# Heat Transfer: Conduction

## Solution of the differential equation for fins

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# Learning goals

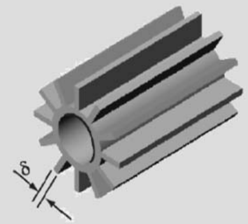
## Fins differential equation:

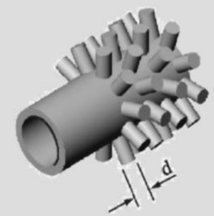
- Homogenization of the fin differential equation
- General solution of the differential equation

$$\frac{\partial^2 \theta}{\partial x^2} - m^2\, \theta(x) = 0$$

## Definition of the fin parameter $m$ :

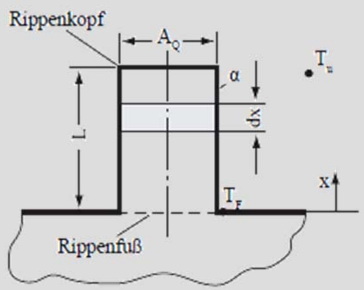
- Interpretation of the fin parameter $m$ for different fin geometries



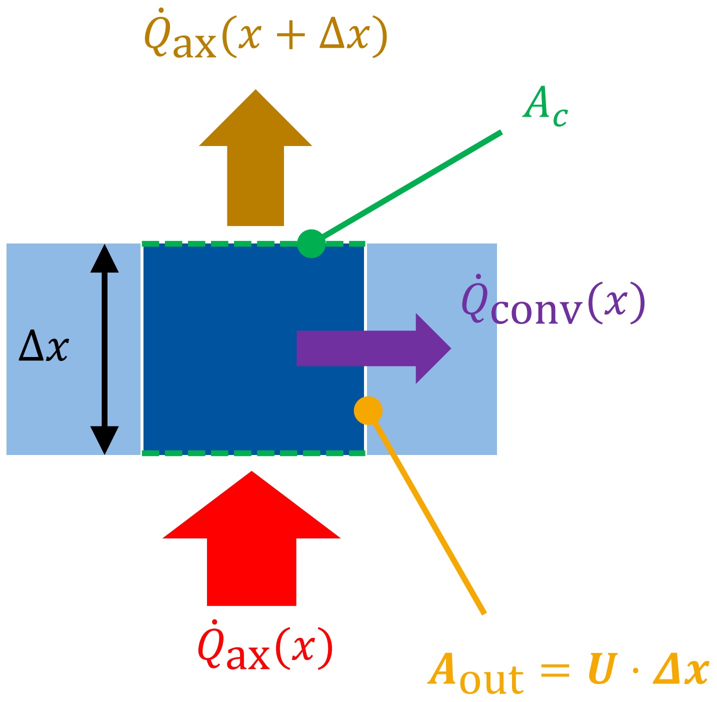




## Boundary conditions:

- Recognition and implementation of different boundary conditions for the fin problem



# Review: Differential equation for fins



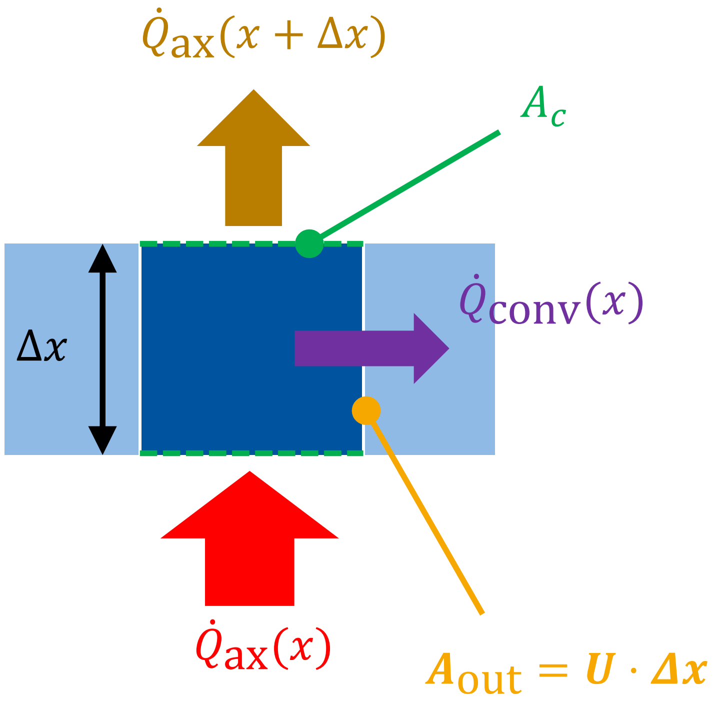

**Inhomogeneous differential equation 2nd order:**

$$\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} = \alpha \cdot U \left(T(x) - T_A\right)$$

