ESE 5420 Homework 4

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Problem 1

b) Written Questions

(i) Random splitting better than sequential splitting in our case

Random splitting is preferred over sequential splitting because it ensures that the model is trained and evaluated on a representative and unbiased dataset. Sequential splitting may lead to biased subsets if the data has an inherent order. The reasons are as follows:

- Representative Distribution: Random splitting ensures that both the training and test sets have a similar distribution, reducing the risk of bias or skewness.
- Avoids Bias: Sequential splitting can introduce bias if the data is ordered (e.g., temporally or by features), which might result in poor generalization to unseen data.
- Breaks Sequential Patterns: Random splitting prevents the model from overfitting to patterns specific to the order of the data, such as trends in time-ordered datasets.
- Improves Generalization: By mixing the data randomly, the model learns to generalize better, as the test set provides a more accurate representation of unseen scenarios.

(ii) Derive the classification rule for the threshold 0.5

To classify using logistic regression with a threshold of 0.5, we derive the rule as follows:

The logistic regression model outputs the probability $P(Y = 1|X)$, given by:

$$P(Y = 1|X) = \frac{1}{1 + \exp(-(\beta_0 + \beta^T X))}.$$

The classification rule is to classify as $+1$ if $P(Y = 1|X) \geq 0.5$. Substituting the formula:

$$\frac{1}{1 + \exp(-(\beta_0 + \beta^T X))} \geq 0.5.$$

Simplify this inequality:

$$\begin{aligned}
1 &\geq 0.5 \cdot (1 + \exp(-(\beta_0 + \beta^T X))) \\
1 &\geq 0.5 + 0.5 \exp(-(\beta_0 + \beta^T X)) \\
0.5 &\geq 0.5 \exp(-(\beta_0 + \beta^T X)) \\
1 &\geq \exp(-(\beta_0 + \beta^T X)).
\end{aligned}$$

Take the natural logarithm of both sides:

$$\ln(1) \geq -(\beta_0 + \beta^T X).$$

Since $\ln(1) = 0$:

$$0 \geq -(\beta_0 + \beta^T X).$$

Multiply through by -1, reversing the inequality:

$$\beta_0 + \beta^T X \geq 0.$$

Thus, the final classification rule is:

$$\hat{y} = \begin{cases} +1 & \text{if } \beta_0 + \beta^T X \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

Problem 2

a) Designing $P(Y = y|X = x)$

To satisfy the given requirements, we need to design the conditional probability $P(Y = y|X = x)$ as follows:

- X is uniformly distributed on the interval $[0, 1]$.
- Y is a binary variable, taking values 0 or 1.
- The conditions are:
  - $P(Y = 0) = P(Y = 1) = 0.5$ (balanced classes),
  - Maximum achievable accuracy of any classifier should be less than or equal to 0.9,
  - The Bayes optimal classifier accuracy should be at least 0.8.

To meet these conditions, we define $P(Y = 1|X = x)$ as a piecewise function:

$$P(Y = 1|X = x) = \begin{cases} 0.9 & \text{if } x > 0.5 \\ 0.1 & \text{if } x \leq 0.5 \end{cases}$$

$$P(Y = 0|X = x) = 1 - P(Y = 1|X = x) = \begin{cases} 0.1 & \text{if } x > 0.5 \\ 0.9 & \text{if } x \leq 0.5 \end{cases}$$

Verification of Requirements

- **Balanced Classes:** Since X is uniformly distributed, we calculate the overall probability of each class Y:

$$P(Y = 1) = 0.9 \cdot 0.5 + 0.1 \cdot 0.5 = 0.5$$

$$P(Y = 0) = 0.1 \cdot 0.5 + 0.9 \cdot 0.5 = 0.5$$

  Thus, $P(Y = 0) = P(Y = 1) = 0.5$, satisfying the balanced classes requirement.

- **Maximum Classifier Accuracy:** For any given x, the maximum probability of $Y = y$ is 0.9, so no classifier can achieve an accuracy greater than 0.9.

- **Bayes Optimal Classifier Accuracy:** The Bayes optimal classifier selects the most probable class for each x. Thus:

  - For $x > 0.5$, it predicts $Y = 1$ with probability 0.9,
  - For $x \leq 0.5$, it predicts $Y = 0$ with probability 0.9.

  The overall accuracy of the Bayes optimal classifier is:

$$0.9 \cdot 0.5 + 0.9 \cdot 0.5 = 0.9$$

  which is greater than the minimum required accuracy of 0.8.

Thus, the final design for $P(Y = y|X = x)$ is:

$$P(Y = 1|X = x) = \begin{cases} 0.9 & \text{if } x > 0.5 \\ 0.1 & \text{if } x \leq 0.5 \end{cases}$$

$$P(Y = 0|X = x) = \begin{cases} 0.1 & \text{if } x > 0.5 \\ 0.9 & \text{if } x \leq 0.5 \end{cases}$$

This design meets all the specified requirements.

Problem 4

a) Write down the expected regret $E[R_T]$ of the algorithm in terms of $E[T_2]$ and Δ

The regret is the difference between the total reward if the optimal arm (arm 1) was always pulled and the reward obtained by the algorithm. Let:

T_1 be the number of times arm 1 is pulled, and T_2 be the number of times arm 2 is pulled.

