Heat Transfer

Boundary Layer Equation – Forced Convection

Prof. Dr.-Ing. Reinhold Kneer
Dr.-Ing. Dr. rer. pol. Wilko Rohlfs
Prof. dr. ir. Kees Venner

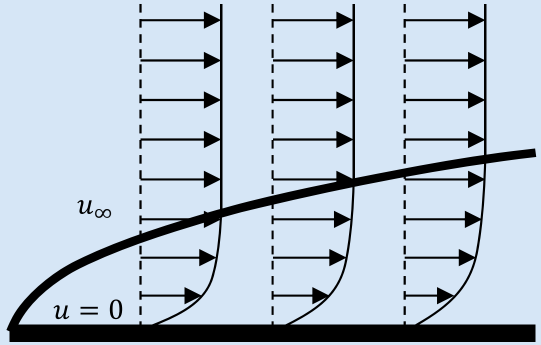
UNIVERSITY OF TWENTE.
Lehrstuhl für Wärme- und Stoffübertragung
RWTH AACHEN UNIVERSITY

Learning Goals

- Boundary layer in forced convection

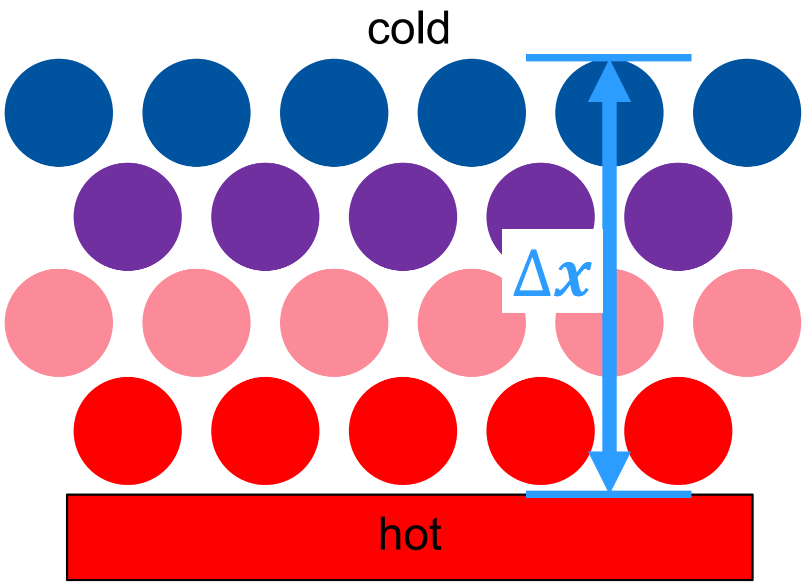
  - ➢ Understanding the boundary layer concept on a flat plate in a constant laminar flow.

  - ➢ Similarity of velocity and temperature profiles in the boundary layer, and the resulting relation between the heat transfer coefficient and the shear stress for this case

