ESE 542 Homework 1

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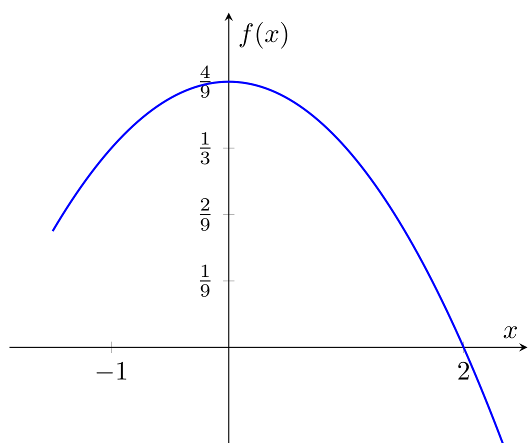
Problem 1

(a)

- At $x = -1$, $f(-1) = \frac{1}{9}(4 - (-1)^2) = \frac{1}{9}(4 - 1) = \frac{1}{3}$. - At $x = 0$, $f(0) = \frac{1}{9}(4 - 0^2) = \frac{4}{9}$. - At $x = 2$, $f(2) = \frac{1}{9}(4 - 2^2) = 0$.

Below is the graph of $f(x)$:



(b)

The cumulative distribution function (CDF) is obtained by integrating the pdf:

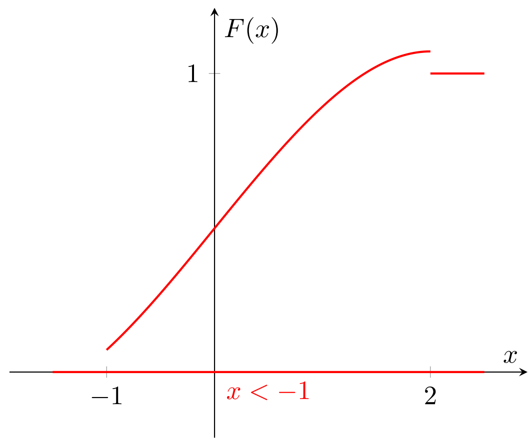
$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

For $x < -1$, $F(x) = 0$, since the probability outside the interval $[-1, 2]$ is zero.

For $x \in [-1, 2]$, calculate:

$$F(x) = \int_{-1}^{x} \frac{1}{9}(4 - t^2)\, dt = \frac{1}{9}\left(4x - \frac{x^3}{3}\right) + \frac{13}{27}.$$

For $x > 2$, $F(x) = 1$.



(c)

calculate the mean $E[X]$,

$$E[X] = \int_{-1}^{2} x \cdot f(x)\, dx = \int_{-1}^{2} x \cdot \frac{1}{9}(4 - x^2)\, dx.$$

$$E[X] = 0.25.$$

To find the variance $\text{Var}(X)$, compute $E[X^2]$:

$$E[X^2] = \int_{-1}^{2} x^2 \cdot f(x)\, dx = 0.6.$$

Then, the variance is given by:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 0.6 - (0.25)^2 = 0.5375.$$

Thus, the mean is 0.25 and the variance is 0.5375.

Problem 2

(a)

calculate $P(X > 3)$:

$$P(X > 3) = \int_{3}^{\infty} \int_{0}^{\infty} x e^{-x(1+y)}\, dy\, dx.$$

First, solve the integral with respect to y:

$$\int_{0}^{\infty} e^{-xy}\, dy = \frac{1}{x}.$$

$$P(X > 3) = \int_3^\infty e^{-x}\, dx.$$

The integral of e^{-x} is:

$$\int_3^\infty e^{-x}\, dx = e^{-3}.$$

Thus, the probability that the lifetime of the first component exceeds 3 is:

$$P(X > 3) = e^{-3} = 0.0498.$$

(b)

1. Marginal pdf of X: The marginal pdf of X is:

$$f_X(x) = \int_0^\infty f(x, y)\, dy = \int_0^\infty xe^{-x(1+y)}\, dy.$$

So, the marginal pdf of X is:

$$f_X(x) = e^{-x}, \quad x \geq 0.$$

2. Marginal pdf of Y: The marginal pdf of Y is:

$$f_Y(y) = \int_0^\infty f(x, y)\, dx = \int_0^\infty xe^{-x(1+y)}\, dx.$$

So, the marginal pdf of Y is:

$$f_Y(y) = \frac{1}{(1 + y)^2}, \quad y \geq 0.$$

3. Independence: To check for independence, compare the joint pdf $f(x, y) = xe^{-x(1+y)}$ with the product of the marginal pdfs:

$$f_X(x) \cdot f_Y(y) = e^{-x} \cdot \frac{1}{(1 + y)^2}.$$

Since these two expressions are not equal, conclude that X and Y are not independent.

(c)

$$P(X > 3 \text{ or } Y > 3) = 1 - P(X \leq 3 \text{ and } Y \leq 3).$$

First, compute $P(X \leq 3 \text{ and } Y \leq 3)$:

$$P(X \leq 3 \text{ and } Y \leq 3) = \int_0^3 \int_0^3 xe^{-x(1+y)}\, dy\, dx.$$

$$\int_0^3 e^{-xy}\,dy = \frac{1-e^{-3x}}{x}.$$

Substituting this back into the original integral:

$$P(X \le 3 \text{ and } Y \le 3) = \int_0^3 e^{-x}(1-e^{-3x})\,dx.$$

$$P(X \le 3 \text{ and } Y \le 3) = \int_0^3 e^{-x}\,dx - \int_0^3 e^{-4x}\,dx.$$

The first integral:

$$\int_0^3 e^{-x}\,dx = 1 - e^{-3} = 0.9502.$$

The second integral:

$$\int_0^3 e^{-4x}\,dx = \frac{1}{4}(1-e^{-12}) = 0.249998.$$

$$P(X \le 3 \text{ and } Y \le 3) = 0.9502 - 0.249998 = 0.7002.$$

Finally, the probability that at least one component exceeds 3 is:

$$P(X > 3 \text{ or } Y > 3) = 1 - 0.7002 = 0.2998.$$

Problem 3

(a)

$$\text{Confidence Interval} = \bar{x} \pm z^* \frac{s}{\sqrt{n}}$$

Where:

- $\bar{x} = 15$
- $z^* = 1.96$
- $s = 4$
- $n = 50$

calculating the margin of error:

$$\text{Margin of error} = z^* \frac{s}{\sqrt{n}} = 1.96 \cdot \frac{4}{\sqrt{50}} = 1.108.$$

$$\text{Confidence Interval} = 15 \pm 1.108 = [13.892, 16.108].$$

Thus, the 95% confidence interval for the population mean is approximately $[13.89, 16.11]$.

(b)

The university claims that students study an average of 14 hours per week. Based on the confidence interval from part (a), need to determine whether this claim is plausible.

The 95% confidence interval calculated is $[13.89, 16.11]$. Since the university's claimed value of 14 hours falls within this interval, can conclude that the claim is plausible.

(c)

The critical value z^* for a 99% confidence interval is $z^* = 2.576$.

$$\text{Margin of error} = 2.576 \cdot \frac{4}{\sqrt{50}} = 1.451.$$

$$\text{Confidence Interval} = 15 \pm 1.451 = [13.549, 16.451].$$

The 99% confidence interval $[13.55, 16.45]$ is wider than the 95% confidence interval $[13.89, 16.11]$.