Institute of Heat and Mass Transfer | RWTH AACHEN UNIVERSITY

Formulary Heat and Mass Transfer

Version 7 from WS 2015

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1. Dimensionless numbers

Dimensionless numbers – Fluid dynamics

$$\mathrm{Gr}_L = \frac{\beta g \rho^2 \left(T_\mathrm{W} - T_\infty\right) L^3}{\eta^2} \qquad \text{(Grashof number)}$$

$$\mathrm{Re}_L = \frac{\rho u L}{\eta} \qquad \text{(Reynolds number)}$$

Dimensionless numbers – Heat transfer

$$\mathrm{Bi}_L = \frac{\alpha L}{\lambda} \qquad \text{(Biot number)}$$

$$\mathrm{Fo} = \frac{a t}{L^2} \quad \text{with} \quad a = \frac{\lambda}{\rho\, c_\mathrm{p}} \qquad \text{(Fourier number)}$$

$$\mathrm{Nu}_L = \frac{\alpha L}{\lambda} \qquad \text{(Nusselt number)}$$

$$\mathrm{Pr} = \frac{\eta c_\mathrm{p}}{\lambda} = \frac{\nu}{a} \qquad \text{(Prandtl number)}$$

$$\mathrm{St}_L = \frac{\mathrm{Nu}}{\mathrm{Re}_L\,\mathrm{Pr}} \qquad \text{(Stanton number)}$$

Dimensionless numbers – Mass transfer

$$\mathrm{Le} = \frac{\lambda}{\rho D\, c_\mathrm{p}} = \frac{a}{D} \qquad \text{(Lewis number)}$$

$$\mathrm{Sc} = \frac{\eta}{\rho D} \qquad \text{(Schmidt number)}$$

$$\mathrm{Sh}_L = \frac{g L}{\rho D} \qquad \text{(Sherwood number)}$$

2. Heat radiation

$$\dot{q}''_{\lambda,\mathrm{b}} = \frac{c_1 \lambda^{-5}}{\exp\left[c_2/(\lambda T)\right] - 1} \qquad \text{(Planck's distribution law)}$$

$$\dot{q}''_{\mathrm{b}} = \int\limits_{\lambda=0}^{\infty} \dot{q}''_{\lambda \mathrm{b}} d\lambda = \sigma T^4 \qquad \text{(Stefan-Boltzmann's law)}$$

$$\lambda_{\max} T = 2898\,\mathrm{\mu m\,K} \qquad \text{(Wien's law of displacement)}$$

with the constants

$$\sigma = 5.67\ 10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2\,K^4}} \qquad \text{(Stefan-Boltzmann constant)}$$

$$c_1 = 3.741\ 10^{-16}\,\mathrm{W\,m^2}$$

$$c_2 = 1.439\ 10^{-2}\,\mathrm{m\,K}$$

λT in µm K	1000.0	1250.0	1500.0	1750.0	2000.0	2500.0
$F(\lambda)$	0.00031	0.00308	0.01283	0.03363	0.06663	0.16115
λT in µm K	3000.0	3500.0	4000.0	5000.0	6000.0	8000.0
$F(\lambda)$	0.27322	0.38250	0.48085	0.63315	0.73715	0.85556

Distribution of black body radiation: $F(\lambda) = \int_0^\lambda \dot{q}''_{\lambda \mathrm{b}} d\lambda \,/\, \sigma T^4$

Properties of radiating bodies

- spectral properties

$$\left.\begin{aligned}\rho\,(\lambda) &\equiv \frac{\dot{q}''_{\lambda\rho}}{\dot{q}''_{\lambda\mathrm{o}}}\\ \alpha\,(\lambda) &\equiv \frac{\dot{q}''_{\lambda\alpha}}{\dot{q}''_{\lambda\mathrm{o}}}\\ \tau\,(\lambda) &\equiv \frac{\dot{q}''_{\lambda\tau}}{\dot{q}''_{\lambda\mathrm{o}}}\end{aligned}\right\}\text{with } \rho(\lambda)+\alpha(\lambda)+\tau(\lambda)=1$$

here: $\dot{q}''_{\lambda\mathrm{o}}$ impacting spectral heat flux

$$\varepsilon\,(\lambda) \equiv \frac{\dot{q}''_{\lambda\varepsilon}}{\dot{q}''_{\lambda\mathrm{b}}}$$

$$\alpha(\lambda) = \varepsilon(\lambda) \qquad \text{(Kirchhoff's law)}$$

- spectrally averaged

$$\varepsilon \equiv \frac{\dot{q}''_{\varepsilon}}{\dot{q}''_{\mathrm{b}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\varepsilon}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{b}}\,d\lambda} \qquad \alpha \equiv \frac{\dot{q}''_{\alpha}}{\dot{q}''_{\mathrm{o}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\alpha}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{o}}\,d\lambda}$$

$$\rho \equiv \frac{\dot{q}''_{\rho}}{\dot{q}''_{\mathrm{o}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\rho}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{o}}\,d\lambda} \qquad \tau \equiv \frac{\dot{q}''_{\tau}}{\dot{q}''_{\mathrm{o}}} \equiv \frac{\int_0^\infty \dot{q}''_{\lambda\tau}\,d\lambda}{\int_0^\infty \dot{q}''_{\lambda\mathrm{o}}\,d\lambda}$$

- special cases

Radiation properties independent of wavelength:

$$\rho+\alpha+\tau=1 \quad \text{and} \quad \alpha=\varepsilon \qquad \text{(Grey body)}$$

$$\alpha=1 \quad \text{and} \quad \alpha=\varepsilon=1 \qquad \text{(Black body)}$$

Spectral radiative properties

$$\rho(\lambda)+\alpha(\lambda)=1 \qquad \text{(Solid body impermeable for radiation)}$$

$$\alpha(\lambda)+\tau(\lambda)=1 \qquad \text{(Gas)}$$

Radiative heat exchange

$$\dot{Q}_{i\to j} = \dot{Q}_i \Phi_{ij} \qquad \text{(Radiative heat flow)}$$

$$\dot{Q}_i = \dot{q}_i'' A_i = \dot{Q}_{i,\mathrm{b}}\,\varepsilon_i + \underbrace{\sum_j \dot{Q}_{j\to i}\,\rho_i}_{\text{Reflection}} + \underbrace{\sum_k \dot{Q}_{k\to i}\,\tau_i}_{\text{Transmission}} \qquad \text{(Surface brightness)}$$

$$\text{with} \quad \dot{Q}_{i,\mathrm{b}} = \dot{q}_{i,\mathrm{b}}'' A_i \qquad \text{(Black body radiation)}$$

$$\Phi_{ij} = \frac{1}{A_i} \int_{A_j} \int_{A_i} \frac{\cos\varphi_i \, \cos\varphi_j}{\pi r^2}\, dA_i \, dA_j \qquad \text{(View factor)}$$

$$A_i \Phi_{ij} = A_j \Phi_{ji} \qquad \text{(Reciprocity relationship)}$$

$$\sum_j \Phi_{ij} = 1 \qquad \text{(Sum rule)}$$

$$\dot{Q}_{\mathrm{i,net}} = \dot{Q}_i - \sum_j \dot{Q}_{j\to i} \qquad \text{(Net radiative heat flow)}$$

$$\dot{Q}_{1\rightleftharpoons 2} = \dot{Q}_{1\to 2} - \dot{Q}_{2\to 1} \qquad \text{(Radiative heat exchange)}$$

$$\begin{aligned}\dot{Q}_{1\rightleftharpoons 2} &= A_1 \Phi_{12}\, \sigma \left[(T_1)^4 - (T_2)^4\right] \qquad \text{(Between two black bodies)} \\ &= A_2 \Phi_{21}\, \sigma \left[(T_1)^4 - (T_2)^4\right]\end{aligned}$$

$$\dot{q}_{1\rightleftharpoons 2}'' = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} \sigma \left(T_1^4 - T_2^4\right) \qquad \text{(Between two grey plates)}$$

- Plates are plane, parallel and infinitely long

$$\dot{Q}_{1\rightleftharpoons 2} = \frac{A_1}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)} \sigma \left(T_1^4 - T_2^4\right) \qquad \text{(Between two grey bodies)}$$

- Body 2 encloses body 1 ($A_2 > A_1$)
- Body 1 convex ($\Phi_{11} = 0$)

View factors of simple geometries

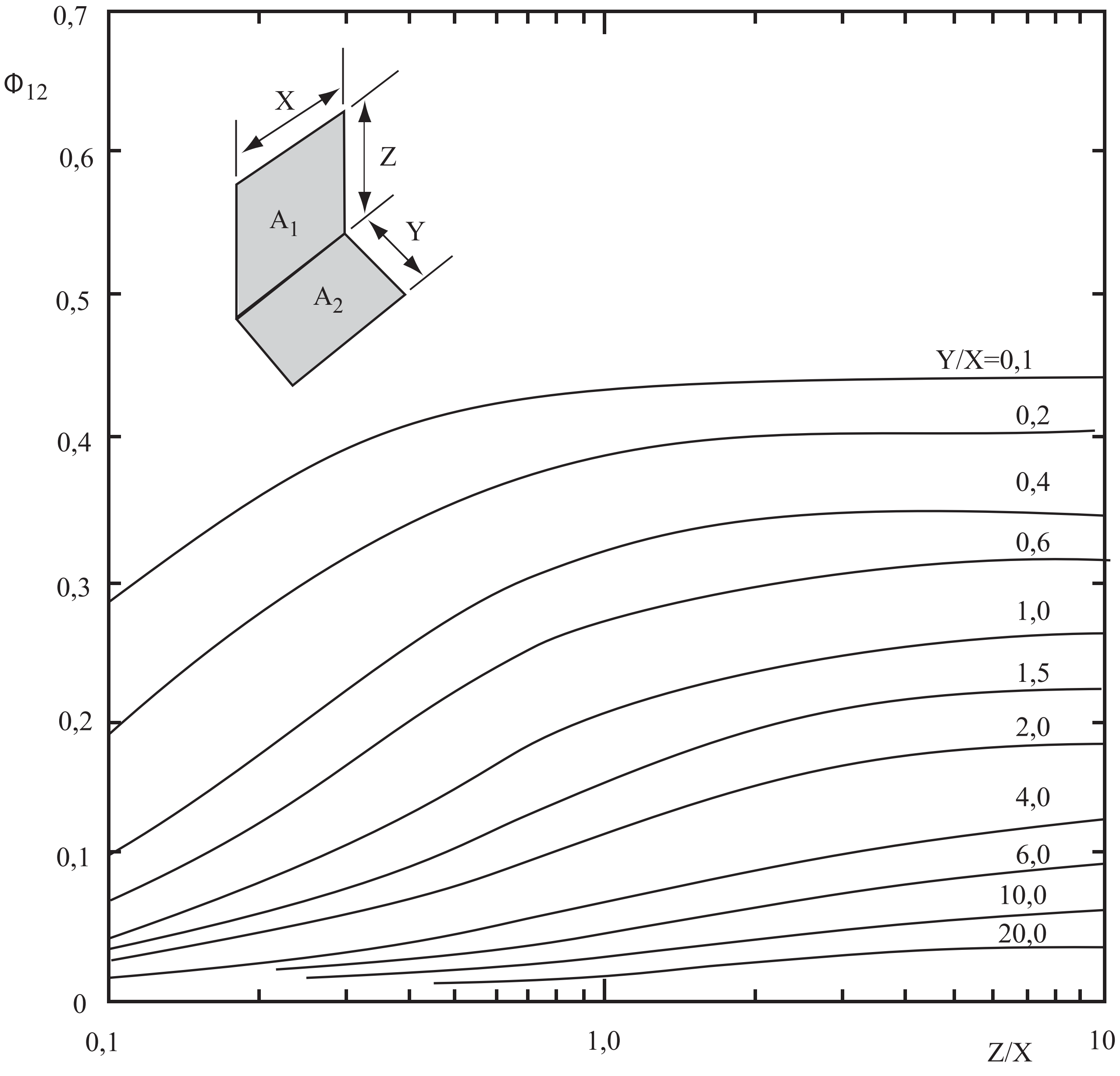


Diagramm 1: View factor of the radiation transfer between perpendicular, rectangular plates

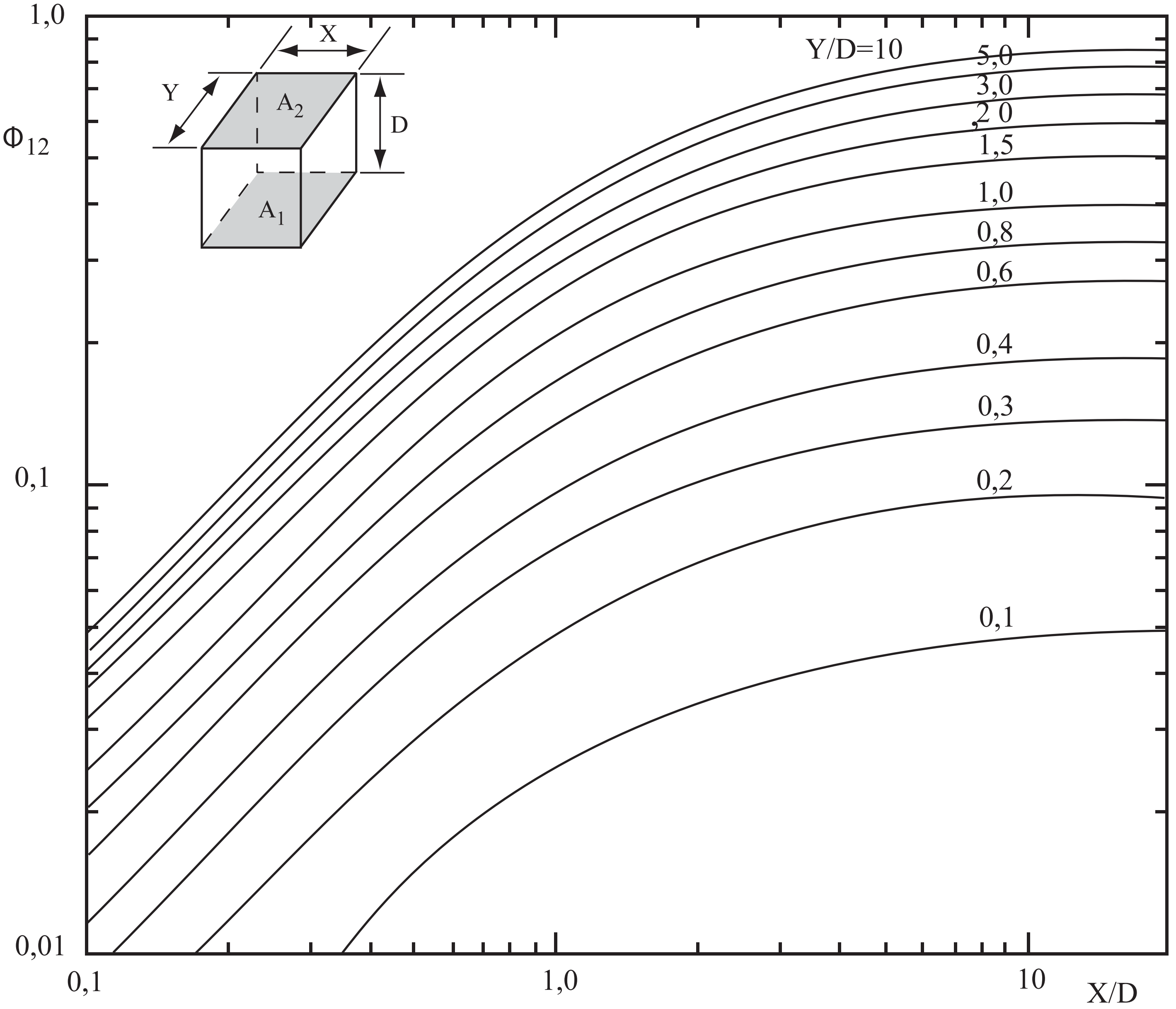


Diagramm 2: View factor of the radiation transfer between parallel, rectangular plates

3. Heat conduction

$$\dot{q}'' = -\lambda \frac{\partial T}{\partial x} \qquad \text{(Fourier's law)}$$

Heat transport equation

- Cartesian coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(\lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) \right] + \dot{\Phi}'''$$

- Cylindrical coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \lambda \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\lambda \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) \right] + \dot{\Phi}'''$$

- Spherical coordinates

$$\rho c \frac{\partial T}{\partial t} = \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \lambda \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\lambda \sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial \Phi} \left(\lambda \frac{\partial T}{\partial \Phi} \right) \right] + \dot{\Phi}'''$$