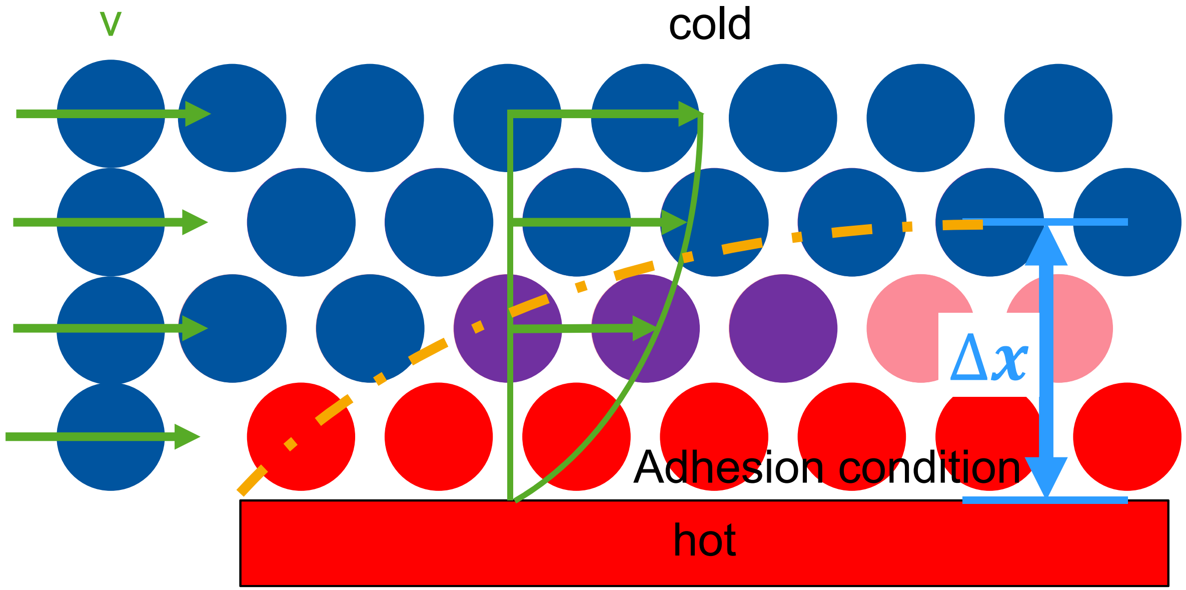


What is a Boundary Layer?

pure Heat Conduction



Heat Conduction + Convection



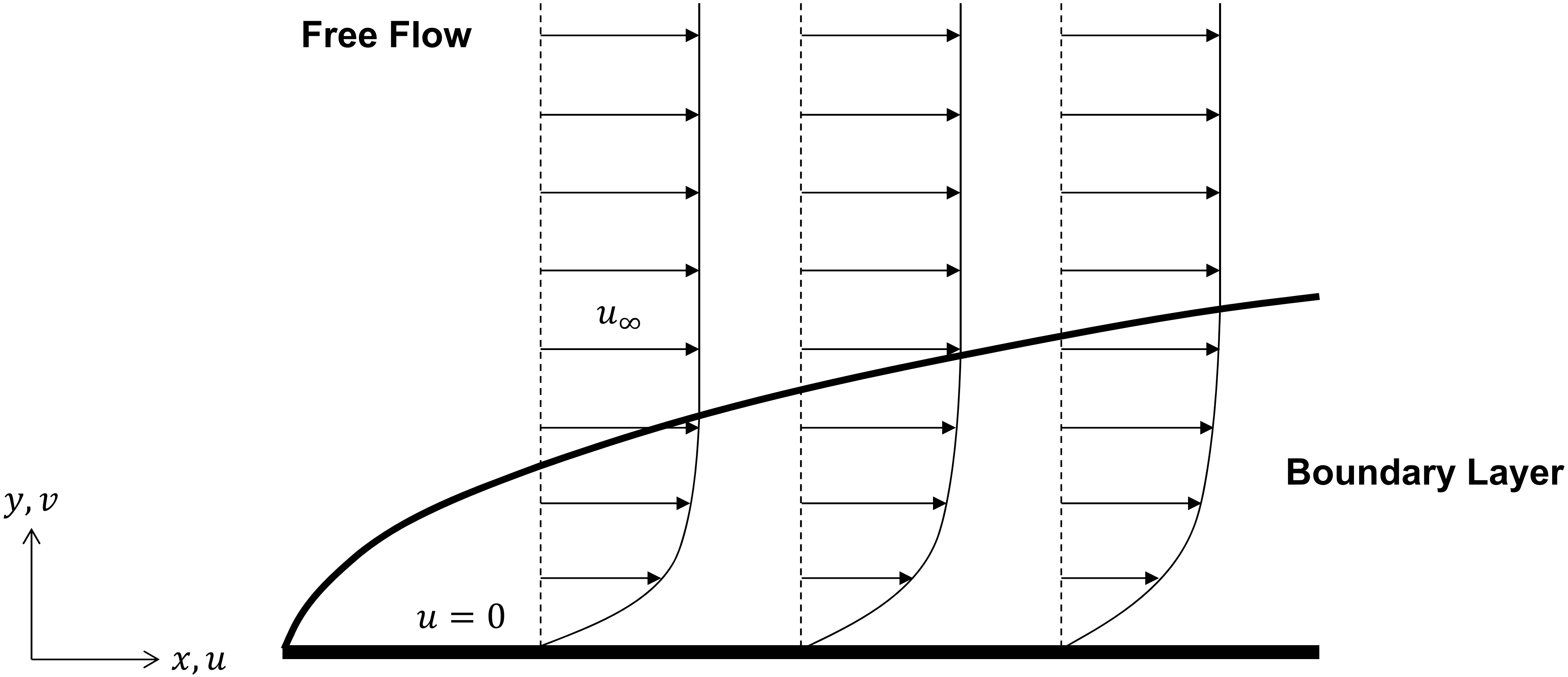
- The "temperature boundary layer" ends where the temperature has approached 99% of the ambient difference temperature.
- Convection increases the heat transfer.

