Heat and Mass Transfer I

Exercise script

Winter Semester
2019/20

"'Heat and Mass Transfer I"' (Exercise script)

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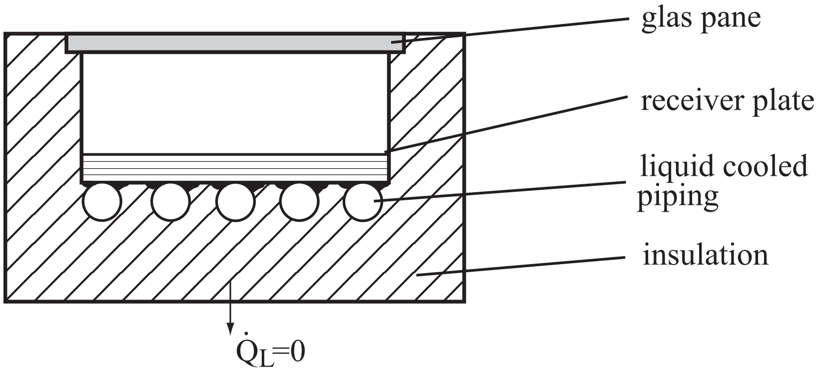
Chapter 1.

Radiation

1.1. Solar panel

The general structure of a solar panel is regarded. The receiver plate (absorption coefficient $\alpha_{\mathrm{R}} = 1$) is placed in a fully insulated box, which is covered by a glass plate. The space between the receiver and the glass plate can be considered as evacuated. The glass has an absoprtion coefficient of $\alpha_{\mathrm{GS}} = 0.05$ for radiation from a source of high temperature (short wavelength radiation) and a transmission coefficient of $\tau_{\mathrm{GS}} = 0.88$; for radiation from a source with a lower temperature (long wavelength radiation) the values are $\alpha_{\mathrm{GL}} = 0.90$ and $\tau_{\mathrm{GL}} = 0.05$, respectively. The emission coefficient of glass at the temperature reached is $\varepsilon_{\mathrm{G}} = 0.84$.

The receiver plate is positioned perpendicularly to the direction of the sunrays. The surroundings, as seen from the receiver, i.e. the glass plate, are considered black bodies. The radiation heat flux from the sun is $\dot{q}''_{\mathrm{S}} = 920\,\mathrm{W/m^2}$, while the ambient heat flux is $\dot{q}''_{\mathrm{A}} = 80\,\mathrm{W/m^2}$. Heat is transfered to a fluid flowing through tubes mounted at the rear of the receiver plate. The contact between the receiver plate and the tubes has no thermal resistance and the fluid flow is such that the receiver plate reaches a constant temperature of approximately $T_{\mathrm{R}} = 60\,°\mathrm{C}$.



Tasks:

a) Which temperature T_G does the glass pane adopt?

b) Quantify the heat gain per unit time and area $\dot{q}''_R$.

c) Quantify the heat gain per unit time and area $\dot{q}''_R$ without the glas pane.

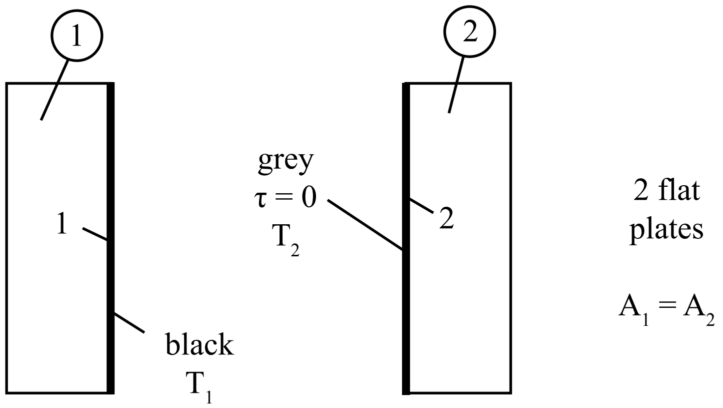
Assumptions:

- Edge effects such as reflections at the side walls of the panel shall be disregarded.
- Heat transfer through conduction and convection shall be disregarded.

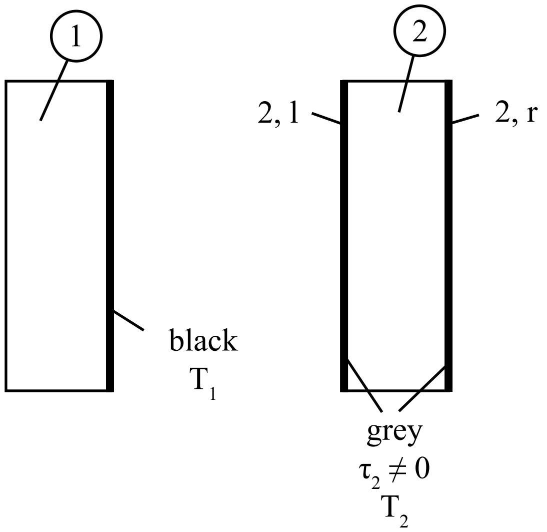
1.2. Surface brightness & energy balances

Determine the necessary surface brightnesses and energy balances for the following geometries. The corresponding view factors are given and all radiative properties $(\rho, \tau, \alpha, \varepsilon)$ are known. The temperature of each object is known and constant. Radiation from the face side of each object and from the surrounding atmosphere can be neglected.

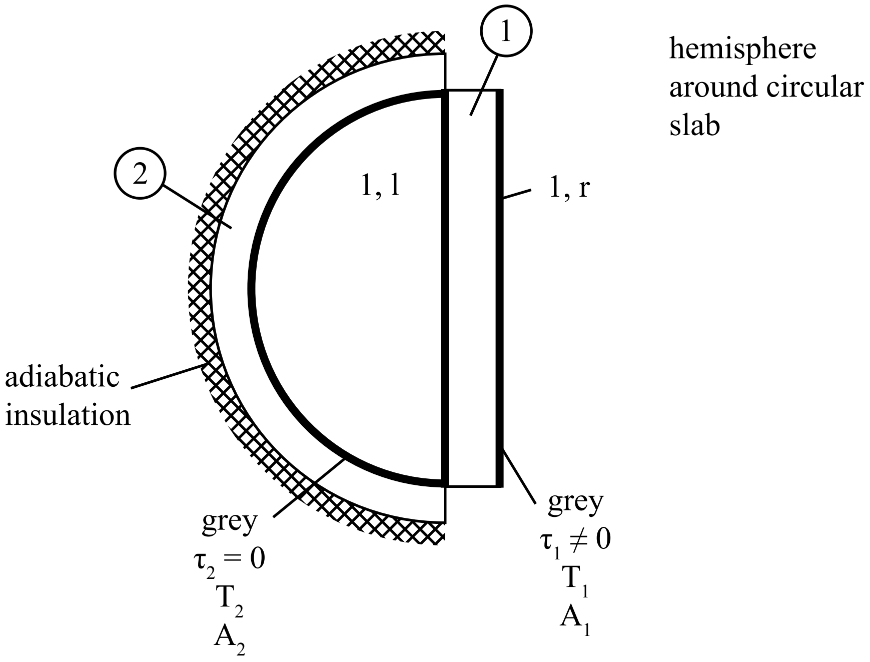
a) Determine the surface brightness for both surfaces.



b) Determine the surface brightness for each surface and formulate the energy balance for the middle body.



c) Determine the surface brightness for each surface and formulate the corresponding energy balance for the hemisphere as well as the circular slab.

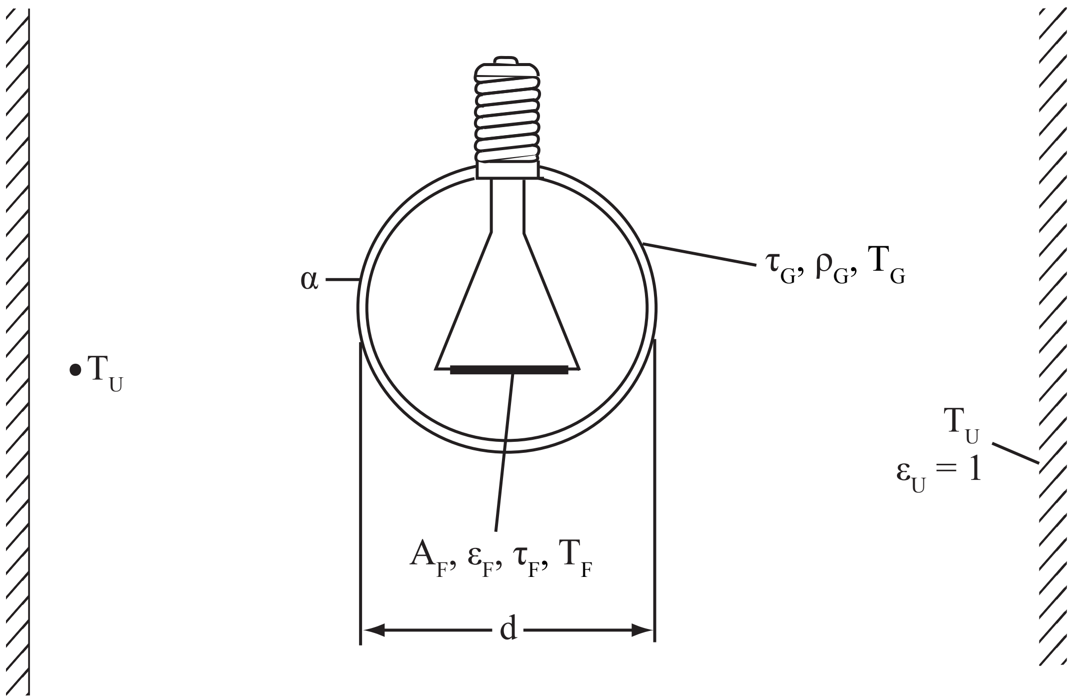


1.3. Spherical, evacuated lightbulb*

For a spherical, evacuated lightbulb the following data are known:

d	30	mm	Diameter of the glass sphere
τ_G	0.97	-	Transmission coefficient of the glass mantle
ρ_G	0	-	Reflection coefficient of the glass mantle
A_F	40	mm^2	Radiation surface of the filament
ε_F	0.25	-	Emission coefficient of the filament surface
τ_F	0	-	Transmission coefficient of the filament surface

The temperature of the glass sphere hanging in the air must not exceed $T_G = 70\,°C$ for an ambient temperature of $T_A = 20\,°C$. The heat transfer coefficient at the surface of the glass is $\alpha = 11\,\mathrm{W/m^2\,K}$.



Tasks: With the given data determine:

a) The temperature of the filament T_{F}.

b) The electric power $\dot{Q}_{\mathrm{el}}$, that the filament can withstand.

c) The wavelength λ_{max} at which the filament emits its maximum radiation heat flux.

d) Which form do the balance equations necessary for the computation of the filament's temperature T_{F} and the electric power $\dot{Q}_{\mathrm{el}}$ take, if the emissivity of the glass sphere itself and the surrounding walls are also taken into account? Calculate the filament's temperature T_{F} and the electric power $\dot{Q}_{\mathrm{el}}$ for this case.

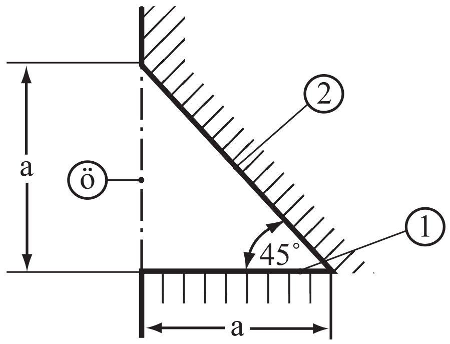
Assumptions:

- The glass body and filament are grey bodies.
- No influence from the lightbulb's bearing.
- The heat resistance of the thin glass mantle is negligible.
- The emission of the glass sphere is negligible (only valid for a)-c)).
- The radiation from the area surrounding the lightbulb is disregarded (only valid for a)-c)).

1.4. Radiation within a wedge-shaped opening

For an infinitely long opening with a wedge-shaped cross section, the following data are known:

The surface (1) is $a = 30\,\mathrm{cm}$ wide, has an emission coefficient of $\varepsilon_1 = 1$ and a temperature of $T_1 = 1000\,\mathrm{K}$. The side comprising the opening, too, is 30 cm wide and perpendicular to surface (1). Surface (2) is a grey body and adiabatically insulated at the back. The space surrounding the opening can be considered to be a black body with a temperature of 0 K. Influences due to convection shall be disregarded.



Tasks:

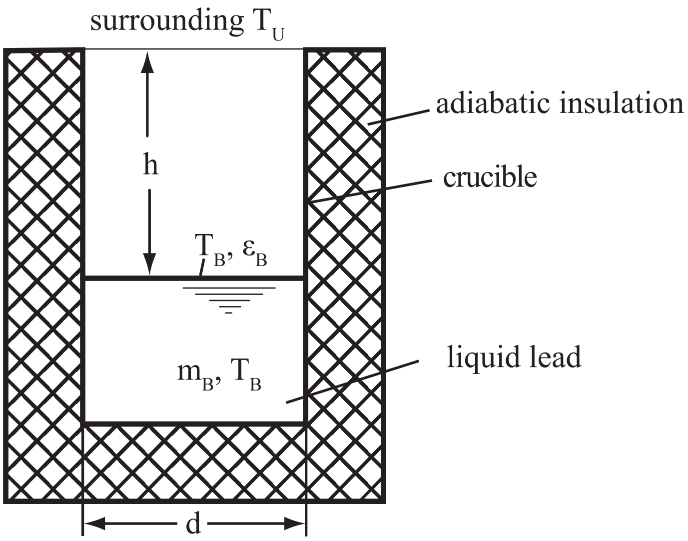
a) Determine the view factors $\Phi_{1,2}$,$\Phi_{2,1}$,$\Phi_{1,\mathrm{o}}$,$\Phi_{2,\mathrm{o}}$

b) Determine the energy loss of surface (1) $\dot{q}'_{1,V}$ and the opening $\dot{q}'_{\mathrm{o},V}$ for a unit length of the opening.

c) Determine the temperature T_2 of surface (2).

1.5. Lead crucible*

A cylindrical crucible contains liquid lead at its melting temperature of $T_\mathrm{L} = 327\,°\mathrm{C}$. The crucible has a diameter of $d = 25\,\mathrm{mm}$, the height between the surface of the liquid lead and the opening of the crucible is $h = 25\,\mathrm{mm}$. The crucible's walls are adiabatically insulated. Using the dimensions d and h a view factor of $\Phi_{L,0} = 0{,}38$ between the surface of the liquid lead and the opening of the crucible is obtained.



Tasks:

a) Determine the radiation heat flux $\dot{Q}_\mathrm{L}$ of the liquid lead, unter the following assumptions:

- The surface temperature of the molten lead is constant and equal to the melting temperature T_L, the surface can be viewed as a grey body with an emissivity of $\varepsilon_\mathrm{L} = 0{,}8$.
- The surroundings of the crucible are black and the ambient temperature is $T_\mathrm{U} = 0\,\mathrm{K}$.
- Convective influences are disregarded.

b) Determine the heat fluxes $\dot{Q}_{\mathrm{L},0}$, $\dot{Q}_{\mathrm{L},\infty}$ for the cases where $h = 0$ und $h = \infty$, respectively, for the same conditions as in a)?

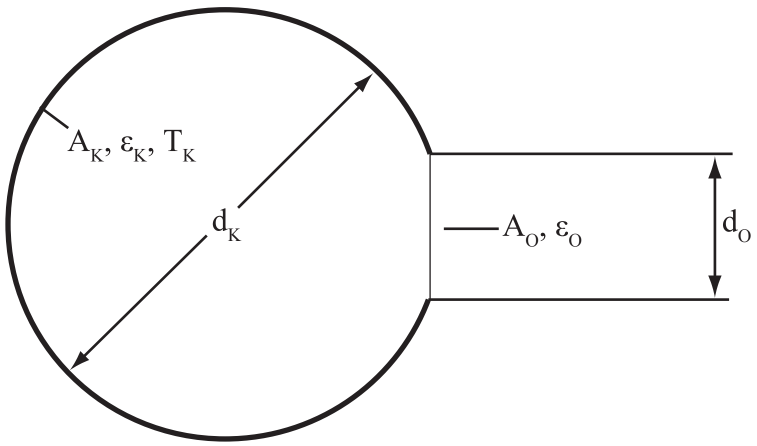
c) For all three cases, $h = 25\,\mathrm{mm}$, 0 und ∞, approximate the time t_S (for the same initial conditions as in a)), for which the melt solidifies. The amount of lead contained within the crucible is $m_L = 90\,\mathrm{g}$, the specific melting enthalpy is $h_L = 23.3\,{}^{\mathrm{kJ}}/_{\mathrm{kg}}$.

1.6. Hollow black body

Black body radiation can be approximated by employing a hollow body of homogeneous temperature with a small opening.

Task: A hollow copper sphere of a homogeneous temperature T_S is $d_S = 20\,\text{cm}$ in diameter. Its internal surfaces are grey and emit diffuse radiation; the emissivity is $\epsilon_S = 0.55$. Determine the diameter of a circular opening d_O necessary to emit radiation of the temperature T_S, equivalent to the sphere's temperature, which differs from the amount of radiation emitted by a black body by exactly $1\,\%$. Consequently the surface area of the opening has an emissivity of $\varepsilon_O = 0.99$.

Assumptions: Interactions of radiation between the sphere or opening and its surroundings can be disregarded ($\epsilon_A = 1$, $T_A = 0\,\text{S}$).

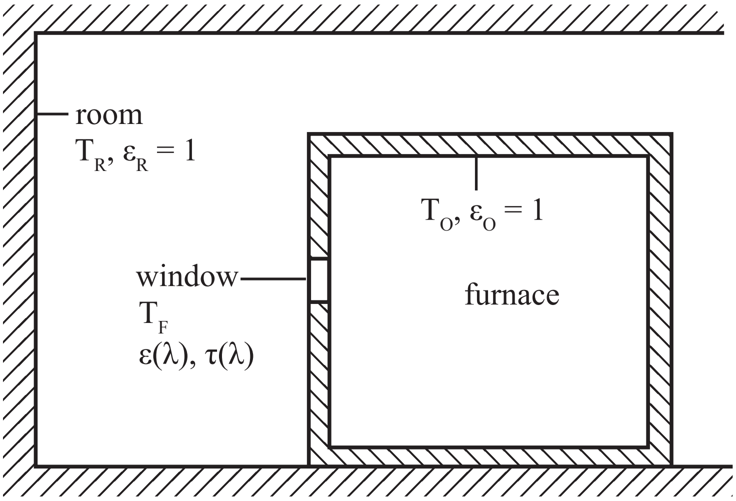


1.7. Furnace with quartz window

The furnace wall contains a quarz window for optical access, of diameter $d = 20\,\text{cm}$. The quarz has the following radiative, wave-length dependent properties:

Wave-length	rad. properties		
$0 \quad < \lambda_S \leq 4\,\mu\text{m}$	$\tau_S = 0.9$	$\varepsilon_S = 0{,}1$	$\rho_S = 0$
$4\,\mu\text{m} < \lambda_L < \infty$	$\tau_L = 0$	$\varepsilon_L = 0{,}8$	$\rho_L = 0.2$

The furnace's inner surface can be regarded as a black body and has a temperature of $T_O = 1000\,\text{K}$. The surrounding of the furnace may also be regarded as a black body with a temperature of $T_R = 300\,\text{K}$.



Known quantities:

$\lambda \cdot T$ [µm K]	1111.1	1222.2	3000	3111.1	3222.2	3555.6	4000
$\dfrac{\int_0^\lambda \dot{q}_s''\lambda(\lambda,T)\mathrm{d}\lambda}{\sigma T^4}$	0.00101	0.00252	0.27322	0.29825	0.32300	0.39445	0.48085

Tasks:

a) Compute the **emitted** radiation of one side of the quarz window per unit area $\dot{q}''_{\mathrm{W},\varepsilon}$ by utilising a energy balance around the window (**do not** calculate the surface brightness).

b) Calculate the radiative heat loss $\dot{Q}_{\mathrm{L}}$ through the quarz window for the quasi-stationary case.

c) Calculate the quarz window's temperature T_{W}.

Annahmen:

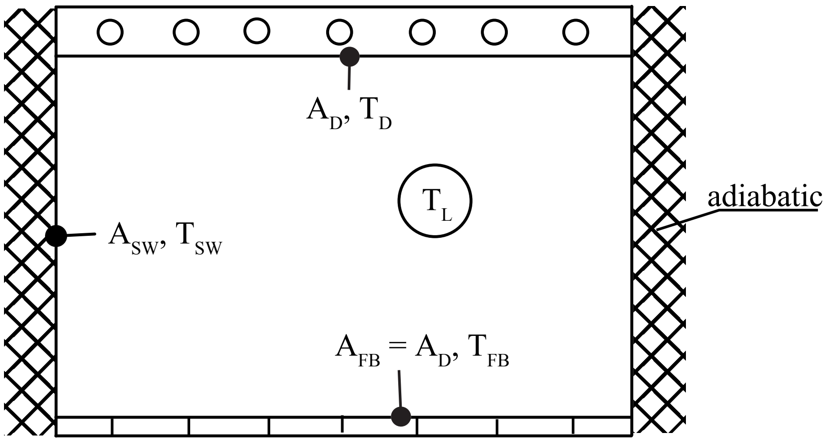
- For the purpose of this task heat transfer through conduction and convection shall be neglected (i.e. the sole heat loss occurs through the quarz window).

1.8. Radiative net heat flux between two plates*

Show how the net radiative heat flux between two parallel, infinitely large plane surfaces changes when radiation shielding with the same emissivity as the outer surfaces is placed between the planes. The space between the planes is evacuated.

1.9. Heated ceiling

The internal space of a building measuring 7 m square by 3.6 m in height is equipped with a radiation heating installed at its ceiling. The ceiling is heated by circulating hot water through embedded piping. The temperature to be reached at the wooden floor is specified as $T_{\mathrm{FB}} = 26\,°\mathrm{C}$. For reasons of comfort, the radiative net heat flux shall measure exactly $\dot{Q}_{\mathrm{S}} = 1750\,\mathrm{W}$ (net radiative heat flux transiting the wooden floor = difference of the total incoming and outgoing radiative energy per unit time). The emission coefficient of all surfaces are assumed to be of value 1.



Tasks:

a) Determine the necessary mean ceiling temperature T_{C} while neglecting the sidewalls. The view factors are $\Phi_{\mathrm{F,C}} = \Phi_{\mathrm{C,F}} = 1$.

b) Determine the mean ceiling temperature T_{C} while considering the influence of the sidewalls. Assume adiabatic sidewalls and a constant air temperature within the room of $T_{\mathrm{R}} = 22\,°\mathrm{C}$, maintained through a continious exchange of air. The heat transfer coefficient of the internal walls has a mean value of $\alpha = 3\,\mathrm{W/m^2K}$. The view factors between ceiling and floor are $\Phi_{\mathrm{D,FB}} = \Phi_{\mathrm{F,C}} = \Phi = 0.41$.

Instructions:

1. First of all determine the sidewall temperature T_{SW}.

2. Treat the sidewalls as<u>one</u> contiguous surface for the prupose of emitted radiation.

3. To solve the equation for T_{SW}, employ following approximation

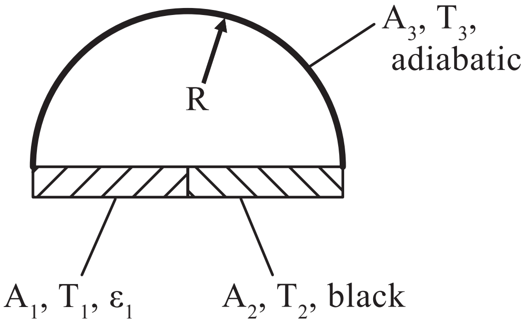
$$\left(\frac{T_{\mathrm{SW}}}{100}\right)^4 - \left(\frac{T_{\mathrm{F}}}{100}\right)^4 \approx \frac{1}{25}\left(\frac{T_{\mathrm{F}}}{100}\right)^3 \cdot (T_{\mathrm{SW}} - T_{\mathrm{F}})$$

c) Determine the total heat flux $\dot{Q}_{\mathrm{H}}$ which needs to be provided by the ceiling heating to meet the conditions postulated in task b).

d) Determine the heat flux $\dot{Q}_{\mathrm{L_F}}$ transmitted through the floor to the surroundings below.

1.10. Cupola

Both semi-circular slabs A_1 and A_2 of the geometric configuration depicted below are conditioned to maintain a constant temperature of $T_1 = 150\,°C$ and $T_2 = 20\,°C$, respectively. Surface A_1 exhibits an emissivity of $\varepsilon_1 = 0.6$, surface A_2 can be considered a black body and the hemispherical surface A_3, situated above the slabs, of radius $R = 3\,m$ is adiabatic. Surfaces A_1 and A_3 are grey bodies and emit difuse radiation. The hemispherical volume is submitted to a vacuum. All surfaces are intransparent to radiation.



Tasks:

a) Compute the amount of heat transfered through radiation between the surfaces A_1 and A_2 (= net radiative flux through surface A_2).

b) Which temperature T_3 is obtained for surface A_3?

Chapter 2.

Heat conduction

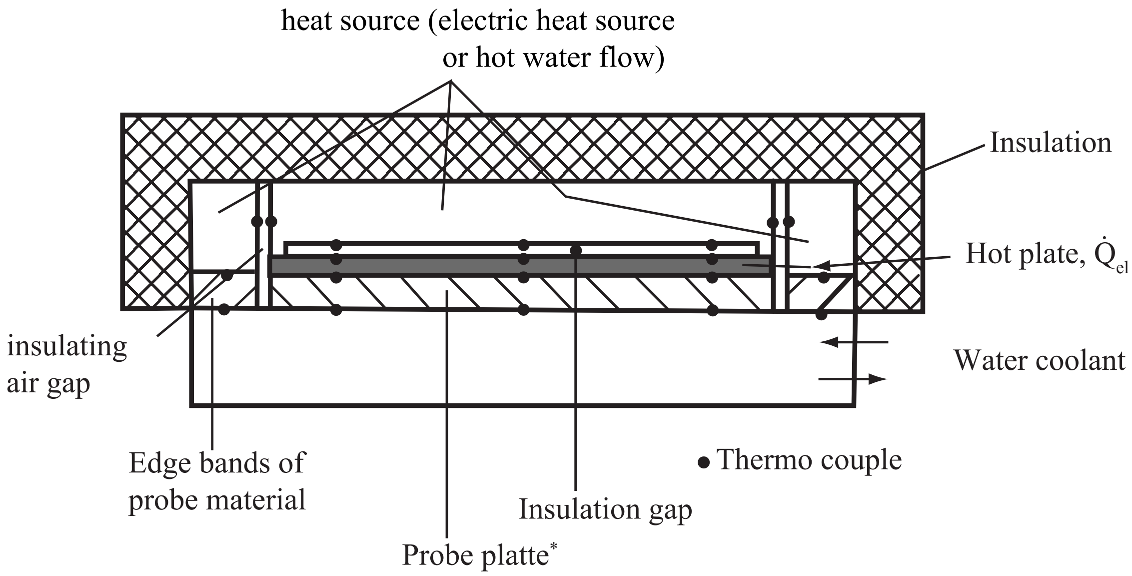
2.1. Transient temperature field*

Derive the partial differential equation for the transient temperature field of an infinitely long cylindrical and isotropic body while taking into account local changes of the conductivity λ. The specific heat capacity c and density ρ remain constant. Also take into account heat sources $\dot{\Phi}'''$ with spatially varying intensities within the regarded body.

2.2. Poensgen device*

To determine the conductivity of solid materials up to 2.5 $\mathrm{W/m\,K}$ a so-called Poensgen device is used. Steady-state one-dimensional heat conduction in plates represents the basis for the measurements.

The plane probe is placed between two copper plates. The first plate is cooled to a homogeneous and constant temperature; while the other plate is electrically heated with constant power. The entire apparatus is surrounded with edge bands and auxiliary heating to minimize heat losses, in order to ensure that the entire power if the heated plate passes through the probe in one direction only. Thermocouples in the insulated space between the auxiliary heating and the heating plate as well as on the probe and edge strips are employed to manipulate the auxiliary heating.



*circular approx 10 cm diameter or square with 30 x 30 cm^2 or 50 x 50 cm^2

The auxiliary heating is correctly adjusted correctly when the thermocouples located within the same plane show equal values. The governing temperature gradient between the upper and lower side of the probe is also measured by thermocouples.

Measurement data	
geometric data	δ_{Probe}, A_{Probe}
current	U, I
temperatures	T_{upper}, T_{lower}

Tasks:

a) Quantify the measuring error induced by airgaps of $\delta_{\mathrm{L}} = 0.05\,\mathrm{mm}$ between both the probe and the cooling plate as well as the probe and the heating plate, if the thermocouples show the temperatures of the respective plates.

   – bei Beton $\lambda = 1.2\,\mathrm{W/m\,K}$

   – bei Kork $\lambda = 0.051\,\mathrm{W/m\,K}$

   The probe thickness measures $\delta_{\mathrm{Probe}} = 20\,\mathrm{mm}$, the thermal conductivity within the gaps reads $\lambda_{\mathrm{L}} = 0.026\,\mathrm{W/m\,K}$.

   – Hints: Determine the ratio of conductivity between the probe and the plate without the influence of the air gaps and that of plate and probe with th air gaps ($\lambda_{\mathrm{P,Korr}} = \lambda_{\mathrm{P,Tats}}$).

b) Name other possible sources for measuring errors and suggest solutions to amend them.

2.3. Temperature profiles in planar walls

Both sides of a planar wall are heated to a constant temperature of T_1 and T_2, respectively; where $T_1 > T_2$.

Tasks:
Qualitatively sketch the temperature profile for steady-state conditions, if

a) the conductivity remains constant or

b) the conductivity is a function of the temperature following the equation $\lambda = \lambda_0(1 + \gamma(T - T_0))$ with $\gamma > 0$ (λ_0 = thermal conductivity at reference temperature T_0)

Justify your sketches in short points.

2.4. The onion layer principle

Measurements of the surface temperature of a planar wall, comprising a $\delta_A = 12.5\,\mathrm{cm}$ thick layer of material A and a $\delta_B = 20\,\mathrm{cm}$ thick layer of material B, yielded following results:

T_A	260	°C	surface temperature of layer A
T_B	32	°C	surface temperature of layer B

After insulating the outer surface of layer B with an layer of $\delta_{ins} = 2.5\,\mathrm{cm}$ thickness, following values were measured:

T_A^*	305	°C	surface temperature of layer A
T_B^*	219	°C	temperature of the contact area of layer B and the insulating layer (previously the surface temperature of layer B)
T_{ins}	27	°C	surface temperature of the insulating layer

Tasks:

a) Determine the transmitted heat flux per unit area $\dot{q}''$ with and without the insulating layer; assume steady-state conditions.

2.5. Fiery furnace*

The wall thickness of a furnace is to be minimised for steady-state conditions, considering the following boundary conditions for the steady-state case: The material for the wall can be chosen from the table below. The outer surface of the oven should comprise a 8 mm thick steel plate to act as a mechanical shield. The furnace wall's outer temperature must never exceed $T_{\mathrm{st}} = 60\,°\mathrm{C}$, while its inner surface must never exceed $T_i = 1000\,°\mathrm{C}$. The maximum permissible loss heat flux is $\dot{q}'' = 1.5\,\mathrm{kW/m^2}$.

The following table list the two possible cases (A) and (B) for the material combinations:

Known quantities:

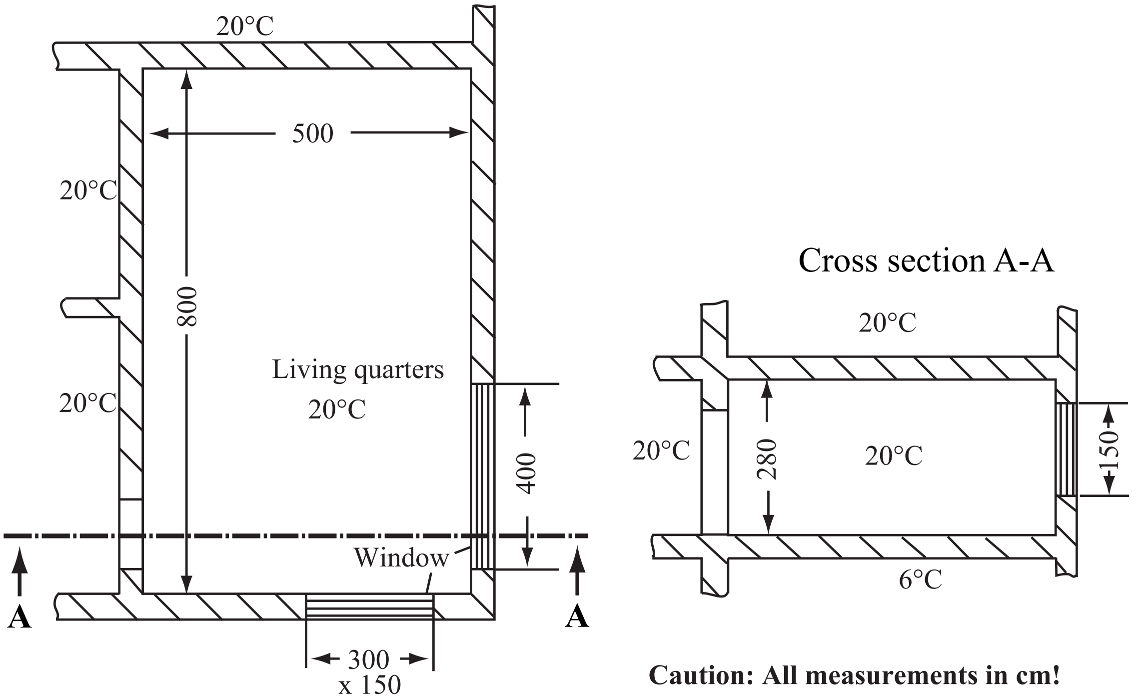
Material	permissible temperature	thermal conductivity λ [W/m K]					
		Case (A)			Case (B)		
		60 °C	800 °C	1200 °C	60 °C	800 °C	1200 °C
Firebrick	1500 °C	0,9	1,3	1,5	0,9	1,3	1,5
Refractory brick	1200 °C				0,4	0,6	0,7
Insulating stone	800 °C	0,2	0,25				
Steel		45			45		

Task:

Determine the wall structure and thickness of each wall layer for both cases while observing the condition of minimising the necessary total thickness.

2.6. Living comfortably*

A living quarter is heated in the winter and its inhabitants are curious, how much heat goes unused during each winter day.



Caution: All measurements in cm!

Task:
Determine the total loss heat flux through the outer surfaces of the sketched quarter for an outer temperature of $T_a = -15\,°C$ and a room temperature of $T_i = 20\,°C$.

Hinweis:

- Assume a one-dimensional heat flux.
- The temperatures of the adjacent rooms and their measurements (in cm) are noted in the sketch. The set up of the walls and the ceilings as well as the thermal conductivity of their respective building materials are given in the table on the next page.

Known quantities:

α_a	20	$^{W}/_{m^2\,K}$	heat transfer coefficient* outer walls
			heat transfer coefficienten* inner walls:
α_i	8	$^{W}/_{m^2\,K}$	- wall surfaces, windows, ceilings if heat flux is directed upwards
	6	$^{W}/_{m^2\,K}$	- ceilings if heat flux is directed downwards
k_F	2.9	$^{W}/_{m^2\,K}$	coefficient of thermal permittivity of the double glazed wooden windows
		<u>outer walls</u>	
$d_{P,A}$	1.5	cm	thickness outer plaster
$\lambda_{P,A}$	0.87	$^{W}/_{m\,K}$	thermal conductivity plaster (outside)
$d_{WD,A}$	2.0	cm	thickness insulation
$\lambda_{WD,A}$	0.09	$^{W}/_{m\,K}$	thermal conductivity insulation
d_Z	34	cm	thickness brickwork
λ_Z	1.05	$^{W}/_{m\,K}$	thermal conductivity brickwork
$d_{P,I}$	1.5	cm	thickness inner plaster
$\lambda_{P,I}$	0.87	$^{W}/_{m\,K}$	thermal conductivity plaster (inside)
		<u>ceilings</u>	
$d_{P,D}$	1.5	cm	plaster
$\lambda_{P,D}$	0.87	$^{W}/_{m\,K}$	thermal conductivity plaster
$d_{WD,D}$	2.0	cm	insulation
$\lambda_{WD,D}$	0.09	$^{W}/_{m\,K}$	thermal conductivity insulation
d_B	20	cm	concrete ceiling
λ_B	1.4	$^{W}/_{m\,K}$	thermal conductivity concrete ceiling
d_E	4	cm	screed
λ_E	0.7	$^{W}/_{m\,K}$	thermal conductivity screed
d_H	1	cm	wooden flooring
λ_H	0.2	$^{W}/_{m\,K}$	thermal conductivity wooden flooring

*sumperimposition of radiation and convection

2.7. Wooden cylinder*

A hollow cylinder of radius r_i, outer radius r_a and length L is heated such that its inner and outer surfaces reach a constant and homogeneously distributed temperautre of T_i and T_a, respectively. The cylinder material has a temperature dependent thermal conductivity according to the following equation:

$$\lambda = \lambda_0(1 + \gamma(T - T_0))$$
$$(\lambda_0 = \text{thermal conductivity at reference temperature } T_0)$$

Tasks:

a) Derive an equation for the heat flux through the wall (mantle) of the cylinder. Compare this equation with that for the case of constant thermal conductivity. For which mean temperature T_m would the thermal conductivity λ have to be introduced in the equation for the heat flux in the case of constant conductivity, if said thermal conductivity λ were to be used to calculate the heat flux for the case of a linear temperature dependency of conductivity?

b) State the equation describing the temperature distribution in the hollow cylinder.

2.8. Brine pipeline

A steel brine pipeline, with an inner diameter of $d_i = 50\,\mathrm{mm}$ and a wall thickness of $\delta = 5\,\mathrm{mm}$, traversing a room with an average air temperature of $T_R = 20\,°\mathrm{C}$ holds brine with an average temperature of $T_B = -20\,°\mathrm{C}$.

Known quantities:

α_i	2300	$\mathrm{W/m^2\,K}$	heat transfer coefficient at the inner side of the pipe
α_o	6	$\mathrm{W/m^2\,K}$	heat transfer coefficient at the outer side of the pipe
λ_R	54	$\mathrm{W/m\,K}$	thermal conductivity steel
λ_{ins}	0.042	$\mathrm{W/m\,K}$	thermal conductivity insulation

Tasks:

a) Determine the thickness of the insulation δ_{ins} such that no condensate is formed at the surface of the insulation is formed, even when the maximum dew point temperature of the surrounding air of $T_{dew} = 15\,°\mathrm{C}$.

b) Determine the amount of heat $\dot{q}'$ absorbed by the brine per unit pipe length and time under the conditions given above.

2.9. Warm-water pipe

A copper warm-water pipe placed in a room of $T_\mathrm{R} = 20\,°\mathrm{C}$ holds water with an average temperature of $T_\mathrm{W} = 80\,°\mathrm{C}$. The copper pipe ($\lambda = 372\,\mathrm{W/mK}$) has an inner diameter of $d_\mathrm{i} = 6\,\mathrm{mm}$ and a wall thickness of $\delta = 1\,\mathrm{mm}$.

Known quantities:

α_i	2300	$\mathrm{W/m^2\,K}$	heat transfer coefficient at the inner side of the pipe
α_a	6	$\mathrm{W/m^2\,K}$	heat transfer coefficient at the outer side of the pipe

Tasks:

a) Determine the heat transfered per unit pipe length for $\dot{q}'$.

   – an uninsulated pipe, and

   – an insulated pipe with a $s = 4\,\mathrm{mm}$ cork layer ($\lambda = 0.042\,\mathrm{W/mK}$).

b) Qualitatively sketch the heat emission profile $\dot{q}'$ as a function of the insulation thickness for different thermal conductivities of the insulation material. Explain the underlying physical principles.

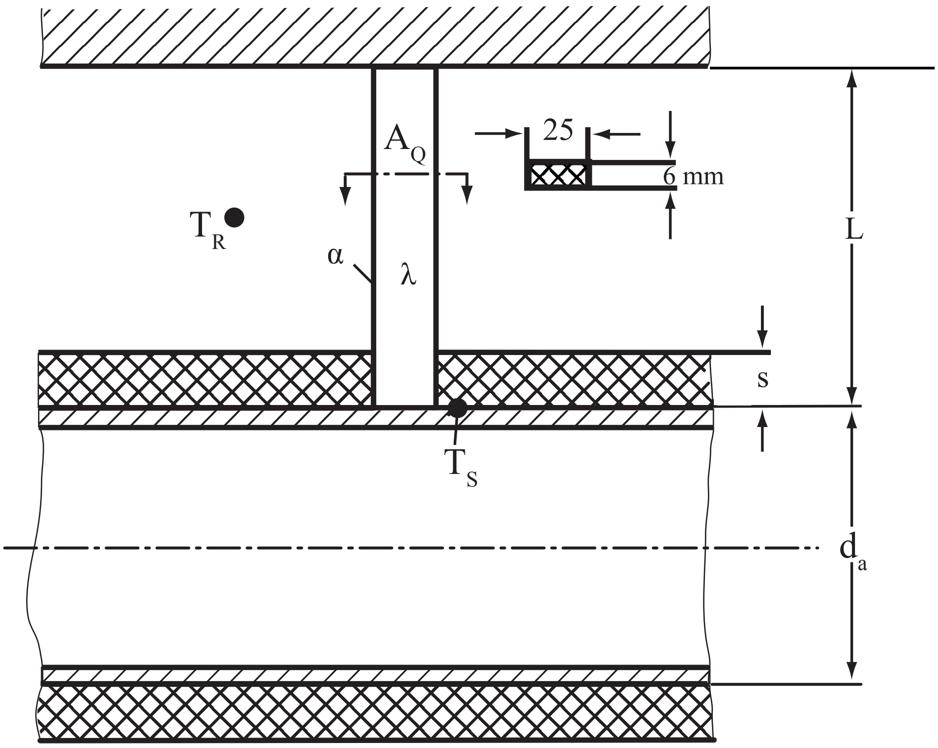
c) Determine the necessary thermal conductivity λ_ins for the insulating material to obtain a general reduction in heat loss.

Assumptions:

- Changes to the heat transfer coefficient at the outer side of the pipe as a function of the diameter are disregarded.

2.10. Pipe fastening*

A pipe containing brine is insulated with cork and fastened to the ceiling with steel bands welded to the pipe. The outer wall temperature of the brine pipe is $T_B = -23{,}5\,°C$, the room has a temperature of $T_R = 20\,°C$.



Known quantities:

d_o	50	mm	outer diameter of the pipe
s	40	mm	insulation thickness
A_Q	25×6	mm	cross-section of the steel band
L	290	mm	length of the steel band
α	6	$^{W}/_{m^2\,K}$	heat transfer coefficient at the steel band's surface
λ	58	$^{W}/_{m\,K}$	thermal conductivity of the steel band

Tasks:

a) Calculate the heat $\dot{Q}$ from one steel band absorbed by the brine.

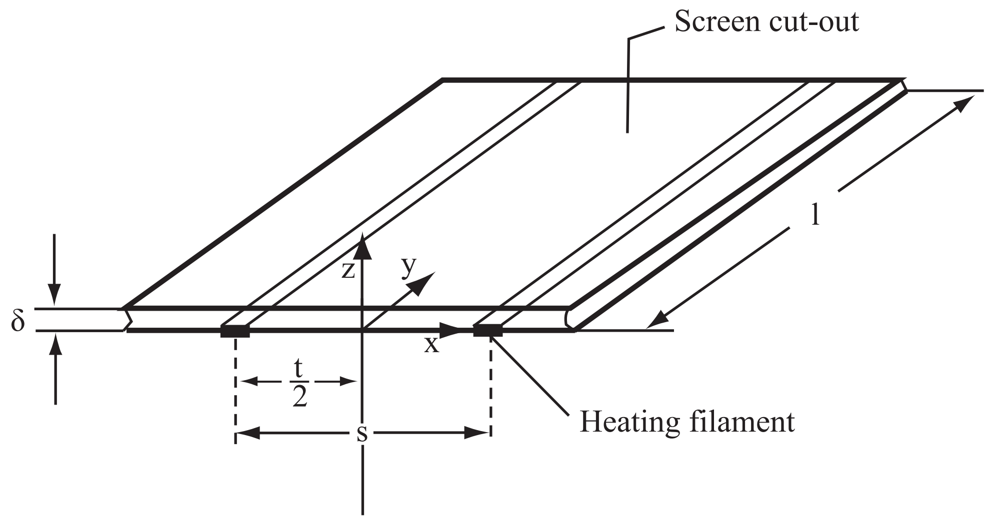
b) Up to which hight h_0 does frost form on the steel ban (h_0 is the distance from the surface of the brine pipe), if the steam content of the air in the surrounding room is above the saturation vapour pressure for the maximum steel band temperature?

Assumptions:

- The temperature distribution in the steel band's cross-section is homogeneous.
- The heat fluxes from the steel bands into both the ceiling and the insulation are negligible.

2.11. Foggy rear window

To prevent dew or frost forming at the inner side of a car's rear window, thin electrical heating elements are embedded within the glass. These are used at low ambient temperatures to feed enough energy in order to maintain a temperature above the dew point for the entire window.



Known quantities:

δ	5	mm	window thickness
s	30	mm	distances between the heating wires
λ	1.16	$\mathrm{W/m\,K}$	thermal conductivity
T_{A}	5	°C	ambient temperature at the outer side
α_{A}	30	$\mathrm{W/m^2\,K}$	heat transfer coefficient at the outer side
T_{I}	20	°C	ambient temperature at the inner side
α_{I}	3	$\mathrm{W/m^2\,K}$	heat transfer coefficient at the inner side
T_{τ}	13	°C	permissible minimum temperature = dew point temperature

Tasks:

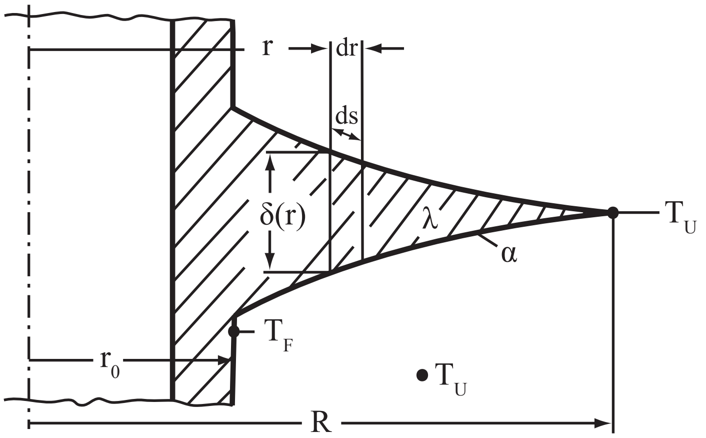
a) Determine the heating power necessary per unit conductor length to avoid reaching the dew point temperature, while observing the following conditions:

Assumptions/instructions:

- Steady-state conditions.
- Edge effects are neglected. Due to the symmetry it is sufficient to regard a window section with to wires – refer to the diagramme above.
- Consider the problem to be one-dimensional. The only significant changes in temperature are in x-direction.
- Homogeneous heat flux over the thicnkess of the window δ.
- For reasons of practicality the coordinate system is to be placed exactly in the middle of two heating wires.

2.12. Circular fin with varying thickness*

A new circular fin is to be designed. The following preceding considerations have been made:



Tasks:

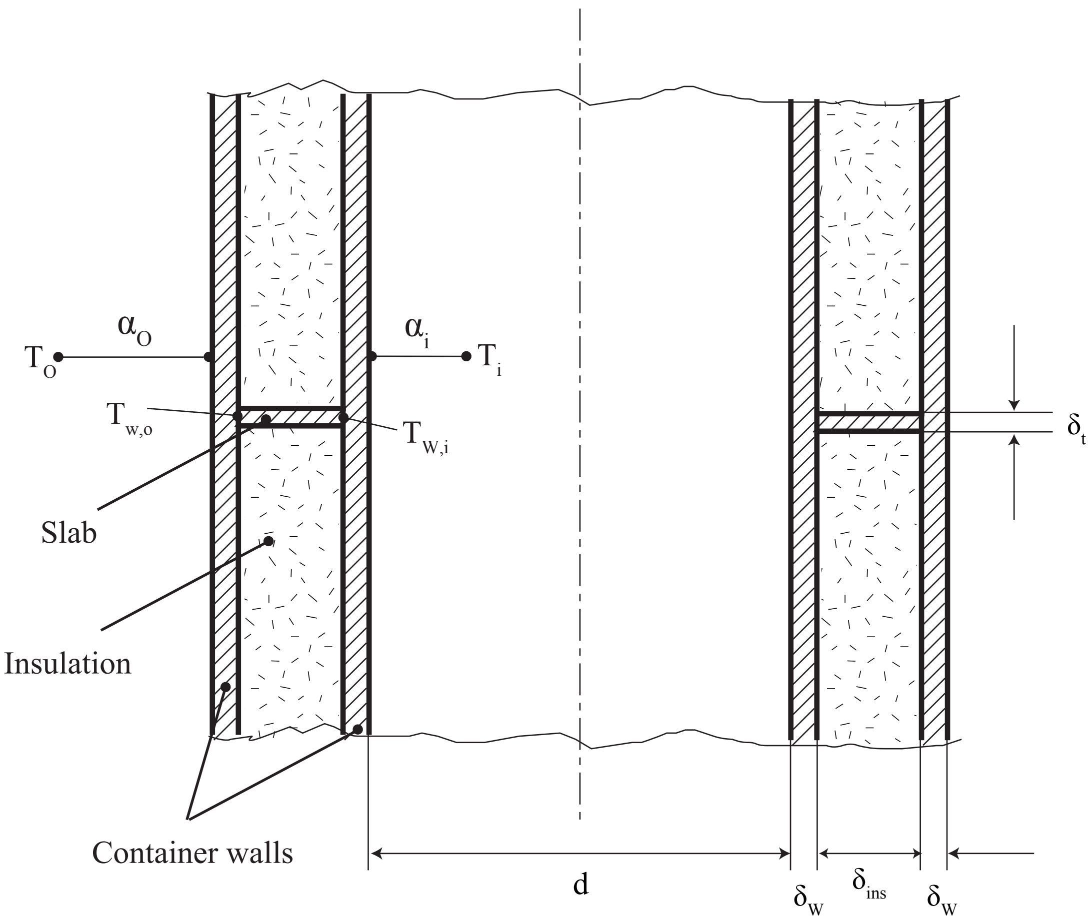
a) State the differential equation of a steady state temperature distribution in a thin circular fin with radially varying thickness $\delta(r)$, assuming a constant thermal conductivity λ of the fin's material and a constant heat transfer coefficient α at the surface of the fin. The ambient temperature T_A is known and constant. The arc elements ds of the fin's flank may be substituted by a radial element dr for simplicity's sake. The temperature differences in the cross section of the fin are assumed to be negligble.

b) Determine the profile $\delta(r)$ of a thin-edged circular fin; regard the conditions given above. The fin is subject to even thermal loads, i.e. a constant heat flux $\dot{q}''_r$ in radial direction. This type of fins necessitates the minimum amount of material to transfer a pre-defined thermal throughput. The inner diamter of the fin is $2 \cdot r_0$, while the outer diamter is $2 \cdot R$. The fin's tip temperature is equal to the ambient temprature.

c) Give the thermal the fin, as designed in b), can transfer if the temperature at the base of the fin is T_B.

2.13. Double walled container*

The following figure shows a section of a double-walled cylindrical container. The walls of the container are made of stainless steel and are interconnected by ligaments of the same material, as shown in the figure below. The distance between the two walls is filled with wool. The container is filled with a fluid of constant temperature. The ambient temperature is also constant.



Known quantities:

d	2	m	container - inner diameter
δ_B	3	mm	container wall thickness
δ_S	1.5	mm	ligament thickness
δ_{ins}	80	mm	insulation thickness
λ_B	16	$^W/_{m\,K}$	thermal conductivity of the container
λ_{ins}	0.04	$^W/_{m\,K}$	thermal conductivity of the insulation
α_L	84	$^W/_{m^2\,K}$	heat transfer coefficient at the outer side
α_F	400	$^W/_{m^2\,K}$	heat transfer coefficient at the inner side
T_F	-40	°C	temperature of the liquid
T_L	20	°C	temperature of the air

Tasks:

a) Derive an equation to calculate the amount of heat transfered from the ligament to the liquid. Neglect any heat transfer between the ligament and the insulation. Also, the thermal resistance of the container walls in radial direction is negligible. Quantify the heat flux absorbed by the liquid under the given conditions.

b) How large is the temperature difference $T_{C,o} - T_{C,i}$ between the outer and inner wall of the container at the ligaments.

c) Which minimum distance between individual ligaments needs to be maintained, so as not to invalidate the assumption that the ligaments do not interact with each other? Criteria for the evaluation should be the decrease of the temperature difference between the container wall and the surroundings. The sphere of influence of each ligament is limited to the point at which the temperature difference has decreased to 1 % of its maximum value.

d) Estimate the total heat flux to the liquid and, hence, the necessary cooling power per unit container length to preserve the temperature of the fluid, if the distance between two ligaments is 1,5 m. Assume no influence of the ligaments on the heat flux through the container.

2.14. High-temperature reactor fuel element

The fuel fora high-temperature nuclear reactor is encased between a graphite sphere 25 mm in diameter and another hollow graphite sphere with an outer diameter of 60 mm and an inner diameter of 31 mm. The fuel rod's specific heat output is $\dot{Q}_{\mathrm{R}} = 2.5\,\mathrm{kW}$. The thermal conductivity of the graphite is $\lambda_{\mathrm{G}} = 126\,{}^{\mathrm{W}}/_{\mathrm{m\,K}}$ (Elektrode graphite), the thermal conductivity of the fueld layer is $\lambda_{\mathrm{B}} = 12\,{}^{\mathrm{W}}/_{\mathrm{m\,K}}$.

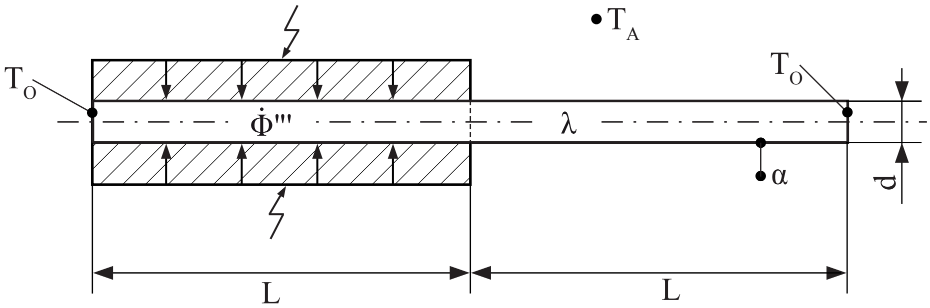
Tasks:

a) Quantify the temperature difference between the fuel rod and its outer surface.

b) Sketch the temperature profile qualitatively.

2.15. Copper rod*

Both ends of a copper rod with a length L and a diameter d are kept at the same temperature T_O. The left half of the rod is insulated against all radial heat losses. An electric heating element generates Joule's heat of heat flux density $\dot{\Phi}'''$. The right half of the rod is subjected to a flow of the ambient air with an air temperature of T_A, yielding a heat transfer coefficient α. The thermal conductivity of the rod is given as λ.



Tasks:

a) Derive the equation for the temperature profile in the rod.

b) Determine the value of $\dot{\Phi}'''$ so that the temperature in the center of the rod equals the temperature T_O at its ends.

c) Calculate $\dot{\Phi}'''$ using the following data: $L = 1\,\mathrm{m}$; $d = 5.2\,\mathrm{mm}$; $T_\mathrm{O} = 120\,°\mathrm{C}$; $T_\mathrm{U} = 100\,°\mathrm{C}$; $\alpha = 6\,\mathrm{W/m^2\,K}$; $\lambda = 372\,\mathrm{W/m\,K}$ for the conditions postulated in b).

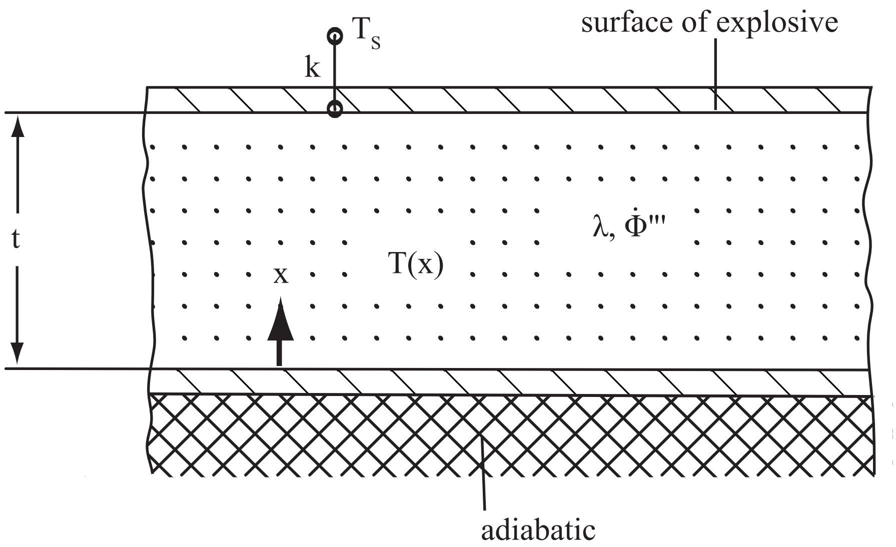
d) Determine the extremes of the temperature distribution for the given values. Give their position and values, additionally, sketch the temperature profile.

Hinweis:

- Place the origin of the coordinate system in the middle of the rod.

2.16. Critical explosive

An explate, stored in the for of a plate, is surrounded by shielding and insulation and is placed on an adiabatic base. The thermal conductivity of the explosive is $\lambda = 0.85\,\mathrm{W/m\,K}$, the overall heat transfer coefficient between the surface of the explosive and the surroundings T_A is $k = 0.2\,\mathrm{W/m^2\,K}$.



The inevitable reaction rate of the explosive causes a steady heat production which depends on the temperature as given by the following relationship:

$$\dot{\Phi}''' = \dot{\Phi}'''_U \cdot (1 + \gamma(T - T_U))$$

Known quantities:

T			temperature of the explosive
T_A			ambient temperature
$\dot{\Phi}'''_U$	0.3	$\mathrm{W/m^3}$	heat source
γ	0.2	$\mathrm{1/K}$	

Task:

a) Determine the thickness of the layer of explosive t_{krit} which will lead to a certain explosion.

Anleitung: First derive an equation for the steady-state temperature profile within the explosive for an arbitrary thickness s. An explosion occurs only if the temperature is premitted to grow infintely.

2.17. Copper sphere

A copper sphere of the diameter $d = 1\,\mathrm{cm}$ and an initial temperature $T_0 = 15\,°\mathrm{C}$ is suddenly exposed to an airflow of temperature T_A. The heat transfer coefficient at the surface of the sphere is $\alpha = 50\,\mathrm{W/m^2\,K}$. The specific heat capicity is $c = 0.419\,\mathrm{kJ/kg\,K}$ and its density is $\rho = 8300\,\mathrm{kg/m^3}$.

Tasks:

a) Determine the time necessary for the sphere's temperature to rise to $T_\mathrm{K} = 17.5\,°\mathrm{C}$, if the surrounding air temperature remains constant at $T_\mathrm{A} = 20\,°\mathrm{C}$.

b) Over time the temperature of the air rises at a rate of $\kappa = 360\,\mathrm{K/h}$, starting from a value of $T_{\mathrm{A},0} = 20\,°\mathrm{C}$.

   – Give the temperature difference of air andsphere $T_\mathrm{A} - T_\mathrm{S}$ after 1 h.

   – Qualitatively sketch the temporal evolution of the air and sphere temperatures.

c) The air temperature fluctuates sinosoidally with an amplitude of $\Theta_{\mathrm{L,max}} = 5\,\mathrm{K}$ and a frequency of $f = \frac{1}{6}\,\mathrm{min}^{-1}$ (angular frequency $\omega = 2\pi f$) around the mean value of $T_{\mathrm{L,m}} = 20\,°\mathrm{C}$.

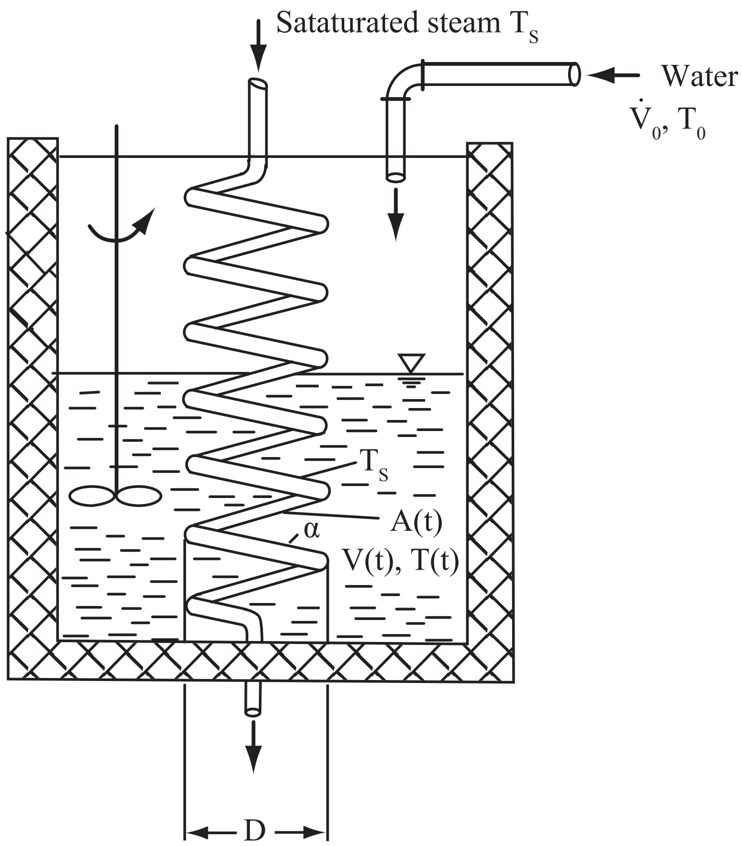
   1) Derive an equation for the temperature of the copper sphere as a function of time.

   2) Determine the amplitude $\Theta_{\mathrm{K,max}}$ of the temperature fluctuation within the sqhere.

   3) Determine the temporal delay with which the temperature oscillation of the sphere lags behind that of the air.

2.18. Stirred tank*

A cylindrical container contains $V_{\text{ges}} = 27\,\text{m}^3$ of water, each minute $\dot{V}_0 = 9\,\text{dm}^3$ water with a temperature of $T_0 = 20\,°\text{C}$ are added. At time $t = 0$ the tnank contains no water. The water within the container is stirred by an agitator to achieve a homogeneous temperature distribution within the container. A coiled tube of surface area A contains water vapour to heat the contents of the container. The coiled tube has $n = 10$ coils and a diameter of $D = 1.2\,\text{m}$. The outer diameter of the tube is $d = 25\,\text{mm}$, the saturation temperature of the steam is $T_S = 105\,°\text{C}$ and the transfer coefficient of the coil (interacting with the water) is $\alpha = 600\,{}^{\text{W}}/_{\text{m}^2\,\text{K}}$. The wetted, and therefore relevant for the heat transfer, surface area is proportional to the amount of water currently within the container.



