ESE 5420 Homework 1

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Problem 1

(a)

At $x = -1$,

$$f(-1) = \frac{1}{9}(4 - (-1)^2) = \frac{1}{9}(4 - 1) = \frac{1}{3}.$$

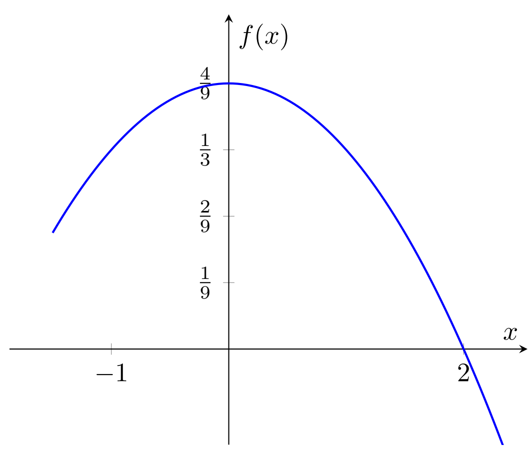
At $x = 0$,

$$f(0) = \frac{1}{9}(4 - 0^2) = \frac{4}{9}.$$

At $x = 2$,

$$f(2) = \frac{1}{9}(4 - 2^2) = 0.$$

Below is the graph of $f(x)$:



(b)

The cumulative distribution function (CDF) is obtained by integrating the pdf:

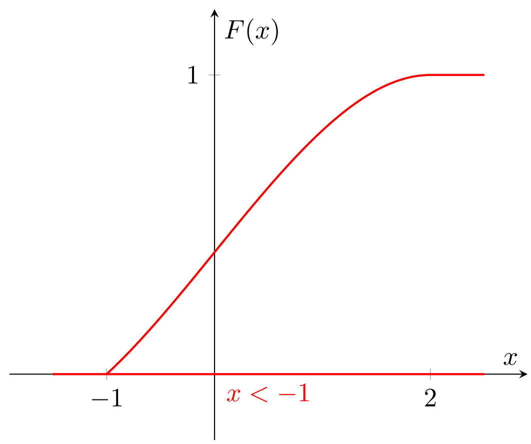
$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

For $x < -1$, $F(x) = 0$, since the probability outside the interval $[-1, 2]$ is zero.

For $x \in [-1, 2]$, calculate:

$$F(x) = \int_{-1}^{x} \frac{1}{9}(4 - t^2)\, dt = \frac{1}{9}\left(4x - \frac{x^3}{3}\right) + \frac{11}{27}.$$

For $x > 2$, $F(x) = 1$.



(c)

calculate the mean $E[X]$,

$$E[X] = \int_{-1}^{2} x \cdot f(x)\, dx = \int_{-1}^{2} x \cdot \frac{1}{9}(4 - x^2)\, dx.$$

$$E[X] = 0.25.$$

To find the variance $\text{Var}(X)$, compute $E[X^2]$:

$$E[X^2] = \int_{-1}^{2} x^2 \cdot f(x)\, dx = 0.6.$$

Then, the variance is given by:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 0.6 - (0.25)^2 = 0.5375.$$

Thus, the mean is 0.25 and the variance is 0.5375.

Problem 2

(a)

calculate $P(X > 3)$:

$$P(X > 3) = \int_3^\infty \int_0^\infty x e^{-x(1+y)} \, dy \, dx.$$

First, solve the integral with respect to y:

$$\int_0^\infty e^{-xy} \, dy = \frac{1}{x}.$$

$$P(X > 3) = \int_3^\infty e^{-x} \, dx.$$

The integral of e^{-x} is:

$$\int_3^\infty e^{-x} \, dx = e^{-3}.$$

Thus, the probability that the lifetime of the first component exceeds 3 is:

$$P(X > 3) = e^{-3} = 0.0498.$$

(b)

1. Marginal pdf of X: The marginal pdf of X is:

$$f_X(x) = \int_0^\infty f(x, y) \, dy = \int_0^\infty x e^{-x(1+y)} \, dy.$$

So, the marginal pdf of X is:

$$f_X(x) = e^{-x}, \quad x \geq 0.$$

2. Marginal pdf of Y: The marginal pdf of Y is:

$$f_Y(y) = \int_0^\infty f(x, y) \, dx = \int_0^\infty x e^{-x(1+y)} \, dx.$$

So, the marginal pdf of Y is:

$$f_Y(y) = \frac{1}{(1+y)^2}, \quad y \geq 0.$$

3. Independence: To check for independence, compare the joint pdf $f(x, y) = x e^{-x(1+y)}$ with the product of the marginal pdfs:

$$f_X(x) \cdot f_Y(y) = e^{-x} \cdot \frac{1}{(1+y)^2}.$$

Since these two expressions are not equal, conclude that X and Y are not independent.