The total reward of the algorithm is:

$$\text{Total Reward} = T_1\mu_1 + T_2\mu_2.$$

The reward from always pulling the optimal arm (arm 1) is:

$$\text{Optimal Reward} = T\mu_1,$$

where T is the total number of rounds.

Thus, the regret is:

$$R_T = \text{Optimal Reward} - \text{Total Reward}.$$

Substitute the rewards:

$$R_T = T\mu_1 - (T_1\mu_1 + T_2\mu_2).$$

Simplify:

$$R_T = (T - T_1)\mu_1 - T_2\mu_2.$$

Since $T_1 + T_2 = T$, this reduces to:

$$R_T = T_2(\mu_1 - \mu_2).$$

Let $\Delta = \mu_1 - \mu_2$. Then:

$$R_T = T_2\Delta.$$

Taking the expected value:

$$\mathbf{E[R_T] = E[T_2] \cdot \Delta}.$$

b) Use the Hoeffding inequality to show that $P[\hat{\mu}_2 > \hat{\mu}_1] \leq 2e^{-m\Delta^2/2}$

We want to bound the probability that $\hat{\mu}_2 > \hat{\mu}_1$, which can be written as:

$$P[\hat{\mu}_2 > \hat{\mu}_1] = P[\hat{\mu}_2 - \hat{\mu}_1 > 0].$$

Step 1: Decompose $\hat{\mu}_2 - \hat{\mu}_1$:

Let $\hat{\mu}_2$ and $\hat{\mu}_1$ represent the empirical mean estimates of arms 2 and 1, respectively. Then:

$$\hat{\mu}_2 - \hat{\mu}_1 = (\hat{\mu}_2 - \mu_2) + (\mu_2 - \mu_1) + (\mu_1 - \hat{\mu}_1).$$

Substitute $\mu_2 - \mu_1 = -\Delta$:

$$\hat{\mu}_2 - \hat{\mu}_1 = (\hat{\mu}_2 - \mu_2) - \Delta + (\mu_1 - \hat{\mu}_1).$$

The event $\hat{\mu}_2 > \hat{\mu}_1$ implies:

$$(\hat{\mu}_2 - \mu_2) + (\mu_1 - \hat{\mu}_1) > \Delta.$$

Step 2: Bound Each Term Using Hoeffding's Inequality:

Hoeffding's inequality states that for independent random variables $X_i \in [0, 1]$, the sample mean $\hat{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$ satisfies:

$$P[|\hat{X} - E[X]| \geq \epsilon] \leq 2e^{-2m\epsilon^2}.$$

Apply Hoeffding's inequality to:

- $P[\hat{\mu}_2 - \mu_2 > \Delta/2]$:
  Here, the deviation ϵ is bounded by $\Delta/2$, so:
  $$P[\hat{\mu}_2 - \mu_2 > \Delta/2] \leq e^{-2m(\Delta/2)^2}.$$
- $P[\mu_1 - \hat{\mu}_1 > \Delta/2]$:
  Similarly, the deviation ϵ is also $\Delta/2$, so:
  $$P[\mu_1 - \hat{\mu}_1 > \Delta/2] \leq e^{-2m(\Delta/2)^2}.$$

Step 3: Combine Bounds:

The event $(\hat{\mu}_2 - \mu_2) + (\mu_1 - \hat{\mu}_1) > \Delta$ requires either $\hat{\mu}_2 - \mu_2 > \Delta/2$ or $\mu_1 - \hat{\mu}_1 > \Delta/2$. Using the union bound:

$$P[\hat{\mu}_2 > \hat{\mu}_1] \leq P[\hat{\mu}_2 - \mu_2 > \Delta/2] + P[\mu_1 - \hat{\mu}_1 > \Delta/2].$$

Substitute the bounds:

$$P[\hat{\mu}_2 > \hat{\mu}_1] \leq e^{-2m(\Delta/2)^2} + e^{-2m(\Delta/2)^2}.$$

Simplify:

$$\mathbf{P[\hat{\mu}_2 > \hat{\mu}_1] \leq 2e^{-m\Delta^2/2}}.$$

c) Use part (b) to show that $E[T_2] \leq m + 2(T - 2m)e^{-m\Delta^2/2}$

The algorithm pulls arm 2 during:

- **The exploration phase:** Both arms are pulled m times.
- **The exploitation phase:** The arm with the higher estimated mean is pulled for $T - 2m$ rounds. Arm 2 is pulled in this phase only if $\hat{\mu}_2 > \hat{\mu}_1$.

