Heat Transfer: Conduction

Unsteady energy conservation equations

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Video overview

Steady state energy conservation equation without sources:

- Steady state 1-D heat conduction without sources
- Steady state ***2-D*** heat conduction without sources

Steady state energy conservation equation with sources:

- Steady state 2-D heat conduction ***with sources***

Transient energy conservation equation:

- ***Transient*** 2-D heat conduction with sources

Transient 3-D energy conservation equation with sources:

- 3-D conservation equation without advection

Learning goals

Internal energy and specific heat:

- Understand the concept of internal energy and the difference to kinetic and potential energy
- Distinguish between the specific heat at constant temperature and constant pressure

Energy balances:

- Setting up energy balances for different cases
- Development of a differential equation from the energy balance using Taylor series expansion
- Establish necessary boundary conditions
- Solving the differential equation for simple cases

Internal energy and specific heat capacity

Internal energy:

U [J] : The internal energy of a thermodynamic system is the energy contained within it. It does not include the kinetic energy of motion of the system as a whole, nor the potential energy of the system as a whole due to external force fields.

Examples for the inner energy are: Thermal energy, chemical bonding energy, …

The internal energy is measured as a difference from a reference zero defined by a standard state. Only the difference between the reference state and the current state is of interest.

Internal energy in heat transfer:

$U(\mathrm{T})$: We consider here changes of the internal energy solely caused by temperature variations and by phase change processes (melting, evaporation, sublimation).

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Internal energy and specific heat capacity

Specific heat capacity:

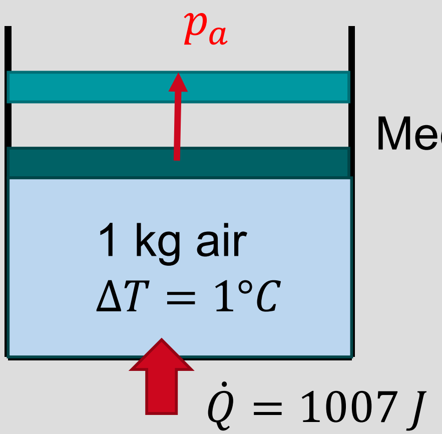
$$\frac{\partial U}{\partial T} = \rho \frac{\partial u}{\partial T} = \rho c$$

with c being the (mass) specific heat capacity

Specific heat capacity for gases:

Addition of heat at variable volume

- constant pressure
- increasing volume $\Delta V > 0$

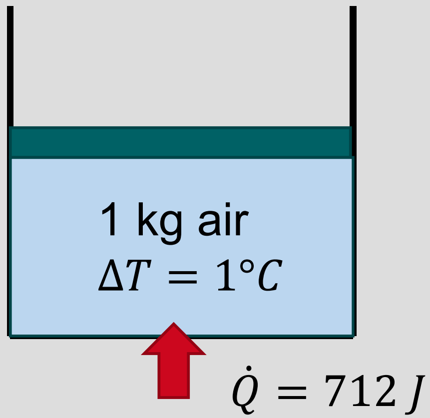


Mechanical work:

$$W = p\Delta V$$

Addition of heat at fixed volume

- constant volume
- increasing pressure



Internal energy and specific heat capacity

Specific heat capacity at 298K and 1 bar:

		$c_p \left[\frac{J}{kgK}\right]$	$c_v \left[\frac{J}{kgK}\right]$
Gases:	Air	1005	718
	Helium	5240	3157
	Hydrogen	14300	10142
Liquids:	Water	4220	
	Oil	1800	
Solids:	Iron	447	
	Wood	1380	