⇒ steeper temperature gradient ⇒ higher heat transfer

Velocity Boundary Layer (Review Fluid Mechanics)

Definition

The thickness of the „Velocity Boundary Layer" δ_u is defined as the location where 99% of the value of the ambient velocity is reached: $u(y = \delta_u) = 0{,}99 u_\infty$

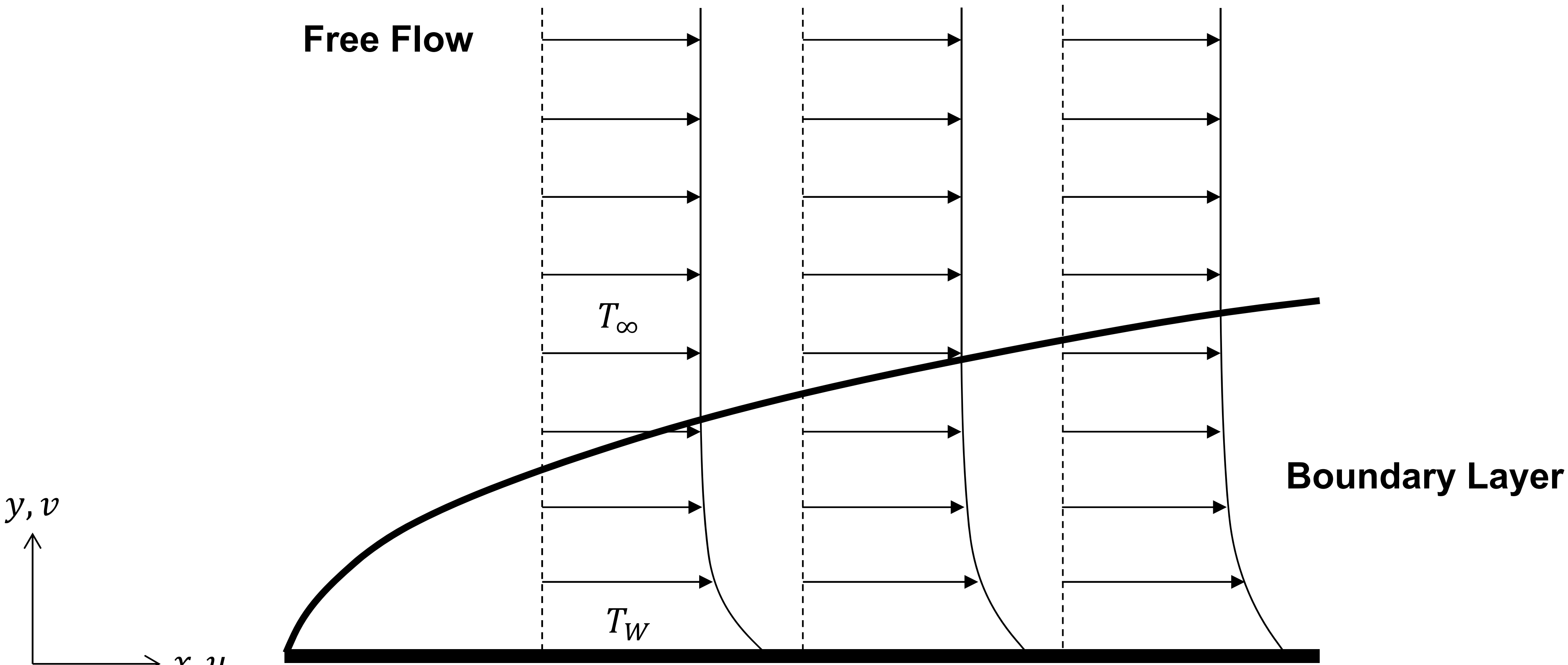


On the wall: adhesion condition

Temperature Boundary Layer

Definition (Analogously)