**Explanation:**

| | |
|---|---|
| $\dot{Q}_{ax}$: | Heat conduction in axial direction |
| $\dot{Q}_{conv}$: | Convective heat dissipation to environment |
| $\Delta x$: | Length of the finite element |
| $A_c$: | Cross-sectional area of the fin |
| $A_{out}$: | Outer surface area (shell area) of the finite element |
| $U$: | Circumference (perimeter) of the fin |
| $T_A$: | Ambient temperature |

# Solution of the differential equation for fins




**Inhomogeneous differential equation 2nd order:**

$$\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} = \alpha \cdot U \left(T(x) - T_A\right)$$

**The constant ambient temperature $T_A$ makes the differential equation inhomogeneous.**

**Apply method of homogenization to solve differential equation**



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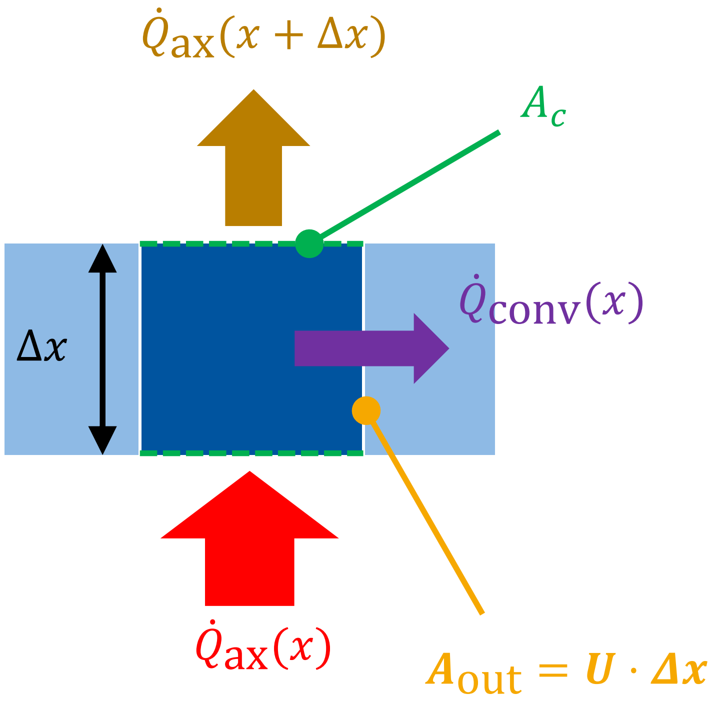
# Homogenization of the differential equation




**Inhomogeneous differential equation 2nd order:**

$$\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} - \alpha \cdot U\,(T\,(x) - T_A) = 0$$

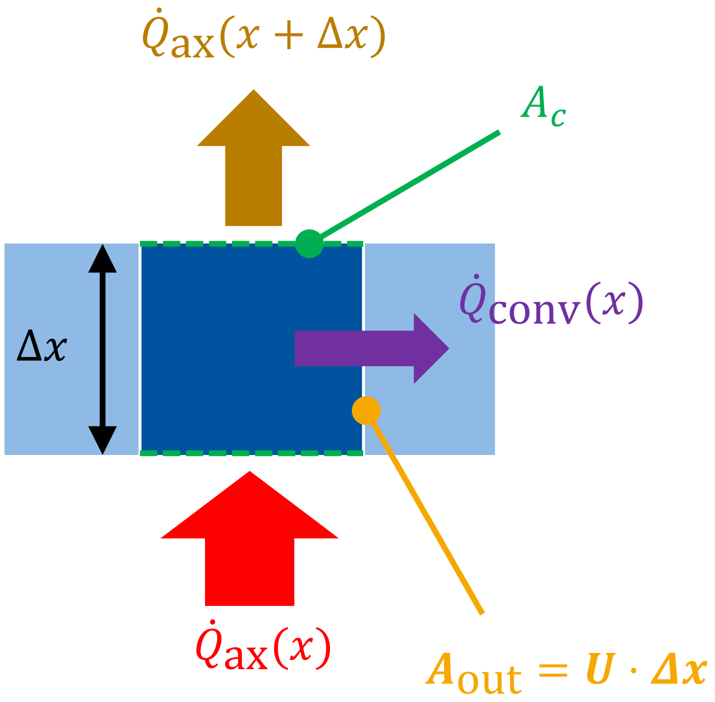
**Homogenization of the equation by introducing the parameter $\theta$ (temperature difference):**