Definition of the internal energy for the transient heat conduction

Energy balance:

$$U = m \cdot c_v \cdot T$$

$$U = \rho \cdot c_v \cdot T \cdot dx \cdot dy \cdot \underbrace{dz}_{1}$$

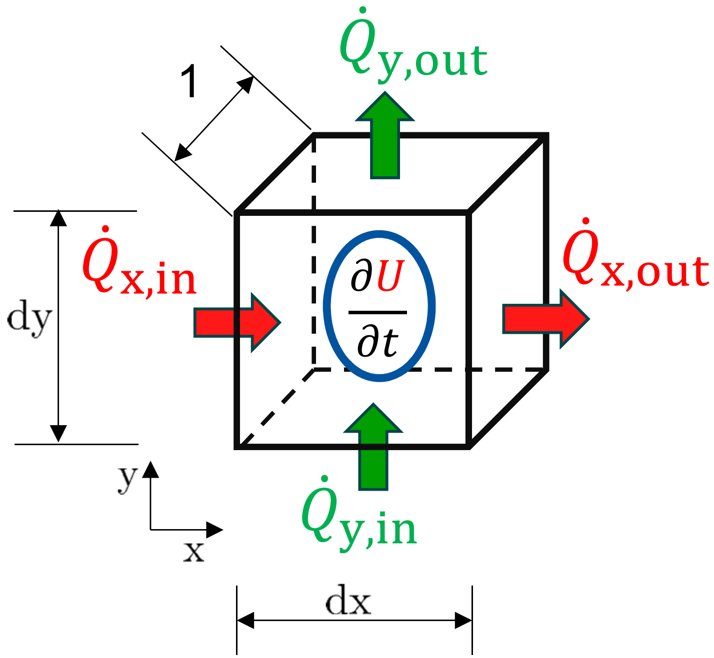
U [J] ▶ Internal energy

$\rho \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$ ▶ Density

$c_v \left[\frac{\mathrm{J}}{\mathrm{kgK}}\right]$ ▶ Specific heat capacity at constant volume

Units – Check:

$$\dot{Q} = \underbrace{\dot{q}''}_{\left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]} \cdot \underbrace{dx \cdot dy}_{[\mathrm{m}^2]} \; [\mathrm{W}] \quad \Rightarrow \frac{\partial U}{\partial t} \quad \left[\frac{\mathrm{J}}{s} = \mathrm{W}\right]$$



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Energy balance for transient 2-D heat conduction with sources

Change of internal energy:

Considered change in internal energy only due to temperature changes! (c_v and ρ are constant)

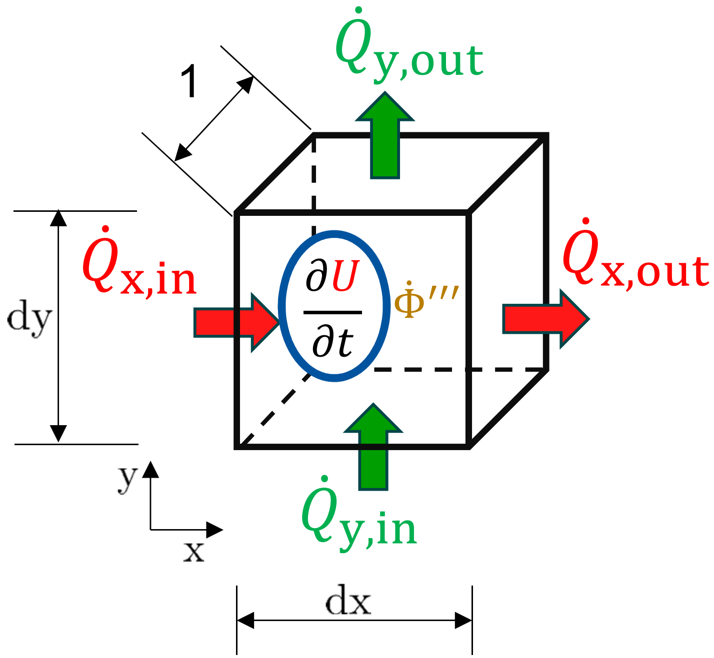
Change within the control volume: $$\frac{\partial U}{\partial t} = \rho \cdot c_v \cdot dx \cdot dy \cdot 1 \cdot \frac{\partial T}{\partial t}$$