The „Temperature Boundary Layer“ thickness δ_T is where the value of 99% of the ambient temperature **difference** is reached : $T(y=\delta_T) - T_\infty = 0{,}99(T_W - T_\infty)$



On the Wall: Given Temperature T_W, here $T_W > T_\infty$

Review: Conservation equations (2D, steady state, incompressible)

Mass Flow

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\delta \ll L, \qquad u \gg v \rightarrow \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

- In the direction of flow, absolute velocity predominates.
- Perpendicular to the surface (normal direction), gradients dominate.

Review: Conservation equations (2D, steady state, incompressible)

Mass Flows

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \gg v \rightarrow \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

Momentum Flow | Pressure | Shear stresses

Momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cancelto{0}{\frac{\partial p}{\partial x}} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\cancel{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \cancel{\nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}$$

No pressure variation accross boundary layer

$$\frac{\partial p}{\partial y} \ll \frac{\partial p}{\partial x} \approx 0$$

Enthalpy Flow | Heat Conduction

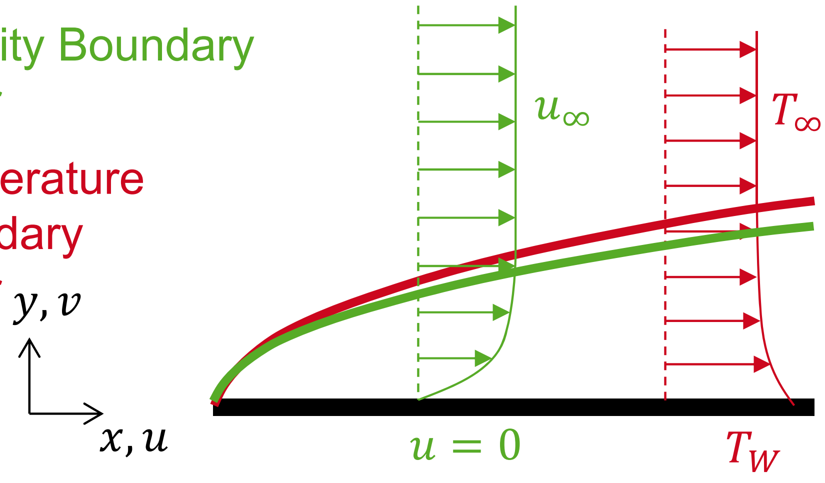
Energy equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\nu}{Pr}\left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2}\right)$$

If $Pr = 1$, the momentum and energy equations are identical.

Prandtl number

Continuity equation



Momentum equation

Reminder: Prandtl number

$$Pr = \frac{\nu}{a} = \frac{\text{Diffusive Momentum transport}}{\text{Diffusive Heat transport}} \begin{array}{l} \rightarrow u \text{ relevant} \\ \rightarrow T \text{ relevant} \end{array}$$

Energy equation

$$Pr = 1$$

$$\Updownarrow$$

Identity between the viscous and the thermal boundary layer

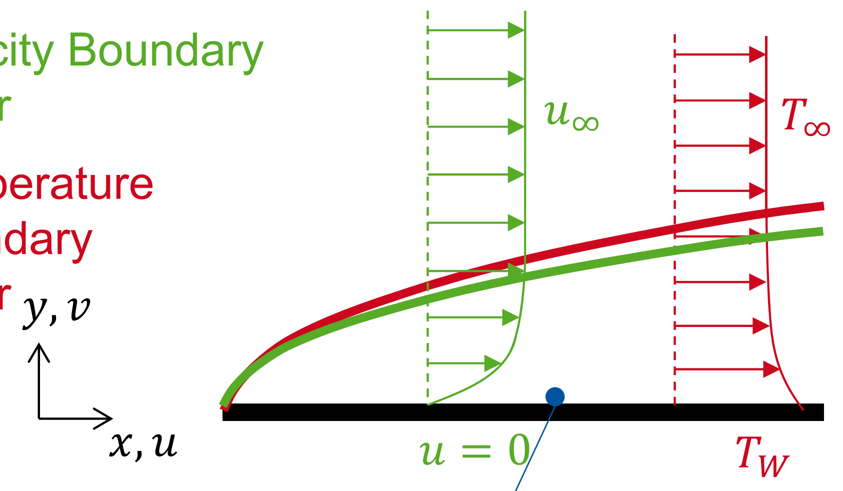
(Thickness $\delta_u = \delta_T$, Gradient $\left.\frac{\partial u}{\partial y}\right|_{y=0} = \left.\frac{\partial T}{\partial y}\right|_{y=0}$ etc.)