Steady state heat conduction in walls without heat sources

$$W = \frac{T_\mathrm{A} - T_\mathrm{B}}{\dot{Q}} \qquad \text{where} \qquad W = \sum_i W_i \qquad \text{(Heat resistance)}$$

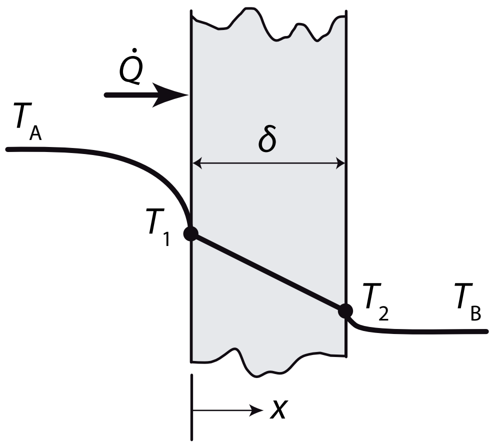
- Plane wall

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = 0 \qquad \text{with BC} \qquad \begin{aligned} T(x=0) &= T_1 \\ T(x=\delta) &= T_2 \end{aligned}$$

$$T = T_1 + \frac{T_2 - T_1}{\delta} x \qquad \text{(Temperature profile)}$$

$$\dot{Q} = -\lambda A \frac{\mathrm{d}T}{\mathrm{d}x} = \lambda A \frac{T_1 - T_2}{\delta} \qquad \text{(Heat flow rate)}$$

$$W = \frac{\delta}{\lambda A} \qquad \text{(Heat resistance)}$$



- Wall consisting of n layers

$$\dot{Q} = \lambda_1 \frac{A}{\delta_1}(T_1 - T_2) = \lambda_2 \frac{A}{\delta_2}(T_2 - T_3) = \cdots = \lambda_n \frac{A}{\delta_n}(T_n - T_{n+1})$$

$$\dot{Q} = \frac{A}{\sum\limits_{i=1}^{n} \frac{\delta_i}{\lambda_i}}(T_1 - T_{n+1}) \qquad \text{(Without conv. heat transfer)}$$

$$\dot{Q} = \frac{A}{\frac{1}{\alpha_A} + \sum\limits_{i=1}^{n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_B}}(T_A - T_B) \qquad \text{(With conv. heat transfer)}$$

- Thick-walled tube

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) = 0 \qquad \text{with BC} \qquad \begin{aligned} T(r = r_1) &= T_1 \\ T(r = r_2) &= T_2 \end{aligned}$$

$$\begin{aligned} T &= T_1 + \ln\left(\frac{r}{r_1}\right)\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \qquad \text{(Temperature profile)} \\ &= T_2 + \ln\left(\frac{r}{r_2}\right)\frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)} \end{aligned}$$

$$\dot{Q} = 2\pi\lambda L \frac{T_1 - T_2}{\ln\left(\frac{r_2}{r_1}\right)} \qquad \text{(Heat flow)}$$

$$W = \frac{1}{2\pi\lambda L}\ln\frac{r_2}{r_1} \quad \text{mit} \quad r_2 > r_1 \qquad \text{(Heat resistance)}$$

- Thick-walled tube consisting of n layers

$$\dot{Q} = 2\pi r L \left(-\lambda_i \frac{\mathrm{d}T}{\mathrm{d}r}\right) = const.$$

$$\dot{Q} = \frac{T_1 - T_{n+1}}{\frac{1}{2\pi L}\sum\limits_{i=1}^{n} \frac{1}{\lambda_i} \ln \frac{r_{i+1}}{r_i}} \qquad \text{(Without conv. heat transfer)}$$

$$\dot{Q} = \frac{2\pi L}{\frac{1}{\alpha_A r_1} + \sum\limits_{i=1}^{n} \frac{1}{\lambda_i} \ln \frac{r_{i+1}}{r_i} + \frac{1}{\alpha_B r_{n+1}}} (T_A - T_B) \qquad \text{(With conv. heat transfer)}$$

Fins

$$\theta = T - T_\mathrm{a} \qquad \text{(Temperature difference)}$$

$$\eta_\mathrm{F} = \frac{\dot{Q}_\mathrm{F}}{\dot{Q}_\mathrm{max}} = \frac{\dot{Q}_\mathrm{F}}{A_0\, \alpha\, \theta_\mathrm{b}} = \frac{\text{transferred heat}}{\text{maximum transferable heat}} \qquad \text{(Efficiency of the fin)}$$

here: A_0 Heat transferring surface

θ_b Fin base temperature

Rod fins and plane fins

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \underbrace{\frac{\alpha U}{\lambda A_c}}_{=m^2} \theta = 0 \quad \text{with} \quad \begin{array}{ll} \text{BC1:} & \theta(x=0) = \theta_\mathrm{b} \\ \text{BC2:} & \text{may vary, see the following:} \end{array}$$

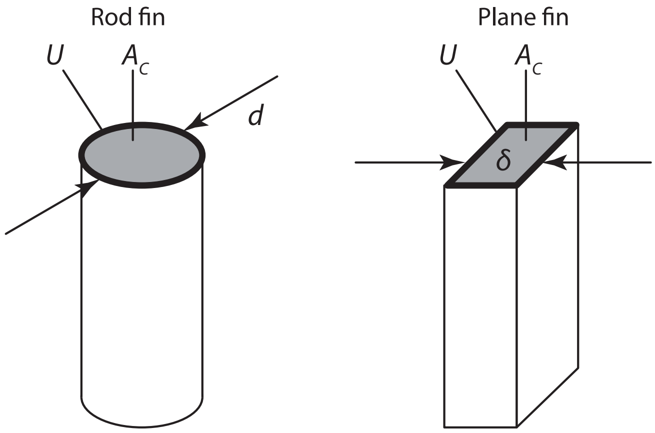
(Differential equation for fins)

$$\theta(x) = A\,\cosh(mx) + B\,\sinh(mx) \qquad \text{(Method of solution)}$$

$$\cdots = C\,\exp(mx) + D\,\exp(-mx)$$

$$m = \sqrt{\frac{\alpha U}{\lambda A_c}} = \sqrt{\frac{4\alpha}{\lambda d}} \qquad \text{(Rod fin)}$$

$$m = \sqrt{\frac{\alpha U}{\lambda A_c}} = \sqrt{\frac{2\alpha}{\lambda \delta}} \qquad \text{(Plane fin)}$$



Boundary condition 2:

- Fins with adiabatic head:

$$\text{BC2:} \quad -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{x=L} = 0$$

$$\theta = \theta_\mathrm{b} \frac{\cosh\left[m\left(L-x\right)\right]}{\cosh\left[mL\right]} \qquad \text{(Temperature profile)}$$

$$\dot{Q} = \lambda\, A_\mathrm{c}\, m\, \theta_\mathrm{b}\, \tanh\left(mL\right) \qquad \text{(Heat flow through the fin)}$$

$$\eta = \frac{\tanh(mL)}{mL} \qquad \text{(Efficiency of the fin)}$$

- Fins with head at ambient temperature (long fins):

$$\text{BC2:} \quad \theta(x = L) = 0$$

- Fins transferring heat at the fin head:

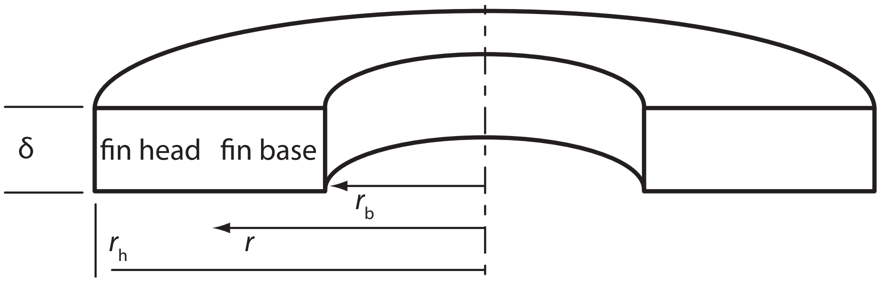
$$\text{BC2:} \quad -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{x=L} = \alpha\, \theta(x = L)$$

Circular fins with adiabatic head

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)-\frac{2\alpha}{\lambda\delta}\,\theta=0 \quad \text{with} \quad \begin{array}{ll}\text{BC1:} & \theta(r=r_\mathrm{b})=\theta_\mathrm{b}\\ \text{BC2:} & -\lambda\frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{r=r_\mathrm{h}}=0\end{array} \qquad \text{(Bessel d.e.)}$$

$$\theta(r)=A\,I_0(mr)+B\,K_0(mr) \qquad \text{(Method of solution)}$$

$$\text{with} \quad m=\sqrt{\frac{2\alpha}{\lambda\delta}}$$



$$\theta(r)=\theta_F\frac{I_0(m\,r)\,K_1(m\,r_\mathrm{h})+I_1(m\,r_\mathrm{h})\,K_0(m\,r)}{I_0(m\,r_\mathrm{b})\,K_1(m\,r_\mathrm{h})+I_1(m\,r_\mathrm{h})\,K_0(m\,r_\mathrm{b})} \qquad \text{(Temperature profile)}$$

$$\dot{Q}=2\pi r_\mathrm{b}\,\lambda\,\delta\,m\,\theta_\mathrm{b}\cdots \qquad \text{(Transferred heat flow)}$$

$$\cdots\frac{I_1(m\,r_\mathrm{h})\,K_1(m\,r_\mathrm{b})-I_1(m\,r_\mathrm{b})\,K_1(m\,r_\mathrm{h})}{I_0(m\,r_\mathrm{b})\,K_1(m\,r_\mathrm{h})+I_1(m\,r_\mathrm{h})\,K_0(m\,r_\mathrm{b})}$$

$$\eta_\mathrm{F}=\frac{2}{mr_\mathrm{b}\left[\left(\frac{r_h}{r_b}\right)^2-1\right]}\cdots \qquad \text{(Efficiency coefficient of fin)}$$

$$\cdots\frac{I_1(m\,r_\mathrm{h})\,K_1(m\,r_\mathrm{b})-I_1(m\,r_\mathrm{b})\,K_1(m\,r_\mathrm{h})}{I_0(m\,r_\mathrm{b})\,K_1(m\,r_\mathrm{h})+I_1(m\,r_\mathrm{h})\,K_0(m\,r_\mathrm{b})}$$

$$\approx\frac{\tanh\left(m\,r_\mathrm{b}\phi\right)}{m\,r_\mathrm{b}\phi} \quad \text{with} \quad \phi=\left(\frac{r_\mathrm{h}}{r_\mathrm{b}}-1\right)\left(1+0{,}35\,\ln\frac{r_\mathrm{h}}{r_\mathrm{b}}\right)$$

Numerical values for the Bessel functions I_0, I_1, K_0 and K_1 $\rightarrow$ Table 9

One-dimensional, unsteady state heat conduction

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho c}\frac{\partial^2 T}{\partial x^2} \qquad \text{(Differential equation)}$$

$$\frac{\partial \theta^*}{\partial t} = a\,\frac{\partial^2 \theta^*}{\partial x^2} \qquad \text{with} \qquad \theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0}$$

- Semi-infinite plate with negligible heat transfer resistance:

$$\mathrm{Bi} = \frac{\alpha\, L}{\lambda} \gg 1$$

$$\left.\begin{array}{r} t = 0 \\ 0 < x < \infty \end{array}\right\} \quad T = T_0 \qquad \theta^* = 0 \qquad \text{(BC1)}$$

$$\left.\begin{array}{r} t > 0 \\ x = 0 \end{array}\right\} \quad T = T_\mathrm{a} \qquad \theta^* = 1 \qquad \text{(BC2)}$$

$$\left.\begin{array}{r} t > 0 \\ x \to \infty \end{array}\right\} \quad T = T_0 \qquad \theta^* = 0 \qquad \text{(BC3)}$$

$$\theta^* = \frac{T - T_0}{T_\mathrm{a} - T_0} = 1 - \mathrm{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}\right) \quad \text{with} \quad \mathrm{Fo} = \frac{at}{x^2} \qquad \text{(Temperature profile)}$$

$$\left.\dot{q}''\right|_{x=0} = \sqrt{\frac{\lambda c \rho}{\pi t}}\,(T_\mathrm{a} - T_0) \qquad \text{(Heat flux)}$$

$$\delta(t) \approx 3{,}6\,\sqrt{at} \qquad \text{(Temperature penetration depth)}$$

- Semi-infinite plate, **non** negligible heat transfer resistance:

$$\left.\begin{array}{r} t>0 \\ x=0 \end{array}\right\} \quad \alpha\left(T_{\mathrm{a}}-T(x=0)\right)=-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} \tag{BC1}$$

$$\theta^{*}=\frac{T-T_{0}}{T_{\mathrm{a}}-T_{0}}=1-\operatorname{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}\right)\cdots \qquad \text{(Temperature profile)}$$

$$\cdots-\left[\exp\left(\mathrm{Bi}_{x}+\mathrm{Fo}\,\mathrm{Bi}_{x}^{2}\right)\right]\left[1-\operatorname{erf}\left(\frac{1}{\sqrt{4\,\mathrm{Fo}}}+\sqrt{\mathrm{Fo}}\,\mathrm{Bi}_{x}\right)\right]$$

$$\text{with} \quad \mathrm{Bi}_{x}=\frac{\alpha x}{\lambda}$$

$$\mathrm{Fo}=\frac{a t}{x^{2}}$$

- Semi-infinite plate, periodically changing surface temperature:

$$\left.\begin{array}{r} t>0 \\ x=0 \end{array}\right\} \quad T\,(x=0)=T_{\mathrm{m}}+\left(T_{\max}-T_{\mathrm{m}}\right)\,\cos\left(2\pi\,t/\tau\right) \tag{BC1}$$

$$\theta^{*}=\frac{T-T_{\mathrm{m}}}{T_{\max}-T_{\mathrm{m}}}=\exp\left(-\sqrt{\frac{\pi x^{2}}{a\tau}}\right)\,\cos\left(\frac{2\pi}{\tau}t-\sqrt{\frac{\pi x^{2}}{a\tau}}\right) \qquad \text{(Temperature profile)}$$

One-dimensional, unsteady heat conduction in simple bodies

$$\frac{T_{\mathrm{m}}-T_{\mathrm{a}}}{T_{0}-T_{\mathrm{a}}} \qquad \text{(Dimensionless temperature in the middle of a body)}$$

$$\frac{T-T_{\mathrm{a}}}{T_{\mathrm{m}}-T_{\mathrm{a}}} \qquad \text{(Dimensionless temperature at position } x \text{ or } r\text{)}$$

$$\frac{Q}{Q_{0}} \quad \text{mit} \quad Q_{0}=m\,c\,\left(T_{0}-T_{\mathrm{a}}\right) \qquad \text{(Dimensionless heat loss)}$$

Determination of temperature profile and heat flow for unsteady conditions
$\rightarrow$ Figures 3 - 11

