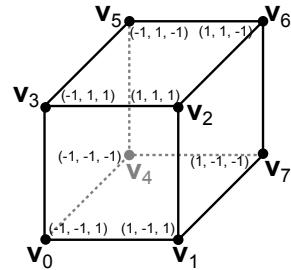


# Chapter 9

## Raycasting polygonal models

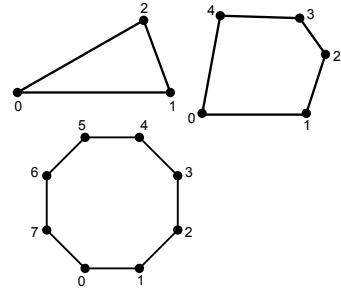
A polygonal model is one in which the geometry is defined by a set of polygons, each specified by the 3D coordinates of their vertices. For example, the cube to the right is specified by its eight vertices,  $\mathbf{v}_0 \dots \mathbf{v}_7$ . With the vertex coordinates given, we see that the cube is centered at the origin, aligned so its sides are parallel to the x-y, y-z, and z-x planes, and each face is a 2x2 square. If we want to raycast such a model, we need two things:

1. a way to determine ray polygon intersections,  
and
2. a way to determine a surface normal at each  
intersection.

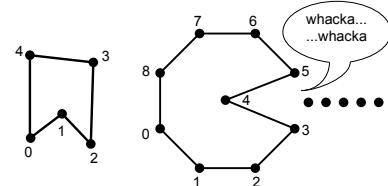


## 9.1 About polygons

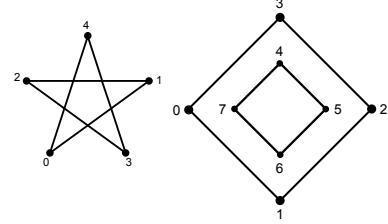
We start the consideration of ray polygon intersection with the definition of a polygon (we will consider only simple polygons). Definition: a polygon is an ordered set of vertices, all lying in a single plane, such that edges connecting the vertices, in order, form a closed loop with no edges crossing. Some example polygons are shown to the right. These are all *convex polygons*, since the internal angles formed by connecting edges at each vertex are all less than  $180^\circ$ .



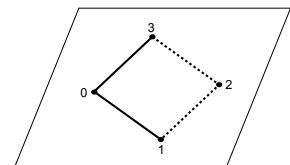
To the right are some more polygons. Note, that even *Pac Man* is a polygon! These are both *concave polygons*, since at least one pair of vertex edges connect to form an internal angle greater than  $180^\circ$ .



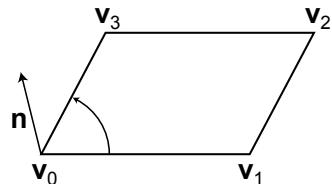
The star and diamond shown to the right are not polygons according to our definition. The star has crossing edges and the diamond consists of two disconnected loops, with one forming a hole in the other. We could expand our definition of a polygon to include these sorts of figures. In fact, they are often called *complex polygons*. But, for our purposes we will not include them for now.



Finally, this is not a polygon at all, since vertex 2 does not lie in the same plane as vertices 0, 1, and 3.



Since a polygon is by definition planar, we can determine the plane the polygon lies in by using any two non-parallel edges to compute the



normal to the plane,

$$\mathbf{n} = \frac{(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_3 - \mathbf{v}_0)}{\|(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_3 - \mathbf{v}_0)\|},$$

and any vertex to complete the plane definition, for example

$$\mathbf{p} = \mathbf{v}_0.$$

So, the distance of any point  $\mathbf{x}$  from the plane of the polygon is

$$d = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{n}, \quad (9.1)$$

and the plane equation is

$$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0. \quad (9.2)$$

## 9.2 Ray-polygon intersection

We can intersect a ray with a polygon in 2 steps:

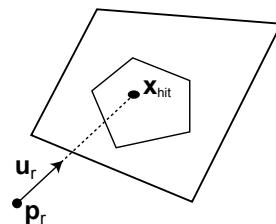
1. find the ray-plane intersection  $\mathbf{x}_{hit}$ , and
2. determine whether or not  $\mathbf{x}_{hit}$  falls inside or outside of the polygon.