Comprehension Questions

What is the difference between the Nusselt and Biot numbers?

What is the relevance of the Prandtl number for Boundary Layer theory?

Determination of Velocity and Temperature profile in the Boundary Layer



Continuity equation

Momentum equation

Energy equation

How can the Velocity and Temperature within boundary layers be quantitatively determined, so that the Shear stress (Velocity gradient) and the heat transfer coefficient (Temperature gradient) on the surface can be determined?

This is what engineers need !!!

Result

Dimensionless wall shear stress (local at x, or average over L)

$$\tau_w = Function\,(\rho, x\ or\ L, u_\infty, \eta, \lambda, c_p\,)$$

Dimensionless heat transfer coefficient (local at x or average L)

$$\alpha = Function\,(\rho, x\ or\ L, u_\infty, \eta, \lambda, c_p)$$

Note, also h is used widely in the literature for the heat transfer coefficient.

This is what engineers need to predict friction and heat transfer for design In pratical applications ! Generally given in Dimensionless Form !

Conservation equations
(2D, steady state, incompressible, plane boundary layer)

Mass Flow

Continuity equation

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$Re_L = \frac{\rho u_\infty L}{\eta}$$

Momentum Flow | Shear stresses

Momentum equation

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

Enthalpy Flow | Heat Conduction

Energy equation

$$u^* \frac{\partial \Theta^*}{\partial x^*} + v^* \frac{\partial \Theta^*}{\partial y^*} = \frac{1}{\underbrace{Re_L Pr}_{Pe}} \frac{\partial^2 \Theta^*}{\partial y^{*2}}$$

Scaling dimensionless variables

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{L}$$

$$u^* = \frac{u}{u_\infty}$$

$$v^* = \frac{v}{u_\infty}$$

$$\Theta^* = \frac{T - T_\infty}{T_W - T_\infty}$$

Solutions – Result for application (correlations)

Dimensionless wall shear stress (local at x, or average over L)

$$\frac{c_f}{2} = \frac{\tau_W}{\rho {u_\infty}^2} = F\,(Re, Pr)$$

Dimensionless heat transfer coefficient (local at x or average L)

$$Nu = \frac{\alpha L}{\lambda} = G(Re, Pr)$$

These “correlations” are what engineers use to predict friction and heat transfer for design In pratical applications ! Different flow problem, different correlation !!! (Here we look at the case of the flat plate in a laminar uniform flow that can be solved analytically)

Exact Solutions (laminar)– Momentum equation (not relevant to the exam)

Momentum equation

Momentum Flow	Shear Stresses
$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*}$	$= \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}}$

Dimensionless wall distance

$$\xi^* = y \left(\frac{\rho u_\infty}{\eta x} \right)^{\frac{1}{2}} = \frac{y}{x} Re_x^{\frac{1}{2}}$$

Stream Function ψ

$$u = \frac{\partial \psi}{\partial y} \;, \quad v = -\frac{\partial \psi}{\partial x}$$

Dimensionless Stream Function f

$$f = \psi \left(\frac{\rho}{\eta u_\infty x} \right)^{\frac{1}{2}}, \qquad f' = \frac{df}{d\xi^*} = \frac{\partial \psi}{\partial \xi^*} \left(\frac{\rho}{\eta u_\infty x} \right)^{\frac{1}{2}}$$

hence

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \xi^*} \cdot \frac{\partial \xi^*}{\partial y} = \frac{\partial \psi}{\partial \xi^*} \cdot \left(\frac{\rho u_\infty}{\eta x} \right)^{\frac{1}{2}} \qquad \Rightarrow f' = \frac{u}{u_\infty}$$