Definition: $\theta(x) = T(x) - T_A$

1. Derivation: $$\frac{\partial \theta(x)}{\partial x} = \frac{\partial T(x)}{\partial x}$$

2. Derivation: $$\frac{\partial^2 \theta(x)}{\partial x^2} = \frac{\partial^2 T(x)}{\partial x^2}$$

# Introduction of the fin parameter m




**Inhomogeneous differential equation 2nd order:**

$$\lambda \cdot A_c \frac{\partial^2 T}{\partial x^2} - \alpha \cdot U\,(T\,(x) - T_A) = 0$$

$$\theta(x) = T(x) - T_A$$

**Substituting $\theta(x)$ into the equation:**

$$\frac{\partial^2 \theta(x)}{\partial x^2} - \underbrace{\frac{\alpha \cdot U}{\lambda \cdot A_c}}_{= m^2 \text{ „Fin parameter“}} \theta(x) = 0$$

$$\frac{\partial^2 \theta(x)}{\partial x^2} - m^2\,\theta(x) = 0$$



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# Fin parameter $m$

## Fin parameter depends on:

- Thermal conductivity of the fin
- Geometry of the fin
- Heat transfer coefficient to the surrounding medium

$$\boxed{m^2 = \frac{\alpha \cdot U}{\lambda \cdot A_c}}$$

## Example of geometry:

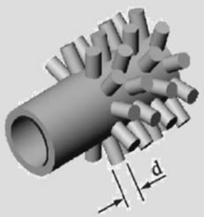
### Pin fins:

Perimeter: $U = \pi\, d$

Cross section area: $A_c = \frac{\pi d^2}{4}$

$$m^2 = \frac{\pi\, d}{\pi d^2/4}$$

$$\boxed{m^2 = \frac{4\,\alpha}{\lambda\, d}}$$



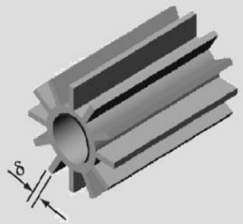

### Plane fins:

Perimeter: $U = 2\,(\,\delta + T)$

Cross section area: $A_c = \delta \cdot T$

$$m^2 = \frac{2\,(\,\delta{+}T)}{\delta \cdot T}$$

For $\delta \ll T$: $\boxed{m^2 \approx \frac{2\,\alpha}{\lambda\delta}}$



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# Derivation of the general solution of the fin equation