Thus:

$$E[T_2] = m + E[\text{Exploitation Phase Pulls for Arm 2}].$$

From part (b), the probability $P[\hat{\mu}_2 > \hat{\mu}_1]$ is bounded by:

$$P[\hat{\mu}_2 > \hat{\mu}_1] \leq 2e^{-m\Delta^2/2}.$$

During the $T-2m$ exploitation rounds, the expected number of pulls for arm 2 is:

$$E[\text{Exploitation Phase Pulls}] = (T-2m)\cdot P[\hat{\mu}_2 > \hat{\mu}_1].$$

Substitute the bound for $P[\hat{\mu}_2 > \hat{\mu}_1]$:

$$E[\text{Exploitation Phase Pulls}] \leq (T-2m)\cdot 2e^{-m\Delta^2/2}.$$

Combine with the exploration phase:

$$\mathbf{E[T_2] \leq m + 2(T-2m)e^{-m\Delta^2/2}}.$$

d) Using the bounds, show that $E[R_T] \leq m\Delta + 2T\Delta e^{-m\Delta^2/2}$

From part (a), the expected regret is:

$$E[R_T] = E[T_2]\cdot\Delta.$$

Substitute the bound for $E[T_2]$ from part (c):

$$E[T_2] \leq m + 2(T-2m)e^{-m\Delta^2/2}.$$

This gives:

$$E[R_T] \leq \Delta\cdot\left(m + 2(T-2m)e^{-m\Delta^2/2}\right).$$

Distribute Δ:

$$E[R_T] \leq m\Delta + 2\Delta(T-2m)e^{-m\Delta^2/2}.$$

Simplify:

$$E[R_T] \leq m\Delta + 2T\Delta e^{-m\Delta^2/2} - 4m\Delta e^{-m\Delta^2/2}.$$

Since $2T\Delta e^{-m\Delta^2/2}$ dominates $-4m\Delta e^{-m\Delta^2/2}$

Since T (the total number of rounds) is typically much larger than m (the number of exploration rounds), the term $2T\Delta e^{-m\Delta^2/2}$ dominates $-4m\Delta e^{-m\Delta^2/2}$. Thus, we simplify to:

$$\mathbf{E[R_T] \leq m\Delta + 2T\Delta e^{-m\Delta^2/2}}.$$

e) Find the m that minimizes this regret bound and show why it is optimal

The regret bound is:

$$E[R_T] \leq m\Delta + 2T\Delta e^{-m\Delta^2/2}.$$

Let:
$$f(m) = m\Delta + 2T\Delta e^{-m\Delta^2/2}.$$

Take the derivative:
$$f'(m) = \Delta - T\Delta^3 e^{-m\Delta^2/2}.$$

Set $f'(m) = 0$:
$$\Delta = T\Delta^3 e^{-m\Delta^2/2}.$$

Simplify:
$$e^{-m\Delta^2/2} = \frac{1}{T\Delta^2}.$$

Take the natural logarithm:
$$-\frac{m\Delta^2}{2} = \ln\left(\frac{1}{T\Delta^2}\right).$$

Solve for m:
$$\mathbf{m} = \frac{\mathbf{2}\ln(\mathbf{T\Delta^2})}{\mathbf{\Delta^2}}.$$

Verify convexity with the second derivative:
$$f''(m) = T\Delta^5 e^{-m\Delta^2/2}.$$

Since $f''(m) > 0$ for all $m > 0$, $f(m)$ is convex, and the solution is a global minimum.

Bonus: Show that for the optimal m, we have the regret bounds

We aim to show the following:
$$E[R_T] \leq \frac{C_1}{\Delta} + \frac{C_2 \log(T\Delta^2)}{\Delta},$$
where C_1 and C_2 are constants. Furthermore, by maximizing the second term, we will show:
$$E[R_T] \leq 1 + C\sqrt{T},$$
where C is some constant.

Step 1: Expected regret bound for optimal m:

From earlier analysis, the expected regret is:
$$E[R_T] \leq m\Delta + 2T\Delta e^{-m\Delta^2/2}.$$

For the optimal m, we have:
$$m = \frac{2\ln(T\Delta^2)}{\Delta^2}.$$

1. **First term ($m\Delta$):**
Substitute m into $m\Delta$:

$$m\Delta = \frac{2\ln(T\Delta^2)}{\Delta^2} \cdot \Delta = \frac{2\ln(T\Delta^2)}{\Delta}.$$

This contributes to the term $\frac{C_2 \log(T\Delta^2)}{\Delta}$.
2. **Second term ($2T\Delta e^{-m\Delta^2/2}$):**
Substitute $m = \frac{2\ln(T\Delta^2)}{\Delta^2}$:

$$2T\Delta e^{-m\Delta^2/2} = 2T\Delta e^{-\ln(T\Delta^2)}.$$

Simplify $e^{-\ln(T\Delta^2)}$:

$$e^{-\ln(T\Delta^2)} = \frac{1}{T\Delta^2}.$$

Substitute back:

$$2T\Delta e^{-m\Delta^2/2} = 2T\Delta \cdot \frac{1}{T\Delta^2} = \frac{2}{\Delta}.$$

This contributes to the term $\frac{C_1}{\Delta}$.
Combine both terms:

$$E[R_T] \leq \frac{2}{\Delta} + \frac{2\ln(T\Delta^2)}{\Delta}.$$

Define constants $C_1 = 2$ and $C_2 = 2$, so:

$$E[R_T] \leq \frac{C_1}{\Delta} + \frac{C_2 \log(T\Delta^2)}{\Delta}.$$

Step 2: Maximizing $\frac{\log(T\Delta^2)}{\Delta}$:
Now we maximize the second term:

$$f(\Delta) = \frac{\log(T\Delta^2)}{\Delta}.$$

Take the derivative of $f(\Delta)$:

$$f'(\Delta) = \frac{d}{d\Delta}\left(\frac{\log(T\Delta^2)}{\Delta}\right).$$

Using the quotient rule:

$$f'(\Delta) = \frac{\Delta \cdot \frac{d}{d\Delta}[\log(T\Delta^2)] - \log(T\Delta^2)}{\Delta^2}.$$