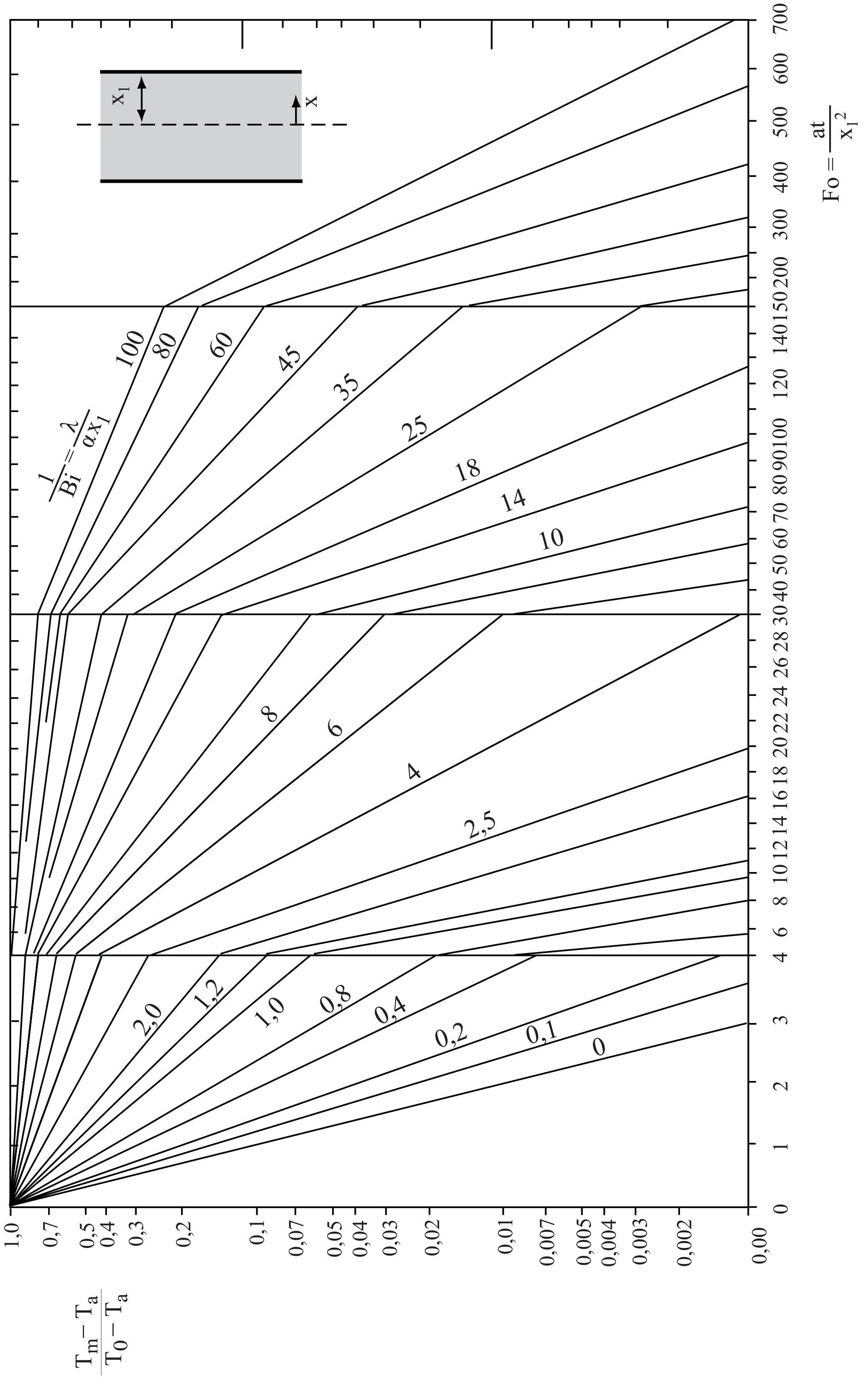


Diagramm 3: Mid-plane temperature of a plate with thickness $2x_1$

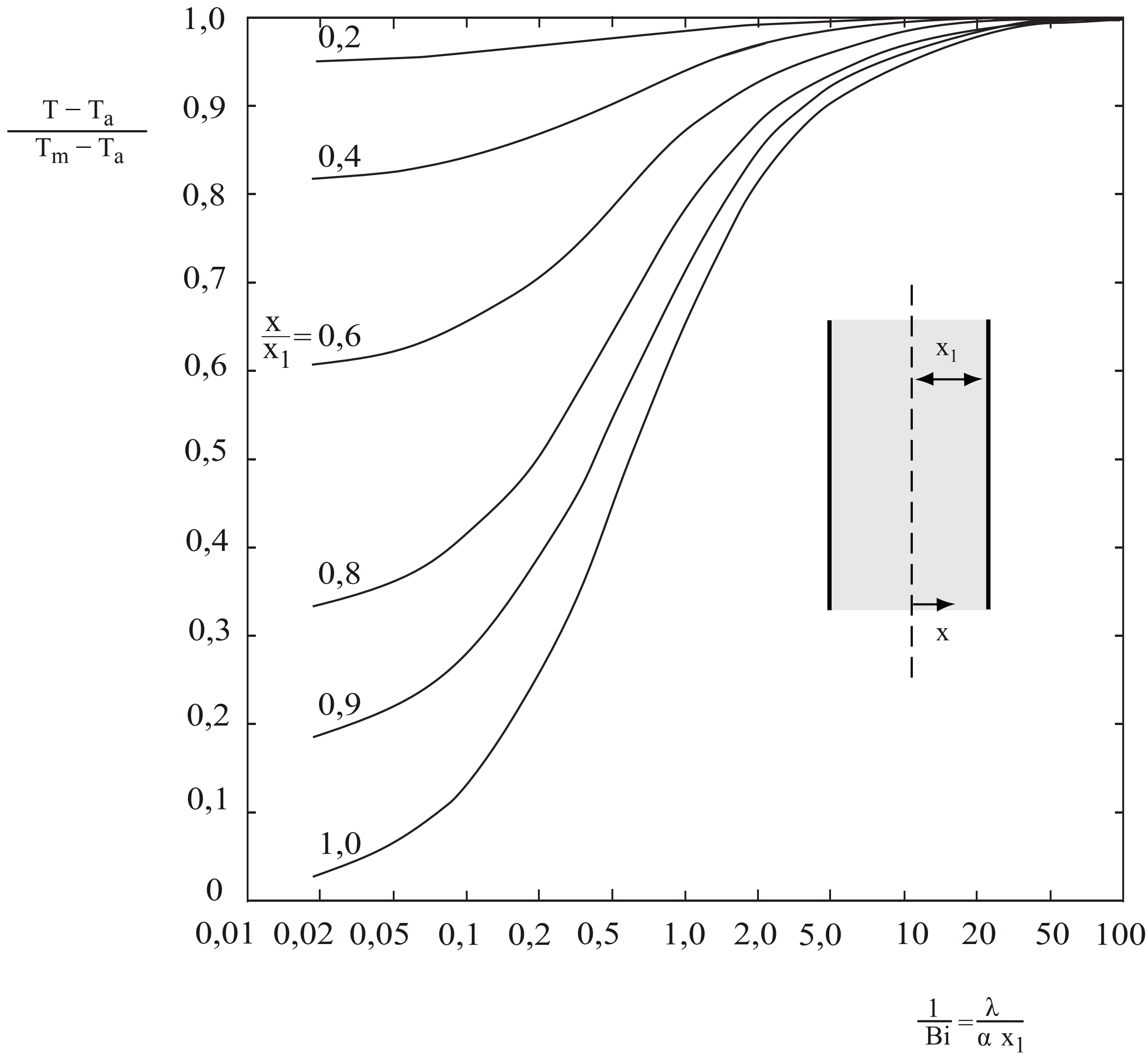


Diagramm 4: Temperature distribution in a plate (valid for Fo > 0,2)

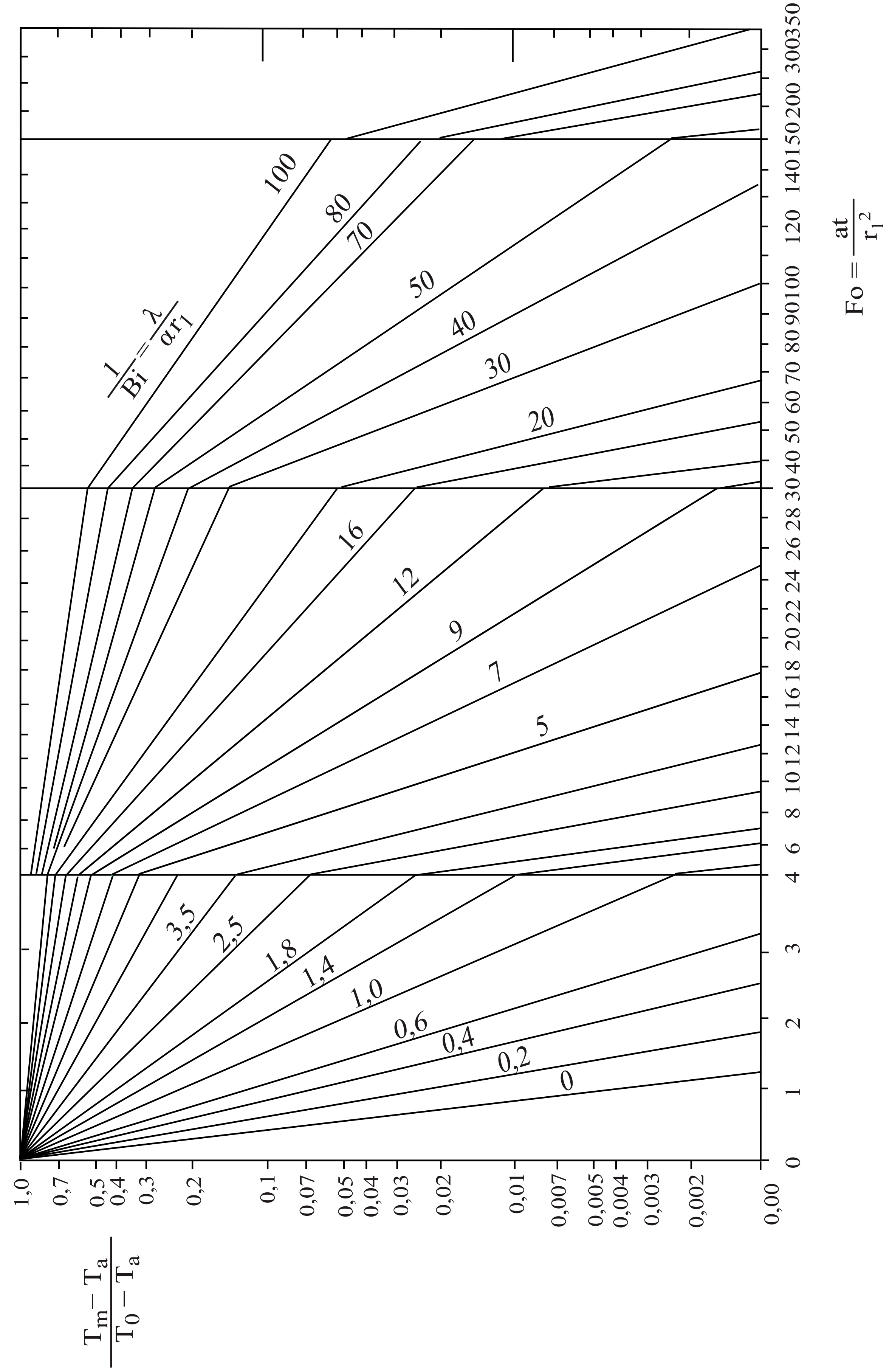


Diagramm 5: Temperature along the axis of a cylinder with radius r_1

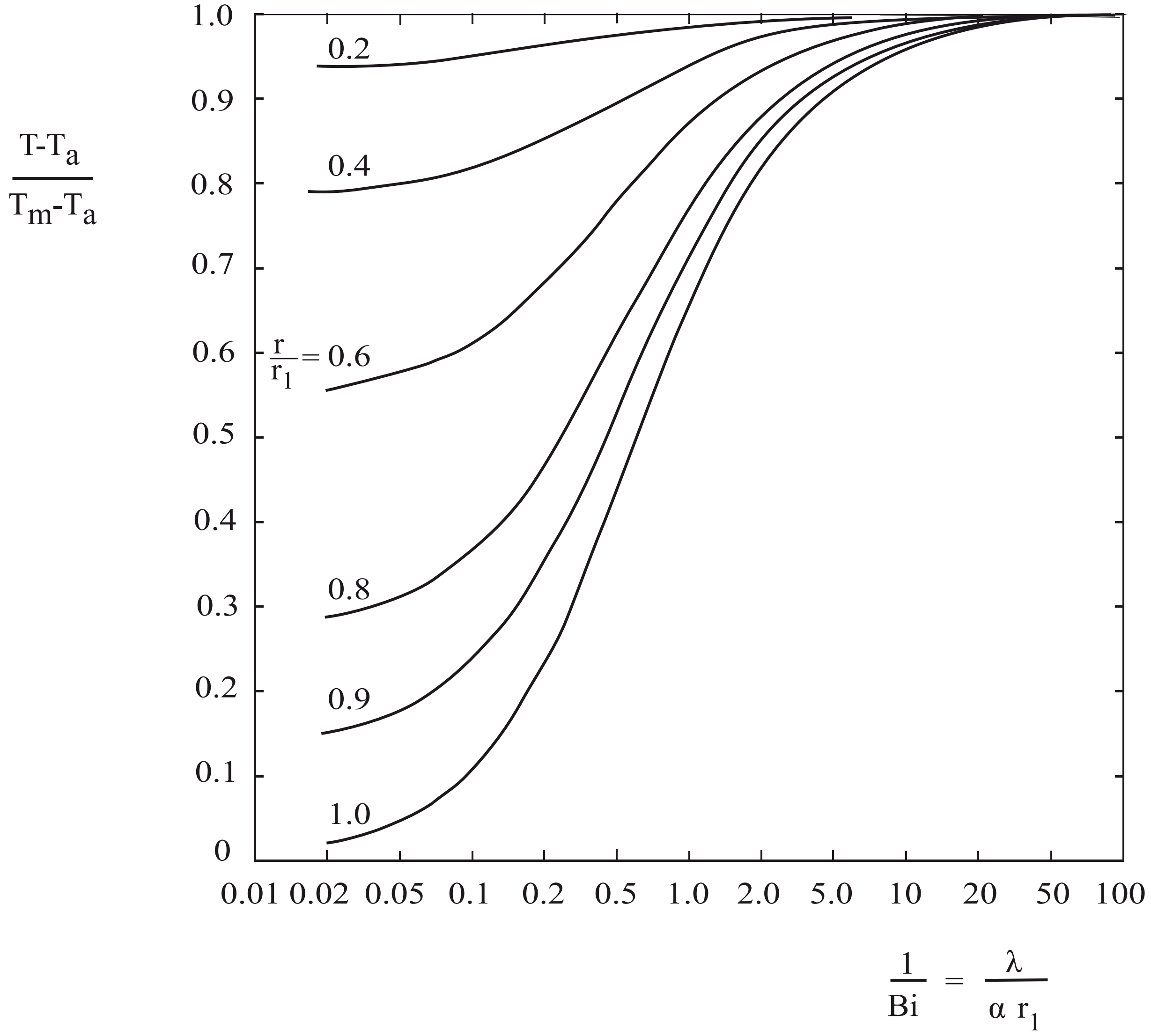


Diagramm 6: Temperature distribution in a cylinder (valid for Fo > 0,2)

