Heat Transfer: Conduction

Steady state, one-dimensional heat conduction
with source

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Video overview

Steady state 1-D heat conduction with source

- How is a source/sink considered?
- Derivation of the differential equation by energy balances

Determination of the temperature profile

- Definition of boundary conditions
- Solution of the differential equation with appropriate boundary conditions

Generalized form of the temperature profile for different geometries

- Final differential equation
- Calculation of maximum and minimum temperature in a body

Video overview

What is a source?

- Heat production within a body
- Examples:
  - (Electrical) heater
  - Battery
  - Fuel element
  - …

What is a sink?

- Heat absorption within a body
- Examples:
  - Endothermic reaction
  - Phase change material
  - Saline heat storage
  - …

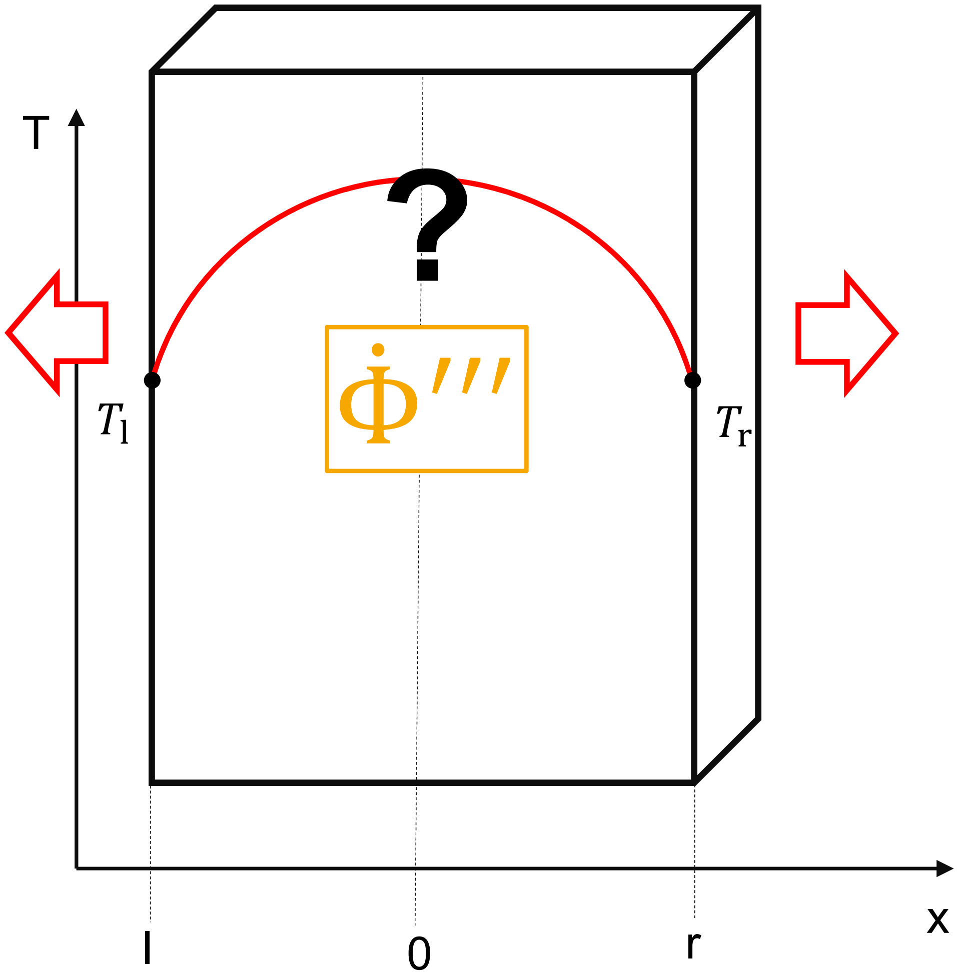
How is the heat flow (of a source/sink) expressed?