The derivative of $\log(T\Delta^2)$ is:

$$\frac{d}{d\Delta}[\log(T\Delta^2)] = \frac{2}{\Delta}.$$

Substitute back:

$$f'(\Delta) = \frac{\Delta \cdot \frac{2}{\Delta} - \log(T\Delta^2)}{\Delta^2}.$$

Simplify:

$$f'(\Delta) = \frac{2 - \log(T\Delta^2)}{\Delta^2}.$$

Set $f'(\Delta) = 0$ to find the critical point:

$$2 - \log(T\Delta^2) = 0.$$

Solve for Δ:

$$\log(T\Delta^2) = 2.$$

Exponentiate both sides:

$$T\Delta^2 = e^2.$$

Solve for Δ:

$$\Delta = \frac{e}{\sqrt{T}}.$$

Step 3: Substituting $\Delta = \frac{e}{\sqrt{T}}$ into $E[R_T]$:
Substitute $\Delta = \frac{e}{\sqrt{T}}$ into the regret bound:

$$E[R_T] \leq \frac{C_1}{\Delta} + \frac{C_2 \log(T\Delta^2)}{\Delta}.$$

1. First term:

$$\frac{C_1}{\Delta} = \frac{C_1}{\frac{e}{\sqrt{T}}} = C_1 \frac{\sqrt{T}}{e}.$$

2. Second term: Substitute $T\Delta^2 = e^2$:

$$\log(T\Delta^2) = \log(e^2) = 2.$$

Substitute $\Delta = \frac{e}{\sqrt{T}}$:

$$\frac{C_2 \log(T\Delta^2)}{\Delta} = \frac{C_2 \cdot 2}{\frac{e}{\sqrt{T}}} = C_2 \frac{2\sqrt{T}}{e}.$$

Combine both terms:

$$E[R_T] \leq C_1 \frac{\sqrt{T}}{e} + C_2 \frac{2\sqrt{T}}{e}.$$

Factor out $\sqrt{T}$:

$$E[R_T] \leq (C_1 + 2C_2) \frac{\sqrt{T}}{e}.$$

Let $C = \frac{C_1 + 2C_2}{e}$, then:

$$E[R_T] \leq C\sqrt{T}.$$

For large T, the regret is dominated by the $\sqrt{T}$ term, so:

$$E[R_T] \leq 1 + C\sqrt{T}.$$

ESE 4020/5420 - Problem Set #5

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```
# These are the only imports you can use for Question 1
import numpy as np
import matplotlib.pyplot as plt
```

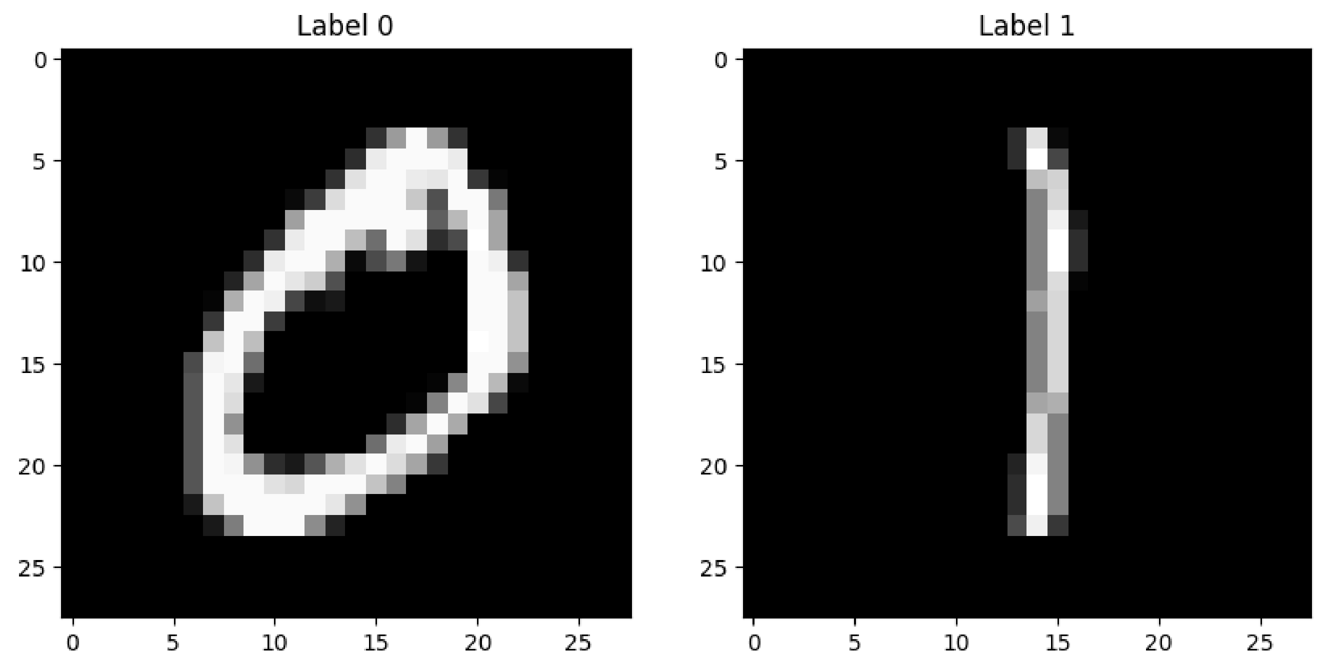
Question 1: Logistic Regression from Scratch

Load the data

```
# TO-DO: Use np.load(...) to read the data then assign the data to X, and the labels to y.
X = np.load('./data-Q1.npy')
y = np.load('./label-Q1.npy')
```

Visualize one input data point as an image for each of label 0 and label 1.