For ray-plane intersection we just replace  $\mathbf{x}$  in Equation 9.2 by the ray equation  $\mathbf{x} = \mathbf{p}_r + t\mathbf{u}_r$  and solve for  $t$ :

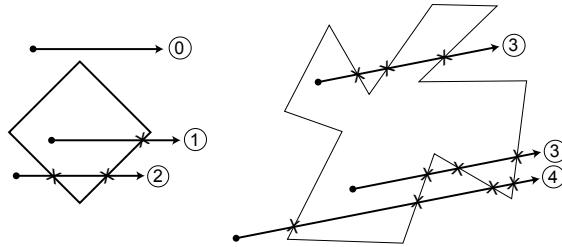
$$t_{hit} = \frac{(\mathbf{p} - \mathbf{p}_r) \cdot \mathbf{n}}{\mathbf{u}_r \cdot \mathbf{n}}.$$

If  $t_{hit} < 0$  then there is no ray-plane intersection. If  $0 \leq t_{hit} < \infty$  then there is an intersection at  $\mathbf{x}_{hit} = \mathbf{p}_r + t_{hit}\mathbf{u}_r$ . Determining if  $\mathbf{x}_{hit}$  is inside or outside of the polygon will tell us if we have a ray polygon intersection or not.

Note that the problem is planar, since all the vertices of the polygon and  $\mathbf{x}_{hit}$  lie in the same plane. With this in mind, a way to make the inside outside test is this. If we shoot a ray lying in the plane from a point inside the polygon it will always intersect an odd number of edges (1, 3, 5, ...). If we shoot a ray from outside the polygon it will always intersect an even number of edges (0, 2, 4, ...).



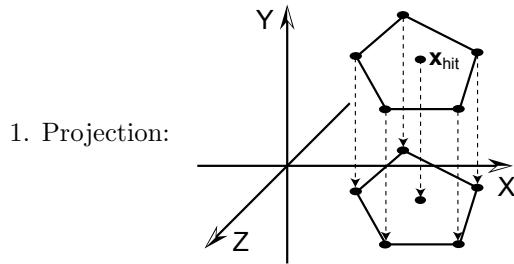
Here are some pictures showing this:



So here is an algorithm:

1. Make the problem 2D by projecting all points into a coordinate plane (like the  $x$ - $y$  plane for example).
2. Translate all points so that  $\mathbf{x}_{\text{hit}}$  is at the origin.
3. Let the positive  $x$  axis be the ray.
4. Count the number of edges that cross the positive  $x$  axis. If this number is odd, then the point lies inside the polygon, if even, then outside.

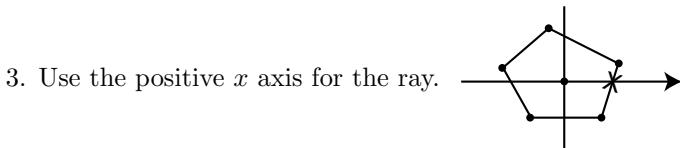
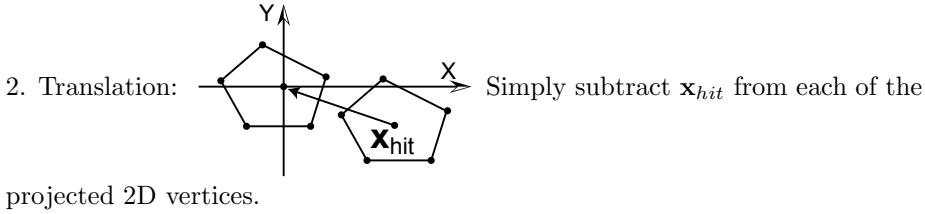
And, here are some details of how this could be implemented:



Find the coordinate plane that has the largest parallel projection (image) of the polygon into the plane and project to that plane by discarding the coordinate orthogonal to the plane:

- to the  $x$ - $y$  plane  $(x, y, z) \implies (x, y)$ ,
- to the  $y$ - $z$  plane  $(x, y, z) \implies (y, z)$ ,
- to the  $z$ - $x$  plane  $(x, y, z) \implies (z, x)$ .

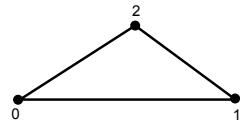
The surface normal to the plane of the polygon will tell you which plane to project to. Think about how this would be done.