- Heat production $\dot{\Phi}'''$ is distributed evenly inside the volume element V

$$\dot{Q} = \dot{\Phi}''' \cdot V \quad \rightarrow \quad \boxed{\dot{\Phi}''' = \frac{\dot{Q}}{V}}$$

sink: $\dot{\Phi}''' = -\frac{\dot{Q}}{V}$

Heat produced by the source can only be released to the outside.



Differential equation:

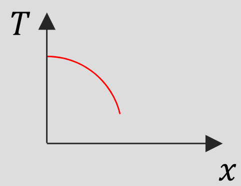
$$0 = \frac{d^2T}{dx^2} + d\dot{\Phi}'''$$

Solution of the differential equation:

$$T = -\frac{\dot{\Phi}'''}{2}\,x^2 + C_1 \cdot x + C_2$$

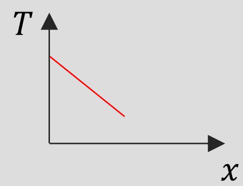
$\dot{\Phi}'''$ > 0 : Parabolic

$$T = -\frac{\dot{\Phi}'''}{2}\,x^2 + C_1 \cdot x + C_2$$



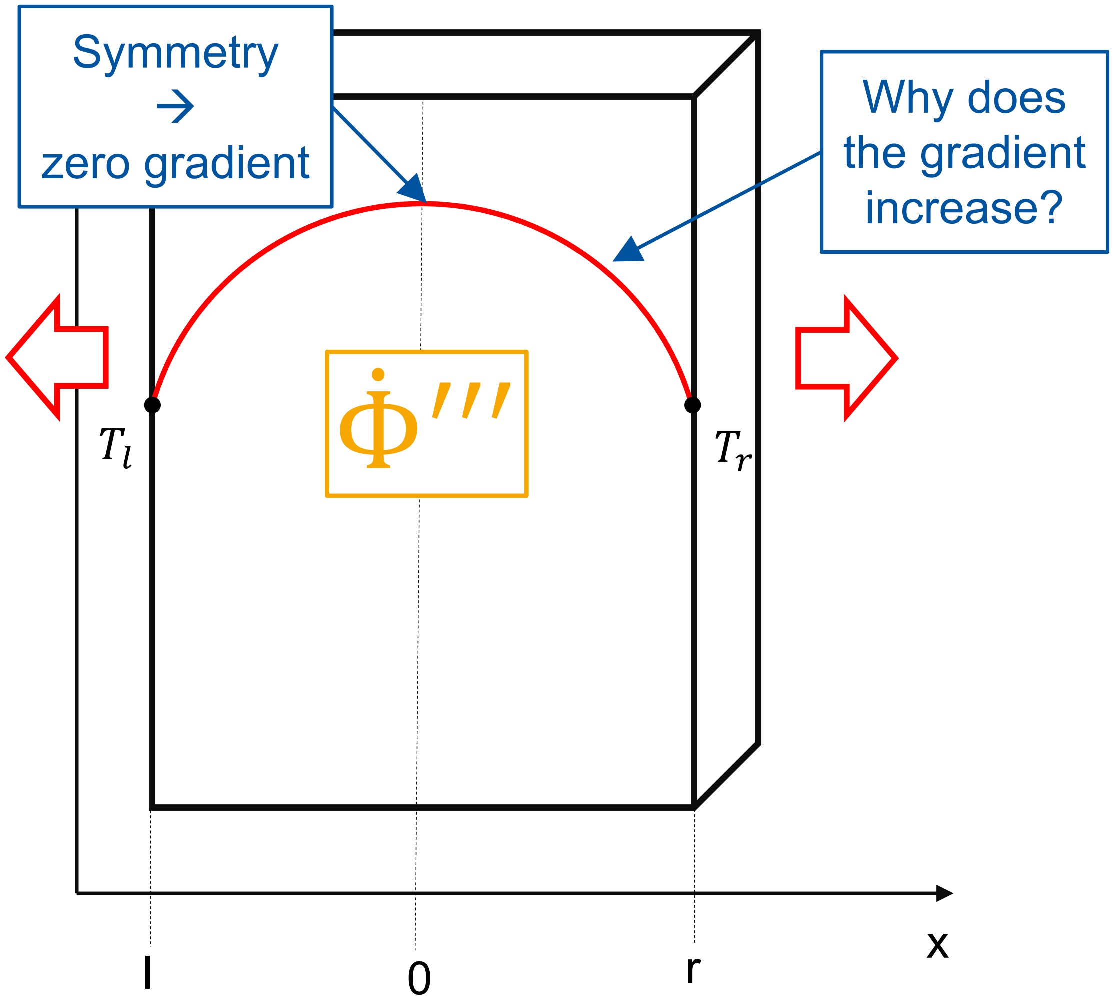
$\dot{\Phi}'''$ = 0 : Linear

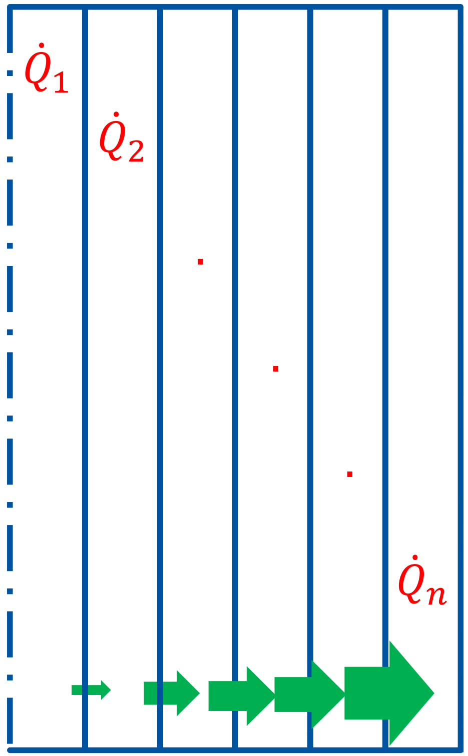
$$T = C_1 \cdot x + C_2$$



Introduction: Temperature profile of a body with source (cartesian coordinates)

Heat produced by the source can only be released to the outside.





