Formulary Heat Transfer: Conduction

Version 1 from 2021

from 7th June 2021

Dimensionless numbers

$$\mathrm{Bi}_L = \frac{\alpha L}{\lambda} \qquad \text{(Biot number)}$$

$$\mathrm{Fo} = \frac{at}{L^2} \quad \text{with} \quad a = \frac{\lambda}{\rho\, c_\mathrm{p}} \qquad \text{(Fourier number)}$$

Heat conduction

$$\dot{q}'' = -\lambda \frac{\partial T}{\partial x} \qquad \text{(Fourier's law)}$$

Heat transport equation

- Cartesian coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right)\right] + \dot{\Phi}'''$$

- Cylindrical coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right)\right] + \dot{\Phi}'''$$

- Spherical coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\lambda \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \Phi}\left(\lambda \frac{\partial T}{\partial \Phi}\right)\right] + \dot{\Phi}'''$$

Steady state heat conduction in walls without heat sources

$$R = \frac{T_\mathrm{A} - T_\mathrm{B}}{\dot{Q}} \qquad \text{where} \qquad R = \sum_i R_i \qquad \text{(Heat resistance)}$$

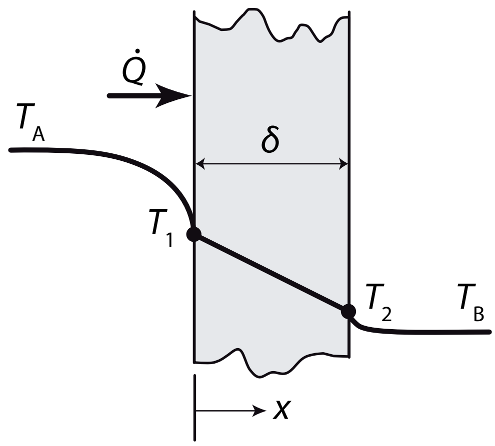
- Plane wall

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 0 \qquad \text{with BC} \qquad \begin{array}{l} T(x=0) = T_1 \\ T(x=\delta) = T_2 \end{array}$$

$$T = T_1 + \frac{T_2 - T_1}{\delta} x \qquad \text{(Temperature profile)}$$

$$\dot{Q} = -\lambda A \frac{\mathrm{d}T}{\mathrm{d}x} = \lambda A \frac{T_1 - T_2}{\delta} \qquad \text{(Heat flow rate)}$$

$$R = \frac{\delta}{\lambda A} \qquad \text{(Heat resistance)}$$



- Wall consisting of n layers

$$\dot{Q} = \lambda_1 \frac{A}{\delta_1} (T_1 - T_2) = \lambda_2 \frac{A}{\delta_2} (T_2 - T_3) = \cdots = \lambda_n \frac{A}{\delta_n} (T_n - T_{n+1})$$

$$\dot{Q} = \frac{A}{\sum_{i=1}^{n} \frac{\delta_i}{\lambda_i}} (T_1 - T_{n+1}) \qquad \text{(Without conv. heat transfer)}$$

$$\dot{Q} = \frac{A}{\frac{1}{\alpha_A} + \sum_{i=1}^{n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_B}} (T_A - T_B) \qquad \text{(With conv. heat transfer)}$$

- Thick-walled tube

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}T}{\mathrm{d}r}\right) = 0 \qquad \text{with BC} \qquad \begin{array}{l} T(r=r_1) = T_1 \\ T(r=r_2) = T_2 \end{array}$$

$$
\begin{aligned}
T &= T_1 + \ln\left(\frac{r}{r_1}\right)\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} && \text{(Temperature profile)} \\
&= T_2 + \ln\left(\frac{r}{r_2}\right)\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}
\end{aligned}
$$

$$
\dot{Q} = 2\pi\lambda L \frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)} \qquad \text{(Heat flow)}
$$

$$
R = \frac{1}{2\pi\lambda L}\,\ln\frac{r_2}{r_1} \quad \text{mit} \quad r_2 > r_1 \qquad \text{(Heat resistance)}
$$

- Thick-walled tube consisting of n layers

$$
\dot{Q} = 2\pi r L\left(-\lambda_i \frac{\mathrm{d}T}{\mathrm{d}r}\right) = const.
$$

$$
\dot{Q} = \frac{T_1 - T_{n+1}}{\frac{1}{2\pi L}\sum_{i=1}^{n}\frac{1}{\lambda_i}\ln\frac{r_{i+1}}{r_i}} \qquad \text{(Without conv. heat transfer)}
$$

$$
\dot{Q} = \frac{2\pi L}{\frac{1}{\alpha_A r_1} + \sum_{i=1}^{n}\frac{1}{\lambda_i}\ln\frac{r_{i+1}}{r_i} + \frac{1}{\alpha_B r_{n+1}}}\,(T_A - T_B) \qquad \text{(With conv. heat transfer)}
$$

Fins

$$
\theta = T - T_{\mathrm{a}} \qquad \text{(Temperature difference)}
$$

$$
\eta_{\mathrm{F}} = \frac{\dot{Q}_{\mathrm{F}}}{\dot{Q}_{\mathrm{max}}} = \frac{\dot{Q}_{\mathrm{F}}}{A_0\,\alpha\,\theta_{\mathrm{b}}} = \frac{\text{transferred heat}}{\text{maximum transferable heat}} \qquad \text{(Efficiency of the fin)}
$$

here: A_0 Heat transferring surface

θ_{b} Fin base temperature

Rod fins and plane fins

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \underbrace{\frac{\alpha U}{\lambda A_c}}_{=m^2}\theta = 0 \quad \text{with} \quad \begin{array}{ll} \text{BC1:} & \theta(x=0) = \theta_\mathrm{b} \\ \text{BC2:} & \text{may vary, see the following:} \end{array}$$

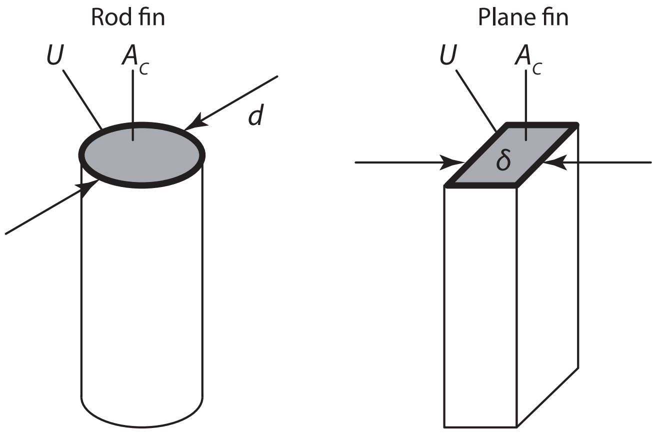
(Differential equation for fins)

$$\theta(x) = A\,\cosh(mx) + B\,\sinh(mx) \qquad \text{(Method of solution)}$$

$$\dots = C\,\exp(mx) + D\,\exp(-mx)$$

$$m = \sqrt{\frac{\alpha U}{\lambda A_c}} = \sqrt{\frac{4\alpha}{\lambda d}} \qquad \text{(Rod fin)}$$

$$m = \sqrt{\frac{\alpha U}{\lambda A_c}} = \sqrt{\frac{2\alpha}{\lambda \delta}} \qquad \text{(Plane fin)}$$