## Inhomogeneous differential equation 2nd order:

$$\frac{\partial^2 \theta(x)}{\partial x^2} - m^2\,\theta(x) = 0$$

## Solution to the inhomogeneous 2nd order differential equation:

**Try:** $\theta(x) = e^{sx} \rightarrow \dfrac{d^2\theta}{dx^2} = s^2 e^{sx}$

$$s^2 e^{sx} - m^2\, e^{sx} = 0$$

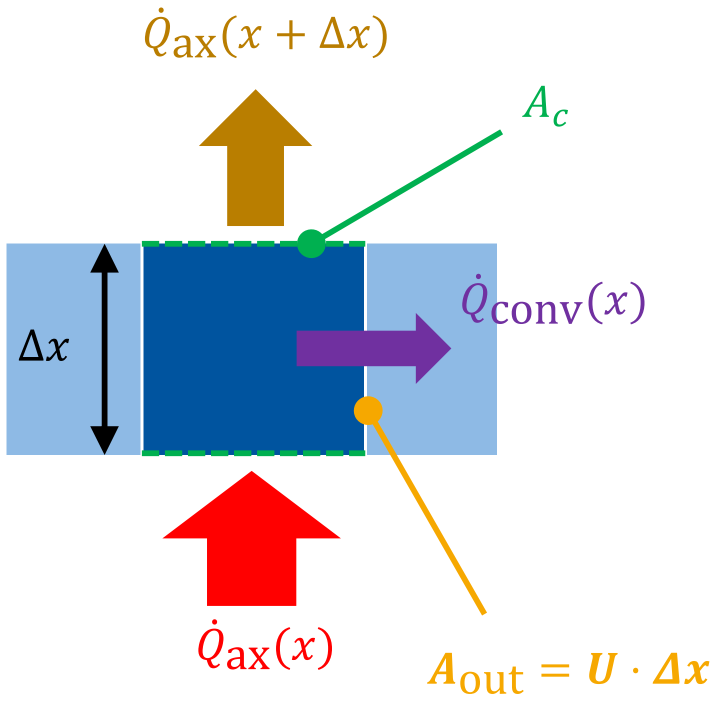
$$(s^2 - m^2) e^{sx} = 0$$

$$\boxed{e^{sx} \neq 0, \quad for\ x \in \mathbb{R}}$$

$$s_{1,2} = \pm\sqrt{m^2}$$

$$\rightarrow \theta(x) = Ce^{s_1 x} + De^{s_2 x} = Ce^{mx} + De^{-mx}$$

# Mathematical relations




**General solution of the fin differential equation:**

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

**Mathematical transformation:**

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

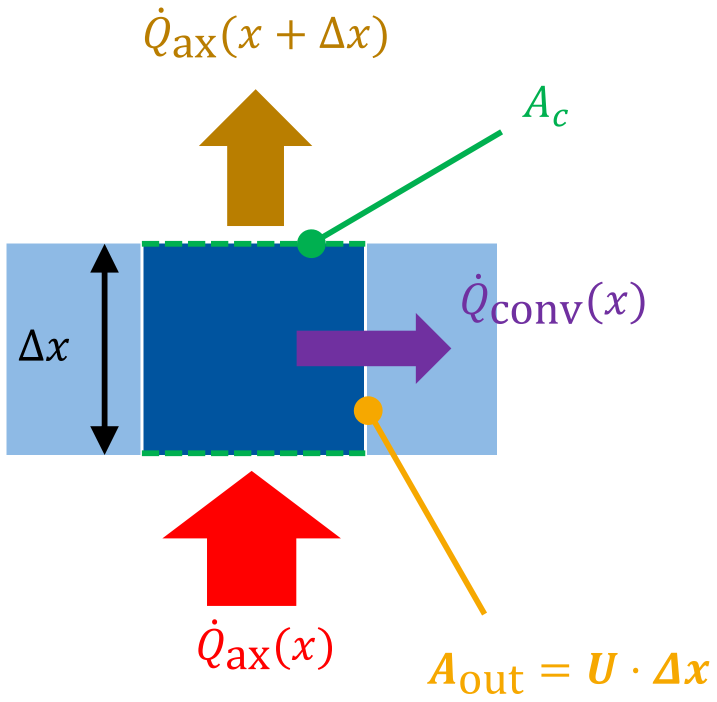
**General solution of the fin differential equation:**

$$\theta(x) = A \cdot \sinh(m \cdot x) + B\cosh(m \cdot x)$$

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

$$\Rightarrow \quad C = \frac{A+B}{2}, \qquad D = \frac{B-A}{2}$$

# General solution of the fin equation




**Inhomogeneous differential equation 2nd order:**

$$\frac{\partial^2 \theta(x)}{\partial x^2} - m^2\,\theta(x) = 0$$

**General solution of the fin differential equation:**

$$\theta(x) = A \cdot \sinh(m \cdot x) + B \cosh(m \cdot x)$$

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

**Diff. eq. 2nd order** → **Two boundary conditions required!**

$A$ , $B$ **or** $C, D$ are unknown constants which depend on the boundary conditions of the problem.



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# Boundary conditions

## Boundary conditions:

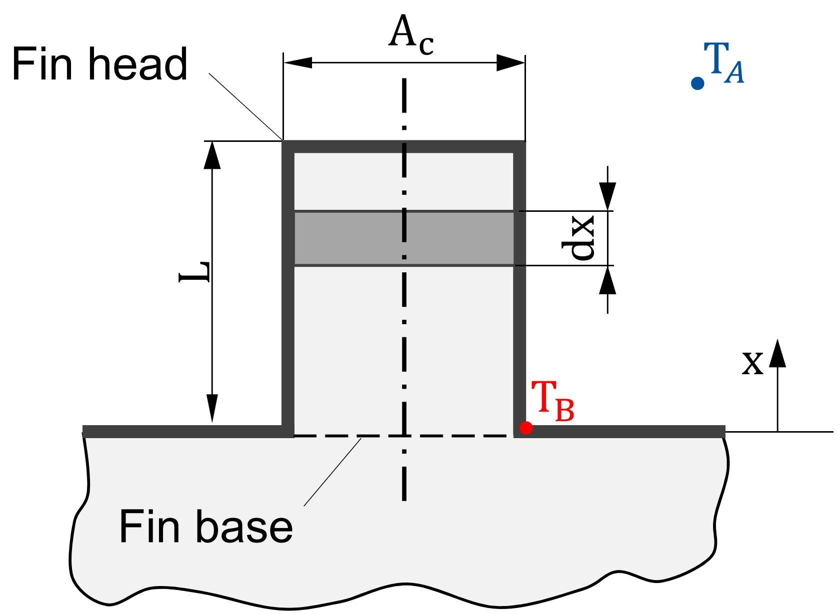
- Usually boundary conditions are defined at the **base** and **head** of the fins.
- The base of the fin is where the fin starts to dissipate heat to the environment by convection.

## Boundary condition at the fin base ($x = 0$):

Known temperature at the fin base:

$$T(x = 0) = T_B$$

$$\theta(x = 0) = T_B - T_A$$



# Boundary conditions

## Boundary conditions:

- Usually, boundary conditions are defined at the base and head of the fins.
- The base of the fin is where the fin starts to dissipate heat to the environment by convection.