```
# TO-DO: The data should be reshaped back to [28 x 28] to be able to visualize it using plt.imshow(
label_0 = np.where(y==0)[0][0]
label_1 = np.where(y==1)[0][0]

plt.figure(figsize=(10,8))

plt.subplot(1,2,1)
plt.imshow(X[label_0].reshape(28,28),cmap='gray')
plt.title('Label 0')


plt.subplot(1,2,2)
plt.imshow(X[label_1].reshape(28,28),cmap='gray')
plt.title('Label 1')

plt.show()
```



Since the data is in between 0 to 255, normalize the data to [0, 1]

```
# TO-DO: Normalize the data
X = X/255.0
```

Set $y_i = +1$ for images originally labeled 0, and $y_i = -1$ for images originally labeled 1.

```
# TO-DO: Convert labels
y = np.where(y==0, 1, -1)
```

Split the data randomly into train and test with a ratio of 80:20.

```
# TO-DO: Create function that takes in (X,y) and splits the data by returning (X_train, y_train, X
def split(X,y):
  np.random.seed(69)
  index = np.arange(X.shape[0])
  np.random.shuffle(index)

  s = int(0.8 * X.shape[0])
  train_ind = index[:s]
  test_ind = index[s:]

  X_train, y_train = X[train_ind], y[train_ind]
  X_test, y_test = X[test_ind], y[test_ind]

  return X_train, y_train, X_test, y_test
```

```
# TO-DO: Call your function to perform the Train-Test Split on our data
X_train, y_train, X_test, y_test = split(X,y)
```

Initialize the coefficients $\beta_0^{(1)}, \vec{\beta}$ using a Normal Distribution of mean 0 and variance 1.

For β_1, initialize all d entries to be $N(0, 1)$)

```
# TO-DO: Initialize all d entries to be sampled from a standard normal distribution
d = X.shape[1]
B = np.random.randn(d)
B_0 = np.random.randn()
```

Compute the loss function

$$L(\beta_0^{(1)}, \vec{\beta}) = \frac{1}{m} \sum_{i=1}^{m} \ln(1 + e^{-y_i(\beta_0 + \Sigma_{j=1}^{d} \beta_j x_{i,j})})$$

where $x_{i,j}$ is the j-th of entry data point $\overrightarrow{\mathbf{x}_i}$

```
# TO DO: Helper function to compute loss
def compute_loss(data, labels, B, B_0):
  lin = np.dot(data, B) + B_0
  inner = np.exp(-labels * lin)
  loss = np.mean((np.log(1+inner)))
  return loss
```

Compute the gradients of the loss function

$$\frac{\partial L}{\partial \beta_0} = d\beta_0 = -\frac{1}{m} \sum_{i=1}^{m} \frac{e^{-y_i \cdot (\beta_0 + \vec{\beta}^T \vec{x_i})}}{1 + e^{-y_i (\beta_0 + \vec{\beta}^T \vec{x_i})}} y_i$$

$$\nabla_\beta L = d\vec{\beta} = -\frac{1}{m} \sum_{i=1}^{m} \frac{e^{-y_i \cdot (\beta_0 + \vec{\beta}^T \vec{x_i})}}{1 + e^{-y_i (\beta_0 + \vec{\beta}^T \vec{x_i})}} y_i x_i$$

```
# TO-DO: Helper function to compute loss
def compute_gradients(data, labels, B, B_0):
    lin = np.dot(data, B) + B_0

    inner = np.exp(-labels * lin)
    probs = inner / (1 + inner)

    grad_B_0 = -np.mean(probs * labels)