4. Count the crossings with the positive  $x$  axis. Think about how to organize this counting to do the minimum number of calculations. Note: you do not necessarily have to calculate where the positive  $x$  axis crosses an edge. You only need to know whether or not it crosses the edge.

### 9.3 A special case: ray-triangle intersection

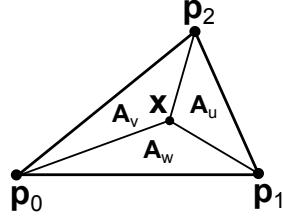
Note that a triangle is always a polygon. Three ordered vertices always lie in a single plane, no edges can cross, and there can be no holes. For this reason, graphics systems use triangles, wherever possible, to represent polygonal surfaces, especially when it comes time to render them. All of the desirable properties of a triangle turn out to be extremely important to writing an efficient and correct rendering algorithm.



For many calculations over triangles, a metric representation called *barycentric coordinates* is very convenient. Barycentric coordinates provide a measuring system that positions a point relative to the edges of the triangle, with a very simple scheme for converting from regular 3D spatial coordinates to barycentric coordinates, and from barycentric to 3D spatial coordinates.

Baycentric coordinates define a position with respect to the positions of the vertices of a triangle. Considering the diagram to the right, the point  $\mathbf{x}$  is internal to the triangle  $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ , whose area is  $A = A_u + A_v + A_w$ . The barycentric coordinates of  $\mathbf{x}$  are named  $u, v$  and  $w$ , and are defined as follows:

$$\begin{aligned} u &= A_u/A, \\ v &= A_v/A, \\ w &= A_w/A = 1 - u - v. \end{aligned}$$



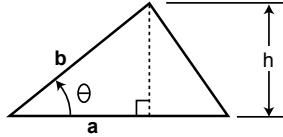
For example:

- if  $\mathbf{x} = \mathbf{p}_0$ ,  $u = 1, v = 0, w = 0$ ,
- if  $\mathbf{x} = \mathbf{p}_1$ ,  $u = 0, v = 1, w = 0$ ,
- if  $\mathbf{x} = \mathbf{p}_2$ ,  $u = 0, v = 0, w = 1$ ,
- if  $\mathbf{x}$  is on the  $\mathbf{p}_1, \mathbf{p}_2$  edge,  $u = 0$ ,
- if  $\mathbf{x}$  is on the  $\mathbf{p}_2, \mathbf{p}_0$  edge,  $v = 0$ ,
- if  $\mathbf{x}$  is on the  $\mathbf{p}_0, \mathbf{p}_1$  edge,  $w = 0$ .

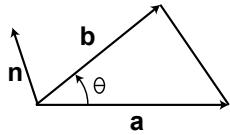
The areas can be found by the trigonometric relation:

$$A = 1/2ab\sin\theta,$$

where  $a$  and  $b$  are two sides of a triangle and  $\theta$  is the angle between them. Note that  $\sin\theta = h/b$ , so  $h = b\sin\theta$ . Therefore,  $A = 1/2ah = 1/2absin\theta$ . We know that the magnitude of the cross product  $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$ , so we can use cross product to determine areas.



We can do even better if we note that

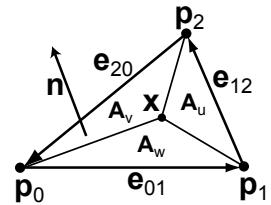


$$\frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} = \mathbf{n},$$

where  $\mathbf{n}$  is the normal to the plane defined by  $\mathbf{a}, \mathbf{b}$  and a common vertex.

Now, referring to the diagram to the right, let

$$\begin{aligned} \mathbf{e}_{01} &= \mathbf{p}_1 - \mathbf{p}_0, \\ \mathbf{e}_{12} &= \mathbf{p}_2 - \mathbf{p}_1, \\ \mathbf{e}_{20} &= \mathbf{p}_0 - \mathbf{p}_2, \\ \mathbf{n} &= (\mathbf{e}_{01} \times \mathbf{e}_{12})/\|\mathbf{e}_{01} \times \mathbf{e}_{12}\|. \end{aligned}$$