Tasks:

a) Derive an equation describing the water temperature as a function of time.

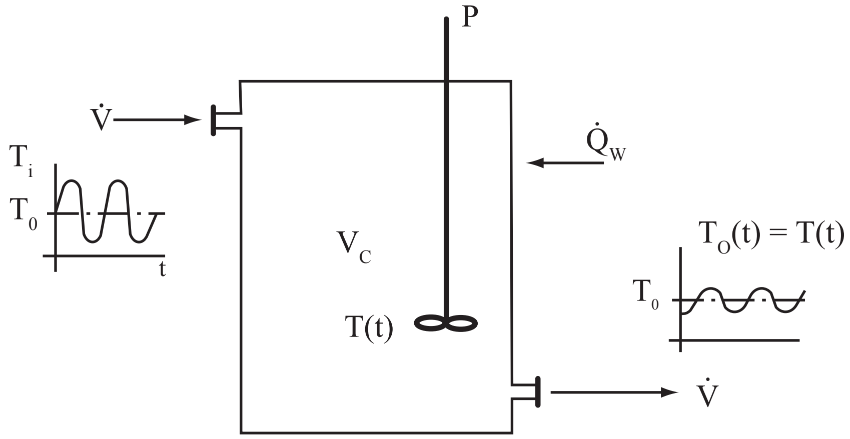
b) Give the water temperature T_{We}, for the point in time when the container is full.

Assumptions:

- The steam vapour temperature within the coiled tube remains constant.
- The material properties can be regarded as constant with respect to the temperature.
- The energy added by the agitator is negligible.
- The heat transfer coefficient is constant.
- The thermal resistance of the coiled tube is negligible.
- The container walls are adiabatically insulated.

2.19. Oscillation problem*

A container with the purpose of compensating temperature fluctuations is subject to a volume flux of $\dot{V} = 1\,\mathrm{m^3/h}$. The temperature of the water entering the container with a temperature T_e oscillating in a sinosoidal fashion. The period of oscillation is $t_s = 0.25\,\mathrm{h}$, the mean temperature T_0 and the amplitude $\theta_{\mathrm{e,max}}$.



Task:
Determine the necessary volume V_C of the container such that the temperature of the water leaving the container fluctuates by 0.1 % at most in repsect to a maximum initial fluctuation amplitude of $\theta_{\mathrm{a,max}}$.

Assumptions:

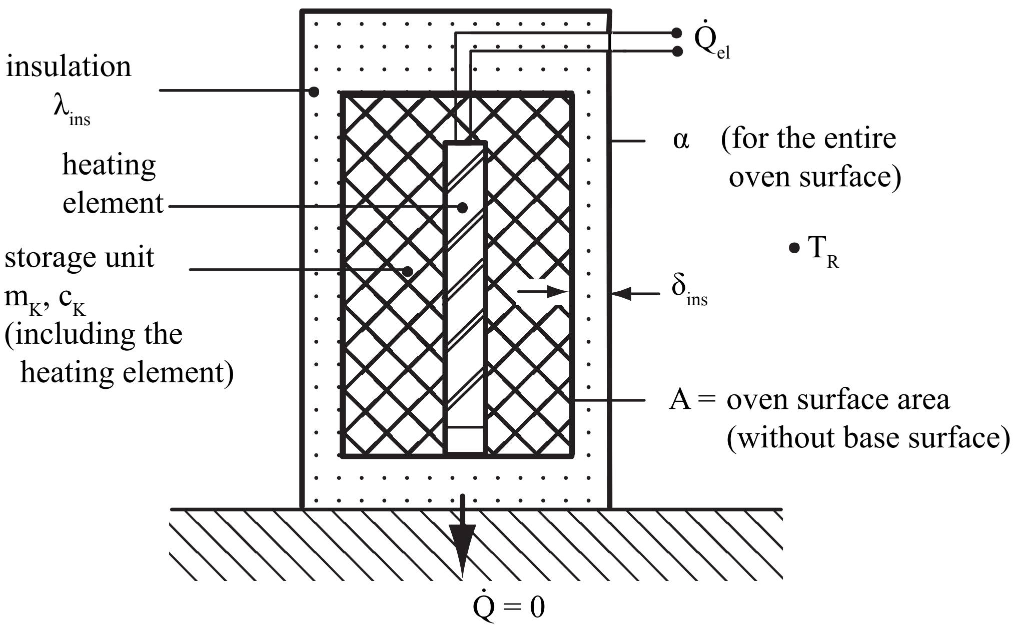
- The liquid within the container is stirred such that a homogeneous temperature is maintained throughout.
- The energy added by the agitator is P. A heat flux $\dot{Q}_\mathrm{W}$ enters the container via its walls from the surroundings.
- Regard the quasi-stationary case, neglecting any instationary phenomena pertaining to the oscillation.

Instructions:
A particular solution of the linear differential equation with constant coefficients can be obtained by using a complimentary function for the perturbation term:

$$\Theta_p = k_1 \sin(\omega t + \varphi) + k_2$$

2.20. Night-storage heater

The follow schematic show the structure of a simple electric night-storage heater. The heater comprises a storage unit, which is surrounded by a layer of insulation (thickness δ_{ins}, thermal conductivity λ_{ins}), and an electric heating element (heating power $\dot{Q}_{\text{el}}$), which is located centrally within the storage unit. The total mass of the storage unit including the heating element is m_{C}, its specific heat capacity is c_{C}.



Known quantities (only valid for tasks c-d):

m_{C}	90	kg	mass of the storage's core inclduing the heating element
c_{C}	1.2	$^{\text{kJ}}/_{\text{kg K}}$	specific heat capacity of the storage's core inclduing the heating element
δ_{ins}	0.04	m	thicnkess of insulation
A	0.8	m^2	total surface area relevant for heat transfer (excluding heater's floor)
α	10	$^{\text{W}}/_{\text{m}^2\,\text{K}}$	heat transfer coefficient at the heater
λ_{ins}	0.08	$^{\text{W}}/_{\text{m K}}$	thermal conductivity of the insulation

Tasks:

a) The oven is operated cyclically. Determine the core temperature $T_\mathrm{C}(t)$ as a function of time (final state) during the heating and cooling cycle, if the warm-up time of the oven is t_L, during which heat is supplied to the core with constant electrical power $\dot{Q}_\mathrm{el}$. The total cycle time is $t_\mathrm{g} = t_\mathrm{L}+t_\mathrm{D} = 24\,\mathrm{h}$ (t_D = Discharge- / Cool-down time). The ambient room temperature T_R, the material properties c_C, λ_ins and the heat transfer coefficient α are known and constant in respect to time. Additional known quantities are m_C, A, δ_ins.
Hint: Set the start of the charging cycle to $t = 0$!

b) Qualitatively sketch the temperature over time for two cycles.

c) Determine the necessary charging time t_L if the amount of heat needed each day is $Q_\mathrm{H} = 13{,}5\,\mathrm{kWh}$ and the power of the heating element is $\dot{Q}_\mathrm{el} = 1.5\,\mathrm{kW}$. Also, give the temperature difference between heating core and ambient temperature at the beginning and end of each charging cycle.
Utilise the values given in the table above.

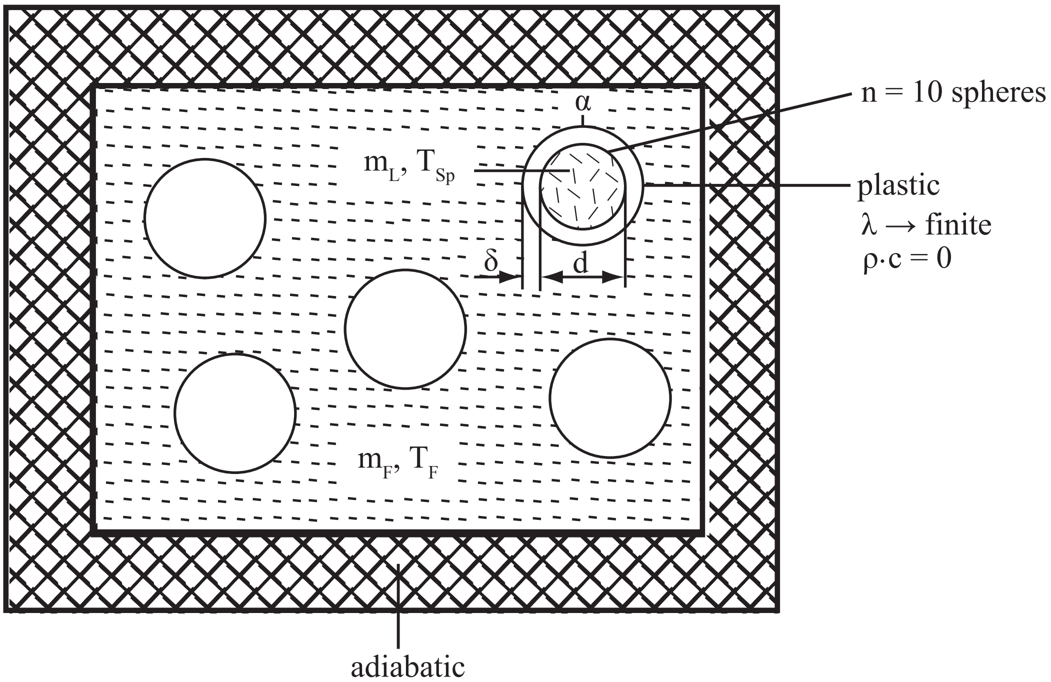
d) Determine the percentage of heat transferred during one cycle of charging and discharging each in respect to the required daily amount of heat Q_H.

Assumptions:

- The thermal resistances of the storage unit including the heating element is negligible in comparison to the other thermal resistances, so that no temperature differences occur in the storage unit.
- The heat capacity of the insulating shell is approximately zero.
- The planar heat flow in the insulating shell must not be neglected, and the surface of the storage unit A (excluding the base) is to be considered as the relevant surface area for heat transfer taking place.

2.21. Ice sphere cooling*

In order to cool a fluid of mass $m_F = 0.5\,\text{kg}$ within an adiabatic container, $n = 10$ identical, hollow plastic spheres are filled with ice of temperature $T_I = 0\,°\text{C}$ and submerged in the fluid. The spheres have a diameter of $d = 20\,\text{mm}$ and a wall thickness $\delta = 0.5\,\text{mm}$. The initial temperature of the fluid is equal to the ambient temperature of $T_A = 20\,°\text{C}$. The mean heat transfer between the surface of the sphere and the fluid is $\alpha = 400\,{}^{\text{W}}/_{\text{m}^2\,\text{K}}$ and constant over time.



Known quantities:

$\rho_F = \rho_W$	1000	$^{\text{kg}}/_{\text{m}^3}$	density fluid/water
$c_F = c_W$	4180	$^{\text{J}}/_{\text{kg K}}$	specific heat capacity fluid/water
ρ_E	917	$^{\text{kg}}/_{\text{m}^3}$	density ice
r	$333 \cdot 10^3$	$^{\text{J}}/_{\text{kg}}$	specific melting enthalpy
λ	0.2	$^{\text{W}}/_{\text{m K}}$	thermal conductivity plastic (sphere)

Tasks:

a) Determine the temperature of the fluid and the contents of the spheres

  – at the point in time t_{m}, when the entire ice has melted (T_{Fm}, T_{ms}), and

  – after a very long time ($T_{\mathrm{F},\infty}$,$T_{\mathrm{K},\infty}$).

b) Find a relation for the time dependent temperature profiles of the spheres' contents $T_{\mathrm{S}}(t)$ and the fluid $T_{\mathrm{F}}(t)$. Begin at the point at which the spheres are submerged in the fluid and end at the point in time at which all ice has melted.

c) Determine the point in time t_{m}, after which there is no more ice contained in the spheres?

d) Pose the differential equations for the temporal evolution of the spheres' and the fluid's temperature in the interval $t_{\mathrm{s}} \leqslant t \leqslant \infty$; also state the initial conditions defining the problem.

e) Determine the equations for the spheres' and the fluid's temperature profiles for $t_{\mathrm{s}} \leqslant t \leqslant \infty$.

f) Qualitatively sketch the temperature profiles as a function in time in the interval $0 \leqslant t \leqslant \infty$.

Assumptions:

- Temperature differences within the liquid and the spheres equalise rapidly.
- The resistance to heat transfer at the spheres' inner surface is negligible.
- The heat capacity of the plastic comprising the spheres and the container in which the set up is contained, shall be neglected.
- As $\delta \ll d$, the curvature of the spheres can be neglected.
- During the melting process the spheres contain a mixture of water and melting ice.

2.22. Contact of semi-infite bodies

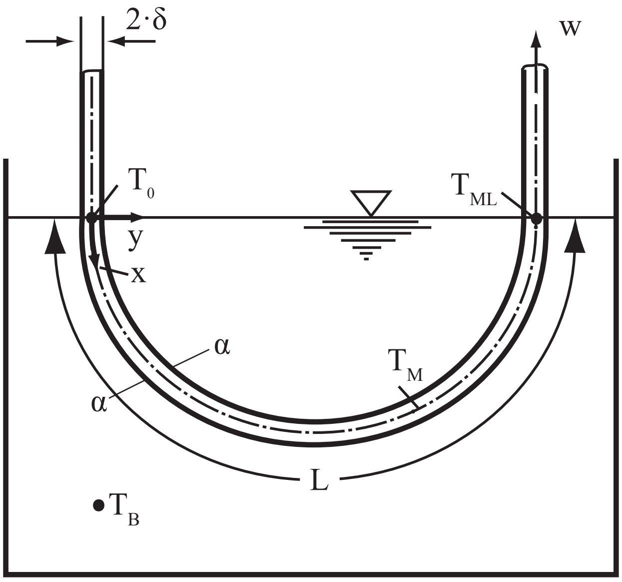
Two semi-infite bodies with different material properties λ, ρ, c and different, but nonetheless homogeneous, temeperatures T_{10} bzw. T_{20} are suddenly put into contact (contact resistance = 0). **Tasks:**

a) How does the temperature develop within both bodies with respect to space and time?

b) Which temperature T_{M} is reached in the contact plane?

2.23. Rolled steel sheet

A rolled sheet of steel of width W and thickness $2 \cdot \delta$ and the material properties λ, ρ, c, is driven through an oil bath of constant temperature T_B wit a constant velocity v. Before entering the bath, the steel sheet has an evenly distributed temperature T_0. The heat transfer coefficient α between the sheet surface and the oil is constant. The leading and trailing edges of the steel sheet are to be considered adiabatic.



Task:
Calculate the velocity v of the sheet, in case that the temperature difference between the center of the steel sheet and the oil bath $T_S - T_B$ after exiting the bath is reduced to half its initial value. Consider the following cases:

a) – No heat conduction in x-direction.
   – The temperature over the sheet's cross-section $2 \cdot \delta \cdot B$ is constant.

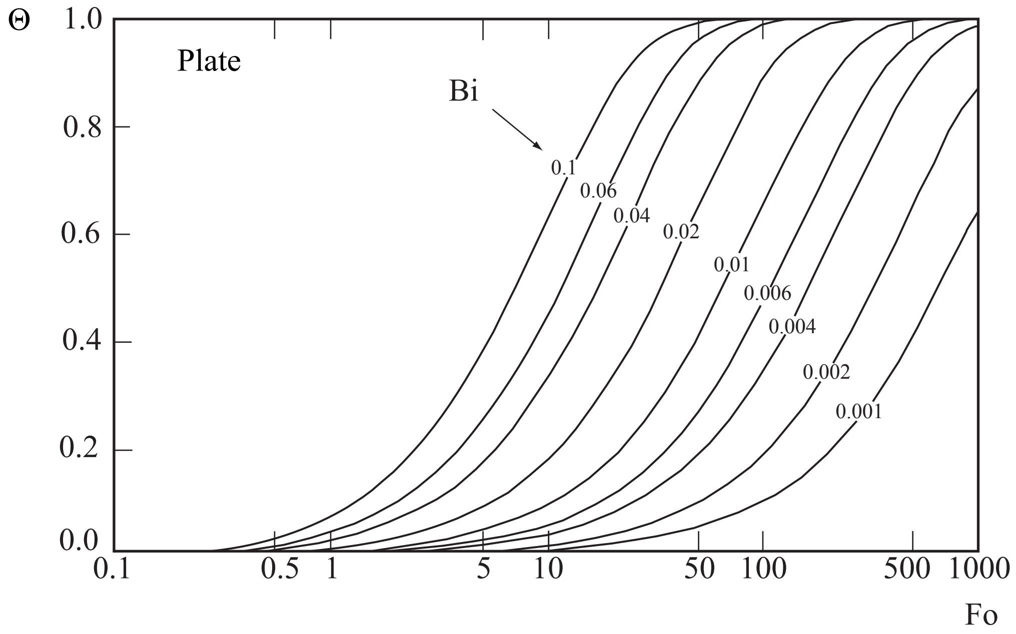
b) – No heat conduction in x-direction.
   – The temperature distribution in y-direction is to be considered (using the following diagramme); with $\Theta = \frac{T(y=0)-T_B}{T_0-T_B}$

Die Krümmung des Stahlbleches kann in beiden Fällen unberücksichtigt bleiben.

Known quantities:

B	1	m	sheet width
$2 \cdot \delta$	20	mm	sheet thickness
L	5	m	submerged length of the sheet
α	400	$^{W}/_{m^2\,K}$	sheet-oil heat transfer coefficient
λ	40	$^{W}/_{m\,K}$	sheet thermal conductivity
c	0.5	$^{kJ}/_{kg\,K}$	sheet specific heat capacity
ρ	8000	$^{kg}/_{m^3}$	sheet density

Hint: Heat loss within the sheet



Chapter 3.

Convection

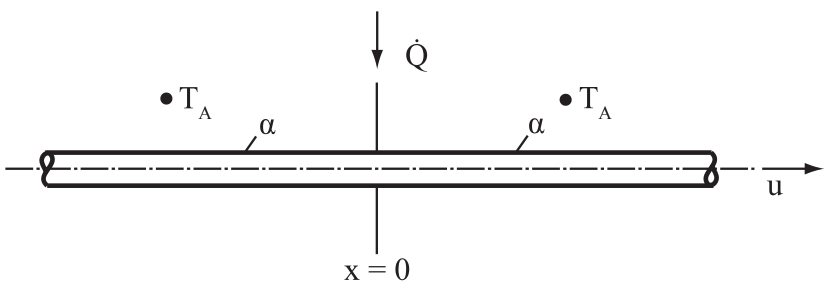
3.1. Hot wire filament*

A wire filament of diameter d is drawn through a fixed heat source with constant veloctity v. The heat flux into the wire aling its circumference is $\dot{Q}$. At a large distance from the heat source, the wire is equal to the ambient temperature T_A. The heat transfer coefficient α between the surface of the wire and its surroundings is known and of constant value. There are no temperature fluctuations in radial direction within the wire. The material properties are assumed to be constant.

Tasks:

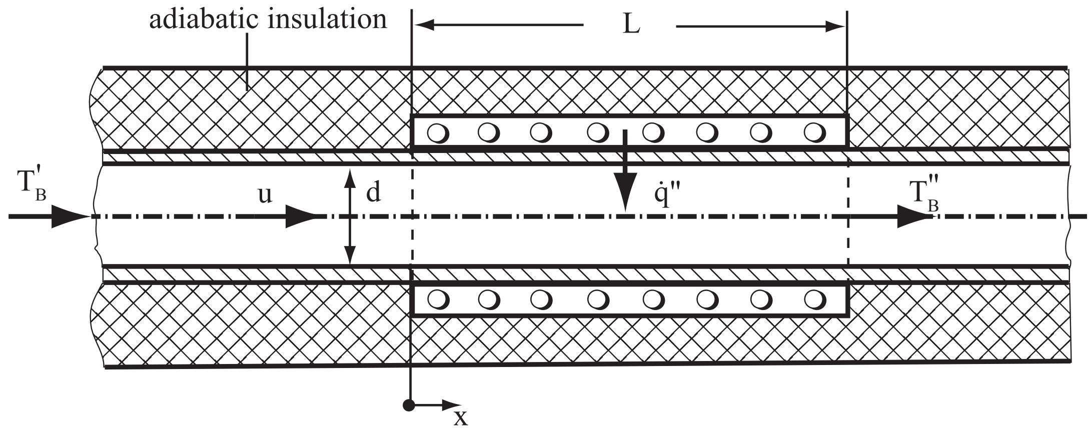
a) Qualitatively sketch the wire's temperature profile before and after it passes the heat source - explain the profile drawn.

b) Derive an equation for the temperature profile in the wire before and after the heat source.



3.2. Lead pipe

In a testing facility molten lead is pumped from a storage container through a pipe of inner diameter $d = 20\,\text{mm}$. The first, very long section of the pipe is adiabatically insulated. The next section of the pipe, length $L = 1\,\text{m}$, is heated with a constant heat flux of $\dot{q}'' = 80\,{}^{\text{kW}}/_{\text{m}^2}$. This section is immediately succeeded by another adiabatically insulated section. The flow velocity is $u = 0.1\,{}^{\text{m}}/_{\text{s}}$. The molten lead inside the storage container has a constant temperature of $T'_\text{B} = 600\,°\text{C}$.



Known quantities:

λ_st	16.7	$^{\text{W}}/_{\text{m K}}$	thermal conductivity lead
ρ	10300	$^{\text{kg}}/_{\text{m}^3}$	density lead
c	147	$^{\text{J}}/_{\text{kg K}}$	heat capacity lead

Tasks:

a) Determine the molten lead's mean exit temperature T''_{L} at the outlet of the heated section.

b) Qualitatively sketch the lead's mean temperature profile, consider the profiles with and without conductive transfer in axial direction, respectively.

c) Derive the equations describing the molten lead's mean temperature profile along the entire length of pipe.

   Approach for the particular solution y_p of the ODE:

$$y'' + ay' + b = 0 : \qquad y_p = -\frac{b}{a} \cdot x$$

d) Determine the ratio of the heat flux which is transmitted thorugh the adiabatic section of piping $x < 0$ through axial conductive transfer in respect to the total amount of heat supplied in the heated section. Discuss the numerical value of your result.

3.3. Flow through wire mesh

In an infinitely extended fluid flow, a thin, well-meshed wire-net is held perpendicular to the direction of flow. THs fluid passes through the mesh with the velocity u. The fluid temperature far away from the mesh is T_0. The wire mesh is electrically heated and emits a constant heat flux per unit area and time $\dot{q}''$ into the fluid. There are no heat sinks behind the wire mesh. **Tasks:**

a) Determine the temperature T_1 the fluid reaches after passing the mesh.

b) Qualitatively sketch the temperature profile of the fluid, consider the case with and without diffusive heat transfer in direction of the flow, respectively. Give both profiles in one diagramme.

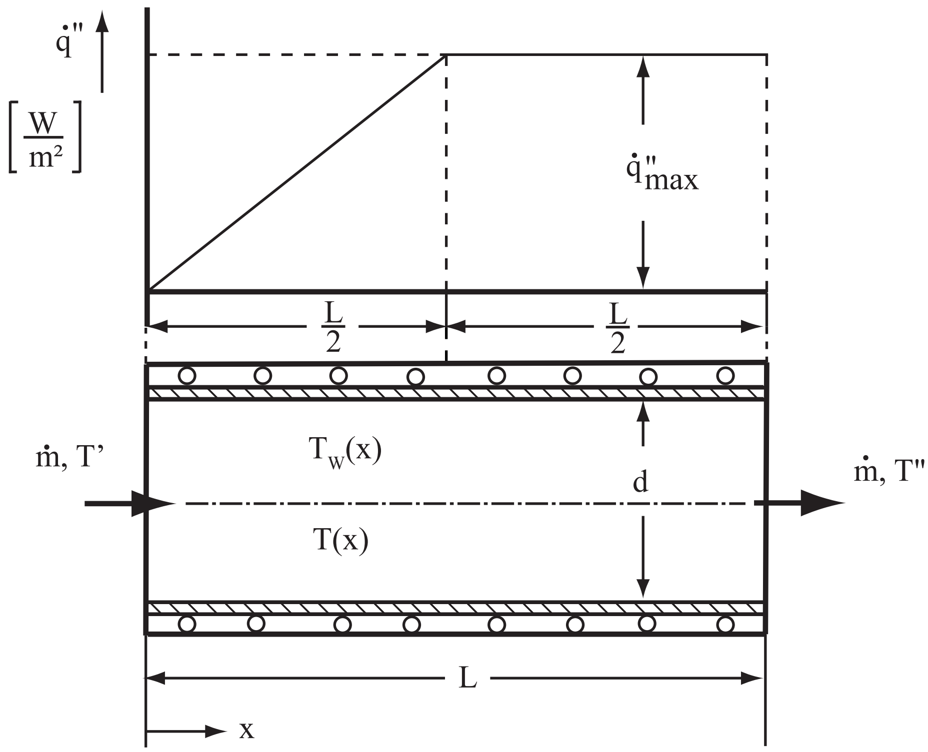
c) Derive an equation for the fluid temperature ahead of the wire mesh while taking into account diffusive heat transfer.

Assumptions:

- The fluid's material properties ρ, c, λ are known and constant.

3.4. Heated pipe

A mass flux $\dot{m}$ (density ρ, spec. heat capacity c, inlet temperature T') passes through a pipe (diameter d, length L). The power distribution of the heated section can be taken from the figure below.



Tasks:

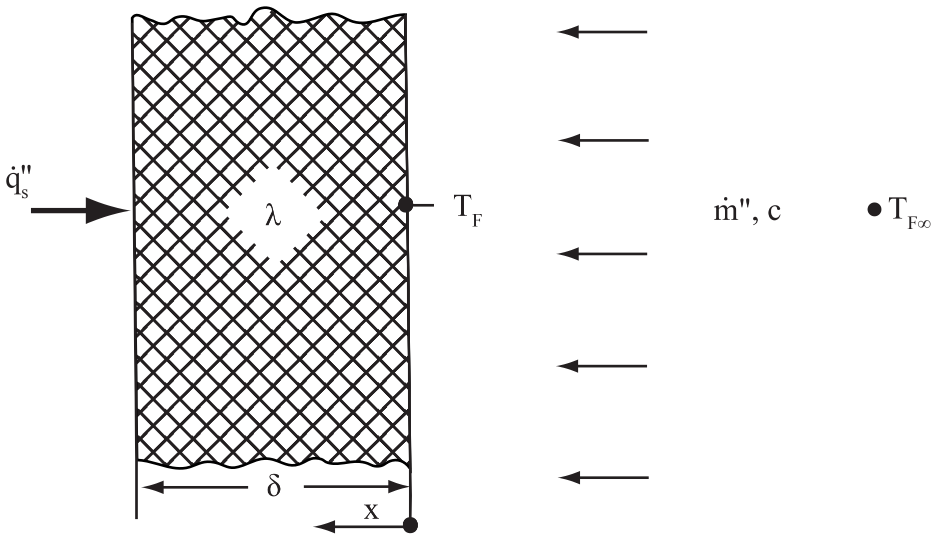
a) Determine the mean pipe outlet temperature T'' of the fluid.
b) Calculate the profile of the mean fluid temperature $T(x)$ in axial direction of the pipe.
c) Present the profile of the pipe's inner wall temperature $T_W(x)$. The heat transfer coefficient α is known and can be considered to be invariant over the pipe's length.
d) Sketch the temperature profiles $T(x)$ and $T_{\mathrm{W}}(x)$ in one diagramme.

Assumptions:

- Axial heat conduction is negligible.

3.5. Absorption in a porous wall*

The surface of a porous wall, impermeable to radiation, absorbs a radiative heat flux of $\dot{q}''_{\mathrm{r}} = 150\,{}^{\mathrm{kW}}/_{\mathrm{m}^2}$ per unit area. The wall is $\delta = 50\,\mathrm{mm}$ and has a thermal conductivity of $\lambda = 8\,{}^{\mathrm{W}}/_{\mathrm{m\,K}}$. For cooling purposes a coolant is circulated through the wall with a specific heat capacity of $c = 1\,{}^{\mathrm{kJ}}/_{\mathrm{kg\,K}}$. The fluid's inlet temperature is $T_{\mathrm{F}} = -15\,°\mathrm{C}$.



Tasks:

a) Determine the temperature profile $T(x)$ for the porous wall.
b) Determine the maximum temperature $T_{\max}$ reached within the wall, if the mass flux per unit area is $\dot{m}'' = 0.6\,{}^{\mathrm{kg}}/_{\mathrm{m}^2\,\mathrm{s}}$.
c) Determine the heat flux $\dot{q}''_{\mathrm{F}}$ per unit area, which is transmitted into the fluid at $x = 0$.
d) Which temperature $T_{\mathrm{F},\infty}$ does the fluid reach far away from the wall?
e) Sketch the temperature profiles for two different mass fluxes and mark each curve.

Assumptions:

- Within the wall the heat flux transmitted thorugh conduction is negligible.
- The local fluid and wall temperatures can be assumed to be identical.

3.6. Vertical pipe

The inner surface of a vertical pipe with an inner diameter of $d = 18\,\mathrm{mm}$ is kept at a temperature of $T_\mathrm{W} = 110\,°\mathrm{C}$ by means of an electric heating. The water's inlet properties are a temperature of$T_0 = 90\,°\mathrm{C}$, a pressure of $p_0 = 1{,}098\,\mathrm{bar}$ and a velocity of $v = 0.8\,\mathrm{m/s}$. At which height in the pipe does the mean water temperature reach the boiling point; assuming that no local boiling takes place at the inner wall's surface?

Assume a mean temperature of $T_\mathrm{mat} = 100\,°\mathrm{C}$ when determining material properties. The pressure drop over the length of pipe due to friction losses in comparison to the hydrostatic pressure drop is negligible.

The boiling temperature depending on the ambient pressure is given in the table below:

p_s [bar]	T_s [°C]
0.9095	97
0.9430	98
0.9776	99
1.0132	100
1.4327	110

The vapour pressure cruve in the relevant interval of pressures can be approximated by the following equation:

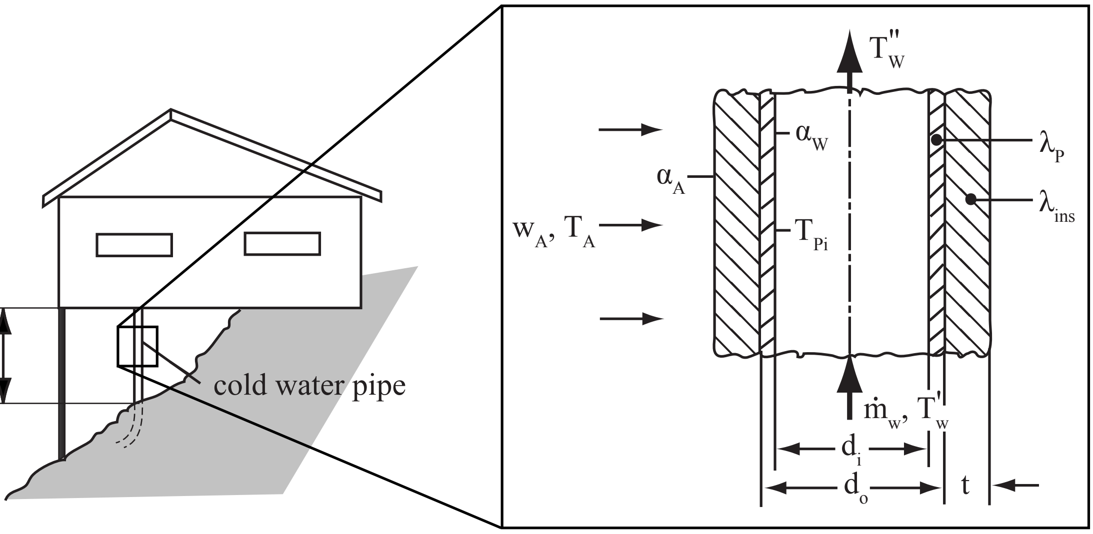
$$T_s = T_\mathrm{W} - A \cdot \exp\left(-\frac{B}{p_\mathrm{W} - p_s}\right)$$

Herein p_W signifies the saturation vapour pressure in bar at a inner pipe wall temperature T_W. The accomodation constants are:

$$A = 37.565\,°\mathrm{C}$$
$$B = 0.56\,\mathrm{bar}$$

3.7. Water mains

The water mains pipe ($d_a = 50\,\text{mm}$ $d_i = 44\,\text{mm}$ $\lambda_R = 60\,\text{W/mK}$) at the slope of a mountain hut hangs freely over a length of$L = 1.54\,\text{m}$. The thickness of the layer of insulation is $s = 20\,\text{mm}$ ($\lambda_{ins} = 0.04\,\text{W/m K}$). The windspeed perpendicular to the pipe is t $w_A = 12\,\text{m/s}$, the ambient air temperature is $T_A = -10\,°\text{C}$. The water has a temperature of $T'_W = 4\,°\text{C}$ at the pipe's inlet.



Known quantities:

λ_L	0.024	W/m K	thermal conductivity air
ν_L	$13 \cdot 10^{-6}$	m²/s	kinematic viscosity air
Pr_L	0.71		Prandtl number air
λ_W	0.57	W/m K	thermal conductivity water
c_W	4200	J/kg K	specific heat capacity water
ρ_W	1000	kg/m³	density water

Tasks:

a) Determine the minimum mass flux necessary to ensure that at no place of the pipe icing occurs. Assume steady-state conditions.

b) Quantify the mass flux for a thermally and and hydrodynamically undeveloped flow. Explain your answer.

Assumptions:

- Axial heat flux within the pipe and the fluid is negligible.
- Arithmetic averaging is sufficient for the purpose of determining the mean temeprature gradient between water and air in the axial direction of the pipe.
- Laminar, hydrodynamically und thermally developed flow.
- The wall temperature can be assumed to be constant for the purpose of deteriming the heat transfer coefficients.

3.8. Heat transfer for a heated plate

A plate measuring $50 \times 50\,\mathrm{cm}^2$ is heated to a constant temperature of $T_\mathrm{P} = 100\,°\mathrm{C}$. From both sides an airflow with the following properties passes over the plate: $T_\mathrm{A} = 20\,°\mathrm{C}$, $p_\mathrm{A} = 1\,\mathrm{bar}$ and $v_\mathrm{A} = 20\,\mathrm{m/s}$.

Tasks:

a) Determine the local heat transfer coeffiecient $\alpha(L)$ at the plate's trailing edge as well as the mean heat transfer coefficient α under the assumption of a laminar flow, in addition to determining both for a turbulent flwo beginning at the plate's leading edge.

b) Which heat flux $\dot{Q}$ is emitted in each case?

c) Determine the amount of heat transfered into the fluid assuming the flow regime is laminar up until a critical Reynolds number of $\mathrm{Re}_\mathrm{crit} = 200\,000$, after which the flow suddenly becomes turbulent.

Assumptions:

- At the onset of the turbulent flow regime the heat transfer coefficient is equal to that which is obtained for a turbulent flow at the palte's leading edge.

Chapter 4.

Radiation and convection

4.1. Thermocouple

Air of temperature $T_{\mathrm{L}} = 100\,°\mathrm{C}$ and pressure $p_{\mathrm{L}} = 1\,\mathrm{bar}$ flows thorugh a pipe of $D = 20\,\mathrm{mm}$, which has a wall temperature of $T_{\mathrm{W}} = 80\,°\mathrm{C}$, with a velocity of $v_{\mathrm{A}} = 2\,\mathrm{m/s}$. A thermocouple of diameter $d = 1\,\mathrm{mm}$ is built into the pipe; the thermocouple is oriented axially to the direction of flow. **Known quantities:**

ϵ_{W}	0.8	emissivity pipe wall
ϵ_{Th}	0.6	emissivity thermocouple
ϵ_{S}	0.04	emissivity protective cladding, mirror coating
ϵ_{S}	0.06	emissivity protective cladding, no coating

Tasks:

a) Determine the temperature T_{Th} reached by a thermocouple without protective cladding.

b) Through which means can the measuring error be reduced?

c) Which temperature T_{Th} is found for a thermocouple with protective cladding? The cladding comprises a mirror coated tube of length $l' = 30\,\mathrm{mm}$ and diameter $d' = 10\,\mathrm{mm}$ while assuming a very low wall strength.

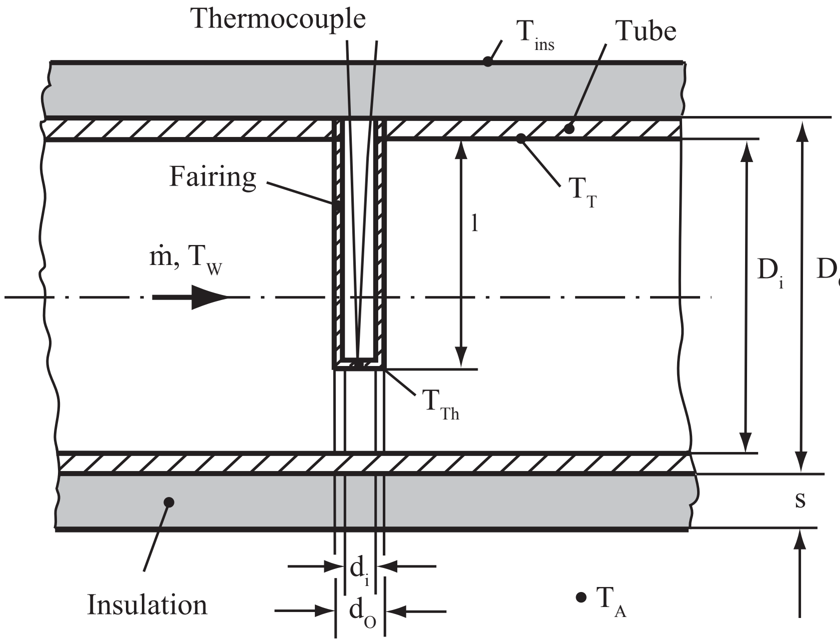
d) Which thermocouple temperature T_{Th} is obtained if the cladding is not mirror coated?

Assumptions:

- Heat losses from the pipe outwards are of such low magnitude that variations of the ambient temperature can be disregarded.
- The view factors between thermocouple and pipe wall, thermocouple and protective cladding, as well as the cladding and the pipe wall can be assumed to be approximately of unity.
- Both the thermocouple and the cladding can be regarded as two flat plates subjected to a paralell flow for the purposes of approximating the heat transfer coefficients. The developing boundary layers shall not intermingle. The straight flow length of the thermocouple be $l = 10\,\mathrm{mm}$.

4.2. Fairing in a pipe*

Water flows through a pipe hanging freely within a room and insulated with glass wool. The water's temperature is recorded by means of a thermocouple. The thermocouple is held within a nozzle which is also fashioned from copper. The thermocouple and the nozzle are in firm contact. The nozzle is perpendicularly soldered to the pipe's inner wall.



Known quantities:

D_i	50	mm	inner pipe diameter
D_O	54	mm	outer pipe diameter
t	10	mm	insulation thickness
d_i	6	mm	inner diameter thermocouple
d_a	8	mm	outer diameter thermocouple
$\dot{m}_\mathrm{W}$	0.2	kg/s	water mass flux
T_W	80	°C	mean water temperature
T_L	15	°C	ambient temperature
λ_iso	0.046	$\mathrm{W}/\mathrm{m\,K}$	thermal conductivity insulation
ϵ_iso	0.9	-	emissivity insulation

The room's walls are of approximately ambient temperature and their surface area is much larger that than of the insulated pipe.

Tasks:

a) Determine the necessary nozzle length l to obtain a measuring error of $T_{\mathrm{W}} - T_{\mathrm{Th}} \leqslant 0.05\,\mathrm{K}$.

b) Diskutieren Sie Maßnahmen für den Stutzeneinbau zur Einhaltung bzw. Verkleinerung des Messfehlers, wenn sich bei der unter a) betrachteten Anordnung eine Eintauchlänge von $l \geq D_{\mathrm{i}}$ ergibt.

Assumptions:

- The calculation of the pipe's inner wall temperature is done without taking into account the nozzle containing the thermocouple.
- The nozzle containing the thermocouple has the same temperature as the wall where it is joined to the wall.
- The heat flux through the inner mantle area of the nozzle, the nozzle's opposing end and the wire connecting the thermocouple is negligible.
- The thermocouple returns the nozzle's temperature.
- Fully developed pipe flow.

4.3. Methanol tank*

Methanol flows from a tank through a pipe of length $L = 18\,\mathrm{m}$, inner diameter $d_\mathrm{i} = 2\,\mathrm{cm}$ and outer diameter $d_\mathrm{a} = 2.4\,\mathrm{cm}$ into a heat exchanger at a rate of $\dot{m} = 1600\,\mathrm{kg/h}$. The pipe is held paralell to the ground of the room with an ambient temperature of $T_\mathrm{A} = 20\,°\mathrm{C}$. The copper pipe is insulated with a $s = 2\,\mathrm{cm}$ layer of cork. The methanol's temperature in the tank and at the heat exchanger inlet are $T_1 = -32.8\,°\mathrm{C}$ and $T_2 = -32.2\,°\mathrm{C}$, respectively.

Known quantities:

ρ_M	850	$\mathrm{kg/m^3}$	density methanol
λ_M	0.221	$\mathrm{W/m\,K}$	thermal conductivity methanol
c_M	2.26	$\mathrm{kJ/K}$	specific heat capacity methanol
η_M	$2.2 \cdot 10^{-3}$	$\mathrm{kg/m\,s}$	dynamic viscosity methanol
λ_K	0.043	$\mathrm{W/m\,K}$	thermal conductivity cork
α_a^*	10	$\mathrm{W/m^2\,K}$	approx. total heat transfer coefficient outer side of pipe

Task:

a) Does the insulation correctly fulfill its purpose or does infiltration of water occur, reducing the insulations effectiveness?

Hinweis:

Do not neglect influences due to radiation.

4.4. Air gap

Regard an infinitely large air gap between two flat planes, horizontally opposed. The surface temperature of the upper plate is $T_1 = 40\,°\mathrm{C}$, that of the lower plate $T_2 = 20\,°\mathrm{C}$. The temperatures are constant over time. All srufaces have an emissivity of $\epsilon = 0.1$. The air gap has a thickness of $\delta = 50\,\mathrm{mm}$. **Tasks:**

a) Determine the apparent thermal conductivity λ_G of the air gap, while considering the influence of radiation.

b) Determine the apparent thermal conductivity λ_G of the air gap, in case a thin radiatively permeable piece of foil is place in the middle of the gap. The emissivity is equal to that of the plates' surfaces.

   Which temperature T_F is obtained for the foil?

c) Derive the relationship for the apparent thermal conductivity in the air gap, in case the gap is divided by n pieces of equidistantly spaced foil. The properties of the foil are identical to those in b).

Hint:
The apparent thermal conductivity in the gap λ_G is obtained for the case when the total heat flux under consideration of all phenomena is instead transmitted through a replacement quantity, i.e. the apparent thermal conductivity.

4.5. Ventilator*

An air mass flux of $\dot{m}_\mathrm{A} = 0.5\,{}^\mathrm{kg}/_\mathrm{h}$ is transported though a plastic tube attached to the outlet of a ventilator; the dew point temperature is $T_\mathrm{dew} = 35\,°\mathrm{C}$. The tube comprises an inner part through which the air flows and an outer part which is insulated with foam. To prevent condensation at the tube's inner wall the air is to be heated so that a temperature difference of 1 K between the wall temperature and the dew point is maintained at all times. The pipe is oriented horizontally and is surroundend by ambient air of $T_\mathrm{A} = 20\,°\mathrm{C}$.

Known quantities:

d_i	10	mm	inner diameter tube
d_a	15	mm	outer diameter tube
D	60	mm	outer diameter insulation
L	1.5	m	tube length
$\rho_\mathrm{L,U}$	1.18	${}^\mathrm{kg}/_\mathrm{m^3}$	density ambient air
$\nu_\mathrm{L,U}$	$15.5 \cdot 10^{-6}$	${}^\mathrm{m^2}/_\mathrm{s}$	kinematic viscosity ambient air
$c_\mathrm{L,U}$	1000	${}^\mathrm{J}/_\mathrm{kg\,K}$	specific heat capacity ambient air
$\lambda_\mathrm{L,U}$	0.026	${}^\mathrm{W}/_\mathrm{m\,K}$	thermal conductivity ambient air
$\mathrm{Pr}_\mathrm{L,U}$	0.71	−	Prandtl-number ambient air
$\rho_\mathrm{L,S}$	1.0	${}^\mathrm{kg}/_\mathrm{m^3}$	density airflow
$\nu_\mathrm{L,S}$	$20 \cdot 10^{-6}$	${}^\mathrm{m^2}/_\mathrm{s}$	kinematic viscosity airflow
$c_{L,S}$	1000	${}^\mathrm{J}/_\mathrm{kg\,K}$	specific heat capacity airflow
$\lambda_\mathrm{L,S}$	0.029	${}^\mathrm{W}/_\mathrm{m\,K}$	thermal conductivity airflow
$\mathrm{Pr}_\mathrm{L,S}$	0.71	−	Prandtl-number airflow
λ_S	0.2	${}^\mathrm{W}/_\mathrm{m\,K}$	thermal conductivity hose
λ_iso	0.04	${}^\mathrm{W}/_\mathrm{m\,K}$	thermal conductivity insulation
η_iso	0.95	−	emissivtiy insulation
α_a	9	${}^\mathrm{W}/_\mathrm{m^2\,K}$	approx. total heat transfer coefficient outer side of hose

Tasks:

a) Determine the heat transfer coefficient α_i at the tube's inner wall as well as the thermal transmittance coefficient k between the airflow and the ambient air with regards to the inner tube wall. Disregard any corrections for viscous effects.

b) Give the equation describing the mean airflow temperature profile and outline the relationship between the airflow and the inner tube wall temperatures at the critical point. Determine the equation describing airflow inlet temperature T'_A under the condition that no condensation occurs.

   Give the airflow inlet temperature T'_A under the conditions stated above.

c) Determine the net heat loss of the airflow $\dot{Q}_L$.

   Check if the mean total heat transfer coefficient α_o at the outer wall was correctly determined. The temperature difference between the tube's surfaces in contact with the ambient air and the ambient air is marginal.

d) After which length of tube L_0 would an electric heating element be required if the inlet temperature were limited to $T'_A* = 60\,°C$ while maintaining the condition of no condensation?

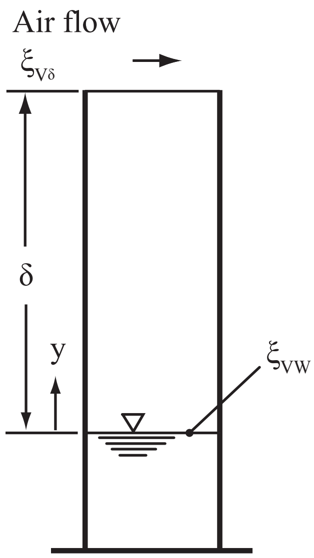
   Give both the minimum electric heating power $\dot{Q}_{el}$ required and the overall heat loss $\dot{Q}_{L,tot}$, again considering the given conditions. The thermal transmissivity k and the heat transfer coefficient α_i can be assumed to be approximately constant over time, as in task a).

Chapter 5.

Mass transfer

5.1. Glass tube

A layer of water with a surface area of $A = 5\,\mathrm{cm}^2$ is contained within a vertical glass tube. The height of the tube from the water's surface to the end of the tube is $\delta = 64\,\mathrm{mm}$. Measurements yield a temperature of $T = 30.8\,°\mathrm{C}$, a total pressure of $p = 0.98\,\mathrm{bar}$ and a rate of evaporation of $25.5\,\mathrm{mg}$. The air surrounding and flowing over the tube does not contain any water.



Known quantities:

p_s	0.0445	bar	saturation pressure at 30.8 °C
r	2429	$^{\mathrm{kJ}}/_{\mathrm{kg}}$	spec. heat of evaporation
R_{D}	461	$^{\mathrm{J}}/_{\mathrm{kgK}}$	gas constant of water
R_{L}	287	$^{\mathrm{J}}/_{\mathrm{kgK}}$	gas constant of air

Tasks:

a) Determine the mass concentration profile of the water vapour in the tube and derive the equation relating the mass flux of evaporated water and the bounding values of the mass concentration.

b) Determine the ratio of the mass transfer coefficient with and without the so-called *Stefan correction*.

c) Calculate the diffusivity of water vapour in air under the given conditions using the data given below.

5.2. Damp wood

A large, $\delta = 75\,\mathrm{mm}$ thick slab of wood contains a water mass fraction of $\xi_0 = 0.218$. If an equilibrium with its surroundings were reached the water mass fraction would be reduced to $\xi_\mathrm{U} = 0.064$. The diffusion coefficient for water in wood is $D = 1.2 \cdot 10^{-9}\,\mathrm{m^2/s}$.

Tasks:

a) Derive the differential equation for the temporal evulotion of the water mass fraction within the wooden slab, while neglecting edge effects.

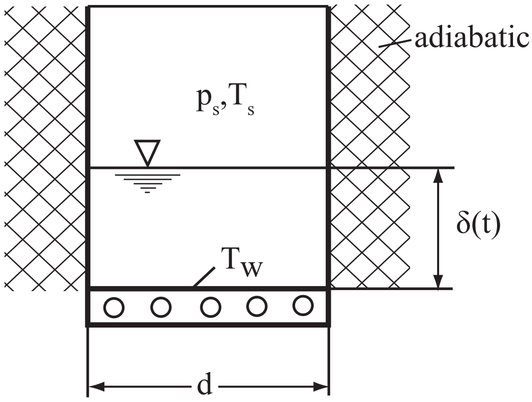
b) Determine the amount of time necessary for the water mass fraction in the centre of the slab to decrease to $\xi_\mathrm{m} = 0.08$.

Hint:

- Consider possible analogies to heat transfer.
- $g \to \infty$

5.3. Condensation of steam

A cylindrical receptacle with a flat base and a diameter of $d = 0.3\,\mathrm{m}$ stands on a flat surface. Through cooling channels in the receptacle's base the temperature is constantly maintained at $T_\mathrm{W} = 20\,°\mathrm{C}$.



The receptacle is suddenly exposed to an air-free atmosphere of saturated water vapour at $p_s = 0.2\,\mathrm{bar}$ and $T_s = 60\,°\mathrm{C}$. The specific enthalpy of evaporation of water under these conditions is $r = 2358\,{}^{\mathrm{kJ}}/_{\mathrm{kg}}$.

Task:
Determine the condensate mass m^*_con accumulated at the receptacle's base after $t^* = 1\,\mathrm{h}$.

Assumptions:

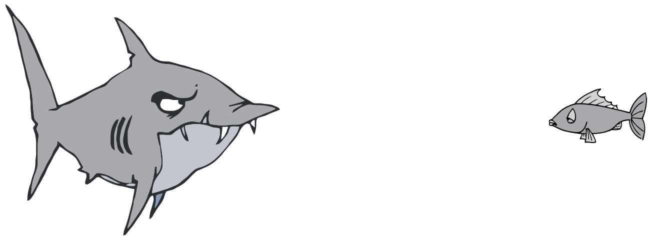
- No condensation at the recptacle's mantle.
- No heat transfer resistance ath the vapour/liquid interface.
- The change in internal energy over time through the formation of a condensate layer is negligible ($\rightarrow$ quasi-stationary temperature profile within the condensate).

Hint:

- T_s is constant.

5.4. Shark attack on Mallorca*

A fish is injured and suffers from blood loss. It is therefore exposed to the possibility of being detected by a shark and eaten. The fish is situated in still water and does not move, thus convective influences on the distribution of blood in the surrounding water are negligible. The shark is able to sense blood in the water as soon as it exceeds a critical concentration of ξ_{crit}.



Tasks:

a) Derive the general transport equation for blood in water for spherical coordinates; use a differential mass balance. Take into regard diffusion, instationary influences and mass sources/sinks, but disregard convection. The relationship is independent of the chosen angle.

b) Explain the analogy between heat and mass transfer by referring to the equation derived in a). Compare the equation derived in a) to the differential equation for the temperature field in spherical coordinates taken from the lecture script:

$$\rho \cdot c \cdot \frac{\partial T}{\partial t} = \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \lambda \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\lambda \sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial \Phi} \left(\lambda \frac{\partial T}{\partial \Phi} \right) \right) + \dot{\Phi}'''$$

Make a table comparing all occuring/analogous variables with each other.

c) In which distance r_{crit} from the fish is the shark able to detect blood in the water? Assume a spherical fish of radius r_{F} and steady-state condictions as well as an isotropic blood loss. At radius r_{F} a blood mass fraction of is ξ_{F} found. State the relationship for the distance r_{K} for all quantities given above.

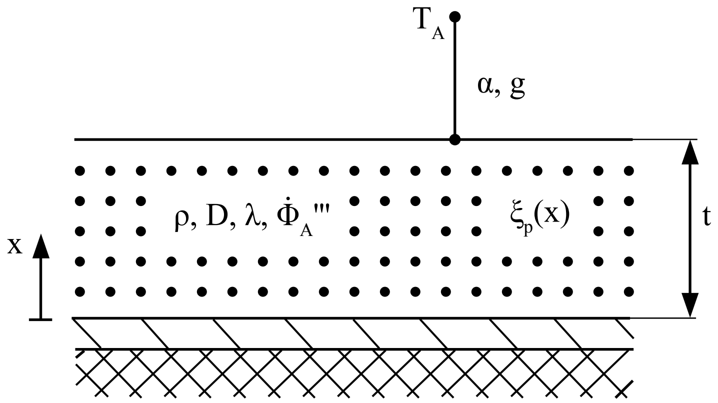
d) The compound found in blood detectable by sharks is an amino acid known as Serine ($C_3H_3(OH)(NH_2)(OOH)$). The Serine mass fraction in blood is ξ_S. The material properties of blood are identical to those of water. Determine – by utilising the following table – the blood mass fraction ξ_F a the fish's skin for the given critical mass fraction ξ_{crit}.

$D_{Serin,Wasser}$	$1.78 \cdot 10^{-7}$	m^2/s	Diffusion coefficient Serine in water
ξ_K	$1.05 \cdot 10^{-15}$	kg_S/kg	Critical blood concentration
$\dot{m}_B$	$2.78 \cdot 10^{-6}$	kg_B/s	Blood mass flux
r_F	0.106	m	Radius of the fish
ρ_{ges}	999	kg/m^3	Densities blood and water
r_K	400	m	Critical distance to the fish
ξ_S	$1.05 \cdot 10^{-4}$	kg_S/kg_B	Mass fraction Serine in blood

5.5. Even more critical explosive*

An explosive as in 2.16 is stored in the shape of a flat plate. In addition to overheating as a cause for detonation, contamination of the explosive can lead to a runaway chemical reaction causing the explosive to detonate. The removal of the upper metal plate makes such a contamination possible. The mass transfer of the explosive into the surrounding air is described by the mass transfer coefficient g. Removing the upper metal plate also invalidates the heat transmittance coefficient k, instead the heat transfer is now characterised by the heat transfer coefficient α. The mass fraction of the reaction product may never exceed a certain threshold as this product does not diffuse within the explosive, thus rendering it unusable. The temperature profile in the explosive is given as:

$$T - T_\mathrm{U} = \frac{1}{\gamma} \cdot \frac{\cos(m \cdot x)}{\cos(m \cdot s) - \frac{\lambda}{\alpha} \cdot m \cdot \sin(m \cdot s)} - \frac{1}{\gamma} \quad \text{mit } m = \sqrt{\frac{\gamma \cdot \dot{\Phi}_\mathrm{U}'''}{\lambda}}$$



Known quantities:

$\dot{\Phi}'''$	$= \dot{\Phi}'''_{\mathrm{U}} \cdot (1 + \gamma(T - T_{\mathrm{U}}))$		Heat production through reaction
$\dot{\Phi}'''_{\mathrm{U}}$	0.3	$\mathrm{W/m^3}$	Source term
γ	0.21	$\mathrm{1/K}$	
T_{U}	20	°C	Ambient temperature
λ	0.85	$\mathrm{W/m\,K}$	Thermal conductivity explosive
Δh	7000	$\mathrm{kJ/kg_p}$	Reaction enthalpy
α	3	$\mathrm{W/m^2\,K}$	Heat transfer coefficient
ρ	3000	$\mathrm{kg_{ges,SP}/m^3}$	Density explosive
D	10^{-9}	$\mathrm{m^2/s}$	Coefficient of diffusivity of the reaction product within the explosive
g	0.001	$\mathrm{kg_{ges,SP}/m^2 s}$	Mass transfer coefficient
$\xi_{\mathrm{P,krit}}$	0.001	$\mathrm{kg_P/kg_{ges,SP}}$	Critical concentration

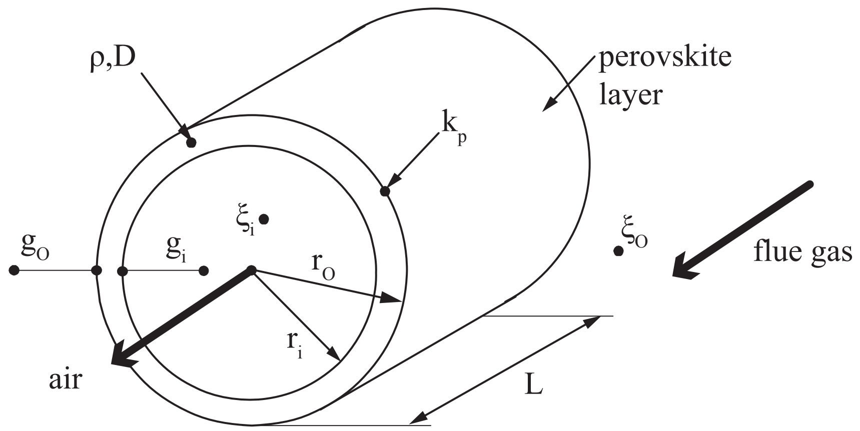
Task:

a) Give the maximum thickness t of the explosive to ensure that does not become contaminated.

5.6. Perowskite*

Some ceramic materials, so-called Perovskites, can be emloyed to enrich flue gasses with oxygen. To achieve this thin perovskite layers are applied to the outer sides of short pipes serving as a carrier. Air is circulated through the pipes. Both oxygen and nitrogen are able to permeate the carrier material, but only oxygen is able to permeate the perovskite layer.

The effective net transport of oxygen through the carrier material is given by the diffusion coefficient D (determined without a perovskite layer) and D^{St} (determined with the perovskite layer). These need to be related to the density ρ. Hot flue gasses circulate around the pipes' outer surfaces; the flue gasses are unable to permeate the perovskite.



Known quantities:

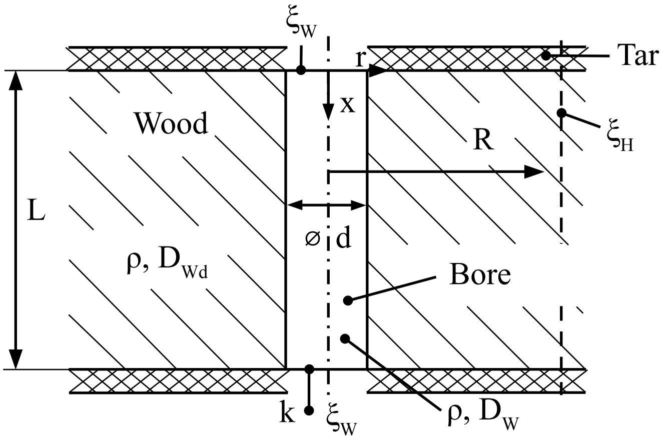
r_i	0.005	m	Inner radius
r_o	0.006	m	Outer radius
k_p	$5 \cdot 10^{-4}$	$\mathrm{kg_L/m^2\,s}$	mass transport coefficient perovskite layer
ρ	1,25	$\mathrm{kg_L/m^3}$	Density [1]
D	$4 \cdot 10^{-7}$	$\mathrm{m^2/s}$	Coefficient of diffusivity without perovskite layer
D^St	$4{,}6 \cdot 10^{-7}$	$\mathrm{m^2/s}$	Coefficient of diffusivity with perovskite layer
ξ_i	0.232	$\mathrm{kg_{O_2}/kg_L}$	mass fraction oxygen (air)
ξ_o	0.08	$\mathrm{kg_{O_2}/kg_{RG}}$	mass fraction oxygen (flue gas)
ρ_RG	1.6	$\mathrm{kg_{RG}/m^3}$	Density flue gas
g_i	0.007	$\mathrm{kg_L/m^2s}$	mass transport coefficient (inner, no Stefan correction)
g_o	0.003	$\mathrm{kg_L/m^2s}$	mass transport coefficient (outer, no Stefan correction)
L	.,05	m	Length of pipe

Tasks:

a) Determine the oxygen mass flux directed into the flue gas? Neglect the Stefan correction.

b) Again determine the mass flux, this time take into account the Stefan correction.

[1] Although the density ρ is chosen for the calculation of the diffusivity in the carrier material, said density is not the total density of the carrier material including trapped air, but that of pure air. This allows the employment of mass fractions ξ, which are all related to the mass of air kg_A. For this reason k_p, g_i and g_o are also related to the mass of the surrounding air. The fact that oxygen diffusing through nitrogen in the carrier material behaves differently than oxygen diffusing through nitrogen alone is taken into account by the different coefficients of diffusivity D and D^St.



5.7. Tarred railway sleeper*

A wooden railway sleeper of thickness L has a bore of diameter d and lays in a puddle of water. The wood is saturated with naphthalene (mass fraction ξ_S), because the sleeper is covered with a thin layer of tar on all sides. The naphthalene in the sleeper is tranported to the bore hole through convective and diffusive transport and then forms a solution with the surrounding water. The mass transfer coefficient of the sleeper at its underside between the lower opening of the bore and place at which ambient conditions, i.e. $\xi_W = 0$, is k. At the sleeper's upper side ambient conditions are reach immediately above its surface. The total density of the sleeper and the water within the bore is ρ. **Known quantities:**

L	0.2	m	Sleeper thickness
d	0.01	m	Bore diameter
ξ_H	10^{-4}	$^{kg_N}/_{kg}$	Naphthalene mass fraction (wood)
k	0.005	$^{kg}/_{m^2\,s}$	Sleeper mass transfer coefficient (underside)
ρ	1000	$^{kg}/_{m^3}$	total density sleeper and water
D_S	$3 \cdot 10^{-10}$	$^{m^2}/_{s}$	Coefficient of diffusivity naphthalene in wood
D_W	$7 \cdot 10^{-6}$	$^{m^2}/_{s}$	Coefficient of diffusivity naphthalene in water
R	0.08	m	Distance from the axis of the bore at which saturated mass fraction is reached

Tasks:

a) Calculate the Biot number. Can ξ be assumed to be constant in radial direction of the bore?

b) Quantify the total naphthalene mass flux leaking from the bore.

c) Quantify the mass flux for the case that the sleeper is perpendicularly immersed in water.

d) Quantify the mass flux under the assumption that the lower end of the bore is impermeable to naphthalene.

e) Qualitatively sketch the function $\xi(x)$ for each of the three cases presented in subtasks b) to d).