    grad_B = -np.dot((probs * labels), data) / len(labels)
    return grad_B_0, grad_B
```

Update the parameters using gradient updates from the train set as:

$$\beta_j \leftarrow \beta_j - 0.05 \cdot d\beta_j$$
$$\beta_0 \leftarrow \beta_0 - 0.05 \cdot d\beta_0$$

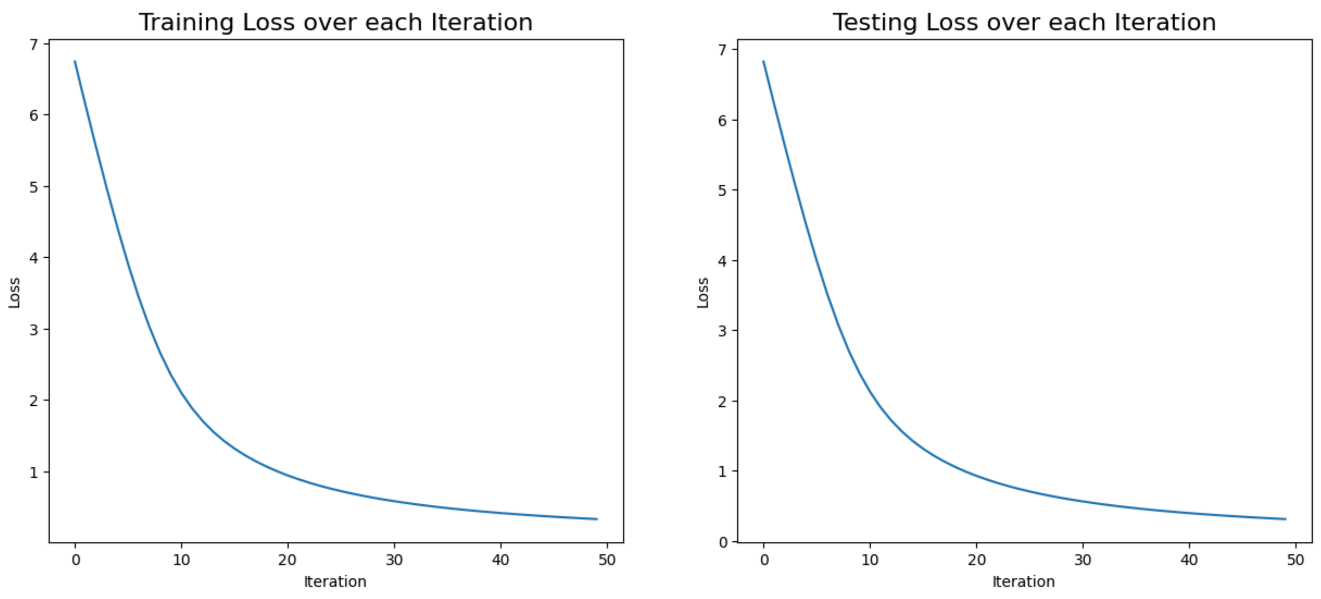
Repeat the process for 50 iterations. You should save your results for each of the 50 epochs in `accuracy_hist`, `train_loss_hist`, and `test_loss_hist`.

```
lr = 0.05

accuracy_hist = []
train_loss_hist = []
test_loss_hist = []

for epoch in range(50):
    grad_B_0, grad_B = compute_gradients(X_train, y_train, B, B_0)
    B = B - (lr * grad_B)
    B_0 = B_0 -  (lr * grad_B_0)
    train_loss = compute_loss(X_train, y_train, B, B_0)
    train_predictions = np.sign(np.dot(X_train, B) + B_0)
    train_accuracy = np.mean(train_predictions == y_train)

    test_loss = compute_loss(X_test, y_test, B, B_0)
    test_predictions = np.sign(np.dot(X_test, B) + B_0)
    test_accuracy = np.mean(test_predictions == y_test)

    train_loss_hist.append(train_loss)
    test_loss_hist.append(test_loss)
    accuracy_hist.append(test_accuracy)

print("Training complete!")
print(f"Final training loss: {train_loss_hist[-1]}")
print(f"Final test loss: {test_loss_hist[-1]}")
print(f"Final testing accuracy: {accuracy_hist[-1]}")
```

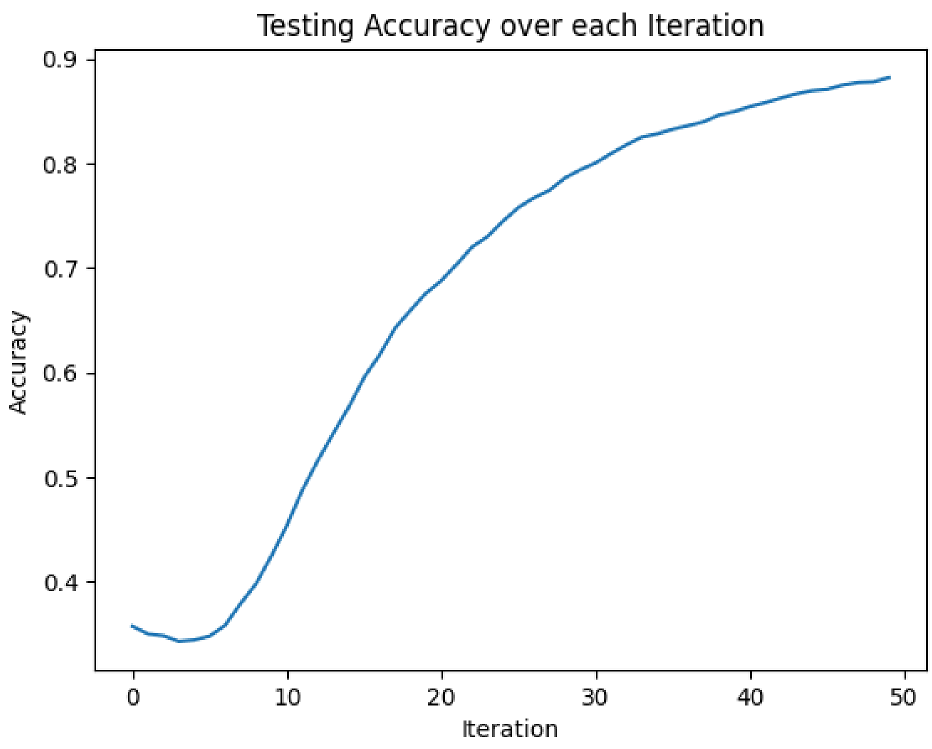
```
Training complete!
Final training loss: 0.32915255073808225
Final test loss: 0.3090091609857607
Final testing accuracy: 0.881935047361299
```

Plot your training and testing loss curves side-by-side below by running the plotting code given below. You don't need to modify these two cells.