(c)

$$P(X > 3 \text{ or } Y > 3) = 1 - P(X \leq 3 \text{ and } Y \leq 3).$$

First, compute $P(X \leq 3 \text{ and } Y \leq 3)$:

$$P(X \leq 3 \text{ and } Y \leq 3) = \int_0^3 \int_0^3 xe^{-x(1+y)}\, dy\, dx.$$

$$\int_0^3 e^{-xy}\, dy = \frac{1 - e^{-3x}}{x}.$$

Substituting this back into the original integral:

$$P(X \leq 3 \text{ and } Y \leq 3) = \int_0^3 e^{-x}(1 - e^{-3x})\, dx.$$

$$P(X \leq 3 \text{ and } Y \leq 3) = \int_0^3 e^{-x}\, dx - \int_0^3 e^{-4x}\, dx.$$

The first integral:

$$\int_0^3 e^{-x}\, dx = 1 - e^{-3} = 0.9502.$$

The second integral:

$$\int_0^3 e^{-4x}\, dx = \frac{1}{4}(1 - e^{-12}) = 0.249998.$$

$$P(X \leq 3 \text{ and } Y \leq 3) = 0.9502 - 0.249998 = 0.7002.$$

Finally, the probability that at least one component exceeds 3 is:

$$P(X > 3 \text{ or } Y > 3) = 1 - 0.7002 = 0.2998.$$

Problem 3

(a)

$$\text{Confidence Interval} = \bar{x} \pm z^* \frac{s}{\sqrt{n}}$$

$$\bar{x} = 15, \quad z^* = 1.96, \quad s = 4, \quad n = 50$$

calculating the margin of error:

$$\text{Margin of error} = z^* \frac{s}{\sqrt{n}} = 1.96 \cdot \frac{4}{\sqrt{50}} = 1.108.$$

$$\text{Confidence Interval} = 15 \pm 1.108 = [13.892, 16.108].$$

Thus, the 95% confidence interval for the population mean is approximately $[13.89, 16.11]$.

(b)

The university claims that students study an average of 14 hours per week. Based on the confidence interval from part (a), need to determine whether this claim is plausible.

The 95% confidence interval calculated is $[13.89, 16.11]$. Since the university's claimed value of 14 hours falls within this interval, can conclude that the claim is plausible.

(c)

The critical value z^* for a 99% confidence interval is $z^* = 2.576$.

$$\text{Margin of error} = 2.576 \cdot \frac{4}{\sqrt{50}} = 1.451.$$

$$\text{Confidence Interval} = 15 \pm 1.451 = [13.549, 16.451].$$

The 99% confidence interval $[13.55, 16.45]$ is wider than the 95% confidence interval $[13.89, 16.11]$.

Problem 4

(a)

$$\hat{p}_1 = 0.12, \quad n_1 = 100, \quad z^* = 1.96$$

The margin of error (ME) is:

$$\text{ME} = z^* \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}} = 1.96 \cdot \sqrt{\frac{0.12(1-0.12)}{100}} = 1.96 \cdot 0.0324 = 0.0635$$

The 95% confidence interval is:

$$\text{Confidence Interval} = \hat{p}_1 \pm \text{ME} = 0.12 \pm 0.0635 = [0.0565, 0.1835]$$

(b)

$$\hat{p}_2 = 0.18, \quad n_2 = 200, \quad z^* = 1.645$$

The margin of error (ME) is:

$$\text{ME} = z^* \cdot \sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} = 1.645 \cdot \sqrt{\frac{0.18(1-0.18)}{200}} = 1.645 \cdot 0.0272 = 0.0447$$

The 90% confidence interval is:

$$\text{Confidence Interval} = \hat{p}_2 \pm \text{ME} = 0.18 \pm 0.0447 = [0.1353, 0.2247]$$

(c)

The formula for the confidence interval for the difference is:

$$(\hat{p}_1 - \hat{p}_2) \pm z^* \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

Using $z^* = 1.96$:

$$\text{SE}_{\text{diff}} = \sqrt{\frac{0.12(1-0.12)}{100} + \frac{0.18(1-0.18)}{200}} = \sqrt{0.001056 + 0.000738} = \sqrt{0.001794} = 0.04236$$

Hence the variance of 0.001794 and a standard error of 0.04236.