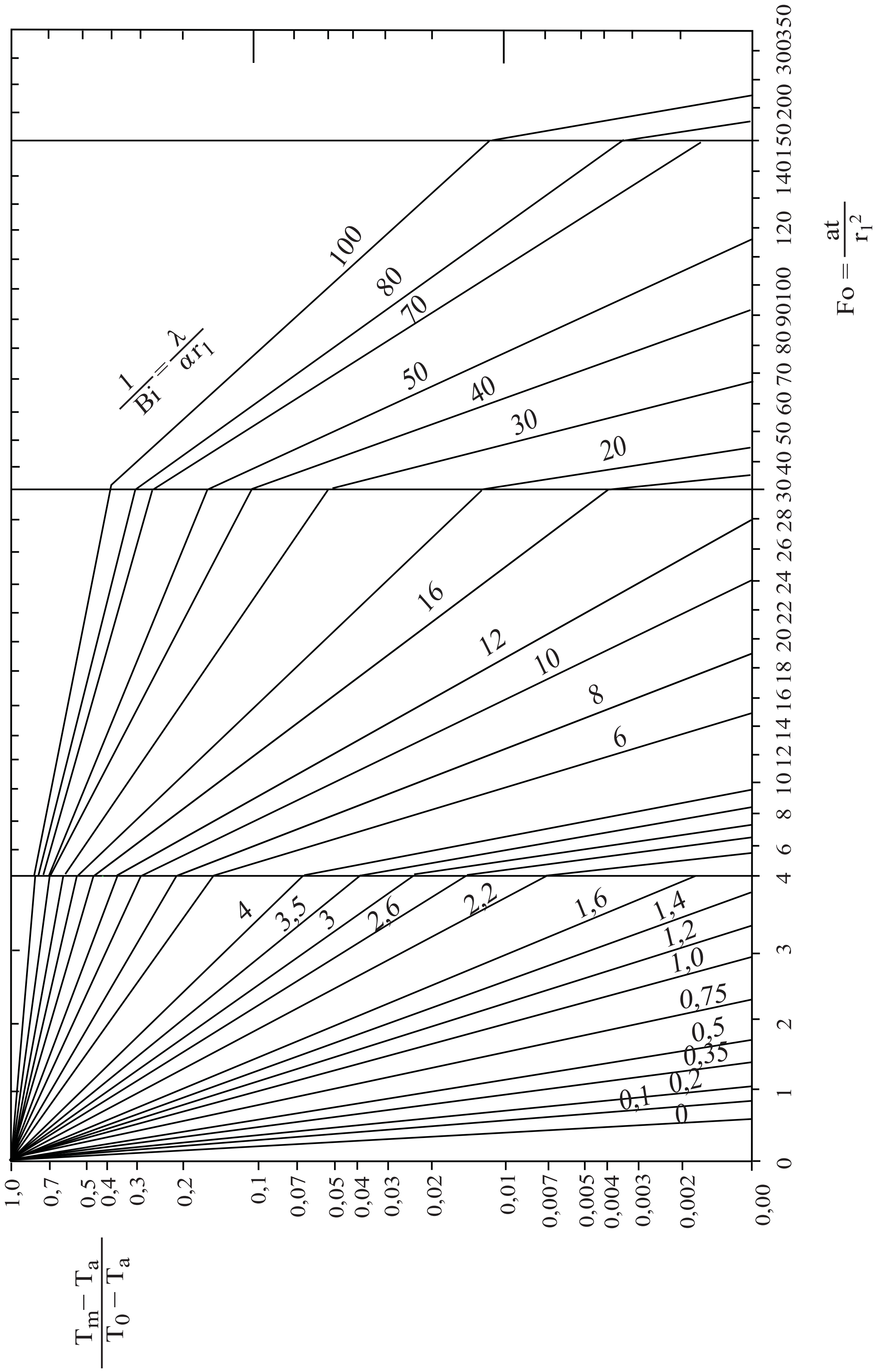


Diagramm 7: Temperature in the centre of a sphere with radius r_1

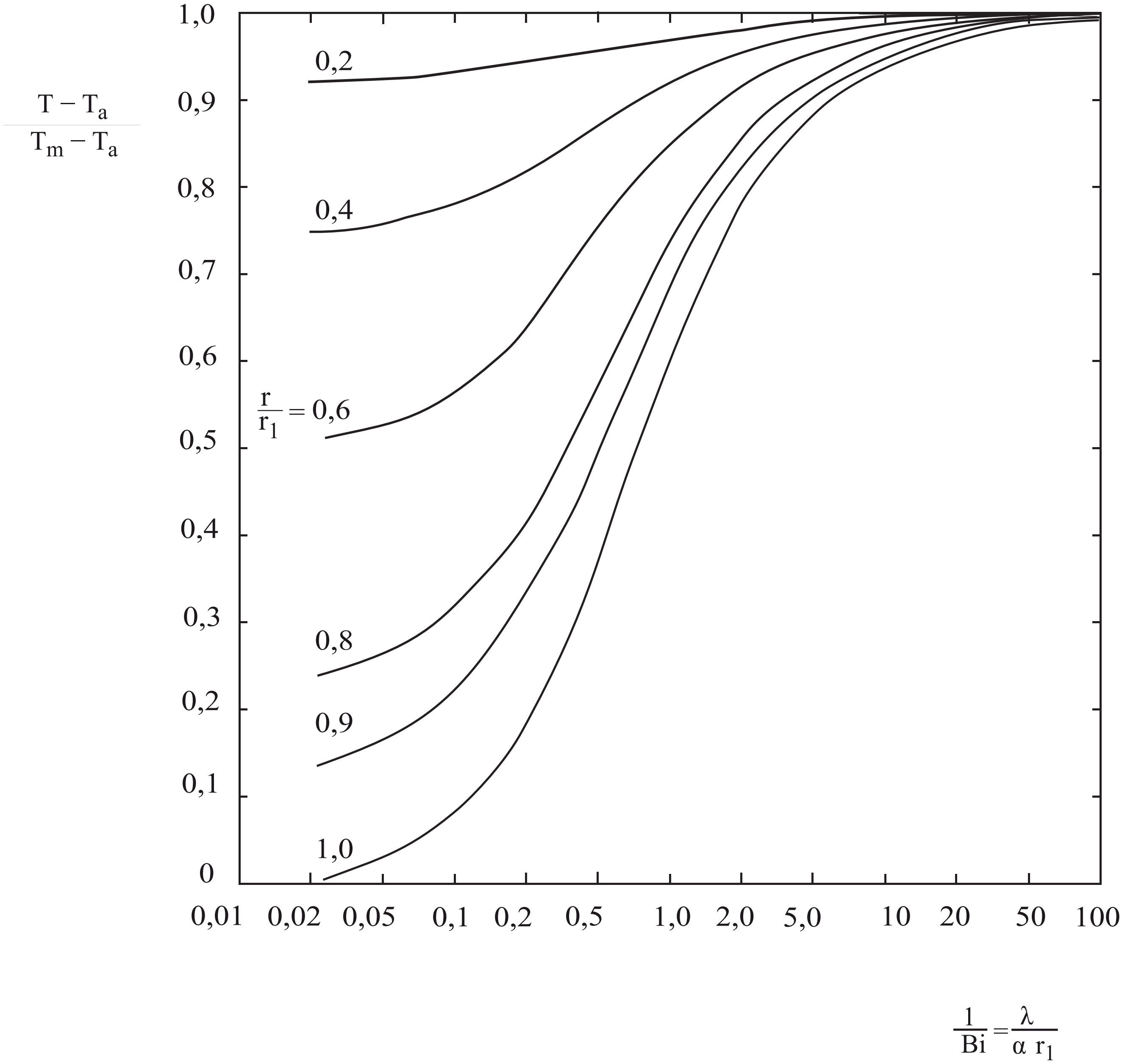


Diagramm 8: Temperature distribution in a sphere (valid for Fo $>$ 0,2)

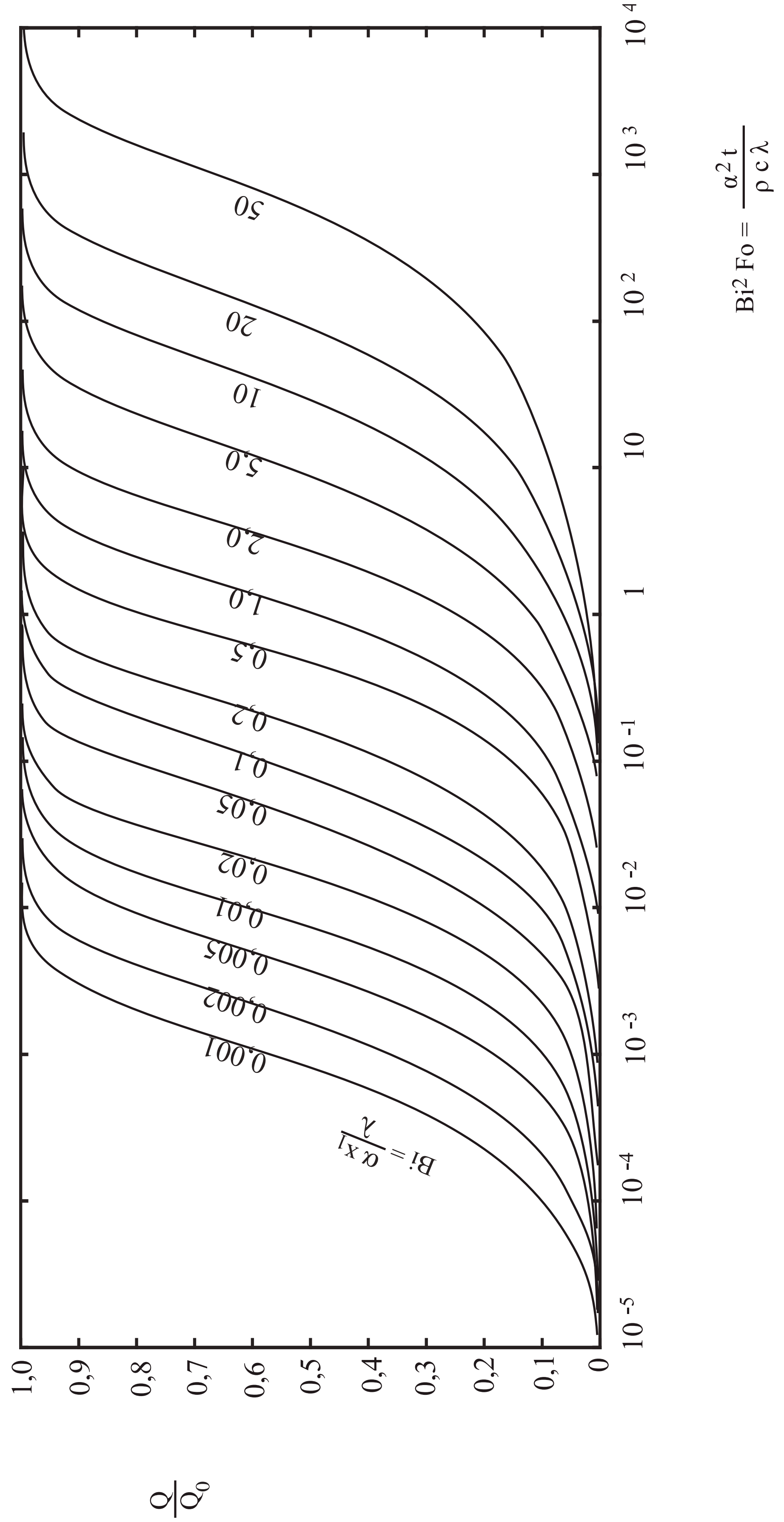


Diagramm 9: Heat loss of a plate

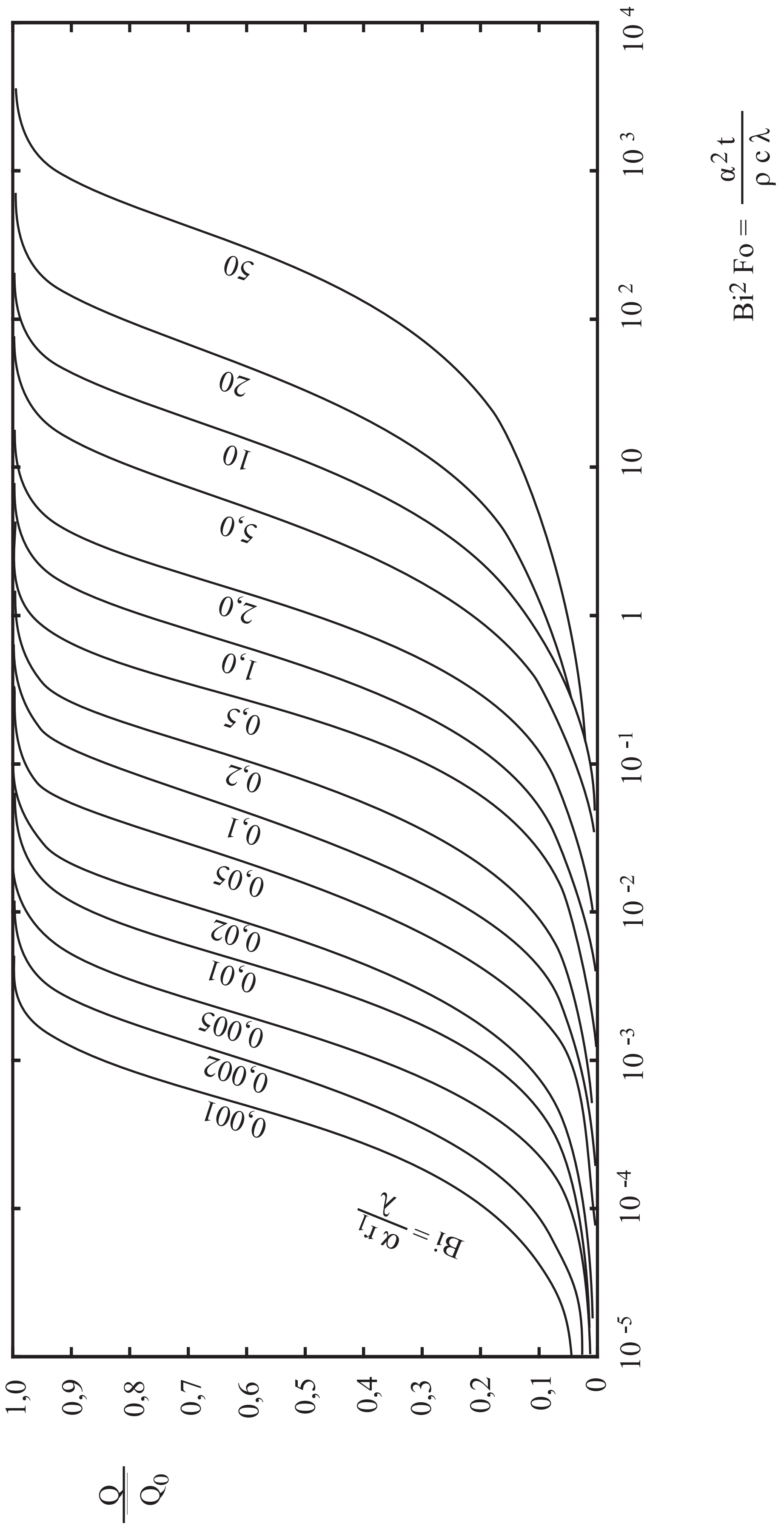


Diagramm 10: Heat loss of a cylinder

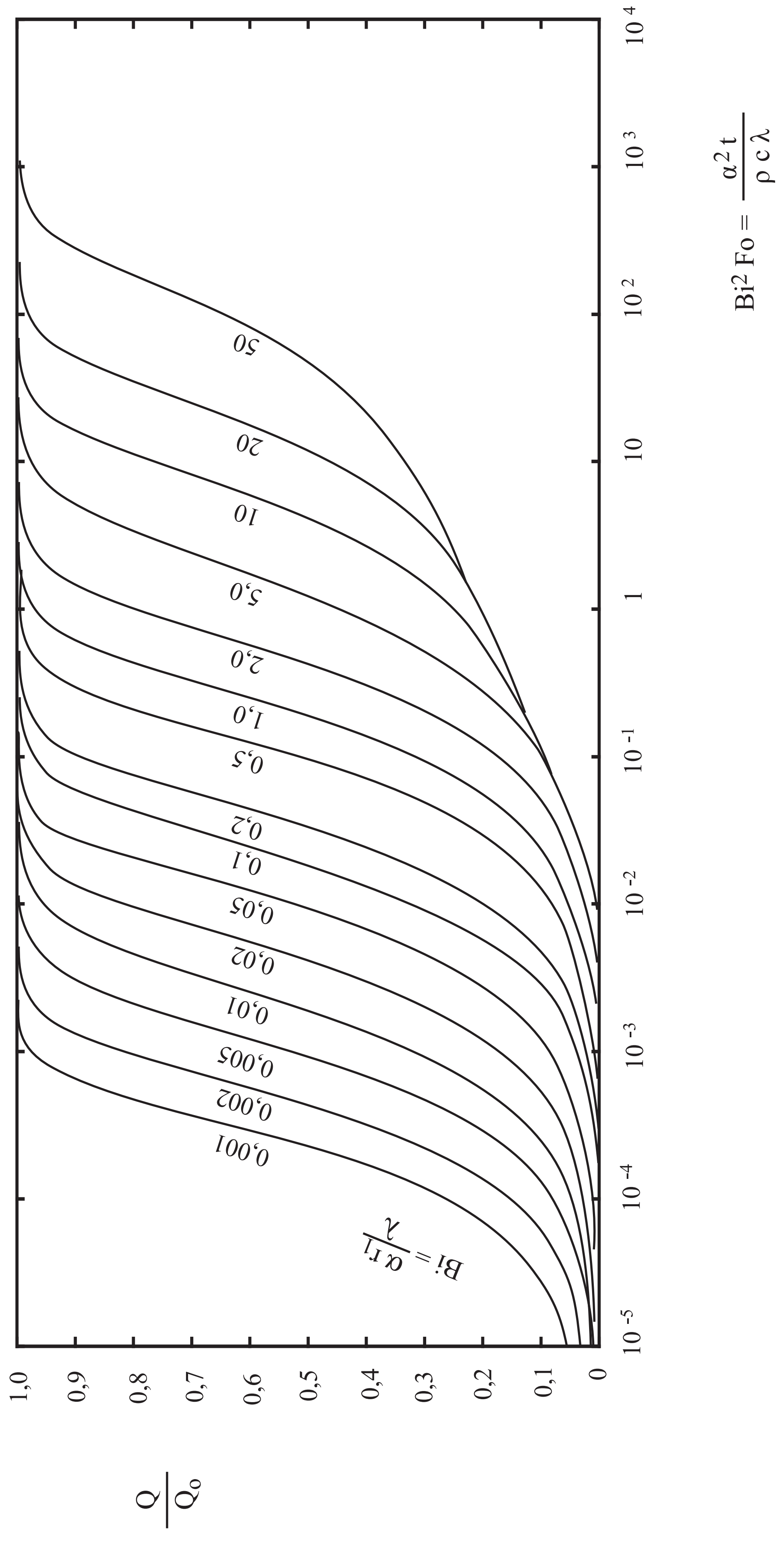


Diagramm 11: Heat loss of a sphere

4. Convection

$$\rho u c_{\mathrm{p}} \frac{\partial T}{\partial x} + \rho v c_{\mathrm{p}} \frac{\partial T}{\partial y} + \rho w c_{\mathrm{p}} \frac{\partial T}{\partial z} = \cdots \qquad \text{(Equation of energy conservation)}$$

$$\cdots = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \dot{\Phi}'''$$

Convective heat transfer

$$\frac{\dot{Q}}{A} = \dot{q}_{\mathrm{w}}'' = \alpha (T_{\mathrm{w}} - T_{\mathrm{fl}}) \qquad \text{(Convective heat flux)}$$

$$W = \frac{1}{\alpha \, A} \qquad \text{(Heat resistance)}$$

$$\alpha = \frac{-\left(\lambda \, \frac{dT}{dy}\right)_{\mathrm{Fluid,\,w}}}{T_{\mathrm{w}} - T_{\mathrm{fl}}} \qquad \text{(Heat transfer coefficient)}$$

$$\overline{\alpha} = \frac{1}{L} \int\limits_0^L \alpha\,(x)\;\mathrm{d}x \qquad \text{(Average h.t. coefficient)}$$