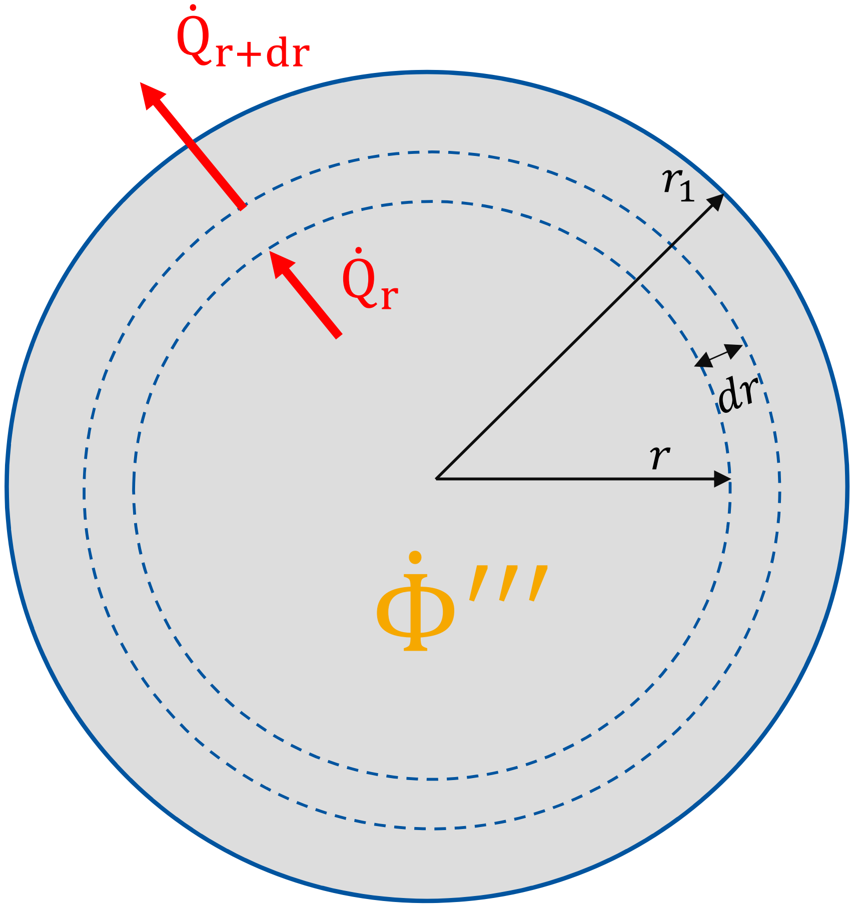
$$\dot{Q}_{\text{Cond},1} = \dot{Q}_1$$

$$\dot{Q}_{\text{Cond},2} = \dot{Q}_1 + \dot{Q}_2$$

$$\vdots$$

$$\dot{Q}_{\text{Cond},n} = \sum \dot{Q}_{\text{Prod}}$$

Derivation of the differential equation
Steady state 1-D heat conduction in cylindrical bodies with source



Cylinder length L

Energy balance around infinitesimal ring element:

$$0 = \dot{Q}_r - \dot{Q}_{r+dr} + d\dot{Q}_{Source}$$

Heat conduction (Fourier) for cylinders:

$$\dot{Q}_r = -\lambda \cdot A(r) \cdot \frac{dT}{dr}$$

Source term:

$$d\dot{Q}_{Source} = \dot{\Phi}''' \cdot dV$$

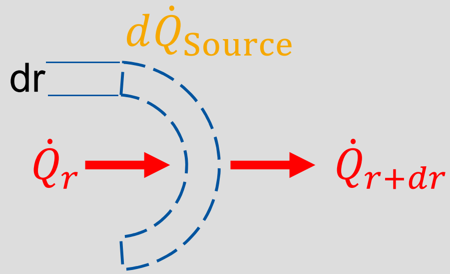
Infinitesimal volume element:

$$dV = 2\,\pi \cdot r \cdot dr \cdot L$$

The diff. equation comes from energy balance with Taylor series expansion and inserted source term:

$$0 = \dot{Q}_r - \dot{Q}_{r+dr} + d\dot{Q}_{\text{Source}}$$

$$0 = \dot{Q}_r - \left(\dot{Q}_r + \frac{d\dot{Q}_r}{dr} \cdot dr\right) + d\dot{Q}_{\text{Source}}$$ (Taylor series expansion)



$$0 = -\frac{d\dot{Q}_r}{dr} \cdot dr + d\dot{Q}_{\text{Source}}$$

$$0 = \cancel{-}\frac{d\left(\cancel{-}\lambda \cdot \cancel{2 \cdot \pi \cdot L} \cdot \left(r\frac{dT}{dr}\right)\right)}{dr} \cdot \cancel{dr} + \dot{\Phi}''' \cdot 2 \cancel{\cdot \pi \cdot L \cdot r} \cdot \cancel{dr}$$

$$\Rightarrow \quad \boxed{0 = \frac{1}{r} \cdot \frac{d}{dr} \cdot \left(r\,\frac{dT}{dr}\right) + \frac{\dot{\Phi}'''}{\lambda}}$$

or

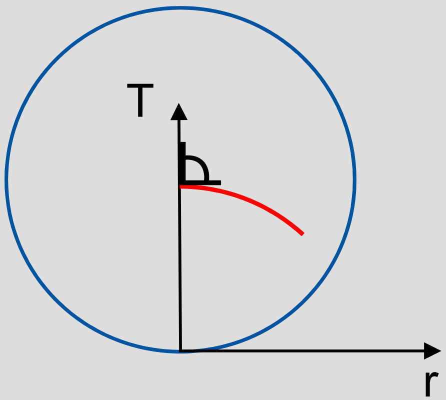
$$\boxed{0 = \frac{d^2T}{dr^2} + \frac{1}{r} \cdot \frac{dT}{dr} + \frac{\dot{\Phi}'''}{\lambda}}$$

