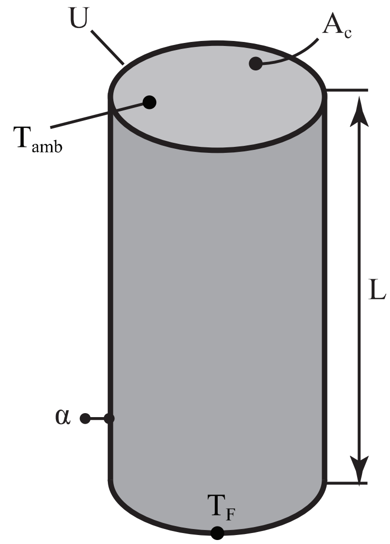
Exercise II.7: (Pin-fin cooling on gas turbine blades ⋆⋆)

A rod fin is used for pin-fin cooling on gas turbine blades. The rod-fin of length L, with a tip temperature equal to the ambient T_{amb}, is given.



Given parameters:

- Fin geometry: U, A_{c}, L
- Fin material properties: λ
- Surface heat transfer coefficient: α
- Fin base temperature and environment temperature: $T_{\mathrm{F}}, T_{\mathrm{amb}}$

Tasks:

a) Derive the heat conduction equation for the given problem.

b) Derive the function of the temperature profile inside the fin.

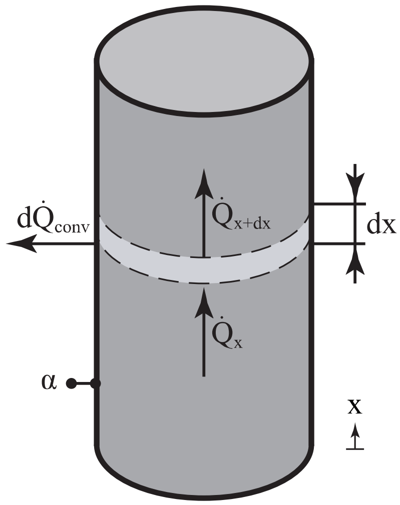
c) Give the expression for the rate of heat loss in terms of the given parameters.

Solution II.7: (Pin-fin cooling on gas turbine blades ⋆⋆)

Task a)

1 Setting up the balance:

Before the calculations, the system must be understood. In this particular scenario, a fin conducts heat through the body and dissipates heat through convection to the surrounding environment.



The heat conduction equation inside the fin is derived from the energy balance of an infinitesimal element within the fin. This balance reads:

$$0 = \underbrace{\dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}}_{\text{Net rate of diffusion}} - \underbrace{d\dot{Q}_{\mathrm{conv}}}_{\text{Convective losses}}. \tag{II.7.1}$$

2 Defining the elements within the balance:

The ingoing rate of heat transfer is derived from Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}. \tag{II.7.2}$$

For an infinitesimal element, the outgoing conductive rate of heat transfer is approximated by the use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x+dx}}}{\partial x} \cdot dx \\ &= -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}\right) \cdot dx. \end{aligned} \tag{II.7.3}$$

Lastly, the rate of heat being lost by convection from the infinitesimal element is written as:

$$\begin{aligned} d\dot{Q}_{\mathrm{conv}} &= \alpha A_{\mathrm{s}} \left(T(x) - T_{\mathrm{amb}}\right) \\ &= \alpha U dx \left(T(x) - T_{\mathrm{amb}}\right). \end{aligned} \tag{II.7.4}$$

Conclusion

3 Inserting and rearranging:

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\alpha U}{\lambda A_c} \cdot \left(T(x) - T_{amb}\right). \tag{II.7.5}$$

Task b)

4 Defining the boundary and/or initial conditions:

The differential equation is solved with the use of two boundary conditions. This appears from the fact that the temperature has undergone two differentiations with respect to x. The temperature at the fin base is:

$$T(x = 0) = T_F, \tag{II.7.6}$$

and he temperature at the fin head is:

$$T(x = L) = T_{amb}. \tag{II.7.7}$$

5 Solving the equation:

To solve the 2^{nd} order differential equation, the equation must be homogenized. The differential equation for a steady-state fin is solved by introducing the temperature difference θ:

$$\theta = T(x) - T_{amb}, \tag{II.7.8}$$

and the fin parameter m:

$$m^2 = \frac{\alpha U}{\lambda A_c}. \tag{II.7.9}$$

Substitution into the differential equation, given in equation (II.7.5), yields:

$$0 = \frac{\partial^2 \theta}{\partial x^2} - m^2\theta. \tag{II.7.10}$$

The boundary conditions must be rewritten in terms of the temperature difference θ:

$$\theta(x = 0) = T_F - T_{amb} = \theta_F, \tag{II.7.11}$$

and:

$$\theta(x = L) = T_{amb} - T_{amb} = 0. \tag{II.7.12}$$

The standard solution for this homogenized differential equation is given as:

$$\theta(x) = A \cdot \cosh(m \cdot x) + B \cdot \sinh(m \cdot x). \tag{II.7.13}$$

The constants A and B are determined by using the boundary conditions. Whereas, constant A yields from:

$$\begin{aligned} \theta(x = 0) &= A \cdot \cosh(0) + B \cdot \sinh(0) = \theta_F \\ \Rightarrow A &= \theta_F, \end{aligned} \tag{II.7.14}$$

and constant B results from:

$$\begin{aligned} \theta(x = L) &= A \cdot \cosh(mL) + B \cdot \sinh(mL) = 0 \\ \Rightarrow B &= -A \cdot \frac{1}{\tanh(mL)}. \end{aligned} \tag{II.7.15}$$

Plugging the constants into the standard solution gives the equation for the fin temperature:

$$\theta(x) = \theta_F \cdot \left[\cosh(m \cdot x) - \frac{\sinh(m \cdot x)}{\tanh(mL)}\right]. \tag{II.7.16}$$

Conclusion

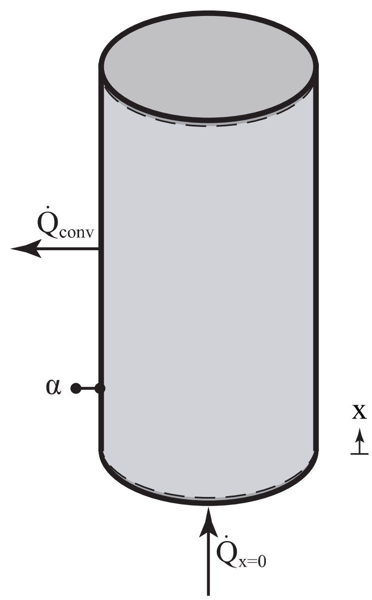
Substitution of temperature difference θ back and rewriting gives the temperature profile inside the fin:

$$T(x) = T_{amb} + (T_F - T_{amb}) \cdot \left[\cosh(m \cdot x) - \frac{\sinh(m \cdot x)}{\tanh(mL)}\right]. \tag{II.7.17}$$

Task c)

1 **Setting up the balance:**

To determine the total rate of heat loss, an energy balance over the entire fin is required. The fin conducts heat through the base at $x = 0$, which is lost by convection to the ambient.



Therefore the energy balance reads:

$$0 = \dot{Q}_{x=0} - \dot{Q}_{conv}. \tag{II.7.18}$$

2 **Defining the elements within the balance:**

The energy balance states that $\dot{Q}_{x=0} = \dot{Q}_{\text{conv}}$. Both are expressed by using Fourier's law:

$$\dot{Q}_{x=0} = \dot{Q}_{\text{conv}} = -\lambda A_{\text{c}} \cdot \left.\frac{\partial T}{\partial x}\right|_{x=0}, \tag{II.7.19}$$

where:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = (T_{\text{F}} - T_{\text{amb}}) \cdot \left(m \cdot \sinh(0) - \frac{m \cdot \cosh(0)}{\tanh(mL)}\right) = -\frac{(T_{\text{F}} - T_{\text{amb}}) \cdot m}{\tanh(mL)}. \tag{II.7.20}$$