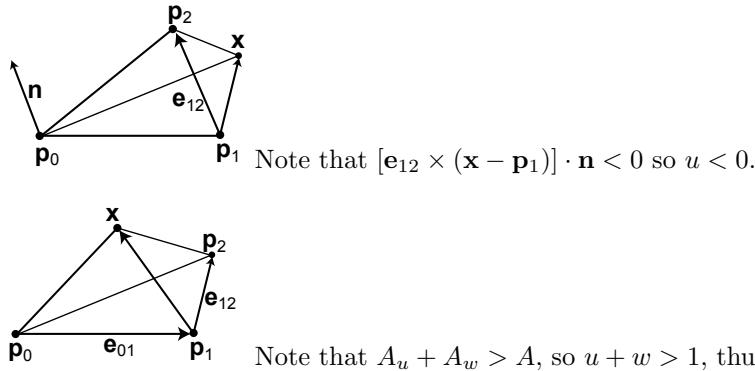
We see that

$$\begin{aligned} A &= \frac{1}{2}(\mathbf{e}_{01} \times \mathbf{e}_{12}) \cdot \mathbf{n}, \\ A_u &= \frac{1}{2}[\mathbf{e}_{12} \times (\mathbf{x} - \mathbf{p}_1)] \cdot \mathbf{n}, \\ A_v &= \frac{1}{2}[\mathbf{e}_{20} \times (\mathbf{x} - \mathbf{p}_2)] \cdot \mathbf{n}. \end{aligned}$$

This gives us

$$u = A_u/A, \quad v = A_v/A, \quad w = 1 - u - v.$$

Note that computation with dot product instead of absolute value gives us a *signed area*. This means that if  $\mathbf{x}$  is outside of a triangle, at least one of the  $u, v, w$  coordinates will be negative, as demonstrated in the following figures.



So given triangle  $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ , a compact way to calculate the barycentric coordinates of 3D point  $\mathbf{x}$  with respect to this triangle consists of the following steps:

$$\begin{aligned} \mathbf{v}_n &= (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_1 - \mathbf{p}_0), \\ A &= \|\mathbf{v}_n\|, \\ \mathbf{n} &= \mathbf{v}_n/A \\ u &= [(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{x} - \mathbf{p}_1)] \cdot \mathbf{n}/A, \\ v &= [(\mathbf{p}_0 - \mathbf{p}_2) \times (\mathbf{x} - \mathbf{p}_2)] \cdot \mathbf{n}/A, \\ w &= 1 - u - v. \end{aligned}$$

Note: the  $A$  in the above equations is actually twice the area of the triangle, but since it only is used in ratios with other areas, this scale factor is cancelled out.

Now our inside/outside test is simply

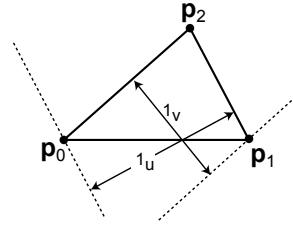
$$u \geq 0, \quad v \geq 0, \quad u + v \leq 1.$$

Given barycentric coordinates  $(u, v, w)$  and vertices  $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$  for a triangle, we can convert back to 3D coordinates by

$$\mathbf{x} = \mathbf{p}_2 + u(\mathbf{p}_0 - \mathbf{p}_2) + v(\mathbf{p}_1 - \mathbf{p}_2).$$

Another way to see this is that  $u, v$  and  $w$  weight vertices to give the position of  $\mathbf{x}$ . By rearranging the equation above, we have

$$\mathbf{x} = u\mathbf{p}_0 + v\mathbf{p}_1 + (1 - u - v)\mathbf{p}_2.$$



This follows directly if we see the  $u$  and  $v$  coordinates as measuring distance from an edge to a parallel line passing through the opposite vertex as shown in the diagram to the left. We see that  $u$  measures the fraction of the full distance from edge  $\mathbf{p}_0, \mathbf{p}_1$  to a parallel line through  $\mathbf{p}_2$ . Defined this way,  $u$  is a per unit measure, so  $u = 0$  on  $\mathbf{p}_0, \mathbf{p}_1$ ,  $u = 1$  on the parallel line. The  $v$  coordinate is similar, measuring distance from the edge  $\mathbf{p}_1, \mathbf{p}_2$ , and  $w$  measures distance from  $\mathbf{p}_2, \mathbf{p}_0$ .