Differential equation derivation
Steady state 1-D heat conduction in cylindrical bodies with source

BC inside $r = 0$:

Zero gradient:

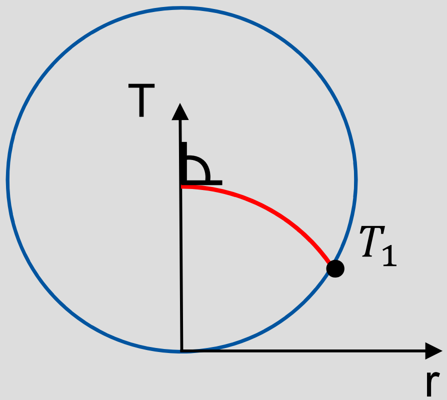
$$\left(\frac{dT}{dr}\right)_{r=0} = 0$$



Boundary condition for cylinder surface $r = r_1$:

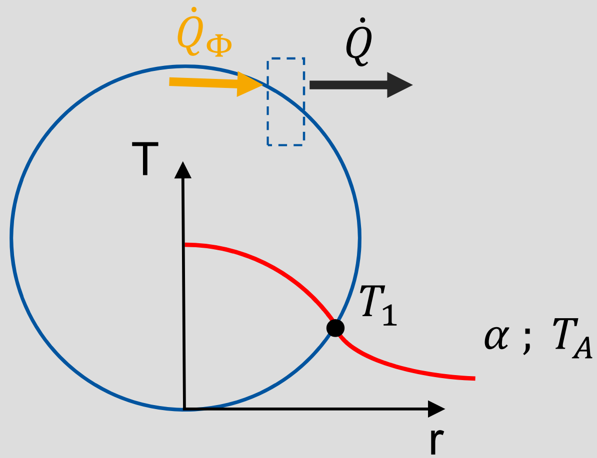
Case 1: a temperature

$$T = T_1$$



Case 2: a heat transfer coefficient α

$$\dot{Q} = 2 \cdot \pi \cdot r_1 \cdot L \cdot \alpha \cdot (T_1 - T_A)$$



If steady state:

$$\dot{Q} = \dot{Q}_\Phi$$

$$2\pi r_1 L\alpha(T_1 - T_A) = \dot{\Phi}''' 2\pi r_1{}^2 L$$

$$\boxed{T_1 = T_A + \frac{r_1}{2}\frac{\dot{\Phi}'''}{\alpha}}$$

Differential equation derivation
Steady state 1-D heat conduction in cylindrical bodies with source

Differential equation:

$$0 = \frac{1}{r} \cdot \frac{d}{dr} \cdot \left(r \frac{dT}{dr} \right) + \frac{\dot{\Phi}'''}{\lambda}$$

1st integration:

$$r \frac{dT}{dr} = -\frac{1}{2} r^2 \frac{\dot{\Phi}'''}{\lambda} + C_1$$

Insert 1st BC $r = 0$; $\frac{dT}{dr} = 0$:

$$r \frac{dT}{dr} = 0 = -\frac{1}{2} r^2 \frac{\dot{\Phi}'''}{\lambda} + C_1$$

$$\Rightarrow \boxed{C_1 = 0}$$

Equation after 1st integration constant inserted:

$$\cancel{r} \frac{dT}{dr} = -\frac{1}{2} r^{\cancel{2}} \frac{\dot{\Phi}'''}{\lambda}$$

Differential equation derivation
Steady state 1-D heat conduction in cylindrical bodies with source