Conclusion

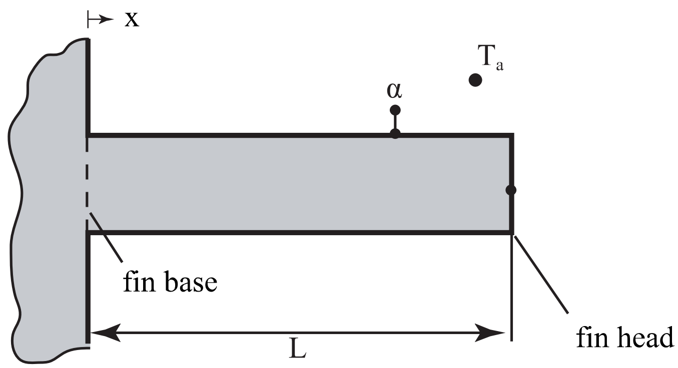
3 **Inserting and rearranging:**

$$\dot{Q}_{x=0} = \dot{Q}_{\text{conv}} = \lambda A_{\text{c}} \frac{(T_{\text{F}} - T_{\text{amb}}) \cdot m}{\tanh(mL)}, \tag{II.7.21}$$

where $m = \sqrt{\frac{\alpha U}{\lambda A_{\text{c}}}}$.

Exercise II.8: (New fin material ⋆⋆)

An electric motor manufacturer is using fins for cooling purposes. He is considering changing the material used for the fins from copper to aluminum. Because the length L of the fin is also modified, the temperature at the fin head remains identical for both materials. However, he does not understand the impact of such a change on the performance of cooling.



Given parameters:

- Thermal conductivity of copper: λ_C
- Thermal conductivity of aluminium: λ_A

Hints:

- The cross-section and the thickness remain unchanged.
- There is no change in the convective heat transfer coefficient.
- The temperature at the fin base does not change.
- For both fins, the heat flow through the head is negligible.

Tasks:

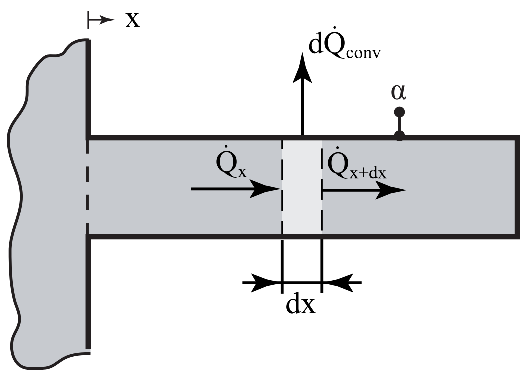
a) Determine the ratio between the heat flow of the aluminum and the copper fin in terms of given parameters.

Solution II.8: (New fin material ⋆⋆)

Task a)

1 Setting up the balance:

Before doing the calculations, the system must be understood. In this particular scenario, a fin conducts heat through the body and dissipates by convection to the surrounding environment.



The heat conduction equation inside the fin is derived based on the energy balance of an infinitesimal element within the fin. This balance reads:

$$0 = \underbrace{\dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}}}_{\text{Net rate of diffusion}} - \underbrace{d\dot{Q}_{\mathrm{conv}}}_{\text{Convective losses}}. \tag{II.8.1}$$

2 Defining the elements within the balance:

The ingoing rate of heat transfer derived from Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}. \tag{II.8.2}$$

For an infinitesimal element, the outgoing conductive rate of heat transfer is approximated by the use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x+dx}}}{\partial x} \cdot dx \\ &= -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}\right) \cdot dx. \end{aligned} \tag{II.8.3}$$

Lastly, the rate of heat being lost by convection from the infinitesimal element is written as:

$$\begin{aligned} d\dot{Q}_{\mathrm{conv}} &= \alpha A_{\mathrm{s}} \left(T(x) - T_{\mathrm{a}}\right) \\ &= \alpha U dx \left(T(x) - T_{\mathrm{a}}\right). \end{aligned} \tag{II.8.4}$$

3 Inserting and rearranging:

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\alpha U}{\lambda A_\mathrm{c}} \cdot \left(T(x) - T_\mathrm{a}\right), \tag{II.8.5}$$

which is the heat conduction equation.