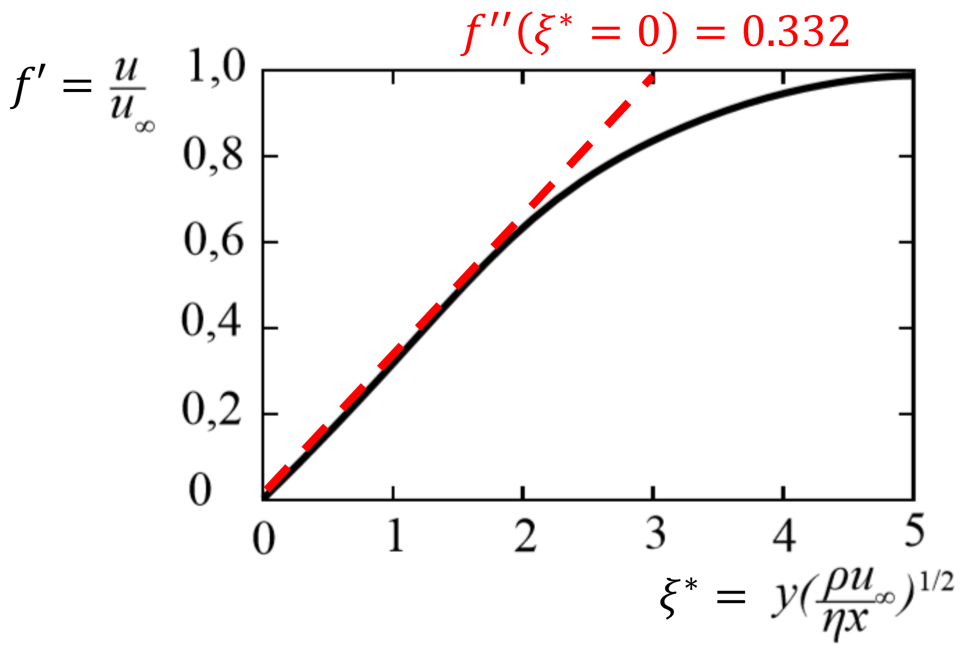
Transformation

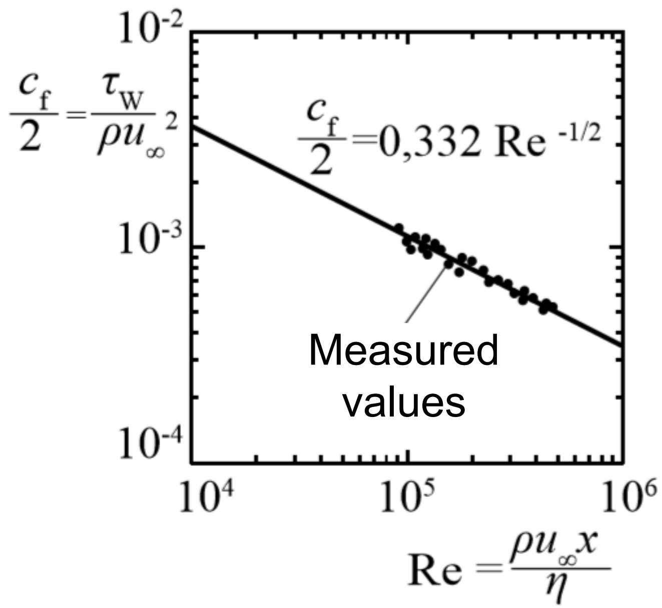
$$2f''' + f'' \cdot f = 0$$

Solution Approach

$f(\xi^*) \Rightarrow$ Velocity Profile $f' \Rightarrow$ Reverse transformation $\left. \frac{\partial u}{\partial y} \right|_{y=0} \Rightarrow$ Wall shear stress τ_W

Exact Solutions – Momentum equation





Dimensionless local wall shear stress

$$\frac{c_f}{2} = \frac{\tau_W}{\rho u_\infty{}^2} = \frac{\eta \left.\frac{\partial u}{\partial y}\right|_{y=0}}{\rho u_\infty{}^2} = \frac{\eta}{\rho u_\infty}\left(\frac{\rho u_\infty}{\eta x}\right)^{\frac{1}{2}} \left.\frac{\partial f'}{\partial \xi^*}\right|_{\xi^*=0} = Re_x{}^{-\frac{1}{2}} f''(\xi^* = 0)$$