```
fig = plt.figure(figsize = (15,6))

# Plot Loss Function
ax1 = fig.add_subplot(121)
ax1.plot(train_loss_hist);
ax1.set_title("Training Loss over each Iteration", fontsize = 16);
ax1.set_xlabel("Iteration");
ax1.set_ylabel("Loss");

# Plot Accuracy Function
ax2 = fig.add_subplot(122)
ax2.plot(test_loss_hist);
ax2.set_title("Testing Loss over each Iteration", fontsize = 16);
ax2.set_xlabel("Iteration");
ax2.set_ylabel("Loss");
```



```
# Plot your accuracy curve below by running the template code below
plt.plot(accuracy_hist);
plt.title("Testing Accuracy over each Iteration");
plt.xlabel("Iteration");
plt.ylabel("Accuracy");
```



Congratulations!!! Now convert this to `.pdf`!

You're done with all the coding questions! Now convert this `.ipynb` into a `.pdf` neatly to prevent cells and figures from splitting across pages and getting weirdly truncated by following these steps:

1. Download your Colab notebook as an `.ipynb` file to save locally on your computer
2. Open it locally via Jupyter Notebook
3. Go to “print preview” mode
4. Print to `.pdf` from there.
5. Merge this PDF with your handwritten notes. You can use https://smallpdf.com/merge-pdf

Leave enough time to ask for help if you're stuck so you don't get hit with the Late Penalty!

Q2 B, C and D

```
import numpy as np
from sklearn.linear_model import LogisticRegression

def generate_data(n, seed=69):
    np.random.seed(seed)
    X = np.random.uniform(0, 1, n)
    probs = np.where(X > 0.5, 0.9, 0.1)
    Y = np.random.binomial(n=1, p=probs)
    return X.reshape(-1, 1), Y

def train_logistic_regression(X, Y):
    model = LogisticRegression()
    model.fit(X, Y)
    return model

def evaluate_model(model, X_test, Y_test):
    Y_pred = model.predict(X_test)
    return np.mean(Y_pred == Y_test) * 100

def bayes_optimal_predict(x):
    return 1 if x > 0.5 else 0

def evaluate_bayes_optimal(X_test, Y_test):
    Y_bayes = np.array([bayes_optimal_predict(x[0]) for x in X_test])
    return np.mean(Y_bayes == Y_test) * 100

# Part (b)
n_train = 100
X_train, Y_train = generate_data(n_train)
model = train_logistic_regression(X_train, Y_train)



# Part (c)
n_test = 100
X_test, Y_test = generate_data(n_test)
lr_accuracy = evaluate_model(model, X_test, Y_test)
bayes_accuracy = evaluate_bayes_optimal(X_test, Y_test)

print("Logistic Regression Accuracy (n=100):", lr_accuracy)
print("Bayes Optimal Classifier Accuracy (n=100):", bayes_accuracy)
```

```
Logistic Regression Accuracy (n=100): 93.0
Bayes Optimal Classifier Accuracy (n=100): 95.0
```

The Bayes optimal classifier outperforms logistic regression because it leverages the exact conditional probabilities from the known data distribution. This allows it to make decisions with maximum accuracy, as it always picks the class with the highest probability based on the true distribution.

In contrast, logistic regression learns an approximation of this relationship from the training data, which may not perfectly capture the true probabilities, especially with a limited sample size. As a result, logistic regression's decision boundary may slightly differ from the optimal one, leading to lower accuracy compared to the Bayes optimal classifier.

```
# Part (d)
n_large = 1000
X_train_large, Y_train_large = generate_data(n_large)
model_large = train_logistic_regression(X_train_large, Y_train_large)

X_test_large, Y_test_large = generate_data(n_large)
lr_accuracy_large = evaluate_model(model_large, X_test_large, Y_test_large)
bayes_accuracy_large = evaluate_bayes_optimal(X_test_large, Y_test_large)

print("Logistic Regression Accuracy (n=1000):", lr_accuracy_large)
print("Bayes Optimal Classifier Accuracy (n=1000):", bayes_accuracy_large)
```