## Boundary condition at the fin head $(x = \mathrm{L})$:

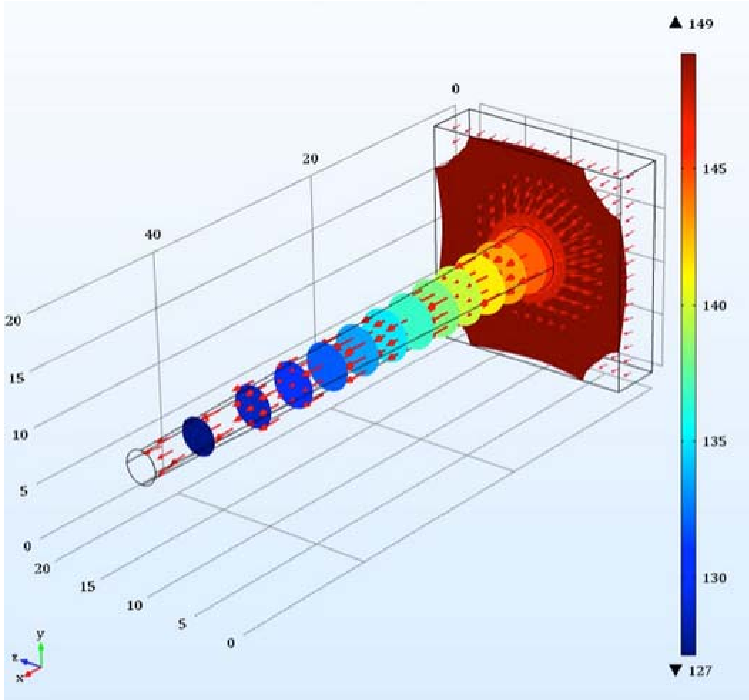
I. Sufficiently long fin:

$$\dot{Q}_{\text{head}} = 0 \quad \Longrightarrow \quad \left.\frac{dT}{dx}\right|_{x=L} = 0$$

identical

II. $A_{\text{Head}} \ll A_{\text{Surface}}$:

$$\dot{Q}_{\text{head}} = 0 \quad \Longrightarrow \quad \left.\frac{dT}{dx}\right|_{x=L} = 0$$

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# Boundary conditions

## Boundary conditions:

- Usually, boundary conditions are defined at the base and head of the fins.
- The base of the fin is where the fin starts to dissipate heat to the environment by convection.

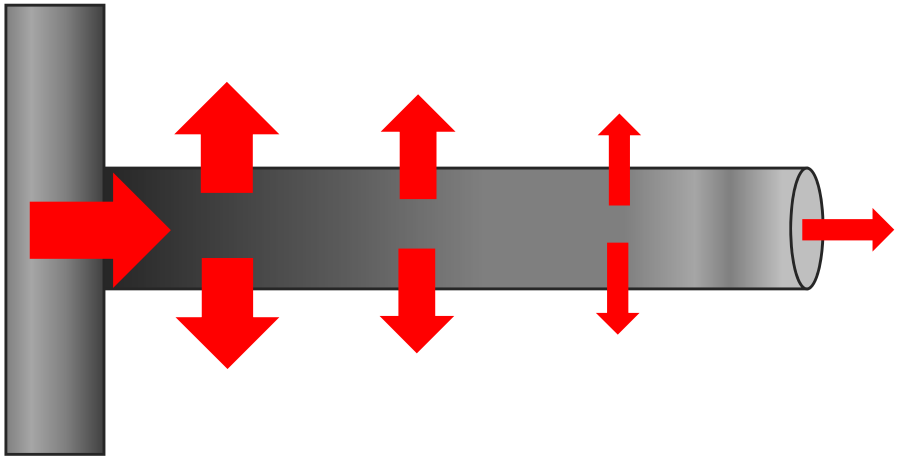
## Boundary condition at the fin head:

III. If heat flow at the head is not negligible:

$$\dot{Q}_{\text{head}} \neq 0$$

$$\dot{Q}_{\text{head}} = \dot{Q}_L = \alpha\, A_c\, \theta_{\text{head}}$$

$$\Rightarrow \quad \dot{Q}_{\text{head}} = \dot{Q}_L = \alpha\, A_c\, (T_{\text{H}} - T_{\text{A}})$$