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Exact Solutions – Energy equation (not relevant to the exam)

Energy equation

Enthalpy Flows — Heat Conduction

$$u^* \frac{\partial \Theta^*}{\partial x^*} + v^* \frac{\partial \Theta^*}{\partial y^*} = \frac{1}{\underbrace{RePr}_{Pe}} \frac{\partial^2 \Theta^*}{\partial y^{*2}}$$

Following a similar procedure as for the Momentum equation

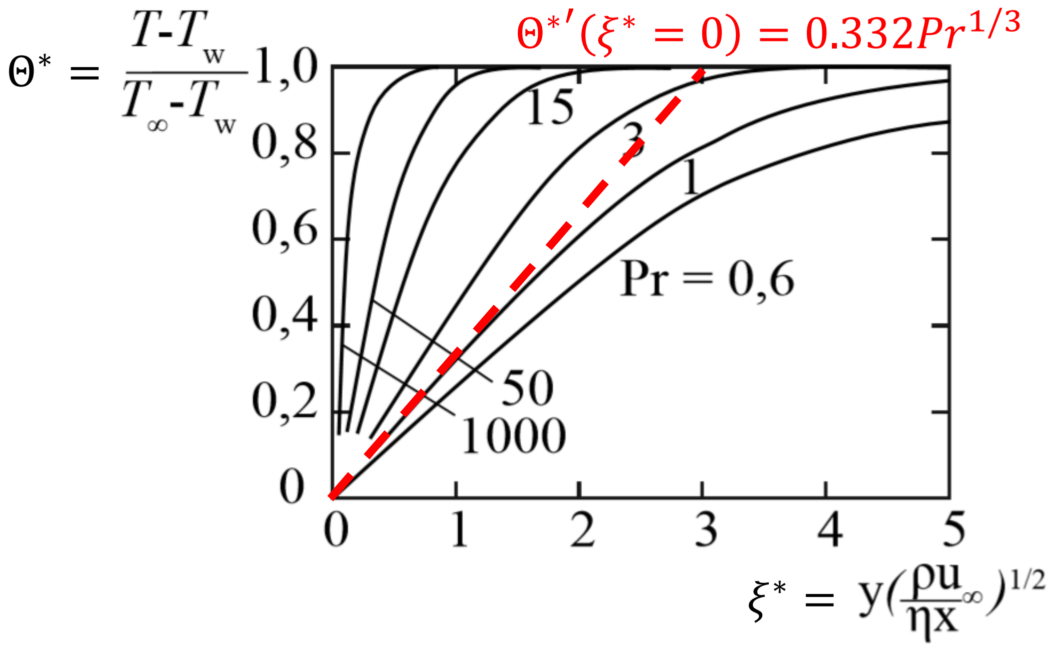
Transformation

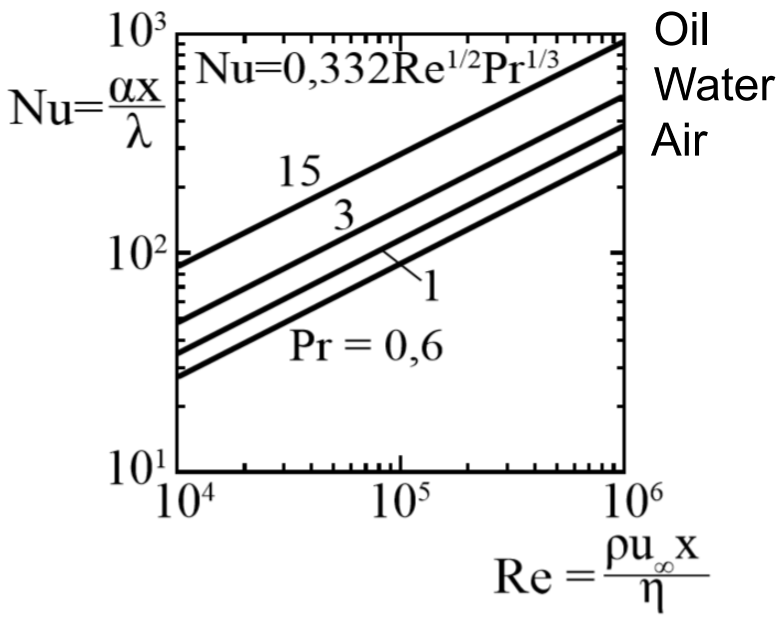
$$\Theta^{*''} + \Theta^{*'} \cdot Pr \cdot f = 0$$