Equation after 1st integration:

$$\frac{dT}{dr} = -\frac{1}{2} r \, \frac{\dot{\Phi}'''}{\lambda}$$

2nd integration:

$$\mathrm{T} = -\frac{1}{4} r^2 \frac{\dot{\Phi}'''}{\lambda} + C_2$$

Insert 2nd BC $r = r_1$: $T(r_1) = T_a + \frac{r_1 \, \dot{\Phi}'''}{2 \, \alpha}$:

$$T_A + \frac{r_1 \, \dot{\Phi}'''}{2 \, \alpha} = -\frac{1}{4} r_1^2 \frac{\dot{\Phi}'''}{\lambda} + C_2$$

2nd integration constant C_2

$$C_2 = T_A + \frac{r_1 \, \dot{\Phi}'''}{2 \, \alpha} + \frac{1}{4} r_1^2 \frac{\dot{\Phi}'''}{\lambda}$$

Differential equation derivation
Steady state 1-D heat conduction in cylindrical bodies with source

Equation after 2nd integration :

$$T(r) = -\frac{1}{4} r^2 \frac{\dot{\Phi}'''}{\lambda} + C_2$$

2nd integration constant C_2 :

$$C_2 = T_A + \frac{r_1 \; \dot{\Phi}'''}{2\,\alpha} + \frac{1}{4} r_1^2 \frac{\dot{\Phi}'''}{\lambda}$$

2nd integration constant C_2 inserted in equation:

$$T(r) = -\frac{1}{4} r^2 \frac{\dot{\Phi}'''}{\lambda} + T_A + \frac{r_1 \; \dot{\Phi}'''}{2\,\alpha} + \frac{1}{4} r^2 \frac{\dot{\Phi}'''}{\lambda}$$

2nd integration constant C_2

$$T(r) = T_u + \frac{r_1^2 \cdot \; \dot{\Phi}'''}{4 \cdot \lambda} \left[1 + \frac{2 \cdot \lambda}{\alpha \cdot r_1} - \left(\frac{r}{r_1} \right)^2 \right]$$

This ist a 2nd order polynomial (Parabola)

Differential equation derivation
Steady state 1-D heat conduction in cylindrical bodies with source

General temperature profile for plate, cylindrical or spherical geometry and symmetry with source :

$$T(\xi) = T_A + \frac{s^2 \cdot \dot{\Phi}'''}{2(n+1)\cdot\lambda}\left[1 + \frac{2\cdot\lambda}{\alpha\cdot s} - \left(\frac{\xi}{s}\right)^2\right]$$

Variables for different geometries:

	Plate (*)	Cylinder	Sphere
ξ	x	r	r
s	δ	r_1	r_1
n	0	1	2

* For the plate ist x is related to the symmetry plane, and $\delta = \frac{1}{2}\cdot(\text{plate thickness})$

Differential equation derivation
Steady state 1-D heat conduction in cylindrical bodies with source

General temperature profile:

$$T(\xi) = T_A + \frac{s^2 \cdot \dot{\Phi}'''}{2(n+1)\cdot\lambda}\left[1 + \frac{2\cdot\lambda}{\alpha\cdot s} - \left(\frac{\xi}{s}\right)^2\right]$$

I) greatest at ξ = 0 (maximum temperature at the body center):

$$T_{\max} = T(\xi = 0) = T_A + \frac{s^2 \cdot \dot{\Phi}'''}{2(n+1)\cdot\lambda}\left[1 + \frac{2\cdot\lambda}{\alpha\cdot s}\right]$$

I) greatest at ξ = 0 (minimum temperature at the body surface):

$$T_{\min} = T(\xi = s) = T_A + \frac{s^2 \cdot \dot{\Phi}'''}{(n+1)\cdot\alpha}$$

$$\boxed{\begin{array}{lll} \alpha \uparrow & \rightarrow & T_{\min} = T_S \downarrow \\ \dot{\Phi}''' \uparrow & \rightarrow & T_{\min} = T_S \uparrow \end{array}}$$

Comprehension questions

What is the temperature profile for cylindrical bodies with sources?

Which different boundary conditions can exist at the cylindrical surface?

How can the minimum and maximum temperatures be determined?