Boundary condition 2:

- Fins with adiabatic head:

$$\text{BC2:} \quad -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{x=L} = 0$$

$$\theta = \theta_\mathrm{b} \frac{\cosh\left[m\left(L-x\right)\right]}{\cosh\left[mL\right]} \qquad \text{(Temperature profile)}$$

$$\dot{Q} = \lambda\, A_\mathrm{c}\, m\, \theta_\mathrm{b}\, \tanh\left(mL\right) \qquad \text{(Heat flow through the fin)}$$

$$\eta = \frac{\tanh(mL)}{mL} \qquad \text{(Efficiency of the fin)}$$

- Fins with head at ambient temperature (long fins):

$$\text{BC2:} \quad \theta(x = L) = 0$$

- Fins transferring heat at the fin head:

$$\text{BC2:} \quad -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{x=L} = \alpha\, \theta(x = L)$$

One-dimensional, unsteady state heat conduction

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c}\frac{\partial^2 T}{\partial x^2} \qquad \text{(Differential equation)}$$

$$\frac{\partial \theta^*}{\partial t} = a\,\frac{\partial^2 \theta^*}{\partial x^2} \qquad \text{with} \qquad \theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0}$$

- Semi-infinite plate with negligible heat transfer resistance:

$$\mathrm{Bi} = \frac{\alpha\, L}{\lambda} \gg 1$$

$$\left.\begin{array}{r} t = 0 \\ 0 < x < \infty \end{array}\right\} \quad T = T_0 \qquad \theta^* = 0 \qquad \text{(BC1)}$$

$$\left.\begin{array}{r} t > 0 \\ x = 0 \end{array}\right\} \quad T = T_\mathrm{a} \qquad \theta^* = 1 \qquad \text{(BC2)}$$

$$\left.\begin{array}{r} t > 0 \\ x \to \infty \end{array}\right\} \quad T = T_0 \qquad \theta^* = 0 \qquad \text{(BC3)}$$

$$\theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}\right) \quad \text{with} \quad \mathrm{Fo} = \frac{at}{x^2} \qquad \text{(Temperature profile)}$$

$$\left.\dot{q}''\right|_{x=0} = \sqrt{\frac{\lambda c \rho}{\pi t}}\,(T_\mathrm{a} - T_0) \qquad \text{(Heat flux)}$$

$$\delta(t) \approx 3{,}6\,\sqrt{at} \qquad \text{(Temperature penetration depth)}$$

- Semi-infinite plate, **non** negligible heat transfer resistance:

$$\left.\begin{array}{r} t>0 \\ x=0 \end{array}\right\} \quad \alpha\left(T_{\mathrm{a}}-T(x=0)\right)=-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} \qquad \text{(BC1)}$$

$$\theta^{*}=\frac{T-T_{0}}{T_{\mathrm{a}}-T_{0}}=1-\operatorname{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}\right)\cdots \qquad \text{(Temperature profile)}$$

$$\cdots-\left[\exp\left(\mathrm{Bi}_{x}+\mathrm{Fo}\,\mathrm{Bi}_{x}^{2}\right)\right]\left[1-\operatorname{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}+\sqrt{\mathrm{Fo}}\,\mathrm{Bi}_{x}\right)\right]$$

$$\text{with} \quad \mathrm{Bi}_{x}=\frac{\alpha x}{\lambda}$$

$$\mathrm{Fo}=\frac{a t}{x^{2}}$$

- Semi-infinite plate, periodically changing surface temperature:

$$\left.\begin{array}{r} t>0 \\ x=0 \end{array}\right\} \quad T\,(x=0)=T_{\mathrm{m}}+\left(T_{\max}-T_{\mathrm{m}}\right)\,\cos\left(2\pi\,t/\tau\right) \qquad \text{(BC1)}$$

$$\theta^{*}=\frac{T-T_{\mathrm{m}}}{T_{\max}-T_{\mathrm{m}}}=\exp\left(-\sqrt{\frac{\pi x^{2}}{a\tau}}\right)\,\cos\left(\frac{2\pi}{\tau}t-\sqrt{\frac{\pi x^{2}}{a\tau}}\right) \qquad \text{(Temperature profile)}$$

One-dimensional, unsteady heat conduction in simple bodies

$$\frac{T_{\mathrm{m}}-T_{\mathrm{a}}}{T_{0}-T_{\mathrm{a}}} \qquad \text{(Dimensionless temperature in the middle of a body)}$$

$$\frac{T-T_{\mathrm{a}}}{T_{\mathrm{m}}-T_{\mathrm{a}}} \qquad \text{(Dimensionless temperature at position } x \text{ or } r\text{)}$$

$$\frac{Q}{Q_{0}} \quad \text{mit} \quad Q_{0}=m\,c\,\left(T_{0}-T_{\mathrm{a}}\right) \qquad \text{(Dimensionless heat loss)}$$