Hints:

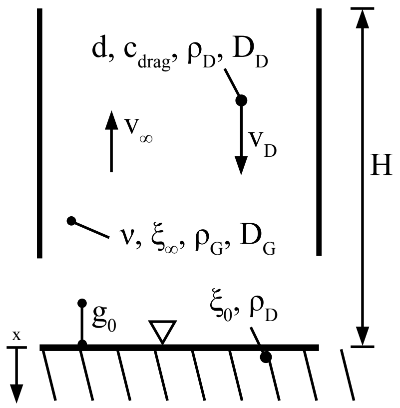
- Assume stead-state conditions.
- No convective transport within the bore.
- No transport of naphthalene in axial direction within the sleeper.

Assumptions:

- The concentration's profile at the interface between wood and water is continuous.
- The influence of the naphthalene concentration on the overall densities is negligible. The coefficients of diffusivity of naphthalene in wood and water, respectively, are D_{S} and D_{W}.
- The bore's diameter is small in comparison to the sleeper's dimensions. Approximately it can be assumed that the saturation concentration ξ_{S} is reached after a distance R from the bore's axis in the wood.

5.8. Scrubber*

Gaseous ethanol containing traces of CO_2 flows through a multistage scrubber to remove the CO_2; for the purposes of this task only a single stage is regarded. The gas flows upwards through a vertical pipe at a velocity of v_∞ while spherical droplets of the scrubbing solvent fall in the opposing direction. The solvent is gathered in a wet sump and reclyced. At the point of injection the solvent contains no CO_2, the CO_2 mass fraction in the free flow is ξ_∞.



Known quantities:

d	10^{-3}	m	Droplet diameter
c_D	0.4	–	Drag coefficient
v_∞	0.05	$\mathrm{m/s}$	Gas velocity
H	0.2	m	Pipe length
g_E	9.81	$\mathrm{m/s^2}$	Gravitational constant
ρ_G	1.86	$\mathrm{kg_G/m^3}$	Gas density
ρ_T	1020	$\mathrm{kg_T/m^3}$	Droplet density
ν	$4.63 \cdot 10^{-6}$	$\mathrm{m^2/s}$	Kinematic viscosity gas
ξ_∞	0.05	$\mathrm{kg_{CO_2}/kg_G}$	Free flow CO_2 mass fraction
D_T	10^{-5}	$\mathrm{m^2/s}$	Coefficient of diffusivity CO_2 in solvent
D_G	$5.4 \cdot 10^{-6}$	$\mathrm{m^2/s}$	Coefficient of diffusivity CO_2 in ethanol
ξ_0	10^{-5}	$\mathrm{kg_{CO_2}/kg_T}$	Initial CO_2 mass fraction in sump
g_0	0.0032	$\mathrm{kg_G/m^2\,s}$	Mass transfer coefficient in sump

Tasks:

a) Determine the terminal velocity v_T of a droplet with diameter d relative to the gas flow. The gas density is neglible in comparison to that of the droplet.

b) Which mass of CO_2 is dissolved an typical droplet (as described in a)) during its fall? The acceleration pahse of the droplet is short in comparison to its time of residence in the pipe and can therefore be neglected.

c) During scrubber shutdown a homogeneous CO_2 mass fraction ξ_0 is reached in the scrubber. After shutdown the sump is no longer actively mixed so that diffusion is the only relevant transport mechanism. Thus the CO_2 transfer is described by the mass transfer coefficient g_0. Which CO_2 mass fraction ξ_S is obtained 10 seconds after shutdown at a depth of $0.005\,\mathrm{m}$?

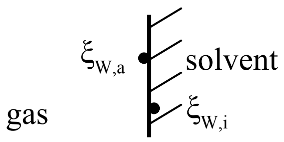
Hinweise:

- Disregard convective influences within individual droplets.
- Due to the small CO_2 mass fractions in the system overall a Stefan correction is obsolete.
- As only a single stage of the scrubber is regarded, ξ_∞ can be assumed to be constant over the total distance traveled by each droplet. This also applies to subtaks c).
- The drag coefficient is given as

$$c_D = \frac{F_D}{A \cdot \frac{1}{2} \cdot \rho_\mathrm{G} \cdot \nu^2} \tag{5.1}$$

  wherein F_D is the induced force of drag and A is the effective exposed area.
- The Henry constant h for the transport CO_2 over the interface of gas and liquid is 1, so that the following relationship at the interface is obtained for the CO_2 mass fractions in the gas ($\xi_\mathrm{W,a}$) and the solvent ($\xi_\mathrm{W,i}$):

$$\xi_\mathrm{W,i} = h \cdot \tfrac{\rho_\mathrm{G}}{\rho_\mathrm{T}} \cdot \xi_\mathrm{W,a} = \tfrac{\rho_\mathrm{G}}{\rho_\mathrm{T}} \cdot \xi_\mathrm{W,a}$$



Part II.

Selected long solutions

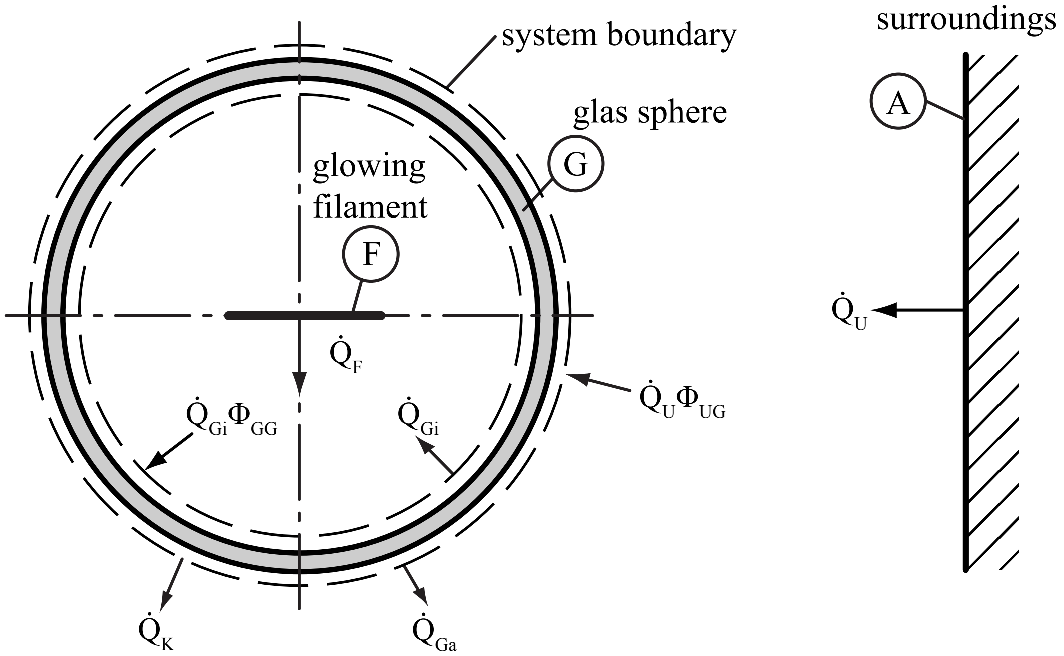
Chapter 1.

Solutions radiative heat tansfer

1.3. Spherical, evacuated lightbulb

a) Problem type:
   Application of radtion laws, surface brightness, radiation balances.

   1. System boundaries and heat balance
      The choice of system boundary relates to the sought-after quantity. As the glowing filament's temperature is to be determined, it would appear logical to choose the system voundaries around the glowing filament. As the power consumption of the filament is unknown, this approach does not yield the desired result. Instead the power consumption is limited by the conditions imposed by the glass sphere. Therefore choosing the glass sphere as the boundary is more practical, as the filament's temperature is included via the radiative heat emission of the filament.



Heat balance around the glass sphere

$$\dot{Q}_A\Phi_{AG} + \dot{Q}_F - \dot{Q}_{Go} - \dot{Q}_{Gi} + \dot{Q}_{Gi}\Phi_{GG} - \dot{Q}_S = 0 \tag{1.1}$$

with

$$\dot{Q}_A, \dot{Q}_F, \dot{Q}_{Gi}, \dot{Q}_{Go} = \text{surface brightness of the corresponding surfaces}$$
$$\dot{Q}_S = \text{heat transfer}$$

2. Determination of necessary quantities

2.1 Surface brightness

Under the condition that self emissivity of the glass sphere and ambient radiation are disregarded, as well as under the inclusion of $\rho_G = 0$ and $\tau_F = 0$ the surface brightness is determined as follows, for each surface:

- Surrounding surface

$$\dot{Q}_{\mathrm{A}} = \dot{q}''_{\mathrm{A}} \cdot A_{\mathrm{A}} = 0 \tag{1.2}$$

- Glowing filament

$$\dot{Q}_{\mathrm{F}} = \dot{q}''_{\mathrm{F}} A_{\mathrm{F}} = \varepsilon_{\mathrm{F}} \cdot \dot{q}''_{\mathrm{BF}} \cdot A_{\mathrm{F}} + ((1-\varepsilon_{\mathrm{F}}) \cdot \dot{q}_{\mathrm{Gi}} A_{\mathrm{G}} \Phi_{\mathrm{GF}} \tag{1.3}$$

- Inner glass sphere surface

$$\dot{Q}_{\mathrm{Gi}} = \dot{q}''_{\mathrm{Gi}} \cdot A_{\mathrm{G}} = 0 \tag{1.4}$$

- Outer glass sphere surface

$$\dot{Q}_{\mathrm{Go}} = \dot{q}''_{\mathrm{Go}} A_{\mathrm{G}} = \tau_{\mathrm{G}} \cdot \dot{q}''_{\mathrm{F}} \cdot A_{\mathrm{F}} \tag{1.5}$$

2.2 Emission of black bodies

$$\dot{q}''_{\mathrm{BF}} = \sigma T_{\mathrm{F}}^4 \tag{1.6}$$

2.3 View factors

Because of $\dot{Q}_{\mathrm{Gi}} = 0$ and $\dot{Q}_{\mathrm{A}} = 0$ no view factors need be determined.

2.4 Heat transfer

$$\dot{Q}_{\mathrm{S}} = \alpha \cdot A_{\mathrm{G}}(T_{\mathrm{G}} - T_{\mathrm{A}}) \tag{1.7}$$

3. Glowing filament temperature

By applying relations 1.2 to 1.5 and 1.7 for the balance equations 1.1 one obtains

$$\varepsilon_{\mathrm{F}} \cdot \dot{q}''_{\mathrm{BF}} A_{\mathrm{F}} - \tau_{\mathrm{G}} \cdot \underbrace{\dot{q}''_{\mathrm{F}} A_{\mathrm{F}}}_{\varepsilon_{\mathrm{F}} \cdot \dot{q}''_{\mathrm{BF}} A_{\mathrm{F}}} - \alpha \cdot A_{\mathrm{G}}(T_{\mathrm{G}} - T_{\mathrm{A}}) = 0$$

$$\rightarrow \dot{q}''_{\mathrm{BF}} = \frac{\alpha \cdot A_{\mathrm{G}}(T_{\mathrm{G}} - T_{\mathrm{A}})}{\varepsilon_{\mathrm{F}} \cdot (1-\tau_{\mathrm{G}}) \cdot A_{\mathrm{F}}}$$

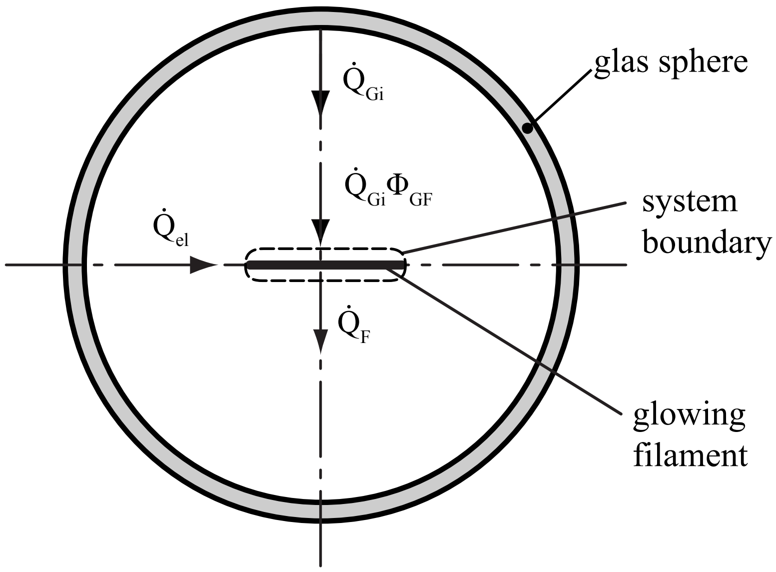
and with eq. 1.6

$$T_{\mathrm{F}}^4 = \frac{\alpha \cdot \pi \cdot d^2 \left(T_{\mathrm{G}} - T_{\mathrm{A}}\right)}{\varepsilon_{\mathrm{F}} \cdot A_{\mathrm{F}} \cdot \sigma \left(1 - \tau_{\mathrm{G}}\right)} \tag{1.8}$$

Numerical values:

$$T_{\mathrm{F}}^4 = 91{,}4 \cdot 10^{12}\,\mathrm{K}^4 \rightarrow T_{\mathrm{F}} = 3092\,\mathrm{K} = 2819\,°\mathrm{C}$$

b) 1. System boundaries and heat balance
The sought-after quantity is the glowing filament's power consumption. Therefore, the system boundary is chosen to encapsulate the filament. After determining the filament's temperature, and therfore also its radiative emission, the power consumption remains the last unknown variable.



Heat balance:

$$\dot{Q}_{\mathrm{el}} + Q_{\mathrm{Gi}}\Phi_{\mathrm{GF}} - \dot{Q}_{\mathrm{F}} = 0 \tag{1.9}$$

2. Determination of necessary quantities
See eq. 1.3 and 1.4.

3. Power consumption glowing filament
   Apply eq. 1.3, 1.4 and 1.6 to 1.9, yielding

$$\dot{Q}_{\mathrm{el}} = \varepsilon_{\mathrm{F}} \cdot \dot{q}''_{\mathrm{BF}} A_{\mathrm{F}} = \varepsilon_{\mathrm{F}} \cdot \sigma \cdot T_F^4 \cdot A_{\mathrm{F}} \tag{1.10}$$

   or with eq. 1.8

$$\dot{Q}_{\mathrm{el}} = \frac{\alpha \cdot \pi \cdot d^2 \,(T_{\mathrm{G}} - T_{\mathrm{A}})}{(1 - \tau_{\mathrm{G}})} \tag{1.11}$$

   Numerical value:

$$\dot{Q}_{\mathrm{el}} = 52\,\mathrm{W} \tag{1.12}$$

c) The relationship between wave-length, for which the maximum radiation emission is found, and the radiator's temperature is given by Wien's law of distribution:

$$\lambda_{\max} \cdot T_{\mathrm{F}} = 2898\,\mathrm{\mu m\,K} \tag{1.13}$$

Thus follows

$$\lambda_{\max} = \frac{2898}{T_{\mathrm{F}}} = 0.94\,\mathrm{\mu m}. \tag{1.14}$$

d) 1. System boundaries and heat balance
   As the task is identical to those posed in a) and b) and no further deliberations concerning the facilitating assumptions for either the glowing filament or ambient radiation have been made, the principal balance equations 1.1 and 1.9 remain unchanged. All difference are found solely in the formulation of the surface brightnesses.

2. Formulation of balance quantities

2.1 Determination of necessary quantities

- Surrounding surface area ($\varepsilon_{\mathrm{A}} = 1$)

$$\dot{Q}_{\mathrm{A}} = \dot{q}''_{\mathrm{SA}} \cdot A_{\mathrm{A}} = \dot{q}''_{\mathrm{A}} \cdot A_{\mathrm{A}} \tag{1.15}$$

- Glowing filament (Eq. 1.3 remains valid)

$$\dot{Q}_{\mathrm{F}} = \dot{q}''_{\mathrm{F}} \cdot A_{\mathrm{F}} = \varepsilon_{\mathrm{F}} \cdot \dot{q}''_{\mathrm{BF}} A_{\mathrm{F}} + (1 - \varepsilon_{\mathrm{F}})\, \dot{q}''_{\mathrm{Gi}} A_{\mathrm{G}} \cdot \Phi_{\mathrm{GF}} \tag{1.16}$$

- Inner glass sphere area

$$\begin{aligned}\dot{Q}_{\mathrm{Gi}} &= \dot{q}''_{\mathrm{Gi}} \cdot A_{\mathrm{G}} \\ &= \varepsilon_{\mathrm{G}} \dot{q}''_{\mathrm{BG}} \cdot A_{\mathrm{G}} + \rho\, (\dot{q}''_{\mathrm{Gi}} A_{\mathrm{G}} \Phi_{\mathrm{GG}} + \dot{q}''_{\mathrm{F}} A_{\mathrm{F}}) + \tau_{\mathrm{G}} \cdot \dot{q}''_{\mathrm{A}} A_{\mathrm{A}} \Phi_{\mathrm{AG}}\end{aligned}$$

With $\rho_{\mathrm{G}} = 0$, $\varepsilon_{\mathrm{G}} = 1 - \tau_{\mathrm{G}}$ and thus

$$\dot{Q}_{\mathrm{Gi}} = \dot{q}''_{\mathrm{Gi}} \cdot A_{\mathrm{G}} = (1 - \tau_{\mathrm{G}})\, \dot{q}''_{\mathrm{BG}} \cdot A_{\mathrm{G}} + \tau_{\mathrm{G}} \cdot \dot{q}''_{\mathrm{A}} A_{\mathrm{A}} \Phi_{\mathrm{AG}} \tag{1.17}$$

- Outer glass sphere area

$$\begin{aligned}\dot{Q}_{\mathrm{Go}} &= \dot{q}''_{\mathrm{Go}} \cdot A_{\mathrm{G}} \\ &= \varepsilon_{\mathrm{G}} \dot{q}''_{\mathrm{BG}} \cdot A_{\mathrm{G}} + \rho_{\mathrm{G}} \cdot \dot{q}''_{\mathrm{A}} A_{\mathrm{A}} \Phi_{\mathrm{AG}} + \tau_{\mathrm{G}}\, (\dot{q}''_{\mathrm{AF}} \cdot A_{\mathrm{F}} + \dot{q}''_{\mathrm{Gi}} A_{\mathrm{G}} \Phi_{\mathrm{GG}})\end{aligned}$$

Mit $\rho_{\mathrm{G}} = 0$, $\varepsilon_{\mathrm{G}} = 1 - \tau_{\mathrm{G}}$ ergibt sich

$$\begin{aligned}\dot{Q}_{\mathrm{Go}} &= \dot{q}''_{\mathrm{Go}} \cdot A_{\mathrm{G}} \\ &= (1 - \tau_{\mathrm{G}})\, \dot{q}''_{\mathrm{BG}} \cdot A_{\mathrm{G}} + \tau_{\mathrm{G}}\, (\dot{q}''_{\mathrm{AF}} \cdot A_{\mathrm{F}} + \dot{q}''_{\mathrm{Gi}} A_{\mathrm{G}} \Phi_{\mathrm{GG}})\end{aligned} \tag{1.18}$$

2.2 Emitted radiation of black bodies

$$\dot{q}''_{\mathrm{BF}} = \sigma \cdot T_{\mathrm{F}}^4 \tag{1.19}$$

$$\dot{q}''_{\mathrm{BG}} = \sigma \cdot T_{\mathrm{G}}^4 \tag{1.20}$$

$$\dot{q}''_{\mathrm{BA}} = \sigma \cdot T_{\mathrm{A}}^4 \tag{1.21}$$

2.3 View factors

The reciprocity relation yields $\Phi_{\mathrm{GU}} = 1$ and $\Phi_{\mathrm{FG}} = 1$

$$\Phi_{\mathrm{AG}} = \frac{A_{\mathrm{G}}}{A_{\mathrm{A}}} \tag{1.22}$$

$$\Phi_{\mathrm{GF}} = \frac{A_{\mathrm{F}}}{A_{\mathrm{G}}} \tag{1.23}$$

and the summation relation yields

$$\Phi_{\mathrm{GG}} = 1 - \Phi_{\mathrm{GF}} \tag{1.24}$$

2.4 Heat transfer (Eq. 1.7 remains valid)

$$\dot{Q}_{\mathrm{K}} = \alpha \cdot A_{\mathrm{G}} \left(T_{\mathrm{G}} - T_{\mathrm{A}}\right) \tag{1.25}$$

3. Equation for the glowing filament's temperature

Applying eq. 1.25, 1.15 and 1.18 as well as eq. 1.1 in addition to solving for $\dot{q}''_{\mathrm{F}} A_{\mathrm{F}}$ while emplyoing Eq. 1.22, 1.23 and 1.24 yields

$$\begin{aligned}(1-\tau_{\mathrm{G}})\,\dot{q}''_{\mathrm{F}} A_{\mathrm{F}} = \alpha \cdot A_{\mathrm{G}} \left(T_{\mathrm{G}} - T_{\mathrm{A}}\right) + (1-\tau_{\mathrm{G}}) \cdot \dot{q}''_{\mathrm{BG}} \cdot A_{\mathrm{G}} + \ldots \\ \ldots + \dot{q}''_{\mathrm{Gi}} A_{\mathrm{G}} \left(\tau_{\mathrm{G}} + (1-\tau_{\mathrm{G}}) \frac{A_{\mathrm{F}}}{A_{\mathrm{G}}}\right) - \dot{q}''_{\mathrm{BA}} A_{\mathrm{G}}\end{aligned}$$

With eq. 1.16 follows

$$\begin{aligned}(1-\tau_{\mathrm{G}})\,\varepsilon_{\mathrm{F}} \cdot \dot{q}''_{\mathrm{F}} A_{\mathrm{F}} = \alpha \cdot A_{\mathrm{G}} \left(T_{\mathrm{G}} - T_{\mathrm{A}}\right) + (1-\tau_{\mathrm{G}}) \cdot \dot{q}''_{\mathrm{BG}} A_{\mathrm{G}} + \ldots \\ \ldots + \dot{q}''_{\mathrm{Gi}} A_{\mathrm{G}} \left(\tau_{\mathrm{G}} + (1-\tau_{\mathrm{G}})\,\varepsilon_{\mathrm{F}} \cdot \frac{A_{\mathrm{F}}}{A_{\mathrm{G}}}\right) - \dot{q}''_{\mathrm{BA}} A_{\mathrm{G}}\end{aligned}$$

After replacing $\dot{q}''_{\mathrm{Gi}} A_{\mathrm{G}}$ with eq. 1.17,

$$\begin{aligned}\dot{q}''_{\mathrm{BF}} = \frac{\alpha \cdot A_{\mathrm{G}} \left(T_{\mathrm{G}} - T_{\mathrm{A}}\right)}{(1-\tau_{\mathrm{G}}) \cdot \varepsilon_{\mathrm{F}} \cdot A_{\mathrm{F}}} + \left(1 - \tau_{\mathrm{G}} + \frac{1+\tau_{\mathrm{G}}}{\varepsilon_{\mathrm{F}}} \cdot \frac{A_{\mathrm{G}}}{A_{\mathrm{F}}}\right) \cdot \dot{q}''_{\mathrm{BG}} - \ldots \\ \ldots - \left(-\tau_{\mathrm{G}} + \frac{1+\tau_{\mathrm{G}}}{\varepsilon_{\mathrm{F}}} \cdot \frac{A_{\mathrm{G}}}{A_{\mathrm{F}}}\right) \cdot \dot{q}''_{\mathrm{BA}}\end{aligned} \tag{1.26}$$

is obtained, following simple algebraic reformulations.

Additively expanding the $\dot{q}_{\mathrm{BA}}$-term $(+1-1)$ and equations 1.19 to 1.21 yields

$$\begin{aligned} T_{\mathrm{F}}^4 = \frac{\alpha \cdot A_{\mathrm{G}}\left(T_{\mathrm{G}}-T_{\mathrm{A}}\right)}{\left(1-\tau_{\mathrm{G}}\right) \cdot \varepsilon_{\mathrm{F}} \cdot \sigma \cdot A_{\mathrm{F}}} + \ldots \\ \ldots + \left(1-\tau_{\mathrm{G}}+\frac{1+\tau_{\mathrm{G}}}{\varepsilon_{\mathrm{F}}} \cdot \frac{A_{\mathrm{G}}}{A_{\mathrm{F}}}\right) \cdot \left(T_{\mathrm{G}}^4-T_{\mathrm{A}}^4\right)+T_{\mathrm{A}}^4. \end{aligned} \tag{1.27}$$

Approximate relations

For

$$\frac{A_{\mathrm{G}}}{A_{\mathrm{F}}} \gg 1 \text{ is}$$
$$\frac{1+\tau_{\mathrm{G}}}{\varepsilon_{\mathrm{F}}} \cdot \frac{A_{\mathrm{G}}}{A_{\mathrm{F}}} \gg 1-\tau_{\mathrm{G}} \text{ respectively } \gg 1$$

Thus follows

$$T_{\mathrm{F}}^4 \approx \frac{1}{\varepsilon_{\mathrm{F}}} \cdot \frac{A_{\mathrm{G}}}{A_{\mathrm{F}}}\left[\frac{\alpha\left(T_{\mathrm{G}}-T_{\mathrm{A}}\right)}{\left(1-\tau_{\mathrm{G}}\right) \cdot \sigma}+\left(1+\tau_{\mathrm{G}}\right) \cdot\left(T_{\mathrm{G}}^4+T_{\mathrm{A}}^4\right)\right] \tag{1.28}$$

For $\frac{A_{\mathrm{G}}}{A_{\mathrm{F}}} \gg 1$ and $\tau_{\mathrm{G}} \approx 1$ eq. 1.27, and 1.28 in eq. 1.8, respectively, transmutes to.

4. Equation for the filament's power consumption

Using. 1.16 in the balance equations 1.9 while using eq. 1.23 yields

$$\dot{Q}_{\mathrm{el}} = \varepsilon_{\mathrm{F}} \cdot A_{\mathrm{F}}\left(\dot{q}_{\mathrm{BF}}''-\dot{q}_{\mathrm{Gi}}''\right)$$

and with eq. 1.17, 1.15 and 1.22

$$\dot{Q}_{\mathrm{el}} = \varepsilon_{\mathrm{F}} \cdot A_{\mathrm{F}}\left(\dot{q}_{\mathrm{BF}}''-\left(1-\tau_{\mathrm{G}}\right) \dot{q}_{\mathrm{BG}}''-\tau_{\mathrm{G}} \dot{q}_{\mathrm{BA}}''\right)$$

With eq. 1.26 one obtains

$$\dot{Q}_{\mathrm{el}} = \frac{\alpha \cdot A_{\mathrm{G}}\left(T_{\mathrm{G}}-T_{\mathrm{A}}\right)}{\left(1-\tau_{\mathrm{G}}\right)}+\left(1+\tau_{\mathrm{G}}\right) \cdot A_{\mathrm{G}} \cdot \sigma\left(T_{\mathrm{G}}^4+T_{\mathrm{A}}^4\right) \tag{1.29}$$

The equation 1.11, derived in subtask b), is an approximate relation, which is obtained from eq. 1.29 for $\tau_\mathrm{G} \to 1$. Furthermore, the remarkable finding that the filament's power consumption is, contrary to its temperature, independent of the surface ration $\frac{A_\mathrm{G}}{A_\mathrm{F}}$ becomes apparent. This finding is illustrated by the following consideration. The glas sphere temperature governing the power consumption is the result of the equilibrium of heat absorbed and heat lost through emission and convection. The electrical power consumed by the filament, the heat flux supplied by the surroundings as well as the emission from the inner sphere's area are all absorbed by the glass sphere, so that the size of the filament's surface area is irrelevant. A increase in the ratio of glass sphere to filament surface area for a constant glass sphere surface area only yields a higher filament temperature without any increase in electrical output.

5. Numerical values:

5.1 Glowing filament temperature:

$$\text{See eq. 1.27 ist } T_\mathrm{F} = 3122\,\mathrm{K} = 2849\,°\mathrm{C} \tag{1.30}$$

The approximate realtion 1.28 yields the same result, as the condition $\frac{A_\mathrm{G}}{A_\mathrm{F}} \gg 1 \left(\frac{A_\mathrm{G}}{A_\mathrm{F}} = 70{,}69\right)$ is well met in this specific case. Also, the divergence from the even further simplified equation . 1.8 (see subtask (a) is irrelevant, as τ_G is very close to unity in this case.

5.2 Power consumption glowing filament

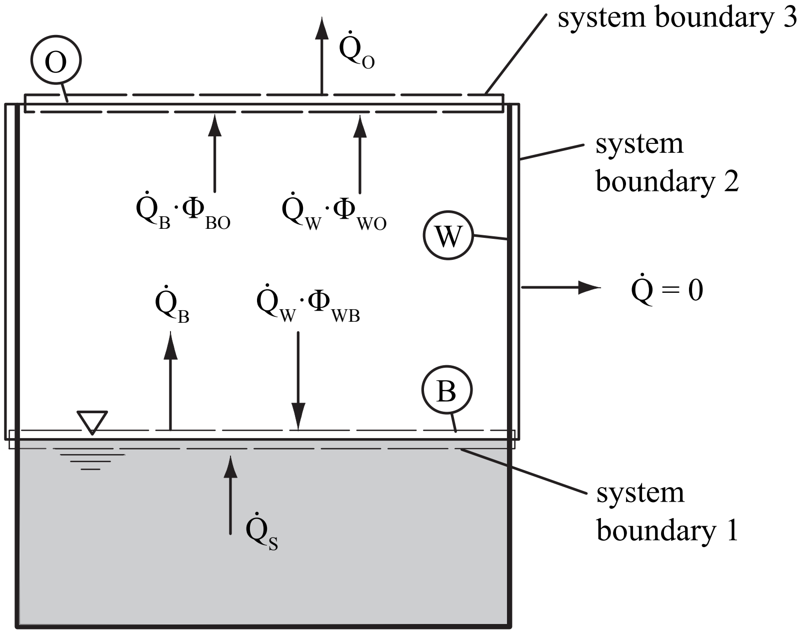
$$\dot{Q}_\mathrm{el} = 54\,\mathrm{W}. \tag{1.31}$$

1.5. Lead crucible

a) Outgoing heat flux $\dot{Q}_\mathrm{K}$

1. Problem type

   Purely radiative heat exchange, calculation by means of surface brightness



2. System boundaries and heat balance

   The outgoing heat flux $\dot{Q}_\mathrm{K}$ is equal to the net heat flux trhough the surface of the molten lead and is obtained by means of a heat balance around the lead's surface (balance system 1)

$$0 = \dot{Q}_\mathrm{K} - \dot{Q}_\mathrm{L} + \dot{Q}_\mathrm{W} \cdot \Phi_\mathrm{WL}$$

$$\rightarrow \dot{Q}_\mathrm{K} = \dot{Q}_\mathrm{L} - \dot{Q}_\mathrm{W} \cdot \Phi_\mathrm{WL} \tag{1.32}$$

with

$$\begin{aligned}\dot{Q}_{\mathrm{L}} &= \text{Surface brightness lead surface}\\ \dot{Q}_{\mathrm{W}} &= \text{ Surface brightness crucible walls}\\ \Phi_{\mathrm{ij}} &= \text{View factor of surface i onto surface j}\end{aligned} \tag{1.33}$$

The adiabatic crucible walls yield a total heat balance of (balance system 2)

$$\dot{Q}_{\mathrm{K}} = \dot{Q}_{\mathrm{Ö}} \tag{1.34}$$

and the balance for the crucible opening reads (balance system 3)

$$-\dot{Q}_{\mathrm{Ö}} + \dot{Q}_{\mathrm{L}} \cdot \Phi_{\mathrm{BÖ}} + \dot{Q}_{\mathrm{W}} \cdot \Phi_{\mathrm{WÖ}} = 0 \tag{1.35}$$

which in turn, with eq. 1.34, yields

$$\dot{Q}_{\mathrm{K}} = \dot{Q}_{\mathrm{L}} \cdot \Phi_{\mathrm{BÖ}} + \dot{Q}_{\mathrm{W}} \cdot \Phi_{\mathrm{WÖ}} \tag{1.36}$$

The unkown heat flux $\dot{Q}_{\mathrm{K}}$ can therefore be determined using eq. 1.32 or (1.36). Here eq. 1.32 is employed.

3. Determination of necessary quantities

   Surface brightness

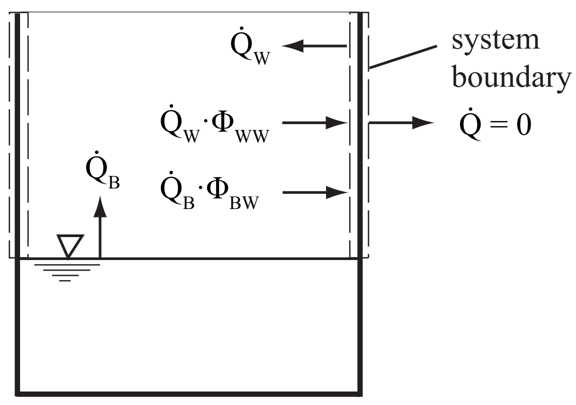
   Lead surface:

$$\dot{Q}_{\mathrm{L}} = \dot{q}''_{\mathrm{L}} A_{\mathrm{L}} = \varepsilon_{\mathrm{L}} \cdot \dot{q}''_{\mathrm{SL}} A_{\mathrm{L}} + (1 - \varepsilon_{\mathrm{L}})\, \dot{q}''_{\mathrm{W}} A_{\mathrm{W}} \cdot \Phi_{\mathrm{WL}} \tag{1.37}$$

$$\dot{q}''_{\mathrm{SL}} = \sigma T_{\mathrm{L}}^4 \tag{1.38}$$

   Curcible walls:

   An approach as in eq. 1.37 (emission and cumulated reflection) does not yield the desired results, as the radiative properties of the crucible's walls are unknown. Here the surface brightness is obtained from the postulation of adiabacy or the radiative equilibrium of the crucible's walls, respectively, and therefore from the heat balance around the crucible itself.



$\dot{Q}_W$
$\dot{Q}_W \cdot \Phi_{WW}$
$\dot{Q}_B$
$\dot{Q}_B \cdot \Phi_{BW}$
system boundary
$\dot{Q} = 0$

It follows

$$0 = \dot{Q}_\mathrm{L} \cdot \Phi_\mathrm{BW} - \dot{Q}_\mathrm{W} + \dot{Q}_\mathrm{W} \cdot \Phi_\mathrm{WW}$$

$$\Leftrightarrow \dot{Q}_\mathrm{W} = \dot{q}''_\mathrm{W} A_\mathrm{W} = \frac{\Phi_\mathrm{BW}}{1 - \Phi_\mathrm{WW}} \dot{q}''_\mathrm{L} A_\mathrm{L} \tag{1.39}$$

$$\tag{1.40}$$

<u>View factors</u>

The view factor $\Phi_\mathrm{BÖ}$ is known, from here on out the remaining view factors are determined. For symmetry reasons follows

$$\Phi_\mathrm{WL} = \Phi_\mathrm{WÖ} \tag{1.41}$$

and with the summation relation

$$1 - \Phi_\mathrm{WW} = \Phi_\mathrm{WÖ} + \Phi_\mathrm{WL} = 2 \cdot \Phi_\mathrm{WL} \tag{1.42}$$

The relation of reciprocity yields

$$\Phi_\mathrm{WL} = \frac{A_\mathrm{L}}{A_\mathrm{W}} \Phi_\mathrm{BW} \tag{1.43}$$

and further application of the summation relation

$$\Phi_\mathrm{BW} = 1 - \Phi_\mathrm{BÖ} \tag{1.44}$$

as $\Phi_\mathrm{BB} = 0$ is.

Eq. 1.42, 1.43 and 1.44 yield the following surface brightness values

$$\dot{q}''_\mathrm{W} A_\mathrm{W} = \frac{1}{2} \cdot \frac{\Phi_\mathrm{BW}}{\Phi_\mathrm{WL} \cdot \dot{q}''_\mathrm{L} \cdot A_\mathrm{L}} \tag{1.45}$$

and insertion into eq. 1.37 as well as solving for $\dot{q}''_\mathrm{L} A_\mathrm{L}$

$$\dot{q}''_\mathrm{L} A_\mathrm{L} = \frac{\varepsilon_\mathrm{L} \cdot \dot{q}''_\mathrm{SL} A_\mathrm{L}}{1 - \frac{1}{2} \cdot (1 - \Phi_\mathrm{BÖ})(1 - \varepsilon_\mathrm{L})} \tag{1.46}$$

4. Determination of outgoing heat flux

   Eq. 1.32 and 1.45 initially yield

$$\dot{Q}_\mathrm{K} = \dot{q}''_\mathrm{L} A_\mathrm{L} \left(1 - \frac{1}{2} \cdot \frac{\Phi_\mathrm{BW}}{\Phi_\mathrm{WL}} \Phi_\mathrm{WL}\right) \tag{1.47}$$

   The insertion of eq. 1.43, 1.46 and 1.38 eventually yields

$$\begin{aligned} \dot{Q}_\mathrm{K} = \frac{1}{2} \dot{q}''_\mathrm{L} A_\mathrm{L} \cdot (1 + \Phi_\mathrm{BÖ}) = \frac{1}{2} \varepsilon_\mathrm{L} A_\mathrm{L} \cdot \sigma \cdot T_\mathrm{L}^4 \cdot \ldots \\ \ldots \cdot \frac{1 + \Phi_\mathrm{BÖ}}{1 - \frac{1}{2} \cdot (1 - \Phi_\mathrm{BÖ})(1 - \varepsilon_\mathrm{L})} \end{aligned} \tag{1.48}$$

   Numerical values:

$$\dot{Q}_\mathrm{K} = \frac{0{,}8}{2} \cdot \frac{\pi \cdot 0{,}025^2}{4} \cdot 5{,}67 \cdot 6^4 \cdot \frac{1{,}38}{1 - 0{,}5 \cdot 0{,}62 \cdot 0{,}2} = 2{,}1\,\mathrm{W} \tag{1.49}$$

b) Limiting cases $h = 0$ and $h \to \infty$

The distance between lead surface and upper end of the crucible's opening solely influences the view factors. The relations are as follows

$$
\begin{aligned}
h &= 0: && \Phi_{\text{BÖ}} = 1 \\
h &\to \infty: && \Phi_{\text{BÖ}} = 0 \quad \Phi_{\text{BW}} = 1
\end{aligned}
$$

Thus the net heat flux through the lead surface reads

$$
\begin{aligned}
h &= 0: && \dot{Q}_{\text{L},0} = \varepsilon_{\text{L}} \cdot A_{\text{L}} \cdot \sigma \cdot T_{\text{L}}^4 \\
h &\to \infty: && \dot{Q}_{\text{L},\infty} = \frac{\varepsilon_{\text{L}}}{1+\varepsilon_{\text{L}}} \cdot A_{\text{L}} \cdot \sigma \cdot T_{\text{L}}^4 && (1.50)
\end{aligned}
$$

$$(1.51)$$

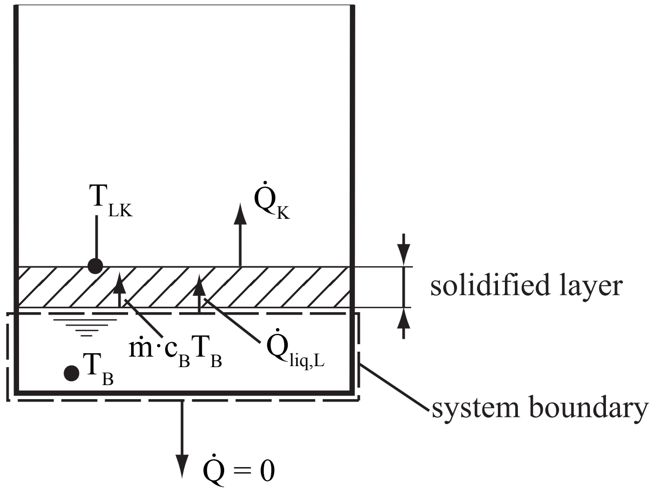
:

$$
\begin{aligned}
\dot{Q}_{\text{L},0} &= 2{,}9\,\text{W} \\
\dot{Q}_{\text{L},\infty} &= 1{,}6\,\text{W}
\end{aligned}
$$

c) Time of solidification

1. System boundaries and heat balance

   Heat balance around the molten lead



$$-\dot{m} \cdot c_{\mathrm{L}} \cdot T_{\mathrm{L}} - \dot{Q}_{\mathrm{Bfl}} = \frac{\mathrm{d}U_{\mathrm{L}}}{\mathrm{d}t} \text{ mit } \dot{m} = \frac{\mathrm{d}m}{\mathrm{d}t} \tag{1.52}$$

2. <u>Determination of necessary quantities</u>

   Under the assumption of a negligible influence of each the heat transfer resistance between melt and the solidified surface as well as the change in internal energy of the solidified layer follows

$$T_{\mathrm{BK}} = T_{\mathrm{L}} \qquad \dot{Q}_{\mathrm{Bfl}} = \dot{Q}_{\mathrm{K}} \tag{1.53}$$

   The internal energy of the molten lead at time t is

$$U_{\mathrm{L}}(t) = (c_{\mathrm{L}} \cdot T_{\mathrm{L}} + h_{\mathrm{L}}) \cdot m(t)$$

   During the process of solidification, e. e. during the *reduction* of melt, T_{L} is constant, therefore

$$\frac{\mathrm{d}U_{\mathrm{L}}}{\mathrm{d}t} = -\left(c_{\mathrm{L}} \cdot T_{\mathrm{L}} + h_{\mathrm{L}}\right) \cdot \dot{m} \tag{1.54}$$

3. <u>Determination of solidification time</u>

E. 1.53 and 1.54 inserted into 1.52, yield

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\dot{Q}_\mathrm{K}}{h_\mathrm{L}} \neq f(t) \tag{1.55}$$

with $\dot{Q}_\mathrm{K}$ as in eq. 1.48 and in both cases in eq. 1.50. The integration from $t = 0$ until $t = t_\mathrm{S}$ reads as follows

$$\int_{m=0}^{m_\mathrm{L}} \mathrm{d}m = \frac{\dot{Q}_\mathrm{K}}{h_\mathrm{L}} \int_{t=0}^{t_\mathrm{S}} \mathrm{d}t \tag{1.56}$$

$$\Rightarrow m_\mathrm{L} = \frac{\dot{Q}_\mathrm{K} \cdot t_\mathrm{S}}{h_\mathrm{L}} \tag{1.57}$$

$$\tag{1.58}$$

solving for t_S yields

$$t_\mathrm{S} = \frac{m_\mathrm{L} \cdot h_\mathrm{L}}{\dot{Q}_\mathrm{K}} \tag{1.59}$$

$h = 0:$	$\dot{Q}_{\mathrm{K},0} = 2.9\,\mathrm{W}$	$t_{\mathrm{E},0} = 0.2\,\mathrm{h}$
$h = 25\,\mathrm{mm}:$	$\dot{Q}_\mathrm{K} = 2.1\,\mathrm{W}$	$t_\mathrm{S} = 0.28\,\mathrm{h}$
$h \to \infty$	$\dot{Q}_{\mathrm{K},\infty} = 1.6\,\mathrm{W}$	$t_{\mathrm{E},\infty} = 0.36\,\mathrm{h}$

1.8. Radiative net heat flux between two plates*

1) Problem type

   Heat exchange between two flat, parallel, infinitely large plates with fixed surface temperatures.

2) Basic equation

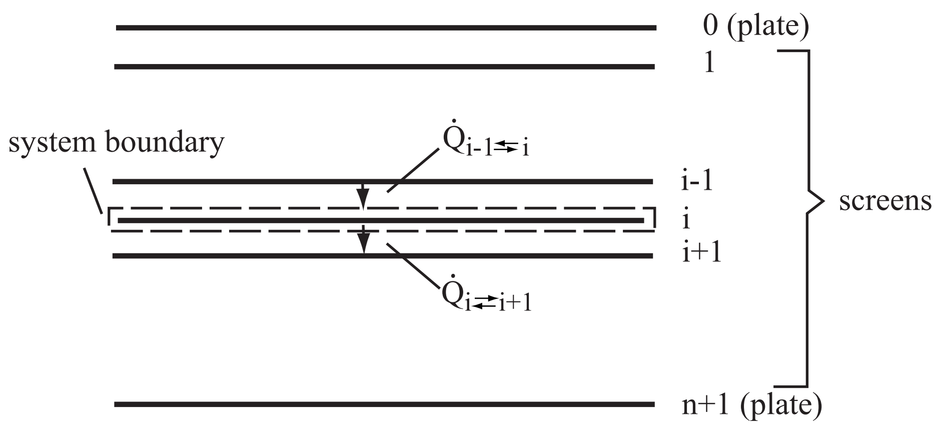
   For the heat exchange between two flat, parallel, impermeable to radiation and infinitely large surfaces A_i and A_{i+1}, it follows $A_{i+1} = A_i = A$.

$$Q_{i\leftrightarrows i+1} = \frac{\sigma}{\frac{1}{\varepsilon_i} + \frac{1}{\varepsilon_{i+1}} - 1} \cdot \left(T_i^4 - T_{i+1}^4\right) \cdot A \tag{1.60}$$

3) Heat exchange between two surfaces A_0 and $A_{\mathrm{n}+1}$

$$Q_{0\leftrightarrows \mathrm{n}+1} = \dot{Q}_0 = \frac{\sigma}{\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_{\mathrm{n}+1}} - 1} \cdot \left(T_0^4 - T_{\mathrm{n}+1}^4\right) \cdot A \tag{1.61}$$

4) Heat exchange between two surfaces A_0 und $A_{\mathrm{n}+1}$ wiht n dazwischen angeordneten, strahlungsundurchlässigen Strahlungsschirmen



The energy balance around the screen i for steady-state conditions and no heat transfer through either conduction or convection ($\rightarrow$ vacuum), yields:

$$\dot{Q}_{i-1\leftrightarrows i} - \dot{Q}_{i\leftrightarrows i+1} = 0$$

where the net heat flux according to Eq. 1.60 is represented by the individual fluxes $\dot{Q}_{i-1\leftrightarrows i} - \dot{Q}_{i\leftrightarrows i+1} = 0$, respectively.

With

$$\dot{Q}_{i-1\leftrightarrows i} = \dot{Q}_{i-1\leftrightarrows i} = \dot{Q}_{\mathrm{n}} = \text{konst.} \tag{1.62}$$

Eq. 1.61 is obtained.

$$\frac{\dot{Q}_{\mathrm{n}}}{A \cdot \sigma} \cdot \left(\frac{1}{\varepsilon_i} + \frac{1}{\varepsilon_{i+1}} - 1\right) = \left(T_i^4 - T_{i+1}^4\right) \tag{1.63}$$

This relation, with the corresponding indices, also holds true for the radiative heat transfer between the two respective surfaces in the system regarded here which comprises $n+2$ surfaces (2 plates and n screens). Thus $n+1$ equations with the unknown interstitial temperatures T_i with $i = 0 \ldots n$ are obtained. The elimination of these temperatures is achieved summation of Eq. 1.63:

$$\frac{\dot{Q}_{\mathrm{n}}}{A \cdot \sigma} \cdot \sum_{i=0}^{n} \left(\frac{1}{\varepsilon_i} + \frac{1}{\varepsilon_{i+1}} - 1\right) = \left(T_0^4 - T_{\mathrm{n}+1}^4\right) \tag{1.64}$$

The exchange of radiation between two infinitely large, planar plates and n screens between the plates is thus:

$$\dot{Q}_{\mathrm{n}} = \frac{\sigma \cdot A}{\sum_{i=0}^{n} \left(\frac{1}{\varepsilon_i} + \frac{1}{\varepsilon_{i+1}} - 1\right)} \cdot \left(T_0^4 - T_{\mathrm{n}+1}^4\right) \tag{1.65}$$

5) Relationship of the equations for the exchange of radiation with and without one or multiple screens

With Eq. 1.65 and 1.61 one obtains,

$$\frac{\dot{Q}_{\mathrm{n}}}{\dot{Q}_0} = \frac{\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_{\mathrm{n}+1}} - 1\right)}{\sum_{i=0}^{n}\left(\frac{1}{\varepsilon_i} + \frac{1}{\varepsilon_{i+1}} - 1\right)} \tag{1.66}$$

If all emittivities are identical ($\varepsilon_i = \varepsilon$), it follows

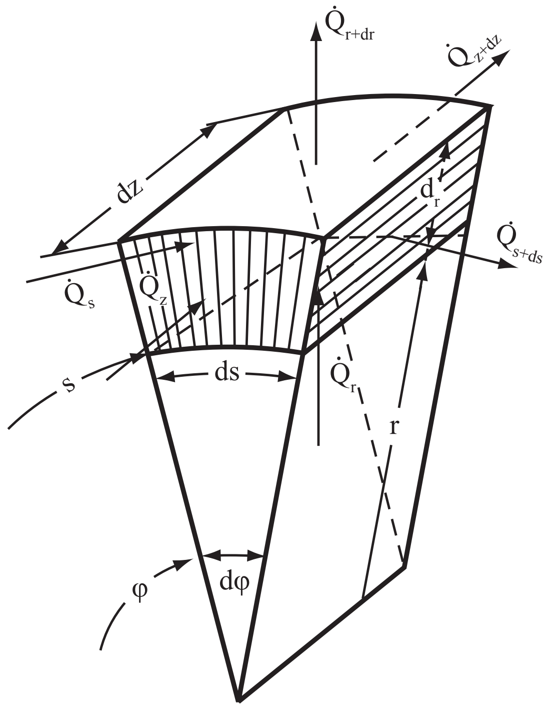
$$\frac{\dot{Q}_{\mathrm{n}}}{\dot{Q}_0} = \frac{\frac{2}{\varepsilon} - 1}{\sum_{i=0}^{n}\left(\frac{2}{\varepsilon} - 1\right)} = \frac{\frac{2}{\varepsilon} - 1}{(n+1)\cdot\left[\frac{2}{\varepsilon} - 1\right]} = \frac{1}{n+1} \quad \text{q.e.d}$$

Chapter 2.

Solutions heat conduction

2.3. Transient temperature fields

The PDE is derived by excising an infinitessimaly small volume element $dr \cdot ds \cdot dz$ from the cylindrical body and posing the heat balance for this volume element.



Through the in r-direction, s-(circumferrential)-direction and z-direction perpendicular surfaces the following heat fluxes, in accordance with the phenomenological approach of Fourier,

$$\begin{aligned}\dot{Q}_r &= -\lambda \cdot \frac{\partial T}{\partial r} \cdot ds \cdot dz \\ \dot{Q}_s &= -\lambda \cdot \frac{\partial T}{\partial s} \cdot dr \cdot dz \\ \dot{Q}_z &= -\lambda \cdot \frac{\partial T}{\partial z} \cdot dr \cdot ds\end{aligned} \tag{2.1}$$

flow into the volume element. While the heat fluxes $ds = r \cdot d\phi$

$$\begin{aligned}\dot{Q}_{\mathrm{r+dr}} &= \dot{Q}_{\mathrm{r}} + \frac{\partial \dot{Q}_{\mathrm{r}}}{\partial r} \cdot dr = \dot{Q}_{\mathrm{r}} - \frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right) dr \cdot d\varphi \cdot dz \\ \dot{Q}_{\mathrm{s+ds}} &= \dot{Q}_{\mathrm{s}} + \frac{\partial \dot{Q}_{\mathrm{s}}}{\partial s} \cdot ds = \dot{Q}_{\mathrm{s}} - \frac{\partial}{\partial s}\left(\lambda\frac{\partial T}{\partial s}\right) r \cdot d\varphi \cdot dr \cdot dz \\ \dot{Q}_{\mathrm{z+dz}} &= \dot{Q}_{\mathrm{z}} + \frac{\partial \dot{Q}_{\mathrm{z}}}{\partial z} \cdot dz = \dot{Q}_{\mathrm{z}} - \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right) dz \cdot dr \cdot r \cdot d\varphi\end{aligned} \tag{2.2}$$

exit the volume element, where $ds = r \cdot d\phi$. It should be noted that the thermal conductivity λ needs to be considered in the differentiation as it is not spatially invariant.

Should te body comprise homogeneously distrubuted heat source of heat flux density $\dot{\Phi}'''$ [$^{\mathrm{W}}/_{\mathrm{m}^3}$], the following heat flux is generated per volume element

$$\dot{\Phi}''' \cdot r \cdot dr \cdot d\varphi \cdot dz \tag{2.3}$$

In accordance with the law of energy conservation the difference in heat per unit time flowing into to and out of the regarded volume element is the amount of energy stored or extracted from the volume element. The amount of heat stored, causing a change in temperature of the volume element, is

$$\rho \cdot c \frac{\partial T}{\partial t} \cdot dr \cdot r \cdot d\varphi \cdot dz \tag{2.4}$$

Thus the relation below ensues

$$\begin{aligned}\underbrace{\dot{Q}_{\mathrm{r}} + \dot{Q}_{\mathrm{s}} + \dot{Q}_{\mathrm{z}}}_{\text{heat influx}} + \underbrace{\dot{\Phi}''' \cdot r \cdot dr \cdot d\varphi \cdot dz}_{\text{heat production}} - \underbrace{(\dot{Q}_{\mathrm{r+dr}} + \dot{Q}_{\mathrm{s+ds}} + \dot{Q}_{\mathrm{z+dz}})}_{\text{heat outflow}} \\ = \underbrace{\rho \cdot c \cdot \frac{\partial T}{\partial t} dr \cdot r \cdot d\varphi \cdot dz}_{\text{heat stored}}\end{aligned}$$

When one inserts these relations into equations 2.1 and 2.2, and divides by the volume element $r \cdot dr \cdot d\varphi \cdot dz$, it follows

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \lambda \cdot \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial s}\left(\lambda \cdot \frac{\partial T}{\partial s}\right) + \frac{\partial}{\partial z}\left(\lambda \cdot \frac{\partial T}{\partial z}\right) + \dot{\Phi}''' = \rho \cdot c \cdot \frac{\partial T}{\partial t} \tag{2.5}$$

Transformation of the differential $\frac{\partial}{\partial s}$:
Here a function $f(s)$ is regarded on the arc $s = r \cdot \varphi$. A partial differentiation yields:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial s} + \frac{\partial f}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial s} \tag{2.6}$$

As the radius is invariant in circumferential direction, it follows

$$\frac{\partial r}{\partial s} = 0 \tag{2.7}$$

Furthermore, with $\varphi = \frac{s}{r}$,

$$\frac{\partial \varphi}{\partial s} = \frac{1}{r} \tag{2.8}$$

Overall,

$$\frac{\partial f}{\partial s} = \frac{1}{r} \cdot \frac{\partial f}{\partial \varphi} \tag{2.9}$$

is obtained, and therefore the differential reads

$$\frac{\partial}{\partial s} = \frac{1}{r} \cdot \frac{\partial}{\partial \varphi} \tag{2.10}$$

Inserted into equation 2.5 the sought-after differential equation for heat conduction in cylinder coordinates is:

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \lambda \cdot \frac{\partial T}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial}{\partial \varphi}\left(\lambda \cdot \frac{\partial T}{\partial s}\right) + \frac{\partial}{\partial z}\left(\lambda \cdot \frac{\partial T}{\partial z}\right) + \dot{\Phi}''' = \rho \cdot c \cdot \frac{\partial T}{\partial t} \tag{2.11}$$

wherein material properties may be spatially and thermally variable. After differentiation of the first term obtained from eq. 2.11 for constant material properties one obtains

$$a \cdot \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{\Phi}'''}{\rho c} = \frac{\partial T}{\partial t} \tag{2.12}$$

$$\text{mit } a = \frac{\lambda}{\rho \cdot c}$$

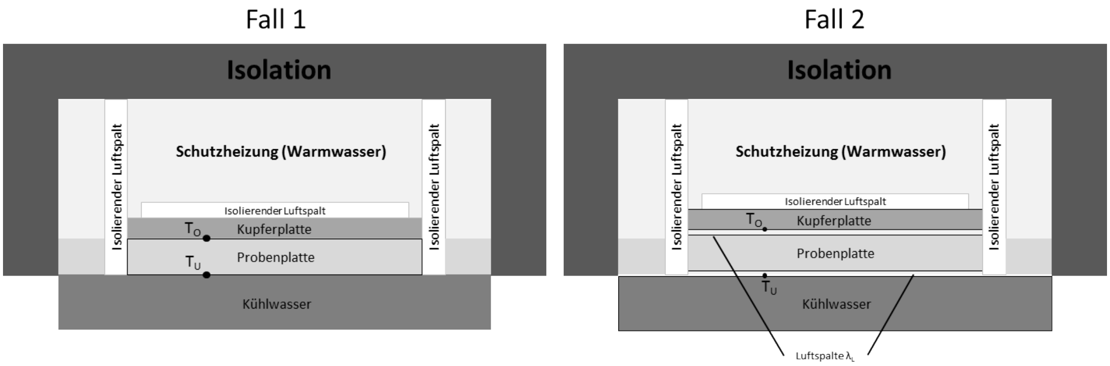
The equation 2.12 can be simplified in many cases. For circular-cylindrical bodies isothermal circles are observed for identical boundary conditions, the temperature is thus independent of the angle φ; i.e. $\frac{\partial T}{\partial \varphi} = 0$. For very long cylinders the leading surfaces influences can be neglected, thus allowing $\frac{\partial T}{\partial z} = 0$ to be set for the entire cylinder under the assumption of constant external conditions. In case of an infinitely long, circular-cylindrical body with constant material properties and identical external conditions for the entire body eq. 2.12 takes the form of

$$\frac{\partial T}{\partial t} = a \cdot \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r}\right) + \frac{\dot{Q}'''}{\rho c} \qquad \text{an.} \tag{2.13}$$

2.2. Poensgen device

a) Problem type

Steady-state heat conduction in multiple layers.



Case 1: no air gaps

$$\dot{Q}_{\mathrm{el}} = \frac{\lambda_{\mathrm{P}}}{\delta_{\mathrm{P}}} \cdot A \cdot (T_{\mathrm{O}} - T_{\mathrm{U}}) \tag{2.14}$$

$$\lambda_{\mathrm{P}} = \frac{\dot{Q}_{\mathrm{el}} \cdot \delta_{\mathrm{P}}}{A \cdot (T_{\mathrm{O}} - T_{\mathrm{U}})} \tag{2.15}$$

Case 2: air gaps above and below the probe plate

$$\dot{Q}_{\mathrm{el}} = \frac{A \cdot (T_{\mathrm{O}} - T_{\mathrm{U}})}{\frac{\delta_{\mathrm{L}}}{\lambda_{\mathrm{L}}} + \frac{\delta_{\mathrm{P}}}{\lambda_{\mathrm{P}}} + \frac{\delta_{\mathrm{L}}}{\lambda_{\mathrm{L}}}} \tag{2.16}$$

$$\delta_{\mathrm{L}} = 0{,}0025 \cdot \delta_{\mathrm{P}} \tag{2.17}$$

$$\lambda_{\mathrm{P,Korr}} = \frac{\lambda_{\mathrm{L}} \cdot \lambda_{\mathrm{P}}}{0{,}005 \cdot \lambda_{\mathrm{P}} + \lambda_{\mathrm{L}}} = \frac{\dot{Q}_{\mathrm{el}} \cdot \delta_{\mathrm{P}}}{A \cdot (T_{\mathrm{O}} - T_{\mathrm{U}})} \tag{2.18}$$

The uncertainty in measurement is related to the ratio of thermal conductivities measured.

$$\frac{\lambda_{\mathrm{P,Korr}}}{\lambda_{\mathrm{P}}} = \frac{\lambda_{\mathrm{L}} \cdot \lambda_{\mathrm{P}}}{\lambda_{\mathrm{P}} \cdot (0{,}005 \cdot \lambda_{\mathrm{P}} + \lambda_{\mathrm{L}})} = \frac{\lambda_{\mathrm{L}}}{0{,}005 \cdot \lambda_{\mathrm{P}} + \lambda_{\mathrm{L}}} \tag{2.19}$$

$$\frac{\lambda_{\mathrm{P,Korr}}}{\lambda_{\mathrm{P}}} = 0.81 \qquad \text{for concrete}$$
$$\frac{\lambda_{\mathrm{P,Korr}}}{\lambda_{\mathrm{P}}} = 0.99 \qquad \text{for cork}$$

b)

causes	corrections
temperature measurement	$\Delta T > 10\,\mathrm{K}$ $\delta > 10\,\mathrm{mm}$ $\lambda < 2.5\,{}^{\mathrm{W}}/_{\mathrm{m\,K}}$
multi-dimensional temperature field	good overlap of auxilliary heating $\lambda > 125\,\mathrm{mm}$
inhomogeneities in the probe plate	control sample
no steady-state	recording of reference temperatures

2.5. Fiery furnace

1. Problem type

   Steady-state heat conduction in multiple flat walls.

2. Optimisation strategy

   Minimum wall thickness while maintaining a minimum overall thermal conductivity. Limit choice by restricting permissible material temperature.

   Conditions for selection:

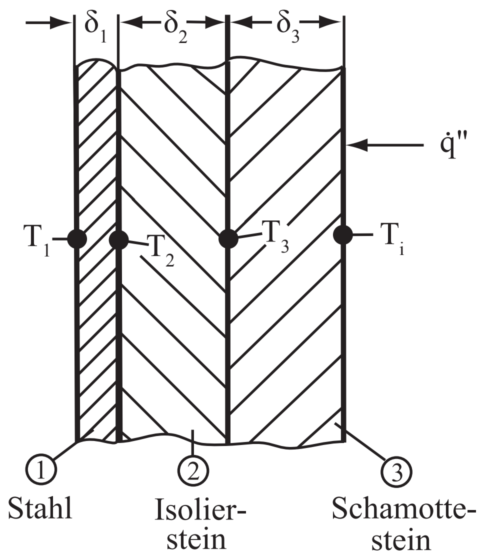
   1. λ_{layer} as small as possible
   2. $T_{\text{layer}} \leq T_{\text{perm}}$

3. Solution

   - **Case A**

     Of the offered materials only firebrick conforms to the condition $T_{\text{perm}} > T_{\text{max}} = T_{\text{i}}$

     The subsequent set-up comprises three layers:



     With $T_3 = T_{\text{insulatingstone,perm}} = T_{3,\text{perm}}$; $T_2 =$ unkown

Determination firebrick thickness

As the temperatures at the boundaries as well as the heat flux are known it is possible to utilise the equation for heat conduction from the formulary wihtout further adjustments.

$$\dot{q}'' = \frac{\lambda}{\delta_3}\left(T_\mathrm{i} - T_\mathrm{perm}\right) \quad \Rightarrow \quad \delta_3 = \frac{\lambda_3}{\dot{q}''} \cdot \left(T_\mathrm{i} - T_\mathrm{2,perm}\right) \tag{2.20}$$

Determination insulating stone thickness:

For steady-state conditions the heat flux $\dot{Q}$ or $\dot{q}''$, respectively, is constant throughout each individual layer. T_2 is unkown, while T_1 is known. Therefore, the steel plate is also included in the system's boundary. Thus one obtains:

$$\dot{q}'' = \frac{T_\mathrm{2,perm} - T_1}{\frac{\delta_2}{\lambda_2} + \frac{\delta_1}{\lambda_1}} \Rightarrow \delta_2 \tag{2.21}$$

Because of $\frac{\delta_1}{\lambda_1} \ll \frac{\delta_2}{\lambda_2}$ follows $T_2 \approx T_1$ and thus

$$\delta_2 = \frac{\lambda_2}{\dot{q}''} \cdot \left(T_\mathrm{2,perm} - T_1\right) \tag{2.22}$$

Total thickness:

$$\delta = \delta_1 + \frac{\lambda_2 \cdot \left(T_\mathrm{2,perm} - T_1\right) + \lambda_3 \cdot \left(T_1 - T_\mathrm{2,perm}\right)}{\dot{q}''} \tag{2.23}$$

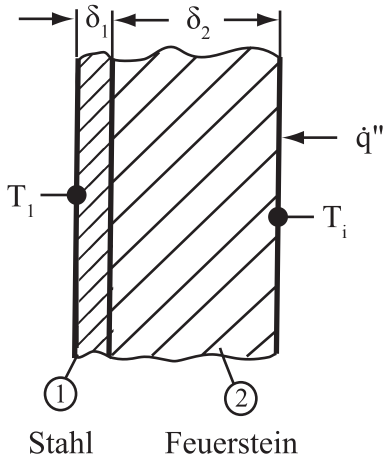
Numerical values:

The thermal conductivities are obtained through linear interpolation

$$\begin{aligned}
\lambda_2 &= f\left(\frac{T_1 + T_\mathrm{2,perm}}{2}\right) = 0.225\,\mathrm{W/m\,K} \\
\lambda_3 &= f\left(\frac{T_i + T_\mathrm{2,perm}}{2}\right) = 1.35\,\mathrm{W/m\,K} \\
\delta &= 0.008 + \frac{0.225 \cdot 740 + 1.35 \cdot 200}{1500} = 0.299\,\mathrm{m} \\
\delta &= 30\,\mathrm{cm} \qquad (\delta_2 = 11.1\,\mathrm{cm}; \delta_3 = 18\,\mathrm{cm})
\end{aligned}$$

- **Fall B**

  The minimum thermal conductivity of any material satisfying the stipulation stated above, $T_{\mathrm{perm}} \geq T_{\max} = T_{\mathrm{i}}$, is refractory brick.