4 Defining the boundary and/or initial conditions:

The temperature has been differentiated twice with respect to x. Therefore, the differential equation must be solved by use of two boundary conditions. The temperature at the fin base is stated as:

$$T(x = 0) = T_\mathrm{B}. \tag{II.8.6}$$

Heat flow through the head is negligible, and thus $\dot{Q}_{x=L} = -\lambda A_\mathrm{c} \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0$ gives:

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} = 0. \tag{II.8.7}$$

5 Solving the equation:

The 2$^\mathrm{nd}$ order differential equation, must be homogenized to solve the equation. The differential equation for a steady-state fin is solved by introducing the temperature difference θ:

$$\theta = T(x) - T_\mathrm{a}, \tag{II.8.8}$$

and the fin parameter m:

$$m^2 = \frac{\alpha U}{\lambda A_\mathrm{c}}. \tag{II.8.9}$$

Substitution into the differential equation, given in equation (II.8.5), yields:

$$0 = \frac{\partial^2 \theta}{\partial x^2} - m^2 \theta. \tag{II.8.10}$$

Furthermore, the boundary conditions must be rewritten in terms of the temperature difference θ:

$$\theta(x = 0) = T_\mathrm{B} - T_\mathrm{amb} = \theta_\mathrm{B}, \tag{II.8.11}$$

and:

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0. \tag{II.8.12}$$

The standard solution for this homogenized differential equation is given as:

$$\theta(x) = A \cdot \cosh\left(m \cdot x\right) + B \cdot \sinh\left(m \cdot x\right). \tag{II.8.13}$$

The constants A and B are determined by using the boundary conditions. Whereas, constant A is derived from:

$$\begin{aligned} \theta(x = 0) &= A \cdot \cosh\left(0\right) + B \cdot \sinh\left(0\right) = \theta_\mathrm{B} \\ &\Rightarrow A = \theta_\mathrm{B}, \end{aligned} \tag{II.8.14}$$

and constant B yields from:

$$\begin{aligned} \left.\frac{\partial \theta}{\partial x}\right|_{x=L} &= A \cdot m \cdot \sinh(mL) + B \cdot m \cdot \cosh(mL) = 0 \\ &\Rightarrow B = -A \cdot \tanh\left(mL\right). \end{aligned} \tag{II.8.15}$$

Plugging in the constants gives the equation for the fin temperature:

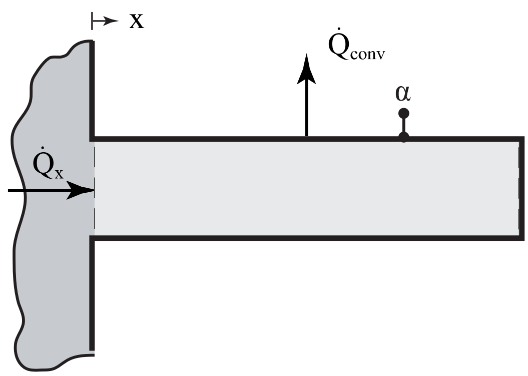
$$\begin{aligned}\theta(x) &= \theta_\mathrm{B}\left[\cosh(m\cdot x) - \tanh(mL)\sinh(m\cdot x)\right] \\ &= \theta_\mathrm{B}\cdot\frac{\cosh\left[m\cdot(L-x)\right]}{\cosh(mL)}.\end{aligned} \tag{II.8.16}$$

Substitution of temperature difference θ back and rewriting the temperature profile inside the fin:

$$T(x) = T_\mathrm{a} + (T_\mathrm{B} - T_\mathrm{a})\left[\cosh(m\cdot x) - \tanh(mL)\sinh(m\cdot x)\right]. \tag{II.8.17}$$

1 **Setting up the balance:**

To determine the total rate of heat loss, an energy balance over the entire fin is required. The fin conducts heat through the base at $x = 0$, which is lost by convection to the ambient.



Therefore the energy balance reads:

$$0 = \dot{Q}_{x=0} - \dot{Q}_\mathrm{conv}. \tag{II.8.18}$$

2 **Defining the elements within the balance:**

The energy balance states that $\dot{Q}_{x=0} = \dot{Q}_\mathrm{conv}$. Both are expressed by the use of Fourier's law:

$$\dot{Q}_{x=0} = \dot{Q}_\mathrm{conv} = -\lambda A_\mathrm{c}\cdot\left.\frac{\partial T}{\partial x}\right|_{x=0}, \tag{II.8.19}$$

where:

$$\begin{aligned}\left.\frac{\partial T}{\partial x}\right|_{x=0} &= (T_\mathrm{B} - T_\mathrm{a})\cdot\left[m\cdot\sinh(0) - m\cdot\tanh(mL)\cdot\cosh(0)\right] \\ &= -(T_\mathrm{B} - T_\mathrm{a})\cdot m\cdot\tanh(mL).\end{aligned} \tag{II.8.20}$$

3 **Inserting and rearranging:**

$$\dot{Q}_{x=0} = \dot{Q}_{\text{conv}} = \lambda A_c m \tanh(mL)(T_B - T_a), \tag{II.8.21}$$

where $m = \sqrt{\frac{\alpha U}{\lambda A_c}}$.