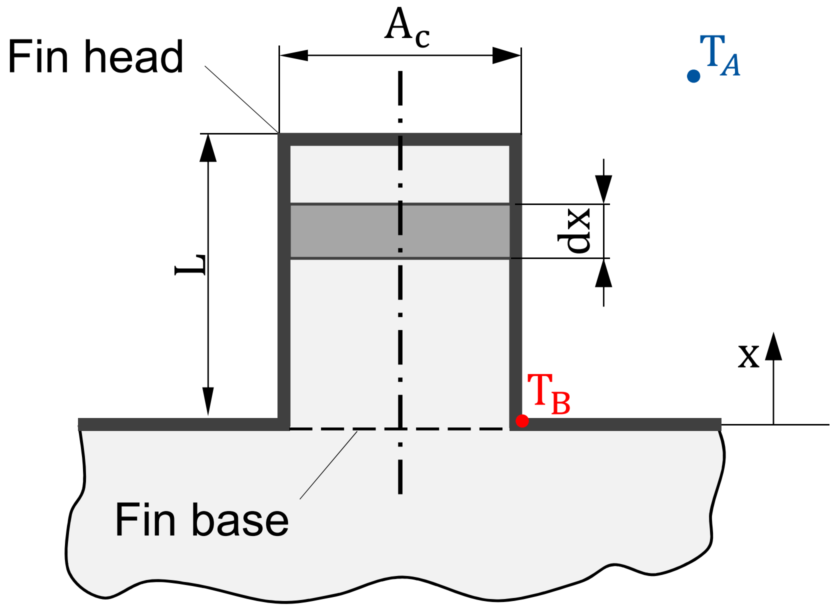


# Replacing boundary conditions and solving the differential equation

## General solution of the diff. equation:

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

$$\frac{\partial\theta(x)}{\partial x} = m \cdot C \cdot e^{m\,x} - m \cdot D \cdot e^{-m\,x}$$




## Replacing boundary conditions:

BC1: Given base temperature at $x = 0$ :

$$\Longrightarrow \quad \theta(x=0) = \theta_B$$

$$\theta_B = C \cdot e^0 + D \cdot e^0$$

$$\theta_B = C + D$$

$$C = \theta_B - D$$

BC2: No heat flow at $x$ = L :

$$\dot{Q}_{Head} = 0 \quad \Longrightarrow \quad \left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

$$m \cdot C \cdot e^{m\,L} - m \cdot D \cdot e^{-m\,L} = 0$$

$$(\theta_B - D) \cdot e^{m\,L} - D \cdot e^{-m\,L} = 0$$

$$\theta_B \cdot e^{m\,L} = D \cdot (e^{m\,L} + e^{-m\,L})$$

$$\Longrightarrow \quad \boxed{\begin{aligned} D &= \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}} \\ C &= \theta_B - \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}} \end{aligned}}$$

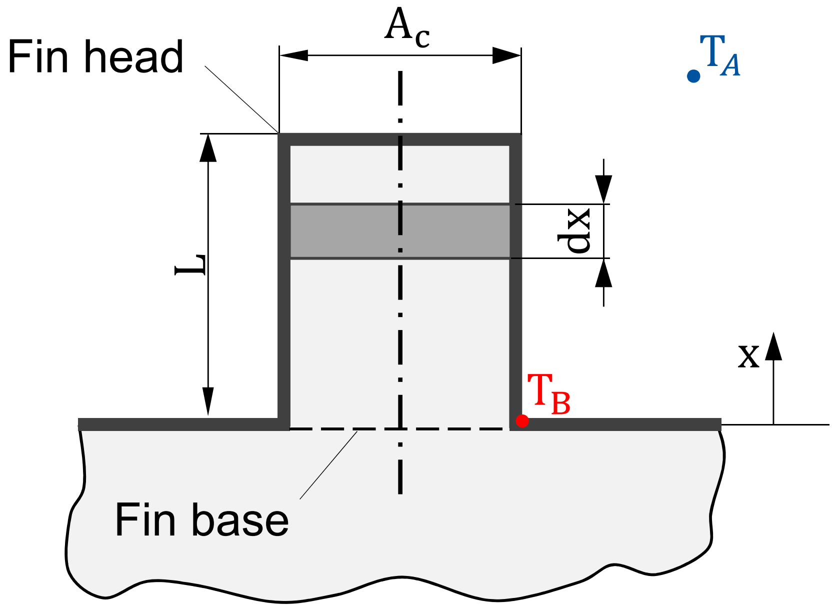
# Replacing boundary conditions and solving the differential equation

## General solution of the diff. equation:

$$\theta(x) = C \cdot e^{m\,x} + D \cdot e^{-m\,x}$$

$$\boxed{C = \theta_B - \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}}}$$

$$\boxed{D = \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}}}$$




## Replace $C$ and $D$ in diff. equation:

$$\theta(x) = \left(\theta_B - \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}}\right) \cdot e^{m\,x} + \theta_B \cdot \frac{e^{m\,L}}{e^{m\,L} + e^{-m\,L}} \cdot e^{-m\,x}$$