As in case A it holds

$$\dot{q}'' = \frac{T_{\mathrm{i}} - T_1}{\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2}} \tag{2.24}$$

and with

$$\frac{\delta_1}{\lambda_1} \ll \frac{\delta_2}{\lambda_2}$$

$$\delta_2 = \frac{\lambda_2}{\dot{q}''} \cdot (T_{\mathrm{i}} - T_1) \tag{2.25}$$

<u>Total thickness:</u>

$$\delta = \delta_1 + \frac{\lambda_2}{\dot{q}''} \cdot (T_{\mathrm{i}} - T_1) \tag{2.26}$$

<u>Numerical values:</u>

Mean thermal conductivity of refractory brick

$$\lambda_2 \left(\frac{T_1 + T_{\mathrm{i}}}{2} \right) = 0.527\,{}^{\mathrm{W}}\!/_{\mathrm{m\,K}}$$

$$\delta = 0.008 + \frac{0.527 \cdot 940}{1500} = 0.338\,\mathrm{m}$$

$$\delta = 33.8\,\mathrm{cm} \qquad (\delta_2 = 33\,\mathrm{cm})$$

2.6. Living comfortably

To determine the overall heat flux leaving the living quarters, all other outgoing fluxes are determined for each individual surface first. Summation subsequently yields the overall heat flux leaving the system.

1. Problem type
   Steady-state, one-dimensional heat conduction through a multi layered flat wall with convective transport inside and out. It holds:

$$\dot{Q} = \frac{A}{\frac{1}{\alpha_o + \sum_{i=0}^{n} \frac{\delta_i}{\lambda_i} + \frac{1}{\lambda_B}}} \cdot (\mathrm{T_i} - \mathrm{T_o}) \tag{2.27}$$

- Floor:

$$\dot{Q}_{\mathrm{V,Floor}} = \frac{5\,\mathrm{m} \cdot 8\,\mathrm{m} \cdot (293.15\,\mathrm{K} - 279.15\,\mathrm{K})}{\frac{1}{6\,\mathrm{W/m^2\,K}} + \frac{0.015\,\mathrm{m}}{0.87\,\mathrm{W/m\,K}} + \frac{0.04\,\mathrm{m}}{0.7\,\mathrm{W/m\,K}} + \frac{0.01\,\mathrm{m}}{0.2\,\mathrm{W/m\,K}} + \ldots \\ \ldots + \frac{0.2\,\mathrm{m}}{1.4\,\mathrm{W/m\,K}} + \frac{0.02\,\mathrm{m}}{0.09\,\mathrm{W/m\,K}} + \frac{1}{6\,\mathrm{W/m^2\,K}}} \tag{2.28}$$

$$= 680.605\,\mathrm{W} \tag{2.29}$$

- Side walls:

$$A_{\mathrm{Window}} = 4\,\mathrm{m} \cdot 1.5\,\mathrm{m} + 3\,\mathrm{m} \cdot 5\,\mathrm{m} = 10.5\,\mathrm{m}^2 \tag{2.30}$$

$$\dot{Q}_{\mathrm{V,Window}} = A_{\mathrm{Window}} \cdot k_{\mathrm{F}} \cdot (T_{\mathrm{i}} - T_{\mathrm{o}}) \tag{2.31}$$

$$= 10.5\,\mathrm{m}^2 \cdot 2.9\,\mathrm{W/m^2\,K} \cdot 35\,\mathrm{K} = 1065.75\,\mathrm{W} \tag{2.32}$$

$$A_{\text{Wall}} = 2.8\,\text{m} \cdot 8\,\text{m} + 2.8\,\text{m} \cdot 5\,\text{m} - A_{\text{Window}} = 25.9\,\text{m}^2 \tag{2.33}$$

$$\dot{Q}_{\text{V,Wall}} = \frac{25.9\,\text{m}^2 \cdot 35\,\text{K}}{\frac{1}{8\,\text{W/m}^2\,\text{K}} + \frac{0.015\,\text{m}}{0.87\,\text{W/m K}} + \frac{0.02\,\text{m}}{0.09\,\text{W/m K}} + \ldots + \frac{0.34\,\text{m}}{1.05\,\text{W/m K}} + \frac{0.015\,\text{m}}{0.87\,\text{W/m K}} + \frac{1}{20\,\text{W/m}^2\,\text{K}}} \tag{2.34}$$

$$= 1199.85\,\text{W} \tag{2.35}$$

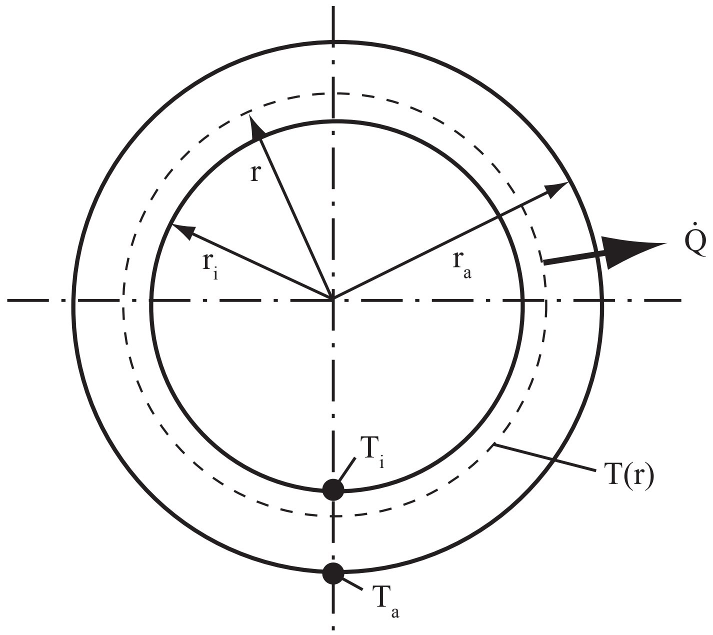
<u>Total heat flux:</u>

$$\dot{Q}_{\text{V,tot}} = 680.605\,\text{W} + 1065.75\,\text{W} + 1199.85\,\text{W} = 2946.2\,\text{W} \tag{2.36}$$

2.7. Wooden cylinder

1. Problem type:
   Steady-state, one-dimensional heat conduction through a curved surface without heat sources.

2. System boundaries and general approach:



As the temperature at the pipe's surface are constant the isothermal in the pipe's cross section are circular, therefore the temperature T is only dependent on the radius r.
For a constant heat flux and no heat sources the heat flux in radial direction is constant.

$$\dot{Q}_\mathrm{r} = \dot{Q} = \text{const.} \tag{2.37}$$

According to Fourier's law the heat flux through the jacket surface of a cylinder of radius r is equal to:

$$\begin{aligned}\dot{Q} &= -\lambda \cdot A \cdot \frac{\mathrm{d}T}{\mathrm{d}r} \\ &= -\lambda \cdot 2\pi r \cdot L \cdot \frac{\mathrm{d}T}{\mathrm{d}r} \\ &= -2\pi L \cdot \lambda_0 \left(1 + \gamma \left(T - T_0\right)\right) \cdot r \cdot \frac{\mathrm{d}T}{\mathrm{d}r}\end{aligned} \tag{2.38}$$

Because of $\dot{Q} = \text{const}$ the ODE can be solved immediately after separating the variables.

3. <u>Boundary conditions:</u>

$$\begin{aligned}T(r = r_\mathrm{i}) &= T_\mathrm{i} \\ T(r = r_\mathrm{o}) &= T_\mathrm{o}\end{aligned} \tag{2.39}$$

4. <u>Solution</u>

   a) Calculation of the heat flux throug the pipe for a variable thermal conductivity.

   Separation of variables:

$$-\frac{\dot{Q}}{2\pi L \cdot \lambda_0} \cdot \frac{\mathrm{d}r}{r} = \left(1 + \gamma \left(T - T_0\right)\right) \cdot \mathrm{d}T \tag{2.40}$$

   Applying the boundary conditions yields:

$$\frac{\dot{Q}}{2\pi L \cdot \lambda_0} \int_{r=r_\mathrm{i}}^{r=r_\mathrm{o}} \frac{\mathrm{d}r}{r} = \int_{T=T_\mathrm{i}}^{T=T_\mathrm{o}} \left(1 + \gamma \left(T - T_0\right)\right) \, \mathrm{d}T \tag{2.41}$$

   After integration the heat flux reads:

$$\begin{aligned}\dot{Q} &= -\frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \left(T_\mathrm{o} - T_\mathrm{i} + \frac{\gamma}{2}\left(\left(T_\mathrm{o} - T_0\right)^2 - \left(T_\mathrm{i} - T_0\right)^2\right)\right) \\ \dot{Q} &= +\frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \left(T_\mathrm{i} - T_\mathrm{o} + \frac{\gamma}{2}\left(T_\mathrm{i}^2 - T_\mathrm{o}^2 - 2T_0 \cdot \left(T_\mathrm{i} - T_\mathrm{o}\right)\right)\right)\end{aligned} \tag{2.42}$$

Heat flux for constant thermal conductivity:

$$\dot{Q} = \frac{2\pi L \cdot \lambda_\mathrm{m}}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot (T_\mathrm{i} - T_\mathrm{o}) \tag{2.43}$$

Through reformulation of eq. 2.42 one obtains

$$\frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot (T_\mathrm{i} - T_\mathrm{o}) \cdot \left(1 + \gamma\left(\frac{T_\mathrm{i} + T_\mathrm{o}}{2} - T_0\right)\right) = \frac{2\pi L \cdot \lambda_\mathrm{m}}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} (T_\mathrm{i} - T_\mathrm{o}) \tag{2.44}$$

Thus follows

$$\begin{aligned} \lambda_\mathrm{m} &= \lambda_0 \cdot \left(1 + \gamma\left(\frac{T_\mathrm{i} + T_\mathrm{o}}{2} - T_0\right)\right) \\ &= \lambda_0 \cdot \left(1 + \gamma\left(T_\mathrm{m} - T_0\right)\right) \end{aligned} \tag{2.45}$$

$$\text{und} \qquad T_\mathrm{m} = \frac{1}{2}\left(T_\mathrm{i} + T_\mathrm{o}\right) \tag{2.46}$$

5. Result
   For a thermal conductivity varying linearly with the temperature the relation for constant thermal conductivities can be utilised as long as the arithmetic mean values of the temperatures at the respective boundaries are used.

b) Temperature profile

Integrating eq. 2.40 within the boundaries

$$\begin{aligned} r &= r_\mathrm{i} & T &= T_\mathrm{i} \\ r &= r & T &= T \end{aligned} \tag{2.47}$$

yields

$$-\frac{\dot{Q}}{2\pi L \cdot \lambda_0} \cdot \ln\left(\frac{r}{r_\mathrm{i}}\right) = T - T_\mathrm{i} + \frac{\gamma}{2}\left(T^2 - T_\mathrm{i}^2 - 2T_0 \cdot (T - T_\mathrm{i})\right) \tag{2.48}$$

Employing the relation for the heat flux laid out in eq. 2.44, 2.45 and 2.46

$$\dot{Q} = \frac{2\pi L \cdot \lambda_0}{\ln\left(\frac{r_o}{r_i}\right)} \cdot \left(1 + \gamma \cdot \left(\frac{T_i + T_o}{2} - T_0\right)\right) \cdot (T_i - T_o) \tag{2.49}$$

one then obtains

$$T^2 + 2\,(T - T_i) \cdot \left(\frac{1}{\gamma} - T_0\right) - T_i^2 + \frac{2}{\gamma} \cdot (T_i - T_o) \cdot \ldots$$
$$\ldots \cdot \left(1 + \gamma\left(\frac{T_i + T_o}{2} - T_0\right)\right) \cdot \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)} = 0$$

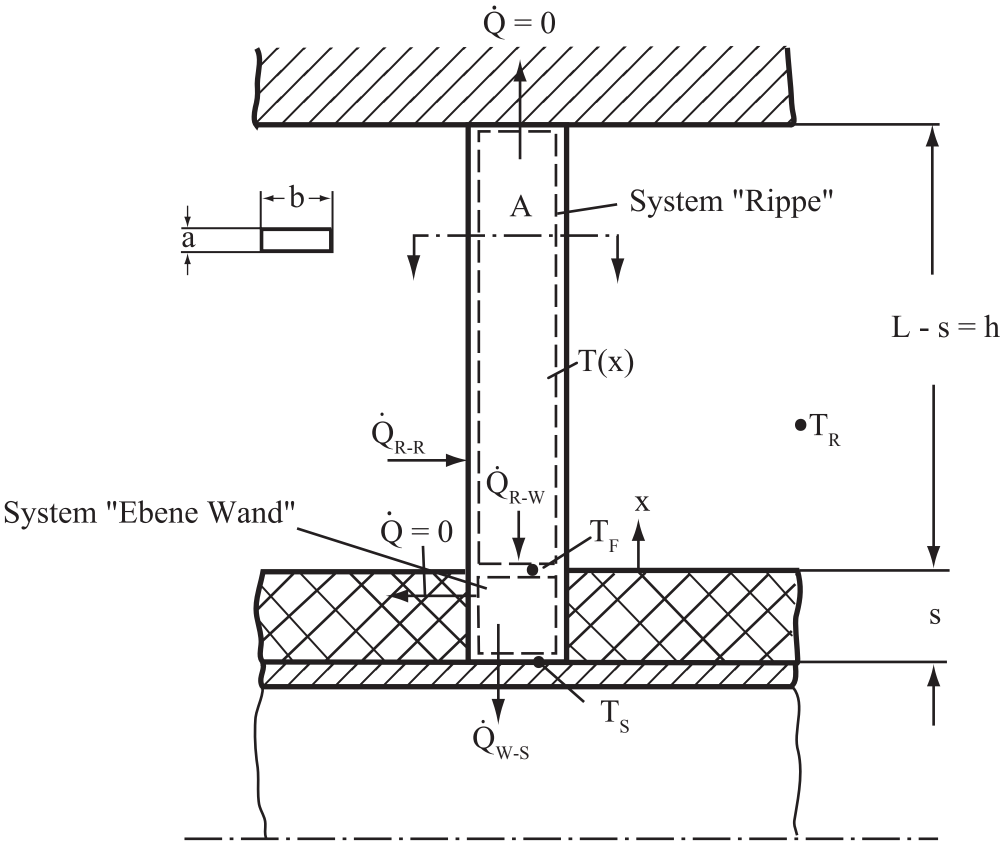
$\rightarrow$ second order polynomial eq. for T.

2.10. Pipe fastening

1. Problem type

   Steady-state heat transfer through a prismatic fin cross-section with heat transfer through a plane wall.

2. System boundary



Due to the different transfer mechanisms two systems need to be investigated. These systems are coupled through the fin base temperature.

3. Heat balances

   System **Find**

$$+\dot{Q}_{\mathrm{F-F}} - \dot{Q}_{\mathrm{F-W}} = 0 \qquad \dot{Q}_{\mathrm{F-F}} = \dot{Q}_{\mathrm{F-W}} \tag{2.50}$$

   System **Plane wall**

$$+\dot{Q}_{\mathrm{F-W}} - \dot{Q}_{\mathrm{W-S}} = 0 \qquad \dot{Q}_{\mathrm{F-W}} = \dot{Q}_{\mathrm{W-S}} = \dot{Q} \tag{2.51}$$

4. Description of balance quantities

   Heat flux through the **fin** under the assumption that the fin's head is adiabatic

$$\begin{aligned} \dot{Q}_{\mathrm{F-W}} &= \eta_{\mathrm{B}} \cdot \dot{Q}_{\mathrm{F,max}} \\ &= \alpha \cdot U \cdot h \cdot (T_{\mathrm{F}} - T_{\mathrm{B}}) \cdot \eta_{\mathrm{F}} \end{aligned} \tag{2.52}$$

   mit

$$\eta_{\mathrm{R}} = \frac{\tanh(m \cdot h)}{m \cdot h} \tag{2.53}$$

$$m = \sqrt{\frac{\alpha \cdot U}{\lambda \cdot A}} = \sqrt{\frac{\alpha \cdot 2\,(a+b)}{\lambda \cdot a \cdot b}} \tag{2.54}$$

   Wärmefluss durch die **Ebene Wall**

$$\dot{Q}_{\mathrm{W-S}} = \frac{\lambda}{s} \cdot A \cdot (T_{\mathrm{B}} - T_{\mathrm{S}}) \tag{2.55}$$

5. Solution

   - Heat flux into the brine

     With $\dot{Q}_{\mathrm{F-W}} = \dot{Q}_{\mathrm{W-S}} = \dot{Q}$ and the elimination of T_{B} follows

$$\dot{Q} = \frac{T_{\mathrm{F}} - T_{\mathrm{S}}}{\frac{1}{\alpha \cdot U \cdot h \cdot \eta_{\mathrm{F}}} + \frac{s}{\lambda \cdot A}} \tag{2.56}$$

- Numerical values:

$$m = 6.54\,{}^{1}\!/\!{}_{\mathrm{m}} \qquad m \cdot h = 1.635$$

$$\eta_{\mathrm{F}} = 0.567$$

$$\dot{Q} = \frac{43.5}{\frac{1}{6\cdot 2\cdot 0.031\cdot 0.25\cdot 0.567} + \frac{0.04}{58\cdot 0.025\cdot 0.006}} = 1.85\,\mathrm{W}$$

- Frost covered length of steel fastening h_0

  Conditions for h_0:

$$T(x = h_0) = T_0 = 0\,°\mathrm{C} \tag{2.57}$$

The determination of h_0 thus necessitates knowledge of the temperature profile within the fin.
For the given conditions it reads

$$\frac{T(x) - T_{\mathrm{F}}}{T_{\mathrm{B}} - T_{\mathrm{F}}} = \frac{\cosh(m(h-x))}{\cosh(m \cdot h)} \tag{2.58}$$

With aforementioned condition follows

$$\cosh(m(h - h_0)) = \frac{T_0 - T_{\mathrm{F}}}{T_{\mathrm{B}} - T_{\mathrm{F}}} \cdot \cosh(m \cdot h) = z \tag{2.59}$$

$$\begin{aligned} h_0 &= h - \frac{1}{m} \cdot \operatorname{arcosh}(z) \\ &= h - \frac{1}{m} \cdot \ln\left(z + \sqrt{z^2 - 1}\right) \end{aligned} \tag{2.60}$$

the fin base temperature T_{B} reamains unkown in this equation. It is easily obtained from eq. 2.55:

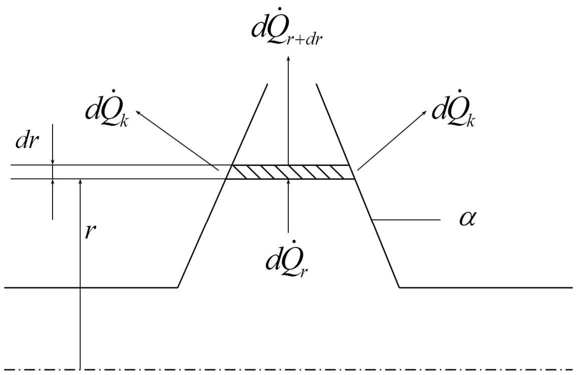
$$T_{\mathrm{B}} = T_{\mathrm{S}} + \frac{\dot{Q} \cdot s}{\lambda \cdot A} \tag{2.61}$$

- Numerical values:

$$
\begin{aligned}
T_{\mathrm{B}} &= -23.5 + \frac{1.85 \cdot 0.04}{58 \cdot 0.025 \cdot 0.006} = -15\,^{\circ}\mathrm{C} \\
z &= \frac{-20}{-35} \cdot \cosh(1.635) = 1.52 \\
h_0 &= 0.25 - \frac{1}{6.54} \ln(2.665) = 0.1\,\mathrm{m} \\
h_0 &= 10\,\mathrm{cm}
\end{aligned}
$$

2.12. Circular fin with varying thickness

a) PDE of steady-state temperature profile



From an energy balance around the infinitesimal arc segment one obtains

$$\dot{Q}_{\mathrm{r}} - \dot{Q}_{\mathrm{r+dr}} = d\dot{Q}_{\mathrm{S}} \tag{2.62}$$

The convective heat flux at the outside of the circular fin can be described by

$$d\dot{Q}_{\mathrm{S}} = \alpha \cdot (T(\mathrm{r}) - T_{\mathrm{o}}) \cdot dA_{\mathrm{S}} \qquad \text{with} \qquad dA_{\mathrm{S}} = 4 \cdot \pi \cdot r \cdot dr \tag{2.63}$$

The conductive heat flux in radial direction immediately follows from Fourier's law.

$$\dot{Q}_{\mathrm{r}} = -\lambda \cdot A(r) \cdot \frac{\mathrm{d}T}{\mathrm{d}r} \qquad \text{with} \qquad A(r) = \delta(r) \cdot 2 \cdot \pi \cdot r \tag{2.64}$$

Using the Taylor expansion $\dot{Q}_{\mathrm{r+dr}}$ is determined.

$$\dot{Q}_{\mathrm{r+dr}} = \dot{Q}_{\mathrm{r}} + \frac{\mathrm{d}\dot{Q}_{\mathrm{r}}}{\mathrm{d}r} \cdot dr \qquad \text{with} \qquad \frac{\mathrm{d}\dot{Q}_{\mathrm{r}}}{\mathrm{d}r} = -\lambda \cdot 2 \cdot \pi \cdot \frac{\mathrm{d}}{\mathrm{d}r}\left(r \cdot \delta \cdot \frac{\mathrm{d}T}{\mathrm{d}r}\right) \tag{2.65}$$

Furthermore, from the postulation of a constant heat flux density within the circular fin it is deduced that the temperature gradient, too, needs to be constant.

$$\dot{q}''_{\mathrm{r}} = \text{const.} \quad \Rightarrow \quad \frac{\mathrm{d}T}{\mathrm{d}r} = \text{const.} \quad \Rightarrow \quad \frac{\mathrm{d}T^2}{\mathrm{d}r^2} = 0 \tag{2.66}$$

Inserting equations 2.63-2.65 into equation 2.62 initially yields

$$\lambda \cdot 2 \cdot \pi \cdot \frac{\mathrm{d}}{\mathrm{d}r}\left(r \cdot \delta \cdot \frac{\mathrm{d}T}{\mathrm{d}r}\right) \cdot dr = \alpha \cdot (T(r) - T_{\mathrm{o}}) \cdot 4 \cdot \pi \cdot r \cdot dr \tag{2.67}$$

and eventually the sought-after PDE for the steady-state temperature distribution.

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r \cdot \delta \cdot \frac{\mathrm{d}T}{\mathrm{d}r}\right) - \frac{2 \cdot \alpha}{\lambda} \cdot r \cdot (T(r) - T_{\mathrm{o}}) = 0 \tag{2.68}$$

b) The fin's profile $\delta(r)$ is unknown.
The solution of eq. 2.68 requires the preceding determination of the temperature profile $T(r)$.

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{\dot{q}_{\mathrm{r}}''}{\lambda} = \text{const.} \tag{2.69}$$

Integration:

$$\mathrm{d}T = -\frac{\dot{q}_{\mathrm{r}}''}{\lambda} \cdot \mathrm{d}r \tag{2.70}$$

$$T(r) = -\frac{\dot{q}_{\mathrm{r}}''}{\lambda} \cdot r + c_1 \tag{2.71}$$

Afterwards c_1 is determined by employing the boundary conditions$T(R) = T_{\mathrm{o}}$.

$$T(r = R) = -\frac{\dot{q}_{\mathrm{r}}''}{\lambda} \cdot R + c_1 = T_{\mathrm{o}} \quad \rightarrow \quad c_1 = T_{\mathrm{o}} + \frac{\dot{q}_{\mathrm{R}}''}{\lambda} \cdot R \tag{2.72}$$

$$T(r) = -\frac{\dot{q}_{\mathrm{r}}''}{\lambda} \cdot r + T_{\mathrm{o}} + \frac{\dot{q}_{\mathrm{r}}''}{\lambda} \cdot R \tag{2.73}$$

After inserting equations 2.69 and 2.73 into 2.68, one obtains:

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}\left(\delta(r)\cdot r\cdot\left(-\frac{\dot{q}_\mathrm{r}''}{\lambda}\right)\right) - \dots$$

$$\dots - \frac{2\cdot\alpha}{\lambda}\cdot r\cdot\left(-\frac{\dot{q}_r''}{\lambda}\cdot r + T_\mathrm{o} + \frac{\dot{q}_\mathrm{r}''}{\lambda}\cdot R - T_\mathrm{o}\right) \tag{2.74}$$

$$\mathrm{d}\left(\delta(r)\cdot r\cdot\left(-\frac{\dot{q}_\mathrm{r}''}{\lambda}\right)\right) = \left(\frac{2\cdot\alpha\cdot r\cdot q_\mathrm{r}''\cdot R}{\lambda^2} - \frac{2\cdot\alpha\cdot r^2\cdot q_\mathrm{r}''}{\lambda^2}\right)\cdot\mathrm{d}r \tag{2.75}$$

$$\delta(r)\cdot r\cdot\left(-\frac{\dot{q}_\mathrm{F}''}{\lambda}\right) = \frac{\alpha\cdot r^2\cdot q_\mathrm{r}''\cdot R}{\lambda^2} - \frac{2\cdot\alpha\cdot r^3\cdot q_\mathrm{r}''}{3\cdot\lambda^2} + c_2 \tag{2.76}$$

$$\delta(r) = -\frac{\alpha\cdot r\cdot R}{\lambda} + \frac{2\cdot\alpha\cdot r^2}{3\cdot\lambda} - \frac{c_2\cdot\lambda}{r\cdot\dot{q}_\mathrm{r}''} \tag{2.77}$$

Boundary condition: Tapered fin $\delta(R) = 0$

$$\delta(R) = -\frac{\alpha\cdot R\cdot R}{\lambda} + \frac{2\cdot\alpha\cdot R^2}{3\cdot\lambda} - \frac{c_2\cdot\lambda}{R\cdot\dot{q}_\mathrm{r}''} = 0 \tag{2.78}$$

$$\Rightarrow c_2 = -\frac{1}{3}\cdot\frac{\alpha\cdot R^3\cdot\dot{q}_\mathrm{r}''}{\lambda^2} \tag{2.79}$$

$$\Rightarrow \delta\left(r\right) = -\frac{\alpha\cdot r\cdot R}{\lambda} + \frac{2}{3}\cdot\frac{\alpha\cdot r^2}{\lambda} + \frac{1}{3}\cdot\frac{R^3\cdot\alpha}{\lambda\cdot r} \tag{2.80}$$

$$= +\frac{\alpha}{3\lambda}\cdot\left(R-r\right)^2\cdot\left(2+\frac{R}{r}\right) \tag{2.81}$$

c)

Approach: The heat flux emitted from the fin's flanks is also transmitted through the fin's base of cross-section A_B.

$$\dot{Q}_\mathrm{B} = -\lambda\cdot A_\mathrm{B}\cdot\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_\mathrm{B} \tag{2.82}$$

$$A_\mathrm{B} = 2\cdot\pi\cdot r_0\cdot\delta(r_0) \quad \text{with} \quad \delta(r_0) = \frac{\alpha}{3\lambda}\cdot\left(R-r_0\right)^2\cdot\left(2+\frac{R}{r_0}\right) \tag{2.83}$$

$$T_\mathrm{B} = T(r_0) = T_\mathrm{o} + \frac{\dot{q}_\mathrm{r}''\cdot(R-r_0)}{\lambda} \tag{2.84}$$

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_\mathrm{B} = -\frac{\dot{q}_\mathrm{r}''}{\lambda} = -\frac{T_\mathrm{B}-T_\mathrm{o}}{R-r_0} \tag{2.85}$$

$$\dot{Q}_{\mathrm{B}} = \lambda \cdot 2 \cdot \pi \cdot r_0 \cdot \frac{\alpha}{3 \cdot \lambda} \cdot (R - r_0)^2 \cdot \left(2 + \frac{R}{r_0}\right) \cdot \frac{T_{\mathrm{B}} - T_{\mathrm{o}}}{R - r_0} \tag{2.86}$$

$$= \alpha \cdot (T_{\mathrm{B}} - T_{\mathrm{o}}) \cdot \frac{2\pi R^2}{3} \cdot \left(1 - \frac{r_0}{R}\right) \cdot \left(1 + \frac{2 \cdot r_0}{R}\right) \tag{2.87}$$

2.13. Double walled container

Problem type Steady-state one-dimensional conductive heat transfer with convective transport inside and out.

a) Approach:
The inner and outer container walls are regard as prismatic fins with an adiabatic fin tip above and below the ligaments. The imaginary fins, therefore, end exactly halfway between two ligaments. The temperature profile and the heat flux transmitted can thus be taken from the formulary (P. 10) immediately.

$$\dot{Q}_{\mathrm{Fin}} = \lambda \cdot A_{\mathrm{Q}} \cdot m \cdot \Theta_{\mathrm{B}} \cdot \tanh(m \cdot L) \tag{2.88}$$

Per ligament each one fin of length L extends upwards and an identical fin also extends downwards (the distance between two ligaments therefore is $2{\cdot}L$). Thus the heat flux transmitted at the in- and outside, respectively, is exactly twice the value of the fin heat flux calculated in equation 2.88.

$$\dot{Q}_{\mathrm{i,o}} = 2 \cdot \dot{Q}_{\mathrm{Fin}} \tag{2.89}$$

The fin's are initially assumed to be infinitely long.

$$\lim_{L\to\infty} \tanh(m \cdot L) = 1 \tag{2.90}$$

In addition the fin parameter m reads, $A = U \cdot \delta_{\mathrm{B}}$, as da $\delta_{\mathrm{B}} \ll d$. Thus follows:

$$m = \sqrt{\frac{\alpha \cdot U}{\lambda_{\mathrm{B}} \cdot A}} = \sqrt{\frac{\alpha}{\lambda_{\mathrm{B}} \cdot \delta_{\mathrm{B}}}} \tag{2.91}$$

Therefore the heat flux transferred to the outer wall from the surroundings is equal to

$$\dot{Q}_{\mathrm{o}} = 2 \cdot \lambda_{\mathrm{B}} \cdot \pi \cdot (d + 2 \cdot \delta_{\mathrm{ins}} + 2 \cdot \delta_{\mathrm{B}}) \cdot \delta_{\mathrm{B}} \cdot \sqrt{\frac{\alpha}{\lambda_{\mathrm{B}} \cdot \delta_{\mathrm{B}}}} \cdot (T_{\mathrm{L}} - T_{\mathrm{BL}}) \tag{2.92}$$

and similiarly the heat flux transferred from the inner wall to the fluid is equal to

$$\dot{Q}_{\mathrm{i}} = 2 \cdot \lambda_{\mathrm{B}} \cdot \pi \cdot d \cdot \delta_{\mathrm{B}} \cdot \sqrt{\frac{\alpha}{\lambda_{\mathrm{B}} \cdot \delta_{\mathrm{B}}}} \cdot (T_{\mathrm{BF}} - T_{\mathrm{B}}) \tag{2.93}$$

These fluxes must be identical to the flux transmitted through the ligament ($\dot{Q}_{\mathrm{o}} = \dot{Q}_{\mathrm{i}} = \dot{Q}_{\mathrm{lig.}}$). To determine the heat flux the ligament is regarded as a curved wall of thickness δ_{L}.

$$\dot{Q}_{\mathrm{lig.}} = \frac{2 \cdot \pi \cdot \delta_{\mathrm{s}}}{\frac{1}{\lambda_{\mathrm{B}}} \cdot \ln\left(\frac{d + 2 \cdot \delta_{\mathrm{B}} + 2 \cdot \delta_{\mathrm{ins}}}{d + 2 \cdot \delta_{\mathrm{B}}}\right)} \cdot (T_{\mathrm{BL}} - T_{\mathrm{BF}}) \tag{2.94}$$

With equations 2.92-2.94 as well as the stipulation of identical heat fluxes, three unknown quantities are represented through three equations. The governing equations for the heat are each of identical appearance.

$$\dot{Q} = \mathrm{const} \cdot \triangle T \tag{2.95}$$

Thus the following simplification follows:

$$\dot{Q} = a \cdot (T_{\mathrm{L}} - T_{\mathrm{BL}}) = i \cdot (T_{\mathrm{BF}} - T_{\mathrm{B}}) = s \cdot (T_{\mathrm{BL}} - T_{\mathrm{BF}}) \tag{2.96}$$

With $T_{\mathrm{BF}} = \dfrac{\dot{Q}}{i} + T_{\mathrm{B}}$ and $T_{\mathrm{BL}} = T_{\mathrm{L}} - \dfrac{\dot{Q}}{a}$ follows, after inserting,

$$\dot{Q} = \frac{(T_{\mathrm{L}} - T_{\mathrm{B}})}{\frac{1}{s} + \frac{1}{i} + \frac{1}{a}} \tag{2.97}$$

When the constants are once again inserted into the equation, it follows:

$$\begin{aligned} \dot{Q} = {} & \frac{(T_{\mathrm{L}} - T_{\mathrm{B}})}{\frac{\ln\left(\frac{d+2\cdot\delta_{\mathrm{B}}+2\cdot\delta_{\mathrm{ins}}}{d+2\cdot\delta_{\mathrm{B}}}\right)}{2 \cdot \pi \cdot \delta_{\mathrm{s}} \cdot \lambda_{\mathrm{B}}} + \frac{1}{2 \cdot \lambda_{\mathrm{B}} \cdot \pi \cdot d \cdot \delta_{\mathrm{B}} \cdot \sqrt{\frac{\alpha}{\lambda_{\mathrm{B}}\cdot\delta_{\mathrm{B}}}}} + \ldots} \\ & \overline{\ldots + \frac{1}{2 \cdot \lambda_{\mathrm{B}} \cdot \pi \cdot (d + 2 \cdot \delta_{\mathrm{ins}} + 2 \cdot \delta_{\mathrm{B}}) \cdot \delta_{\mathrm{B}} \cdot \sqrt{\frac{\alpha}{\lambda_{\mathrm{B}}\cdot\delta_{\mathrm{B}}}}}} \end{aligned} \tag{2.98}$$

$$= 92.93\,\mathrm{W} \tag{2.99}$$

b) The temperature drop can immediately determined from equation 2.94 by inserting the heat flux obtained in subtask a) and solving for $\triangle T$.

$$T_{\mathrm{BL}} - T_{\mathrm{BF}} = \frac{\dot{Q} \cdot \frac{1}{\lambda_{\mathrm{B}}} \cdot \ln\left(\frac{d + 2 \cdot \delta_{\mathrm{B}} + 2 \cdot \delta_{\mathrm{ins}}}{d + 2 \cdot \delta_{\mathrm{B}}}\right)}{2 \cdot \pi \cdot \delta_{\mathrm{s}}} \tag{2.100}$$

$$= 47.3\,\mathrm{K} \tag{2.101}$$

c) From the description of the subtask it is known that the temperature at $x = L$ is equal to one percent of the maximum temperature. With the temperature profile for fins with adiabatic fin tips, taken from the formulary, one obtains the following approach:

$$\Theta(x = L) = \Theta_{\max} \cdot \frac{\cosh(m \cdot (L - L))}{\cosh(m \cdot L)} = \frac{\Theta_{\max}}{\cosh(m \cdot L)} \stackrel{!}{=} \frac{\Theta_{\max}}{100} \tag{2.102}$$

$$L = \frac{\operatorname{arcosh}(100)}{m} = 0.41\,\mathrm{m} \tag{2.103}$$

The necessary distance between two ligaments is therefore equal to $2 \cdot L = 0.82\,\mathrm{m}$.

d) Approach:
Heat conduction through a multi-layered curved wall with convective heat transfer for $2 \cdot L = 1.5\,\mathrm{m}$.

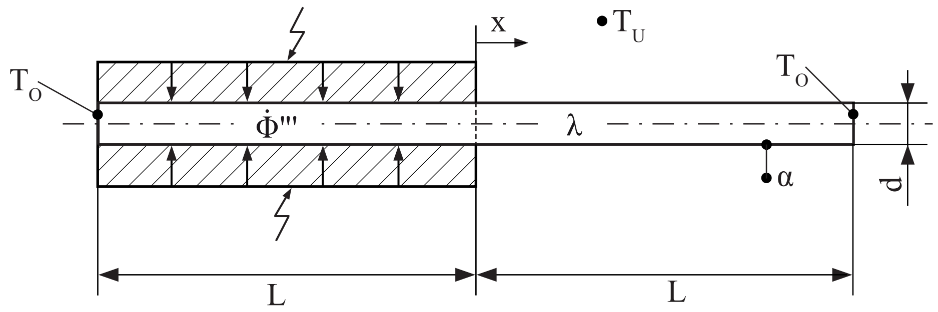
$$\frac{\dot{Q}_{\mathrm{Wall}}}{L} = \frac{\pi \cdot (T_{\mathrm{L}} - T_{\mathrm{B}})}{\frac{1}{\alpha_{\mathrm{B}} \cdot d} + \frac{\ln\left(\frac{d+2\delta_{\mathrm{B}}}{d}\right)}{2 \cdot \lambda_{\mathrm{B}}} + \frac{\ln\left(\frac{d+2\delta_{\mathrm{B}}+2\delta_{\mathrm{ins}}}{d+2\delta_{\mathrm{B}}}\right)}{2 \cdot \lambda_{\mathrm{ins}}} + \ldots} \atop {\ldots + \frac{\ln\left(\frac{d+4\delta_{\mathrm{B}}+2\delta_{\mathrm{ins}}}{d+2\delta_{\mathrm{B}}+2\delta_{\mathrm{ins}}}\right)}{2 \cdot \lambda_{\mathrm{B}}} + \frac{1}{\alpha_{\mathrm{L}} \cdot (d + 4 \cdot \delta_{\mathrm{B}} + 2 \cdot \delta_{\mathrm{ins}})}} \tag{2.104}$$

The heat transfer through the container's wall, including the insulation, remains unperturbed by the heat flux transported through the ligaments. The total heat flux transmitted is therefore the sum of both individual heat fluxes.e transportierten Wärmestrom. Den insgesamt transportierten Wärmestrom erhält man somit durch Addition beider Wärmeströme.

$$\frac{\dot{Q}_{\text{ges}}}{L} = \frac{\frac{\dot{Q}_{\text{Wall}}}{L} \cdot 1{,}5m + \dot{Q}_{\text{lig.}}}{1{,}5\,\text{m}} = 247.1\,{}^{\text{W}}\!/_{\text{m}} \tag{2.105}$$

2.15. Copper rod

1. Problem type Steady-state, one-dimensional heat conduction with heat sources (left half of rod) and steady-state, one-dimensional heat conduction within a fin (right half of rod).

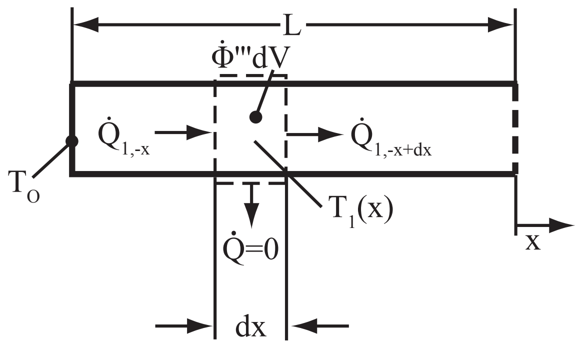


a)

2. System boundary and heat balances

As the thermal boundary conditions differ at the right (convective heat transfer) and left (adiabatic insulation $\Rightarrow \dot{Q} = 0$) side of the rod, both halves of the rod need to be regarded separately from each other. The coupling of both individual systems is ensured by the boundary conditions at $x = 0$.

- Heat balance for an element of the rod's left half:

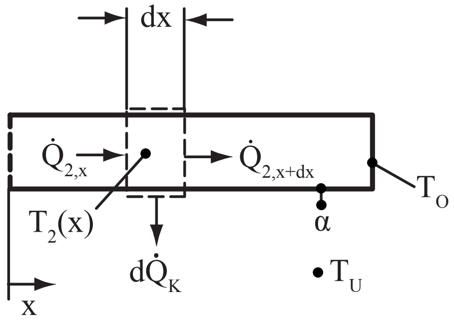


$$\dot{Q}_{1,-x} - \dot{Q}_{1,-x+dx} + \dot{\Phi}'''dV = 0$$

Systemelement:

Cylindrical slab of thickness dx (no heat flux and therefore no temperature differences in radial direction due to adiabatic insulation in radial direction).

- Heat balance for an element of the rod's right half:



$$\dot{Q}_{2,x} - \dot{Q}_{2,x+dx} - d\dot{Q}_K = 0$$

Systemelement:

Cylindrical slab of thickness dx (Assumption: heat transfer resistance in rod cross-section $\ll$ heat transfer resistance at rod circumference (Bi $\ll$ 1) no temperature difference within rod cross-section).

3. Formulation of balance quantities and differential equations for the temperature profile

- Left half of rod

$$\dot{Q}_{1,-\mathrm{x}} = -\lambda \cdot \frac{\pi d^2}{4} \cdot \frac{\mathrm{d}T_1}{\mathrm{d}x}, \qquad \dot{Q}_{1,-\mathrm{x}+\mathrm{dx}} = \dot{Q}_{1,-\mathrm{x}} + \frac{\mathrm{d}\dot{Q}_{1,-\mathrm{x}}}{\mathrm{d}x} dx$$

$$\dot{\Phi}''' dV = \dot{\Phi}''' \frac{\pi d^2}{4} dx$$

Thus becomes

$$\frac{\mathrm{d}^2 T_1}{\mathrm{d}x^2} = -\frac{\dot{\Phi}'''}{\lambda} \tag{2.106}$$

- Right half of rod

$$\dot{Q}_{2,\mathrm{x}} = -\lambda \cdot \frac{\pi d^2}{4} \cdot \frac{\mathrm{d}T_2}{\mathrm{d}x}, \qquad \dot{Q}_{2,\mathrm{x+dx}} = \dot{Q}_{2,\mathrm{x}} + \frac{\mathrm{d}\dot{Q}_{2,\mathrm{x}}}{\mathrm{d}x} dx$$
$$d\dot{Q}_K = \alpha \cdot \pi d \cdot dx \, (T_2 - T_\mathrm{U})$$

Thus becomes

$$\frac{\mathrm{d}^2 T_2}{\mathrm{d}x^2} - \frac{4 \cdot \alpha}{\lambda \cdot d} \, (T_2 - T_\mathrm{U}) = 0 \tag{2.107}$$

4. Boundary conditions
   1. $T_1(x = -L) = T_\mathrm{O}$
   2. $T_1(x = 0) = T_2(x = 0)$
   3. The heat flux flowing from the rod's left half at $x = 0$ must flow into the rod's right half.

   $$\left(\frac{\mathrm{d}T_1}{\mathrm{d}x}\right)_{x=0} = \left(\frac{\mathrm{d}T_2}{\mathrm{d}x}\right)_{x=0}$$

   4. $T_2(x = L) = T_\mathrm{O}$
5. Solution of the differential equation
   - Left half

   Integrating eq. 2.106 twice yields

   $$T_1 = -\frac{\dot{\Phi}'''}{2 \cdot \lambda} \cdot x^2 + c_1 \cdot x + c_2 \tag{2.108}$$

   - Right half

   Linear second order differential equation witg constant coefficients; homogenisable by introducing overtemperature $\Theta_2 = T_2 - T_\mathrm{U}$

   $$\Rightarrow T_2 = T_\mathrm{U} + c_3 \cdot \sinh(mx) + c_4 \cdot \cosh(mx) \tag{2.109}$$
   $$\text{mit } m = \sqrt{\frac{4 \cdot \alpha}{\lambda \cdot d}}$$

- Determination of integration constants

  BC 1 yields: $T_\mathrm{O} = -\frac{\dot{\Phi}''' \cdot L^2}{2\lambda} - c_1 \cdot L + c_2$

  BC 2 yields: $c_2 = T_\mathrm{A} + c_4$

  BC 3 yields: $c_1 = m \cdot c_3$

  BC 4 yields: $T_\mathrm{O} = T_\mathrm{A} + c_3 \cdot \sinh(m \cdot L) + c_4 \cdot \cosh(m \cdot L)$

  A set of linear equations is obtained for the constants c_1 to c_4; its solution yields the following relation:

$$
\begin{aligned}
c_1 &= \frac{1}{L} \cdot \frac{(T_\mathrm{O} - T_\mathrm{A}) - \left(T_\mathrm{O} - T_\mathrm{A} + \frac{\dot{\Phi}'''}{2\lambda} L^2\right) \cdot \cosh(m \cdot L)}{\frac{\sinh(m \cdot L)}{m \cdot L} + \cosh(m \cdot L)} \\
c_2 &= T_\mathrm{A} + \frac{(T_\mathrm{O} - T_\mathrm{A}) + \left(T_\mathrm{O} - T_\mathrm{A} + \frac{\dot{\Phi}'''}{2\lambda} L^2\right) \cdot \frac{\sinh(m \cdot L)}{m \cdot L}}{\frac{\sinh(m \cdot L)}{m \cdot L} + \cosh(m \cdot L)} \\
c_3 &= \frac{c_1}{m} \\
c_4 &= c_2 - T_\mathrm{A}
\end{aligned}
$$

  Thus the temperature profile in the entire rod is defined for a given heat source flux density $\dot{\Phi}'''$.

b) Determination of heat source flux density $\dot{\Phi}'''$ under the condition $T_1(x = 0) = T_2(x = 0) = T_\mathrm{O}$.

   For a prefined heat source flux density a the temperature in the rod's centre is reached due to the constraints imposed by the boundary conditions. To obtain a certain temperature value, the source flux density needs to be adjusted accordingly.

   Through the condition $T_1(x = 0) = T_2(x = 0) = T_\mathrm{O}$, boundary condition 2 is further constrained. Thus follows Durch die Bedingung $T_1(x = 0) = T_2(x = 0) = T_\mathrm{O}$ wird die Randbedingung 2 noch weiter festgelegt. Es ist dann

$$
c_2 = T_\mathrm{A} + c_4 = T_\mathrm{O} \tag{2.110}
$$

With the relation for the temperature profile the stipulation for the source flux density $\dot{\Phi}'''$ is obtained.

$$c_2 = T_\mathrm{O} = T_\mathrm{A} + \frac{T_\mathrm{O} - T_\mathrm{A} + \left(T_\mathrm{O} - T_\mathrm{A} + \frac{\dot{\Phi}'''}{2\lambda} L^2\right) \cdot \frac{\sinh(mL)}{mL}}{\frac{\sinh(mL)}{mL} + \cosh(mL)} \tag{2.111}$$

Solving for $\dot{\Phi}'''$ yields

$$\dot{\Phi}''' = \frac{2 \cdot \lambda \cdot m}{L} \cdot (T_\mathrm{O} - T_\mathrm{A}) \cdot \frac{\cosh(mL) - 1}{\sinh(mL)} \tag{2.112}$$

and with the addition theorems for hyperbolic functions

$$\dot{\Phi}''' = \frac{2 \cdot \lambda \cdot m}{L} \cdot (T_\mathrm{O} - T_\mathrm{A}) \cdot \tanh\left(\frac{mL}{2}\right) \tag{2.113}$$

c) Numerical values:

$$m = \sqrt{\frac{4 \cdot \alpha}{\lambda \cdot d}} = 3.52\,{}^{1}\!/\!_{\mathrm{m}}$$
$$\dot{\Phi}''' = 49.4\,{}^{\mathrm{kW}}\!/\!_{\mathrm{m}^3}$$

d) Location of extreme values for $T_1(x = 0) = T_2(x = 0) = T_\mathrm{O}$

For reasons of symmetry the locus of the maximum temperature is

$$x_\mathrm{max} = -\frac{L}{2} \tag{2.114}$$

similiarly the locus of the minimum temperature is

$$x_\mathrm{min} = \frac{L}{2}. \tag{2.115}$$

Temperature profile for $T_1(x = 0) = T_2(x = 0) = T_\mathrm{O}$

With $c_2 = T_\mathrm{O}$ and BC 1 one easily obtains for the left half of the rod

$$T_1 = -\frac{\dot{\Phi}''' \cdot L^2}{2\lambda} \left(\left(\frac{x}{L}\right)^2 + \frac{x}{L}\right) + T_\mathrm{O} \tag{2.116}$$

In the rod's right half the heat flux is zero at $x = \frac{L}{2}$ due to reasons of symmetry. I.e. the right half of the rod can be regarded as a fin of height $\frac{L}{2}$ and an overtemperature of $(T_{\mathrm{O}} - T_{\mathrm{A}})$ at its base. According to eq. (3.39) of the lecture script one obtains:

$$T_2 = T_{\mathrm{A}} + (T_{\mathrm{O}} - T_{\mathrm{A}}) \cdot \frac{\cosh\left(m\left(\frac{L}{2} - x\right)\right)}{\cosh\left(\frac{mL}{2}\right)} \tag{2.117}$$

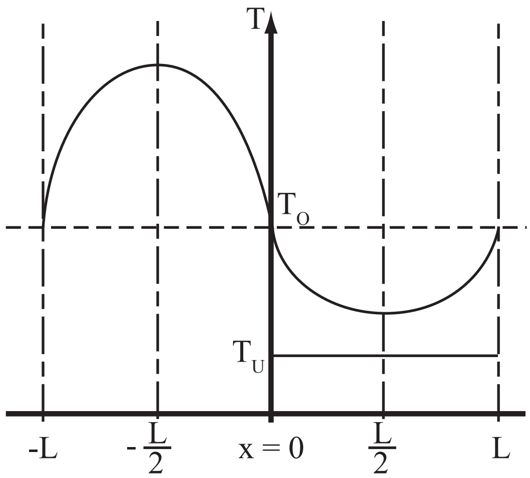
The same result is obtained after a few reformulations when taking into account the constants determined in a) and inserting these into the relation for $\dot{\Phi}'''$ of subtask b).

Maximum temperature

$$T_{\max} = T_1\left(x = -\frac{L}{2}\right) = T_{\mathrm{O}} + \frac{\dot{\Phi}''' \cdot L^2}{8\lambda} = 136.6\,^{\circ}\mathrm{C} \tag{2.118}$$

Minimum temperature

$$T_{\min} = T_2\left(x = \frac{L}{2}\right) = T_U + \frac{T_{\mathrm{O}} - T_{\mathrm{A}}}{\cosh\left(\frac{mL}{2}\right)} = 106.7\,^{\circ}\mathrm{C} \tag{2.119}$$

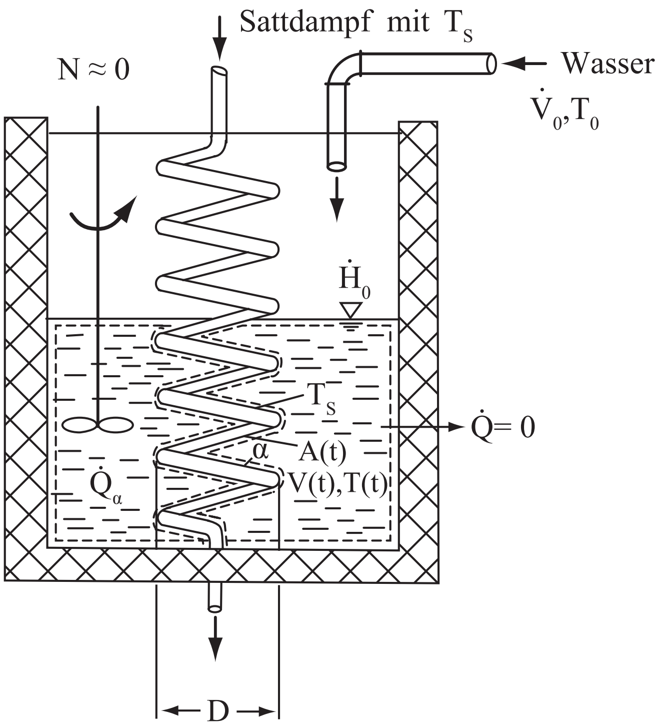


2.18. Stirred tank

1. Problem type:

   Instationary heat transfer with negligible local temperature differences (Heating up and cooling down phases)

2. System boundary and heat balance



Amount of liquid $\rho V(t)$ within the tank. Energy stored within the agitator the coiled pipe and in the tank's wall as well as energy imparted through the agitator are negligible.
For reasons of practicality the system boundary is placed around the coiled pipe, as its surface temperature is T_C known.
An unknown mass flux of saturated steam or condensate, respectively, enters the system at the location where the system boundary intersects the coil.

Heat balance

$$+\dot{Q}_\alpha + \dot{H}_0 = \frac{d(m \cdot u)}{dt} \tag{2.120}$$

3. Formulation of balance quantities and differential equations for the temperature profile

   Temporal change in internal energy within the system boundary

$$\frac{\mathrm{d}(m \cdot u)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho V \cdot u\right) = \rho u \frac{\mathrm{d}V}{\mathrm{d}t} + \rho \cdot V \frac{\mathrm{d}u}{\mathrm{d}t} \tag{2.121}$$

   The specific internal energy u can be described with $c_v = c_p = c$ and $\rho = \text{const.}$ (incompressible liquid) as follows

$$u = u_0(T_{\mathrm{Bez}}) + c \cdot (T - T_{\mathrm{ref}}) \tag{2.122}$$

   As no phase change takes place,

$$u_0(T_{\mathrm{ref}}) = 0 \qquad \text{und} \quad T_{\mathrm{ref}} = 0 \tag{2.123}$$

   can be set to:

$$\frac{\mathrm{d}(m \cdot u)}{\mathrm{d}t} = \rho c \frac{\mathrm{d}V}{\mathrm{d}t} \cdot T + \rho c \cdot V \cdot \frac{\mathrm{d}T}{\mathrm{d}t} \tag{2.124}$$

   Enthalpy influx

$$\dot{H}_0 = \rho \cdot \dot{V}_0 \cdot c \cdot T_0 \tag{2.125}$$

   Convective heat influx crossing system boundary:

$$\dot{Q}_\alpha = \alpha \cdot A\left(T_{\mathrm{C}} - T\right) \tag{2.126}$$

   The amount of liquid contained within the tank and the surface area involved in heat transfer at any point in time is given by

$$V = \dot{V}_0 \cdot t \tag{2.127}$$

$$A = \frac{A_{\mathrm{ges}}}{V_{\mathrm{ges}}} \cdot V = \frac{A_{\mathrm{ges}}}{V_{\mathrm{tot}}} \cdot \dot{V}_0 \cdot t \tag{2.128}$$

Therefore, one obtains, with minimal adjustments, the following ODE

$$\frac{\alpha \cdot A_{\text{tot}}}{\rho \cdot c \cdot V_{\text{tot}}} \cdot t \cdot (T_{\text{C}} - T) - (T - T_0) = t \cdot \frac{\mathrm{d}T}{\mathrm{d}t} \tag{2.129}$$

The introduction of the dimensionless variables

$$\Theta^* = \frac{T - T_0}{T_{\text{C}} - T_0}, \qquad t^* = \frac{t}{\tau} \qquad \text{with} \quad \tau = \frac{\rho \cdot c \cdot V_{\text{tot}}}{\alpha \cdot A_{\text{tot}}} \tag{2.130}$$

leads to the ODE

$$\frac{\mathrm{d}\Theta^*}{\mathrm{d}t^*} + \left(1 + \frac{1}{t^*}\right) \cdot \Theta^* = 1 \tag{2.131}$$

3. Initial condition

$$T(t = 0) = T_0 \quad \Rightarrow \quad \Theta^*(t^* = 0) = 0 \tag{2.132}$$

4. Solutionof the differential equation

Type of ODE: inhomogeneous, linear, first order

$$\begin{aligned}
\Theta^* &= \exp\left(-\int \left(1 + \frac{1}{t^*}\right) \mathrm{d}t^*\right) \cdot \ldots \\
&\qquad \ldots \cdot \left(c + \int 1 \cdot \exp\left(\int \left(1 + \frac{1}{t^*}\right) \mathrm{d}t^*\right) \mathrm{d}t^*\right) \\
&= \exp(-t^*) \cdot \exp(-\ln t^*) \cdot \left(c + \int \exp(t^*) \cdot \exp(\ln t^*)\, \mathrm{d}t^*\right) \\
&= \frac{\exp(-t^*)}{t^*} \cdot (c + t^* \cdot \exp(t^*) - \exp(t^*)) \\
&= 1 + \frac{c \cdot \exp(-t^*) - 1}{t^*}
\end{aligned} \tag{2.133}$$

Determination of integration constants

If one formally inserts the initial condition into the general solution, an undefined expression is obtained. This solution in turn leads to the stipulation that, in order to adhere to the initial condition,

$$\Theta^*(t^* = 0) = 0$$

the integration constant would have to meet the following constraint

$$\lim_{t^* \to 0} \frac{c \cdot \exp\left(-t^*\right) - 1}{t^*} = -1 \tag{2.134}$$

A finite value in the limit can only be obtained, if the numerator is of identical order as the denominator for $t^* \to 0$. Then one obtains, according to the law of Bernoulli-L'Hospital,

$$\begin{aligned} \lim_{t^* \to 0} \frac{c \cdot \exp\left(-t^*\right) - 1}{t^*} &= \lim_{t^* \to 0} \frac{-c \cdot \exp\left(-t^*\right)}{1} \stackrel{?}{=} -1 \\ \Rightarrow c &= 1! \end{aligned} \tag{2.135}$$

The equation for the temporal evolution of the temperature profile thus reads

$$\Theta^* = 1 + \frac{\exp\left(-t^*\right) - 1}{t^*} \tag{2.136}$$

5. Determination of the water temperature's numerical value

For the time after the tank has been filled, it holds:

$$V_{\mathrm{tot}} = \dot{V}_0 \cdot t_{\mathrm{tot}} \qquad\qquad t_{\mathrm{tot}} = \frac{V_{\mathrm{tot}}}{\dot{V}_0} \tag{2.137}$$

Thus transmutes

$$\frac{T_{\mathrm{We}} - T_0}{T_{\mathrm{C}} - T_0} = 1 + \frac{\exp(\left(-\frac{\alpha \cdot A_{\mathrm{tot}}}{\rho \cdot c \cdot V_0}\right) - 1}{\frac{\alpha \cdot A_{\mathrm{tot}}}{\rho \cdot c \cdot \dot{V}_0}} \tag{2.138}$$

Material properties of water at approx. 50 °C:

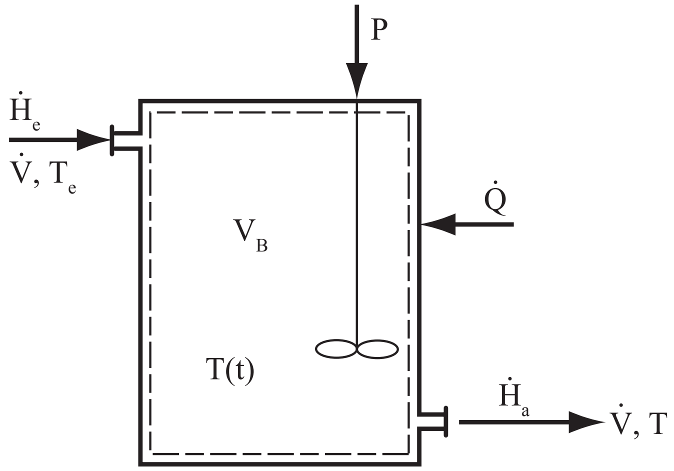
$$
\begin{aligned}
\rho &= 988\,{}^{\mathrm{kg}}\!/\!_{\mathrm{m^3}}, \\
c &= 4.184\,{}^{\mathrm{kJ}}\!/\!_{\mathrm{kg\,K}} \\
\Rightarrow \frac{\alpha \cdot A_{\mathrm{tot}}}{\rho \cdot c \cdot \dot{V}_0} &= \frac{\alpha \cdot \pi^2 \cdot d \cdot D \cdot n}{\rho \cdot c \cdot \dot{V}_0} = 2.865 \\
\Rightarrow \frac{T_{\mathrm{We}} - T_0}{T_{\mathrm{C}} - T_0} &= 0.671 \\
\Rightarrow T_{\mathrm{We}} &= 77\,{}^{\circ}\mathrm{C}
\end{aligned}
$$

2.19. Oscillation problem

1. Problem type

   Periodic heat transfert in a stirred tank.

2. System boundary and heat balance



As local temperature differences are equalised through the tank's agitator the balance system can be chosen to be identical to the tank volume V_T.

Heat balance:

$$\dot{H}_\mathrm{e} + P + \dot{Q}_\mathrm{W} - H_\mathrm{o} = \frac{\mathrm{d}U}{\mathrm{d}t} \tag{2.139}$$

3. Balance quantities and differential equations for the temperature profile

   Enthalpy influx: $\dot{H}_\mathrm{i} = \dot{V} \cdot \rho \cdot c \cdot T_\mathrm{i}$

   Enthalpy outflux: $\dot{H}_\mathrm{o} = \dot{V} \cdot \rho \cdot c \cdot T$

   The specific reference enthalpy has been set to $h_\mathrm{ref}(T_\mathrm{ref}) = 0$.