The rate of heat losses for the aluminum and copper fin are expressed respectively as:

$$\dot{Q}_A = \lambda_A A_c m_A \tanh(m_A L_A)(T_B - T_a), \tag{II.8.22}$$

and:

$$\dot{Q}_c = \lambda_c A_c m_c \tanh(m_c L_c)(T_B - T_a). \tag{II.8.23}$$

However, the lengths are not the same. The ratio of the length of both fins is found by the tip temperature:

$$T(x = L_A) = T_a + (T_B - T_a)\left[\cosh(m_A L_A) - \tanh(m_A L_A)\sinh(m_A L_A)\right]. \tag{II.8.24}$$

$$T(x = L_C) = T_a + (T_B - T_a)\left[\cosh(m_C L_C) - \tanh(m_C L_C)\sinh(m_C L_C)\right]. \tag{II.8.25}$$

Equalling the two expressions for the tip temperature of the aluminum and copper fin:

$$m_A L_A = m_C L_C. \tag{II.8.26}$$

Finally, determining the ratio between the heat flows, substituting the fin parameter, and canceling all identical terms, the following is obtained:

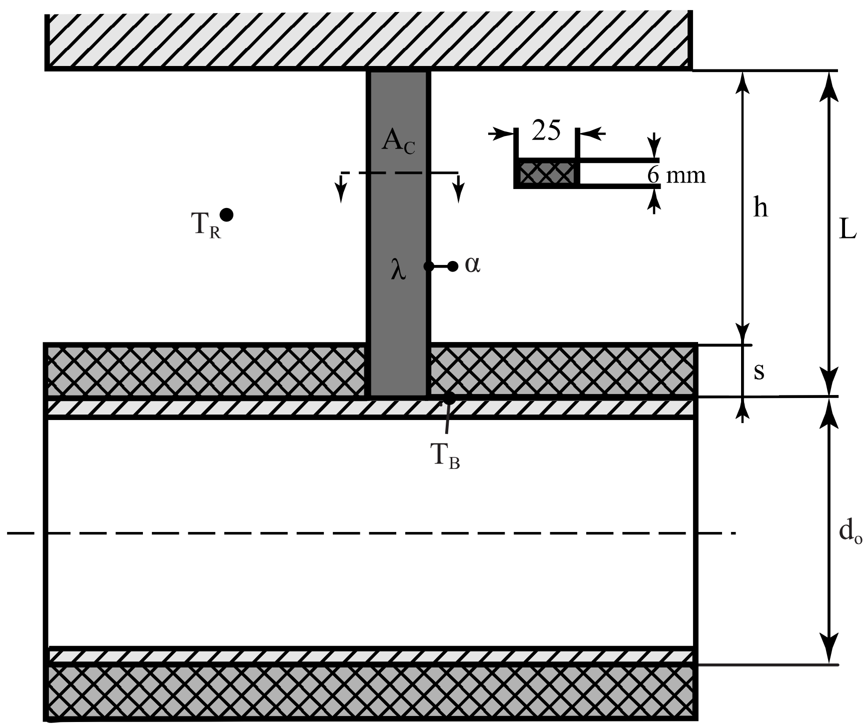
$$\begin{aligned}\frac{\dot{Q}_A}{\dot{Q}_C} &= \frac{\lambda_A A_c \sqrt{\frac{\alpha U}{\lambda_A A_c}}\theta_B \tanh(m_A L_A)}{\lambda_C A_c \sqrt{\frac{\alpha U}{\lambda_C A_c}}\theta_B \tanh(m_C L_C)} \\ &= \sqrt{\frac{\lambda_A}{\lambda_C}}.\end{aligned} \tag{II.8.27}$$

Conclusion

If the electric motor manufacturer transitions from using copper to aluminum as the material, the impact on the heat flow of aluminum with respect to the copper situation changes with a factor $\sqrt{\frac{\lambda_A}{\lambda_C}}$.

Exercise II.9: (Pipe fastening ⋆ ⋆ ⋆)

A pipe containing brine is insulated with cork and fastened to the ceiling with steel bands welded to the pipe.



