to establish the temperature profile of the fin in the axial direction.



1) Setting up an energy balance:

The temperature profile can be derived from the conduction equation. The conduction equation results from the energy balance around an infinitesimal element in the system:

$$\frac{dU}{dt} = \sum \dot{Q}_{\mathrm{in}} - \sum \dot{Q}_{\mathrm{out}}$$

Which for the steady-state case will be:

$$0 = \dot{Q}_{\mathrm{cond}}(x) - \dot{Q}_{\mathrm{cond}}(x+dx) - \dot{Q}_{\mathrm{conv}}(x)$$

2) Defining the fluxes:

The ingoing conductive flux can be described by use of Fourier's law:

$$\dot{Q}_{\mathrm{cond}}(x) = -\lambda A_c \frac{dT}{dx} = -\lambda a^2 \frac{dT}{dx}$$

The outgoing conductive flux for an infinitesimal element can be approximated by use of Taylor series:

$$\dot{Q}_{\mathrm{cond}}(x+dx) = \dot{Q}_{\mathrm{cond}}(x) + \frac{d\dot{Q}_{\mathrm{cond}}(x)}{dx} \cdot dx$$

Furthermore, the loss due to convection for the infinitesimal element can be described by use of Newton's law of cooling:

$$\dot{Q}_{\mathrm{conv}}(x) = \alpha A_{\mathrm{s}} \left(T(x) - T_{\mathrm{amb}}\right) = 4\alpha a dx \left(T(x) - T_{\mathrm{amb}}\right)$$

3) Inserting and rearranging:

Inserting the found fluxes into the balance yields:

$$\lambda a^2 \frac{d^2T}{dx^2} \cdot dx - 4\alpha a dx \left(T(x) - T_{\mathrm{amb}}\right)$$

$$\boxed{\rightarrow \frac{d^2T}{dx^2} - \frac{4\alpha}{\lambda a}\left(T(x) - T_{\mathrm{amb}}\right)}$$