   Change in internal energy: $\frac{\mathrm{d}U}{\mathrm{d}t} = V_\mathrm{T} \cdot \rho \cdot c \cdot \frac{\mathrm{d}T}{\mathrm{d}t}$

Thus

$$\dot{V} \cdot \rho \cdot c \cdot (T_\mathrm{i} - T) + P + \dot{Q}_\mathrm{W} = V_\mathrm{T} \cdot \rho \cdot c \frac{\mathrm{d}T}{\mathrm{d}t} \tag{2.140}$$

With

$$T_\mathrm{i} = T_0 + \Theta_\mathrm{e,max} \cdot \sin(\omega t) \tag{2.141}$$

the ODE reads

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\dot{V}}{V_{\mathrm{T}}}(T - T_0) = \frac{\dot{V}}{V_{\mathrm{T}}}\Theta_{\mathrm{e,max}} \cdot \sin(\omega t) + \frac{P + \dot{Q}_{\mathrm{W}}}{\rho \cdot c \cdot V_{\mathrm{T}}} \tag{2.142}$$

Introducing a few substitutions

$$\tau = \frac{V_{\mathrm{T}}}{\dot{V}} \qquad \kappa = \frac{P + \dot{Q}_{\mathrm{W}}}{\rho \cdot c \cdot V_{\mathrm{T}}}$$
$$\Theta = T - T_0$$

yields

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} + \frac{1}{\tau} \cdot \Theta = \frac{\Theta_{\mathrm{e,max}}}{\tau} \cdot \sin(\omega t) + \kappa \tag{2.143}$$

4. Solution of the differential equation

The generall solution of this first order ODE comprises a homogeneous and a particalur contribution

$$\Theta = \Theta_{\mathrm{h}} + \Theta_{\mathrm{p}} \tag{2.144}$$

The homogeneous term describes the transient process from an initial state $\rightarrow \exp\left(\frac{-t}{\tau}\right)$. The state to be investigated here has fully passed the transient state ($t \rightarrow \infty$) and thus the homogeneous part of the solution is irrelevant. Furthermore, the initial state is irrelevant and can no longer be determined, in any case. The temperature is governed purely through the solution particular contribution.

Approach for the particular solution as a perturbation term within the differential equation:

$$\Theta = k_1 \cdot \sin(\omega t + \varphi) + k_2 \tag{2.145}$$

The quantities k_1, k_2 and φ need to be determined such that the approach satisfies the stipulation of the ODE. One obtains

$$\begin{aligned} \omega k_1 \cdot \underbrace{\cos(\omega t + \varphi)}_{\cos(\omega t)\cdot\cos(\varphi) - \sin(\omega t)\cdot\sin(\varphi)} + \frac{1}{\tau} \cdot k_1 \cdot \underbrace{\sin(\omega t + \varphi)}_{\sin(\omega t)\cdot\cos(\varphi) + \cos(\omega t)\cdot\sin(\varphi)} + \frac{k_2}{\tau} = \\ = \frac{\Theta_{\mathrm{e,max}}}{\tau} \cdot \sin(\omega t) + \kappa \end{aligned} \tag{2.146}$$

A comparison of coefficients yields

Constant: $\frac{k_2}{\tau} = \kappa \Rightarrow k_2 = \tau \cdot \kappa$

$\sin(\omega t)$: $-\omega k_1 \cdot \sin(\varphi) + \frac{1}{\tau} \cdot k_1 \cdot \sin(\varphi) = \frac{\Theta_{\text{i,max}}}{\tau}$

$\cos(\omega t)$: $\omega k_1 \cdot \cos(\varphi) + \frac{1}{\tau} \cdot k_1 \cdot \sin(\varphi) = 0 \Rightarrow \tan(\varphi) = -\omega\tau$

Solving the last two equations for $\sin(\varphi)$ and $\cos(\varphi)$ yields

$$\begin{aligned} -\sin(\varphi) &= \frac{\Theta_{\text{i,max}}}{k_1 \cdot \tau} \cdot \frac{\omega}{\omega^2 + \frac{1}{\tau^2}} \\ \cos(\varphi) &= \frac{\Theta_{\text{i,max}}}{k_1 \cdot \tau^2} \cdot \frac{1}{\omega^2 + \frac{1}{\tau^2}} \end{aligned}$$

Squaring the equations and adding them produces:

$$\begin{aligned} \sin^2(\varphi) + \cos^2(\varphi) = 1 &= \frac{\Theta^2_{\text{i,max}}}{k_1^2 \cdot \left(\omega^2 + \frac{1}{\tau^2}\right)^2} \cdot \left(\left(\frac{\omega}{\tau}\right)^2 + \frac{1}{\tau^4}\right) \\ &= \frac{\Theta^2_{\text{i,max}}}{k_1^2 \cdot \tau^2} \cdot \frac{1}{\omega^2 + \frac{1}{\tau^2}} \end{aligned}$$

The unknown adjustment coefficient therefore are

$$\begin{aligned} \varphi &= -\arctan(\omega) \cdot \frac{V_{\text{T}}}{\dot{V}} \\ k_1 = \Theta_{\text{o,max}} &= \frac{\Theta_{\text{i,max}}}{\tau} \cdot \frac{1}{\sqrt{\omega^2 + \frac{1}{\tau^2}}} \\ &= \frac{\Theta_{\text{i,max}}}{\sqrt{1 + \omega^2\tau^2}} = \frac{\Theta_{\text{i,max}}}{\sqrt{1 + \left(\frac{\omega \cdot V_{\text{T}}}{\dot{V}}\right)^2}} \\ \rightarrow k_2 = \kappa \cdot \tau &= \frac{P + \dot{Q}_{\text{W}}}{\rho \cdot c \cdot \dot{V}} = \Delta T_0 \end{aligned}$$

ΔT_0 represents the increase in the temporal mean value of the liquid's temperature from ingestion into until expulsion from the tank.

5. Determination of the necessary tank volume

Stipulation:

$$\frac{\Theta_{\mathrm{o,max}}}{\Theta_{\mathrm{i,max}}} = p = 0.001 \tag{2.147}$$

Amplitude of temperature oscillation at point of expulsion

$$\Theta_{\mathrm{o,max}} = k_1 = \Theta_{\mathrm{i,max}} \frac{1}{\sqrt{1 + \left(\frac{\omega V_{\mathrm{T}}}{\dot{V}}\right)^2}}$$

$$\frac{\Theta_{\mathrm{o,max}}}{\Theta_{\mathrm{i,max}}} = p = \frac{1}{\sqrt{1 + \left(\frac{\omega V_{\mathrm{T}}}{\dot{V}}\right)^2}}$$

Solving for V_{T} yields

$$\omega = \frac{2\pi}{t_{\mathrm{C}}}$$

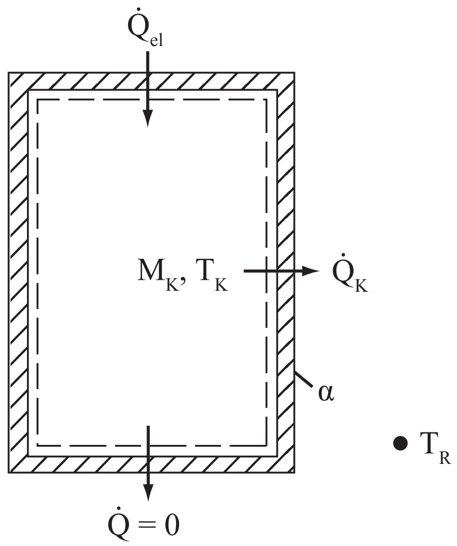
$$V_{\mathrm{T}} = \frac{\dot{V} \cdot t_{\mathrm{C}}}{2\pi} \cdot \sqrt{\frac{1}{p^2} - 1} = 39.8\,\mathrm{m}^3$$

2.20. Night-storage heater

1. Problem type

   Instationary heat conduction with heat transfer, heating and cooling.

2. System boundary



   System:
   As no temperature differences occur within the storage core, the system's boundary can be drawn around the storage core. The storage capacity of the heater's insulation is negligible such that no additional balance needs to be introduced. Heat balance:

$$\dot{Q}_{\mathrm{el}} - \dot{Q}_{\mathrm{C}} = \frac{\mathrm{d}U_{\mathrm{C}}}{\mathrm{d}t} \tag{2.148}$$

3. Balance quantities and differential equations

   Heat flux from storage core to surroundings:

$$\dot{Q}_{\mathrm{C}} = k \cdot A \cdot (T_{\mathrm{C}} - T_{\mathrm{F}}) \tag{2.149}$$

Coefficient of thermal transmissivity:

$$\frac{1}{k} = \frac{1}{\alpha} + \frac{\delta_{\text{iso}}}{\lambda_{\text{iso}}} \tag{2.150}$$

Temporal evolution of storage core's internal energy. As no phase change takes place, it holds:

$$\frac{\mathrm{d}U_{\mathrm{C}}}{\mathrm{d}t} = M_{\mathrm{C}} \cdot c_{\mathrm{C}} \cdot \frac{\mathrm{d}T_{\mathrm{C}}}{\mathrm{d}t} \tag{2.151}$$

Thus

$$M_{\mathrm{C}} \cdot c_{\mathrm{C}} \cdot \frac{\mathrm{d}T_{\mathrm{C}}}{\mathrm{d}t} = \dot{Q}_{\mathrm{el}} - k \cdot A \cdot (T_{\mathrm{C}} - T_{\mathrm{F}}) \tag{2.152}$$

With the temporal constant

$$\tau = \frac{M_{\mathrm{C}} \cdot c_{\mathrm{C}}}{k \cdot A} \tag{2.153}$$

the steady-state storage core temperature for continous loading $\left(\frac{\mathrm{d}T_{\mathrm{C}}}{\mathrm{d}t} = 0\right)$

$$T_{\mathrm{K,stat}} = T_{\mathrm{F}} + \frac{\dot{Q}_{\mathrm{el}}}{k \cdot A} \tag{2.154}$$

and by introducing the overtemperature

$$\Theta_{\mathrm{C}} = T_{\mathrm{C}} - T_{\mathrm{F}} \tag{2.155}$$

the differential equations result.

Charging cycle: $0 < t < t_{\mathrm{L}}$

$$\frac{\mathrm{d}\Theta_{\mathrm{C,L}}}{\mathrm{d}t} + \frac{1}{\tau}\Theta_{\mathrm{C,L}} = \frac{\dot{Q}_{\mathrm{el}}}{M_{\mathrm{C}} \cdot c_{\mathrm{C}}} = \frac{1}{\tau} \cdot (T_{\mathrm{K,stat}} - T_{\mathrm{F}}) \tag{2.156}$$

Discharging cycle: $t_{\mathrm{L}} < t < t_{\mathrm{g}}$

$$\frac{\mathrm{d}\Theta_{\mathrm{C,D}}}{\mathrm{d}t} + \frac{1}{\tau}\Theta_{\mathrm{C,D}} = 0 \tag{2.157}$$

4. Initial conditions

For a quasi steady-state it holds

$$T_{\mathrm{C,L}}(t=0) = T_{\mathrm{C,D}}(t=t_{\mathrm{g}}) \qquad \text{bzw.} \quad \Theta_{\mathrm{C,L}}(t=0) = \Theta_{\mathrm{C,D}}(t=t_{\mathrm{g}})$$
$$T_{\mathrm{C,L}}(t=t_{\mathrm{L}}) = T_{\mathrm{C,D}}(t=t_{\mathrm{L}}) \qquad \text{bzw.} \quad \Theta_{\mathrm{C,L}}(t=t_{\mathrm{L}}) = \Theta_{\mathrm{C,D}}(t=t_{\mathrm{L}})$$

5. Solution of the differential equations

   The ODE for the charging cycle resembles the general case and comprises the discharging cycle for $\dot{Q}_{\mathrm{el}} = 0$.

   General solution of the ODE:

$$\Theta_{\mathrm{C}} = c \cdot \exp\left(-\frac{t}{\tau}\right) + \frac{\dot{Q}_{\mathrm{el}}}{k \cdot A} \tag{2.158}$$

   Charging cycle

$$\Theta_{\mathrm{C,L}} = c_{\mathrm{L}} \cdot \exp\left(-\frac{t}{\tau}\right) + \frac{\dot{Q}_{\mathrm{el}}}{k \cdot A} \tag{2.159}$$

   Discharging cycle

$$\Theta_{\mathrm{C,D}} = c_{\mathrm{i}} \cdot \exp\left(-\frac{t}{\tau}\right) \tag{2.160}$$

   Determination of integration constants

   From stipulation 1 follows

$$c_{\mathrm{L}} + \frac{\dot{Q}_{\mathrm{el}}}{k \cdot A} = c_{\mathrm{i}} \cdot \exp\left(-\frac{t_{\mathrm{g}}}{\tau}\right) \tag{2.161}$$

   and from stipulation 2

$$c_{\mathrm{L}} + \exp\left(-\frac{t_{\mathrm{L}}}{\tau}\right) + \frac{\dot{Q}_{\mathrm{el}}}{k \cdot A} = c_{\mathrm{i}} \cdot \exp\left(-\frac{t_{\mathrm{L}}}{\tau}\right) \tag{2.162}$$

Solving with $t_\mathrm{i} = t_\mathrm{g} - t_\mathrm{L}$ yields

$$c_\mathrm{L} = -\frac{\dot{Q}_\mathrm{el}}{k \cdot A} \cdot \frac{1 - \exp\left(-\frac{t_\mathrm{i}}{\tau}\right)}{1 - \exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \qquad c_\mathrm{i} = -\frac{\dot{Q}_\mathrm{el}}{k \cdot A} \cdot \frac{1 - \exp\left(\frac{t_\mathrm{L}}{\tau}\right)}{1 - \exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \tag{2.163}$$

Temporal evolution of temperature profiles

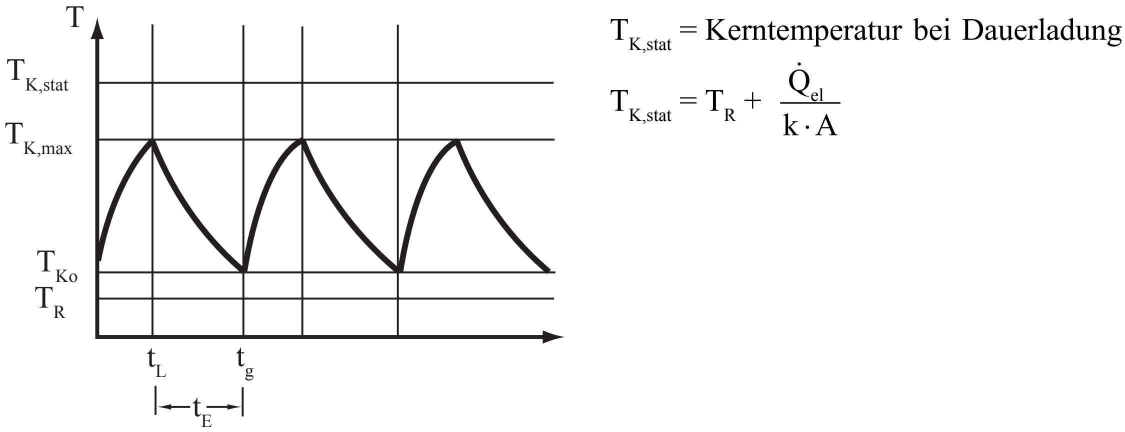
Charging cycle

$$\Theta_\mathrm{C,L} = \frac{\dot{Q}_\mathrm{el}}{k \cdot A} \cdot \left(1 - \frac{1 - \exp\left(-\frac{t_\mathrm{i}}{\tau}\right)}{1 - \exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \cdot \exp\left(-\frac{t}{\tau}\right)\right) \tag{2.164}$$

Discharging cycle

$$\Theta_\mathrm{C,D} = \frac{\dot{Q}_\mathrm{el}}{k \cdot A} \cdot \frac{1 - \exp\left(-\frac{t_\mathrm{L}}{\tau}\right)}{1 - \exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \cdot \exp\left(-\frac{t - t_\mathrm{L}}{\tau}\right) \tag{2.165}$$

b) Sketch of the temperature profile



c) Charging temperature

Amount of heat required per day $Q_\mathrm{H} = 13.5\,\mathrm{kWh}$. This amount of heat needs to be stored during the charging cycle. Considering the heating element's power consumption $\dot{Q}_\mathrm{el}$, one obtains:

$$t_\mathrm{L} = \frac{Q_\mathrm{H}}{\dot{Q}_\mathrm{el}} = 9\,\mathrm{h} \tag{2.166}$$

Core overtemperatures

Initiation of charging

$$\begin{aligned}\Theta_{\mathrm{K},0} &= \Theta_{\mathrm{C,L}}(t=0) = \Theta_{\mathrm{C,D}}(t=t_\mathrm{g}) \\ &= \frac{\dot{Q}_\mathrm{el}}{k \cdot A} \cdot \frac{\exp\left(+\frac{t_\mathrm{L}}{\tau}\right) - 1}{\exp\left(+\frac{t_\mathrm{g}}{\tau}\right) - 1}\end{aligned} \tag{2.167}$$

Cessation of charging

$$\Theta_\mathrm{max} = \Theta_{\mathrm{C,L}}(t=t_\mathrm{L}) = \Theta_{\mathrm{C,D}}(t=t_\mathrm{L}) = \frac{\dot{Q}_\mathrm{el}}{k \cdot A} \cdot \frac{1-\exp\left(-\frac{t_\mathrm{L}}{\tau}\right)}{1-\exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \tag{2.168}$$

Numerical values

$$\begin{aligned}\frac{1}{k} &= 0.1 + \frac{0.04}{0.08} = 0.6 \quad \Rightarrow \quad k = 1.67\,{}^\mathrm{W}/_{\mathrm{m^2\,K}} \\ \tau &= \frac{90 \cdot 1{,}2 \cdot 10^3}{1.67 \cdot 0.8 \cdot 3600} = 22.5\,\mathrm{h} \\ \Theta_{\mathrm{C},0} &= 290\,\mathrm{K} \\ \Theta_{\mathrm{C,max}} &= 566\,\mathrm{K}\end{aligned}$$

d) Heat output during discharging cycle

$$
\begin{aligned}
Q_{\mathrm{H,E}} &= k \cdot A \cdot \int_{t_\mathrm{L}}^{t_\mathrm{g}} \Theta_{\mathrm{C,D}} \mathrm{d}t \\
&= \dot{Q}_{\mathrm{el}} \cdot \frac{1 - \exp\left(-\frac{t_\mathrm{L}}{\tau}\right)}{1 - \exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \cdot \int_{t_\mathrm{L}}^{t_\mathrm{g}} \exp\left(-\frac{t - t_\mathrm{L}}{\tau}\right) \mathrm{d}t \\
&= \dot{Q}_{\mathrm{el}} \cdot \tau \cdot \frac{1 - \exp\left(-\frac{t_\mathrm{L}}{\tau}\right)}{1 - \exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \cdot \left(1 - \exp\left(-\frac{t_\mathrm{i}}{\tau}\right)\right)
\end{aligned}
$$

Percentage of heat emitted during charging and discharging cycles

Discharging cycle

$$
\frac{Q_{\mathrm{H,E}}}{Q_\mathrm{H}} \cdot 100\,\% = 100 \cdot \frac{Q_{\mathrm{H,E}}}{t_\mathrm{L} \cdot \dot{Q}_{\mathrm{el}}} = \frac{\tau}{t_\mathrm{L}} \cdot \frac{1 - \exp\left(-\frac{t_\mathrm{L}}{\tau}\right)}{1 - \exp\left(-\frac{t_\mathrm{g}}{\tau}\right)} \cdot \left(1 - \exp\left(-\frac{t_\mathrm{i}}{\tau}\right)\right) \tag{2.169}
$$

Charging cycle

$$
\frac{Q_{\mathrm{H,L}}}{Q_\mathrm{H}} \cdot 100\,\% = 100 \cdot \frac{Q_\mathrm{H} - Q_{\mathrm{H,E}}}{Q_\mathrm{H}} = 100 \cdot \left(1 - \frac{Q_{\mathrm{H,E}}}{Q_\mathrm{H}}\right) \tag{2.170}
$$

Numerical values

$$
\frac{Q_{\mathrm{H,E}}}{Q_\mathrm{H}} \cdot 100\,\% = 61\,\% \qquad \frac{Q_{\mathrm{H,L}}}{Q_\mathrm{H}} \cdot 100\,\% = 39\,\% \tag{2.171}
$$

2.21. Ice sphere cooling

- Problem type

  Instationary heat transfer by conduction, phase change, heating and cooling processes.

a) Temperatures of the liquid and the sphere's contents:

$t = t_\mathrm{m}$:
At the point in time t_m, the ice within the spheres has melted in its entirity. Additional heating of the water now contained in the spheres has, thus, not yet taken place. The temperature of the spheres can be easily determined.

$$T_\mathrm{s,m} = 0\,^\circ\mathrm{C} \tag{2.172}$$

Taking into consideration the ice's melting enthalpy the amount of heat extracted from the liquid is calculated.

$$Q_\mathrm{I,m} = m_\mathrm{Ice} \cdot r = 12787.2\,\mathrm{J} \tag{2.173}$$

The container is adiabatically insulated. A heat balance around the spheres, therefore, immediately yields the liquid's temperature difference.

$$\triangle Q_\mathrm{L,m} = Q_\mathrm{I,m} = m_\mathrm{B} \cdot c_\mathrm{B} \cdot \triangle T_\mathrm{L,m} \tag{2.174}$$

$$\triangle T_\mathrm{L,m} = \frac{\triangle Q_\mathrm{L,m}}{m_\mathrm{B} \cdot c_\mathrm{B}} = \frac{-12787.2\,\mathrm{J}}{4180\,\mathrm{J/kg\,K} \cdot 0.5\,\mathrm{kg}} = -6.12\,\mathrm{K} \tag{2.175}$$

$$T_\mathrm{L,m} = T_\mathrm{A} + \triangle T_\mathrm{L,m} = 293.15\,\mathrm{K} - 6.12\,\mathrm{K} = 287.03\,\mathrm{K} \widehat{=} 13.88\,^\circ\mathrm{C} \tag{2.176}$$

$t = t_\infty$:
At t_∞ thermal equilibrium is reached. I.e. the temperature of the water surrounding the spheres and the temperature of the liquid within the spheres have equalised and no further heat transfer takes place($T_{\mathrm{F},\infty} = T_{\mathrm{W},\infty} = T_\infty$).

Furthermore, the amount of heat absorbed by the water must be equal to that emitted by the liquid.

$$m_\mathrm{B} \cdot c_\mathrm{B} \cdot (T_\infty - T_\mathrm{L,m}) = m_\mathrm{W} \cdot c_\mathrm{W} \cdot (T_\mathrm{s,m} - T_\infty) \tag{2.177}$$

$$T_\infty = \frac{m_\mathrm{W} \cdot c_\mathrm{W} \cdot T_\mathrm{s,m} + m_\mathrm{B} \cdot c_\mathrm{B} \cdot T_\mathrm{L,m}}{m_\mathrm{W} \cdot c_\mathrm{W} + m_\mathrm{B} \cdot c_\mathrm{B}} \qquad \text{with} \qquad m_\mathrm{W} = m_\mathrm{Ice} \tag{2.178}$$

$$T_\infty = \frac{0.0384\,\mathrm{kg} \cdot 4180\,{}^\mathrm{J}/_\mathrm{kg\,K} \cdot 273.15\,\mathrm{K} + 0.5\,\mathrm{kg} \cdot 4180\,{}^\mathrm{J}/_\mathrm{kg\,K} \cdot 287.03\,\mathrm{K}}{0.0384\,\mathrm{kg} \cdot 4180\,{}^\mathrm{J}/_\mathrm{kg\,K} + 0.5\,\mathrm{kg} \cdot 4180\,{}^\mathrm{J}/_\mathrm{kg\,K}} \tag{2.179}$$

$$T_\infty = 286.042\,\mathrm{K} \widehat{=} 12.892\,^\circ\mathrm{C} \tag{2.180}$$

b) Temporal evolution of the temperature profiles of the spheres as well as the liquid up until $t = t_\mathrm{m}$:

Approach: The heat flux transmitted through conduction is equal to the change in internal energy of the liquid.

$$\dot{Q} = n \cdot k \cdot A_\mathrm{Kugel} \cdot (T_\mathrm{S}(t) - T_\mathrm{B}(t)) = m_\mathrm{T} \cdot c_\mathrm{T} \frac{\mathrm{d}T_\mathrm{B}(t)}{\mathrm{d}t} \tag{2.181}$$

The spheres' surface area each is equal to $A_\mathrm{Kugel} = \pi \cdot d^2$. Furthermore, shall be $T_\mathrm{S}(t) = \mathrm{const.} = 273.15\,\mathrm{K}$ within the interval $t_0 \leq t \leq t_\mathrm{C}$. Thus follows:

$$\frac{\mathrm{d}T_\mathrm{T}}{\mathrm{d}t} = \frac{n \cdot k \cdot \pi \cdot d^2}{m_\mathrm{T} \cdot c_\mathrm{T}} \cdot (T_\mathrm{C} - T_\mathrm{T}) = \frac{1}{\tau_\mathrm{T}} \cdot (T_\mathrm{C} - T_\mathrm{T}) \tag{2.182}$$

The ODE is solved by employing the usual assumptions:
$\Theta_\mathrm{T} = T_\mathrm{T} - T_\mathrm{C}$, as well as the approach from p. 52 in the formulary for a first order linear ODE.

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{1}{\tau_\mathrm{T}} \cdot \Theta_\mathrm{T} = 0 \tag{2.183}$$

$$\Theta_\mathrm{B} = \exp\left(-\int \frac{1}{\tau_\mathrm{B}} \mathrm{d}t\right) \cdot (c + \int (0 \cdot \exp\left(+\int \frac{1}{\tau_\mathrm{B}} \mathrm{dt}\right)) \mathrm{d}t) \tag{2.184}$$

$$\Theta_{\mathrm{B}} = \exp\left(-\int \frac{1}{\tau_{\mathrm{B}}}\mathrm{d}t\right) \cdot c \tag{2.185}$$

The temporal evolution of the temperature profile reads:

$$\frac{T_{\mathrm{T}} - T_{\mathrm{C}}}{T_{\mathrm{A}} - T_{\mathrm{C}}} = \exp\left(-\frac{t}{\tau_{\mathrm{B}}}\right) \tag{2.186}$$

c) The time t_{m} after which the ice has melted in its entirity is unkown and to be determined.

Approach:

All ice has melted when $\mathrm{T_F} = \mathrm{T_F(t_s)} = 13.9$ °C. If the temporal evolution of the temperature profile from b) is employed it holds:

$$\ln\left(\frac{T_{\mathrm{B}} - T_{\mathrm{E}}}{T_{\mathrm{o}-T_{\mathrm{E}}}}\right) \cdot (-\tau_{\mathrm{B}}) = t \tag{2.187}$$

$$\frac{1}{k} = \frac{1}{\alpha} + \frac{\delta}{\lambda} = \frac{1}{400\,\mathrm{W/m^2\,K}} + \frac{0.0005\,\mathrm{m}}{0.2\,\mathrm{W/m\,K}} \qquad k = 200\,\mathrm{W/m^2\,K} \tag{2.188}$$

$$\tau_{\mathrm{B}} = \frac{m_{\mathrm{B}} \cdot c_{\mathrm{B}}}{n \cdot k \cdot \pi \cdot d^2} = \frac{0.5\,\mathrm{kg} \cdot 4180\,\mathrm{J/kg\,K}}{10 \cdot 200\,\mathrm{W/m^2\,K}\pi\,(0.02\,\mathrm{m} + 0.001\,\mathrm{m})^2} = 754\,\mathrm{s} \tag{2.189}$$

$$t_{\mathrm{m}} = -754\,\mathrm{s}\ln\frac{13.9\,\mathrm{K}}{20\,\mathrm{K}} = 274\,\mathrm{s} = 4.6\,\mathrm{min} \tag{2.190}$$

d) The temperature differential equations of the liquid as well as the water within the spheres for the interval $t_{\mathrm{m}} \leq t \leq \infty$ are to be determined.

Differential equation for the liquid:

$$m_{\mathrm{B}} \cdot c_{\mathrm{B}} \cdot \frac{\mathrm{d}T_{\mathrm{B}}}{\mathrm{d}t} = k \cdot n \cdot \pi \cdot d_{\mathrm{S}}^2 \cdot (T_{\mathrm{S}} - T_{\mathrm{B}}) \tag{2.191}$$

$$\frac{\mathrm{d}T_{\mathrm{B}}}{\mathrm{d}t} - \frac{k \cdot n \cdot \pi \cdot d_{\mathrm{S}}^2}{m_{\mathrm{B}} \cdot c_{\mathrm{B}}} \cdot (T_{\mathrm{S}} - T_{\mathrm{B}}) = 0 \tag{2.192}$$

$$\frac{\mathrm{d}T_{\mathrm{B}}}{\mathrm{d}t} - \frac{1}{\tau_{\mathrm{B}}} \cdot (T_{\mathrm{S}} - T_{\mathrm{B}}) = 0 \tag{2.193}$$

Differential equation for the water within the spheres:

$$\frac{\pi \cdot d^3}{6} \cdot \rho_{\mathrm{I}} \cdot c_{\mathrm{W}} \cdot \frac{\mathrm{d}T_{\mathrm{S}}}{\mathrm{d}t} = -k \cdot \pi \cdot (d + 2 \cdot \delta)^2 \cdot (T_{\mathrm{S}} - T_{\mathrm{B}}) \tag{2.194}$$

$$\frac{\mathrm{d}T_{\mathrm{S}}}{\mathrm{d}t} + \frac{6 \cdot k \cdot \pi \cdot d^2 \cdot \left(1 + \frac{2 \cdot \delta}{d}\right)^2}{\pi \cdot d^3 \cdot \rho_{\mathrm{I}} \cdot c_{\mathrm{W}}} \cdot (T_{\mathrm{S}} - T_{\mathrm{B}}) = 0 \tag{2.195}$$

$$\frac{\mathrm{d}T_{\mathrm{S}}}{\mathrm{d}t} + \frac{1}{\tau_{\mathrm{S}}} (T_{\mathrm{S}} - T_{\mathrm{B}}) = 0 \tag{2.196}$$

$$\frac{\mathrm{d}T_{\mathrm{S}}}{\mathrm{d}t} = -\mu \cdot \frac{\mathrm{d}T_{\mathrm{B}}}{\mathrm{d}t} \tag{2.197}$$

$$\text{with} \qquad \mu = \frac{\tau_{\mathrm{B}}}{\tau_{\mathrm{S}}} \qquad \text{sowie} \qquad \tau_{\mathrm{S}} = \frac{\rho_{\mathrm{I}} \cdot \ c_{\mathrm{W}} \cdot d}{6 \cdot k \cdot \left(1 + \frac{2 \cdot \delta}{d}\right)^2} \tag{2.198}$$

e) The temperature profiles of the liquid and the water within the spheres are to be determined within the interval $t_{\mathrm{m}} \leq t \leq \infty$.
Temperature profile of the water within the spheres:

Intergration of the ODE from subtask d) within the interval of t_{m} and t yields:

$$\int_{T_{\mathrm{s,m}}}^{T_{\mathrm{S}}} \mathrm{d}\tilde{T}_{\mathrm{S}} = -\mu \cdot \int_{T_{\mathrm{L,m}}}^{T_{\mathrm{B}}} \mathrm{d}\tilde{T}_{\mathrm{B}} \tag{2.199}$$

$$T_{\mathrm{S}} - T_{\mathrm{s,m}} = -\mu\,(t) \cdot (T_{\mathrm{T}} - T_{\mathrm{L,m}}) \tag{2.200}$$

$$T_{\mathrm{S}}(t) = T_{\mathrm{I}} - \mu \cdot (T_{\mathrm{B}}(t) - T_{\mathrm{L,m}}) \tag{2.201}$$

Temperature profile of the liquid: $T_{\mathrm{S}}(t)$ is inserted into the ODE for $T_{\mathrm{B}}(t)$. The subsequent solution yields:

$$\frac{\mathrm{d}T_{\mathrm{B}}}{\mathrm{d}t} = \frac{1}{\tau_{\mathrm{B}}} \cdot (T_{\mathrm{I}} - \mu \cdot (T_{\mathrm{B}} - T_{\mathrm{L,m}}) - T_{\mathrm{B}}) \tag{2.202}$$

$$\frac{\mathrm{d}T_{\mathrm{B}}}{\mathrm{d}t} + \frac{1}{\tau_{\mathrm{B}}} \cdot (1+\mu) \cdot T_{\mathrm{B}} = \frac{1}{\tau_{\mathrm{B}}} \cdot (T_{\mathrm{I}} + \mu \cdot T_{\mathrm{L,m}}) \tag{2.203}$$

$$T_{\mathrm{B}} = c \cdot \exp\left(-(1+\mu) \cdot \frac{t}{\tau_{\mathrm{B}}}\right) + \frac{T_{\mathrm{I}} + \mu \cdot T_{\mathrm{L,m}}}{1+\mu} \tag{2.204}$$

With the initial condition $T_{\mathrm{B}}(t_{\mathrm{m}}) = T_{\mathrm{L,m}}$ one obtains:

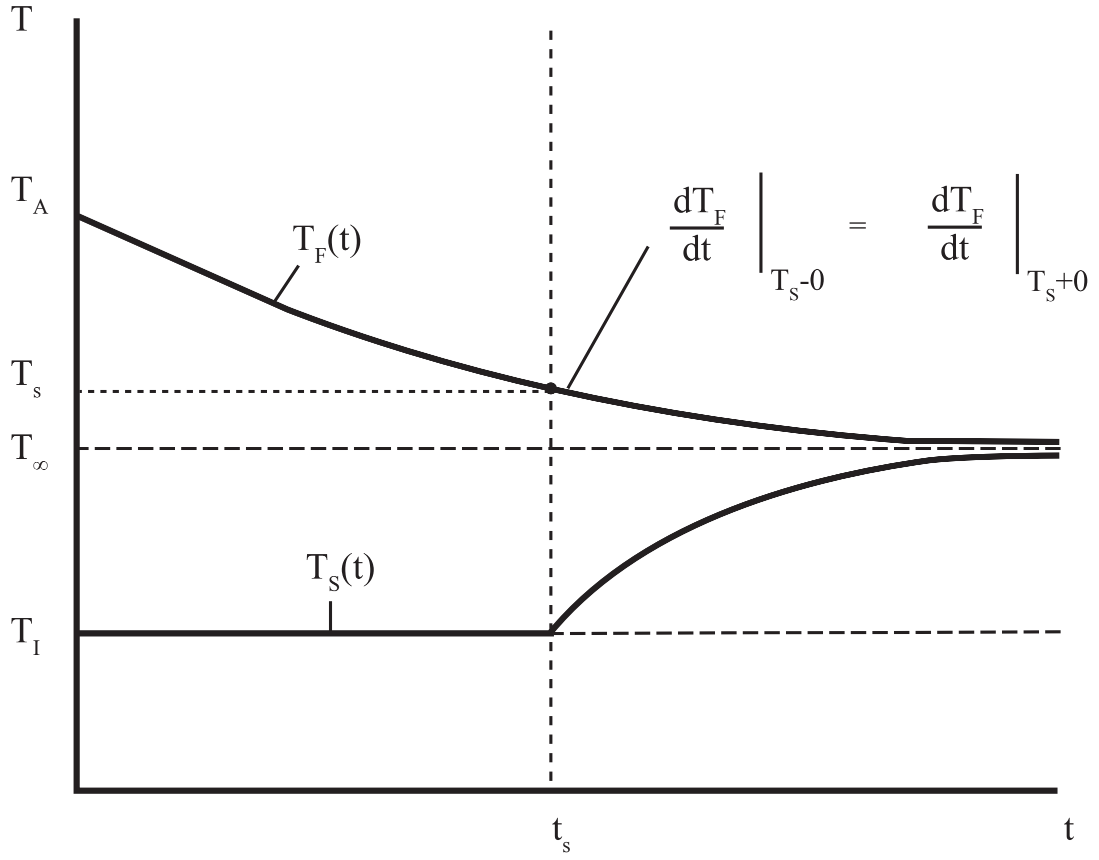
$$T_{\mathrm{L,m}} = c \cdot \exp\left(-(1+\mu) \cdot \frac{t_{\mathrm{m}}}{\tau_{\mathrm{B}}}\right) + \frac{T_{\mathrm{I}} + \mu \cdot T_{\mathrm{L,m}}}{1+\mu} \tag{2.205}$$

$$c = \frac{T_{\mathrm{L,m}} - T_{\mathrm{I}}}{1+\mu} \cdot \exp\left((1+\mu) \cdot \frac{t_{\mathrm{m}}}{\tau_{\mathrm{B}}}\right) \tag{2.206}$$

$$T_{\mathrm{B}} = \frac{T_{\mathrm{L,m}} - T_{\mathrm{I}}}{1+\mu} \cdot \exp\left(-(1+\mu) \cdot \frac{t - t_{\mathrm{m}}}{\tau_{\mathrm{B}}}\right) + \frac{T_{\mathrm{I}} + \mu \cdot T_{\mathrm{L,m}}}{1+\mu} \tag{2.207}$$

$$T_{\mathrm{B}}(t) = T_{\mathrm{L,m}} - \frac{T_{\mathrm{L,m}} - T_{\mathrm{I}}}{1+\mu} \cdot \left(1 - \exp\left(-(1+\mu) \cdot \frac{t - t_{\mathrm{m}}}{\tau_{\mathrm{B}}}\right)\right) \tag{2.208}$$

f) Temperature profiles



Chapter 3.

Solutions convection

3.1. Hot wire filament

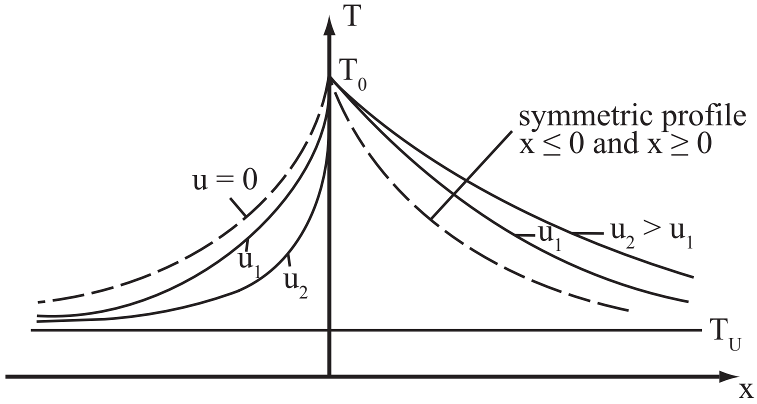
1. Problem type:

   Description of the process depending on the choice of frame of reference:

   - System moving with velocity u: instationary, one-dimensional heat conduction within a resting and heat transfer at its surface.
   - Fixed system: steady-state, one-dimensional heat conduction in a moving body (fluid) with heat transfer at the body's bounds; employed here.

a)

2. Sketch of the temperature profile



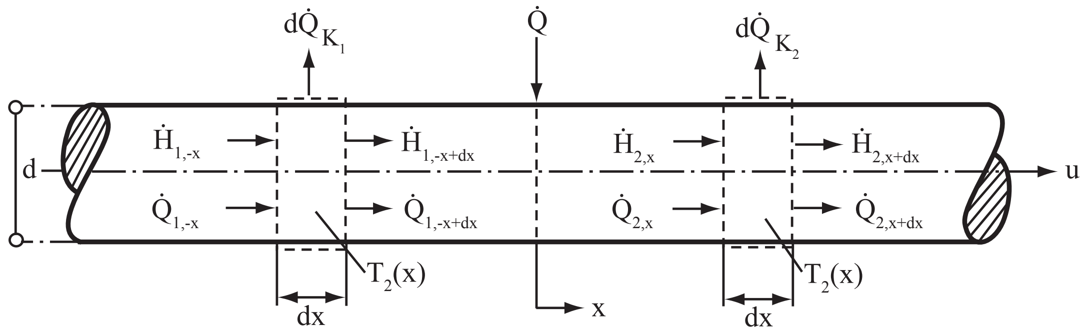
Justification for the asymmetric profile for $u > 0$:

The two most important mechanisms of transport are heat conduction and convection are opposed on the left-hand side ($x < 0$) and oriented in equal direction on the right-hand side ($x > 0$).

b)

3. System boundaries and energy balance:

Due to the contrasting directions of the transport mechanisms and the discontinuous heat flux profile at $x = 0$, respectively, both sides $x < 0$ and $x > 0$ need to be investigated separately.



Energy balances for elements $\frac{\pi d^2}{4} \cdot dx$:

$$x < 0 : 0 = \dot{H}_{1,-\mathrm{x}} + \dot{Q}_{1,-\mathrm{x}} - \dot{H}_{1,-\mathrm{x+dx}} - \dot{Q}_{1,-\mathrm{x+dx}} - d\dot{Q}_{\mathrm{B},1} \tag{3.1}$$

$$x > 0 : 0 = \dot{H}_{2,\mathrm{x}} + \dot{Q}_{2,\mathrm{x}} - \dot{H}_{2,\mathrm{x+dx}} - \dot{Q}_{2,\mathrm{x+dx}} - d\dot{Q}_{\mathrm{B},2} \tag{3.2}$$

4. Description of balancing quantities and differential equations for the Temperature profile

$$x < 0 : \dot{Q}_{1,-\mathrm{x}} = -\lambda \cdot \frac{\pi d^2}{4} \cdot \frac{\mathrm{d}T_1}{\mathrm{d}x} \qquad \dot{Q}_{1,-\mathrm{x+dx}} = \dot{Q}_{1,-\mathrm{x}} + \frac{\mathrm{d}\dot{Q}_{1,-\mathrm{x}}}{\mathrm{d}x} dx$$

$$\dot{H}_{1,-\mathrm{x}} = \rho \cdot c \cdot u \cdot \frac{\pi d^2}{4} \cdot T_1 \qquad \dot{H}_{1,-\mathrm{x+dx}} = \dot{H}_{1,-\mathrm{x}} + \frac{\mathrm{d}\dot{H}_{1,-\mathrm{x}}}{\mathrm{d}x} dx$$

$$d\dot{Q}_{\mathrm{B},1} = \alpha \cdot \pi d \cdot dx \cdot (T_1 - T_\mathrm{U})$$

$$\Rightarrow \frac{\mathrm{d}^2 T_1}{\mathrm{d}x^2} - \frac{u}{a} \cdot \frac{\mathrm{d}T_1}{\mathrm{d}x} - m^2 (T_1 - T_\mathrm{U}) \tag{3.3}$$

with

$$a = \frac{\lambda}{\rho \cdot c} \qquad \text{und} \qquad m = \sqrt{\frac{4 \cdot \alpha}{\lambda \cdot d}}$$

$$\begin{aligned} x > 0 : \dot{Q}_{2,\mathrm{x}} &= -\lambda \cdot \frac{\pi d^2}{4} \cdot \frac{\mathrm{d}T_2}{\mathrm{d}x} & \dot{Q}_{2,\mathrm{x+dx}} &= \dot{Q}_{2,\mathrm{x}} + \frac{\mathrm{d}\dot{Q}_{2,\mathrm{x}}}{\mathrm{d}x} dx \\ \dot{H}_{2,\mathrm{x}} &= \rho \cdot c \cdot u \cdot \frac{\pi d^2}{4} \cdot T_2 & \dot{H}_{2,\mathrm{x+dx}} &= \dot{H}_{2,\mathrm{x}} + \frac{\mathrm{d}\dot{H}_{2,\mathrm{x}}}{\mathrm{d}x} dx \\ d\dot{Q}_{\mathrm{K},2} &= \alpha \cdot \pi d \cdot dx \cdot (T_2 - T_{\mathrm{U}}) \end{aligned}$$

$$\Rightarrow \frac{\mathrm{d}^2 T_2}{\mathrm{d}x^2} - \frac{u}{a} \cdot \frac{\mathrm{d}T_2}{\mathrm{d}x} - m^2 \left(T_2 - T_{\mathrm{U}}\right) \tag{3.4}$$

Introduction of the overtemperatures

$$\Theta_1 = T_1 - T_{\mathrm{U}} \qquad \Theta_2 = T_2 - T_{\mathrm{U}} \tag{3.5}$$

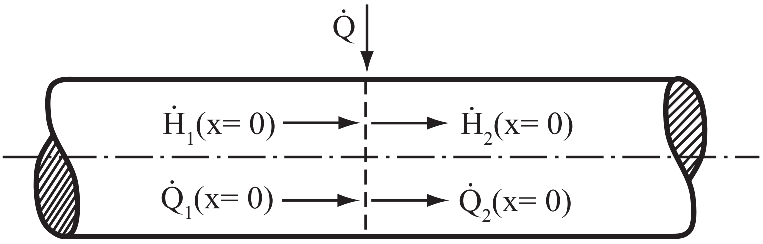
yields

$$x < 0 : \qquad \tfrac{\mathrm{d}^2\Theta_1}{\mathrm{d}x^2} - \tfrac{u}{a} \cdot \tfrac{\mathrm{d}\Theta_1}{\mathrm{d}x} - m^2 \cdot \Theta_1 = 0 \tag{3.6}$$

$$x > 0 : \qquad \tfrac{\mathrm{d}^2\Theta_2}{\mathrm{d}x^2} - \tfrac{u}{a} \cdot \tfrac{\mathrm{d}\Theta_2}{\mathrm{d}x} - m^2 \cdot \Theta_2 = 0 \tag{3.7}$$

5. Boundary conditions

    1. $\Theta_1(x = -\infty) = T_1(-\infty) - T_{\mathrm{U}} = 0$
    2. $\Theta_2(x = -\infty) = T_2(\infty) - T_{\mathrm{U}} = 0$
    3. $\Theta_1(x = 0) = \Theta_2(x = 0) = \Theta_0$
    4. Bilanz für den Drahtquerschnitt $x = 0$:



$$\dot{Q} + \dot{H}_1(x=0) + \dot{Q}_1(x=0) - \dot{H}_2(x=0) - \dot{Q}_2(x=0) = 0 \tag{3.8}$$

Because of $H_1(0) = H_2(0)$, it holds:

$$\begin{aligned} \dot{Q} &= -\dot{Q}_1(x=0) + \dot{Q}_2(x=0) \\ \dot{Q} &= \lambda \cdot \frac{\pi d^2}{4} \cdot \left[\frac{\mathrm{d}T_1}{\mathrm{d}x}\Big|_{x=0} - \frac{\mathrm{d}T_2}{\mathrm{d}x}\Big|_{x=0} \right] \end{aligned} \tag{3.9}$$

6. Solution of the differential equations

Type of differential equation: linear, second order, constant coefficients. The approach yields

$$x < 0: \qquad \Theta_1 = c_1 \cdot \exp(\mu_1 x) + c_2 \cdot \exp(\mu_2 x) \tag{3.10}$$

$$x > 0: \qquad \Theta_2 = c_3 \cdot \exp(\mu_1 x) + c_4 \cdot \exp(\mu_2 x) \tag{3.11}$$

with

$$\mu_1 = \frac{u}{2a} + \sqrt{\left(\frac{u}{2a}\right)^2 + m^2} \qquad > 0! \tag{3.12}$$

$$\mu_2 = \frac{u}{2a} - \sqrt{\left(\frac{u}{2a}\right)^2 + m^2} \qquad < 0! \tag{3.13}$$

Determination of the integration constants:

From BC 1. follows $\quad \mu_2 x \to +\infty \quad c_2 = 0 \quad$ and

from BC 2. follows $\quad \mu_1 x \to +\infty \quad c_3 = 0$

Thus follows

$$\Theta_1 = c_1 \cdot \exp(\mu_1 x) \quad (3.14)$$
$$\Theta_2 = c_4 \cdot \exp(\mu_2 x) \quad (3.15)$$

BC 3. yields:

$$c_1 = c_4$$

and BC 4:

$$\begin{aligned} \dot{Q} &= \lambda \cdot A_{\mathrm{Q}} \cdot (\mu_1 \cdot c_1 - \mu_2 \cdot c_4) \\ &= \lambda \cdot A_{\mathrm{Q}} \cdot c_1 \cdot (\mu_1 - \mu_2) \\ &= 2\lambda \cdot A_{\mathrm{Q}} \cdot c_1 \cdot \sqrt{\left(\frac{u}{2a}\right)^2 + m^2} \end{aligned}$$

$$\Longrightarrow \qquad c_1 = c_4 = \Theta_0 = \frac{\dot{Q}}{2 \cdot \lambda \cdot A_{\mathrm{Q}} \cdot \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}} \quad (3.16)$$

7. Temperature profiles

$$x < 0: \qquad \Theta_1 = \frac{2\dot{Q}}{\lambda \cdot \pi d^2 \cdot \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}} \cdot \ldots \quad (3.17)$$
$$\ldots \cdot \exp\left(\left(\frac{u}{2a} + \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}\right) \cdot x\right) \quad (3.18)$$

$$x > 0: \qquad \Theta_2 = \frac{2\dot{Q}}{\lambda \cdot \pi d^2 \cdot \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}} \cdot \ldots \quad (3.19)$$
$$\ldots \cdot \exp\left(\left(\frac{u}{2a} - \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}\right) \cdot x\right) \quad (3.20)$$

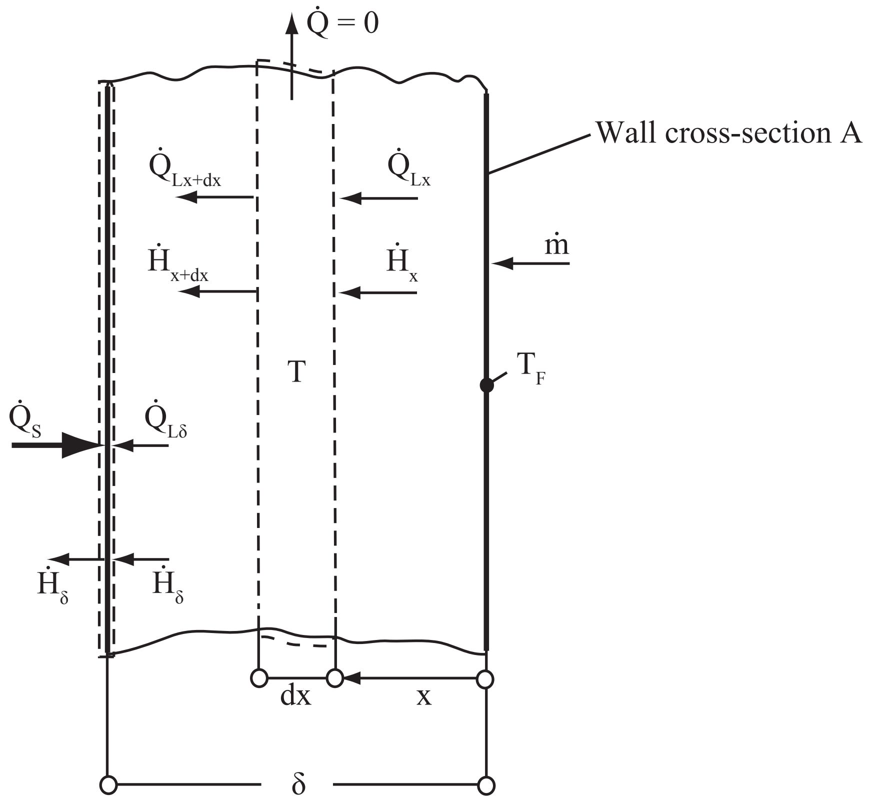
3.5. Absorption in a porous wall

a) 1. Problem type:

Steady-state, one-dimensional heat transfer through conduction and convection.

2. System boundary

The temperature profile within the wall is to be determined. Therefore a local/internal balance should be chosen. For negligible heat fluxes at the wall's edge faces a one-dimensional temperature profile is obtained. Thus follows the choice of balancing element:



3. Energy balance

$$\dot{Q}_{\mathrm{L,x}} + \dot{H}_{\mathrm{x}} - \dot{Q}_{\mathrm{L,x+dx}} - \dot{H}_{\mathrm{x+dx}} = 0 \tag{3.21}$$

With

$$\dot{H}_{\mathrm{x+dx}} = \dot{H}_{\mathrm{x}} + \frac{\mathrm{d}\dot{H}_{\mathrm{x}}}{\mathrm{d}x} dx \tag{3.22}$$

$$\dot{Q}_{\mathrm{L,x+dx}} = \dot{Q}_{\mathrm{L,x}} + \frac{\mathrm{d}\dot{Q}_{\mathrm{L,x}}}{\mathrm{d}x} dx \tag{3.23}$$

one obtains

$$\frac{\mathrm{d}\dot{Q}_{\mathrm{L,x}}}{\mathrm{d}x} + \frac{\mathrm{d}\dot{H}_{\mathrm{x}}}{\mathrm{d}x} = 0 \tag{3.24}$$

4. Description of balancing quantities and differential equation

Heat conduction in the wall

$$\dot{Q}_{\mathrm{L,x}} = -\lambda \cdot A \cdot \frac{\mathrm{d}T}{\mathrm{d}x} \tag{3.25}$$

Enthalpy flux

$$\dot{H}_{\mathrm{x}} = H_0 + A \cdot \dot{m}'' \cdot c \cdot T \tag{3.26}$$

Eq. 3.25 and Eq. 3.26inserted into Eq. 3.24 yields the differential equation:

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} - \frac{\dot{m}'' \cdot c}{\lambda} \cdot \frac{\mathrm{d}T}{\mathrm{d}x} = 0 \tag{3.27}$$

5. Boundary conditions

Location $x = 0$:

$$T(x = 0) = T_{\mathrm{F}} \tag{3.28}$$

Location $x = \delta$:

The heat balance for the wall cross-section A yields

$$\begin{aligned}\dot{Q}_{\mathrm{L},\delta} &+ \dot{Q}_{\mathrm{S}} + \dot{H}_{\delta} - \dot{H}_{\delta} = 0 && (3.29)\\ \Rightarrow \dot{Q}_{\mathrm{S}} &= -\dot{Q}_{\mathrm{L},\delta} = \lambda \cdot A \cdot \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=\delta} && (3.30)\\ \dot{q}_{\mathrm{s}}'' &= \lambda \cdot \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=\delta} && (3.31)\end{aligned}$$

6. <u>Solution of the differential equation</u>

   Type: Linear, homogeneous, 2. order with constant coefficients

   Approach $T = \exp(\mu x)$ yields:

$$\mu_1 = 0, \qquad \mu_2 = \frac{\dot{m}'' \cdot c}{\lambda} \tag{3.32}$$

   General solution

$$T = c_1 + c_2 \cdot \exp\left(\frac{\dot{m}'' \cdot c}{\lambda} \cdot x\right) \tag{3.33}$$

   <u>Determination of integration constants</u>

   From BC 3.28 can be deduced:

$$c_1 = T_{\mathrm{F}} - c_2 \tag{3.34}$$

   BC 3.31 yields:

$$\begin{aligned}\dot{q}_{\mathrm{s}}'' = \lambda \cdot \frac{\dot{m}'' \cdot c}{\lambda} \cdot c_2 \cdot \exp\left(\frac{\dot{m}'' \cdot c}{\lambda} \cdot \delta\right)\\ \Rightarrow c_2 = \frac{\dot{q}_{\mathrm{s}}''}{\dot{m}'' \cdot c} \cdot \exp\left(-\frac{\dot{m}'' \cdot c}{\lambda} \cdot \delta\right)\end{aligned} \tag{3.35}$$

Temperature profile

$$T = T_\mathrm{F} - \frac{\dot{q}''_\mathrm{s}}{\dot{m}'' \cdot c} \cdot \exp\left(-\frac{\dot{m}'' \cdot c}{\lambda} \cdot \delta\right) \cdot \left(1 - \exp\left(\frac{\dot{m}'' \cdot c}{\lambda} \cdot x\right)\right) \tag{3.36}$$

b) Maximum temperature in the wall

$$\begin{aligned} T_\mathrm{max} &= T(x = \delta) \\ &= T_\mathrm{F} + \frac{\dot{q}''_\mathrm{s}}{\dot{m}'' \cdot c} \cdot \left(1 - \exp\left(-\frac{\dot{m}'' \cdot c \cdot \delta}{\lambda}\right)\right) \end{aligned} \tag{3.37}$$

Numerical values:

$$T_\mathrm{max} = -15 + \frac{150 \cdot 10^3}{0.6 \cdot 1000} \cdot (1 - \exp(-3.75)) = 229\,^\circ\mathrm{C}$$

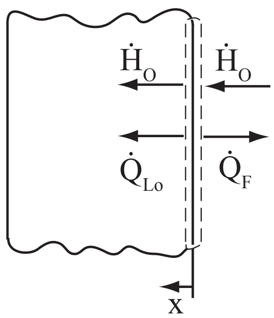
c) Heat flux into the coolant at x = 0

1. System boundary and energy balance

   The balancing boundaries are to be placed such that they cross the unknown heat flux and other energy fluxes are known.

   Alternative #1

   Balance boundary: Cross-section A at $x = 0$



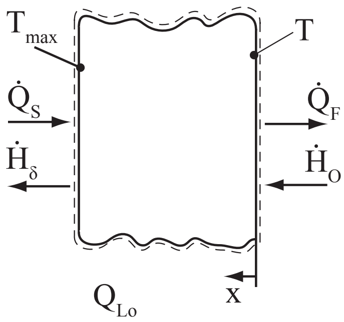
Balance:

$$\dot{H}_0 - \dot{H}_0 - \dot{Q}_{\mathrm{L},0} - \dot{Q}_{\mathrm{F}} = 0 \tag{3.38}$$

$$\dot{q}''_{\mathrm{F}} = -\dot{q}''_{\mathrm{L},0} \tag{3.39}$$

Alternative #2

Balance boundary: entire wall



Balance:

$$\dot{Q}_{\mathrm{s}} - \dot{Q}_{\mathrm{F}} + \dot{H}_0 - \dot{H}_\delta = 0 \tag{3.40}$$

$$\dot{q}''_{\mathrm{F}} = \dot{q}''_{\mathrm{s}} + \dot{h}''_0 - \dot{h}''_\delta \tag{3.41}$$

2. Description of the balance quantities and governing equation

Alternative #1

Heat flux through conduction at $x = 0$:

$$\dot{q}''_{\mathrm{C0}} = -\lambda \cdot \frac{\mathrm{d}T}{\mathrm{d}x}\Big|_{x=0}$$

With Eq. 3.35 andEq. 3.38 follows

$$\dot{q}''_{\mathrm{F}} = \lambda \cdot \frac{\dot{q}''_{\mathrm{s}}}{\dot{m}'' \cdot c} \cdot \exp\left(-\frac{\dot{m}'' \cdot c \cdot \delta}{\lambda}\right) \cdot \frac{\dot{m}'' \cdot c}{\lambda} \cdot 1 \tag{3.42}$$

$$= \dot{q}''_{\mathrm{s}} \cdot \exp\left(-\frac{\dot{m}'' \cdot c \cdot \delta}{\lambda}\right) \tag{3.43}$$

Alternative #2

Enthalpy fluxes at $x = 0$ and$x = \delta$

$$h''_0 = \dot{m}'' \cdot c \cdot T_{\mathrm{F}} \tag{3.44}$$

$$h''_\delta = \dot{m}'' \cdot c \cdot T(x = \delta) = \dot{m}'' \cdot c \cdot T_{\max} \tag{3.45}$$

Thus Eq. 3.39 is transmuted to

$$\dot{q}''_{\mathrm{F}} = \dot{q}''_{\mathrm{s}} + \dot{m}'' \cdot c \cdot (T_{\mathrm{F}} - T_{\max}) \tag{3.46}$$

and with Eq. 3.36

$$\dot{q}''_{\mathrm{F}} = \dot{q}''_{\mathrm{s}} - \dot{q}''_{\mathrm{s}} \cdot \left(1 - \exp\left(-\frac{\dot{m}'' \cdot c \cdot \delta}{\lambda}\right)\right) = \dot{q}''_{\mathrm{s}} \cdot \exp\left(-\frac{\dot{m}'' \cdot c \cdot \delta}{\lambda}\right) \tag{3.47}$$

3. Numerical values

$$\dot{q}''_{\mathrm{F}} = 150 \cdot \exp(-3.75) = 3.5 \, \mathrm{kW/m^2}$$

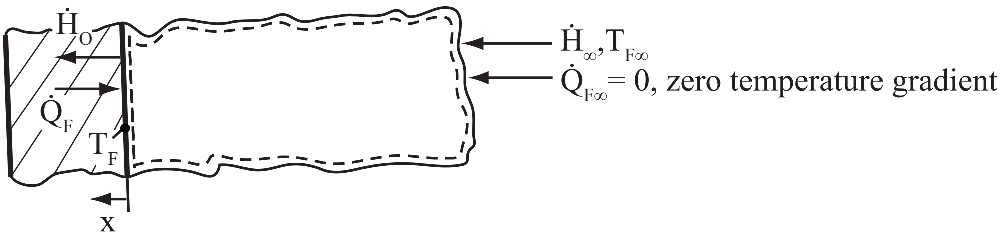
d) Coolant temperature before the wall ($x = -\infty$)

1. System boundary and energy balance

The choice for the balancing boundary is chosen in identical manner as in subtask c). Again to possible approaches arise.