Given parameters:

- Outer diameter of the pipe: d_o = 50 mm
- Insulation thickness: s = 40 mm
- Cross-section of the steel band: $A_c = 25 \times 6$ mm
- Length of the steel band: L = 290 mm
- Heat transfer coefficient at the steel band's surface: $\alpha = 6\ \mathrm{W/m^2K}$
- Thermal conductivity of the steel band: $\lambda = 58\ \mathrm{W/mK}$
- Temperature outer wall of the brine pipe: $T_B = -23.5\ °C$
- Temperature of the room: $T_R = 20\ °C$

Hints:

- The temperature distribution in the steel band's cross-section is homogeneous.
- The heat fluxes from the steel bands into both the ceiling and the insulation are negligible.

Tasks:

a) Calculate the heat $\dot{Q}$ from one steel band absorbed by the brine.

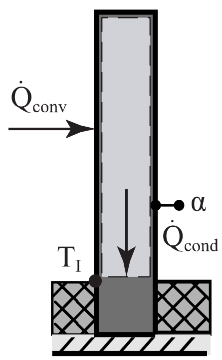
b) Up to which height h_0 does frost form on the steel ban (h_0 is the distance from the surface of the pipe's insulation layer), if the steam content of the air in the surrounding room is above the saturation vapor pressure for the maximum steel band temperature?

Solution II.9: (Pipe fastening ⋆ ⋆ ⋆)

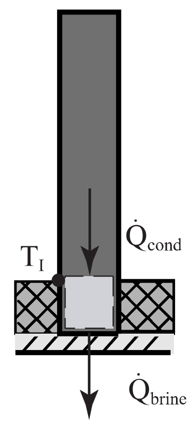
Task a)

1 **Setting up the balance:**

The steel band can be seen as two interconnected systems in series. The initial segment of the steel band, referred to as the fin part, heats up due to convection. This heat is conducted through the body, and at a specific section of the steel band, no convection can occur. Here only heat conduction takes place, and this part of the system is called the plane wall part.



(a) Fin part.



(b) Plane wall part.

The respective energy balances of the fin and plane wall part read:

$$0 = \dot{Q}_{\text{conv}} - \dot{Q}_{\text{cond}}, \tag{II.9.1}$$

and:

$$0 = \dot{Q}_{\text{cond}} - \dot{Q}_{\text{brine}}. \tag{II.9.2}$$

2 **Defining the elements within the balance:**

Heat transfer from the steel band into the ceiling and the insulation is negligible, indicating that the fin's head is adiabatic, and all heat conducted through the fin is transferred to the plane wall part. For a fin with an adiabatic head, the heat transfer rate is expressed as:

$$\dot{Q}_{\text{conv}} = \lambda A_{\text{c}} m \theta_{\text{I}} \tanh(mh), \tag{II.9.3}$$

with:

$$\begin{aligned} m &= \sqrt{\frac{\alpha U}{\lambda A}} = \sqrt{\frac{2\alpha(a+b)}{\lambda ab}} \\ &= \sqrt{\frac{6\ \left(\frac{\text{W}}{\text{m}^2\text{K}}\right)\cdot 2\cdot(0.025+0.006)\ (\text{m})}{58\ \left(\frac{\text{W}}{\text{mK}}\right)\cdot 0.025\ (\text{m})\cdot 0.006\ (\text{m})}} = 6.54\ \left(\text{m}^{-1}\right), \end{aligned} \tag{II.9.4}$$

and:

$$\theta = T_{\text{R}} - T(x). \tag{II.9.5}$$

Note that $\theta \neq T(x) - T_{\text{R}}$, because otherwise the direction of the heat flow is incorrect

Furthermore, $\dot{Q}_{\mathrm{conv}}$ incorporates the interface temperature T_{I}, which is still unknown. Hence, a second definition for the rate of heat transfer is necessary to determine this temperature.

The heat conduction through the plane wall part can be described by use of Fourier's law.

$$\begin{aligned}\dot{Q}_{\mathrm{brine}} &= -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} \\ &= \lambda A_{\mathrm{c}} \frac{T_{\mathrm{I}} - T_{\mathrm{B}}}{s}.\end{aligned} \tag{II.9.6}$$

The balances yield that $\dot{Q}_{\mathrm{conv}} = \dot{Q}_{\mathrm{cond}} = \dot{Q}_{\mathrm{brine}}$.