Determination of temperature profile and heat flow for unsteady conditions

$\rightarrow$ Figures 3 - 11

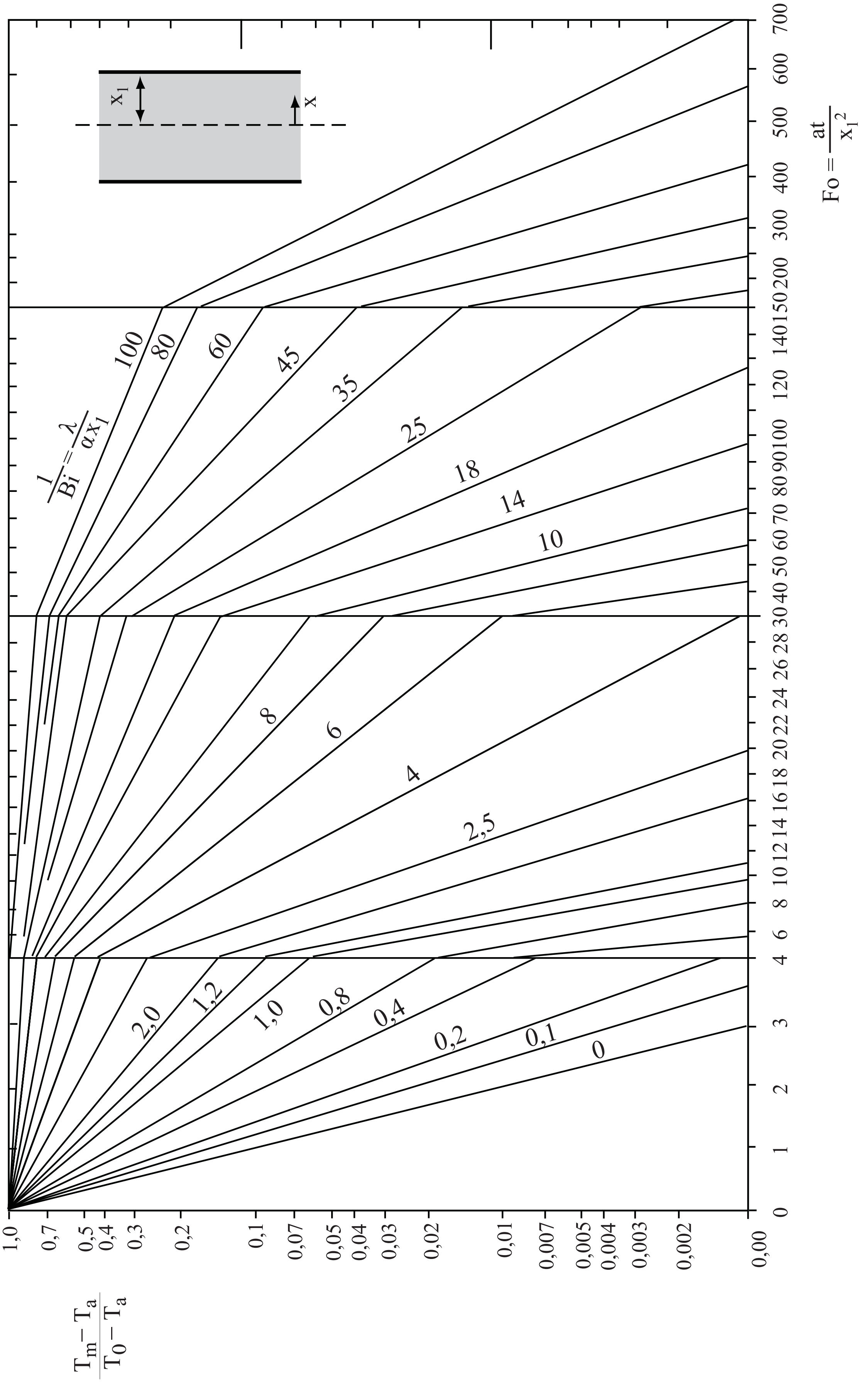


Diagramm 1: Mid-plane temperature of a plate with thickness $2x_1$

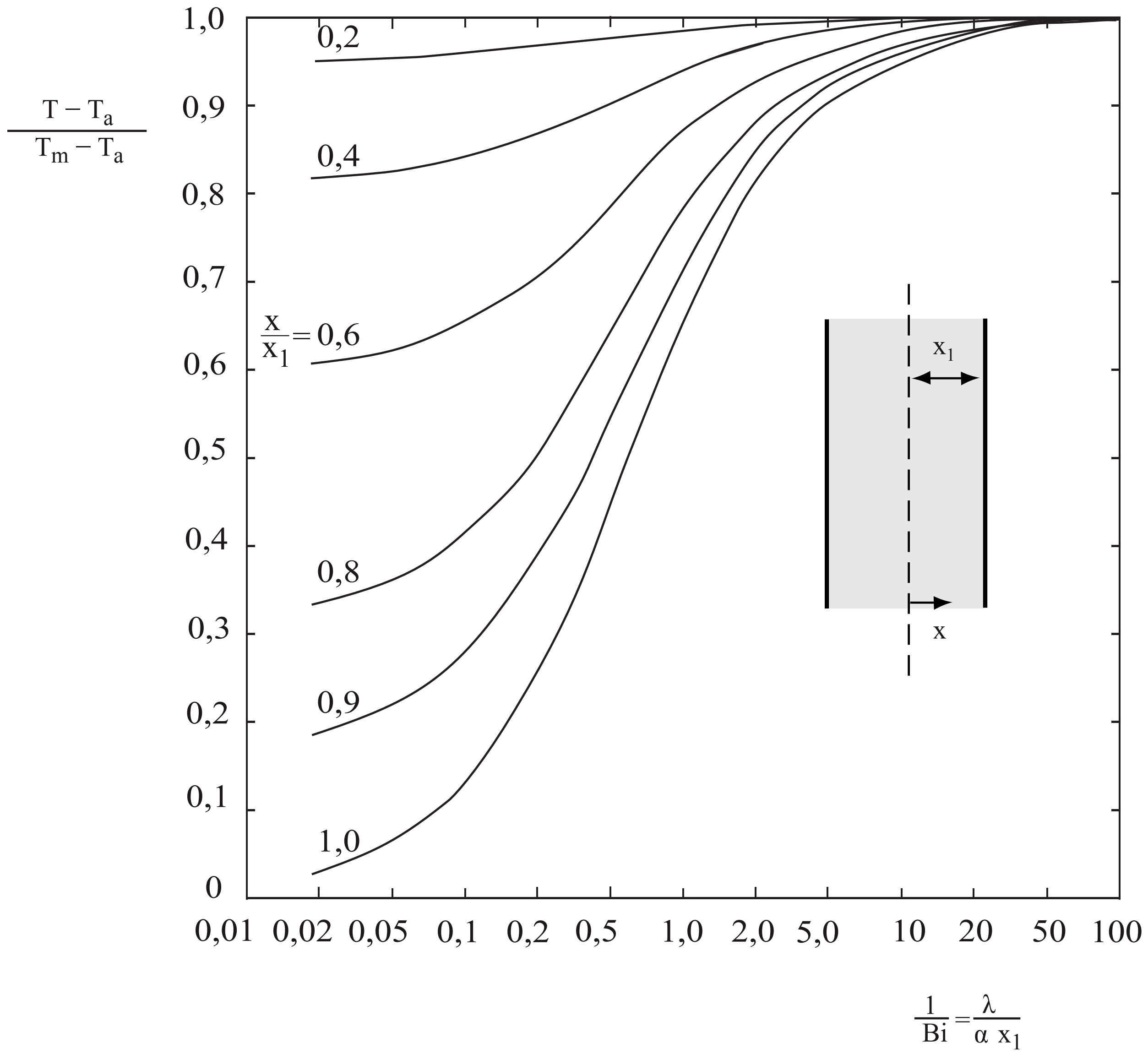


Diagramm 2: Temperature distribution in a plate (valid for Fo > 0,2)