Alternative #1

Balance boundary: Fluidstrombereich $0 \leq x \leq -\infty$

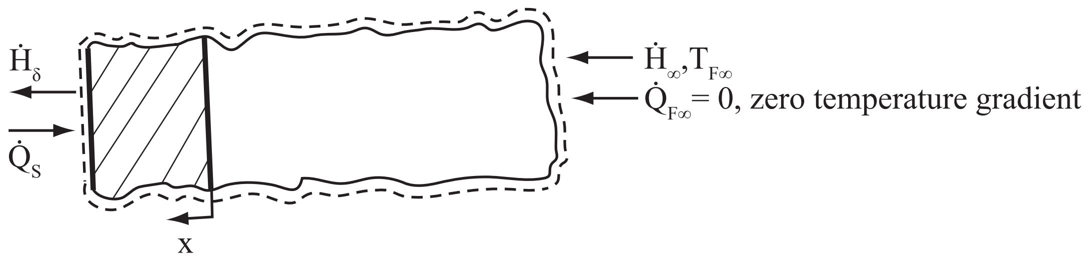


Balance:

$$
\begin{aligned}
+\dot{Q}_{\mathrm{F}} - \dot{H}_0 + \dot{H}_\infty &= 0 \\
\dot{h}''_\infty &= \dot{h}''_0 - \dot{q}''_{\mathrm{F}}
\end{aligned} \tag{3.48}
$$

Alternative #2

Balance boundary: Wall and area of fluid flux $0 \leq x \leq -\infty$



Balance:

$$
\begin{aligned}
+\dot{Q}_{\mathrm{s}} - \dot{H}_\delta + \dot{H}_\infty &= 0 \\
\dot{h}''_\infty &= \dot{h}''_\delta - \dot{q}''_{\mathrm{F}}
\end{aligned} \tag{3.49}
$$

2. Description of balance quantities and governing equation

Enthalpy fluxes:

$$\dot{h}''_{\infty} = \dot{m}'' \cdot c \cdot T_{\mathrm{F}\infty} \tag{3.50}$$

$$\dot{h}''_{0} \quad , \dot{h}''_{\delta} \quad \text{see Eq. 3.45 and Eq. 3.45}$$

Thus one obtains, in accordance with

Alternative #1

$$T_{\mathrm{F}\infty} = T_{\mathrm{F}} - \frac{\dot{q}''_{\mathrm{F}}}{\dot{m}'' \cdot c} \tag{3.51}$$

and with Eq. 3.42

$$T_{\mathrm{F}\infty} = T_{\mathrm{F}} - \frac{\dot{q}''_{\mathrm{s}}}{\dot{m}'' \cdot c} \cdot \exp\left(-\frac{\dot{m} \cdot c}{\lambda}\right) \tag{3.52}$$

and according to

Alternative #2

$$T_{\mathrm{F}\infty} = T_{\mathrm{max}} - \frac{\dot{q}''_{\mathrm{s}}}{\dot{m}'' \cdot c} \tag{3.53}$$

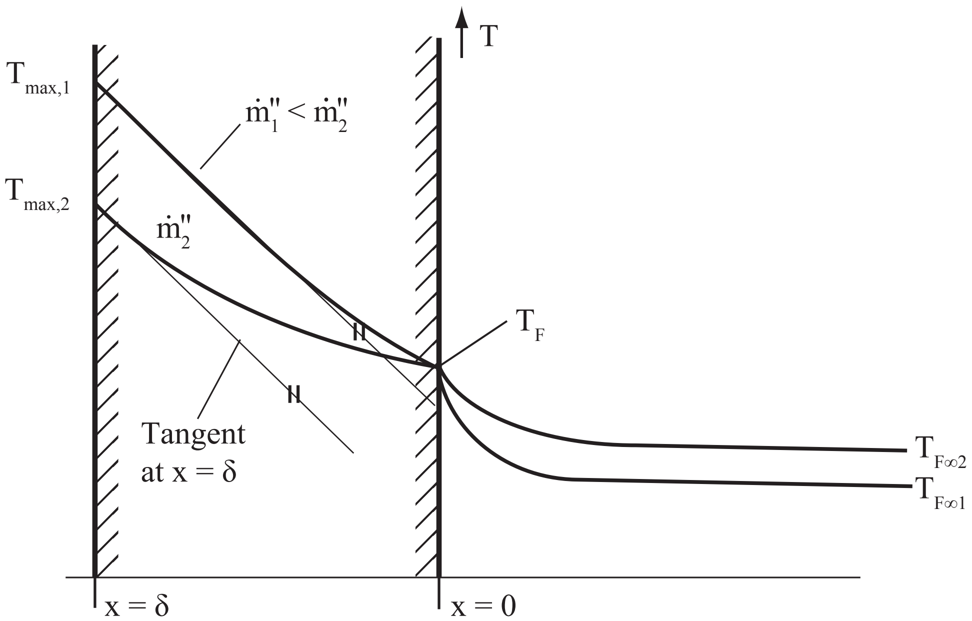
and with Eq. 3.36

$$\begin{aligned} T_{\mathrm{F}\infty} &= T_{\mathrm{F}} + \frac{\dot{q}''_{\mathrm{s}}}{\dot{m}'' \cdot c}\left(1 - \exp\left(-\frac{\dot{m}'' \cdot c}{\lambda} \cdot \delta\right) - 1\right) \\ &= T_{\mathrm{F}} - \frac{\dot{q}''_{\mathrm{s}}}{\dot{m}'' \cdot c} \exp\left(-\frac{\dot{m}'' \cdot c}{\lambda} \cdot \delta\right) \end{aligned} \tag{3.54}$$

3. Numerical values

$$T_{\mathrm{F}} = -15 - \frac{150 \cdot 10^3}{0.6 \cdot 10^3} \exp(-3.75) = -21\,^{\circ}\mathrm{C}$$

e) Sketch of the temperature profile



Chapter 4.

Solutions radiation and convection

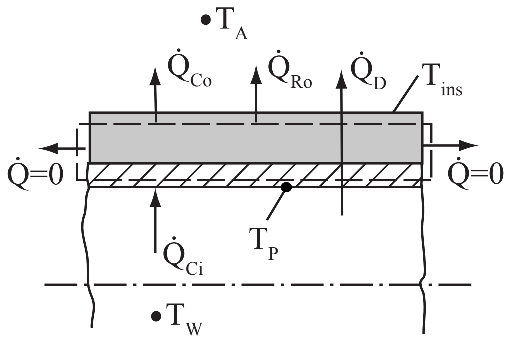
4.2. Fairing in a pipe

a) 1. Problem type and analysis

The measuring error is caused by the fairing for the thermocouple which acts as a fin, which in turn is the cause for a temperature difference between flow and fairing.
$\longrightarrow$ heat conduction in a fin.
Cause for the outwardly directed heat flux is the temperature gradient between the pipe flow and its surroundings, due to which the, in accordance with tha change in the ratio of the pipe's resistance to heat transfer and the inner resistance to heat transfer, the fluid temperature differs from the pipe's inner surface temperature. This pipe temperature is is forced upon the fairing's base and is therefore the driving potential for the temperature profile of the fairing and the unknown temperature at the fairing's tip. It is thus necessary to determine this temperature first.
$\longrightarrow$ heat transfer through a single-layer pipe, coupled with radiative heat transfer at the outer surface.

2. Governing equation for temperature of the pipe's inner wall

System boundary and heat balance



The balance's boundaries are chosen such that only the unknown temperature T_{P} as well as known temperatures or heat fluxes, respectively, are included. The practical choice of boundary includes the inner pipe surface and utilises the heat tranfer relation for multi-layered pipes. For the case presented this is only possible under the condition that the heat transfer mechanisms at the pipe's outer surface of radiation and convection are bundled into single overall heat flux which manifests itself only later on in the calculations. For this reason the boundaries are initally drawn between the inner and outer surfaces of the pipe. For steady-state conditions one thus obtains

$$\dot{Q}_{\mathrm{Ci}} - \dot{Q}_{\mathrm{Co}} - \dot{Q}_{\mathrm{Ro}} = 0 \tag{4.1}$$

If one bundles $\dot{Q}_{\mathrm{Co}}$ and $\dot{Q}_{\mathrm{Ro}}$ into a resulting overall heat flux of standard definition

$$\dot{Q}_{\mathrm{Co}} + \dot{Q}_{\mathrm{Ro}} = \dot{Q}_{\mathrm{Co,tot}} = \alpha_{\mathrm{o,tot}} \cdot \pi \left(D_{\mathrm{o}} + 2s\right) \cdot L \cdot \left(T_{\mathrm{ins}} - T_{\mathrm{o}}\right) \tag{4.2}$$

one obtains

$$\dot{Q}_{\mathrm{Ci}} = \dot{Q}_{\mathrm{Co,tot}} = \dot{Q}_{\mathrm{D}} \tag{4.3}$$

wherein the penetrating heat flux $\dot{Q}_{\mathrm{D}}$ is to be described by means of $\alpha_{\mathrm{o,tot}}$.

Description of balance quantities

- internal heat transfer

$$\dot{Q}_{\mathrm{Ci}} = \alpha_{\mathrm{i}} \cdot \pi \cdot D_{\mathrm{i}} \cdot L \left(T_{\mathrm{W}} - T_{\mathrm{P}}\right) \tag{4.4}$$

- total outer heat transfer (with respect to the radiative heat transfer the set-up can be regarded as an enclosed surface, wherein $\frac{A_{\mathrm{ins}}}{A_{\mathrm{Raumwände}}} \approx 0$)

$$\begin{aligned}\dot{Q}_{\mathrm{Co,tot}} = \pi \left(D_{\mathrm{o}} + 2s\right) \cdot L \cdot \ldots \\ \ldots \cdot \left\{\alpha_{\mathrm{o}} \left(T_{\mathrm{ins}} - T_{\mathrm{o}}\right) + \epsilon_{\mathrm{ins}} \cdot \sigma \cdot \left(T_{\mathrm{ins}}^4 - T_{\mathrm{o}}^4\right)\right\}\end{aligned} \tag{4.5}$$

- penetrating heat flux

$$\dot{Q}_{\mathrm{D}} = \frac{\pi \cdot L \left(T_{\mathrm{W}} - T_{\mathrm{o}}\right)}{\frac{1}{\alpha_{\mathrm{i}} D_{\mathrm{i}}} + \frac{1}{2\lambda_{\mathrm{P}}} \ln\left(\frac{D_{\mathrm{o}}}{D_{\mathrm{i}}}\right) + \frac{1}{2\lambda_{\mathrm{ins}}} \ln\left(\frac{D_{\mathrm{o}}+2s}{D_{\mathrm{o}}}\right) + \frac{1}{\alpha_{\mathrm{o,tot}} \cdot \left(D_{\mathrm{o}}+2s\right)}} \tag{4.6}$$

With the defining equation Eq. 4.2 the total heat transfer coefficient follows from Eq. 4.5 and

$$\alpha_{\mathrm{o,tot}} = \alpha_{\mathrm{o}} + \underbrace{\epsilon_{\mathrm{ins}} \cdot \sigma \cdot \frac{T_{\mathrm{ins}}^4 - T_{\mathrm{o}}^4}{T_{\mathrm{ins}} - T_{\mathrm{o}}}}_{\alpha_{\mathrm{Ro}}} \tag{4.7}$$

<u>Equation for the pipe's inner wall temperature</u>

From Eq. 4.3 and 4.4 one obtains

$$T_{\mathrm{P}} = T_{\mathrm{W}} - \frac{\dot{Q}_{\mathrm{D}}}{L} \cdot \frac{1}{\alpha_{\mathrm{i}} \cdot \pi D_{\mathrm{i}}} \tag{4.8}$$

where $\frac{\dot{Q}_{\mathrm{D}}}{L}$ can be determined by means of Eq. 4.6. In Eq. 4.8 and 4.6 the heat transfer coefficients α_{i}, α_{o}, and $\alpha_{\mathrm{o,tot}}$, respectively, are unknown. While α_{i} can be determined from the flow characteristics, as the temperature influence for the underlying case of forced convection is negligible, the determination of $\alpha_{\mathrm{o,tot}}$ requires an iterative method, as both the heat transfer coefficient for free convection α_{o} as well as the radiative heat transfer coefficient α_{Ro} is dependent on the unknown temperature T_{ins} whose governing equation cannot be solved analytically.

3. Calculation of the heat transfer coefficients and the insulation's surface temperature

Inner pipe wall: forced pipe flow

Temperature for which the material properties are determined $T_{\mathrm{st}} = T_{\mathrm{W}} = 80\,°\mathrm{C}$

Material properties water:

$$\rho_{\mathrm{W}} = 972\,{}^{\mathrm{kg}}/_{\mathrm{m}^2} \qquad \mathrm{Pr}_{\mathrm{W}} = 2.22$$
$$\lambda_{\mathrm{W}} = 0.669\,{}^{\mathrm{W}}/_{\mathrm{m\,K}} \qquad \nu_{\mathrm{W}} = 0.364 \cdot 10^{-6}\,{}^{\mathrm{m}^2}/_{\mathrm{s}}$$

Flow state:

$$\mathrm{Re}_{\mathrm{D_i}} = \frac{u_{\mathrm{m}} \cdot D_{\mathrm{i}}}{\nu_{\mathrm{W}}} = \frac{4 \cdot \dot{m}_{\mathrm{W}}}{\pi \cdot D_{\mathrm{i}} \cdot \rho_{\mathrm{W}} \cdot \nu_{\mathrm{W}}} = 14400 \quad \rightarrow \text{ turbulente Strömung} \tag{4.9}$$

Thermal boundary condition: constant wall temperature

Heat transfer law HTL.14:

$$\overline{\mathrm{Nu}_{\mathrm{D_i}}} = \frac{\alpha_{\mathrm{i}} \cdot D_{\mathrm{i}}}{\lambda_{\mathrm{W}}} = 0.0235 \left(\mathrm{Re}_{\mathrm{D_i}}^{0.8} - 230\right) \left(1.8 \cdot \mathrm{Pr}_{\mathrm{W}}^{0.3} - 0.8\right) \cdot \ldots$$
$$\ldots \cdot \left(1 + \left(\frac{D_{\mathrm{i}}}{L}\right)^{\frac{2}{3}}\right) \left(\frac{\eta_{\mathrm{W}}}{\eta_{\mathrm{P}}}\right)^{0.14} \tag{4.10}$$

As the pipe flow is fully developed and no major temperature differences between flow and bounding pipe are to be expected ($\eta_{\mathrm{W}} \approx \eta_{\mathrm{P}}$), it holds:

$$\overline{\mathrm{Nu}_{\mathrm{D_i}}} = 0.0235 \left(\mathrm{Re}_{\mathrm{D_i}}^{0.8} - 230\right) \left(1.8 \cdot \mathrm{Pr}_{\mathrm{W}}^{0.3} - 0.8\right) = 66 \tag{4.11}$$
$$\rightarrow \quad \alpha_{\mathrm{i}} = 884\,{}^{\mathrm{W}}/_{\mathrm{m}^2\,\mathrm{K}} \tag{4.12}$$

Outer pipe wall: free convection and radiation

Due to the reciprocating relationship between heat transfer coefficient and surface temperature, this relationship must first be posed by means of a heat balance which solely contains these unknown quantities. Such a balance is represented in Eq. 4.3

$$\dot{Q}_{\mathrm{Co,tot}} = \dot{Q}_{\mathrm{D}} \tag{4.13}$$

in combination with Eqns. 4.5 and 4.6 or 4.2 and 4.6, respectively. To solve this equation can one can either estimate an initial value for $\alpha_{\mathrm{o,tot}}$ or T_{ins}, calculate the other quantity and then check the estimated quantity. Here the method is presented for an estimate of $\alpha_{\mathrm{o,tot}}$. The necessary governing equation for the temperature T_{ins} can be obtained from Eq. 4.3 and 4.2.

$$T_{\mathrm{ins}} = T_{\mathrm{o}} + \frac{\dot{Q}_{\mathrm{D}}}{L} \cdot \frac{1}{\alpha_{\mathrm{o,tot}} \cdot \pi \left(D_{\mathrm{o}} + 2s\right)} \tag{4.14}$$

with $\frac{\dot{Q}_{\mathrm{D}}}{L}$ from Eq. 4.6

1. Estimate: $\alpha^*_{\mathrm{o,tot}} = 10\,\mathrm{W/m^2\,K}$

$$\text{Eq. 4.6}: \left(\frac{\dot{Q}_{\mathrm{D}}}{L}\right)^* = \frac{\pi \cdot 65}{\frac{1}{884 \cdot 0.05} + \frac{1}{2 \cdot 372}\ln\left(\frac{54}{50}\right) + \frac{1}{2 \cdot 0.046}\ln\left(\frac{74}{55}\right) + \frac{1}{10 \cdot 0.074}}$$
$$= \frac{\pi \cdot 65}{3.45 + 1.35} = 42.6\,\mathrm{W/m}$$
$$\rightarrow \text{Eq. 4.14}: \; T_{\mathrm{ins}} = 15 + \frac{42.6}{10 \cdot \pi \cdot 0.074} = 33.3\,^{\circ}\mathrm{C}$$

Convective heat transfer: free convection at an upright cylinder

Temperature for which the material properties are determined: $T_{\mathrm{st}} = 0.5 \cdot (T^*_{\mathrm{ins}} + T_{\mathrm{o}}) = 24\,^{\circ}\mathrm{C}$

Material properties of air:

$$\nu_{\mathrm{o}} = 15.68 \cdot 10^{-6}\,\mathrm{m^2/s} \qquad \lambda_{\mathrm{o}} = 0.026\,\mathrm{W/m\,K} \qquad \mathrm{Pr}_{\mathrm{o}} = 0.71$$

Flow state:

$$\mathrm{Gr}_{\mathrm{D_a+2s}} \cdot \mathrm{Pr} = \frac{g \cdot \beta \cdot (T^*_{\mathrm{ins}} - T_{\mathrm{o}}) \cdot (D_{\mathrm{o}} + 2s)^3}{\nu_{\mathrm{o}}^2} \cdot \mathrm{Pr}_{\mathrm{o}} = 7.07 \cdot 10^5 \quad (4.15)$$

$$\left(\beta = \frac{1}{T_{\mathrm{st}}} \text{ bei idealen Gasen}\right)$$

Heat transfer law HTL.20:

$$\overline{\mathrm{Nu}}^*_{\mathrm{d_o}} = \frac{\alpha^*_{\mathrm{o}}\,(D_{\mathrm{o}} + 2s)}{\lambda_{\mathrm{o}}} = 0.53 \cdot (\mathrm{Gr}_{\mathrm{D_a+2s}} \cdot \mathrm{Pr})^{\frac{1}{4}} = 15.4 \quad (4.16)$$

$$\alpha^*_{\mathrm{o}} = 5{,}4\,\mathrm{W/m^2\,K} \quad (4.17)$$

→ Overall heat transfer coefficient (Eq. 7)

$$\alpha_{\mathrm{o,tot}} = 5.4 + 0.9 \cdot 5.67 \cdot \frac{3.063^4 - 2.88^4}{18.3} = 10.8\,\mathrm{W/m^2\,K} \quad (4.18)$$

Comparison of estimate and calculations:

Estimate: $\alpha^*_{\mathrm{o,tot}} = 10\,\mathrm{W/m^2\,K}$

Check: $\alpha_{\mathrm{o,tot}} = 10.8\,\mathrm{W/m^2\,K}$

General agreement is sufficient.

The next iteration step with $\alpha^*_{\mathrm{o}} = 10.8\,\mathrm{W/m^2\,K}$ yields

$$\left(\frac{\dot{Q}_{\mathrm{D}}}{L}\right)^* = 43.4\,\mathrm{W/m} \qquad \alpha^*_{\mathrm{o}} = 5.33\,\mathrm{W/m^2\,K}$$

$$T^*_{\mathrm{ins}} = 32.2\,°\mathrm{C} \qquad \alpha^*_{\mathrm{o,tot}} = 10.7\,\mathrm{W/m^2\,K}$$

$$\alpha_{\mathrm{o,tot}} = 10.7\,\mathrm{W/m^2\,K} = \alpha^*_{\mathrm{o}}$$

4. Calculation of the pipe's inner wall temperature

With Eq. 4.6 follows

$$\frac{\dot{Q}_{\mathrm{D}}}{L} = \frac{\pi \cdot 65}{3.45 + \frac{1}{10.7 \cdot 0.074}} = 43.3\,\mathrm{W/m} \quad (4.19)$$

and with Eq. 4.8

$$T_\mathrm{P} = 80 - \frac{43.3}{884 \cdot \pi \cdot 0.05} = 79.71\,^\circ\mathrm{C} \tag{4.20}$$

$$T_\mathrm{W} - T_\mathrm{P} = 0.31\,\mathrm{K} \tag{4.21}$$

5. Determination of thermo-couple's fairing length

Constraints on fairing dimensions and solution

The immersed length of the fairing is to be determined such that the temperature diffrence between the flow and the fairing at the pipe's inner wall (location of fastening) of

$$T_\mathrm{W} - T_\mathrm{P} = 0.31\,\mathrm{K}$$

is equal to the temperature difference at the fairing's tip

$$T_\mathrm{W} - T_\mathrm{Th} \leq 0.05\,\mathrm{K}.$$

The thermo-couple's fairing can be regarded as a prismatic fin with a base temperature of T_P and a negligible heat flux through the fin's tip (see task description, Assumption 3). Thus, the length l can be determined from the known correlation with the temperature profile along the axis of the fin

$$\frac{\Theta(x)}{\Theta_\mathrm{F}} = \frac{\cosh\left(m(l-x)\right)}{\cosh\left(m \cdot l\right)}. \tag{4.22}$$

Equation for the immersed length of the fairing

with Eq. 4.22 follows

$$\frac{T_\mathrm{W} - T_\mathrm{Th}}{T_\mathrm{W} - T_\mathrm{P}} = \frac{T_\mathrm{W} - T(x=l)}{T_\mathrm{W} - T_\mathrm{P}} = \frac{1}{\cosh\left(m \cdot l\right)} \tag{4.23}$$

solving for l yields

$$\cosh\left(m \cdot l\right) = \frac{T_{\mathrm{W}} - T_{\mathrm{P}}}{T_{\mathrm{W}} - T_{\mathrm{Th}}} \tag{4.24}$$

$$l = \frac{1}{m} \cdot \operatorname{arcosh} \frac{T_{\mathrm{W}} - T_{\mathrm{P}}}{T_{\mathrm{W}} - T_{\mathrm{Th}}} \tag{4.25}$$

$$= \frac{1}{m} \cdot \ln\left[\frac{1}{m} \cdot \frac{T_{\mathrm{W}} - T_{\mathrm{P}}}{T_{\mathrm{W}} - T_{\mathrm{Th}}} + \sqrt{\left(\left(\frac{T_{\mathrm{W}} - T_{\mathrm{P}}}{T_{\mathrm{W}} - T_{\mathrm{Th}}}\right)^2 - 1\right)}\right] \tag{4.26}$$

with

$$m = \sqrt{\frac{\alpha_{\mathrm{d_o}} \cdot \pi \cdot d_{\mathrm{o}}}{\lambda_{\mathrm{Cu}} \cdot \frac{\pi}{4}\left(d_{\mathrm{o}}^2 - d_{\mathrm{i}}^2\right)}} = \sqrt{\frac{4 \cdot \alpha_{\mathrm{d_o}} \cdot d_{\mathrm{o}}}{\lambda_{\mathrm{Cu}} \cdot \left(d_{\mathrm{o}}^2 - d_{\mathrm{i}}^2\right)}} \tag{4.27}$$

The heat transfer coeffiecient $\alpha_{\mathrm{d_o}}$ is as yet unknown.

Calculation of the heat transfer coefficient at the fairing's surface

Problem categorisation:	cylinder forcibly exposed to flow
Temperature for which the material properties are determined:	$T_{\mathrm{st}} = 0.5\left(T_{\mathrm{W}} + \overline{T_{\mathrm{Stutzen}}}\right) \approx T_{\mathrm{W}} = 80\,^{\circ}\mathrm{C}$
Material properties of water:	as under point 3

Flow state:

$$\mathrm{Re}_{\mathrm{d_o}} = \frac{w \cdot d_{\mathrm{o}}}{\nu_{\mathrm{W}}} = \frac{4 \cdot \dot{m}_{\mathrm{W}}}{\pi \cdot D_{\mathrm{i}}^2 \cdot \rho_{\mathrm{W}}} \cdot \frac{d_{\mathrm{o}}}{\nu_{\mathrm{W}}} = 2300 \tag{4.28}$$

$$\rightarrow \text{HTL.7:}\ \mathrm{Nu}_{\mathrm{d_o}} = \frac{\alpha_{\mathrm{d_o}} \cdot d_{\mathrm{o}}}{\lambda_{\mathrm{W}}} = 0.683 \cdot \mathrm{Re}_{\mathrm{d_o}}^{0.466} \cdot \mathrm{Pr}_{\mathrm{W}}^{0.4} \tag{4.29}$$

$$\mathrm{Nu}_{\mathrm{d_o}} = 34.7 \qquad \alpha_{\mathrm{d_o}} = 2900\,\mathrm{W/m^2\,K} \tag{4.30}$$

<u>Calculation of immersed length</u>

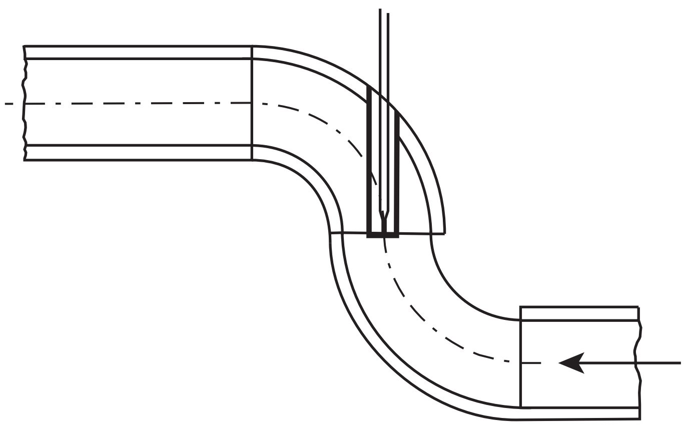
$$m = \sqrt{\frac{4 \cdot 2900 \cdot 0{,}008}{399 \cdot (0.008^2 - 0.006^2)}} = 91.14\,{}^1\!/\!\mathrm{m} \qquad (4.31)$$

$$\frac{T_\mathrm{W} - T_\mathrm{P}}{T_\mathrm{W} - T_\mathrm{Th}} = \frac{0{,}31}{0.05} = 6.2 \qquad (4.32)$$

$$l = \frac{1.82}{91.14} = 0.020\,\mathrm{m} \quad \rightarrow \quad l = 20\,\mathrm{mm} \qquad (4.33)$$

b) Possible measures include:

- Thicker insulation of the pipe. This reduced the heat flux through the pipe and therefore also the driving potential of the temperature difference between the fluid flow and the pipe's inner surface which is proportional to the measurign error (Eq. 4.23).
- Offset/diagonal instalation of the fairin and thermo-couple. Thus, the effective length exposed to the flow increases while heat transfer coefficient decreases (due to an increasing proportion of the flow being a boundary layer flow) which counteracts the first effect; overall, the influence of the greater length exposed to the flow dominates.
- Instalation of the fairing in a pipe bend.



4.3. Methanol tank

1. Problem type:

   Heat transfer for free and forced convection within a pipe gemoetry. Radiation effects are neglected.

2. Approach to the solution and system boundary

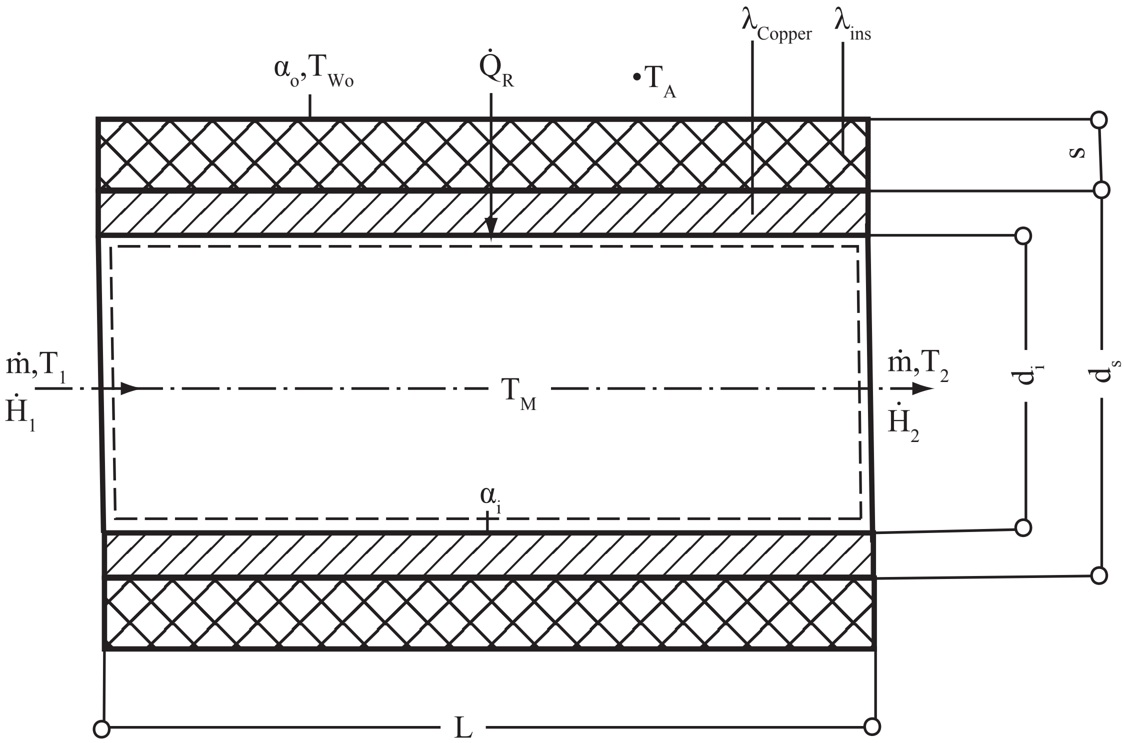
   The temperature change of the methanol results from the heat flux which passes through the insulated pipe wall from the outside. Thus, the changes in the enthalpy flux within the methanol$\Delta \dot{H}$ and heat flux through the pipe $\dot{Q}_\mathrm{P}$ reach an equilibrium for steady-state conditions. Therefore, two possible answers exist to the question whether the insulation is undamaged.

   1. One determines the insulation's thermal conductivity λ_ins from the known heat flux through the pipe and compares it to the thermal conductivity of cork. In case that $\lambda_\mathrm{ins} > \lambda_\mathrm{cork}$, the insulation is damp (as $\lambda_\mathrm{water} > \lambda_\mathrm{cork}$!).

   2. One determines the heat flux $\dot{Q}_\mathrm{P,cork}$ obtained for a thermal conductivity of $\lambda_\mathrm{cork} = 0.043\,\mathrm{W/m\,K}$ for the insulation and compares is to the actual heat flux $\dot{Q}_\mathrm{P}$. In case that $\dot{Q}_\mathrm{P,cork} < \dot{Q}_\mathrm{P}$ the insulation has become damp.

   Here approach 1 is illustrated.

   For reasons of practicality the system boundaries are chosen such that all energy fluxes are either known or can be easily substited and that only the sought-after thermal conductivity λ_ins is unknown. Thus, all temperatures or temperature gradients at the system boundary must be known. This condition is best fulfilled by the system boundary encompassing the methanol mass flux.



3. Energy balance and description of balance quantities

$$\dot{H}_1 - \dot{H}_2 + \dot{Q}_\mathrm{P} = 0 \tag{4.34}$$

Enthalpy fluxes

$$\dot{H}_1 = \dot{m} \cdot c \cdot T_1 \tag{4.35}$$

$$\dot{H}_2 = \dot{m} \cdot c \cdot T_2 \tag{4.36}$$

Heat transfer through the pipe

$$\dot{Q}_\mathrm{P} = \frac{\pi \cdot L \left(T_\mathrm{o} - T_\mathrm{M}\right)}{\frac{1}{\alpha_\mathrm{i} d_\mathrm{i}} + \frac{1}{2\lambda_\mathrm{Cu}} \ln\left(\frac{d_\mathrm{o}}{d_\mathrm{i}}\right) + \frac{1}{2\lambda_\mathrm{ins}} \ln\left(\frac{d_\mathrm{o}+2s}{d_\mathrm{o}}\right) + \frac{1}{\alpha_\mathrm{o} \cdot (d_\mathrm{o}+2s)}} \tag{4.37}$$

4. Governing equation for λ_ins

Combination of 4.34 to 4.37 yields

$$\begin{aligned}\frac{\pi \cdot L}{\dot{m} \cdot c} \cdot \frac{T_\mathrm{o} - T_\mathrm{M}}{T_2 - T_1} = \frac{1}{\alpha_\mathrm{i} d_\mathrm{i}} + \frac{1}{2\lambda_\mathrm{Co}} \ln\left(\frac{d_\mathrm{o}}{d_\mathrm{i}}\right) + \dots \\ \dots + \frac{1}{2\lambda_\mathrm{ins}} \ln\left(\frac{d_\mathrm{o} + 2s}{d_\mathrm{o}}\right) + \frac{1}{\alpha_\mathrm{o} \cdot (d_\mathrm{o} + 2s)}\end{aligned} \tag{4.38}$$

The sought-after thermal conductivity aside, the following quantities in this equation are unknown:

$$T_\mathrm{M}, \alpha_\mathrm{i} \text{ and } \alpha_\mathrm{o} \tag{4.39}$$

$T_\mathrm{o} - T_\mathrm{M}$ poses the driving potential for the heat transfer in the underlying case. As the temperature change in axial direction of the flow is small in comparison to the gradient present, once can employ the arithmetic mean as an averaging tool.

$$T_\mathrm{o} - T_\mathrm{M} = T_\mathrm{o} - \frac{1}{2}\left(T_1 + T_2\right) \tag{4.40}$$

Within the confines of the pipe the flow is subjected to forced convection for which the heat transfer is mainly dependent on the velocity and the current flow state. The temperature's influence is generally weak.

For the flow at the outside of the pipe, heat transfer is driven by the temperature difference $T_\mathrm{o} - T_\mathrm{Wo}$. As the temperature T_Wo is unknown, an estimate for either it or the heat transfer coefficient α_o needs to be given.

5. Determination of heat transfer coefficients

Inner pipe wall: pipe flow, forced convection

Temperature for which the material properties are determined: $T_\mathrm{st} = \frac{T_1 + T_2}{2} = -32.5\,°\mathrm{C}$

→ corresponding material properties: see task description

Flow state:

$$\mathrm{Re} = \frac{u_\mathrm{m} \cdot d_\mathrm{i} \cdot \rho}{\eta} = \frac{4 \cdot \dot{m} \cdot d_\mathrm{i} \cdot \rho}{\rho \cdot \pi \cdot d_\mathrm{i}^2 \cdot \eta} = \frac{4 \cdot \dot{m}}{\pi \cdot d_\mathrm{i} \cdot \eta}$$
$$= 12860 \quad \rightarrow \text{turbulent flow} \tag{4.41}$$

Thermal boundary condition: $\approx$ constant wall temperature

Heat transfer law HTL.14:

$$\overline{\mathrm{Nu}_{\mathrm{d_i}}} = \frac{\alpha_\mathrm{i} \cdot d_\mathrm{i}}{\lambda} = 0.0235 \cdot \left(\mathrm{Re}_{\mathrm{d_i}}^{0.8} - 230\right) \cdot \left(1{,}8 \cdot \mathrm{Pr}^{0.3} - 0.8\right) \cdot \ldots$$
$$\ldots \cdot \left(1 + \left(\frac{d_\mathrm{i}}{L}\right)^{\frac{2}{3}}\right) \cdot \left(\frac{\eta}{\eta_\mathrm{W}}\right)^{0.14} \tag{4.42}$$

As no significant temperature differences over the pipe's cross-section are to be expected and pipe inlet effects have diminished due to $d_\mathrm{i} \ll L$, one can set

$$\frac{\eta}{\eta_\mathrm{W}} \approx 1 \qquad \text{and} \qquad 1 + \left(\frac{d_\mathrm{i}}{L}\right)^{\frac{2}{3}} \approx 1. \tag{4.43}$$

.

$$\begin{aligned} \overline{\mathrm{Nu}_{\mathrm{d_i}}} &= 0.0235 \cdot \left(\mathrm{Re}_{\mathrm{d_i}}^{0.8} - 230\right) \cdot \left(1.8 \cdot \mathrm{Pr}^{0.3} - 0.8\right) \\ \mathrm{Pr} &= \frac{\nu}{a} = \frac{\eta \cdot c_\mathrm{p}}{\lambda} = 22.5 \\ \overline{\mathrm{Nu}_{\mathrm{d_i}}} &= 152 \\ \alpha_i &= 1.677\,\mathrm{kW/m^2\,K} \end{aligned}$$

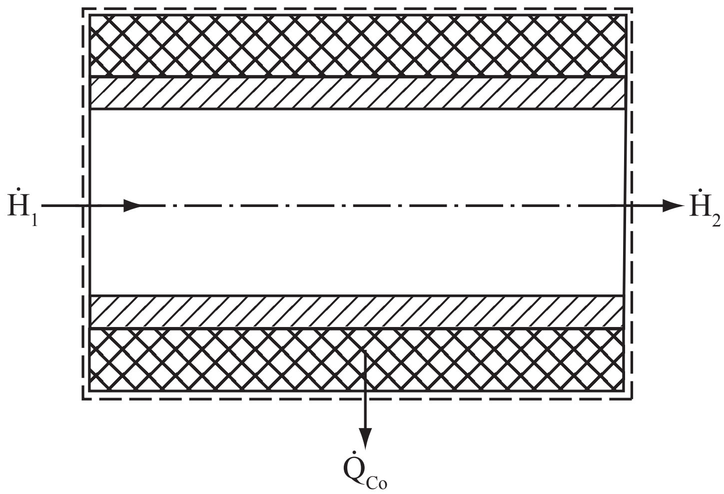
Outer pipe surface: surrounding flow on (horizontal) pipe, free convection

As laid out under point 3, the determination of T_{Wo} or α_{o} necessitates an iterative approach. Here:

$$\alpha_{\mathrm{o}}^{*} = 10\,\mathrm{W}/\mathrm{m}^2\,\mathrm{K} \tag{4.44}$$

To check the value for α_{o} it is necessary first to determine the surface temperature T_{Wo}. With a suitable choice of the system boundary the surface temperature is immediately obtained from an energy balance.

System boundary



If one regards α_{o} to denote the overall heat transfer coefficient, by including the radiative heat transfer between the pipe's outer jacket surface with its surroundings, it follows Versteht man unter α_{o} den Gesamtwärmeübergangskoeffizienten, schließt man also den Strahlungsaustausch zwischen der äußeren Mantelfläche and der Umgebung ein, so gilt

$$\dot{H}_1 - \dot{H}_2 = \dot{Q}_{\mathrm{Co}} \tag{4.45}$$

with the enthalpy fluxes according to 4.35 and 4.36 and with the heat transfer equation

$$\dot{Q}_{\mathrm{Co}} = \alpha_{\mathrm{o}}^{*} \cdot \pi \cdot (d_{\mathrm{o}} + 2s) \cdot L \cdot (T_{\mathrm{Wo}}^{*} - T_{\mathrm{o}}) \tag{4.46}$$

Thus follows

$$\begin{aligned} T_{\mathrm{Wo}} &= T_{\mathrm{o}} - \frac{\dot{m} \cdot c \cdot (T_2 - T_1)}{\alpha_{\mathrm{o}}^* \cdot \pi \cdot (d_{\mathrm{o}} + 2s) \cdot L} \\ T_{\mathrm{Wo}}^* &= 3.35\,^{\circ}\mathrm{C} \end{aligned}$$

Nachrechnung des Wärmeübergangskoeffizienten

Konvektiver Wärmeübergang:

Temperature for which the material properties are determined: $T_{\mathrm{st}} = \frac{T_{\mathrm{Wo}}^* + T_{\mathrm{o}}}{2} = 11.7\,^{\circ}\mathrm{C}$

Stoffwerte von Luft:

$$\nu = 14.6 \cdot 10^{-6}\,\mathrm{m^2/s} \qquad \lambda = 0.0251\,\mathrm{W/m\,K} \qquad \mathrm{Pr_o} = 0.71 \tag{4.47}$$

Flow state:

$$\begin{aligned} \mathrm{Gr} \cdot \mathrm{Pr} &= \frac{g \cdot \beta\,(T_{\mathrm{o}} - T_{\mathrm{Wo}}^*) \cdot (d_{\mathrm{o}} + 2s)^3}{\nu^2} \cdot \mathrm{Pr_o} \\ &= \frac{g \cdot (T_{\mathrm{o}} - T_{\mathrm{Wo}}^*) \cdot (d_{\mathrm{o}} + 2s)^3}{T_{\mathrm{o}} \cdot \nu^2} \cdot \mathrm{Pr_o} \\ &= 0.49 \cdot 10^6 \;\rightarrow\; \text{laminar flow} \end{aligned} \tag{4.48}$$

Thermal boundary condition: $\approx$ constant wall temperature

Heat transfer law HTL.20

$$\overline{\mathrm{Nu}_{\mathrm{d_o+2s}}} = \frac{\alpha_{\mathrm{K}}\,(d_{\mathrm{o}} + 2s)}{\lambda} = 0.53 \cdot (\mathrm{Gr}_{\mathrm{d_o+2s}} \cdot \mathrm{Pr})^{\frac{1}{4}} = 14 \tag{4.49}$$

$$\alpha_{\mathrm{K}} = \frac{12.9 \cdot 0.0251}{0.0645} = 5.5\,\mathrm{W/m^2\,K} \tag{4.50}$$

Heat transfer through radiation:

To make a reasonable approximation, the pipe is assumed to be a black body. The following estimate for the heat transfer coefficient is obtained

$$\alpha_S = 4\sigma \cdot \left(T^*_{Wo}\right)^3 = 4.8\,\mathrm{W/m^2\,K} \tag{4.51}$$

Overall heat transfer coefficient

$$\begin{aligned} \alpha_o &= \alpha_K + \alpha_S = 10.3\,\mathrm{W/m^2\,K} && (4.52)\\ \Rightarrow \quad \alpha_o &\approx \alpha^*_o = 10\,\mathrm{W/m^2\,K} \qquad T_{Wo} \approx T^*_{Wo} && (4.53) \end{aligned}$$

6. Determination of the insulation's thermal conductivity

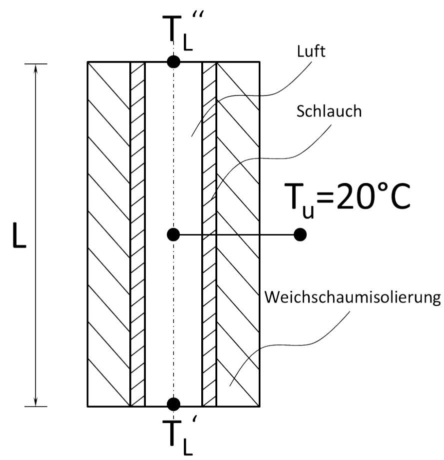
For the conditions present in the underlying case, the heat transfer coefficient at the pipe's inner wall and the resistance to heat transfer within the copper pipe can be neglected when compared to the other resistances to heat transfer. In accordance with Eq. 4.38, it follows

$$\begin{aligned} \lambda_{ins} &= \frac{\frac{1}{2} \cdot \ln\left(\frac{d_o+2s}{d_o}\right)}{\frac{\pi \cdot L}{\dot{m} \cdot c} \cdot \frac{T_o - 0{,}5 \cdot (T_1+T_2)}{T_2 - T_1} - \frac{1}{\alpha_o \cdot (d_o+2s)}} \\ &= \frac{0.5 \cdot \ln\left(\frac{0.064}{0.024}\right)}{\frac{\pi \cdot 18 \cdot 3600}{1600 \cdot 2260} \cdot \frac{52.5}{0.6} - \frac{1}{10 \cdot 0.064}} \\ &= 0.15\,\mathrm{W/m\,K} \end{aligned} \tag{4.54}$$

$\lambda_{ins} > \lambda_{cork} \qquad \Rightarrow \qquad$ Isolierung ist durchfeuchtet!

4.5. Ventilator

1. Problem type Heat transfer for forced pipe-flow convection.



a) The heat transfer coefficient at the tube inner wall in addition to the overall coefficient of heat transmissivity k are to be determined.
To determine the heat transfer coefficient it is necessary to determine the Nusselt number. First the flow velocity is determined in order to calculate the Reynolds number.
Determination of the flow velocity:

$$u = \frac{\dot{\mathrm{m}}_\mathrm{o}}{\rho} \cdot \frac{4}{\mathrm{d}_\mathrm{i}^2 \cdot \pi} = \frac{0.5\,\mathrm{kg}}{3600\,\mathrm{s}} \cdot \frac{1\mathrm{m}^3}{1\,\mathrm{kg}} \frac{4}{(0.01\,\mathrm{m})^2 \cdot \pi} = 1.768\,\mathrm{m/s} \tag{4.55}$$

Thus the Reynolds number reads:

$$\mathrm{Re} = \frac{\rho \cdot u \cdot d_\mathrm{i}}{\eta} = \frac{u \cdot d_\mathrm{i}}{\nu} = \frac{1.768\,\mathrm{m/s} \cdot 0.01\,\mathrm{m}}{20 \cdot 10^{-6}\,\mathrm{m^2/s}} = 884 < 2300 \tag{4.56}$$

As the task description stipulates the negligilibity of viscous forces, it holds:

$$\frac{\eta}{\eta_\mathrm{W}} = 1 \tag{4.57}$$

The Nusselt number can thus be determined from HTL.13 for a hydrodynamically and thermally undeveloped laminar flow.

$$\mathrm{Nu} = 3.66 + \frac{0.0677 \cdot (884 \cdot 0.71 \cdot \frac{0.01}{1.5})^{1.33}}{1 + 0.1 \cdot 0.71 \cdot (884 \cdot \frac{0.01}{1.5})^{0.83}} = 4.0069 \approx 4 \tag{4.58}$$

$$\alpha_\mathrm{i} = \frac{\mathrm{Nu} \cdot \lambda_\mathrm{o}}{d_\mathrm{i}} = \frac{4 \cdot 0.029\,\mathrm{W/m\,K}}{0.01\,\mathrm{m}} = 11.6\,\mathrm{W/m^2\,K} \tag{4.59}$$

The heat flux transmitted from the ambient air can be described by a combination of heat conduction as well as convective heat transfer at the inner and outer walls.

$$\dot{Q} = \frac{d_\mathrm{i} \cdot \pi \cdot L \cdot \triangle T}{\frac{1}{\alpha_\mathrm{i} \cdot d_\mathrm{i}} + \frac{1}{2} \cdot \left(\frac{1}{\lambda_\mathrm{s}} \cdot \ln\left(\frac{d_\mathrm{o}}{d_\mathrm{i}}\right) + \frac{1}{\lambda_\mathrm{ins}} \cdot \ln\left(\frac{D}{d_\mathrm{o}}\right)\right) + \frac{1}{\alpha_\mathrm{o} \cdot D}} \tag{4.60}$$

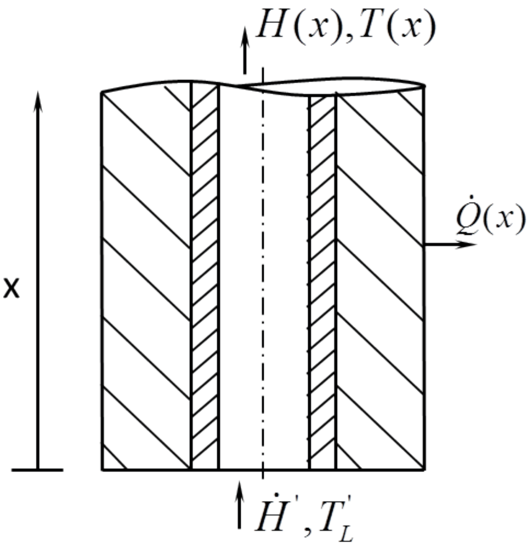
When utilising the heat transfer coefficient the relation simplifies to:

$$\frac{\dot{Q}}{A_\mathrm{i}} = \frac{k \cdot \pi \cdot d_\mathrm{i} \cdot L \cdot \triangle T}{\pi \cdot d_\mathrm{i} \cdot L} \tag{4.61}$$

A comparision of coefficients yields the governing equation for the heat transfer coeffiecient k.

$$\begin{aligned} k &= \left[\frac{1}{\alpha_\mathrm{i}} + \frac{d_\mathrm{i}}{2} \cdot \left(\frac{1}{\lambda_\mathrm{s}} \cdot \ln\left(\frac{d_\mathrm{o}}{d_\mathrm{i}}\right) + \frac{1}{\lambda_\mathrm{ins}} \cdot \ln\left(\frac{D}{d_\mathrm{o}}\right)\right) + \frac{d_\mathrm{i}}{\alpha_\mathrm{o} \cdot D}\right]^{-1} \\ &= \left[\frac{1}{11.6\,\mathrm{W/m^2 \cdot K}} + \frac{0.01\,\mathrm{m}}{2} \cdot \left(\frac{1}{0.2\,\mathrm{W/m\,K}} \cdot \ln\left(\frac{0.015}{0.01}\right) + \dots\right.\right. \\ &\qquad \left.\left. \dots + \frac{1}{0.04\,\mathrm{W/m\,K}} \cdot \ln\left(\frac{0.06}{0.015}\right)\right) + \frac{0.01\,\mathrm{m}}{9\,\mathrm{W/m^2\,K} \cdot 0.06\,\mathrm{m}}\right]^{-1} \\ &= 3.47\,\mathrm{W/m^2\,K} \end{aligned} \tag{4.62}$$

b) Profile of mean air flow temperature



According to the first law of thermodynamics the heat flux transmitted at the location x is equal to the difference in enthalpy relative to the hose inlet

$$\dot{Q}(x) = \dot{H}' - \dot{H}(x) = \dot{m} \cdot c_{\mathrm{o}} \cdot (T_{\mathrm{o}}' - T_{\mathrm{o}}(x)) \tag{4.63}$$

Additionally, it holds:

$$\dot{Q}(x) = k \cdot A_{\mathrm{i}}(x) \cdot \triangle T \tag{4.64}$$

with

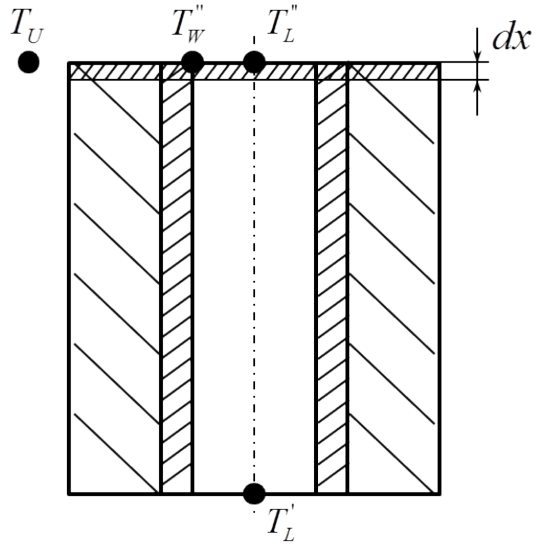
$$\triangle T = (T_{\mathrm{o}} - T_{\mathrm{U}})_{\mathrm{m}} = \frac{((T_{\mathrm{o}}' - T_{\mathrm{U}}) - (T_{\mathrm{o}}\,(x) - T_{\mathrm{U}})}{\ln\left(\dfrac{T_{\mathrm{o}}' - T_{\mathrm{U}}}{T_{\mathrm{o}}\,(x) - T_{\mathrm{U}}}\right)} \tag{4.65}$$

Comparing 4.63 and 4.64 and solving for T yields the temperature profile.

$$\dot{m} \cdot c_{\mathrm{o}} \cdot (T_{\mathrm{o}}' - T_{\mathrm{o}}(x)) = \frac{k \cdot \pi \cdot d_{\mathrm{i}} \cdot x \cdot (T_{\mathrm{o}}' - T_{\mathrm{o}}(x))}{\ln\left(\dfrac{T_{\mathrm{o}}' - T_{\mathrm{U}}}{T_{\mathrm{o}}(x) - T_{\mathrm{U}}}\right)} \tag{4.66}$$

$$\frac{T_{\mathrm{o}}(x) - T_{\mathrm{U}}}{T_{\mathrm{o}}' - T_{\mathrm{U}}} = \left[\exp\left(\frac{k \cdot \pi \cdot d_{\mathrm{i}} \cdot L}{\dot{m} \cdot c_{\mathrm{o}}} \cdot \frac{x}{L}\right)\right]^{-1} = \exp\left(-\frac{k \cdot \pi \cdot d_{\mathrm{i}} \cdot L}{\dot{m} \cdot c_{\mathrm{o}}} \cdot \frac{x}{L}\right) \tag{4.67}$$

In subtask b) the relation for the airflow and hose wall temperature at the critical location is unknown; no condensation may occur within the hose. The airflow is continuously cooled over the entire length of the hose. Thus, the critical location is at the hose's inner wall at the end of the hose (Index $''$).



The circumferential heat flux $q'_{\mathrm{W}\to\mathrm{C}}$ between wall and ambient air along the length dx can be expressed as follows:

$$\dot{q}'_{\mathrm{W}\to\mathrm{C}} = \frac{dx}{\frac{1}{k} - \frac{1}{\alpha_{\mathrm{i}}}} \cdot (T''_{\mathrm{W}} - T_{\mathrm{A}}) \tag{4.68}$$

It must be identical to the heat flux transferred from the ambient air $q'_{\mathrm{L}\to\mathrm{A}}$

$$\dot{q}'_{\mathrm{L}\to\mathrm{U}} = k \cdot dx \cdot (T''_{\mathrm{o}} - T_{\mathrm{A}}) = \frac{dx}{\frac{1}{k} - \frac{1}{\alpha_{\mathrm{i}}}} \cdot (T'' - T_{\mathrm{A}}) = \dot{q}'_{\mathrm{W}\to\mathrm{U}} \tag{4.69}$$

Thus results the temporal relationship between wall and hose wall temperature.

$$T''_{\mathrm{o}} - T_{\mathrm{A}} = \frac{1}{1 - \frac{k}{\alpha_{\mathrm{i}}}} \cdot (T''_{\mathrm{W}} - T_{\mathrm{A}}) \tag{4.70}$$

Solving 4.67 for $T_{\mathrm{o}}(x) - T_{\mathrm{A}}$, evaluating for $x = L$ and comparing with 4.70 yields the sought-after air inlet temperature.

$$T''_{\mathrm{o}} - T_{\mathrm{A}} = \frac{1}{1 - \frac{k}{\alpha_{\mathrm{i}}}} \cdot (T''_{\mathrm{W}} - T_{\mathrm{A}}) = \exp\left(-\frac{k \cdot \pi \cdot d_{\mathrm{i}} \cdot L}{\dot{m} \cdot c_{\mathrm{o}}} \cdot \frac{x}{L}\right) \cdot (T'_{\mathrm{o}} - T_{\mathrm{A}}) \quad (4.71)$$

$$T'_{\mathrm{o}} = \frac{\exp\left(\frac{k \cdot \pi \cdot d_{\mathrm{i}} \cdot L}{\dot{m} \cdot c_{\mathrm{o}}}\right)}{1 - \frac{k}{\alpha_{\mathrm{i}}}} \cdot (T''_{\mathrm{W}} - T_{\mathrm{A}}) + T_{\mathrm{A}} \quad (4.72)$$

c) Heat loss $\dot{Q}_{\mathrm{o}}$ of the airflow The heat flux of the air flow is equal to the enthalpy difference between hose in- and outlet.

$$\dot{Q}_{\mathrm{o}} = \dot{m} \cdot c_{\mathrm{o}} \cdot (T''_{\mathrm{o}} - T'_{\mathrm{o}}) \quad (4.73)$$

The task specifies that the wall temperature at the cirtical location must lie 1 K above the dew point temperature. With this specification the air temperature can be determined with help of eq. 4.70.

$$T''_{\mathrm{o}} = \frac{T''_{\mathrm{W}} - T_{\mathrm{A}}}{1 - \frac{k}{\alpha_{\mathrm{i}}}} + T_{\mathrm{A}} \quad (4.74)$$

$$= \frac{(309.15\,\mathrm{K} - 293.15\,\mathrm{K})}{1 - \frac{3.5}{11.6}} + 293.15\,\mathrm{K} = 316.06\,\mathrm{K} \mathrel{\hat{=}} 42.91\,^{\circ}\mathrm{C} \quad (4.75)$$

Equation 4.72 yields the air inlet temperature:

$$T'_{\mathrm{o}} = \frac{\exp\left(\frac{3.5\,\mathrm{W/m^2\,K} \cdot \pi \cdot 0.01\,\mathrm{m} \cdot 1.5\,\mathrm{m}}{0.5\,\mathrm{kg/3600\,s} \cdot 1000\,\mathrm{J/kg\,K}}\right)}{1 - \frac{3.5}{11.6}} \cdot (309.15\,\mathrm{K} - 293.15\,\mathrm{K}) + 293.15\,\mathrm{K}$$
$$= 368.28\,\mathrm{K} \quad (4.76)$$

allowing the heat flux lost over the hose's length to be calculated.

$$\dot{Q}_{\mathrm{o}} = 0.5\,\mathrm{kg/3600\,s} \cdot 1000\,\mathrm{J/kg\,K} \cdot (368.28\,\mathrm{K} - 316.06\,\mathrm{K}) = 7.253\,\mathrm{W} \quad (4.77)$$

$$\rightarrow \quad T_{\mathrm{W,ins}} = T_{\mathrm{A}} + \frac{\dot{Q}_{\mathrm{o}}}{\alpha_{\mathrm{o}} \cdot \pi \cdot D \cdot L} = 296\,\mathrm{K} \quad (4.78)$$

with hint of negligible temperature differences the following linearisation is applicable:

$$T_\mathrm{o}^4 - T_\mathrm{b}^4 \approx 4 \cdot T_\mathrm{b}^3 \cdot (T_\mathrm{o} - T_\mathrm{b}) \tag{4.79}$$

The heat transfer coefficient for radiation thus reads

$$\alpha_\mathrm{o,Rad} \cdot (T_\mathrm{W,ins} - T_\mathrm{A}) = \epsilon_\mathrm{ins} \cdot \sigma \cdot 4 \cdot T_\mathrm{A}^3 \cdot (T_\mathrm{W,ins} - T_\mathrm{A}) \tag{4.80}$$

$$\alpha_\mathrm{o,Rad} = \epsilon_\mathrm{ins} \cdot \sigma \cdot 4 \cdot T_\mathrm{A}^3 \tag{4.81}$$

$$\alpha_\mathrm{o,Rad} = 0.95 \cdot 5.67 \cdot 10^{-8}\,\mathrm{W/m^2\,K^4} \cdot 4 \cdot (293.15\,\mathrm{K})^3 = 5.43\,\mathrm{W/m^2\,K} \tag{4.82}$$

Determination of the convective and overall heat transfer coefficients

$$\mathrm{Gr}\cdot\mathrm{Pr} = \frac{g\cdot(T_{\mathrm{W,ins}}-T_{\mathrm{A}})\cdot D^3}{T_{\mathrm{A}}\cdot\nu_{\mathrm{o}}}\cdot\mathrm{Pr_o} = 60910.89 \tag{4.83}$$

$$\text{HTL.20: }\mathrm{Nu} = 0.53\cdot(\mathrm{Gr}\cdot\mathrm{Pr})^{0.25} = 8.32 \tag{4.84}$$

$$\alpha_{\mathrm{o,conv.}} = \frac{\lambda_{\mathrm{o}}}{D}\cdot\ \mathrm{Nu} = 3.608\,\mathrm{W/m^2\,K} \tag{4.85}$$

$$\alpha_{\mathrm{overall}} = \alpha_{\mathrm{o,Rad}}+\alpha_{\mathrm{o,conv.}} = 9.04\,\mathrm{W/m^2\,K} \tag{4.86}$$

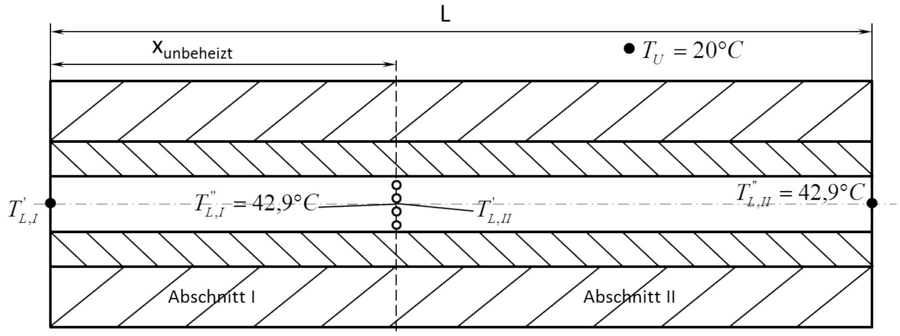
$$\tag{4.87}$$

Alternative solution for the radiative heat transfer coefficient

$$\varepsilon_{\mathrm{ins}}\cdot\sigma\cdot\left(T^4_{\mathrm{W,ins}}-T^4_{\mathrm{U}}\right) = \alpha_{\mathrm{o,Rad}}\cdot(T_{\mathrm{W,ins}}-T_{\mathrm{A}}) \tag{4.88}$$

$$\alpha_{\mathrm{o,Rad}} = \frac{\varepsilon_{\mathrm{ins}}\cdot\left(T^4_{\mathrm{W,ins}}-T^4_{\mathrm{U}}\right)}{T_{\mathrm{W,ins}}-T_{\mathrm{U}}} = 5.6\,\mathrm{W/m^2\,K} \tag{4.89}$$

d) Installation of an electric heating element due to the limitation of the tube inlet temperature $T_A^{'*}$



The heating element needs to be installed at the location x where the air temperature without heating would fall under the stipulated minimum temperature of ($T_{\mathrm{A,min}} = 42.91\,°\mathrm{C}$ (see 4.75)). An appropriate governing equation can be determined through solving equation 4.66 for x.

$$x_{\mathrm{unheated}} = \frac{-\ln\left(\dfrac{T_{\mathrm{A,min}} - T_{\mathrm{A}}{}^{*}}{T'_{\mathrm{o}}} - T_{\mathrm{A}}\right) \cdot \dot{m} \cdot c_{\mathrm{o}}}{k \cdot \pi \cdot d_{\mathrm{i}}} \tag{4.90}$$

$$= \frac{-\ln\left(\frac{22.91}{40}\right) \cdot 0.5\,{}^{\mathrm{kg}}/_{3600\,\mathrm{s}} \cdot 1000\,{}^{\mathrm{J}}/_{\mathrm{kg\,K}}}{3.5\,{}^{\mathrm{W}}/_{\mathrm{m^2\,K}} \cdot \pi \cdot 0.01\,\mathrm{m}} = 0.704\,\mathrm{m} \tag{4.91}$$

Determination of the required power input for the heating elements:
It holds: $\dot{Q}_{\mathrm{L,II}} = -\dot{Q}_{\mathrm{el}}$ where $\dot{Q}_{\mathrm{L,II}}$ signifies the loss heat flux in section II. To determine the loss heat flux $\dot{Q}_{\mathrm{V,II}}$ the required temperature of the air flow immediately after the heating element needs to be determined. The air temperature can be determined by employing equation 4.67.

$$\frac{T''_{\mathrm{L,II}} - T_{\mathrm{A}}}{T'_{\mathrm{L,II}} - T_{\mathrm{A}}} = \exp\left(\frac{k \cdot \pi \cdot d_{\mathrm{i}} \cdot (L - x_{\mathrm{unheated}})}{\dot{m}_{\mathrm{o}} \cdot c_{\mathrm{o}}}\right) \tag{4.92}$$

$$T'_{\mathrm{L,II}} = T_{\mathrm{A}} + (T''_{\mathrm{L,II}} - T_{\mathrm{A}}) \cdot \exp\left(\frac{k \cdot \pi \cdot d_{\mathrm{i}} \cdot (L - x_{\mathrm{unheated}})}{\dot{m}_{\mathrm{o}} \cdot c_{\mathrm{o}}}\right) = 336.25\,\mathrm{K} \tag{4.93}$$

The necessary heating power input thus reads:

$$\begin{aligned}\dot{Q}_{\mathrm{el}} &= \dot{m}_{\mathrm{o}} \cdot c_{\mathrm{o}} \cdot \left(T'_{\mathrm{L,II}} - T''_{\mathrm{L,II}}\right)\\ &= 0.5\,{}^{\mathrm{kg}}\!/_{3600\,\mathrm{s}} \cdot 1000\,{}^{\mathrm{J}}\!/_{\mathrm{kg\,K}} \cdot (336.25\,\mathrm{K} - 316.05\,\mathrm{K})\\ &= 2.8\,\mathrm{W}\end{aligned} \tag{4.94}$$

The heat flux lost in section I can be determined analogously to that of section II

$$\begin{aligned}\dot{Q}_{\mathrm{L,I}} &= \dot{m} \cdot c_{\mathrm{o}} \cdot (T''_{\mathrm{L,I}} - T'_{\mathrm{L,I}})\\ &= 0.5\,{}^{\mathrm{kg}}\!/_{3600\,\mathrm{s}} \cdot 1000\,{}^{\mathrm{J}}\!/_{\mathrm{kg\,K}} \cdot (316.06\,\mathrm{K} - 333.15\,\mathrm{K})\\ &= -2.375\,\mathrm{W}\end{aligned} \tag{4.95}$$

$$\dot{Q}_{\mathrm{L,total}} = \dot{Q}_{\mathrm{L,I}} + \dot{Q}_{\mathrm{L,II}} = -5.175\,\mathrm{W} \tag{4.96}$$

Chapter 5.