Boundary layer equations (Approximation with linear velocity profile)

$$\frac{\delta_u}{x} \approx \sqrt{\frac{12\,\eta}{\rho\,u_\infty x}} = \sqrt{\frac{12}{\mathrm{Re}_x}} \qquad \text{(Thickness of the velocity boundary layer)}$$

$$\frac{\delta_T}{\delta_u} \approx \left(\frac{\lambda}{\eta c_{\mathrm{p}}}\right)^{1/3} = \frac{1}{\mathrm{Pr}^{1/3}} \qquad \text{(Thickness of the temperature boundary layer)}$$

5. Heat transfer correlations

$$\Delta T_{\mathrm{ln}} = (T_{\mathrm{w}} - T_{\mathrm{fl}})_{\mathrm{m}} = \frac{\Delta T_{\mathrm{I}} - \Delta T_{\mathrm{O}}}{\ln \frac{\Delta T_{\mathrm{I}}}{\Delta T_{\mathrm{O}}}} \qquad \text{(Logarithmic temperature difference)}$$

$$\dot{Q}_{\mathrm{m}} = \bar{\alpha}\, A\, (T_{\mathrm{w}} - T_{\mathrm{fl}})_{\mathrm{m}} \qquad \text{(Average heat flow)}$$

Forced convection flow along surfaces

$$\mathrm{Nu}_x = f\,(\mathrm{Re}_x, \mathrm{Pr}, \ldots) \qquad \text{(Nusselt-correlation)}$$

$$T_{\mathrm{prop.}} = \frac{T_{\mathrm{w}} + T_\infty}{2} \qquad \text{(Temperature for determination of properties)}$$

- **Flat plate – laminar flow, isothermal surface (1)**

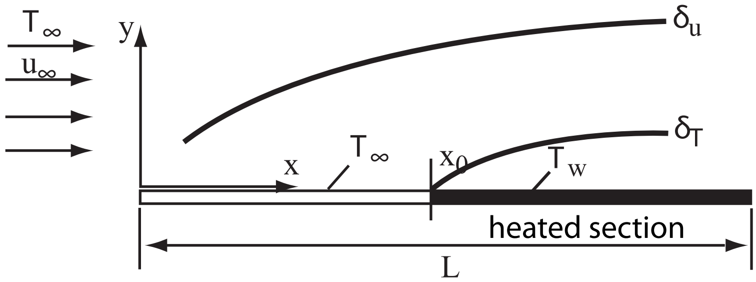
  ($0{,}6 < \mathrm{Pr} < 10$ and $\mathrm{Re}_x < \mathrm{Re}_{x,\,\mathrm{crit}} \approx 2 \cdot 10^5$)

$$\mathrm{Nu}_x = 0{,}332\ \mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3} \qquad \text{(HTC.1)}$$

$$\overline{\mathrm{Nu}}_L = 0{,}664\ \mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3} \qquad \text{(HTC.2)}$$

- **Flat plate – laminar boundary layer flow, isothermal surface (2)**

  Heating or cooling starts at $x = x_0$



($0{,}6 < \mathrm{Pr} < 10$ and $\mathrm{Re}_x < \mathrm{Re}_{x,\,\mathrm{crit}} \approx 2 \cdot 10^5$)

$$\mathrm{Nu}_x = 0{,}332\ \mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3} \qquad \text{(HTC.3)}$$

$$\overline{\mathrm{Nu}}_L = \frac{L}{L - x_0}\frac{1}{\lambda}\int\limits_{x_0}^{L} \alpha(x)\mathrm{d}x$$

$$= 0{,}664\ \mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3}\frac{\left[1 - \left(\dfrac{x_0}{L}\right)^{3/4}\right]^{2/3}}{\left[1 - \dfrac{x_0}{L}\right]} \qquad \text{(HTC.4)}$$

- **Flat plate – turbulent boundary layer flow, isothermal surface**

  ($\mathrm{Re}_{L,\,\mathrm{crit}} \approx 2 \cdot 10^5$ and $5 \cdot 10^5 < \mathrm{Re} < 10^7$)

$$\mathrm{Nu}_x = 0{,}0296\ \mathrm{Re}_x^{0{,}8}\mathrm{Pr}^{0{,}43} \qquad \text{(HTC.5)}$$

$$\overline{\mathrm{Nu}}_L \approx 0{,}036\ \mathrm{Pr}^{0{,}43}\left(\mathrm{Re}_L^{0{,}8} - 9400\right) \qquad \text{(HTC.6)}$$

- **Cylinders in a flow parallel to their longitudinal axis**

  If the diameter of the body is much greater compared to the thickness of the boundary layer, cylinders in longitudinal flow can be regarded as flat plates.

- **Cylinders in a flow perpendicular to their longitudinal axis**

$$\overline{\mathrm{Nu}}_d = C\ \mathrm{Re}_d^m\mathrm{Pr}^{0{,}4} \qquad \text{(HTC.7)}$$

Re_d	C	m
0,4 – 4	0,989	0,330
4 – 40	0,911	0,385
40 – 4000	0,683	0,466
4000 – 40000	0,193	0,618
40000 – 400000	0,0266	0,805

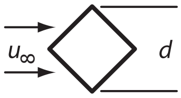
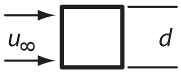
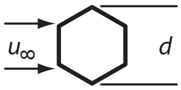
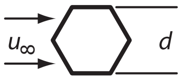
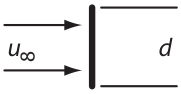
HTC.8 can be used as an alternative to HTC.7:

$$\overline{\mathrm{Nu}}_d = \left[0{,}40\ \mathrm{Re}_d^{1/2} + 0{,}06\ \mathrm{Re}_d^{2/3}\right]\mathrm{Pr}^{0{,}4}\left(\frac{\eta_\infty}{\eta_\mathrm{w}}\right)^{1/4} \qquad \text{(HTC.8)}$$

here: $T_{\mathrm{prop.}} = T_\infty$

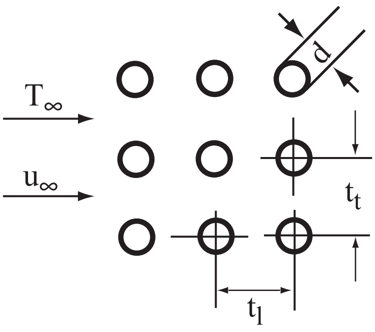
- **Mean heat transfer for non circular cylinders**

$$\overline{\mathrm{Nu}}_d = C\ \mathrm{Re}_d^m \mathrm{Pr}^{0,4} \tag{HTC.9}$$

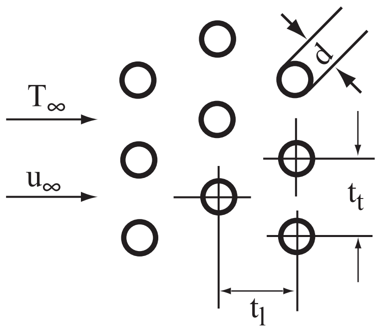
Geometry	Re_d	C	m
	$5 \cdot 10^3 - 1 \cdot 10^5$	0,246	0,588
	$5 \cdot 10^3 - 1 \cdot 10^5$	0,102	0,675
	$5 \cdot 10^3 - 1{,}95 \cdot 10^4$	0,160	0,638
	$1{,}95 \cdot 10^4 - 1 \cdot 10^5$	0,0385	0,782
	$5 \cdot 10^3 - 1 \cdot 10^5$	0,153	0,638
	$4 \cdot 10^3 - 1{,}5 \cdot 10^4$	0,228	0,731

- **Flow perpendicular to plain tube bundles**

$$\overline{\mathrm{Nu}}_d = 0{,}287\ \mathrm{Re}_d^{0,6}\ \mathrm{Pr}^{0,36}\ f_\mathrm{e} \tag{HTC.10}$$



in-line arrangement



staggered arrangement

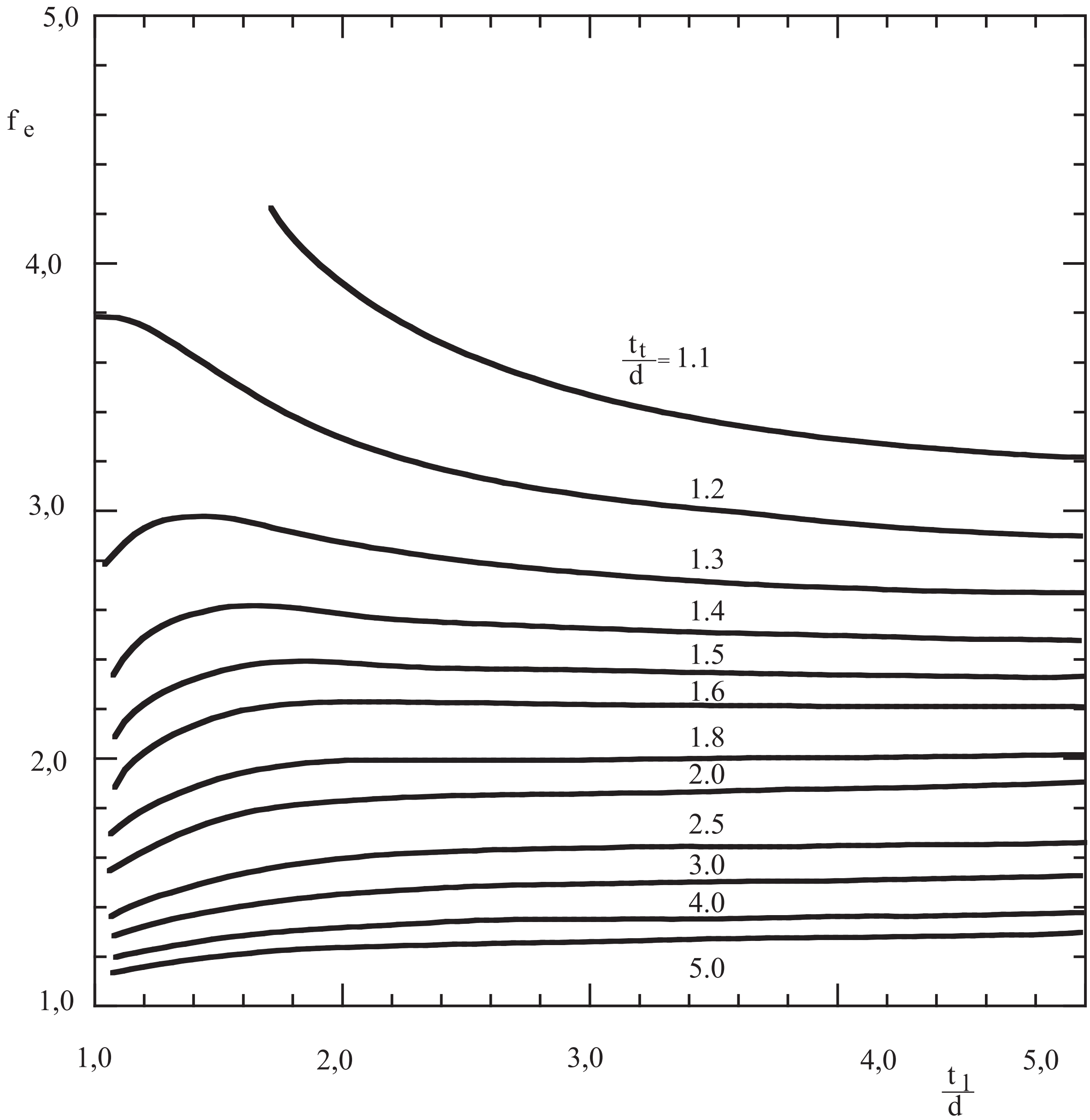


Figure 1: Arrangement factor, in-line arrangement

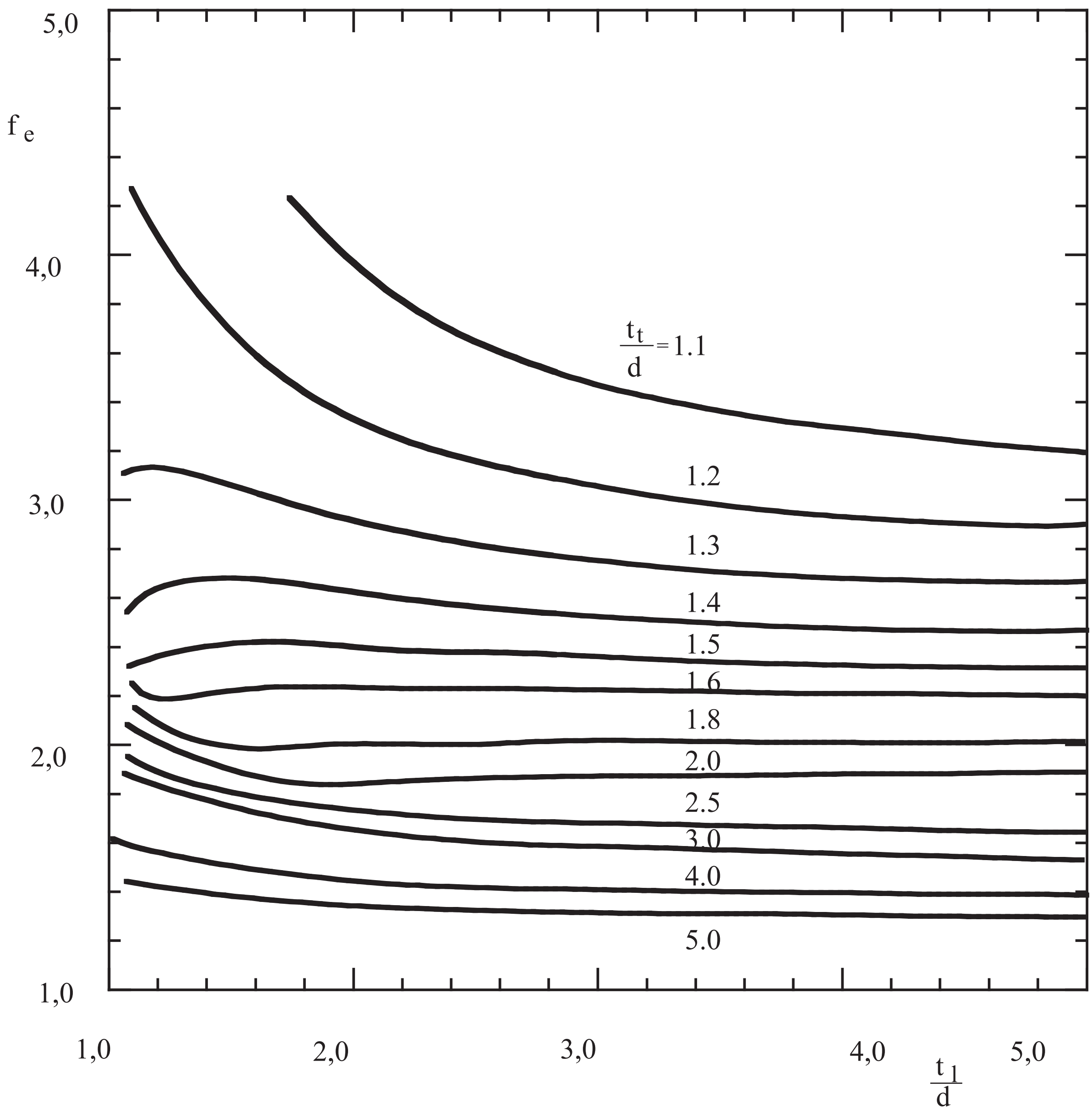


Figure 2: Arrangement factor, staggered arrangement

- **Heat transfer from a surrounding fluid to spheres in the stream**

  ($0{,}7 < \mathrm{Pr} < 380$ and $3{,}5 < \mathrm{Re}_d < 8 \cdot 10^4$)

$$\overline{\mathrm{Nu}}_d = 2 + \left(0{,}4\ \mathrm{Re}_d^{1/2} + 0{,}06\ \mathrm{Re}_d^{2/3}\right)\ \mathrm{Pr}^{0{,}4}\ \left(\frac{\eta_\infty}{\eta_\mathrm{w}}\right)^{1/4} \qquad \text{(HTC.11)}$$

here: $T_{\text{prop.}} = T_\infty$

Forced convection in tubes, internal flow

$$\mathrm{Nu}_x = f\left(\mathrm{Re}_x, \mathrm{Pr}, \ldots\right) \qquad \text{(Nusselt-correlation)}$$

$$T_\mathrm{mat} = \frac{T_{\mathrm{fl,\,O}} + T_{\mathrm{fl,\,I}}}{2} \qquad \text{(Material property determination temperature)}$$