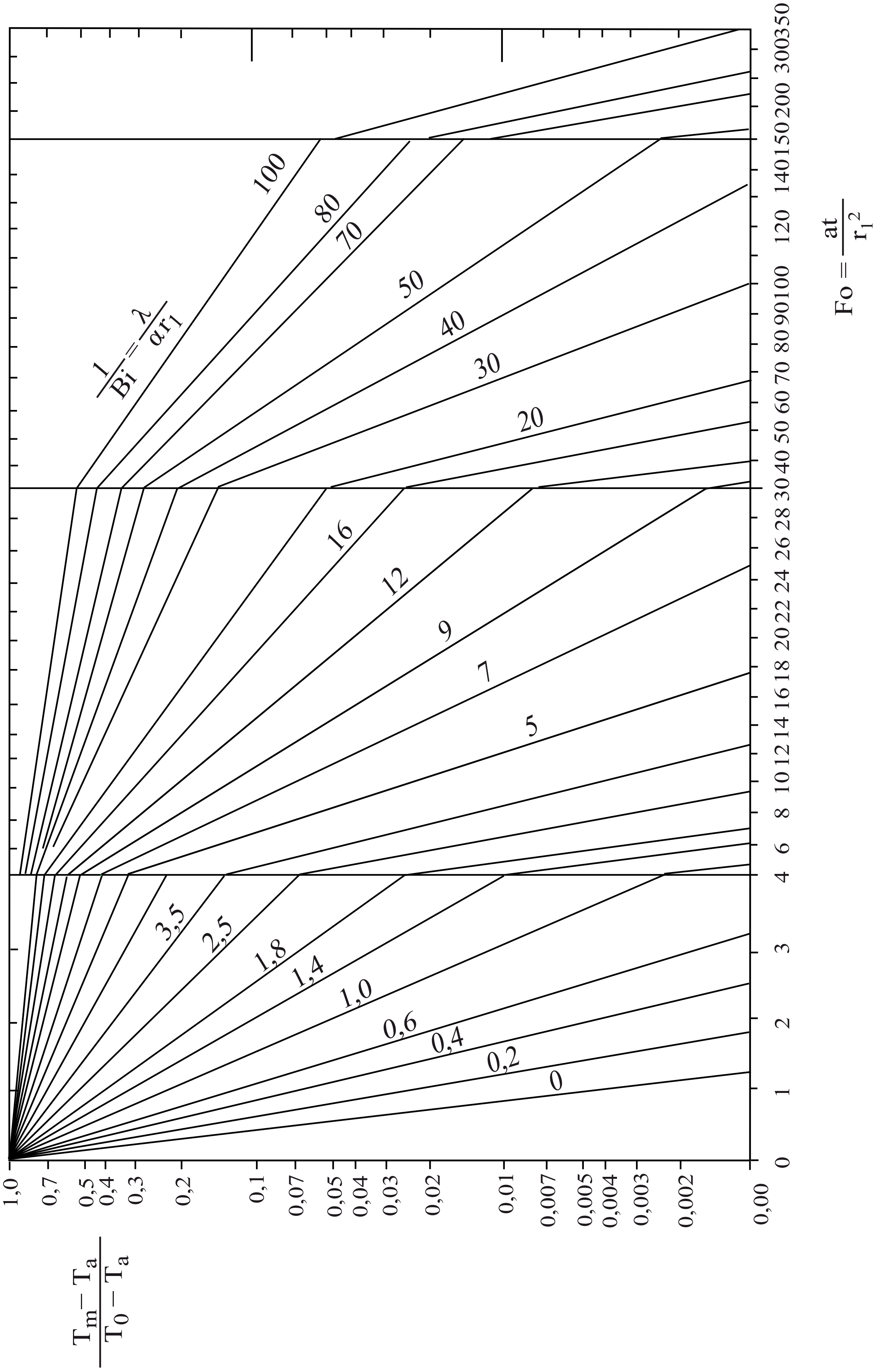


Diagramm 3: Temperature along the axis of a cylinder with radius r_1

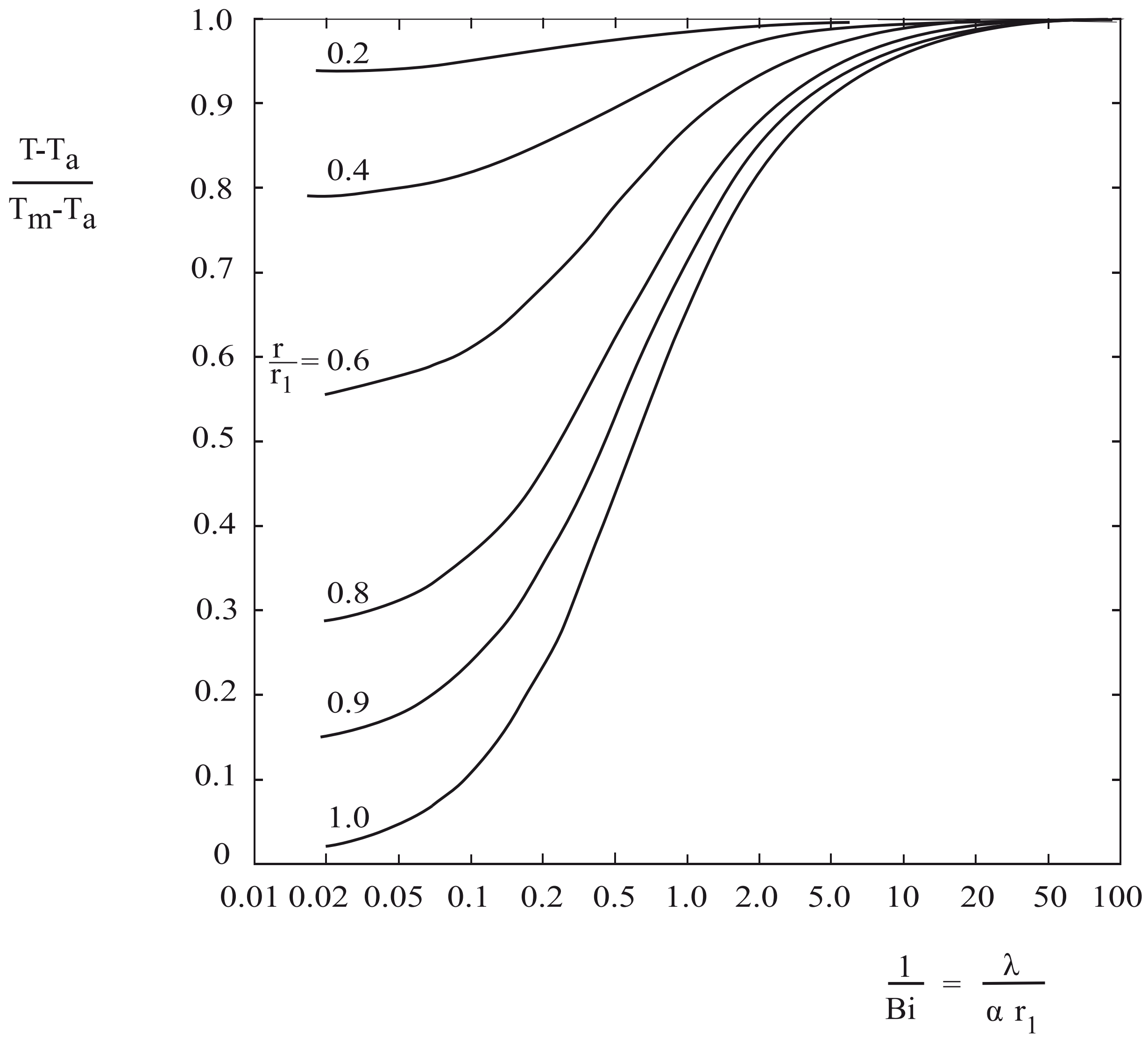


Diagramm 4: Temperature distribution in a cylinder (valid for Fo > 0,2)