Solutions mass transfer

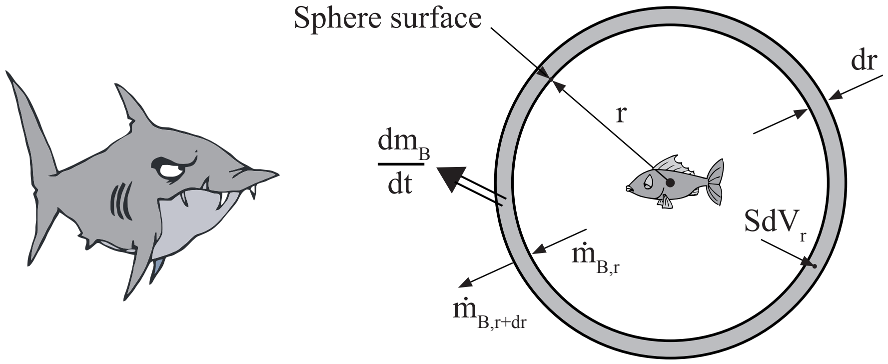
5.4. Shark attack on Mallorca

1. Problem type

   Diffusion in radially symmetric geometry, thus spherical coordinates are chosen. Convection is neglected.

2. Solution

   a) In the differential mass balance for the derivation of the ODE the fundamental mechanisms of diffusion as well as the temporal dependency of the blood flow (index B) as well as possible source terms need to be taken into account (see illustration).



For the individual terms of the balance it holds

$$\dot{m}_{\mathrm{B,r}} = j''_{\mathrm{B}} \cdot A_{\mathrm{r}} = -\rho_{\mathrm{tot}} \cdot D_{\mathrm{B,W}} \cdot \frac{\partial \xi_{\mathrm{B}}}{\partial r} \cdot A_{\mathrm{r}} \text{ (Fick's 1. Law)} \tag{5.1}$$

$$\dot{m}_{\mathrm{B,r+dr}} = \dot{m}_{\mathrm{B,r}} + \frac{\partial \dot{m}_{\mathrm{B,r}}}{\partial r} dr \qquad \text{(Taylor expansion)} \tag{5.2}$$

$$\frac{\partial m_{\mathrm{B}}}{\partial t} = \rho_{\mathrm{tot}} \cdot dV_{\mathrm{r}} \cdot \frac{\partial \xi_{\mathrm{B}}}{\partial t} \tag{5.3}$$

$$S \cdot dV_{\mathrm{r}} = S \cdot A_{\mathrm{r}} \cdot dr = S \cdot 4\pi r^2 \cdot dr \qquad [S] = {}^{\mathrm{kg_B}}/_{\mathrm{m^3 s}} \tag{5.4}$$

wherein S is a volume specific source terms. For the balance it holds:

$$\dot{m}_{\mathrm{B,r}} - \dot{m}_{\mathrm{B,r+dr}} + S \cdot dV_{\mathrm{r}} - \frac{\partial m_{\mathrm{B}}}{\partial t} = 0 \tag{5.5}$$

$$\begin{aligned} \frac{\partial}{\partial r}\left(\rho_{\mathrm{tot}} \cdot D_{\mathrm{B,W}} \cdot \frac{\partial \xi_{\mathrm{B}}}{\partial r} \cdot 4\pi r^2\right) dr + \ldots \\ \ldots + S \cdot 4\pi r^2 dr - \rho_{\mathrm{tot}} \cdot dV_{\mathrm{r}} \cdot \frac{\partial \xi_{\mathrm{B}}}{\partial t} = 0 \end{aligned} \tag{5.6}$$

$$\rho_{\mathrm{tot}} \cdot \frac{\partial \xi_{\mathrm{B}}}{\partial t} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(\rho_{\mathrm{tot}} \cdot D_{\mathrm{D,W}} \cdot r^2 \cdot \frac{\partial \xi_{\mathrm{B}}}{\partial r}\right) + S \tag{5.7}$$

b) The analogy of heat and mass transfer becomes apparent for the comparison with the general equation for heat conduction in spherical coordinates:

$$\begin{aligned} \rho \cdot c_p \cdot \frac{\partial T}{\partial t} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \cdot \lambda \cdot \frac{\partial T}{\partial r}\right) + \ldots \\ \ldots + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\lambda \cdot \sin\theta \cdot \frac{\partial T}{\partial \theta}\right) + \ldots \\ \ldots + \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial}{\partial \phi}\left(\lambda \cdot \frac{\partial T}{\partial \phi}\right) + \dot{\Phi}''' \end{aligned} \tag{5.8}$$

or $\left(\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial \phi} = 0\right)$ if no radial or azimuthal dependency on angles exists, respectively:

$$\rho \cdot c_p \frac{\partial T}{\partial t} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \cdot \lambda \cdot \frac{\partial T}{\partial r} \right) + \dot{\Phi}''' \tag{5.9}$$

The units chosen in this set of equations are the units of the conservation quantities mass and energy, with reference to volume and time, thus they read [$^{\mathrm{kg_B}}/_{\mathrm{m^3 s}}$] and [$^{\mathrm{J}}/_{\mathrm{m^3 s}}$]. With respect to the mass transfer equation it must be noted that the mass conserved of the material regarded here is not equal to the overall mass. This aösp explains the existence of the mass source terms. E.g. if a chemical reaction takes place within a system this is reflected in a negative mass source term for the educts and a positive source terms for the reaction's porducts, repectively.

The analogy becomes even clearer if one stipulates constant material properties and normalises the equations for the temporal change of the dependent variable:

$$\frac{\partial \xi_{\mathrm{B}}}{\partial t} = D_{\mathrm{B,W}} \cdot \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \cdot \left(r^2 \cdot \frac{\partial \xi_{\mathrm{B}}}{\partial r} \right) + \frac{S}{\rho_{\mathrm{tot}}} \tag{5.10}$$

$$\frac{\partial T}{\partial t} = a \cdot \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \cdot \frac{\partial T}{\partial r} \right) + \frac{\dot{\Phi}'''}{\rho_{\mathrm{tot}} \cdot c_p} \tag{5.11}$$

The individual quantities can now be compared as follows:

Meaning	Mass transfer (purely diffusive)	Heat conduction
Independent variables	r,t	r,t
Dependent variables (Potential)	ξ_B [$\mathrm{kg_B/kg}$]	T [K]
Source term	S [$\mathrm{kg_B/kg}$]	$\dot{\Phi}'''\left[\frac{J}{m^3 s}\right] = [\mathrm{W/m^3}]$
Description of the relation between temporal change in potential and curvature of the potential field through a proportional constant	$D_{B,W}$ [$\mathrm{m^2/s}$] (Diffusion coefficient)	$a = \frac{\lambda}{\rho \cdot c_p}$ [$\mathrm{m^2/s}$]
Factor for conversion from potential (ξ_B, T) to the volume dependent conservation quantity (Blood mass, energy)	ρ_{tot} [$\mathrm{kg/m^3}$]	$\rho \cdot c_p$ [$\mathrm{J/m^3K}$]

c) $\frac{\partial \xi_B}{\partial t} = 0$ is set, as steady-state conditions are present. It holds $S = 0$, as $r > r_F$ is considered for the solution of the task and no mass sources are present.

$$\Rightarrow \qquad \frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \xi_B}{\partial r}\right) \tag{5.12}$$

Through integration one obtains

$$\frac{\partial \xi_B}{\partial r} = \frac{c_1}{r^2} \quad \text{und} \quad \xi_B = -\frac{c_1}{r} + c_2 \tag{5.13}$$

With the 1. Fick las, applied for the fish's skin (state F),

$$\dot{m}_B = -4\pi r_F^2 \cdot \rho_{tot} \cdot D_{B,W} \cdot \left.\frac{\partial \xi_B}{\partial r}\right|_{r=r_F} \tag{5.14}$$

and the boundary conditions

$$\left.\frac{\partial \xi_{\mathrm{B}}}{\partial r}\right|_{r=r_{\mathrm{F}}} = -\frac{\dot{m}_{\mathrm{B}}}{4\pi r_{\mathrm{F}}^2 \cdot \rho_{\mathrm{tot}} \cdot D_{\mathrm{B,W}}} = \frac{c_1}{r_{\mathrm{F}}^2} \qquad \xi_B(r = r_{\mathrm{F}}) = \xi_{\mathrm{F}} = -\frac{c_1}{r_{\mathrm{F}}} + c_2 \tag{5.15}$$

One obtains

$$c_1 = \frac{\dot{m}_{\mathrm{B}}}{4\pi \cdot \rho_{\mathrm{tot}} \cdot D_{\mathrm{B,W}}} \quad \text{und} \quad c_2 = \xi_{\mathrm{F}} - \frac{\dot{m}_{\mathrm{B}}}{4\pi r_{\mathrm{F}} \cdot \rho \mathrm{tot.} \cdot D_{\mathrm{B,W}}} \tag{5.16}$$

and thus

$$\xi = \xi_{\mathrm{F}} + B\left(\frac{1}{r} - \frac{1}{r_{\mathrm{F}}}\right) \quad \text{mit} \quad B \equiv \frac{\dot{m}_{\mathrm{B}}}{4\pi \cdot \rho_{\mathrm{tot}} \cdot D_{\mathrm{B,W}}} \tag{5.17}$$

With $\xi(r_{\mathrm{crit}}) = \xi_{\mathrm{crit}}$ one eventually obtains the required safe distance

$$r_{\mathrm{crit}} = \frac{1}{(\xi_{\mathrm{crit}} - \xi_{\mathrm{F}})\, B^{-1} + r_{\mathrm{F}}^{-1}} \tag{5.18}$$

d) Approach:

As the critical values ξ_{crit} und r_{crit} are valid only for Serine and not for blood, the Serine concentration at the fish's skin ξ_{SF} is determined through the diffusion of Serine in water. Thus, blood has to be replaced with Serine for all perviously derived formulae. The diffusion of the remaining blood fractions in water is irrelevant as they behave like water, in accordance with the task's description. The diffusion of Serine within the blood is irrelevant, too, as the Serine is homogeneously distributed within the blood.

$$\xi_{\mathrm{SF}} = \xi - B\left(\frac{1}{r} - \frac{1}{r_{\mathrm{F}}}\right) \tag{5.19}$$

$$\begin{aligned} B &= \frac{\xi_{\mathrm{S}}\dot{m}_{\mathrm{B}}}{4\pi\cdot\rho_{\mathrm{tot}}\cdot D_{\mathrm{B,W}}} = \frac{1{,}05\cdot10^{-4}\,{}^{\mathrm{kg_S}}\!/\!_{\mathrm{kg_B}}\cdot 2{,}78\cdot10^{-6}\,{}^{\mathrm{kg_B}}\!/\!_{\mathrm{s}}}{4\pi\cdot 999\,{}^{\mathrm{kg}}\!/\!_{\mathrm{m^3}}\cdot 1.78\cdot10^{-7}\,{}^{\mathrm{kg_S m}}\!/\!_{\mathrm{kg}}} \\ &= 1.302\cdot10^{-7}\,{}^{\mathrm{kg_S m}}\!/\!_{\mathrm{kg}} \end{aligned} \tag{5.20}$$

$$\begin{aligned} \xi_{\mathrm{SF}} &= \xi_{\mathrm{crit}} - B\left(\frac{1}{r_{\mathrm{crit}}} - \frac{1}{r_{\mathrm{F}}}\right) \\ &= 1.05\cdot10^{-15}\,{}^{\mathrm{kg_S}}\!/\!_{\mathrm{kg}} - 1.302\cdot10^{-7}\,{}^{\mathrm{kg_S m}}\!/\!_{\mathrm{kg}}\cdot\left(\frac{1}{400\,\mathrm{m}} - \frac{1}{0.106\,\mathrm{m}}\right) \\ &= 1.228\cdot10^{-6}\,{}^{\mathrm{kg_S}}\!/\!_{\mathrm{kg}} \end{aligned} \tag{5.21}$$

Calculation of the sought-after blood fraction from the Serine fraction:

$$\xi_{\mathrm{F}} = \frac{\xi_{\mathrm{SF}}}{\xi_{\mathrm{S}}} = \frac{1.228\cdot10^{-6}\,{}^{\mathrm{kg_S}}\!/\!_{\mathrm{kg}}}{1{,}05\cdot10^{-4}\,{}^{\mathrm{kg_S}}\!/\!_{\mathrm{kg_B}}} = 0.0117\,{}^{\mathrm{kg_B}}\!/\!_{\mathrm{kg}} \tag{5.22}$$

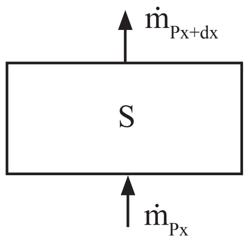
5.5. Even more critical explosive

1. Problem type

   One-dimensional steady-state diffusion with source terms.

2. Solution

   a) Balance around a differential volume element:



$$0 = \dot{m}_{\mathrm{P,x}} - \dot{m}_{\mathrm{P,x+dx}} + S \cdot dV \quad (5.23)$$

$$0 = -\frac{\mathrm{d}\dot{m}_{\mathrm{P,x}}}{\mathrm{d}x} dx + S \cdot A \cdot dx \quad (5.24)$$

$$\dot{m}_{\mathrm{P,x}} = -\rho \cdot D \cdot A \cdot \frac{\mathrm{d}\xi_P}{\mathrm{d}x} \qquad \text{(1. Ficksches Gesetz)} \quad (5.25)$$

$$\Rightarrow \quad 0 = \rho \cdot D \cdot \frac{\mathrm{d}^2\xi_{\mathrm{P}}}{\mathrm{d}x^2} + S \quad (5.26)$$

Calculation of the source term S:

The given temperature profile is valid under the condition of a temperature dependent heat source term, as outlined in the task description. Therefore, the local source term can be deduced from the resulting temperature prodile. The local heat production not only influences the local temperature, but also the source term's strength of the product P in the mass balance, which is immediately correlated to the heat production by means of the reaction enthalpy:

$$S=\frac{\dot{\Phi}'''}{\Delta h}=\frac{\dot{\Phi}'''_{\mathrm{A}}}{\Delta h}\left(1+\gamma\left(T-T_{\mathrm{A}}\right)\right)$$
$$=\frac{\dot{\Phi}'''_{\mathrm{A}}}{\Delta h}\cdot\frac{\cos(m\cdot x)}{\cos(m\cdot s)-\frac{\lambda}{\alpha}\cdot m\cdot\sin(m\cdot s)} \tag{5.27}$$

insert into ODE:

$$\Rightarrow\frac{\mathrm{d}^2\xi_{\mathrm{P}}}{\mathrm{d}x^2}=-\frac{\dot{\Phi}'''_{\mathrm{A}}}{\Delta h}\cdot\frac{1}{\cos(m\cdot s)-\frac{\lambda}{\alpha}\cdot m\cdot\sin(m\cdot s)}\cdot\frac{1}{\rho\cdot D}\cdot\cos(m\cdot x)$$
$$\equiv\underbrace{B}_{\text{const.}}\cdot\cos(m\cdot x) \tag{5.28}$$

Integration of the ODE:

$$\Rightarrow\frac{\mathrm{d}\xi_{\mathrm{P}}}{\mathrm{d}x}\;=\;\frac{B}{m}\cdot\sin(m\cdot x)+c_1 \tag{5.29}$$
$$\Rightarrow\xi_{\mathrm{P}}\;=\;-\frac{B}{m^2}\cdot\cos(m\cdot x)+c_1\cdot x+c_2 \tag{5.30}$$

Boundary condtions:

1. Impermeable lower metal jacket surface:

   $\left.\frac{\mathrm{d}\xi_{\mathrm{P}}}{\mathrm{d}x}\right|_{x=0}=\frac{B}{m}\cdot\sin(m\cdot 0)+c_1=0\Rightarrow c_1=0$

2. Mass transfer at the upper plate:

   $g\cdot\left(\xi_{\mathrm{P}}(x=s)-\xi_{\mathrm{A}}\frac{\rho_{\mathrm{L}}}{\rho}\right)=-\rho D\cdot\left.\frac{\mathrm{d}\xi_{\mathrm{P}}}{\mathrm{d}x}\right|_{x=s}$ [1]

[1] A note concerning mass fractions in systems with phases of differing densities, such as explosive and air in this case: The ξ_{P}-values used here have the unit $\mathrm{kg_P}/\mathrm{kg_{tot,SP}}$, wherein the index tot,SP specifies the total mass of the explosive including the diffusing reaction product. (The influence of the local product concentration on the total density is neglected in this case.) The mass fraction ξ_{A} generally needs to be, although it describes the reaction product's concentration in the surronding air, converted from the unit $\mathrm{kg_P}/\mathrm{kg_{tot,A}}$ to $\mathrm{kg_P}/\mathrm{kg_{tot,SP}}$, as a difference needs to be calculated for the second boundary condition. Thsi happens under the assumption of a continous concentration function over the phase boundary with a density ratio of $\frac{\rho_{\mathrm{A}}}{\rho}$. As $\xi_{\mathrm{A}}=0$ the further conversion can be omitted.

As no reaction product enters the free-stream, it holds $\xi_{\mathrm{A}} = 0$:

$$\begin{aligned}
\Rightarrow c_2 &= -\frac{\dot{\Phi}'''_{\mathrm{A}}}{\Delta h} \cdot \frac{\cos(m \cdot s) - \frac{\rho \cdot D}{g} \cdot m \cdot \sin(m \cdot s)}{\cos(m \cdot s) - \frac{\lambda}{\alpha} \cdot m \cdot \sin(m \cdot s)} \cdot \frac{1}{\rho \cdot D} \cdot \frac{1}{m^2} \\
&= -\frac{B}{m^2} \cdot \cos(m \cdot x) - \ldots \\
&\qquad \ldots - \frac{\dot{\Phi}'''_{\mathrm{A}}}{\Delta h} \cdot \frac{\cos(m \cdot s) - \frac{\rho \cdot D}{g} \cdot m \cdot \sin(m \cdot s)}{\cos(m \cdot s) - \frac{\lambda}{\alpha} \cdot m \cdot \sin(m \cdot s)} \cdot \frac{1}{\rho \cdot D} \cdot \frac{1}{m^2} \\
&= \frac{\dot{\Phi}'''_{\mathrm{A}}}{\Delta h} \cdot \frac{1}{\rho \cdot D} \cdot \frac{1}{m^2} \cdot \ldots \\
&\qquad \ldots \cdot \frac{\cos(m \cdot x) - \cos(m \cdot s) + \frac{\rho \cdot D}{g} \cdot m \cdot \sin(m \cdot s)}{\cos(m \cdot s) - \frac{\lambda}{\alpha} \cdot m \cdot \sin(m \cdot s)}
\end{aligned} \tag{5.31}$$

Numerical values:

$$m = \sqrt{\frac{\gamma \cdot \dot{\Phi}''''_{\mathrm{A}}}{\lambda}} = \sqrt{\frac{0.2\,{}^{1}/_{\mathrm{K}} \cdot 0.3\,{}^{\mathrm{W}}/_{\mathrm{m}^3}}{0.85\,{}^{\mathrm{W}}/_{\mathrm{mK}}}} = 0.2657\,{}^{1}/_{\mathrm{m}} \tag{5.32}$$

$$\begin{aligned}
\Rightarrow \xi_{\mathrm{P}} &= 0.20238\,{}^{\mathrm{kg_P}}/_{\mathrm{kg}} \cdot \ldots \\
&\ldots \cdot \frac{\cos(0.2657\,{}^{1}/_{\mathrm{m}} \cdot x) - \cos(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s) + 7.971 \cdot 10^{-4} \cdot \sin(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s)}{\cos(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s) - 0.07528 \cdot \sin(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s)}
\end{aligned}$$

As the highest product fraction is encountered at $x = 0$, it needs to be lower than the critical mass fraction $\xi_{\mathrm{P,crit}} = 0.001$ here, to ensure that at no location within the plate the critical fraction is reached or even exceeded.

$$\begin{aligned}
\xi_{\mathrm{P}}(x = 0) &= 0.001 \\
&= 0.20238\,{}^{\mathrm{kg_P}}/_{\mathrm{kg}} \cdot \ldots \\
&\ldots \cdot \frac{1 - \cos(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s_{\max}) + 7.971 \cdot 10^{-4} \cdot \sin(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s_{\max})}{\cos(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s_{\max}) - 0.07528 \cdot \sin(0.2657\,{}^{1}/_{\mathrm{m}} \cdot s_{\max})}
\end{aligned}$$

$s_{\max}$ can be determined iteratively:

$$s_{\max} = 0.373\,\mathrm{m} \tag{5.33}$$

5.6. Perowskite

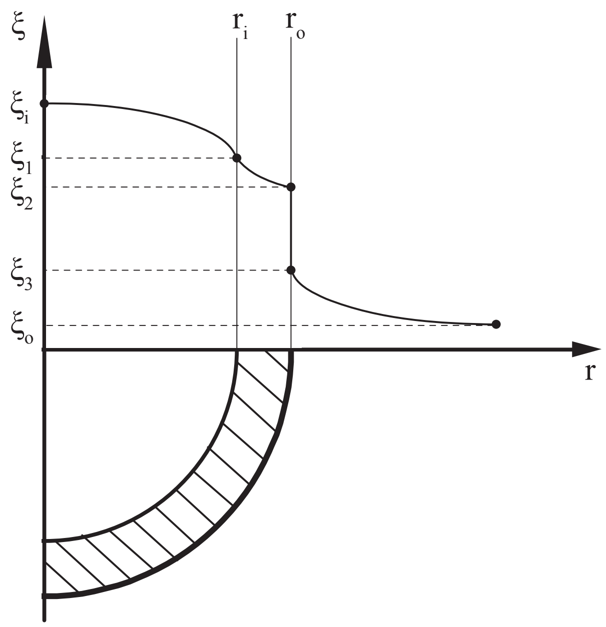
1. Problem type

   Mass transfer throguh a pipe wall: convective transfer inside and out, diffusion in a curved wall, transfer coefficient at a layer of unknown thickness, application of the Stefan correction at a semi-permeable surface.

2. Solution

   a) As the pipe is short relative to it's diameter, the mass fraction's dependence on the axial coordinate can be neglected. The transport of oxygen in radial direction is determined through four factors:

Mechanism	Described by
Convective transport at the inner pipe wall	g_i
Diffusion in the curved pipe walll	$\rho \cdot D$ resp. $\rho \cdot D^\mathrm{St}$
Transport through the Perowskite layer	k_p
(Thickness unknown)	
Convective transport at the outer pipe wall	g_o



Thus, it initially holds:

$$\begin{aligned}
\dot{m}_{O_2} &= A_\mathrm{i} \cdot g_\mathrm{i} \cdot (\xi_\mathrm{i} - \xi_1) && \text{(Inner side)}\\
&= A_\mathrm{i} \cdot k_\mathrm{T} \cdot (\xi_1 - \xi_2) && \text{(Pipe wall)}\\
&= A_\mathrm{o} \cdot k_\mathrm{p} \cdot (\xi_2 - \xi_3) && \text{(Perowskite)}\\
&= A_\mathrm{o} \cdot g_\mathrm{o} \cdot (\xi_3 - \xi_a) = A_\mathrm{o} \cdot g_\mathrm{o} \cdot \left(\xi_3 - \xi_\mathrm{o}^* \cdot \frac{\rho_\mathrm{RG}}{\rho_\mathrm{L}}\right) && \text{(Außenseite)}
\end{aligned}$$

$$A_\mathrm{i} = 2 \cdot \pi \cdot r_\mathrm{i} \cdot L \qquad\qquad A_\mathrm{o} = 2 \cdot \pi \cdot r_\mathrm{o} \cdot L$$

The exact definition of the effective transport coefficient k_T in the pipe's wall in relation to ρ and D. For this the differential equation for steady-state mass transfer without source terms in cylindrical coordinates is regarded (analoguous to equation 3.7b in the lecture script; in this case steady-state, no axial - due to the short pipe- or azimuthal/angular dependency, no source terms):

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r \cdot \frac{\mathrm{d}\xi}{\mathrm{d}r}\right) = 0 \tag{5.34}$$

Through integration one obtains:

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} = \frac{c_1}{r} \qquad \text{und} \qquad \xi = c_1 \cdot \ln r + c_2 \tag{5.35}$$

Inserting the boundary conditions leads to

$$\xi_1 = c_1 \cdot \ln r_\mathrm{i} + c_2 \tag{5.36}$$

$$\xi_2 = c_1 \cdot \ln r_\mathrm{o} + c_2 \tag{5.37}$$

Through subtraction of the equations one obtains:

$$\xi_1 - \xi_2 = c_1 \cdot \ln\left(\frac{r_\mathrm{i}}{r_\mathrm{o}}\right) \quad \Leftrightarrow \quad c_1 = \frac{\xi_1 - \xi_2}{\ln\left(\frac{r_\mathrm{i}}{r_\mathrm{o}}\right)}$$

$$\Rightarrow \quad \frac{\mathrm{d}\xi}{\mathrm{d}r} = \frac{\xi_1 - \xi_2}{\ln\left(\frac{r_\mathrm{i}}{r_\mathrm{o}}\right)} \cdot \frac{1}{r} \tag{5.38}$$

The diffusive mass flux through the pipe wall can now be express though Fick's 1. law:

$$\begin{aligned} \dot{m}_{\mathrm{O}_2} &= -A \cdot \rho \cdot D \cdot \frac{\mathrm{d}\xi}{\mathrm{d}r} = -A_\mathrm{i} \cdot \rho \cdot D \cdot \frac{\mathrm{d}\xi}{\mathrm{d}r}\Big|_{r=r_\mathrm{i}} \\ &= -A_\mathrm{i} \cdot \rho \cdot D \cdot \frac{\xi_1 - \xi_2}{\ln\left(\frac{r_\mathrm{i}}{r_\mathrm{o}}\right)} \cdot \frac{1}{r_\mathrm{i}} = A_\mathrm{i} \cdot k_\mathrm{T} \cdot (\xi_1 - \xi_2) \end{aligned} \tag{5.39}$$

$$\Rightarrow k_\mathrm{T} = -\frac{\rho \cdot D}{\ln \frac{r_\mathrm{i}}{r_\mathrm{o}}} \cdot \frac{1}{r_\mathrm{i}} = \frac{\rho \cdot D}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} \cdot \frac{1}{r_\mathrm{i}} \tag{5.40}$$

The four equations above can be employed to describe the mass flux, are used to eliminate the unknown mass fractions ξ_1, ξ_2, and ξ_3 and to determine the mass flux in dependence of the known mass fractions ξ_i and ξ_o:

$$\dot{m}_{\mathrm{O}_2} = \frac{2 \cdot \pi \cdot L \cdot (\xi_\mathrm{i} - \xi_\mathrm{o})}{\frac{1}{g_\mathrm{i} \cdot r_\mathrm{i}} + \frac{1}{k_\mathrm{T} \cdot r_\mathrm{i}} + \frac{1}{k_\mathrm{P} \cdot r_\mathrm{o}} + \frac{1}{g_\mathrm{o} \cdot r_\mathrm{o}}} = \frac{2 \cdot \pi \cdot L \cdot (\xi_\mathrm{i} - \xi_\mathrm{o})}{\frac{1}{g_\mathrm{i} \cdot r_\mathrm{i}} + \frac{1}{\rho \cdot D} \ln \frac{r_\mathrm{o}}{r_\mathrm{i}} + \frac{1}{k_\mathrm{P} \cdot r_\mathrm{o}} + \frac{1}{g_\mathrm{o} \cdot r_\mathrm{o}}} \tag{5.41}$$

With the given numerical values, one obtains

$$\dot{m}_{\mathrm{O}_2} = 5.206 \cdot 10^{-8}\ \mathrm{kg_{O_2}/s} \tag{5.42}$$

b) The Stefan correction must be taken into account for semi-permeable surfaces. This is especially true for the underlying case: The inner surface of the Perowskity layer is impermeable for the nitrogen contained in the air, whereas the outer surface is impermeable for the flue gas. Therefore,

the Stefan correction needs to be applied both to the transport of oxygen in the wall's Perwoskite layer as well as to the transport of oxygen from the Perowskite layer into the flue gas.
Additionally, the pipe's inner surface de facto also is semi-permeable: As the net nitrogen flux over the boundary is zero for steady-state conditions, the Stefan correction is also valid here. Thus, it holds for the unknown oxygen mass flux:

$$\begin{aligned}
\dot{m}_{O_2}^{St} &= A_\mathrm{i} \cdot g_\mathrm{i} \cdot \frac{\xi_\mathrm{i} - \xi_1^\mathrm{St}}{1 - \xi_1^\mathrm{St}} && \text{(Inner side)} \\
&= A_\mathrm{i} \cdot k_\mathrm{T}^\mathrm{St} \cdot \left(\xi_1^\mathrm{St} - \xi_2^\mathrm{St}\right) && \text{(Pipe wall)} \qquad k_\mathrm{T}^\mathrm{St} = \frac{\rho \cdot D^\mathrm{St}}{\ln \frac{r_\mathrm{o}}{r_\mathrm{i}}} \cdot \frac{1}{r_\mathrm{i}} \\
&= A_\mathrm{o} \cdot g_\mathrm{o} \cdot \frac{\xi_3^\mathrm{St} - \xi_\mathrm{o}}{1 - \xi_3^\mathrm{St}} && \text{(Outer side)}
\end{aligned}$$

For the transport within the pipe wall the diffusion coefficient D^St is to be employed. A correction by means of the Stefan factor $\frac{1}{1-\xi}$ as applied to the inner and outer pipe walls is not possible here as the underlying mass transfer mechanism is not binary, but tertiary (involving three components) diffusion. None of these components can be described by any of ther remaining components to aid a formulation as binary diffusion:

- Oxygen is the diffusing quantity.
- Nitrogen is the quantity which cannot permeate the Perowskite layer; thus necessetating the Stefan correction.
- The substrate/carrier material does not diffuse and is therefore not subjected to the Stefan correction.

Instead of a complete elimination of the quantities ξ_1^St and ξ_3^St from the set of equations, a partial elimination is more practical: In the preceeding calculation g_i, k_T and g_o are replaced by $\frac{g_\mathrm{o}}{1-\xi_1^\mathrm{St}}$, k_T^St and $\frac{g_\mathrm{o}}{1-\xi_3^\mathrm{St}}$. ξ_2^St is eliminated just as it was previously. Analoguously one obtains:

$$\dot{m}_{\mathrm{O}_2}^\mathrm{St} = \frac{2 \cdot \pi \cdot L \cdot (\xi_\mathrm{i} - \xi_\mathrm{o})}{\frac{1-\xi_1^\mathrm{St}}{g_\mathrm{i} \cdot r_\mathrm{i}} + \frac{1}{\rho \cdot D^\mathrm{St}} \ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right) + \frac{1}{k_\mathrm{p} \cdot r_\mathrm{o}} + \frac{1-\xi_3^\mathrm{St}}{g_\mathrm{o} \cdot r_\mathrm{o}}} \tag{5.43}$$

With this approach the sought-after mass flux can be determined iteratively. THe calculation of the mass fractions ξ_1^St and ξ_3^St is done with the equations

$$\xi_1^\mathrm{St} = \frac{\xi_\mathrm{i} - \frac{\dot{m}_{\mathrm{O}_2}^\mathrm{St}}{g_\mathrm{i} \cdot A_\mathrm{i}}}{1 - \frac{\dot{m}_{\mathrm{O}_2}^\mathrm{St}}{g_\mathrm{i} \cdot A_\mathrm{i}}} \quad \text{und} \quad \xi_3^\mathrm{St} = \xi_2^\mathrm{St} - \frac{\dot{m}_{\mathrm{O}_2}^\mathrm{St}}{A_\mathrm{i} \cdot k_\mathrm{p}} \cdot \frac{r_\mathrm{i}}{r_\mathrm{o}}$$

$$\text{mit} \quad \xi_2^\mathrm{St} = \xi_1^\mathrm{St} - \frac{\dot{m}_{\mathrm{O}_2}^\mathrm{St}}{k_\mathrm{T}^\mathrm{St} \cdot A_\mathrm{i}} \tag{5.44}$$

If the mass flux calculated in a) is chosen as the initial value, the iteration runs as follows:

$\dot{m}_{\mathrm{O}_2}^\mathrm{St}$ $[10^{-8}\ \mathrm{kg_{O_2}/s}]$	ξ_1^St $[\mathrm{kg_{O_2}/kg}]$	ξ_2^St $[\mathrm{kg_{O_2}/kg}]$	ξ_3^St $[\mathrm{kg_{O_2}/kg}]$
$\dot{m}_{\mathrm{O}_2} = 5.206$	0.2283	0.1758	0.1206
5.645	0.2280	0.1711	0.1112
5.640	0.2280	0.1711	0.1113
5.640	0.2280	0.1711	0.1113

Due to the relatively high oxygen concentration within the system, the omission of the Stefan correction yields an error of approx. 8 %. If k_p were even smaller in relation to the three remaining, Stefan-corrected transport coefficients, the arising error would be even larger.

5.7. Tarred railway sleepr

1. Problem type

   Steady-state diffusive mass transfer with varying boundary conditions.

2. Solution

   a) The radial differences in concentration within the bore can be neglected. as long as they are significantly smaller than the Naphthalene concentration in the wood. A measure for the ratio is given in the Biot number Bi. For the underlying case it is defined as:

   $$\mathrm{Bi} = \frac{g \cdot \frac{d}{2}}{\rho \cdot D_{\mathrm{Wood}}} \tag{5.45}$$

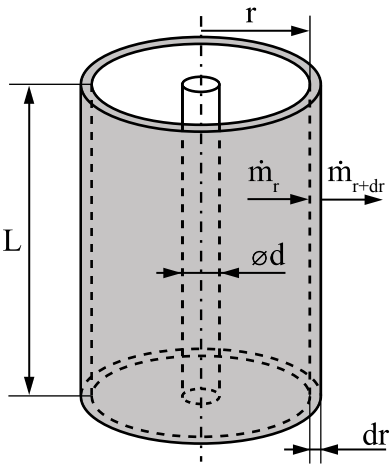
   where g is the effective mass transfer coefficient describing the naphthalene transport within the wood (from R to $\frac{d}{2}$).For the characteristical length a radial measure is to be chosen, the thickness of the plate is an inappropriate choice as the mass transfer to be investigated by means of the Biot number takes place in radial direction. The characteristical length is to be chosen as $\frac{d}{2}$ for the maximum ξ-difference is found in the bore between its centre and its wall. The radial mass transfer within the bore is given by the term $\rho \cdot D_W$.

   When $\rho \cdot D_W$ grows larger, the radial resistance to diffusion in the bore diminishes alongside the radial ξ-difference. The smaller g becomes, the larger the difference between ξ at the locations $\frac{d}{2}$ and R grows. Therefore, the radial difference in the bore can be neglected in comparison to that in the wood for small Biot numbers.

   The quantity g is initially unknown. It is determined as follows:

   $$g \cdot (\xi_W ood - \xi_{\frac{d}{2}}) = +\rho \cdot D_W ood \cdot \left.\frac{\mathrm{d}\xi}{\mathrm{d}r}\right|_{r=\frac{d}{2}} \tag{5.46}$$

   The left and right sides of the balance pose two different formulations for the specific jacket surface mass flux of bore. (To the left the definition of the mass transfer coefficient, to the right Fick's 1. Law.) For the right hand side of the equation the wood-side is regarded to avoid the need to use the water-sided gradient if the mass fraction function $\xi(x,r)$ - whose calculation through the omission of the radial dependence was to be evitated. A positive sign is to be chosen for Fick's Law as the naphthalene flux is oriented in negative coordinate direction and its gradient is thus positive.



The diffusion of the naphthalene within the wood is given by

$$\dot{m}_r - \dot{m}_{r+dr} = 0 \quad \Leftrightarrow \quad \dot{m}_r - \dot{m}_r - \frac{\mathrm{d}\dot{m}_r}{\mathrm{d}r} dr = 0 \tag{5.47}$$

$$\Rightarrow \quad -\frac{\mathrm{d}}{\mathrm{d}r}\left(-\rho \cdot D_W ood \cdot 2\pi r \cdot L \cdot \frac{\mathrm{d}\xi}{\mathrm{d}r}\right) dr = 0 \tag{5.48}$$

$$\Leftrightarrow \quad \rho \cdot D_W ood \cdot 2\pi \cdot L \cdot \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) dr = 0 \tag{5.49}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(r \cdot \frac{\mathrm{d}\xi}{\mathrm{d}r}\right) = 0 \tag{5.50}$$

The axial length is set to be L; equally, it could have been set to Δx or dx. Overall the choice is irrelevant as any diffusion in axial direction is to be omitted wherefore the mass flux is independent of x. This is also expressed by the fact that L is not represented in the fully simplified equation.

Here a negative sign is once again chosen for Fick's law as theTaylor expansion is done in positive coordinate direction. The boundary conditions dictate the balance quantity's (the mass flux's) sign in the final ODE, thus the sign of no importance for the derivation of the ODE.

Thus follows:

$$\Rightarrow \quad \frac{\mathrm{d}\xi}{\mathrm{d}r} = \frac{c_1}{r} \tag{5.51}$$

$$\Rightarrow \quad \xi = c_1 \cdot \ln(r) + c_2 \tag{5.52}$$

Two mass fractions are chosen as boundary conditions - the axial coordinate remains unspecified:

$$\xi\left(r = R\right) = \xi_{\text{Wood}} \tag{5.53}$$

$$\xi\left(r = \frac{d}{2}\right) = \xi_{\frac{d}{2}} \tag{5.54}$$

The first boundary condition is independent of x, the value $\xi_{\frac{d}{2}}$ represents a generic mass fraction at the location $r = \frac{d}{2}$ for any (unspecified) x-value, and is thus always valid.

Inserting the boundary conditions yields

$$c_1 = \frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln\left(\frac{2R}{d}\right)} \qquad c_2 = \xi_{\text{Wood}} - \ln\left(R\right) \cdot \frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln\left(\frac{2R}{d}\right)} \tag{5.55}$$

Thus follows for g:

$$g = \frac{\rho \cdot D_{\text{Wood}} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}r}\Big|_{r=\frac{d}{2}}}{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}} = \frac{\rho \cdot D_{\text{Wood}} \cdot \frac{2}{d} \cdot c_1}{\xi_{\text{Wood}} - c_1 \ln\left(r = \frac{d}{2}\right) + c_2} \tag{5.56}$$
$$= \frac{\rho \cdot D_{\text{Wood}} \cdot \frac{2}{d} \cdot \frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln\left(\frac{2R}{d}\right)}}{\xi_{\text{Wood}} - \left(\frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln\left(\frac{2R}{d}\right)} \ln\left(\frac{d}{2}\right) + \xi_{\text{Wood}} - \ln R \cdot \frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln \frac{2R}{d}}\right)}$$
$$= \frac{\rho \cdot D_{\text{Wood}} \cdot \frac{2}{d} \cdot \frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln\left(\frac{2R}{d}\right)}}{-\frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln\left(\frac{2R}{d}\right)} \ln\left(\frac{d}{2}\right) + \ln(R) \cdot \frac{\xi_{\text{Wood}} - \xi_{\frac{d}{2}}}{\ln\left(\frac{2R}{d}\right)}}$$
$$= \frac{\rho \cdot D_{\text{Wood}} \cdot \frac{2}{d}}{-\ln\left(\frac{d}{2}\right) + \ln R}$$
$$= \frac{\rho \cdot D_{\text{Wood}}}{\ln\left(\frac{2R}{d}\right)} \cdot \frac{2}{d} = \frac{1000\,{}^{\text{kg}}/_{\text{m}^3} \cdot 3 \cdot 10^{-10}\,{}^{\text{m}^2}/_{\text{s}}}{\ln \frac{2 \cdot 0{,}08\,\text{m}}{0{,}01\,\text{m}}} \cdot \frac{2}{0{,}01\,\text{m}} = 2{,}164 \cdot 10^{-5}\,{}^{\text{kg}}/_{\text{m}^2\text{s}}$$

This result is independent of all and any ξ-values, i.e. not determining $\xi_{\frac{d}{2}}$ was justified.

The Biot number is

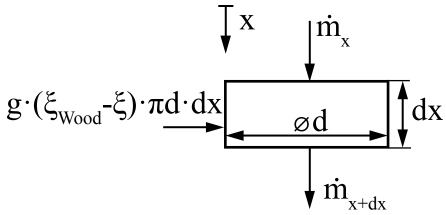
$$\text{Bi} = \frac{g \cdot \frac{d}{2}}{\rho \cdot D_{\text{Wood}}} = \frac{2.164 \cdot 10^{-5}\,{}^{\text{kg}}/_{\text{m}^2\text{s}} \cdot \frac{0.01\,\text{m}}{2}}{1000\,{}^{\text{kg}}/_{\text{m}^3} \cdot 7 \cdot 10^{-6}\,{}^{\text{m}^2}/_{\text{s}}} = 1.546 \cdot 10^{-5} \tag{5.57}$$

Therefore, the radial difference in concentration is indeed negligible.

b) The total mass flux comprises the fluxes both at the top and bottom of the bore. These can be determined through

$$\dot{m}_{\text{o}} = +\rho \cdot D_{\text{Wood}} \cdot \frac{\pi}{4} d^2 \cdot \frac{\mathrm{d}\xi}{\mathrm{d}x}\Big|_{x=0} \tag{5.58}$$
$$\dot{m}_{\text{o}} = k \cdot \frac{\pi}{4} d^2 \cdot \left(\xi\,(x = L) - \xi_{\text{Wood}}\right) \tag{5.59}$$

The sign of Fick's law is once again positive, as has been expounded above.



The knowledge of the naphthalene mass fraction within the bore is required. The function can be determined through a differential balance:

$$0 = \dot{m}_x - \dot{m}_{x+dx} + g \cdot (\xi_{\text{Wood}} - \xi) \cdot \pi d \cdot dx \tag{5.60}$$

$$\Leftrightarrow 0 = \rho \cdot D_{\text{Wood}} \cdot \frac{\pi}{4} d^2 \cdot \frac{\mathrm{d}^2 \xi}{\mathrm{d}x^2} + g \cdot (\xi_{\text{Wood}} - \xi) \cdot \pi d \tag{5.61}$$

With

$$\xi^* = \xi - \xi_{\text{Wood}} \tag{5.62}$$

one obtains

$$\Rightarrow \quad \frac{\mathrm{d}^2 \xi^*}{\mathrm{d}x^2} - \underbrace{\frac{4 \cdot g}{\rho \cdot D_{\text{Wood}} \cdot d}}_{m^2} \cdot \xi^* = 0 \tag{5.63}$$

This differential equation can be solved with the following approach

$$\xi^* = c_3 \cdot \sinh(m \cdot x) + c_4 \cdot \cosh(m \cdot x) \tag{5.64}$$

The valid boundary conditions are

$$x = 0: \qquad \xi^* = \xi_{\text{Wood}} - \xi_{\text{Wood}} = \xi^*_{\text{Wood}} \tag{5.65}$$

$$x = L: \quad k \cdot (\xi\,(x = L) - \xi_{\text{Wood}}) = -\rho \cdot D_{\text{Wood}} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}x}\Big|_{x=L} \tag{5.66}$$

$$\Rightarrow \quad k \cdot (\xi^*\,(x = L) + \xi_{\text{Wood}} - \xi_{\text{Wood}}) = -\rho \cdot D_{\text{Wood}} \cdot \frac{\mathrm{d}\xi^*}{\mathrm{d}x}\Big|_{x=L} \tag{5.67}$$

Inserting yields

$$\xi^*_{\text{Wood}} = c_3 \cdot \sinh(0) + c_4 \cdot \cosh(0) \quad \Rightarrow \quad c_4 = \xi^*_{\text{Wood}} \tag{5.68}$$

and with

$$\frac{\mathrm{d}\xi^*}{\mathrm{d}x} = c_3 \cdot m \cdot \cosh(m \cdot x) + c_4 \cdot m \cdot \sinh(m \cdot x), \tag{5.69}$$

$$\begin{aligned}
&k \cdot ([c_3 \cdot \sinh(m \cdot L) + c_4 \cdot \cosh(m \cdot L)] + \xi_{\text{Wood}} - \xi_{\text{Wood}}) \\
&\quad = -\rho \cdot D_{\text{Wood}} \cdot (c_3 \cdot m \cdot \cosh(m \cdot L) + c_4 \cdot m \cdot \sinh(m \cdot L)) \\
&\Leftrightarrow k \cdot (c_3 \cdot \sinh(m \cdot L) + \xi^*_{\text{Wood}} \cdot \cosh(m \cdot L) - \xi^*_{\text{Wood}}) \\
&\quad = -\rho \cdot D_{\text{Wood}} \cdot (c_3 \cdot m \cdot \cosh(m \cdot L) + \xi^*_{\text{Wood}} \cdot m \cdot \sinh(m \cdot L)) \\
&\Rightarrow c_3 = -\xi^*_{\text{Wood}} \cdot \frac{\sinh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m}[\cosh(m \cdot L) - 1]}{\cosh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m} \sinh(m \cdot L)}
\end{aligned} \tag{5.70}$$

is obtained.

Thus holds

$$\begin{aligned}
\xi^* = \xi^*_{\text{Wood}} \cdot \Bigg[&\cosh(m \cdot x) - \dots \\
&\dots - \frac{\sinh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m}[\cosh(m \cdot L) - 1]}{\cosh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m} \sinh(m \cdot L)} \cdot \sinh(m \cdot x) \Bigg]
\end{aligned} \tag{5.71}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\xi^*}{\mathrm{d}x} = \xi^*_{\text{Wood}} \cdot m \cdot \Bigg[&\sinh(m \cdot x) - \dots \\
&\dots - \frac{\sinh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m}[\cosh(m \cdot L) - 1]}{\cosh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m} \sinh(m \cdot L)} \cdot \cosh(m \cdot x) \Bigg]
\end{aligned} \tag{5.72}$$

and therefore

$$\dot{m}_o = -\rho \cdot D_{\text{Wood}} \cdot \frac{\pi}{4} d^2 \cdot \xi^*_{\text{Wood}} \cdot m \cdot \ldots$$
$$\ldots \cdot \frac{\sinh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m}\left[\cosh(m \cdot L) - 1\right]}{\cosh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m}\sinh(m \cdot L)} \tag{5.73}$$
$$\dot{m}_o = k \cdot \frac{\pi}{4} d^2 \cdot (\xi(x = L) - \xi_{\text{Wood}}) \tag{5.74}$$
$$= k \cdot \frac{\pi}{4} d^2 \cdot (\xi^*(x = L) + \xi_{\text{Wood}} - \xi_{\text{Wood}}) \tag{5.75}$$
$$= k \cdot \frac{\pi}{4} d^2 \cdot (\xi^*(x = L) - \xi^*_{\text{Wood}}) \tag{5.76}$$
$$= k \cdot \frac{\pi}{4} d^2 \cdot \xi^*_{\text{Wood}} \cdot (\cosh(m \cdot L) - 1) - k \cdot \frac{\pi}{4} d^2 \cdot \xi^*_{\text{Wood}} \cdot \ldots$$
$$\ldots \cdot \frac{\sinh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m}\left[\cosh(m \cdot L) - 1\right]}{\cosh(m \cdot L) + \frac{k}{\rho \cdot D_{\text{Wood}} \cdot m}\sinh(m \cdot L)} \sinh(m \cdot L) \tag{5.77}$$

With

$$\xi^*_{\text{Wood}} = \xi_{\text{Wood}} - \xi_{\text{Wood}} = -10^{-4}\, \text{kg}_\text{N}/\text{kg} \tag{5.78}$$
$$m = \sqrt{\frac{4 \cdot g}{\rho \cdot D_{\text{Wood}} \cdot d}}$$
$$= \sqrt{\frac{4 \cdot 2.164 \cdot 10^{-5}\, \text{kg}/\text{m}^2\text{s}}{1000\, \text{kg}/\text{m}^3 \cdot 7 \cdot 10^{-6}\, \text{m}^2/\text{s} \cdot 0.01\, \text{m}}}$$
$$= 1.112\, 1/\text{m} \tag{5.79}$$

$$\dot{m}_o = 1.2563 \cdot 10^{-11}\, \text{kg}_\text{N}/\text{s} \tag{5.80}$$
$$\dot{m}_o = 8.3433 \cdot 10^{-13}\, \text{kg}_\text{N}/\text{s} \tag{5.81}$$
$$\dot{m} = \dot{m}_o + \dot{m}_o$$
$$= 1.3397 \cdot 10^{-11}\, \text{kg}_\text{N}/\text{s} \tag{5.82}$$

is obtained.

c) To determine the mass flux for an impermeable underside, $\dot{m}_o = 0$ and $k = 0$ in the equation for $\dot{m}_o$ are set:

$$\begin{aligned} \dot{m}_o^* &= -\rho \cdot D_{\text{Wood}} \cdot \frac{\pi}{4} d^2 \cdot \xi_{\text{Wood}}^* \cdot m \cdot \frac{\sinh(m \cdot L)}{\cosh(m \cdot L)} \\ &= -\rho \cdot D_{\text{Wood}} \cdot \frac{\pi}{4} d^2 \cdot \xi_{\text{Wood}}^* \cdot m \cdot \tanh(m \cdot L) \end{aligned} \tag{5.83}$$

The similarity of the derived formula to rquation Eq. 3.33a in the lecture script stands out which was derived for the condition of an adiabatic find tip, as analoguously holds true here ($k = 0$). The negative sign results from the definition of ξ_{Wood}^* and the fact that the naphthalene mass flux is directed to the "‘fin's base"’ (the upper orifice) and not, as is a heat flux in a cooling fin, in the opposing direction.

The mass flux is equal to

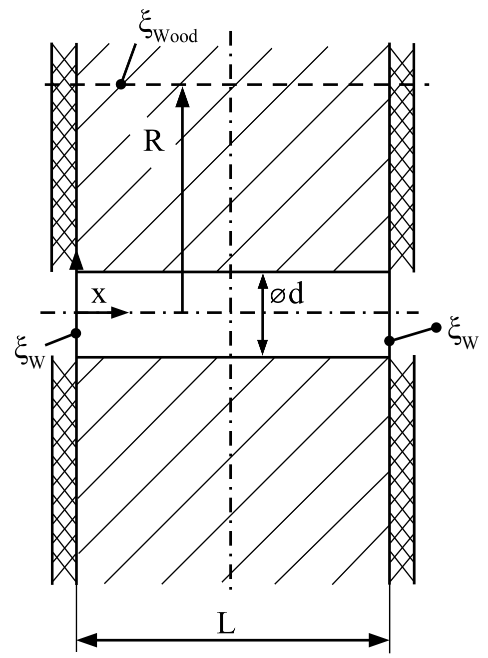
$$\dot{m}_o^* = 1.3377 \cdot 10^{-11}. \tag{5.84}$$

This result barely deviates from the result obtained in a). The reason for this is that the reduction of the overall mass flux, yielded by the omission of the mass flux at the underside, is nearly compensated **for the given values**. The naphthalene „accumulates“at the now impermeable lower surface of the bore, whereby the gradient at the surface becomes ≥ 0, and the overall concentration rises in turn causing the gradient at the upper surface (i.e. the mass flux) to increase (see subtask e)).

The existence of both opposing effects poses another argument in favour of the assumption of a fin with an adiabatic fin tip; at least with respect to the calculation of the mass and cooling heat fluxes, respectively.

d) For the calculation of this particular case the differential equation derived in b) can be solved implementing a modified second boundary condition ($\xi(x = L) = \xi_{\text{Wood}}$).

A more practical approach, however, is to utilised the underlying geometry:

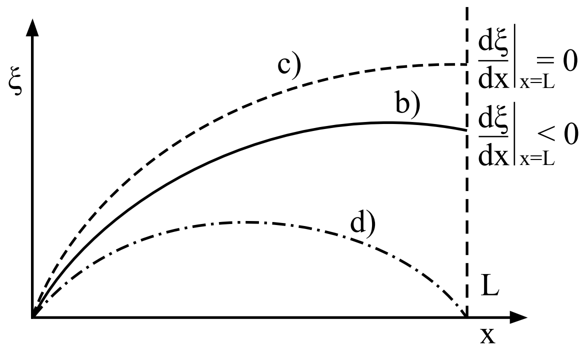


As the same boundary condition is now valid on the left- and right-hand side and all relevant parameters (ρ, D_{Wood}, g, d) are independent of x, the mass fraction profile of ξ has to be symmetrical around $x = \frac{L}{2}$. Due to the symmetry a zero-gradient is encountered at $x = \frac{L}{2}$. Both symmetrical halves of the bore resemble to fins of length $\frac{L}{2}$ with an adiabatic boundary condition at the fin's tip and over whose bases ($x = 0$ and $x = L$, respectively,) half of the total mass flux is dissipated.

$$\dot{m}_{\text{o}}^{**} = -2 \cdot \rho \cdot D_{\text{Wood}} \cdot \frac{\pi}{4} d^2 \cdot \xi_{\text{Wood}}^{*} \cdot m \cdot \tanh\left(m \cdot \frac{L}{2}\right) = 1.3541 \cdot 10^{-11}\ {}^{\text{kg}_\text{N}}\!/\!_\text{s} \tag{5.85}$$

Here too, two opposing effects underlie, again (**in this case**) yielding a nealy constant overall mass flux: The reduction of the fin length and the fact that, de facto, two fin's are available for the mass transfer.

e)



5.8. Scrubber

1. Problem type

   Unsteady convective and diffusive mass transfer in spherical and plane geometries.

2. Solution

   a) The terminal velocity can be determined through a force balance around the droplet. The flow resistance is governed by the unknown flow velocity relative to the gaseous phase:

$$0 \overset{!}{=} m_{\mathrm{T}} \cdot a = F_G - F_W = m_{\mathrm{T,eff}} \cdot g_{\mathrm{E}} - c_W \cdot A \cdot \frac{1}{2} \cdot \rho_{\mathrm{G}} \cdot \nu_{\mathrm{T}}^2 \tag{5.86}$$

$$\Leftrightarrow c_W \cdot \frac{\pi}{4} \cdot d^2 \cdot \frac{1}{2} \cdot \rho_{\mathrm{G}} \cdot \nu_{\mathrm{T}}^2 = (\rho_{\mathrm{T}} - \rho_{\mathrm{G}}) \cdot \frac{\pi}{6} \cdot d^3 \cdot g_{\mathrm{E}} \tag{5.87}$$

$$\Rightarrow \nu_{\mathrm{T}} = \sqrt{\frac{4}{3} \cdot \frac{d \cdot g_{\mathrm{E}}}{c_W} \cdot \frac{\rho_{\mathrm{T}} - \rho_{\mathrm{G}}}{\rho_{\mathrm{G}}}} \tag{5.88}$$

   Thus one obtains:

$$\nu_{\mathrm{T}} = \sqrt{\frac{4}{3} \cdot \frac{10^{-3}\,\mathrm{m} \cdot 9.81\,\mathrm{m/s^2}}{0.4} \cdot \frac{1020\,\mathrm{kg/m^3} - 1.86\,\mathrm{kg/m^3}}{1.86\,\mathrm{kg/m^3}}} = 424\,\mathrm{m/s} \tag{5.89}$$

   b) The transition of CO_2 from the gaseous to the liquid phase of the droplet is governed both by convective effects from without as well as diffusive effects within the droplet. The underlying problem is of transient and one-dimensional nature described in spherical coordinates, for which no analytical solution is presented in the lecture script. Therefore, a *Heisler* diagramme is utilised. In this case figure 3.35 („Heat loss of a sphere"). This diagramme can also be utilised to model the CO_2-transport for the droplet, as the analogy between heat and mass transfer holds true in this case, too. The change in sign (intake instead of loss) is irrelevant in this case, as the diagramme only indicates the relative change in a balancing process; as is the case. The sign is determined through the difference in the potentials ξ and T at two locations each.

How much CO_2 is transfered until the sump is reached, depends on the duration of the fall as well as on the quality of the convective and diffusive mass transfer processes. This is indicated by the Fourier or Biot number, respectively:

$$\mathrm{Fo} = \frac{D_\mathrm{T} \cdot t}{\left(\frac{d}{2}\right)^2} \qquad \mathrm{Bi} = \frac{g \cdot \frac{d}{2}}{\rho_\mathrm{T} \cdot D_\mathrm{T}} \cdot \frac{\rho_\mathrm{T}}{\rho_\mathrm{G}} = \frac{g \cdot \frac{d}{2}}{\rho_\mathrm{G} \cdot D_\mathrm{T}} \; {}^{2} \tag{5.90}$$

While the Fourier number takes into account the fall's duration and thus the temporal evolution of the CO_2 mass, the Biot number provides a met-

[2]The common notation of the Biot number needs to be amended by the factor $\frac{\rho_\mathrm{T}}{\rho_\mathrm{G}}$ in this case. The explanation for this is that Bi without this correction factor would not be dimensionless but would have the unit $\frac{kg_G}{kg_T}$ instead. This inturn would imply that the similarity theory would not be applicable.
The necessity of the correction becomes apparent if one poses, analoguously to the derivation of the Biot number in the context of heat transfer in chapter 3.5.3, the gas- and droplet-sided CO_2 balance:

$$g \cdot (\xi_{\mathrm{CO_2,G,W}} - \xi_\infty) = -\rho_\mathrm{T} \cdot D_\mathrm{T} \frac{\mathrm{d}\xi_{\mathrm{CO_2,T}}}{\mathrm{d}r}\Big|_W$$

As the mass of the CO_2 in the gaseous phase is related to the gas and to that of the droplet in the liquid phase, it holds for the assumtpion of a continuous concentration profile at the phase boundary/interface ($h = 1$):

$$\xi_{\mathrm{CO_2,G}} \cdot \rho_\mathrm{G} = \xi_{\mathrm{CO_2,T}} \cdot \rho_\mathrm{T}$$

and therefore

$$\begin{aligned} g \cdot (\xi_{\mathrm{CO_2,G,W}} - \xi_\infty) &= -\rho_\mathrm{T} \cdot D_\mathrm{T} \cdot \frac{\mathrm{d}\left(\frac{\rho_\mathrm{G}}{\rho_\mathrm{T}} \cdot \xi_{\mathrm{CO_2,G,W}}\right)}{\mathrm{d}r} \\ &= -\rho_\mathrm{T} \cdot D_\mathrm{T} \cdot \frac{\mathrm{d}\left(\frac{\rho_\mathrm{G}}{\rho_\mathrm{T}} \cdot \xi_{\mathrm{CO_2,G,W}} - \overbrace{\frac{\rho_\mathrm{G}}{\rho_\mathrm{T}} \cdot \xi_\infty}^{\text{konst.}}\right)}{\mathrm{d}r} \end{aligned}$$

With $r \equiv \frac{2 \cdot r}{d}$ and $\xi^*_{\mathrm{Wood}} \equiv \frac{\xi_{\mathrm{CO_2,G,W}} - \xi_\infty}{\Delta\xi}$ one obtains:

$$\frac{\mathrm{d}\xi^*_{\mathrm{Wood}}}{\mathrm{d}r^*} = -\underbrace{\frac{g \cdot \frac{d}{2}}{\rho_\mathrm{T} \cdot D_\mathrm{T}} \cdot \frac{\rho_\mathrm{T}}{\rho_\mathrm{G}}}_{\mathrm{Bi}} \cdot \xi^*_{\mathrm{Wood}}$$

ric for the strength of convective and diffusive mass transport **relative to each other**. The **absolute** quality of the mass transport is as yet undefined. This in turn is achieved by the Fourier number, i.e. the parameter D_T included in its definition.

The characteristic length in the Fourier number is, in accordance with the definition of the parameters Bi and $\mathrm{Bi}^2 \cdot \mathrm{Fo}$ in the diagramme, the droplet's radius and not its diameters. The reasoning behind this is that the radius is equal to the maximum penetration depth of CO_2. The initial calculation of Fo necessitates the droplet's fall duration; the relevant velocity is determined in the absolute reference system.

$$t = \frac{H}{\nu_\mathrm{T} - \nu_\infty} = \frac{0.2\,\mathrm{m}}{(4.24 - 0.05)\,^\mathrm{m}/_\mathrm{s}} = 0.0477\,\mathrm{s} \tag{5.91}$$

$$\Rightarrow \quad \mathrm{Fo} = \frac{10^{-5}\,^{\mathrm{m}^2}/_\mathrm{s} \cdot 0.0477\,\mathrm{s}}{\left(\frac{10^{-3}}{2}\mathrm{m}\right)^2} = 1.909 \tag{5.92}$$

The calculation of the Biot number necessitates knowledge of the mass transfer coefficient g. It results from a relation for the Sherwood number Sh with regard to the given geometry and flow conditions:

$$\begin{aligned}\mathrm{Sh} &= \frac{g \cdot d}{\rho_\mathrm{G} \cdot D_\mathrm{G}} \\ &= 2 + \left(0.4 \cdot \mathrm{Re}_\mathrm{d}^{\frac{1}{2}} + 0.06 \cdot \mathrm{Re}_\mathrm{d}^{\frac{2}{3}}\right) \mathrm{Sc}^{0.4} \cdot \left(\frac{\eta_\infty}{\eta_\mathrm{Wood}}\right)^{\frac{1}{4}} \qquad (\text{WÜK.11})^3\end{aligned} \tag{5.93}$$

[3] In general the influence of the CO_2 mass fraction's gradient within the droplet boundary onto the dimensionless quantities Re, Sc and Sh needs to be taken into account employing the arithmetic mean values across the the boundary layer.

$$\rho = \frac{\rho_\infty + \rho_\mathrm{Wood}}{2} \qquad \rho_x = \frac{p_{\mathrm{tot.},x} - p_{\mathrm{CO_2},x}}{R_\mathrm{Eth} T} + \frac{p_{\mathrm{CO_2},x}}{T_{\mathrm{CO_2}} T} \qquad x = \infty, W$$

The influence manifests in ρ_G, which is also included in the definitions of Re and Sc ($\nu = \frac{\eta}{\rho}$). In the case presented constant values for ρ_G and ν are stipulated. Here th influence of the gas composition can neglected for two reasons:

As the problem at hand is iso-thermal and the influence of the change in CO_2 concentration within the boundary layer on the viscosity is negligible, it holds $\eta_\infty = \eta_{\mathrm{Wood}}$. Re and Sc are defined as

$$\mathrm{Re_d} = \frac{\nu_{\mathrm{T}} \cdot d}{\nu} = \frac{4.24\,\mathrm{m/s} \cdot 10^{-3}\,\mathrm{m}}{4.63 \cdot 10^{-6}\,\mathrm{m^2/s}} = 915.8 \tag{5.94}$$

$$\mathrm{Sc} = \frac{\nu}{D_{\mathrm{G}}} = \frac{4.63 \cdot 10^{-6}\,\mathrm{m^2/s}}{5.4 \cdot 10^{-6}\,\mathrm{m^2/s}} = 0.857 \tag{5.95}$$

$\mathrm{Re_d}$ as well as Sc meet the conditions for the applicability of HTL.11. Thus holds:

$$\mathrm{Sh} = 2 + \left(0.4 \cdot 915.8^{\frac{1}{2}} + 0.06 \cdot 915.8^{\frac{2}{3}}\right) 0.857^{0.4} = 18.70 \tag{5.96}$$

$$\begin{aligned} g &= \frac{\mathrm{Sh} \cdot \rho_{\mathrm{G}} \cdot D_{\mathrm{G}}}{d} \\ &= \frac{18.70 \cdot 1.86\,\mathrm{kg_G/m^3} \cdot 5.4 \cdot 10^{-6}\,\mathrm{m^2/s}}{10^{-3}\,\mathrm{m}} = 0.188\,\mathrm{kg_G/m^2s} \end{aligned} \tag{5.97}$$

$$\mathrm{Bi} = \frac{g \cdot \frac{d}{2}}{\rho_{\mathrm{G}} \cdot D_{\mathrm{T}}} = \frac{0.188\,\mathrm{kg_G/m^2s} \cdot 5 \cdot 10^{-4}\,\mathrm{m}}{1.86\,\mathrm{kg_G/m^3} \cdot 10^{-5}\,\mathrm{m^2/s}} = 5.05 \tag{5.98}$$

1. The maximum CO_2 mass fraction in the system is only 5 %. Therefore, the maximum $\Delta\xi_{CO_2}$ between states ∞ and W can also only reach a value of 0.05. Because of

$$p_{CO_2,x} = \frac{1}{1 + \frac{1-\xi_{CO_2,x}}{\xi_{CO_2,x}} \frac{T_{\mathrm{Eth}}}{T_{CO_2}}} p_{\mathrm{tot.},x}$$

only small changes in the partial pressures $p_{\mathrm{tot.,x}} - p_{\mathrm{CO_2,x}}$ and $p_{\mathrm{CO_2,x}}$ relative to the overall pressure $p_{\mathrm{tot.,x}}$ may occur.