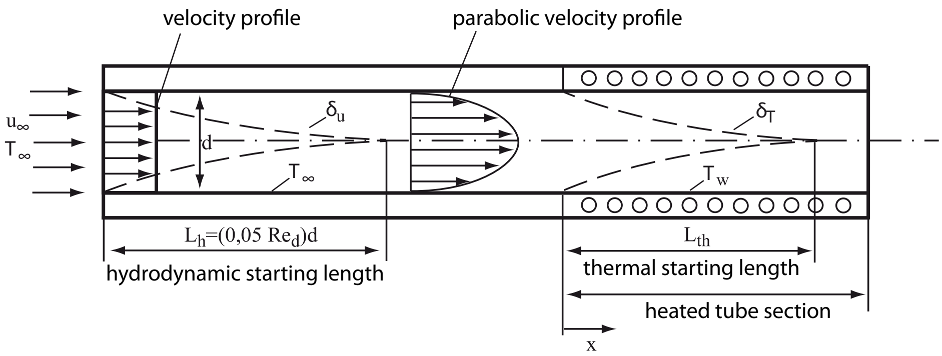
$$d_h = 4\frac{A_\mathrm{c}}{U} \qquad \text{(Hydraulic mean diameter)}$$

here: A_c cross-section area

U wetted perimeter

- **Laminar flow in tubes – isothermal surface (1)**

  Fully developed flow at the start of the heat transferring section of a tube ($\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300$)



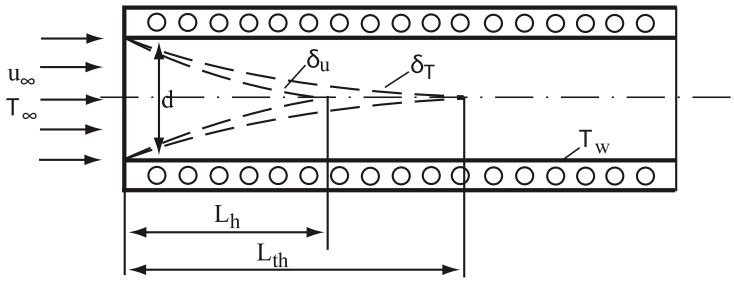
$$\overline{\mathrm{Nu}}_d = \left(3{,}66 + \frac{0{,}19\left(\mathrm{Re}_d\mathrm{Pr}\frac{d}{L}\right)^{0{,}8}}{1 + 0{,}117\left(\mathrm{Re}_d\mathrm{Pr}\frac{d}{L}\right)^{0{,}467}}\right)\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0{,}14} \qquad \text{(HTC.12)}$$

$$\frac{L_\mathrm{th}}{d} \approx 0{,}05\ \mathrm{Re}_d\mathrm{Pr} \qquad \text{(Thermal starting length)}$$

After L_th, the Nusselt number has a final value of $\overline{\mathrm{Nu}}_\infty = 3{,}66\,(\eta/\eta_\mathrm{W})^{0{,}14}$.

- **Laminar flow in tubes – isothermal surface (2)**

  Simultaneous hydrodynamic and thermal start ($\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300$)



$$\overline{\mathrm{Nu}}_d = \left(3{,}66 + \frac{0{,}0677\left(\mathrm{Re}_d \mathrm{Pr}\frac{d}{L}\right)^{1{,}33}}{1 + 0{,}1\mathrm{Pr}\left(\mathrm{Re}_d\frac{d}{L}\right)^{0{,}83}}\right)\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0{,}14} \tag{HTC.13}$$

$$\frac{L_\mathrm{th}}{d} \approx 0{,}05\,\mathrm{Re}_d\mathrm{Pr} \qquad \text{(Thermal starting length)}$$

  After L_th, the Nusselt number has a final value of $\overline{\mathrm{Nu}}_\infty = 3{,}66\,(\eta/\eta_\mathrm{W})^{0{,}14}$.

- **Laminar flow in tubes – impressed heat flow**

  If instead of the wall temperature, the heat flow at the wall remains constant, then the heat transfer coefficients have values increased by 20%.

- **Turbulent flow in tubes – isothermal surface**

  Simultaneous hydrodynamic and thermal start
  ($\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300$, $\mathrm{Re}_d > 2300$, $0{,}6 < \mathrm{Pr} < 500$ and $L/d > 1$)

$$\overline{\mathrm{Nu}}_d = 0{,}0235\left(\mathrm{Re}_d^{0{,}8} - 230\right)\left(1{,}8\mathrm{Pr}^{0{,}3} - 0{,}8\right)\left(1 + \left(\frac{d}{L}\right)^{2/3}\right)\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0{,}14} \tag{HTC.14}$$

  Simplified Nusselt law for the *fully developed* turbulent pipe flow
  ($\mathrm{Re}_{d,\,\mathrm{crit}} \approx 2300$, $3000 < \mathrm{Re}_d < 10^5$ and $L/d > 40$)

$$\overline{\mathrm{Nu}}_d = 0{,}027\mathrm{Re}_d^{0{,}8}\mathrm{Pr}^{1/3}\left(\frac{\eta}{\eta_\mathrm{w}}\right)^{0{,}14} \tag{HTC.15}$$

- **Turbulent flow in tubes – impressed heat flow**

  The heat transfer coefficients at impressed heat flows are comparable to the coefficients obtained at constant wall temperatures.

Natural convection

$$\mathrm{Nu}_x = f\left(\mathrm{Gr}_x, \mathrm{Pr}, \ldots\right) \qquad \text{(Nusselt-correlation)}$$

$$T_{\text{prop.}} = \frac{T_\mathrm{w} + T_\infty}{2} \qquad \text{(Temperature for determination of properties)}$$

$$\beta = \frac{1}{T_\infty} \qquad \text{(Isobaric expansion coefficient)}$$

- **Vertical plate – laminar boundary layer flow, isothermal surface**

$$\mathrm{Nu}_x = 0{,}508\left(\frac{\mathrm{Pr}}{0{,}952 + \mathrm{Pr}}\right)^{1/4}\left(\mathrm{Gr}_x\mathrm{Pr}\right)^{1/4} \qquad \text{(HTC.16)}$$

$$\overline{\mathrm{Nu}}_L = C\left(\mathrm{Gr}_L\mathrm{Pr}\right)^{1/4} \qquad \text{(HTC.17)}$$

for $\quad \mathrm{Gr}_L\mathrm{Pr} < \mathrm{Gr}_{L,crit}\mathrm{Pr} = 4 \cdot 10^9$

Pr	0,003	0,01	0,03	0,72	1	2	10	100	1000	∞
C	0,182	0,242	0,305	0,516	0,535	0,568	0,620	0,653	0,665	0,670

- **Vertical plate – laminar boundary layer flow, impressed heat flow**

$$\mathrm{Nu}_x = 0{,}60\left(\mathrm{Gr}_x^*\mathrm{Pr}\right)^{1/5} \qquad \text{(HTC.18)}$$

for $\quad 10^5 < \mathrm{Gr}_x^* < 10^{11}$

with $\quad \mathrm{Gr}_x^* = \mathrm{Gr}_x\mathrm{Nu}_x = \dfrac{\rho^2 g \beta \dot{q}_w'' x^4}{\lambda \eta^2}$

- **Vertical plate – turbulent boundary layer flow, isothermal surface**

$$\overline{\mathrm{Nu}}_L = 0{,}13\left(\mathrm{Gr}_L Pr\right)^{1/3} \qquad \text{(HTC.19)}$$

for $\quad 10^9 < \left(\mathrm{Gr}_L\mathrm{Pr}\right) < 10^{12}$

- **Vertical cylinder – laminar and turbulent boundary layer flow**

  For diameter-length-ratios of $d/L > 35\,\mathrm{Gr}_L^{-1/4}$, the relationships valid for the vertical plate can be applied.

- **Horizontal cylinder – isothermal surface**

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_d = 0{,}53\,(\mathrm{Gr}_d\mathrm{Pr})^{1/4} \tag{HTC.20}$$
$$\text{for} \quad 10^4 < \mathrm{Gr}_d\mathrm{Pr} < 10^9$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_d = 0{,}13\,(\mathrm{Gr}_d\mathrm{Pr})^{1/3} \tag{HTC.21}$$
$$\text{for} \quad 10^9 < \mathrm{Gr}_d\mathrm{Pr} < 10^{12}$$

- **Horizontal plate – isothermal surface**

  Free upper side with $T_\mathrm{W} > T_\infty$ or free lower side with $T_\mathrm{W} < T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0{,}54\,(\mathrm{Gr}_L\mathrm{Pr})^{1/4} \tag{HTC.22a}$$
$$\text{for} \quad 2\cdot 10^4 < \mathrm{Gr}_L\mathrm{Pr} < 8\cdot 10^6$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_L = 0{,}15\,(\mathrm{Gr}_L\mathrm{Pr})^{1/3} \tag{HTC.23a}$$
$$\text{for} \quad 8\cdot 10^6 < \mathrm{Gr}_L\mathrm{Pr} < 10^{11}$$

  Free upper side with $T_\mathrm{W} < T_\infty$ or free lower side with $T_\mathrm{W} > T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0{,}27\,(\mathrm{Gr}_L\mathrm{Pr})^{1/4} \tag{HTC.24a}$$
$$\text{for} \quad 10^5 < \mathrm{Gr}_L\mathrm{Pr} < 10^{11}$$

- **Horizontal plate – impressed heat flow**

  Free upper side with $T_\mathrm{W} > T_\infty$ or free lower side with $T_\mathrm{W} < T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0{,}13\,(\mathrm{Gr}_L\mathrm{Pr})^{1/3} \tag{HTC.22b}$$
$$\text{for} \quad \mathrm{Gr}_L\mathrm{Pr} < 2\cdot 10^8$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_L = 0{,}16\,(\mathrm{Gr}_L\mathrm{Pr})^{1/3} \tag{HTC.23b}$$
$$\text{for} \quad 2\cdot 10^8 < \mathrm{Gr}_L\mathrm{Pr} < 10^{11}$$

Free upper side with $T_{\mathrm{W}} < T_\infty$ or free lower side with $T_{\mathrm{W}} > T_\infty$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_L = 0{,}58\,(\mathrm{Gr}_L\mathrm{Pr})^{1/5} \tag{HTC.24b}$$

$$\text{for} \quad 10^6 < \mathrm{Gr}_L\mathrm{Pr} < 10^{11}$$

- **Fluid layers between isothermal, vertical walls**

  Height/distance ratio $3{,}1 < H/s < 42{,}2$

$$\text{heat conduction only:} \qquad \overline{\mathrm{Nu}}_s = 1 \quad \text{for} \quad \mathrm{Gr}_s < 2 \cdot 10^3$$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_s = 0{,}20\,(H/s)^{-1/9}\,(\mathrm{Gr}_s\mathrm{Pr})^{1/4} \tag{HTC.25}$$

$$\text{for} \quad 2 \cdot 10^3 < \mathrm{Gr}_s < 2 \cdot 10^4$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_s = 0{,}071\,(H/s)^{-1/9}\,(\mathrm{Gr}_s\mathrm{Pr})^{1/3} \tag{HTC.26}$$

$$\text{for} \quad 2 \cdot 10^5 < \mathrm{Gr}_s < 10^7$$

- **Fluid layers between isothermal, horizontal walls**

$$\text{heat conduction only:} \qquad \overline{\mathrm{Nu}}_s = 1 \quad \text{for} \quad \mathrm{Gr}_s < 2 \cdot 10^3$$

$$\text{laminar:} \qquad \overline{\mathrm{Nu}}_s = 0{,}21\,(\mathrm{Gr}_s\mathrm{Pr})^{1/4} \tag{HTC.27}$$

$$\text{for} \quad 2 \cdot 10^3 < \mathrm{Gr}_s < 3{,}2 \cdot 10^5$$

$$\text{turbulent:} \qquad \overline{\mathrm{Nu}}_s = 0{,}075\,(\mathrm{Gr}_s\mathrm{Pr})^{1/3} \tag{HTC.28}$$

$$\text{for} \quad 3{,}2 \cdot 10^5 < \mathrm{Gr}_s < 10^7$$

If heated from above, the relationships for heat conduction only are valid.

7. Mass transfer

$$j'' = -\rho\, D \frac{\partial \xi}{\partial x} \qquad \text{(Fick's law)}$$

$$\xi_i = \frac{\rho_i}{\rho_{total}} \qquad \text{(Mass fraction)}$$

$$\sum j'' = 0 \qquad \text{(Sum of all diffusion fluxes)}$$

$$\sum \xi = 1 \qquad \text{(Sum of all mass concentrations)}$$

Transport equation

$$\rho u \frac{\partial \xi}{\partial x} + \rho v \frac{\partial \xi}{\partial y} + \rho w \frac{\partial \xi}{\partial z} = \cdots$$

$$\cdots = \frac{\partial}{\partial x}\left(\rho D \frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho D \frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho D \frac{\partial \xi}{\partial z}\right) + \dot{m}'''$$

Mass transfer on a surface

$$\frac{\dot{m}}{A} = \dot{m}'' = g\,(\xi_O - \xi_\infty) \qquad \text{(Convective mass flux)}$$

$$\mathrm{Sh}_x = f\,(\mathrm{Re}_x, \mathrm{Sc}, \ldots) \qquad \text{(Mass transfer laws)}$$

compare to: $$\mathrm{Nu}_x = f\,(\mathrm{Re}_x, \mathrm{Pr}, \ldots) \qquad \text{(Heat transfer laws)}$$

Mass transfer on a semi-permeable surface (e.g. evaporation)

$$\dot{m}'' = g \cdot \frac{\xi_O - \xi_\infty}{1 - \xi_O} \qquad \text{(Mass flux with Stefan correction)}$$

Analogy between heat and mass transfer

$$\left.\begin{aligned} \mathrm{Sh}_x &= C\,\mathrm{Re}_x^m\,\mathrm{Sc}^n \\ \mathrm{Nu}_x &= C\,\mathrm{Re}_x^m\,\mathrm{Pr}^n \end{aligned}\right\}\ \frac{\mathrm{Sh}_x}{\mathrm{Nu}_x} = \left(\frac{\mathrm{Sc}}{\mathrm{Pr}}\right)^n$$

$$\mathrm{Le} = \frac{\mathrm{Sc}}{\mathrm{Pr}} = \frac{\lambda}{\rho D\, c_\mathrm{p}} \qquad \text{(Lewis number)}$$

$$\mathrm{Le} = 1 \quad \rightarrow \quad \frac{g}{\alpha / c_\mathrm{p}} = 1 \qquad \text{(Lewis relation)}$$

Heat transfer		Mass transfer	
Temperature	T	Mass fraction	ξ
Heat flux	$\dot{q}''$	Mass flux	j''
Vol. heat capacity	$\rho\, c_\mathrm{p}$	Density	ρ
Heat conductivity	λ		$\rho\, D$
Thermal diffusivity	$a = \lambda / \rho\, c_\mathrm{p}$	Coefficient of diffusion	D
Convective heat- and masstransport			
Heat transfer coefficient	α	Mass transfer coefficient.	g
Nusselt number	$\mathrm{Nu}_x = \alpha\, x / \lambda$	Sherwood number	$\mathrm{Sh}_x = g\, x / \rho\, D$
Prandtl number	$\mathrm{Pr} = \eta\, c_\mathrm{p} / \lambda$	Schmidt number	$\mathrm{Sc} = \eta / \rho D$

Appendix A – Properties of various materials

Tabelle 1: Metals at 20°C

	ρ	c	λ	a
	10^3 kg/m^3	kJ/kg K	W/m K	10^{-6} m^2/s
Aluminum	2,70	0,888	237	98,80
Lead	11,34	0,129	35	23,90
Chromium	6,92	0,440	91	29,90
Iron	7,86	0,452	81	22,80
Gold	19,26	0,129	316	127,20
Copper	8,93	0,382	399	117,00
Magnesium	1,74	1,020	156	87,90
Manganese	7,42	0,473	21	6,00
Molybdenum	10,20	0,251	138	53,90
Sodium	9,71	1,220	133	11,20
Nickel	8,85	0,448	91	23,00
Platinum	21,37	0,133	71	25,00
Silver	10,50	0,235	427	173,00
Titanium	4,50	0,522	22	9,40
Wolfram	19,00	0,134	173	67,90
Zinc	7,10	0,387	121	44,00
Tin, white	7,29	0,225	67	40,80
Bronze	8,80	0,377	62	18,70
Cast iron	7,80	0,540	42. . . 50	10. . . 12
Carbon steel (<0,4% C)	7,85	0,465	42. . . 50	12. . . 15
Cr-Ni-steel (X12CrNi 18,8)	7,80	0,500	15	3,80