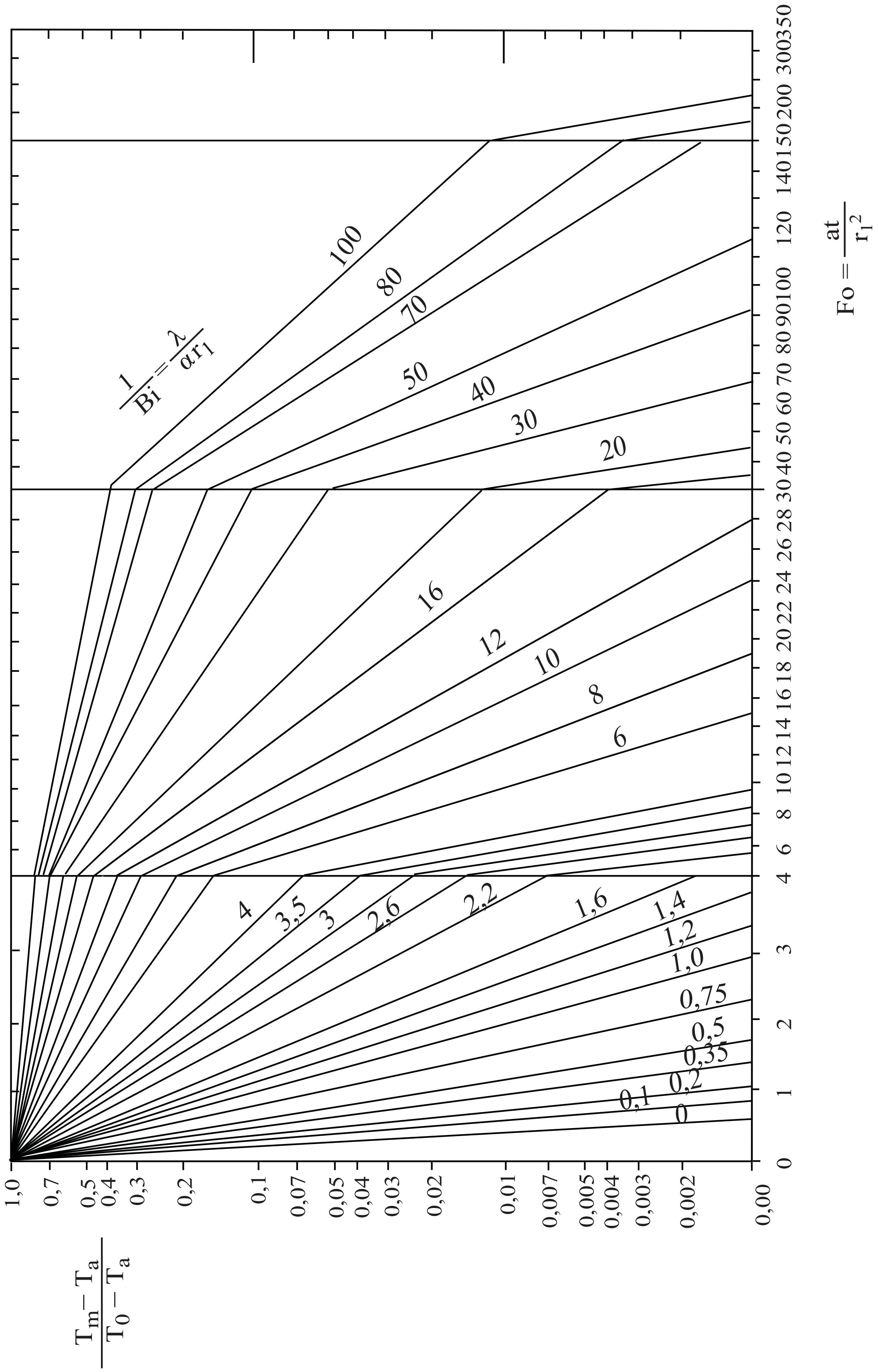


Diagramm 5: Temperature in the centre of a sphere with radius r_1

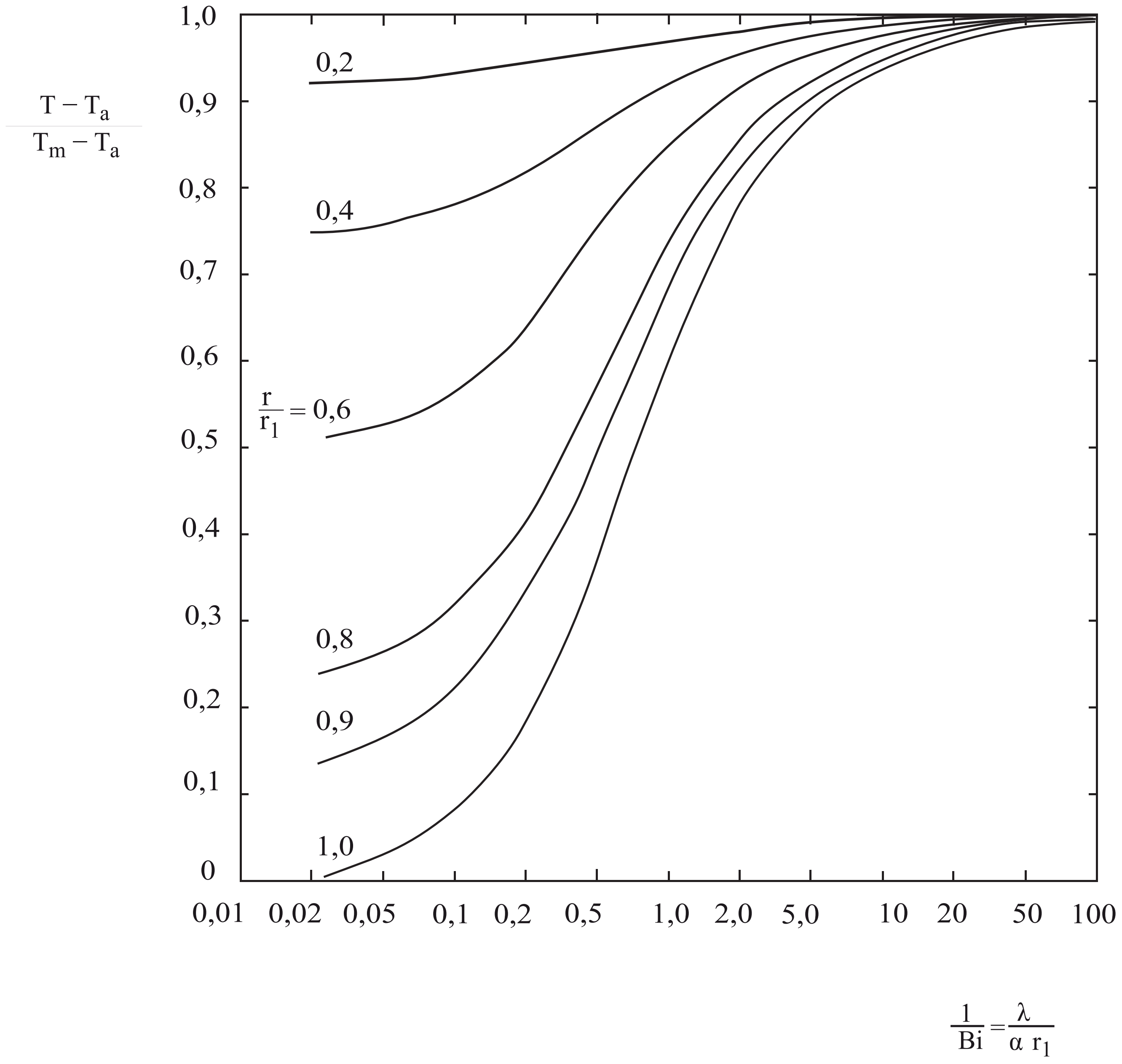


Diagramm 6: Temperature distribution in a sphere (valid for Fo > 0,2)