Energy balance:

$$\frac{\partial U}{\partial t} = \left(\dot{Q}_{\mathrm{x,in}} - \dot{Q}_{\mathrm{x,out}}\right) + \left(\dot{Q}_{\mathrm{y,in}} - \dot{Q}_{\mathrm{y,out}}\right) + \dot{\Phi}''' \cdot V$$

The change of the heat flows and the source term are on the right side of the energy balance. **(Cause)**

The temporal change of the internal energy is on the left side of the energy balance **(Effect)**



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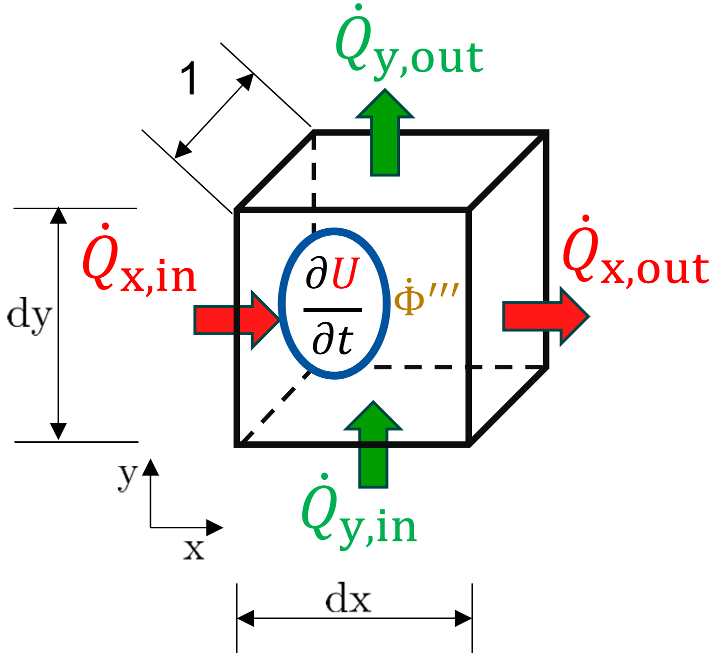
Definition of all terms results in a diff. equation:

$$\frac{\partial U}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \dot{\Phi}'''$$

$$\rho \cdot c_v \cdot \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \dot{\Phi}'''$$

$$\Rightarrow \frac{\partial T}{\partial t} = \frac{1}{\rho \cdot c_v} \left[\lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \dot{\Phi}'''\right]$$



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3-D transient energy conservation equation with sources

Energy balance:

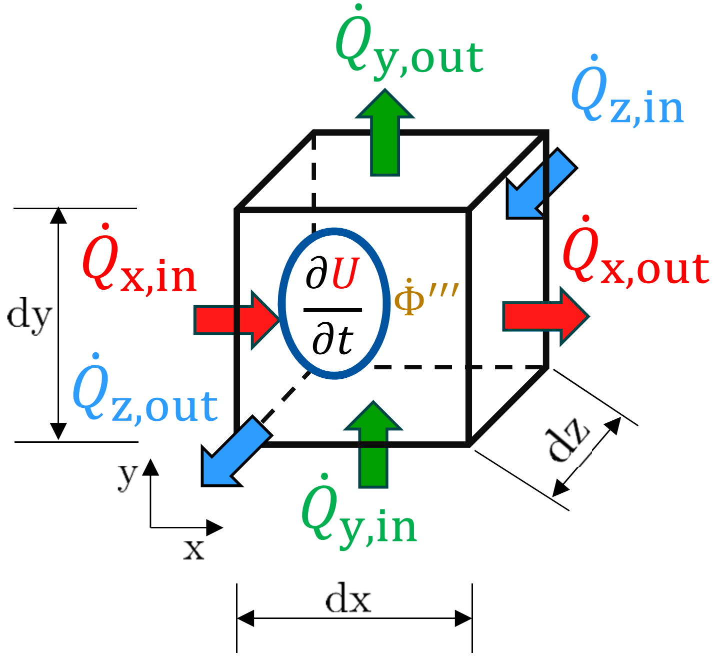
$$\frac{\partial U}{\partial t} = \left(\dot{Q}_{\mathrm{x,in}} - \dot{Q}_{\mathrm{x,out}}\right) + \left(\dot{Q}_{\mathrm{y,in}} - \dot{Q}_{\mathrm{y,out}}\right) + \left(\dot{Q}_{\mathrm{z,in}} - \dot{Q}_{\mathrm{z,out}}\right) + \dot{\Phi}''' \cdot V$$