Tabelle 2: Non-metal solids at 20°C

	ρ	c	λ	a
	10^3 kg/m^3	kJ/kg K	W/m K	10^{-6} m^2/s
Acryl glass	1,18	1,44	0,184	0,108
Asphalt	2,12	0,92	0,7	0,36
Concrete	2,1	0,88	1	0,54
Ice (water 0ï¿½C)	0,917	2,04	2,25	1,203
Soil coarse gravel	2,04	1,84	0,52	0,14
Sand, dry	1,65	0,8	0,27	0,2
Sand, wet	1,75	1	0,58	0,33
Clay	1,45	0,88	1,28	1
Glass.				
window	2,48	0,7	0,87	0,5
mirror	2,7	0,8	0,76	0,35
quarz	2,21	0,73	1,4	0,87
Glass wool	1,2	0,66	0,046	0,58
Gypsum	1	1,09	0,51	0,47
Granite	2,75	0,89	2,9	1,18
Cork	0,19	1,88	0,041	0,115
Marble	2,6	0,8	2,8	1,35
Mortar	1,9	0,8	0,93	0,61
Paper	0,7	1,2	0,12	0,14
Polyethylene	0,92	2,3	0,35	0,17
Polytetrafluorethylene	2,2	1,04	0,23	0,1
PVC	1,38	0,96	0,15	0,11
Porcelain (95ï¿½C)	2,4	1,08	1,03	0,4
Hard coal	1,35	1,26	0,26	0,15
Fir wood (radial)	0,415	2,72	0,14	0,12
Plaster	1,69	0,8	0,79	0,58
Bricks	1,6. . . 1,8	0,84	0,38. . . 0,52	0,28. . . 0,34

Tabelle 3: Liquids at 1 bar

	T	ρ	c	λ	ν	a	Pr
	°C	10^3 kg/m^3	kJ/kg K	W/m K	10^{-6} m^2/s	10^{-6} m^2/s	1
Nitrogen	-190	0,861	1,988	0,161	0,321	0,0939	3,42
Water	0	0,9998	4,218	0,561	1,793	0,133	13,48
	20	0,9982	4,181	0,598	1,004	0,1434	7,001
	40	0,9922	4,177	0,631	0,658	0,1521	4,3280
	60	0,9832	4,184	0,654	0,475	0,1591	2,983
	80	0,9718	4,197	0,67	0,365	0,1643	2,221
	99,63	0,9586	4,216	0,679	0,295	0,168	1,757
Aqueous non-organic solution							
21% NaCl	-10	1,187	3,312	0,528	4,02	0,136	29,5
Benzene	20	0,879	1,738	0,154	0,74	0,101	7,33
Methanol	20	0,792	2,495	0,22	0,737	0,111	6,57
Fuel oil	20	0,819	2	0,116	1,82	0,0709	25,7
	100	0,766	2,38	0,104	0,711	0,0572	12,4
Mercury	20	13,55	0,139	9,3	0,115	4,9	0,023

Tabelle 4: Gases at 1 bar

	T	ρ	c	λ	ν	a	Pr
	°C	kg/m^3	kJ/kg K	10^{-3} W/m K	10^{-6} m^2/s	10^{-6} m^2/s	1
Air	-200	5,106	1,186	6,886	0,979	1,137	0,8606
	-100	2,019	1,011	16,2	5,829	7,851	0,7423
	0	1,275	1,006	24,18	13,52	18,83	0,7179
	20	1,188	1,007	25,69	15,35	21,47	0,7148
	40	1,112	1,007	27,16	17,26	24,24	0,7122
	80	0,9859	1,01	30,01	21,35	30,14	0,7083
	100	0,9329	1,012	31,39	23,51	33,26	0,707
	200	0,7356	1,026	37,95	35,47	50,3	0,7051
	400	0,517	1,069	49,96	64,51	90,38	0,7137
	600	0,3986	1,116	61,14	99,63	137,5	0,7247
	800	0,3243	1,155	71,54	140,2	191	0,7342
	1000	0,2734	1,185	80,77	185,9	249,2	0,7458
Steam	100	0,5896	2,042	25,08	20,81	20,83	0,999
	200	0,4604	1,975	33,28	35,14	36,6	0,96
	400	0,3223	2,07	54,76	75,86	82,07	0,9243
	600	0,2483	2,203	79,89	131,4	146,1	0,8993
	800	0,2019	2,343	107,3	199,9	226,8	0,8816
	1000	0,1702	2,478	163,3	280	323,2	0,8665
Hydrogen	0	0,0886	14,24	176	95	139	0,68
	50	0,0748	14,36	202	126	188	0,67
	100	0,0649	14,44	229	159	244	0,65
Carbon dioxide	0	1,95	0,829	14,3	7,1	8,86	0,8
	50	1,648	0,875	17,8	9,8	12,3	0,8
	100	1,428	0,925	21,3	12,4	16,1	0,8
Helium	27	0,1625	5,193	155,7	122,6	184,5	0,655

Tabelle 5: Binary diffusion coefficients from gases

	T	D
	K	10^{-4} m^2/s
Air – CO_2	276	0,144
	317	0,179
Air – C_2H_5OH	313	0,147
Air – He	276	0,632
Air – H_2O	313	0,292
CO_2 – H_2O	307	0,201
He – H_2O	352	1,136
H_2 – H_2O	307	0,927
CH_4 – H_2O	352	0,361

Tabelle 6: Binary diffusion coefficients from diluted aqueous solutions

	T	D
	K	10^{-9} m^2/s
CH_4 – H_2O	275	0,85
	333	3,55
CO_2 – H_2O	298	2
CH_3OH – H_2O	288	1,26
C_2H_5OH – H_2O	288	1
O_2 – H_2O	298	2,4
N_2 – H_2O	298	2,6
H_2 – H_2O	298	6,3

Tabelle 7: Emissivity of various solids (Total emissivity ε, Emissivity in normal direction of the surface ε_n)

Surface	T K	ε_n	ε	**Surface**	T K	ε_n	ε
Metals				Zinc, highly polished poliert	500		0,045
Alumium, plain	443	0,039	0,049		600		0,055
… polished	373	0,095		Iron plate, galvanized			
… heavily oxidized	366	0,2		… plain	301	0,228	
	777	0,31		… grey oxidized	297	0,276	
Aluminum oxide	550	0,63		Tin, non oxidized	298		0,043
	1100	0,26			373	0,05	
	1089	0,052		**Non-Metals**			
Chromium, polished	423	423	423	Asbestos, paper	296	0,96	
				… Papier	311	0,93	
Gold. highly polished	500	0,018			644	0,94	
	900	900		Concrete, rough	273 – 366		0,94
Copper, polished	293	0,03		Roofing felt	294	0,91	
… struck	293	0,037		Gips	293	0,8 – 0,9	
… black oxidized	293	0,78		Glas	293	0,94	
… oxidized	403	0,76		Quartz (7 mm thick)	555	0,93	
Inconel, rolled	1089		0,69		1111	0,47	
… sandblasted	1089		0,79	Rubber	293	0,92	
Iron and steel,				Wood			
… highly polished	450	0,052		Oak, planed	273 – 366		0,9
… polished	700	0,144		Beech	343	0,94	0,91
	1300	0,377		Ceramics			
… sanded	293	0,242		White Al_2O_3	366		0,9
Cast iron, polished	473	0,21		Carbon			
Cast steel, polished	1044	0,52		… not oxidized	298		0,81
	1311	0,56			773		0,79
Iron sheet				… Fibers	533		0,95
… heavy rusty	292	0,685		… Graphite	373		0,76
… rolled	294	0,657		Corundum, rough	353	0,85	0,84
Cast iron,				Coating, colors:			
… oxidized at 866 K	472	0,64		Oil paint black	366		0,92
	872	0,78		… green	366		0,95
Steel,				… red	366		0,97
… oxidized at 866 K	472	0,79		… white	373		0,94
	872	0,79		Coating. white	373	0,925	
Brass, not oxidized	298	0,035		… flat black	353		0,97
	373	0,035		Bakelite coating	353	0,935	
… oxidized	473	0,61		Mennig color	373	0,93	
	873	0,59		Radiator (acc. to VDI-74)	373	0,925	
	1673	0,17		Enamel, white on iron	292	0,897	
Nickel, not oxidized	298	0,045		Marble			
	373	0,06		light grey. polished	273 – 366		0,9
	873	0,478		Paper	273		0,92
… oxidized	473	0,37			366		0,94
Platinum	422	0,022		Porcelain, white	295		0,924
	1089	0,123		Clay, glassy	298		0,9
Mercury,				… flat	298		0,93
… not oxidized	298	0,1		Water	273	0,95	
	373	0,12			373	0,96	
Silver, polished	311	0,022		Ice, smooth with water	273	0,966	0,92
	644	0,031		… rough surface	273	0,985	
Wolfram	298		0,024	Bricks, red	273 – 366		0,93
	1273		0,15				
	1773		0,23				

Appendix B – Mathematical formulary

Error function

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int\limits_{\xi=0}^{\xi=\eta} \exp(-\xi^2) d\xi$$

$$\operatorname{erfc}(\eta) = 1 - \operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int\limits_{\xi=\eta}^{\xi=\infty} \exp(-\xi^2) d\xi$$

Important characteristics

$$\operatorname{erf}(\infty) = 1 \quad \operatorname{erf}(-\eta) = -\operatorname{erf}(\eta) \quad \frac{d}{d\eta}\left[\operatorname{erf}(\eta)\right] = \frac{2}{\sqrt{\pi}} \exp(-\eta^2)$$

Tabelle 8: Evaluation of the Error function

η	$\operatorname{erf}(\eta)$	$\operatorname{erfc}(\eta)$	$^2/_{\sqrt{\pi}} \exp(-\eta^2)$
0	0	1	1,128
0,05	0,056	0,944	1,126
0,1	0,112	0,888	1,117
0.15	0,168	0,832	1,103
0,2	0,223	0,777	1,084
0,25	0,276	0,724	1,060
0,3	0,329	0,671	1,031
0,35	0,379	0,621	0,998
0,4	0,428	0,572	0,962
0,45	0,475	0,525	0,922
0,5	0,520	0,480	0,879
0,55	0,563	0,437	0,834
0,6	0,604	0,396	0,787
0,65	0,642	0,378	0,740
0,7	0,678	0,322	0,691
0,75	0,711	0,289	0,643
0,8	0,742	0,258	0,595
0,85	0,771	0,229	0,548
0,9	0,797	0,203	0,502
0,95	0,821	0,179	0,458
1	0,843	0,157	0,415
1,1	0,880	0,120	0,337
1,2	0,910	0,090	0,267
1,3	0,934	0,066	0,208
1,4	0,952	0,048	0,159
1,5	0,966	0,034	0,119
1,6	0,976	0,024	0,087
1,7	0,984	0,016	0,063
1,8	0,989	0,011	0,044
1,9	0,993	0,007	0,030
2	0,995	0,005	0,021

Bessel functions

Tabelle 9: Evaluation of the Bessel functions of 1. and 2. mode

x	$I_0(x)$	$I_1(x)$	$2/\pi \cdot K_0(x)$	$2/\pi \cdot K_1(x)$
0	1	0	∞	∞
0,2	1,0100	0,1005	1,1160	3,0410
0,4	1,0404	0,2040	0,7095	1,3910
0,6	1,0920	0,3137	0,4950	0,8294
0,8	1,1665	0,4329	0,3599	0,5486
1	1,2661	0,5652	0,2680	0,3832
1,2	1,3937	0,7147	0,2028	0,2768
1,4	1,5534	0,8861	0,1551	0,2043
1,6	1,7500	1,0848	0,1197	0,1532
1,8	1,9896	1,3172	0,9290 10^{-1}	0,1163
2	2,2796	1,5906	0,7251	0,8904 10^{-1}
2,2	2,6291	1,9141	0,5683	0,6869
2,4	3,0493	2,2981	0,4470	0,5330
2,6	3,5533	2,7554	0,3527	0,4156
2,8	4,1573	3,3011	0,2790	0,3254
3	4,8808	3,9534	0,2212	0,2556
3,2	5,7472	4,7343	0,1757	0,2014
3,4	6,7848	5,6701	0,1398	0,1592
3,6	8,0277	6,7028	0,1114	0,1261
3,8	9,5169	8,1404	0,8891 10^{-2}	0,9999 10^{-2}
4	11,302	9,7595	0,7105	0,7947
4,2	13,443	11,706	0,5684	0,6327
4,4	16,010	14,046	0,4551	0,5044
4,6	19,093	16,863	0,3648	0,4027
4,8	22,794	20,253	0,2927	0,3218
5	27,240	24,336	0,2350	0,2575
5,2	32,584	29,254	0,1888	0,2062
5,4	39,009	35,182	0,1518	0,1653
5,6	46,738	42,328	0,1221	0,1326
5,8	56,038	50,946	0,9832 10^{-3}	0,1064
6	67,234	61,342	0,7920	0,8556 10^{-3}
6,2	80,718	73,886	0,6382	0,6879
6,4	96,962	89,026	0,5146	0,5534
6,6	116,54	107,31	0,4151	0,4455
6,8	140,14	129,38	0,3350	0,3588
7	168,59	156,04	0,2704	0,2891
7,2	202,92	188,25	0,2184	0,2331
7,4	244,34	227,18	0,1764	0,1880
7,6	294,33	274,22	0,1426	0,1517
7,8	354,69	331,10	0,1153	0,1424
8	427,56	399,87	0,9325 10^{-4}	0,9891 10^{-4}
8,2	515,59	483,05	0,7543	0,7991
8,4	621,94	583,66	0,6104	0,6458
8,6	750,46	705,38	0,4941	0,5220
8,8	905,80	852,66	0,4000	0,4221
9	1.093,0	1.030,90	0,3239	0,3415
9,2	1.320,7	1.246,70	0,2624	0,2763
9,4	1.595,3	1.507,90	0,2126	0,2236
9,6	1.927,5	1.824,10	0,1722	0,1810
9,8	2.329,4	2.207,10	0,1396	0,1465
10	2.815,7	2.671,00	0,1131	0,1187

Particular functions

$$\begin{aligned}
\sin(x \pm y) &= \sin(x) \cdot \cos(y) \pm \cos(x) \cdot \sin(y) \\
\cos(x \pm y) &= \cos(x) \cdot \cos(y) \mp \sin(x) \cdot \sin(y) \\
\sin(2x) &= 2 \cdot \sin(x) \cdot \cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \\
\sinh(x) &= \frac{\exp(x) - \exp(-x)}{2} \\
\cosh(x) &= \frac{\exp(x) + \exp(-x)}{2} \\
\sinh(x \pm y) &= \sinh(x) \cdot \cosh(y) \pm \cosh(x) \cdot \sinh(y) \\
\cosh(x \pm y) &= \cosh(x) \cdot \cosh(y) \pm \sinh(x) \cdot \sinh(y) \\
\sinh(2x) &= 2 \cdot \sinh(x) \cdot \cosh(x) \\
\cosh(2x) &= \sinh^2(x) + \cosh^2(x) = 2\cosh^2(x) - 1 \\
\operatorname{artgh}(x) &= \frac{1}{2} \ln \frac{1+x}{1-x} \quad \text{mit} \quad (|x| < 1) \\
\operatorname{arsinh}(x) &= \ln\left(x + \sqrt{x^2+1}\right) \\
\operatorname{arcosh}(x) &= \ln\left(x + \sqrt{x^2-1}\right) \quad \text{mit} \quad (|x| \geq 1) \\
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp\left(-\eta^2\right) d\eta \quad \text{mit} \quad \operatorname{erf}(\infty) = 1 \\
\operatorname{erfc}(x) &= 1 - \operatorname{erf}(x) \\
\exp\left(\ln(x)\right) &= x \\
\lg(x) &= \frac{\ln(x)}{\ln(10)}
\end{aligned}$$

Series

- Arithmetic series

$$s = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \frac{n}{2}(2a + (n-1)d)$$

- Geometric Series

$$s = a + aq + aq^2 + \cdots + aq^{n-1} = \sum_{\nu=0}^{n-1} aq^\nu = a\,\frac{1-q^n}{1-q}$$

$$\text{Infinite series:} \qquad s = \sum_{\nu=0}^{\infty} aq^\nu = a\,\frac{1}{1-q} \quad \text{for} \quad |q| < 1$$