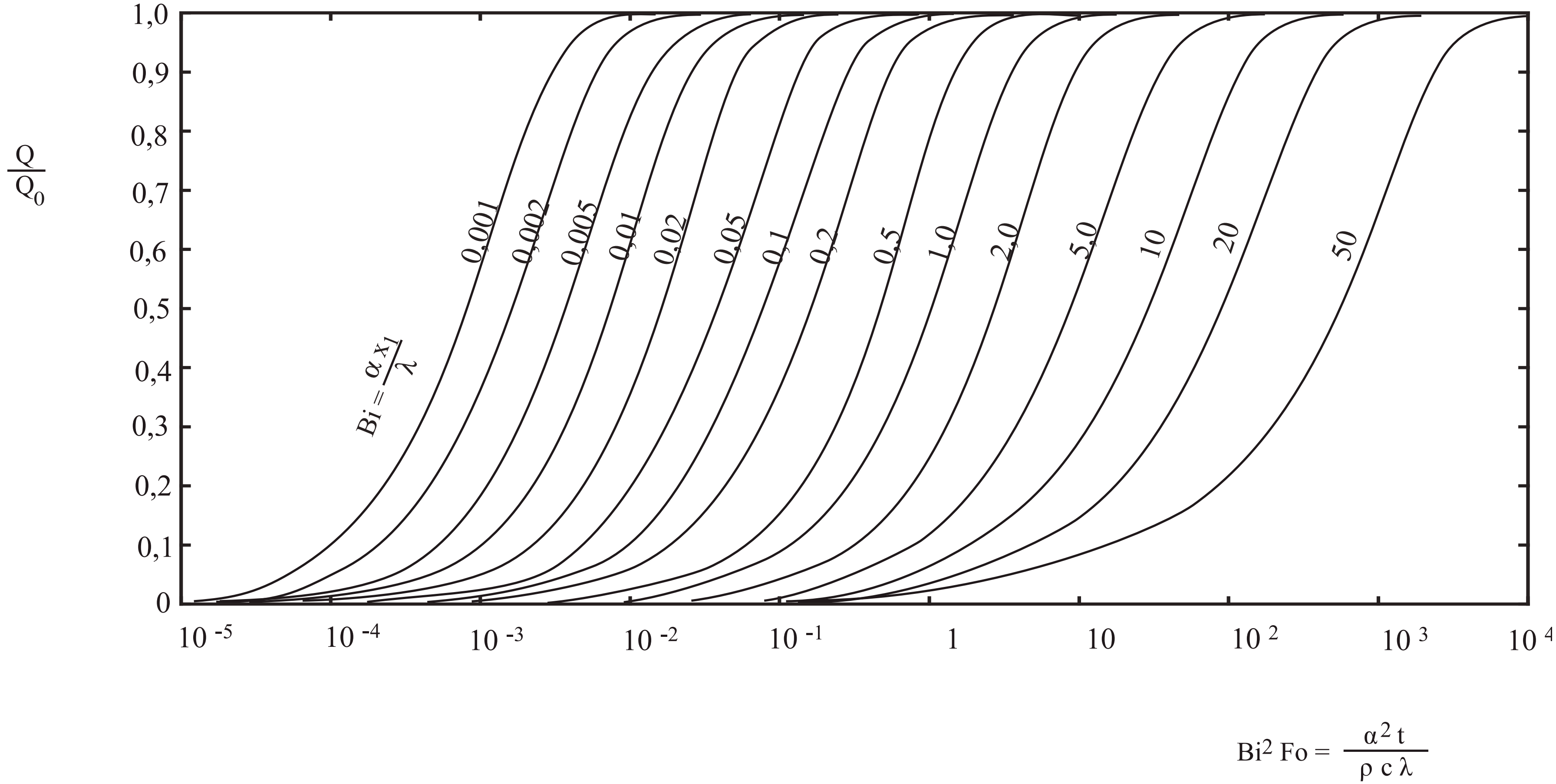


Diagramm 7: Heat loss of a plate

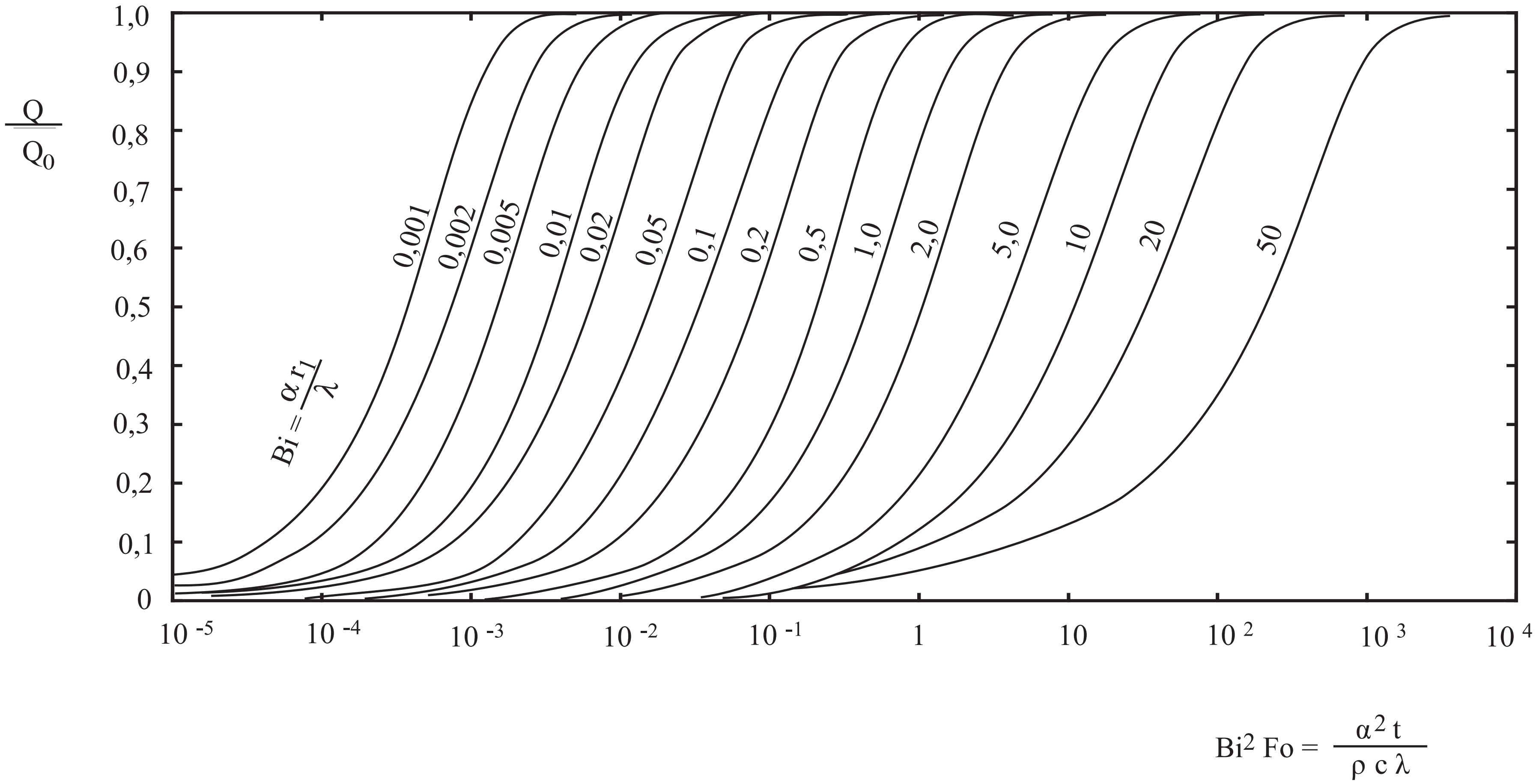


Diagramm 8: Heat loss of a cylinder

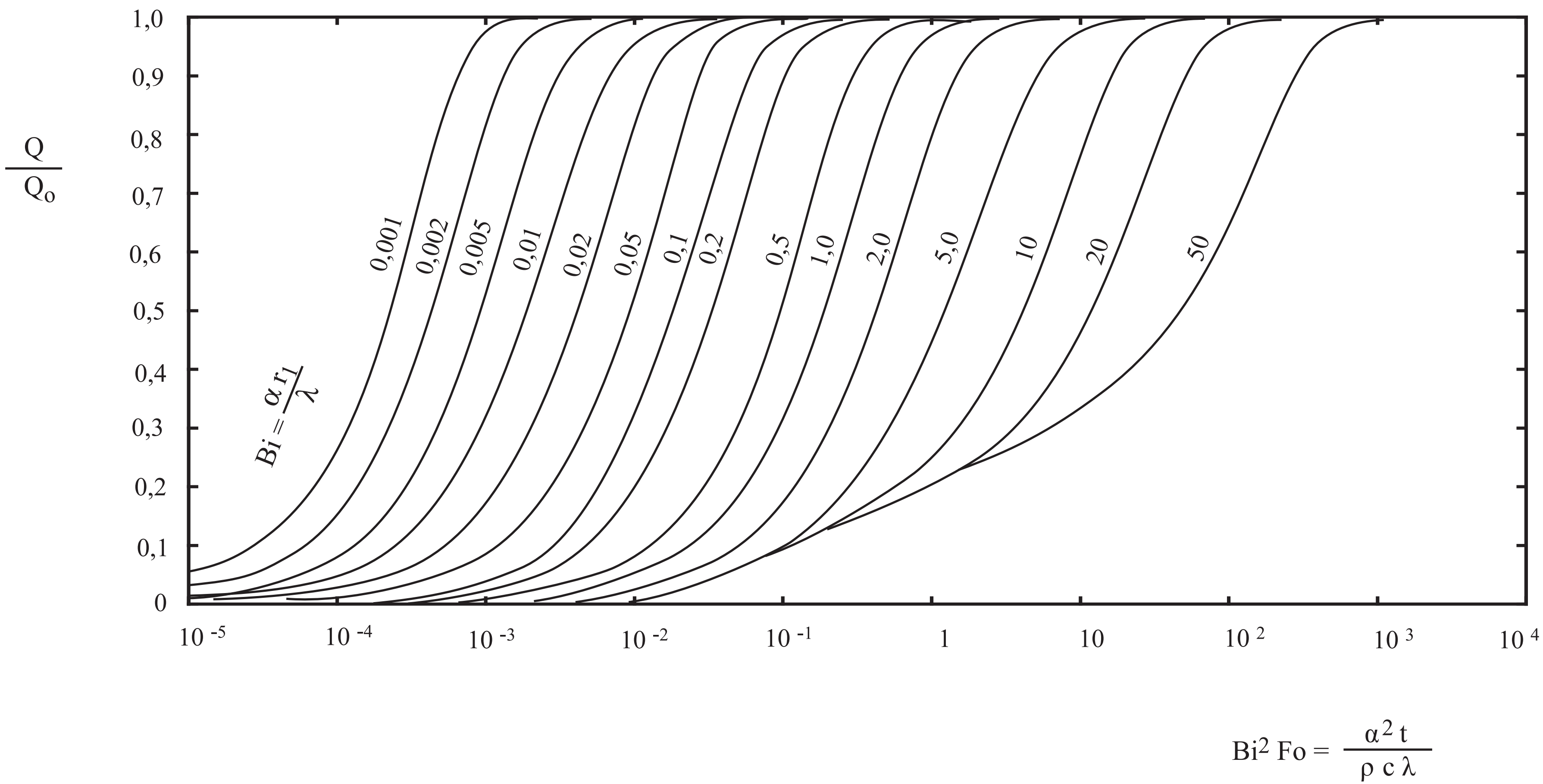


Diagramm 9: Heat loss of a sphere

Appendix A – Properties of various materials

Tabelle 1: Metals at 20°C

	ρ	c	λ	a
	10^3 kg/m^3	kJ/kg K	W/m K	10^{-6} m^2/s
Aluminum	2,70	0,888	237	98,80
Lead	11,34	0,129	35	23,90
Chromium	6,92	0,440	91	29,90
Iron	7,86	0,452	81	22,80
Gold	19,26	0,129	316	127,20
Copper	8,93	0,382	399	117,00
Magnesium	1,74	1,020	156	87,90
Manganese	7,42	0,473	21	6,00
Molybdenum	10,20	0,251	138	53,90
Sodium	9,71	1,220	133	11,20
Nickel	8,85	0,448	91	23,00
Platinum	21,37	0,133	71	25,00
Silver	10,50	0,235	427	173,00
Titanium	4,50	0,522	22	9,40
Wolfram	19,00	0,134	173	67,90
Zinc	7,10	0,387	121	44,00
Tin, white	7,29	0,225	67	40,80
Bronze	8,80	0,377	62	18,70
Cast iron	7,80	0,540	42…50	10…12
Carbon steel (<0,4% C)	7,85	0,465	42…50	12…15
Cr-Ni-steel (X12CrNi 18,8)	7,80	0,500	15	3,80

Tabelle 2: Non-metal solids at 20°C

	ρ	c	λ	a
	10^3 kg/m^3	kJ/kg K	W/m K	10^{-6} m^2/s
Acryl glass	1,18	1,44	0,184	0,108
Asphalt	2,12	0,92	0,7	0,36
Concrete	2,1	0,88	1	0,54
Ice (water 0ï¿½C)	0,917	2,04	2,25	1,203
Soil coarse gravel	2,04	1,84	0,52	0,14