Resulting 3-D temperature profile:

$$\frac{\partial \mathrm{U}}{\partial \mathrm{t}} = \lambda\left(\frac{\partial^2 \mathrm{T}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{y}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{z}^2}\right) + \dot{\Phi}'''$$

$$\rho \cdot c_v \cdot \frac{\partial \mathrm{T}}{\partial \mathrm{t}} = \lambda\left(\frac{\partial^2 \mathrm{T}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{y}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{z}^2}\right) + \dot{\Phi}'''$$

$$\Rightarrow \frac{\partial \mathrm{T}}{\partial \mathrm{t}} = \frac{1}{\rho \cdot c_v}\left[\lambda\left(\frac{\partial^2 \mathrm{T}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{y}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{z}^2}\right) + \dot{\Phi}'''\right]$$



Overview of the differential equations of all cases

1-D steady state without sources:

$$0 = -\lambda \cdot A \frac{\partial^2 T}{\partial x^2}$$

2-D steady state without sources (Laplace):

$$0 = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

2-D steady state with sources (Poisson):

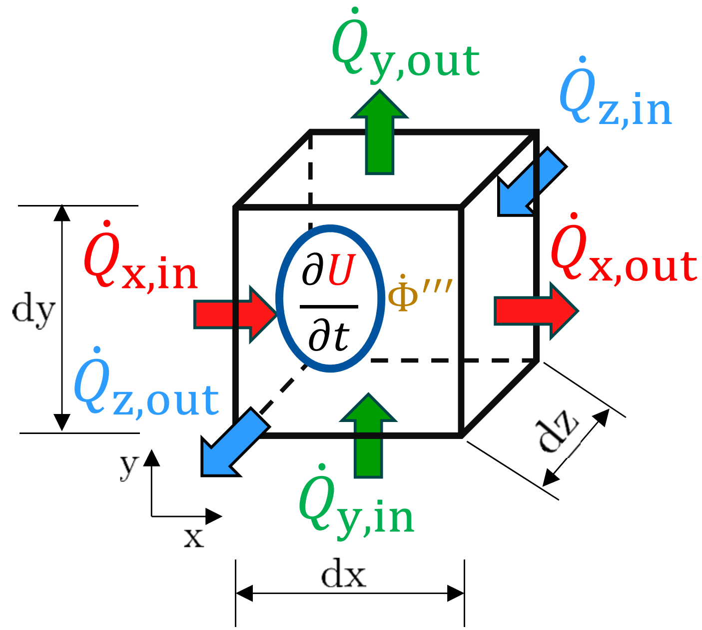
$$0 = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \dot{\Phi}'''$$

2-D transient with sources:

$$\Rightarrow \frac{\partial T}{\partial t} = \frac{1}{\rho \cdot c_v} \left[\lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \dot{\Phi}''' \right]$$

3-D transient with sources (general diff. eq.):

$$\Rightarrow \frac{\partial T}{\partial t} = \frac{1}{\rho \cdot c_v} \left[\lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \dot{\Phi}''' \right]$$



Comprehension questions

What is the steady state temperature profile for a homogeneous, one-dimensional, flat wall without heat sources?

Under which conditions does Poisson's equation become Laplace's equation (heat conduction)?