- Taylor-series

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

- Power series of particular functions

$$\begin{aligned}
(1 \pm x)^m &= 1 \pm mx + \frac{m(m-1)}{2!}x^2 \pm \frac{m(m-1)(m-2)}{3!}x^3 + ... \\
&= \sum_{\nu=0}^{m} \binom{m}{\nu} x^\nu \quad \text{with} \quad \binom{m}{\nu} = \frac{m!}{\nu!(m-\nu)!} \\
\exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - ... = \sum_{\nu=0}^{\infty} (-1)^{\nu+1}\,\frac{x^\nu}{\nu} \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{\nu=0}^{\infty} (-1)^\nu\,\frac{x^{2\nu+1}}{(2\nu+1)!} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{\nu=0}^{\infty} (-1)^\nu\,\frac{x^{2\nu}}{2\nu!}
\end{aligned}$$

Differentiation

- Product rule
$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = [f\cdot g]' = f'\cdot g + g'\cdot f$$
- Quotient rule
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \left[\frac{f}{g}\right]' = \frac{f'\cdot g - g'\cdot f}{g^2}$$
- Chain rule
$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = \left[f\left(g(x)\right)\right]' = \frac{df}{dg}\cdot\frac{dg}{dx}$$
- Total differential of function $z = f(x,y)$
$$dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
- Derivation of a composite function of more than one variable $z = f(x,y)$ with $x(t)$ and $y(t)$
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Derivatives of elementary functions

$$\begin{array}{rcl}
\text{Function} & \xrightarrow{d/dx(\ldots)} & \text{Derivative} \\
x^n & & n\,x^{n-1} \\
\exp(x) & & \exp(x) \\
\ln(x) & & \dfrac{1}{x} \\
a^x & & a^x\ln(a) \\
\sin(x) & & \cos(x) \\
\cos(x) & & -\sin(x) \\
\tan(x) & & \dfrac{1}{\cos^2(x)} \quad = \quad 1+\tan^2(x) \\
\cot(x) & & -\dfrac{1}{\sin^2(x)} \quad = -(1+\cot^2(x))
\end{array}$$

$$\begin{array}{rcll}
\text{Function} & \xrightarrow{d/dx(...)} & \text{Derivative} & \\
\sinh(x) & & \cosh(x) & \\
\cosh(x) & & \sinh(x) & \\
\tanh(x) & & \dfrac{1}{\cosh^2(x)} & = \quad 1 - \tanh^2(x) \\
\coth(x) & & -\dfrac{1}{\sinh^2(x)} & \\
\lg(x) & & \dfrac{1}{\ln(10)}\dfrac{1}{x} &
\end{array}$$

Indefinite expressions

Rule of Bernoulli de l'Hospital

- Indefinite expressions of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

- Indefinite expressions of form $0 \cdot \infty$ with $f(x_0) = 0$ and $g(x_0) = \infty$

$$\lim_{x \to x_0} f(x) \cdot g(x) = \lim_{x \to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \lim_{x \to x_0} \frac{f'(x)}{\left(\frac{1}{g(x)}\right)'}$$

- Indefinite expressions of form $\infty - \infty$

$$\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} \frac{\left(\frac{1}{g(x)} - \frac{1}{f(x)}\right)'}{\left(\frac{1}{g(x)} \cdot \frac{1}{f(x)}\right)'}$$

Integration

- Indefinite Integral

$$\int f(x)dx = F(x) + C \quad \text{with the primitive function} \quad F(x)$$

- Definite Integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = -\int_b^a f(x)dx = F(b) - F(a)$$

- Differentiation of an integral by its upper limit

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$
$$\frac{d}{dx}\int_a^x f(x,t)dt = \int_a^x \frac{\partial f(x,t)}{\partial x}dt + f(x,t)$$

Integration rules

- Integration by parts

$$\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)dx$$

- Substitution formula

$$\int f(x)dx \quad \text{Substitution of} \quad x = g(t) \quad \rightarrow \quad t = h(x)$$

  - Indefinite Integration

$$\int f(x)dx = \int f(g(t)) \cdot g'(t)dt + C$$

  - Definite Integration

$$\int_a^b f(x)dx = \int_{h(a)}^{h(b)} f(g(t)) \cdot g'(t)dt$$

Primitives (antiderivatives) of elementary functions

Integrand	→	Primitive
$f(x)$	$\rightarrow$	$F(x) = \int f(x)dx$
x^n		$\dfrac{x^{n+1}}{n+1}$
$\dfrac{1}{x}$		$\ln\|x\|$
$\sin(x)$		$-\cos(x)$
$\cos(x)$		$\sin(x)$
$\dfrac{1}{\sin^2(x)}$		$-\mathrm{ctan}(x)$
$\tan(x)$		$-\ln\|\cos(x)\|$
$\cot(x)$		$\ln\|\sin(x)\|$
$\dfrac{1}{a^2+x^2}$		$\dfrac{1}{a}\mathrm{arc}\tan\dfrac{x}{a}$
$\dfrac{1}{a^2-x^2}$		$\dfrac{1}{a}\mathrm{arc}\tanh\dfrac{x}{a}$
$(\|x\| < a)$		$= \dfrac{1}{2a}\ln\left\|\dfrac{a+x}{a-x}\right\|$
$\dfrac{1}{x^2-a^2}$		$-\dfrac{1}{a}\mathrm{arc}\coth\dfrac{x}{a}$
$(\|x\| > a)$		$= \dfrac{1}{2a}\ln\left\|\dfrac{x-a}{x+a}\right\|$
$\dfrac{1}{a^4-x^4}$		$\dfrac{1}{4a^3}\ln\left\|\dfrac{a+x}{a-x}\right\|\cdots$
		$+\dfrac{1}{2a^3}\mathrm{arc}\tan\dfrac{x}{a}$

Integrand	→	Primitive
$f(x)$	$\rightarrow$	$F(x) = \int f(x)dx$
$\exp(x)$		$\exp(x)$
a^x		$\dfrac{a^x}{\ln(a)}$
$\sinh(x)$		$\cosh(x)$
$\cosh(x)$		$\sinh(x)$
$\dfrac{1}{\sinh^2(x)}$		$-\mathrm{ctanh}(x)$
$\tanh(x)$		$\ln\|\cosh(x)\|$
$\coth(x)$		$\ln\|\sinh(x)\|$
$\dfrac{1}{\sqrt{a^2+x^2}}$		$\mathrm{arc}\sinh\dfrac{x}{a}$
		$= \ln\|x+\sqrt{x^2+a^2}\|$
$\dfrac{1}{\sqrt{a^2-x^2}}$		$\mathrm{arc}\sin\dfrac{x}{a}$
$\dfrac{1}{\sqrt{x^2-a^2}}$		$\mathrm{arc}\cosh\dfrac{x}{a}$
		$= \ln\|x+\sqrt{x^2-a^2}\|$
$\sqrt{x^2 \pm a^2}$		$\dfrac{x}{2}\sqrt{x^2 \pm a^2}\cdots$
		$\pm\dfrac{a^2}{2}\ln\|x+\sqrt{x^2 \pm a^2}\|$

plus an integration constant C in every case

Integration methods

- Rational functions

$\int \frac{Q(x)}{P(x)}dx$ with the polynomials $Q(x)$, $P(x)$ $\rightarrow$ expansion into partial fraction

- Irrational functions

$$R(\ldots) = \text{rational function of } (\ldots)$$

$$\int R(\sinh(x), \cosh(x))dx \quad \rightarrow \quad \text{Substitute} \quad t = \tanh\left(\frac{x}{2}\right)$$

$$\sinh(x) = \frac{2t}{1-t^2}$$

$$\cosh(x) = \frac{1+t^2}{1-t^2}$$

$$dx = \frac{2}{1-t^2}dt$$

$$\int R(\sinh(x)) \cdot \cosh(x)dx \quad \rightarrow \quad \text{Substitute} \quad t = \sinh(x)$$

$$\int R(\cosh(x)) \cdot \sinh(x)dx \quad \rightarrow \quad \text{Substitute} \quad t = \cosh(x)$$

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{1/n}\right] dx \quad \rightarrow \quad \text{Substitute} \quad t = \left(\frac{ax+b}{cx+d}\right)^{1/n}$$

$$\int R\left(x, \sqrt{ax^2+2bx+c}\right) dx \quad \text{with} \quad ac - b^2 \neq 0$$

$$ac - b^2 > 0 \,,\, a > 0 \quad \rightarrow \quad \text{Substitute} \quad \sinh(t) = \frac{ax+b}{\sqrt{ac-b^2}}$$

$$ac - b^2 < 0 \,,\, a > 0 \quad \rightarrow \quad \text{Substitute} \quad \cosh(t) = \frac{ax+b}{\sqrt{b^2-ac}}$$

$$ac - b^2 < 0 \,,\, a < 0 \quad \rightarrow \quad \text{Substitute} \quad \sinh(t) = \frac{ax+b}{\sqrt{b^2-ac}}$$

- Trigonometric functions

$$\int R(\sin(x), \cos(x))dx \quad \rightarrow \quad \text{Substitute} \qquad t = \tan\left(\frac{x}{2}\right)$$

$$\sin(x) = \frac{2t}{1+t^2}$$

$$\cos(x) = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2}{1+t^2}dt$$

$$\int R(\sin(x)) \cdot \cos(x)dx \quad \rightarrow \quad \text{Substitute} \qquad t = \sin(x)$$

$$\int R(\cos(x)) \cdot \sin(x)dx \quad \rightarrow \quad \text{Substitute} \qquad t = \cos(x)$$

- Other transcendental functions

$$\int R\left(\exp(x)\right) dx \quad \rightarrow \quad \text{Substitute} \qquad t = \exp(x)$$

$$\int R\left(\ln(x)\right) dx \quad \rightarrow \quad \text{Substitute} \qquad t = \ln x$$

$$\cdots \quad \rightarrow \quad \text{partial Integration}$$

Application of single or multiple integration by parts on integrals of the following form:

$$\int \exp(ax)\sin(bx)dx^* \qquad \int P(x)\exp(ax)$$

$$\int \exp(ax)\cos(bx)dx^* \qquad \int P(x)\ln(bx)$$

$$\int \exp(ax)\sinh(bx)dx^* \qquad \int P(x)\sin(bx)$$

$$\int \exp(ax)\sinh(bx)dx^* \qquad \int P(x)\cos(bx)$$

$$\int P(x)\sinh(bx)$$

$$\int P(x)\cosh(bx)$$

*) The multiple application of the integration by parts leads to the original integral: Solving an algebraic equation

Differential equations

The order of a differential equation is the one of the highest included derivative in the equation. If a specific initial state is determined besides the already given ordinary boundary conditions of a differential equation, the problem is called Initial value problem.

Ordinary differential equations of first order

- Separable types

  Type: $y' = f(x) \cdot g(y)$ $\rightarrow$ Solution: $\int \frac{dy}{g(y)} = \int f(x)dx + C$

  Type: $y' = f(ax + by + c)$ $\rightarrow$ Substitute: $z = ax + by + c$

  $\cdots$ $\rightarrow$ separable equation: $z' = a + b\ f(z)$

- Linear differential equation

  The dependent variable y and its derivatives appear in its first power only. Coefficients $f(x)$ are not constant, inhomogeneous if $g(x) \neq 0$

  Type: $y' + y\ f(x) = g(x)$

  $\rightarrow$ Solution: $y = \exp\left(-\int f(x)dx\right)\left(C + \int g(x)\exp\left(\int f(x)dxdx\right)\right)$

- Bernoulli differential equation

  Type: $y' + y\ f(x) = g(x)y^n$ $\rightarrow$ Substitute: $z = y^{1-n}$

  $z' = (1-n)y^{-n}y'$

  $\cdots$ $\rightarrow$ linear equation: $z' + (1-n)f(x)z \cdots$

  $\cdots = (1-n)g(x)$

Ordinary differential equations of higher order

Ordinary differential equations of high order always can be converted into a system of ordinary differential equations of first order. If a ordinary differential equation y_1 is of order n, auxiliary functions are introduced as follows:

$$\begin{aligned} y_1^{'} &= y_2 \\ y_2^{'} &= y_3 \\ &\vdots \\ y_{n-1}^{'} &= y_n \\ y_n^{'} &= f\left(x, y_1, y_2, y_3, \ldots, y_n\right) \end{aligned}$$

Therefor a system of n ordinary differential equations of first order is set, which can be solved like described in the text above.

Linear differential equations of second order with constant coefficients

$$\text{Type:}\quad y'' + a_1 y' + a_0 y = f(x) \quad \rightarrow \quad \text{Solution:}\quad y = y_{\mathrm{H}} + y_{\mathrm{P}}$$

Homogeneous solution

Complimentary function $y_{\mathrm{H}} = \exp(\mu x)$, determination of roots of characteristic polynomial which is derived for the homogeneous solution:

$$\mu^2 + a_1\mu + a_0 = 0 \quad \rightarrow \quad \mu_{1/2} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$$

Case-by-case analysis of the roots:

- $\left(\left(\frac{a_1}{2}\right)^2 - a_0\right) > 0 \quad \rightarrow \quad \mu_1 \neq \mu_2$ (two real roots)

  Approach: $y_{\mathrm{H}} = C_1 \exp(\mu_1 x) + C_2 \exp(\mu_2 x)$

  $$a_1 = 0,\ a_0 < 0 \quad \rightarrow \quad \mu_{1/2} = \pm\sqrt{-a_0},\ \mu = \sqrt{-a_0}$$

  Approach: $y_{\mathrm{H}} = C_1 \sinh(\mu x) + C_2 \cosh(\mu x)$

- $\left(\left(\frac{a_1}{2}\right)^2 - a_0\right) = 0 \quad \rightarrow \quad \mu_{1/2} = -\frac{a_1}{2}$ (one double root)

  Approach: $y_{\mathrm{H}} = \exp\left(-\frac{a_1}{2}\,x\right)\ (C_1 + C_2\,x)$

- $\left(\left(\frac{a_1}{2}\right)^2 - a_0\right) < 0 \quad \rightarrow \quad \mu_{1/2} = -\frac{a_1}{2} \pm i\sqrt{a_0 - \left(\frac{a_1}{2}\right)^2}$ (conj. complex root)

  Approach: $y_{\mathrm{H}} = \exp(-\frac{a_1}{2}\,x)\cdots$

  $$\cdots\left(C_1\,\cos\left(x\sqrt{a_0 - \left(\frac{a_1}{2}\right)^2}\right) + C_2\,\sin\left(x\sqrt{a_0 - \left(\frac{a_1}{2}\right)^2}\right)\right)$$

  $$a_1 = 0,\ a_0 > 0 \quad \rightarrow \quad \mu_{1/2} = \pm\, i\,\sqrt{a_0},\ \mu = \sqrt{a_0}$$

  Approach: $y_{\mathrm{H}} = C_1 \sin(\mu x) + C_2 \cos(\mu x)$

Particular solution – Particular form of perturbation term

$$f(x) = \exp(kx)\ (P_n(x)\cos(\omega x) + Q_n(x)\sin(\omega x))$$

Approach in form of perturbation term:

$$y_{\mathrm{P}} = \exp(kx)\ (M_n(x)\cos(\omega x) + N_n(x)\sin(\omega x)) \cdot x^m$$

- For $\omega \neq 0$ always both polynomials M_n and N_n have to be considered in the approach of a solution, for $\omega = 0$ N_n is omitted.
- The polynomials M_n and N_n always have to be completely considered in the approach of a solution, i.e. no coeffizient is expected to be zero.

Case-by-case analysis for m on basis of roots of the characteristic polynomial:

$$\begin{array}{rrcl} & \mu_1 = \mu_2 \overset{!}{=} k & \rightarrow & m = 2 \\ & \mu_{1/2} \overset{!}{=} k \pm i\omega & \rightarrow & m = 1 \\ \text{apart from that:} & \mu_{1/2} \neq k \pm i\omega & \rightarrow & m = 0 \end{array}$$

Application of the perturbation term $f(x)$ and the Ansatz y_{P} to the differential equation. The coefficients of the polynomials $M_n(x)$ and $N_n(x)$ are calculated by comparison of coefficients.

Particular solution – In general

Solution of the precipitation integral

$$y_{\mathrm{P}}(x) = \int_0^x y_{\mathrm{H},0}(x-\xi) \cdot f(\xi) d\xi$$

Here $y_{\mathrm{H},0}$ is a particular homogeneous solution with the initial-/boundary conditions $y_{\mathrm{H}}(0) = 1$ and $y'_{\mathrm{H}}(0) = 1$.