Sand, dry	1,65	0,8	0,27	0,2
Sand, wet	1,75	1	0,58	0,33
Clay	1,45	0,88	1,28	1
Glass.				
window	2,48	0,7	0,87	0,5
mirror	2,7	0,8	0,76	0,35
quarz	2,21	0,73	1,4	0,87
Glass wool	1,2	0,66	0,046	0,58
Gypsum	1	1,09	0,51	0,47
Granite	2,75	0,89	2,9	1,18
Cork	0,19	1,88	0,041	0,115
Marble	2,6	0,8	2,8	1,35
Mortar	1,9	0,8	0,93	0,61
Paper	0,7	1,2	0,12	0,14
Polyethylene	0,92	2,3	0,35	0,17
Polytetrafluorethylene	2,2	1,04	0,23	0,1
PVC	1,38	0,96	0,15	0,11
Porcelain (95ï¿½C)	2,4	1,08	1,03	0,4
Hard coal	1,35	1,26	0,26	0,15
Fir wood (radial)	0,415	2,72	0,14	0,12
Plaster	1,69	0,8	0,79	0,58
Bricks	1,6…1,8	0,84	0,38…0,52	0,28…0,34

Tabelle 3: Liquids at 1 bar

	T	ρ	c	λ	ν	a	Pr
	°C	10^3 kg/m^3	kJ/kg K	W/m K	10^{-6} m^2/s	10^{-6} m^2/s	1
Nitrogen	-190	0,861	1,988	0,161	0,321	0,0939	3,42
Water	0	0,9998	4,218	0,561	1,793	0,133	13,48
	20	0,9982	4,181	0,598	1,004	0,1434	7,001
	40	0,9922	4,177	0,631	0,658	0,1521	4,3280
	60	0,9832	4,184	0,654	0,475	0,1591	2,983
	80	0,9718	4,197	0,67	0,365	0,1643	2,221
	99,63	0,9586	4,216	0,679	0,295	0,168	1,757
Aqueous non-organic solution							
21% NaCl	-10	1,187	3,312	0,528	4,02	0,136	29,5
Benzene	20	0,879	1,738	0,154	0,74	0,101	7,33
Methanol	20	0,792	2,495	0,22	0,737	0,111	6,57
Fuel oil	20	0,819	2	0,116	1,82	0,0709	25,7