(d)

$$\hat{d} = \hat{p}_1 - \hat{p}_2 = 0.12 - 0.18 = -0.06,$$

and the standard error calculated in part (c) is:

$$SE(\hat{d}) = 0.042.$$

confidence interval is:

$$\hat{d} \pm z \cdot SE(\hat{d}),$$

where z is the critical value corresponding to the confidence level.

99% confidence interval

For a 99% confidence level, $z = 2.576$:

$$\text{Confidence Interval} = -0.06 \pm 2.576 \cdot 0.042 = -0.06 \pm 0.108.$$

Thus, the 99% confidence interval is:

$$(-0.168, 0.048).$$

95% confidence interval

For a 95% confidence level, $z = 1.96$:

$$\text{Confidence Interval} = -0.06 \pm 1.96 \cdot 0.042 = -0.06 \pm 0.082.$$

Thus, the 95% confidence interval is:

$$(-0.142, 0.022).$$

90% confidence interval

For a 90% confidence level, $z = 1.645$:

$$\text{Confidence Interval} = -0.06 \pm 1.645 \cdot 0.042 = -0.06 \pm 0.069.$$

Thus, the 90% confidence interval is:

$$(-0.129, 0.009).$$

At the 99% and 95% confidence levels, the intervals include zero, so there is no clear evidence that p_1 is different from p_2. However, at the 90% confidence level, the interval is just barely excluding zero, providing weak evidence that p_1 and p_2 may differ.

Problem 5

(a)

Given:

$$p = 0.25, \quad n = 100, \quad z^* = 1.96$$

find δ such that $P(|\hat{p} - p| \geq \delta) \approx 0.025$.

The sample proportion $\hat{p}$ follows a normal distribution under the Central Limit Theorem (CLT) with a standard error:

$$\text{Standard Error} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.25(1-0.25)}{100}} = 0.0433$$

Using $z^* = 1.96$ for a 95% confidence level, the margin of error (which in this case is δ) is:

$$\delta = z^* \cdot \text{Standard Error} = 1.96 \cdot 0.0433 = 0.0849$$

Thus, $\delta \approx 0.0849$, meaning there is a 2.5

(b)

Given:

$$\hat{p} = 0.25, \quad n = 100, \quad z^* = 1.96$$

$$\text{Confidence Interval} = \hat{p} \pm z^* \cdot \text{Standard Error}$$

We already computed the standard error as 0.0433. The margin of error is:

$$\text{Margin of Error} = 1.96 \cdot 0.0433 = 0.0849$$

Thus, the confidence interval is:

$$\text{Confidence Interval} = 0.25 \pm 0.0849 = [0.1651, 0.3349]$$

The true proportion $p = 0.25$ lies within this interval, so the confidence interval contains the true value.

Problem 6

We are given that each person eats 0, 1, or 2 sandwiches with the following probabilities:

$$P(0) = \frac{1}{4}, \quad P(1) = \frac{1}{2}, \quad P(2) = \frac{1}{4}$$

The expected number of sandwiches consumed by one person is:

$$E[X] = 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

Thus, on average, each person eats 1 sandwich.

Next, the variance of sandwiches consumed by one person is calculated as:

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

First, calculate $E[X^2]$:

$$E[X^2] = 0^2 \cdot P(0) + 1^2 \cdot P(1) + 2^2 \cdot P(2) = 0 + \frac{1}{2} + 4 \cdot \frac{1}{4} = 1.5$$

Thus, the variance is:

$$\text{Var}(X) = 1.5 - 1^2 = 1.5 - 1 = 0.5$$

For $n = 64$ friends, the total expected number of sandwiches is:

$$E[\text{Total Sandwiches}] = n \cdot E[X] = 64 \cdot 1 = 64$$

The total variance is:

$$\text{Var}(\text{Total}) = n \cdot \text{Var}(X) = 64 \cdot 0.5 = 32$$

The total standard deviation is:

$$\text{Standard Deviation} = \sqrt{32} = 5.66$$

We need to ensure no shortage with 95% confidence. Since we only care about the upper tail (ordering enough sandwiches), we use the one-tailed z-value $z^* = 1.645$.

Using the z-value for 95% confidence, $z^* = 1.645$,

$$\text{Total Sandwiches Needed} = E[\text{Total Sandwiches}] + z^* \cdot \text{Standard Deviation}$$

Substituting the values:

$$\text{Total Sandwiches Needed} = 64 + 1.645 \cdot 5.66 = 64 + 9.31 = 73.31$$

To ensure that the probability of no shortage is at least 0.95, round up to 74 sandwiches. Thus, the number of sandwiches needed is 74.