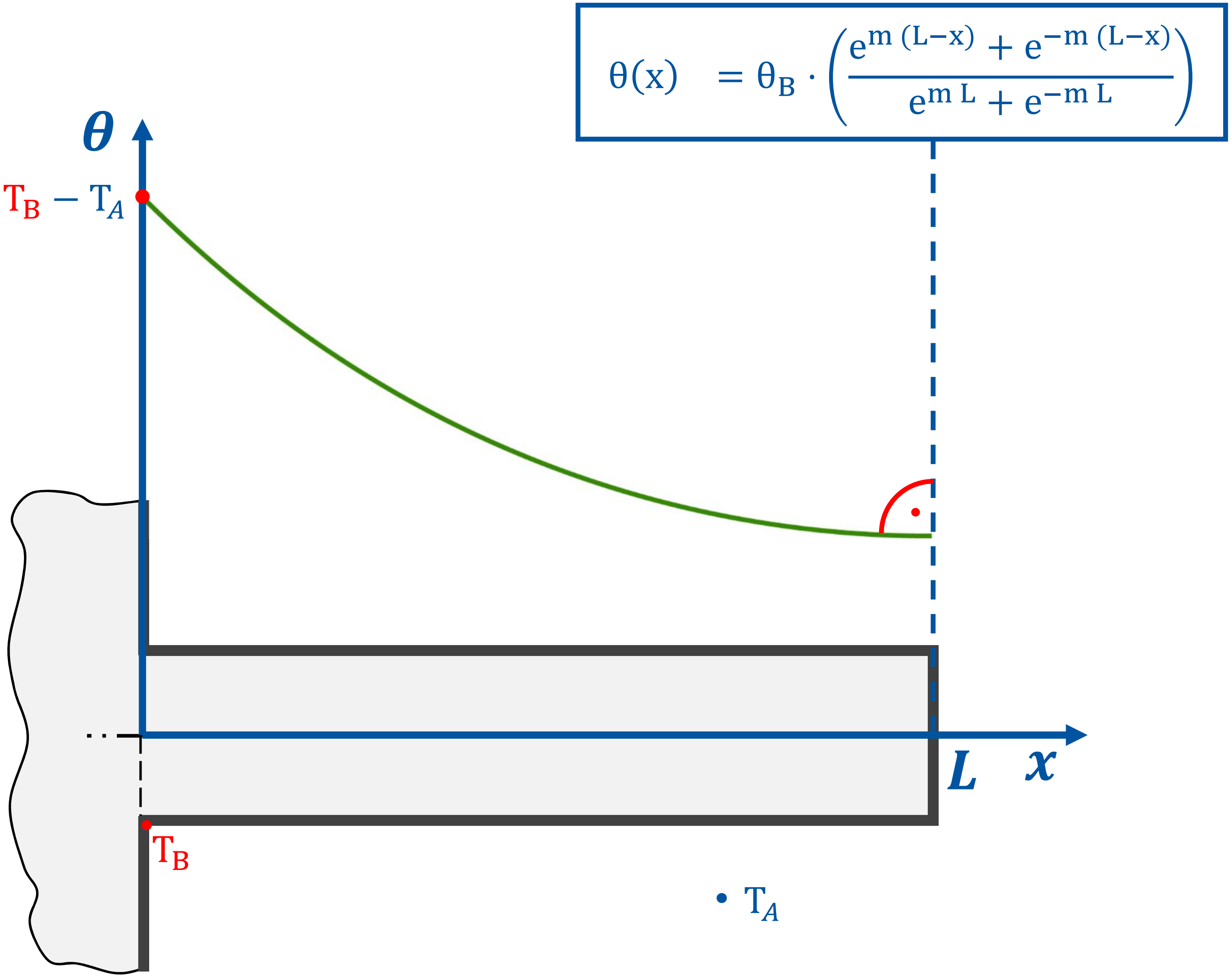
Mathematically reformulation and simplification:

$$\boxed{\theta(x) = \theta_B \cdot \left(\frac{e^{m\,(L-x)} + e^{-m\,(L-x)}}{e^{m\,L} + e^{-m\,L}}\right)}$$

Alternatively:

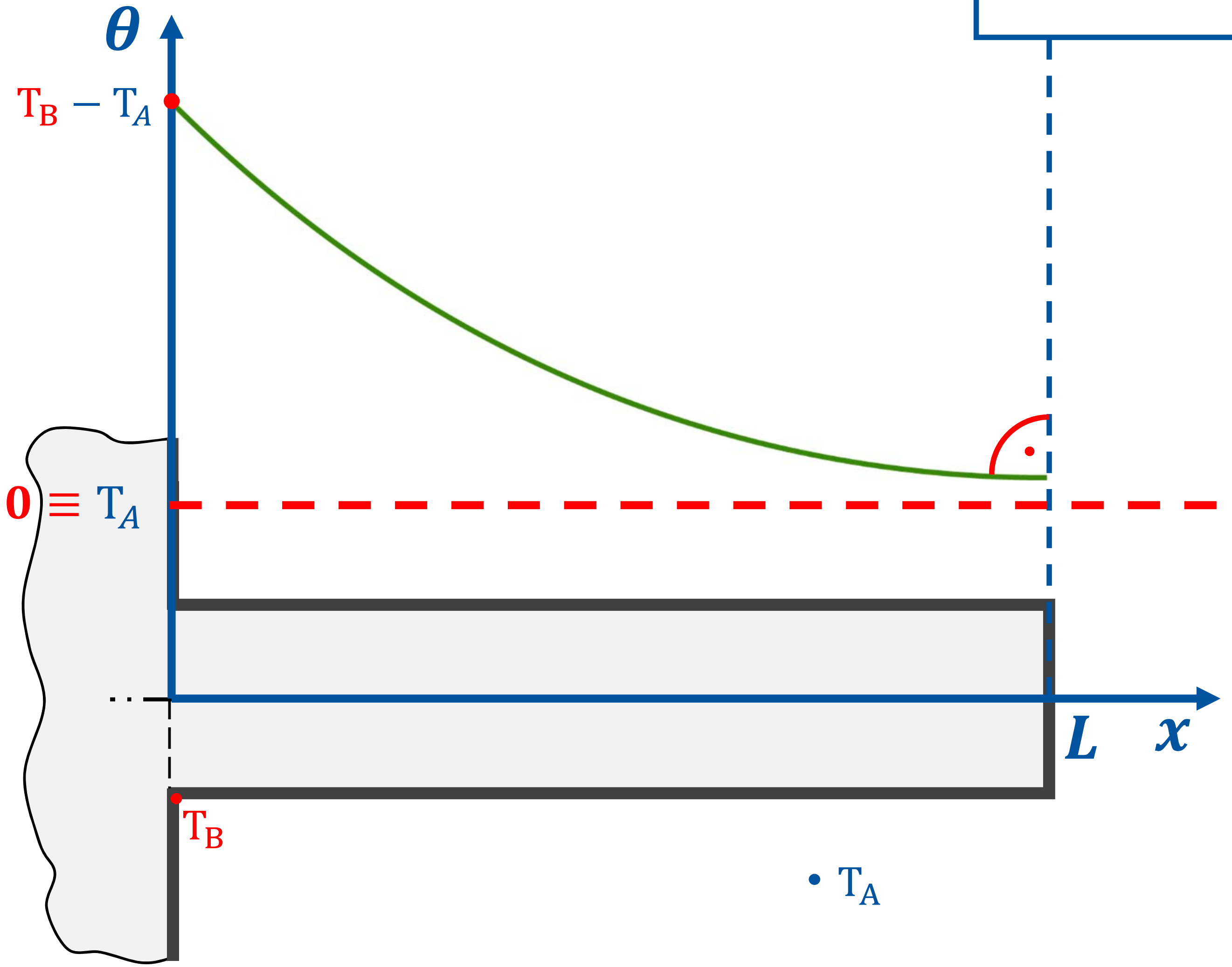
$$\boxed{\theta(x) = \theta_B \cdot \left(\frac{\cosh(m(L-x))}{\cosh(mL)}\right)}$$

# Temperature profile in the fin



## Temperature profile in the fin

$$\theta(x) \quad = \theta_B \cdot \left( \frac{e^{m\,(L-x)} + e^{-m\,(L-x)}}{e^{m\,L} + e^{-m\,L}} \right)$$




Is the head temperature $T_\mathrm{H}$ equal to the ambient temperature when $\dot{Q}_\mathrm{Head} = 0$?

For x = L :

$$\theta(L) \qquad = \theta_B \cdot \left( \frac{e^{0} + e^{-0}}{e^{m\,L} + e^{-m\,L}} \right)$$

$$= \theta_B \cdot \left( \frac{2}{e^{m\,L} + e^{-m\,L}} \right)$$

**Conclusion:**
Even with $\dot{\boldsymbol{Q}}_\mathrm{head} = \mathbf{0}$ the head temperature $\boldsymbol{T}_\mathrm{H}$ is always above the ambient temperature and only approaches to it.

# Comprehension questions

**Which approach can be used to solve the inhomogeneous fin differential equation?**

**Which quantities influence the fin parameter $m$ ?**

**Which common boundary conditions can be used to solve the fin temperature profile?**