Solution Approach

Temperature Profile Θ^* $\Rightarrow$ Reverse transformation $\left.\frac{\partial T}{\partial y}\right|_{y=0}$ $\Rightarrow$ Heat transfer coefficient α

Exact Solutions – Energy equation





Dimensionless local heat transfer coefficient

$$Nu_x = \frac{\alpha x}{\lambda} = \frac{\left(\lambda \frac{\partial T}{\partial y}\Big|_{y=0}\right) x}{\lambda} = x \left(\frac{\rho u_\infty}{\eta x}\right)^{\frac{1}{2}} \frac{\partial \Theta^*}{\partial \xi^*}\Big|_{\xi^*=0} = Re_x^{\frac{1}{2}} \Theta^{*\prime}(\xi^* = 0)$$

Exact Solutions (laminar flow)

Dimensionless velocity and temperature profiles are identical

$$\left.\frac{\partial f'}{\partial y}\right|_{y=0} = \left.\frac{\partial \Theta^*}{\partial y}\right|_{y=0} \quad \text{if } Pr = 1$$

Approximation (local)

$$Nu = \underbrace{0{,}332\, Re^{\frac{1}{2}}}_{\frac{c_f}{2} Re^{\frac{1}{2}}} Pr^{\frac{1}{3}} \Rightarrow Nu = Nu(Re, Pr), \text{ and related to friction as: } Nu = \frac{c_f}{2} Re Pr^{\frac{1}{3}}$$

$\boxed{\text{Reynolds based on x !}}$

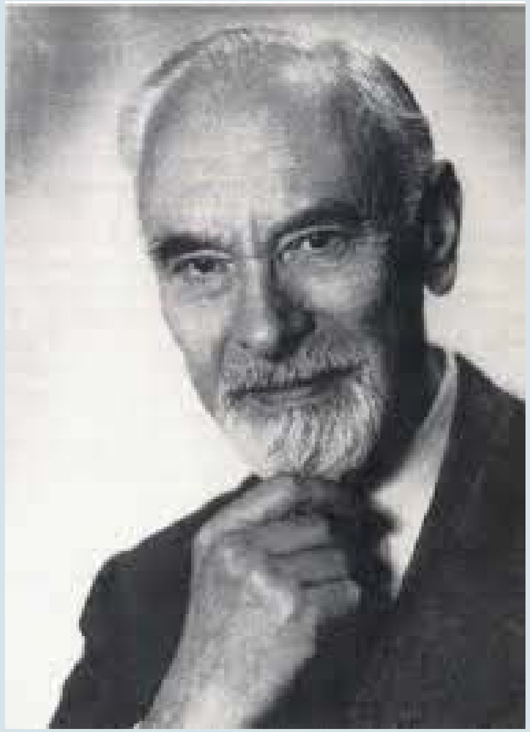
Local and average heat transfer coefficient

Along the entire Plate $\quad \bar{\alpha} = \frac{1}{L} \int_0^L \alpha(x) dx$

$$\overline{Nu} = \frac{\bar{\alpha} L}{\lambda} = 0.664\, Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

$\boxed{\text{Reynolds based on L !}}$

Names to remember:

Ludwig Prandtl	(Paul Richard) Heinrich Blasius
	
1875-1953	1883-1970
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Comprehension Questions

What is the difference between the Nusselt and Biot numbers?

What is the relevance of the Prandtl number for Boundary Layer theory?

If there is an identity between the thickness of the Flow Boundary Layer and the Temperature Boundary Layer ($\delta_u = \delta_T$), what is the relationship for the Nusselt number? (not relevant to the exam)

$$Pr = 1 \Longrightarrow Nu = \frac{c_f}{2} Re Pr^{\frac{1}{3}} \sim Re \sim u_\infty$$

(In the case described, the Nusselt number is directly proportional to the Reynolds number and, by extension, to the flow velocity.)