3 **Inserting and rearranging:**

$$\begin{aligned}T_{\mathrm{I}} &= \frac{T_{\mathrm{R}} sm \tanh(mh) + T_{\mathrm{B}}}{sm \tanh(mh) + 1} \\ &= \frac{20\ (^\circ\mathrm{C}) \cdot 0.04\ (\mathrm{m}) \cdot 6.54\ \left(\mathrm{m}^{-1}\right) \cdot \tanh\left[6.54\ \left(\mathrm{m}^{-1}\right) \cdot (0.29 - 0.04)\ \ (\mathrm{m})\right] - 23.5\ (^\circ\mathrm{C})}{0.04\ (\mathrm{m}) \cdot 6.54\ (\mathrm{m}^{-1}) \cdot \tanh\left[6.54\ (\mathrm{m}^{-1}) \cdot (0.29 - 0.04)\ (\mathrm{m})\right] + 1} = -15\ (^\circ\mathrm{C})\,.\end{aligned} \tag{II.9.7}$$

Now that the temperature T_{I} at the interface between the fin system and the plane wall system has been established, all parameters are substituted to determine the rate of heat transfer towards the brine:

$$\begin{aligned}\dot{Q}_{\mathrm{brine}} &= \lambda A_{\mathrm{c}} \frac{T_{\mathrm{I}} - T_{\mathrm{B}}}{s} \\ &= 56 \left(\frac{\mathrm{W}}{\mathrm{mK}}\right) \cdot (0.025 \cdot 0.006)\ \left(\mathrm{m}^2\right) \cdot \frac{(-15 + 23.5)\ (\mathrm{K})}{0.04\ (\mathrm{m})} = 1.85\ (\mathrm{W}).\end{aligned} \tag{II.9.8}$$

Conclusion

The heat from a single steel band absorbed by the brine is 1.85 W.

Task b)

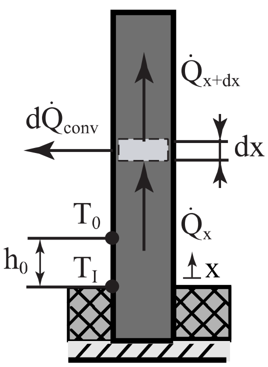
Frost forms at a temperature below 0 °C, therefore the following condition needs to be satisfied to determine the height h_0 up to which frost forms:

$$T(x = h_0) = T_0 = 0\,(^{\circ}\mathrm{C})\,. \tag{II.9.9}$$

To find this height, the temperature profile inside the fin part needs to be determined.

1 **Setting up the balance:**

In this scenario, a fin that conducts heat through the body and heats up by convection from the surrounding environment is being worked with.



The heat conduction equation inside the fin is derived based on the energy balance of an infinitesimal element within the fin. This balance reads:

$$0 = \dot{Q}_{\mathrm{x}} - \dot{Q}_{\mathrm{x+dx}} - d\dot{Q}_{\mathrm{conv}}. \tag{II.9.10}$$

Note that the energy balance assumes heat conduction in the positive x-direction and heat loss by convection. However, the steel band will receive heat due to convection and conduct in the negative x-direction towards the brine. This direction is considered when defining the fluxes.

2 **Defining the elements within the balance:**

The ingoing rate of heat transfer is found using Fourier's law:

$$\dot{Q}_{\mathrm{x}} = -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}. \tag{II.9.11}$$

For an infinitesimal element, the outgoing conductive rate of heat transfer is approximated by the use of the Taylor series expansion:

$$\begin{aligned} \dot{Q}_{\mathrm{x+dx}} &= \dot{Q}_{\mathrm{x}} + \frac{\partial \dot{Q}_{\mathrm{x+dx}}}{\partial x} \cdot dx \\ &= -\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-\lambda A_{\mathrm{c}} \frac{\partial T}{\partial x}\right) \cdot dx. \end{aligned} \tag{II.9.12}$$

Lastly, the rate of heat being lost by convection from the infinitesimal element is written as:

$$\begin{aligned} d\dot{Q}_{\mathrm{conv}} &= \alpha A_{\mathrm{s}} \left(T(x) - T_{\mathrm{R}}\right) \\ &= \alpha U dx \left(T(x) - T_{\mathrm{R}}\right). \end{aligned} \tag{II.9.13}$$

3 **Inserting and rearranging:**

$$0 = \frac{\partial^2 T}{\partial x^2} - \frac{\alpha U}{\lambda A_c} \cdot (T(x) - T_R)\,, \tag{II.9.14}$$

which is the heat conduction equation.