	100	0,766	2,38	0,104	0,711	0,0572	12,4
Mercury	20	13,55	0,139	9,3	0,115	4,9	0,023

Tabelle 4: Gases at 1 bar

	T	ρ	c	λ	ν	a	Pr
	°C	kg/m^3	kJ/kg K	10^{-3} W/m K	10^{-6} m^2/s	10^{-6} m^2/s	1
Air	-200	5,106	1,186	6,886	0,979	1,137	0,8606
	-100	2,019	1,011	16,2	5,829	7,851	0,7423
	0	1,275	1,006	24,18	13,52	18,83	0,7179
	20	1,188	1,007	25,69	15,35	21,47	0,7148
	40	1,112	1,007	27,16	17,26	24,24	0,7122
	80	0,9859	1,01	30,01	21,35	30,14	0,7083
	100	0,9329	1,012	31,39	23,51	33,26	0,707
	200	0,7356	1,026	37,95	35,47	50,3	0,7051
	400	0,517	1,069	49,96	64,51	90,38	0,7137
	600	0,3986	1,116	61,14	99,63	137,5	0,7247
	800	0,3243	1,155	71,54	140,2	191	0,7342
	1000	0,2734	1,185	80,77	185,9	249,2	0,7458
Steam	100	0,5896	2,042	25,08	20,81	20,83	0,999
	200	0,4604	1,975	33,28	35,14	36,6	0,96
	400	0,3223	2,07	54,76	75,86	82,07	0,9243
	600	0,2483	2,203	79,89	131,4	146,1	0,8993
	800	0,2019	2,343	107,3	199,9	226,8	0,8816
	1000	0,1702	2,478	163,3	280	323,2	0,8665
Hydrogen	0	0,0886	14,24	176	95	139	0,68
	50	0,0748	14,36	202	126	188	0,67
	100	0,0649	14,44	229	159	244	0,65
Carbon dioxide	0	1,95	0,829	14,3	7,1	8,86	0,8
	50	1,648	0,875	17,8	9,8	12,3	0,8
	100	1,428	0,925	21,3	12,4	16,1	0,8
Helium	27	0,1625	5,193	155,7	122,6	184,5	0,655