2. The molecular weight of ethanol ($46\,\mathrm{g/mol}$) and CO_2 ($44\,\mathrm{g/mol}$) differ only very little. Thus, R_{Eth} and R_{CO_2}, too, differ very little. Therefore, the gas composition's influence on the overall density is also limited:

$$\lim_{R_{CO_2} \to R_{\mathrm{Eth}}} \frac{\rho_\infty + \rho_{\mathrm{Wood}}}{2} = \frac{p_{\mathrm{ges},\infty}}{R_{\mathrm{Eth}} \cdot T} = \frac{p_{\mathrm{ges,W}}}{R_{\mathrm{Eth}} \cdot T} = \rho \neq \rho(\xi_{CO_2})$$

The Biot number is approx. 5. Thus, the change of ξ is negligible neiter within the gaseous phase nor within the droplet. For the parameter on the abscissa in figure 3.35, it holds:

$$\mathrm{Bi}^2 \cdot \mathrm{Fo} = 5.05^2 \cdot 1.909 = 48.7 \tag{5.99}$$

For the given combination of Bi and Fo $\frac{Q}{Q_0} = 1$ is obtained from figure 3.35. This means that the total amount of heat which **could** theoretically be transferred in the given time frame, actually **is** transferred in good approximation. Applied to mass transfer this implies that the overall mass which **could** theoretically be transferred in the given time frame, approximately **is** transferred ($\frac{m}{m_0} = 1$). The definition of m_0 remains to be given.

In the fringe case of an infinitely long fall duration and under the asumption of a continuous concentration profile along the phase boundary the balancing process describing the mass transfer would reach a state characterised by the retention of identical amounts of CO_2 both within the droplet and the gaseous atmosphere per volume unit (due to $h = 1$). This does not imply the identity of the mass fraction within the droplet and gaseous atmosphere as the mass of CO_2 is related to differing total masses (that of the gas and that of the liquid).

The Second Law of Thermodynamics postulates that the concentrations, i.e. the **molecules per unit volume**, needs to be identical everywhere in a state of equilibrium for the assumptions made earlier. Thus holds

$$c_{\mathrm{CO_2,G}}\Big|_{t\to\infty} = c_{\mathrm{CO_2,T}}\Big|_{t\to\infty} \qquad \left[\frac{\mathrm{mol_{CO_2}}}{\mathrm{m^3}}\right] \tag{5.100}$$

$$\Rightarrow M_{\mathrm{CO_2}} \cdot c_{\mathrm{CO_2,G}}\Big|_{t\to\infty} = M_{\mathrm{CO_2}} \cdot c_{\mathrm{CO_2,T}}\Big|_{t\to\infty} \qquad \left[\frac{\mathrm{kg_{CO_2}}}{\mathrm{m^3}}\right] \tag{5.101}$$

$$\Leftrightarrow \rho_{\mathrm{CO_2,G}}\Big|_{t\to\infty} = \rho_{\mathrm{CO_2,T}}\Big|_{t\to\infty} \tag{5.102}$$

$$\Leftrightarrow \rho_{\mathrm{G}} \cdot \xi_{\mathrm{CO_2,G}}\Big|_{t\to\infty} = \rho_T \cdot \xi_{\mathrm{CO_2,T}}\Big|_{t\to\infty} \tag{5.103}$$

$$\Leftrightarrow \xi_{\mathrm{CO_2,T}}\Big|_{t\to\infty} = \frac{\rho_{\mathrm{G}}}{\rho_{\mathrm{T}}} \cdot \xi_{\mathrm{CO_2,G}}\Big|_{t\to\infty} = \frac{\rho_G}{\rho_{\mathrm{T}}} \cdot \xi_\infty \tag{5.104}$$

The mass of CO_2 contained within the drop afer an infinitely long period of time is thus defined as

$$\begin{aligned} m_{\mathrm{CO_2,T}}\Big|_{t\to\infty} &= \xi_{\mathrm{CO_2,T}}\Big|_{t\to\infty} \cdot m_{\mathrm{T}} \\ &= \frac{\rho_{\mathrm{G}}}{\rho_{\mathrm{T}}} \cdot \xi_\infty \cdot \frac{\pi}{6} \cdot d^3 \cdot \rho_{\mathrm{T}} = \rho_{\mathrm{G}} \cdot \xi_\infty \cdot \frac{\pi}{6} \cdot d^3. \end{aligned}$$

Because of

$$m_{\mathrm{CO_2,T}}\Big|_{t\to\infty} = 0$$

it holds:

$$\begin{aligned} m_0 &= m_{\mathrm{CO_2,T}}\Big|_{t\to\infty} - m_{\mathrm{CO_2,T}}\Big|_{t=0} = \rho_{\mathrm{G}} \cdot \xi_\infty \cdot \frac{\pi}{6} d^3 \\ m = m_0 &= 1.86\,{}^{\mathrm{kg_G}}\!/_{\mathrm{m^3}} \cdot 0.05\,{}^{\mathrm{kg_{CO_2}}}\!/_{\mathrm{kg_G}} \cdot \frac{\pi}{6} \cdot (10^{-3}\,\mathrm{m})^3 \\ &= 4.87 \cdot 10^{-11}\,\mathrm{kg_{CO_2}} \end{aligned}$$

c) As the sump is very large, the solvent can be regarded as a semi-infinite body. The CO_2 concentration at its surface is unknown and changes over time. Therefore, the resistance to mass transfer within the gaseous phase may not be omitted. According to Schneider equation 3.66 holds:

$$\frac{\xi - \xi_0}{\xi_{\infty,\mathrm{S}} - \xi_0} = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4 \cdot \mathrm{Fo}}}\right) - \exp\left(\mathrm{Bi} + \mathrm{Bi}^2 \cdot \mathrm{Fo}\right) \cdot \ldots \\ \ldots \cdot \left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{4 \cdot \mathrm{Fo}}} + \sqrt{\mathrm{Fo}}\,\mathrm{Bi}\right)\right] \tag{5.105}$$

The term on the left hand side of equation 5.105 describes the difference in mass fractions **within the sump**, relative to the maximum diffference in mass fractions $\xi_{\infty,\mathrm{S}} - \xi_0$ possible **within the sump**. The value $\xi_{\infty,\mathrm{S}}$ is not equal to ξ_∞ in the free-stream, which would be reached at the inner side of the phase boundary (i.e. within the sump), if ξ_∞ were encountered on the opposite side of the interface (for $g_0 \to \infty$). Thus, it reads $\xi_{\infty,\mathrm{S}} = \frac{\rho_\mathrm{G}}{\rho_\mathrm{T}} \cdot \xi_\infty$.

$$\mathrm{Fo} = \frac{D_\mathrm{T} \cdot t}{x^2} = \frac{10^{-5}\,\mathrm{m^2/s} \cdot 10\,\mathrm{s}}{(0.005\,\mathrm{m})^2} = 4 \tag{5.106}$$

$$\begin{aligned} \mathrm{Bi} &= \frac{g_0 \cdot x}{\rho_\mathrm{T} \cdot D_\mathrm{T}} \cdot \frac{\rho_\mathrm{T}}{\rho_\mathrm{G}} \\ &= \frac{g_0 \cdot x}{\rho_\mathrm{G} \cdot D_\mathrm{T}} = \frac{0.0032\,\mathrm{kg_G/m^2s} \cdot 0.005\,\mathrm{m}]}{1.86\,\mathrm{kg_G/m^3} \cdot 10^{-5}\,\mathrm{m^2/s}} = 0.861 \end{aligned} \tag{5.107}$$

$$\Rightarrow \frac{1}{\sqrt{4 \cdot \mathrm{Fo}}} = 0.25 \Rightarrow \mathrm{erf}\left(\frac{1}{\sqrt{4 \cdot \mathrm{Fo}}}\right) = 0.276 \tag{5.108}$$

$$\mathrm{Bi} + \mathrm{Bi}^2 \cdot \mathrm{Fo} = 3.8256 \tag{5.109}$$

$$\sqrt{\mathrm{Fo}}\,\mathrm{Bi} = 1.72184 \Rightarrow \mathrm{erf}\left(\frac{1}{\sqrt{4 \cdot \mathrm{Fo}}} + \sqrt{\mathrm{Fo}}\,\mathrm{Bi}\right) = 0.9944 \tag{5.110}$$

$$\Rightarrow \frac{\xi_\mathrm{S} - \xi_0}{\frac{\rho_\mathrm{G}}{\rho_\mathrm{T}}\xi_\infty - \xi_0} = 1 - 0.276 - \exp(3.8256) \cdot [1 - 0.9944] = 0.467 \tag{5.111}$$

$$\Leftrightarrow \xi_\mathrm{S} = 4.791 \cdot 10^{-5}\,\mathrm{kg_{CO_2}/kg_T} \tag{5.112}$$

Part III.

Results

Chapter 1.

Results radiative heat transfer

1.1. Solar panel

a) $T_{\mathrm{G}} = 24\,°\mathrm{C}$

b) $\dot{q}''_{\mathrm{Nutz}} = 522\,\mathrm{W/m^2}$

c) $\dot{q}''_{\mathrm{Nutz},0} = 303\,\mathrm{W/m^2}$

1.2. Surface brightness values & energy balances

a) $\dot{Q}_1 = A_1 \cdot \sigma T_1^4$

$\dot{Q}_2 = A_2 \cdot \sigma T_2^4 \cdot \varepsilon_2 + \dot{Q}_1 \Phi_{12} \rho_2$

b) $\dot{Q}_1 = A_1 \sigma T_1^4$

$\dot{Q}_{2\mathrm{Li}} = A_2 \cdot \sigma T_2^4 \cdot \varepsilon_2 + \dot{Q}_1 \Phi_{12\mathrm{Li}} \rho_2 + \dot{Q}_3 \Phi_{32\mathrm{Re}} \tau_2$

$\dot{Q}_{2\mathrm{Re}} = A_2 \cdot \sigma T_2^4 \cdot \varepsilon_2 + \dot{Q}_1 \Phi_{12\mathrm{Li}} \tau_2 + \dot{Q}_3 \Phi_{32\mathrm{Re}} \rho_2$

$\dot{Q}_3 = A_3 \sigma T_3^4$

outer energy balance: $\dot{Q}_1 \Phi_{12\mathrm{Li}} + \dot{Q}_3 \Phi_{32\mathrm{Re}} - \dot{Q}_{2\mathrm{Li}} - \dot{Q}_{2\mathrm{Re}} = 0$

inner energy balance: $\dot{Q}_1 \Phi_{12\mathrm{Li}} \alpha_2 + \dot{Q}_3 \Phi_{32\mathrm{Re}} \alpha_2 - 2 \cdot A_2 \cdot \sigma T_2^4 \varepsilon_2 = 0$

c) $\dot{Q}_{1\mathrm{Re}} = A_1 \sigma T_1^4 \varepsilon_1 + \dot{Q}_2 \Phi_{21\mathrm{Li}} \tau_1$

$\dot{Q}_{1\mathrm{Li}} = A_1 \sigma T_1^4 \varepsilon_1 + \dot{Q}_2 \Phi_{21\mathrm{Li}} \rho_1$

$$\dot{Q}_2 = \frac{\epsilon_2 \sigma T_2^4 A_2 + \rho_2 \dot{Q}_{1Li} \phi_{12}}{1 - \rho \phi_{22}}$$

Halbkugel:

outer energy balance: $\dot{Q}_{1\mathrm{Li}}\Phi_{1\mathrm{Li}2} + \dot{Q}_2\Phi_{22} - \dot{Q}_2 = 0$

inner energy balance: $\dot{Q}_{1\mathrm{Li}}\Phi_{1\mathrm{Li}2}\alpha_2 + \dot{Q}_2\Phi_{22}\alpha_2 - A_2 \cdot \sigma T_2^4 \varepsilon_2 = 0$

Platte:

outer energy balance: $\dot{Q}_2\Phi_{21\mathrm{Li}} - \dot{Q}_{1\mathrm{Li}} - \dot{Q}_{1\mathrm{Re}} = 0$

inner energy balance: $\dot{Q}_2\Phi_{21\mathrm{Li}}\alpha_1 - 2 \cdot A_1 \cdot \sigma T_1^4 \varepsilon = 0$

1.3. Spherical, evacuated lightbulb

a) $T_\mathrm{F} = 3092\,\mathrm{K} \mathrel{\hat{=}} 2819\,°\mathrm{C}$

b) $\dot{Q}_\mathrm{el} = 52\,\mathrm{W}$

c) $\lambda_\mathrm{max} = \frac{2898}{T_\mathrm{F}} = 0.94\,\mathrm{\mu m}$

d) $T_\mathrm{F} = 3122\,\mathrm{K} \mathrel{\hat{=}} 2849\,°\mathrm{C}$

$\dot{Q}_\mathrm{el} = 54\,\mathrm{W}$

1.4. Radiation within a wedge-shaped orifice

a) $\Phi_{12} = \frac{1}{\sqrt{2}}$

$\Phi_{1\ddot{\mathrm{O}}} = 1 - \frac{1}{\sqrt{2}}$

$\Phi_{21} = \Phi_{2\ddot{\mathrm{O}}} = \frac{1}{2}$

b) $\dot{q}'_{1\dot{V}} = \dot{q}'_{\ddot{\mathrm{O}}\dot{V}} = 11\,\mathrm{kW/m}$

c) $T_2 = 840\,\mathrm{K}$

1.5. Lead crucible

a) $\dot{Q}_{\mathrm{K}} = 2.1\,\mathrm{W}$

b) $\dot{Q}_{\mathrm{K},0} = 2.9\,\mathrm{W}$

$\dot{Q}_{\mathrm{K},\infty} = 1.6\,\mathrm{W}$

c) $\left.\begin{array}{l}\dot{Q}_{\mathrm{K},0} = 2.9\,\mathrm{W} \\ t_{\mathrm{E},0} = 0.2\,\mathrm{h}\end{array}\right\}$ für $h = 0$

$\left.\begin{array}{l}\dot{Q}_{\mathrm{K}} = 2.1\,\mathrm{W} \\ t_{\mathrm{E}} = 0.28\,\mathrm{h}\end{array}\right\}$ für $h = 25\,\mathrm{mm}$

$\left.\begin{array}{l}\dot{Q}_{\mathrm{K},\infty} = 1.6\,\mathrm{W} \\ t_{\mathrm{E},\infty} = 0.36\,\mathrm{h}\end{array}\right\}$ für $h \to \infty$

1.6. Hollow black body

$d_{\mathrm{O}} = 4.4\,\mathrm{cm}$

1.7. Furnace with quartz window

a) $\dot{q}''_{\mathrm{F}\epsilon} = 13.3\,\mathrm{kW/m^2}$

b) $\dot{Q}_V = 1.2\,\mathrm{kW}$

c) $T_{\mathrm{F}} = 798\,\mathrm{K}$

1.8. Radiative net heat flux between two plates

Please regard the solutions provided.

1.9. Heated ceiling

a) $T_{\mathrm{D}} = 32\,°\mathrm{C}$

b) $T_{\mathrm{SW}} = 27.4\,°\mathrm{C}$

$T_{\mathrm{D}} = 37.6\,°\mathrm{C}$

c) $\dot{Q}_{\mathrm{H}} = 5.7\,\mathrm{kW}$

d) $\dot{Q}_{\mathrm{L,FB}} = 1162\,\mathrm{W}$

1.10. Cupola

a) $\dot{Q} = 7.4\,\mathrm{kW}$

b) $T_3 = 359\,\mathrm{K}$

Chapter 2.

Results conductive heat transfer

2.1. Transient temperature fields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\lambda\cdot\frac{\partial T}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\cdot\lambda\cdot\frac{\partial T}{\partial\varphi}\right)+\frac{\partial}{\partial z}\left(\lambda\cdot\frac{\partial T}{\partial z}\right)+\dot{\Phi}'''=\rho\cdot c\cdot\frac{\partial T}{\partial t}$$

2.2. Poensgen device

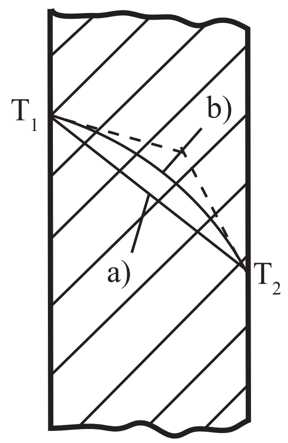
a) $\frac{\lambda_{\mathrm{P}}}{\lambda_{\mathrm{P,corr}}} = 0.81$ for concrete

$= 0.99$ for cork

b)

causes	corrections
temperature measurement	$\Delta T > 10\,\mathrm{K}$ $\delta > 10\,\mathrm{mm}$ $\lambda < 2.5\,{}^{\mathrm{W}}/_{\mathrm{m\,K}}$
multi-dimensional temperature field	good overlap of auxilliary heating $\lambda > 125\,\mathrm{mm}$
inhomogeneities in the probe plate	control sample
no steady-state	recording of reference temperatures

2.3. Temperature profiles in planar walls



2.4. The onion layer principle

$$\dot{q}''_{\mathrm{wo,ins}} = 1527\,\mathrm{W/m^2}$$
$$\dot{q}''_{\mathrm{w,ins}} = 576\,\mathrm{W/m^2}$$

2.5. Feuerofen

The wall thickness is smallest for a choice of construction materials with the smallest thermal conductivities possible. δ_1 = steel, δ_2 = insulating stone, δ_3 = firebrick, δ_4 = refractory brick.

Case A) $\delta_1 = 8\,\mathrm{mm}$
$\delta_2 = 111\,\mathrm{mm}$
$\delta_4 = 180\,\mathrm{mm}$
$\delta = 299\,\mathrm{mm}$

Case B) $\delta_1 = 8\,\mathrm{mm}$
$\delta_3 = 330\,\mathrm{mm}$
$\delta = 338\,\mathrm{mm}$

2.6. Living comfortably

$$\dot{Q}_{V,\mathrm{tot}} = 2946\,\mathrm{W}$$

2.7. Hollow cylinder

a) See the solutions provided

b) $T^2 + 2 \cdot (T - T_\mathrm{i}) \cdot \left(\frac{1}{\gamma} - T_0\right) - T_\mathrm{i}^2 + \ldots$

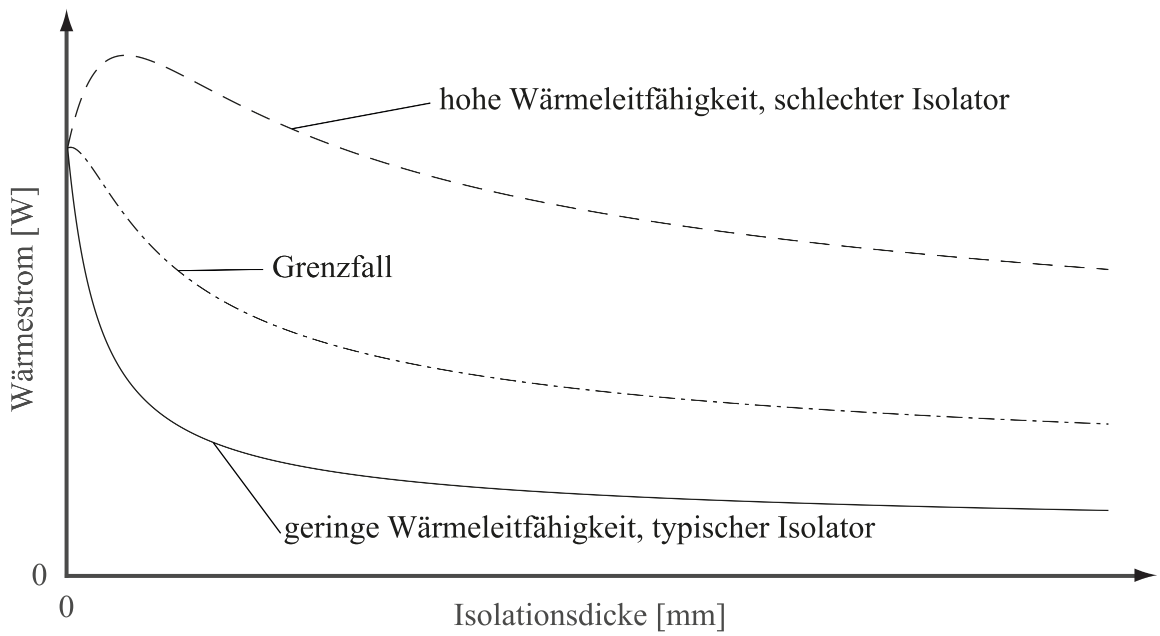
$$\ldots \frac{2}{\gamma} \cdot (T_\mathrm{i} - T_\mathrm{o}) \cdot \left(1 + \gamma \cdot \left(\frac{T_\mathrm{i} + T_\mathrm{o}}{2} - T_0\right)\right) \cdot \frac{\ln\left(\frac{r}{r_\mathrm{i}}\right)}{\ln\left(\frac{r_\mathrm{o}}{r_\mathrm{i}}\right)} = 0$$

2.8. Brine pipeline

a) $\delta_{\text{ins}} = 34\,\text{mm}$
b) $\dot{q}' = 12\,{}^{\text{W}}/_{\text{m}}$

2.9. Warm-water pipe

a) $\dot{q}'_{\text{wo,ins}} = 9\,{}^{\text{W}}/_{\text{m}}$
$\dot{q}''_{\text{w,ins}} = 10\,{}^{\text{W}}/_{\text{m}}$
b)



c) $\lambda_{\text{iso}} \leq 0.024\,{}^{\text{W}}/_{\text{m K}}$

2.10. Pipe fastening

a) $\dot{Q} = 1.85\,\mathrm{W}$
b) $h_0 = 10\,\mathrm{cm}$

2.11. Foggy rear window

$\dot{q}'_{\mathrm{H}} = 8\,{}^{\mathrm{W}}/_{\mathrm{m}}$

2.12. Circular fin with varying thickness

a) $\frac{d}{dr}\left(r \cdot \delta \cdot \frac{dT}{dr}\right) - \frac{2\alpha}{\lambda} \cdot r \cdot (T - T_{\mathrm{U}}) = 0$
b) $\delta\left(r\right) = \frac{\alpha}{3\lambda} \cdot (R - r)^2 \cdot \left(2 + \frac{R}{r}\right)$
c) $\dot{Q} = \alpha\left(T_{\mathrm{F}} - T_{\mathrm{U}}\right) \cdot \frac{2\pi R^2}{3} \cdot \left(1 - \frac{r_0}{R}\right) \cdot \left(1 + \frac{2r_0}{R}\right)$

2.13. Dooouble-walled container

a) $\frac{d}{dr}\left(r \cdot \delta \cdot \frac{dT}{dr}\right) - \frac{2\alpha}{\lambda} \cdot r \cdot (T - T_{\mathrm{U}}) = 0$
$Bi_{\mathrm{F}} = \frac{\alpha_{\mathrm{F}} \cdot \delta_{\mathrm{B}}}{\lambda_{\mathrm{B}}}$ $\quad Bi_{\mathrm{L}} = \frac{\alpha_{\mathrm{L}} \cdot \delta_{\mathrm{B}}}{\lambda_{\mathrm{B}}}$
$Bi_{\mathrm{F}} = 0.075$ $\quad Bi_{\mathrm{L}} = 0.0015$ $\quad \rightarrow \quad \dot{Q} = 93\,\mathrm{W}$
b) $T_{\mathrm{BL}} - T_{\mathrm{BF}} = 47\,\mathrm{K}$
c) Minimum ligament spacing: $2 \cdot L_\infty = 0.82\,\mathrm{m}$
d) $\frac{\dot{Q}_{\mathrm{ges}}}{L} = 247\,{}^{\mathrm{W}}/_{\mathrm{m}}$

2.14. High-temperature reactor fuel element

$T_{\mathrm{i}} - T_{\mathrm{O}} = 159\,\mathrm{K}$

2.15. Copper rod

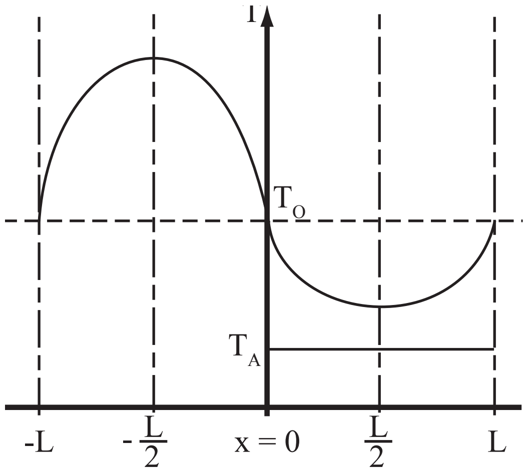
a) Siehe Musterlösung zu dieser Aufgabe

b) $\dot{\Phi}''' = \frac{2\cdot\lambda\cdot m}{L} \cdot (T_\mathrm{O} - T_\mathrm{U}) \cdot \tanh\left(\frac{m\cdot L}{2}\right)$

c) $\dot{\Phi}''' = 49.4\,{}^{\mathrm{kW}}/_{\mathrm{m}^3}$

d) $T_\mathrm{max} = 136.6\,°\mathrm{C}$

$T_\mathrm{min} = 106.7\,°\mathrm{C}$



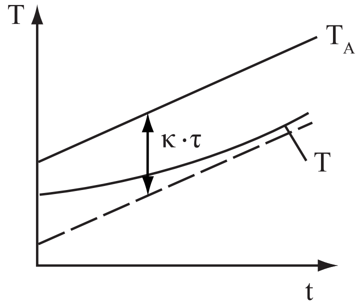
2.16. Critical explosive

$$s_\mathrm{krit} = \sqrt{\frac{\lambda}{\dot{\Phi}'''_\mathrm{U}\cdot\gamma}} \cdot \arctan\frac{k}{\sqrt{\dot{\Phi}'''_\mathrm{U}\cdot\gamma\cdot\lambda}} = 2.7\,\mathrm{m}$$

2.17. Copper sphere

a) $t_\mathrm{K} = 80\,\mathrm{s}$

b) $T_\mathrm{L} - T_\mathrm{K} = 11.6\,\mathrm{K}$



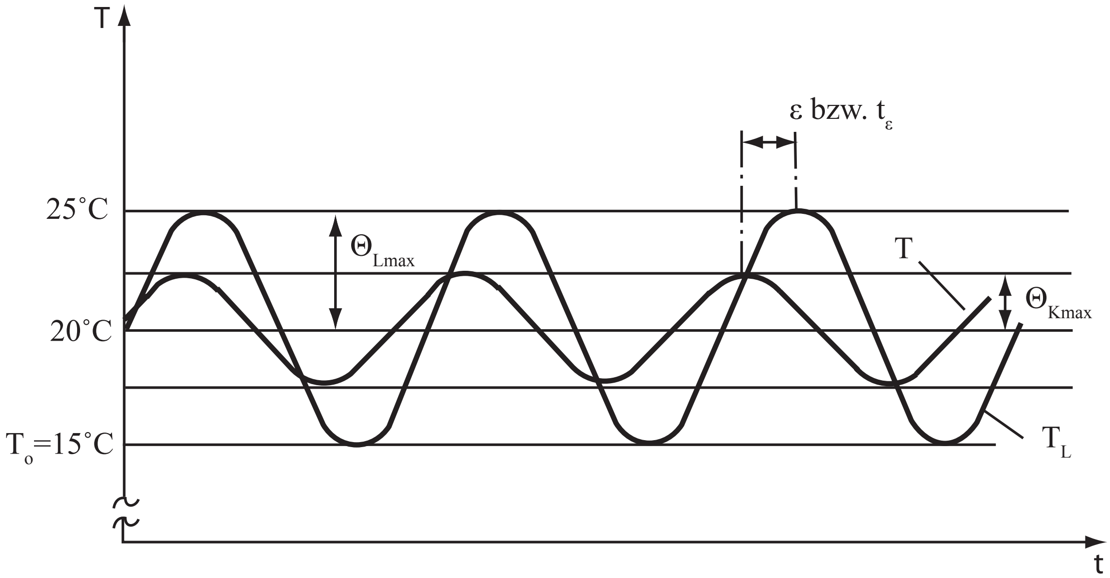
c) $\frac{T-T_{\mathrm{L,m}}}{\Theta_{\mathrm{L,max}}} = \left[\frac{T_0-T_{\mathrm{L,m}}}{\Theta_{L,max}} + \frac{\omega\cdot\tau}{1+(\omega\tau)^2}\right] \cdot \exp\left(-\frac{t}{\tau}\right) + \frac{1}{\sqrt{1+(\omega\tau)^2}} \cdot \sin\left(\omega\tau + \epsilon\right)$

$\tau = \frac{\rho\cdot c\cdot d}{6\cdot\alpha}$

$\epsilon = -\arctan(\omega\tau) = -\omega\cdot t_\epsilon$

$\Theta_{\mathrm{K,max}} = \frac{\Theta_{\mathrm{L,max}}}{\sqrt{1+(\omega\tau)^2}} = 2.2\,\mathrm{K}$

$t_\epsilon = 1.1\,\mathrm{min}$



2.18. Stirred tank

a) $\Theta^* = 1 + \frac{\exp(-t^*)-1}{t^*}$

b) $T_{\mathrm{We}} = 77\,°\mathrm{C}$

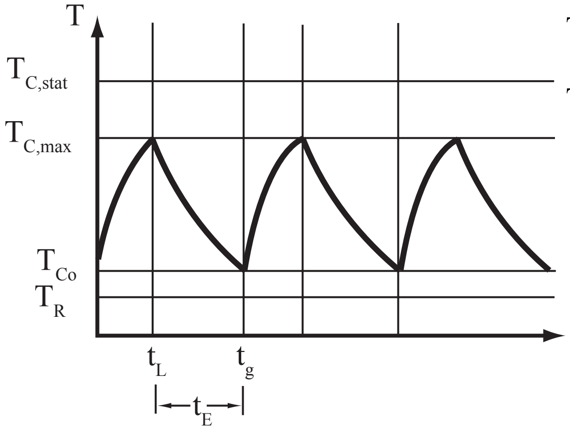
2.19. Oscillation problem

$V_{\mathrm{B}} = \frac{\dot{V} \cdot t_{\mathrm{s}}}{2\pi} \cdot \sqrt{\frac{1}{p^2} - 1} = 39.8\,\mathrm{m}^3$
mit $p = \frac{\Theta_{\mathrm{a,max}}}{\Theta_{\mathrm{e,max}}} = 0.001$

2.20. Night-storage heater

a) Charging cycle:
$\Theta_{\mathrm{KL}} = \frac{\dot{Q}_{\mathrm{el}}}{k \cdot A} \cdot \left(1 - \frac{1-\exp\left(-\frac{t_{\mathrm{E}}}{\tau}\right)}{1-\exp\left(-\frac{t_{\mathrm{g}}}{\tau}\right)} \cdot \exp\left(-\frac{t-t_{\mathrm{L}}}{\tau}\right)\right)$
Discharging cycle:
$\Theta_{\mathrm{KL}} = \frac{\dot{Q}_{\mathrm{el}}}{k \cdot A} \cdot \left(\frac{1-\exp\left(-\frac{t_{\mathrm{L}}}{\tau}\right)}{1-\exp\left(-\frac{t_{\mathrm{g}}}{\tau}\right)} \cdot \exp\left(-\frac{t-t_{\mathrm{L}}}{\tau}\right)\right)$

b) Sketch of the temperature profile:



$T_{C,stat}$ = Core temperature, continuous heating

$T_{C,stat} = T_R + \frac{\dot{Q}_{el}}{k \cdot A}$

c) $t_{\mathrm{L}} = \frac{Q_{\mathrm{H}}}{\dot{Q}_{\mathrm{el}}} = 9h$
$\Theta_{\mathrm{K0}} = \Theta_{\mathrm{KL}}\,(t=0) = \Theta_{\mathrm{KE}}\,(t=t_{\mathrm{g}}) = 290\,\mathrm{K}$
$\Theta_{\mathrm{max}} = \Theta_{\mathrm{KL}}\,(t=t_{\mathrm{L}}) = \Theta_{\mathrm{KE}}\,(t=t_{\mathrm{L}}) = 566\,\mathrm{K}$

d) $\frac{Q_{\mathrm{HL}}}{Q_{\mathrm{H}}} \cdot 100\,\% = 39\,\%$

2.21. Ice sphere cooling

a) $T_{\mathrm{Ks}} = T_{\mathrm{E}} = 0\,°\mathrm{C}$

$T_{\mathrm{Fs}} = 13.9\,°\mathrm{C}$

$T_{\mathrm{F},\infty} = T_{\mathrm{K},\infty} = 12.9\,°\mathrm{C}$

b) $T_{\mathrm{K}}(t) = T_{\mathrm{E}}$

$\frac{T_{\mathrm{F}}(t)-T_{\mathrm{E}}}{T_{\mathrm{U}}-T_{\mathrm{E}}} = \exp\left(-\frac{t}{\tau_{\mathrm{F}}}\right)$

mit $\tau_{\mathrm{F}} = \frac{\dot{m}_{\mathrm{F}}\cdot c_{\mathrm{F}}}{n\cdot k\cdot\pi\cdot d^2}$

c) $t_{\mathrm{s}} = 4.6\,\mathrm{min}$

d) $\frac{dT_{\mathrm{F}}}{dt} + \frac{1}{\tau_{\mathrm{F}}}\cdot(T_{\mathrm{F}} - T_{\mathrm{K}}) = 0$

$\frac{dT_{\mathrm{K}}}{dt} = -\mu\cdot\frac{dT_{\mathrm{F}}}{dt}$

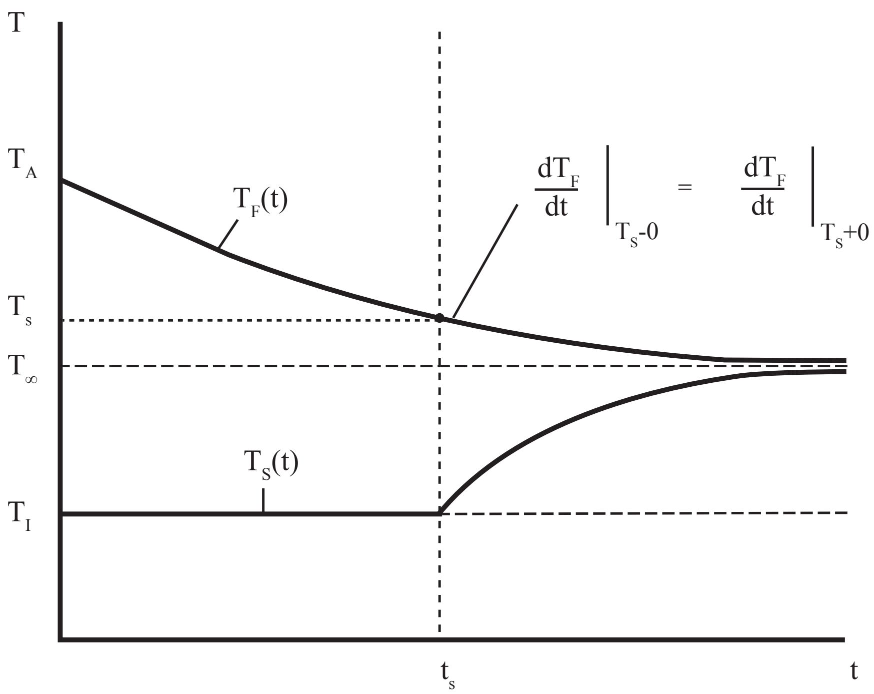
mit $\mu = \frac{\tau_{\mathrm{F}}}{\tau_{\mathrm{K}}}$

$\tau_{\mathrm{K}} = \frac{\rho_{\mathrm{E}}\cdot c_{\mathrm{W}}\cdot d}{6k\cdot\left(1+\frac{2\delta}{d}\right)^2}$

e) $T_{\mathrm{K}}(t) = T_{\mathrm{E}} - \mu\cdot(T_{\mathrm{F}}(t) - T_{\mathrm{Fs}})$

$T_{\mathrm{F}}(t) = T_{\mathrm{Fs}} - \frac{T_{\mathrm{Fs}}-T_{\mathrm{E}}}{1+\mu}\cdot\left(1-\exp\left(-(1+\mu)\cdot\frac{t-t_{\mathrm{s}}}{\tau_{\mathrm{F}}}\right)\right)$

f)



2.22. Contact of semi-infinite bodies

a) $\frac{T_1-T_{10}}{T_{10}-T_{20}} = -\frac{1}{1+\frac{b_1}{b_2}} \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{a_1 \cdot t}}\right)$ $\quad x \geq 0$

$\frac{T_2-T_{20}}{T_{10}-T_{20}} = \frac{1}{1+\frac{b_2}{b_1}} \cdot \operatorname{erfc}\left(\frac{|x|}{2\sqrt{a_2 \cdot t}}\right)$ $\quad x \leq 0$

$\frac{T_M-T_{20}}{T_{10}-T_{20}} = \frac{1}{1+\frac{b_2}{b_1}}$ $\quad b_1 = \sqrt{(\lambda \cdot \rho \cdot c)_1}$

$b_2 = \sqrt{(\lambda \cdot \rho \cdot c)_2}$

2.23. Rolled steel sheet

a) $w = \frac{\alpha \cdot L}{\rho \cdot c \cdot \delta} \cdot \frac{1}{\ln 2} = 0.072\,\mathrm{m/s}$

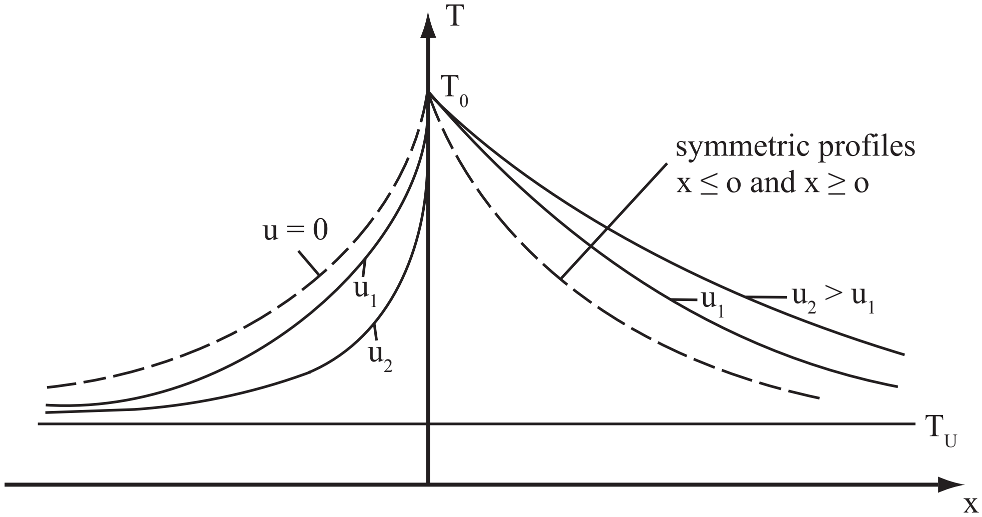
b) $w = \frac{1}{7.3} \cdot \frac{\lambda}{\rho \cdot c} \cdot \frac{L}{\delta^2} = 0.068\,\mathrm{m/s}$

Chapter 3.

Results convective heat transfer

3.1. Hot wire filament

a) Sketch of the temperature profile:



b) for $x < 0$:

$$\Theta_1 = \frac{2\dot{Q}}{\lambda \cdot \pi d^2 \cdot \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}} \cdot \exp\left(\frac{u}{2a} + \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}\right) \cdot x$$

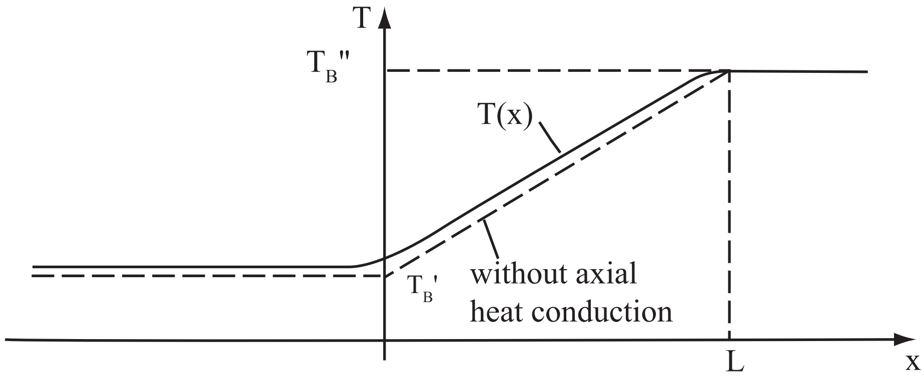
for $x > 0$:

$$\Theta_2 = \frac{2\dot{Q}}{\lambda \cdot \pi d^2 \cdot \sqrt{\left(\frac{u}{2a}\right)^2 + m^2}} \cdot \exp\left(\frac{u}{2a} - s\sqrt{\left(\frac{u}{2a}\right)^2 + m^2}\right) \cdot x$$

3.2. Lead pipe

a) $T''_{\mathrm{B}} = 706\,°\mathrm{C}$

b)



c) $\frac{T-T'_{\mathrm{B}}}{T''_{\mathrm{B}}-T'_{\mathrm{B}}} = \frac{1}{\beta} \cdot (1 - \exp -\beta) \cdot \exp\left(\beta \cdot \frac{x}{L}\right) \qquad x < 0$

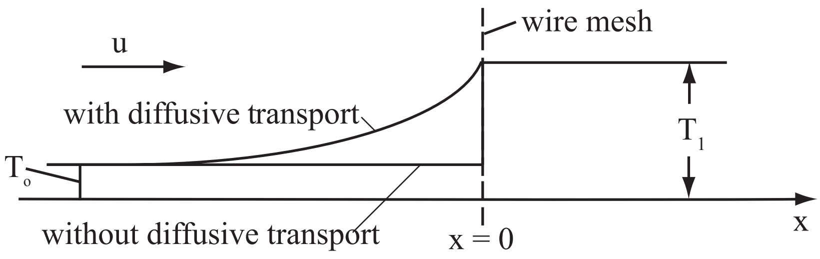
$\frac{T-T'_{\mathrm{B}}}{T''_{\mathrm{B}}-T'_{\mathrm{B}}} = \frac{1}{\beta} \cdot \left(1 - \exp\left(-\beta \cdot \left(1 - \frac{x}{L}\right)\right)\right) + \frac{x}{L} \qquad x \geq 0$

$\beta = \frac{u \cdot L}{a}$

d) $\frac{\left|\dot{Q}_{\mathrm{L}}\right|}{\dot{Q}} = \frac{1-\exp -\beta}{\beta} \approx 10^{-4} \qquad \beta = \mathrm{Pé} = \text{Péclét-Number}$

3.3. Flow through wire mesh

a) $T_1 = T_0 + \frac{\dot{q}''}{\rho \cdot c \cdot u}$



b)

c) $T = T_0 + \frac{\dot{q}''}{\rho \cdot c \cdot u} \cdot \exp\left(\frac{\rho \cdot c \cdot u}{\lambda} \cdot x\right)$ $\qquad x \leq 0$

3.4. Heated pipe

a) $T'' = T' + \frac{3}{4} \cdot \frac{\dot{q}''_{\max} \cdot A}{\dot{m} \cdot c}$

b) $0 \leq x \leq \frac{L}{2}$:

$T = T' + \frac{\dot{q}''_{\max} \cdot A}{\dot{m} \cdot c} \cdot \left(\frac{x}{L}\right)^2$ $\qquad A = \pi \cdot d \cdot L$

$\frac{L}{2} \leq x \leq L$:

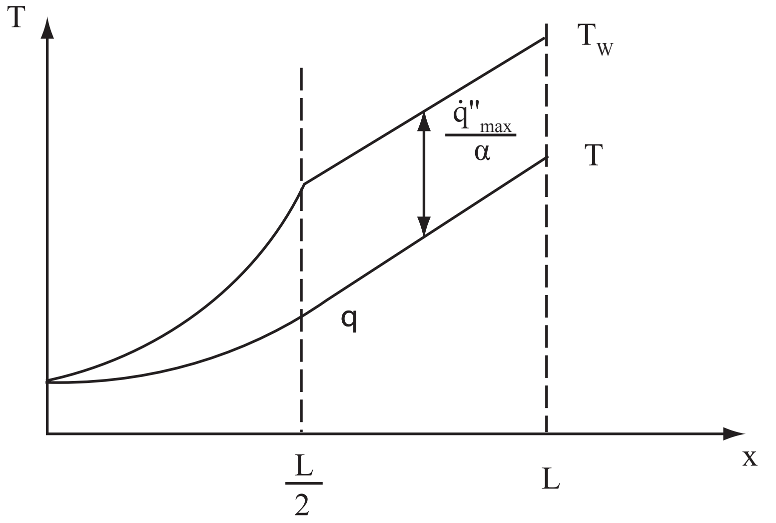
$T = T' + \frac{\dot{q}''_{\max} \cdot A}{\dot{m} \cdot c} \cdot \left(\frac{x}{L} - \frac{1}{4}\right)$

c) $0 \leq x \leq \frac{L}{2}$:

$T_{\mathrm{W}} = T' + \frac{\dot{q}''_{\max}}{\alpha} \cdot \left[2 + \frac{\alpha \cdot A}{\dot{m} \cdot c} \cdot \frac{x}{L}\right] \cdot \frac{x}{L}$

$\frac{L}{2} \leq x \leq L$:

$T_{\mathrm{W}} = T' + \frac{\dot{q}''_{\max}}{\alpha} \cdot \left[1 + \frac{\alpha \cdot A}{\dot{m} \cdot c} \cdot \left(\frac{x}{L} - \frac{1}{4}\right)\right]$



3.5. Absorption in a porous wall

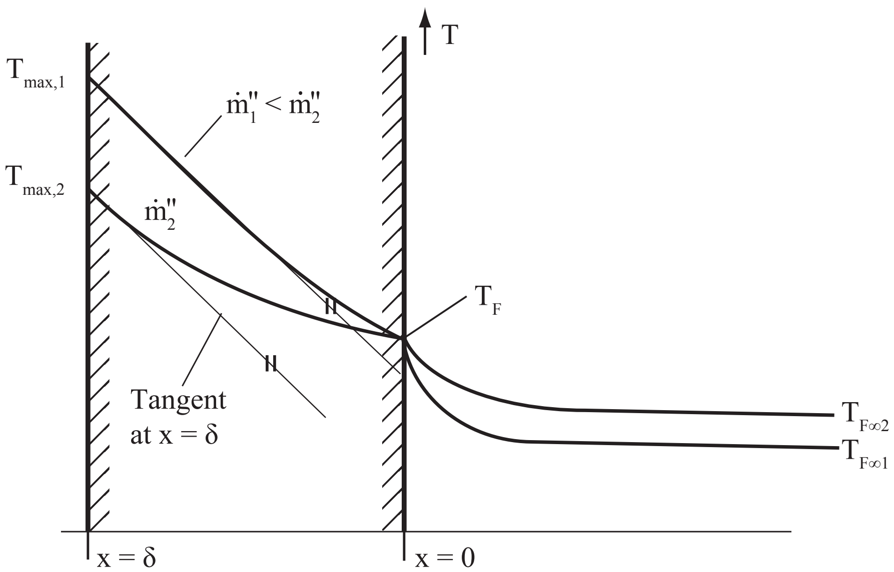
a) $T = T_{\mathrm{F}} - \frac{\dot{q}''_{\mathrm{s}}}{\dot{m}''\cdot c} \cdot \exp\left(-\frac{\dot{m}''\cdot c}{\lambda}\cdot\delta\right)\cdot\left(1-\exp\left(-\frac{\dot{m}''\cdot c}{\lambda}\cdot x\right)\right)$

b) $T_{\max} = -15 + \frac{150\cdot 10^3}{0{,}6\cdot 1000}\cdot\left(1-\exp\left(-3.75\right)\right) = 229\,°\mathrm{C}$

c) $\dot{q}''_{\mathrm{F}} = 150\cdot\exp\left(-3{,}75\right) = 3.5\,\mathrm{kW}/\mathrm{m}^2$

d) $T_{\mathrm{F}} = -15 - \frac{150\cdot 10^3}{0{,}6\cdot 1000}\cdot\exp\left(-3.75\right) = -21\,°\mathrm{C}$

e)



3.6. Vertical pipe

$H_s = 1.33$
$T_s = 99\,°\mathrm{C}$

3.7. Water mains

a) $\alpha_\mathrm{L} = 56.4\,\mathrm{W/m^2\,K}$
$T''_\mathrm{W} = 0.64\,°\mathrm{C}$
$\alpha_\mathrm{W} = 47.4\,\mathrm{W/m^2\,K}$
$\dot{m}_\mathrm{W} = 0.5\,\mathrm{g/s}$

b) A lower water mass flux is required.
Justification: For an undeveloped heat transfer α_W takes on larger local and mean values. The heat flux at the pipe's end defined by the inner pipe wall temperature T''_Pi can thus be transmitted at lower temperature differences, i.e. the water can be cooled to lower temperatures. As the heat insulation layer poses the dominating resistance to heat transfer, the overall heat loss increases only marginally through the improved water-side heat transfer. From larger values of $\dot{Q} \approx const.$ and ΔT_W follow lower values of $\dot{m}_\mathrm{W}$. This can also be proven analytically!

3.8. Heat transfer for a heated plate

a) $\alpha_{\mathrm{lam}}(L) = 12\,\mathrm{W/m^2\,K}$

$\alpha_{\mathrm{turb}}(L) = 55\,\mathrm{W/m^2\,K}$

$\overline{\alpha}_{\mathrm{lam}} = 24\,\mathrm{W/m^2\,K}$

$\overline{\alpha}_{\mathrm{turb}} = 68\,\mathrm{W/m^2\,K}$

b) $\dot{Q}_{\mathrm{lam}} = 976\,\mathrm{W}$

$\dot{Q}_{\mathrm{turb}} = 2720\,\mathrm{W}$

c) See Eq. (HTL. 6)

$\dot{Q} = 1980\,\mathrm{W}$

Instructions for the use of (HTL.1) and (HTL.5)

$\dot{Q} = 2036\,\mathrm{W}$

Chapter 4.

Results convective and radiative heat transfer

4.1. Thermocouple

a) $T_{\mathrm{L}} - T_{\mathrm{Th}} = 2\,\mathrm{K}$
b) Increase in impinging velocity, size reduction of the thermocouple, mirror coating, protective cladding
c) $T_{\mathrm{L}} - T_{\mathrm{Th}} = 0.01\,\mathrm{K}$

4.2. Fairing in a pipe

a) $l = 0.020\,\mathrm{m}$

b) Possible solutions:
- Thicker pipe insulation. Thus, the heat flux through the pipe is reduced which in turn reduces the temperature gradient, to which the measuring error is directly proportional (see eq. 11), between wall and fluid.
- Angled insertion of the fairing. Thus, the immersed length increases while the heat transfer coefficient at the fairing decreases simultaneously as a result of the increased immersed length (larger boundary layer thickness) - an effect counteracting the greater immersed length. Although, overall the effect of the greater immersed length dominates.
- Installation of the fairing in a pipe bend.

4.3. Methanol tank

a) $\lambda_{\mathrm{ins}} > \lambda_{\mathrm{Cork}} \longrightarrow$ The insulation is damp!

4.4. Luftspalt

a) $\left(\frac{\lambda_{\mathrm{S}}}{\lambda_{\mathrm{L}}}\right)_0 = 1.63$

b) $\left(\frac{\lambda_{\mathrm{S}}}{\lambda_{\mathrm{L}}}\right)_1 = 1.31$

c) $T_{\mathrm{F}} = 30.1\,°\mathrm{C}$

d) $\left(\frac{\lambda_{\mathrm{S}}}{\lambda_{\mathrm{L}}}\right)_n = 1 + \frac{1}{n+1} \cdot \left[\left(\frac{\lambda_{\mathrm{S}}}{\lambda_{\mathrm{L}}}\right)_0 - 1\right]$

4.5. Beatmungsgerät

a) $\alpha_\mathrm{i} = 11.6\,\mathrm{W/m^2\,K}$

$k = 3{,}5\,\mathrm{W/m^2 K}$

b) Air temperature profile:

$\frac{T_\mathrm{L}-T_\mathrm{U}}{T'_\mathrm{L}-T_\mathrm{U}} = \exp\left(-\frac{k\cdot\pi d_\mathrm{i}\cdot L}{\dot{m}_\mathrm{L}\cdot c_\mathrm{L}}\cdot\frac{x}{L}\right)$

Relationship between air and hose-wall temperature:

$T''_\mathrm{L} - T_\mathrm{U} = \frac{1}{1-\frac{k}{\alpha_\mathrm{i}}}\cdot(T''_\mathrm{W} - T_\mathrm{U})$

Necessary air inlet temperature:

$T'_\mathrm{L} = T_\mathrm{U} + \frac{\exp\left(\frac{k\cdot\pi d_\mathrm{i}\cdot L}{\dot{m}_\mathrm{L}\cdot c_\mathrm{L}}\right)}{1-\frac{k}{\alpha_\mathrm{i}}}\cdot(T''_\mathrm{W} - T_\mathrm{U})$

c) $\dot{Q}_\mathrm{L} = 7.2\,\mathrm{W}$

Check outer overall heat transfer coefficient:

$\alpha_\mathrm{a,Konv} = 3.6\,\mathrm{W/m^2\,K}$

$\alpha_\mathrm{a,Str} = 5.5\,\mathrm{W/m^2\,K}$

bei $T_\mathrm{iso,a} = 22.8\,°\mathrm{C}$

d) $L_0 = 0.7\,\mathrm{m}$ ($T''_\mathrm{L} = 42.9\,°\mathrm{C}$)

$\dot{Q}_\mathrm{el} = 2.8\,\mathrm{W}$

$\dot{Q}_V = 5.2\,\mathrm{W}$

Chapter 5.

Results mass transfer

5.1. Glass tube

a) $\xi_{\mathrm{D}}(x) = 1 + (\xi_{\mathrm{D0}} - 1) \cdot \cdots \cdot \left(\frac{\xi_{\mathrm{D}\delta}-1}{\xi_{\mathrm{D0}}-1}\right)^{\frac{x}{\delta}}$

$\dot{m}_{\mathrm{D,Stefan}} = \rho_{\mathrm{tot}} \cdot \frac{D}{\delta} \cdot A_{\mathrm{Q}} \cdot \ln \frac{1-\xi_{\mathrm{D}\delta}}{1-\xi_{\mathrm{D0}}}$

b) $\frac{\dot{m}_{\mathrm{D,Stefan}}}{\dot{m}_{\mathrm{D,Diff}}}$ mit $\dot{m}_{\mathrm{D,Stefan}}$ from a) and

$\dot{m}_{\mathrm{D,Diff}} = \rho_{\mathrm{tot}} \cdot \frac{D}{\delta} \cdot A_{\mathrm{Q}}(\xi_{\mathrm{D0}} - \xi_{\mathrm{D}\delta})$

c) $\xi_{\mathrm{D}\delta} = 0$

$\xi_{\mathrm{D0}} = \frac{\frac{p_s}{R_{\mathrm{D}}}}{\frac{p_s}{R_{\mathrm{D}}} + \frac{(p-p_s)}{R_{\mathrm{L}}}} = 0.0288$

$\rho_{\mathrm{tot}} \approx \frac{p}{R_{\mathrm{L}} \cdot T} = 1.12\,\mathrm{kg}/\mathrm{m}^3$

$\dot{m}_{\mathrm{D,Stefan}} = \frac{25.5 \cdot 10^{-6}\,\mathrm{kg}}{3600\,\mathrm{s}} = 7{,}08 \cdot 10^{-9}\,\mathrm{kg}/\mathrm{s}$

$D = \frac{\frac{\dot{m}_{\mathrm{D,Stefan}}}{\rho_{\mathrm{tot}}} \cdot \frac{\delta}{A}}{\ln \frac{1-\xi_{\mathrm{D}\delta}}{1-\xi_{\mathrm{D0}}}} = 27.7 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$

5.2. Damp wood

a) $\frac{\mathrm{d}\xi_{\mathrm{W}}}{\mathrm{d}t} = D \cdot \frac{\mathrm{d}^2\xi_{\mathrm{W}}}{\mathrm{d}x^2}$

b) $t = \frac{\delta^2}{4 \cdot D} = 13.56\,\mathrm{Tage}$

5.3. Condensation of steam

$\delta = \left[2 \cdot \frac{\lambda \Delta T}{\rho_W \Delta h_v} \cdot t\right]^{\frac{1}{2}}$
$\Delta h_v = r = 2358\,{}^{\mathrm{kJ}}/_{\mathrm{kg\,K}}$
$\lambda \approx \lambda(40\,°\mathrm{C}) = 0.631\,{}^{\mathrm{W}}/_{\mathrm{m\,K}}$
$\rho_W \approx 1000\,{}^{\mathrm{kg}}/_{\mathrm{m^3}}$
$\delta(t = 3600\,\mathrm{s}) = 8.78\,\mathrm{mm}$

$\Rightarrow m = \rho_W \frac{\pi}{4} D^2 \delta = 0.621\,\mathrm{kg}$

5.4. Shark attack on Mallorca

a) $\rho_{\mathrm{tot}} \cdot \frac{\partial \xi_B}{\partial t} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(\rho_{\mathrm{tot}} \cdot D_{B,W} \cdot r^2 \cdot \frac{\partial \xi_B}{\partial r}\right) + S$
b) see long solution
c) $r_K = \frac{1}{\frac{\xi_K - \xi_F}{B} + \frac{1}{r_F}}$
d) $\xi_F = \frac{\xi_{SF}}{\xi_S} = \frac{1.228 \cdot 10^{-6}\,\mathrm{kg_S/kg}}{1.05 \cdot 10^{-4}\,\mathrm{kg_S/kg}} = 0.0117\,{}^{\mathrm{kg_B}}/_{\mathrm{kg}}$

5.5. Even more critical explosive

a) $s_{\max} = 0.373\,\mathrm{m}$

5.6. Perowskite

a) $\dot{m}_{O_2} = 5.206 \cdot 10^{-8}\,{}^{\mathrm{kg_{O_2}}}/_{\mathrm{s}}$
b) $\dot{m}_{O_2,St} = 5.640 \cdot 10^{-8}\,{}^{\mathrm{kg_{O_2}}}/_{\mathrm{s}} \longrightarrow$ Due to the relatively high concentration of oxygen the omission of the Stefan correction incurs an error of $\approx 8\,\%$.

5.7. Tarred railway sleeper

a) $\mathrm{Bi} = \frac{g \cdot \frac{d}{2}}{\rho \cdot D_\mathrm{W}} = \frac{2.164 \cdot 10^{-5}\,\mathrm{kg/m^2s} \cdot \frac{0.01\,\mathrm{m}}{2}}{1000\,\mathrm{kg/m^3} \cdot 7 \cdot 10^{-6}\,\mathrm{m^2/s}} = 1.546 \cdot 10^{-5}$

Radial differences in concentration are negiglible (See long solution for justification).

b) $\dot{m} = \dot{m}_O + \dot{m}_U$

$= 1.2563 \cdot 10^{-11}\,\mathrm{kg_N/s} + 8.3433 \cdot 10^{-13}\,\mathrm{kg_N/s} = 1.3397 \cdot 10^{-11}\,\mathrm{kg_N/s}$

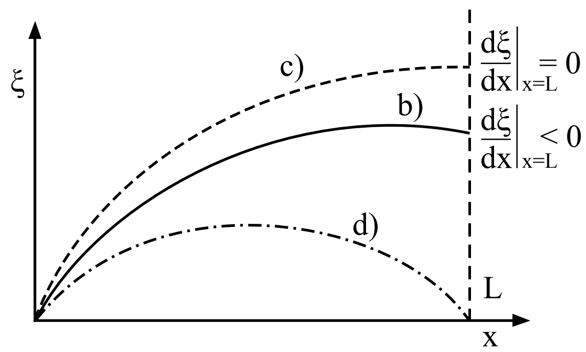
$\dot{m}_\mathrm{O}$, $\dot{m}_\mathrm{U}$ = mass flux at the lower or upper side, respectively, of the bore.

c) Underside of the bore impermeable $\longrightarrow \dot{m}_\mathrm{U} = 0$

$\dot{m}_\mathrm{O}^* = 1.3377 \cdot 10^{-11}\,\mathrm{kg_N/s}$

d) $\dot{m}_\mathrm{O}^{**} = 1.3541 \cdot 10^{-11}\,\mathrm{kg_N/s}$

e)



5.8. Scrubber

a) $v_\mathrm{T} = \sqrt{\frac{4}{3} \cdot \frac{10^{-3}\,\mathrm{m} \cdot 9.81\,\mathrm{m/s^2}}{0{,}4} \cdot \frac{1020\,\mathrm{kg/m^3} - 1.86\,\mathrm{kg/m^3}}{1.86\,\mathrm{kg/m^3}}} = 4.24\,\mathrm{m/s}$

b) $m = m_0 = 1.86\,\mathrm{kg_G/m^3} \cdot 0.05\,\mathrm{kg_{CO_2}/kg_G} \cdot \frac{\pi}{6} \cdot \left(10^{-3}\,\mathrm{m}\right)^3 = 4.87 \cdot 10^{-11}\,\mathrm{kg_{CO_2}}$

c) $\xi_\mathrm{S} = 4.791 \cdot 10^{-5}\,\mathrm{kg_{CO_2}/kg_T}$