4 **Defining the boundary and/or initial conditions:**

The differential equation is solved using two boundary conditions to x. The temperature of the steel band at the surface of the pipe's insulation layer is:

$$T(x = 0) = T_I. \tag{II.9.15}$$

Heat flow through the ceiling is negligible, and thus $\dot{Q}_{x=h} = -\lambda A_c \left.\frac{\partial T}{\partial x}\right|_{x=h} = 0$ yields:

$$\left.\frac{\partial T}{\partial x}\right|_{x=h} = 0. \tag{II.9.16}$$

5 **Solving the equation:**

The 2nd order differential equation must be homogenized to solve the equation. The differential equation for a steady-state fin is solved by introducing the temperature difference θ

$$\theta = T(x) - T_R, \tag{II.9.17}$$

and the fin parameter m:

$$m^2 = \frac{\alpha U}{\lambda A_c}. \tag{II.9.18}$$

Substitution into the differential equation, given in equation (II.9.14), yields:

$$0 = \frac{\partial^2 \theta}{\partial x^2} - m^2 \theta. \tag{II.9.19}$$

Furthermore, the boundary conditions are rewritten in terms of the temperature difference θ as well:

$$\theta(x = 0) = T_I - T_{amb} = \theta_I, \tag{II.9.20}$$

and:

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0. \tag{II.9.21}$$

The standard solution for this homogenized differential equation is given as:

$$\theta(x) = A \cdot \cosh(m \cdot x) + B \cdot \sinh(m \cdot x). \tag{II.9.22}$$

The constants A and B are determined by using the boundary conditions. Whereas, constant A yields from:

$$\begin{aligned} \theta(x = 0) &= A \cdot \cosh(0) + B \cdot \sinh(0) = \theta_I \\ &\Rightarrow A = \theta_I, \end{aligned} \tag{II.9.23}$$

and constant B yields from:

$$\begin{aligned} \left.\frac{\partial \theta}{\partial x}\right|_{x=h} &= A \cdot m \cdot \sinh(mh) + B \cdot m \cdot \cosh(mh) = 0 \\ &\Rightarrow B = -A \cdot \tanh(mh). \end{aligned} \tag{II.9.24}$$

Plugging in the constants yields the equation for the fin temperature:

$$\theta(x) = \theta_\mathrm{I}\left[\cosh(m\cdot x) - \tanh(mh)\sinh(m\cdot x)\right] \\ = \theta_\mathrm{I}\cdot\frac{\cosh\left[m\cdot(h-x)\right]}{\cosh(mh)}. \tag{II.9.25}$$

Substitution of temperature difference θ back and rewriting yields the temperature profile inside the fin:

$$T(x) = T_\mathrm{R} + (T_\mathrm{I} - T_\mathrm{R})\cdot\frac{\cosh\left[m\cdot(h-x)\right]}{\cosh(mh)}. \tag{II.9.26}$$

Substitution of the frost temperature into the temperature profile of the fin and rewriting yields:

$$T(x = h_0) = T_\mathrm{R} + (T_\mathrm{I} - T_\mathrm{R})\cdot\frac{\cosh\left[m\cdot(h-h_0)\right]}{\cosh(mh)} = T_0 \\ \Rightarrow h_0 = h - \frac{1}{m}\cosh^{-1}\left[\frac{T_0 - T_\mathrm{R}}{T_\mathrm{I} - T_\mathrm{R}}\cosh(mh)\right] \\ = 0.25\ (\mathrm{m}) - \frac{1}{0.25\ (\mathrm{m})}\cosh^{-1}\left[\frac{(0-20)\ (^\circ\mathrm{C})}{(-15-20)\ (^\circ\mathrm{C})}\cosh\left(6.54\ \left(\mathrm{m}^{-1}\right)\cdot 0.25\ (\mathrm{m})\right)\right] = 0.1\ (\mathrm{m}). \tag{II.9.27}$$

Conclusion

Up to 0.1 m from the surface of the pipe's insulation layer frost will form on the steel ban.