```
Logistic Regression Accuracy (n=1000): 89.8
Bayes Optimal Classifier Accuracy (n=1000): 90.3
```

Yes, the results are notably different when comparing n=100 versus n=1000 samples. The larger dataset produces more stable and theoretically aligned results, with the logistic regression achieving 89.8% accuracy and the Bayes optimal classifier reaching 90.3%. This represents a significant improvement from the n=100 case, where we saw more variability and generally lower performance.

The improved performance with n=1000 can be attributed to several key factors. First, the larger sample size allows the logistic regression model to better estimate its parameters, reducing the impact of random variations in the training data. Second, with more data points, both classifiers can better approximate the true underlying probability distribution P(Y|X). The Bayes optimal classifier's accuracy of 90.3% is particularly telling as it's very close to the theoretical maximum of 0.9 that we designed in part (a), while the logistic regression's performance of 89.8% demonstrates that with sufficient data, it can nearly match the optimal classifier's performance despite its simpler linear decision boundary assumption.

3(a)

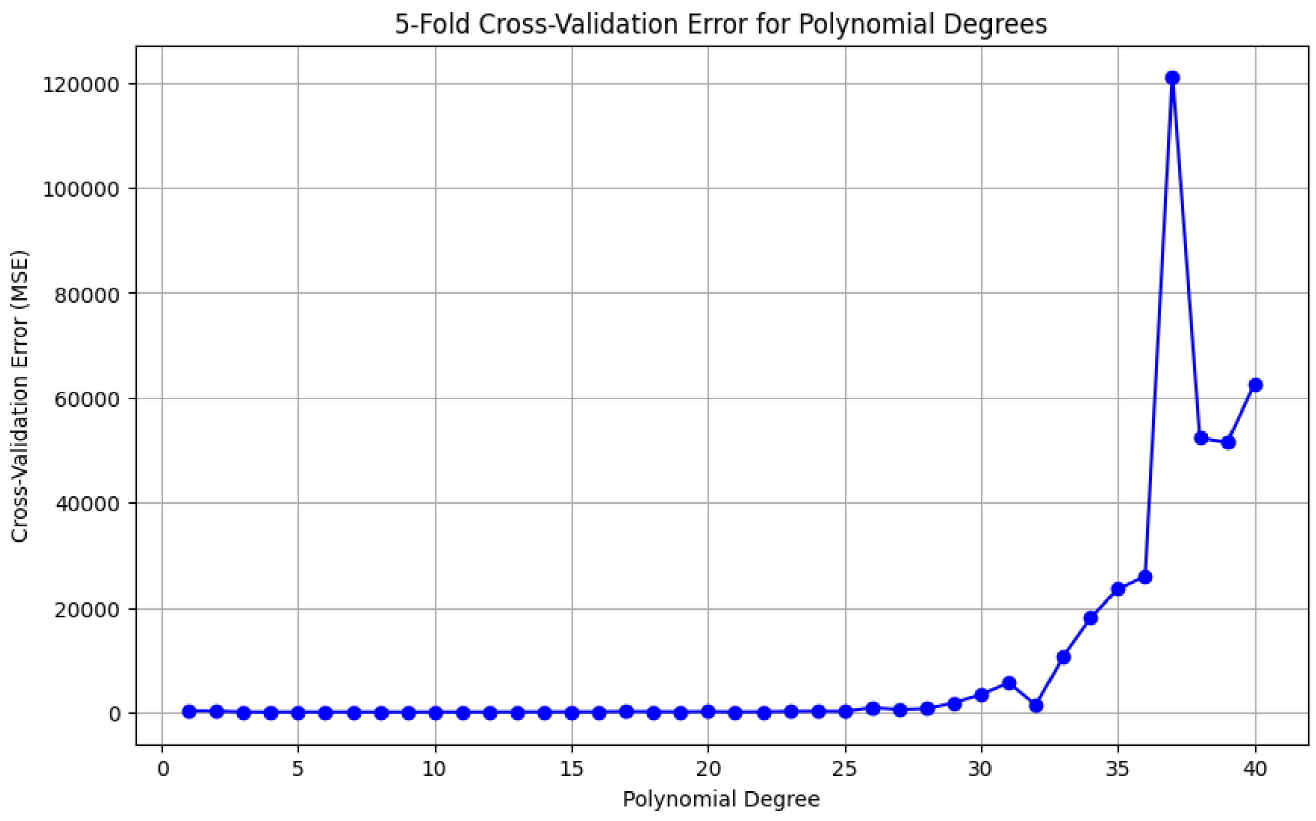
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

```
data = pd.read_csv('poly_data.csv', names=['X', 'Y'], header=None)
X = data[['X']].values
y = data['Y'].values
```

```
from sklearn.model_selection import KFold
from sklearn.preprocessing import PolynomialFeatures
from sklearn.linear_model import LinearRegression
from sklearn.metrics import mean_squared_error
```

```
degrees = range(1, 41)
k = 5
kf = KFold(n_splits=k, shuffle=True, random_state=69)

cv_errors = []

for degree in degrees:
    poly = PolynomialFeatures(degree)
    X_poly = poly.fit_transform(X)

    fold_errors = []

    for train_index, val_index in kf.split(X_poly):
        X_train, X_val = X_poly[train_index], X_poly[val_index]
        y_train, y_val = y[train_index], y[val_index]
        # print(len(X_train), len(y_train))
        model = LinearRegression()
        model.fit(X_train, y_train)

        y_val_pred = model.predict(X_val)
        fold_error = mean_squared_error(y_val, y_val_pred)
        fold_errors.append(fold_error)

    avg_cv_error = np.mean(fold_errors)
    cv_errors.append(avg_cv_error)
```

```
plt.figure(figsize=(10, 6))
plt.plot(degrees, cv_errors, marker='o', linestyle='-', color='b')
plt.xlabel('Polynomial Degree')
plt.ylabel('Cross-Validation Error (MSE)')
plt.title(f'{k}-Fold Cross-Validation Error for Polynomial Degrees')
plt.grid(True)
plt.show()
```



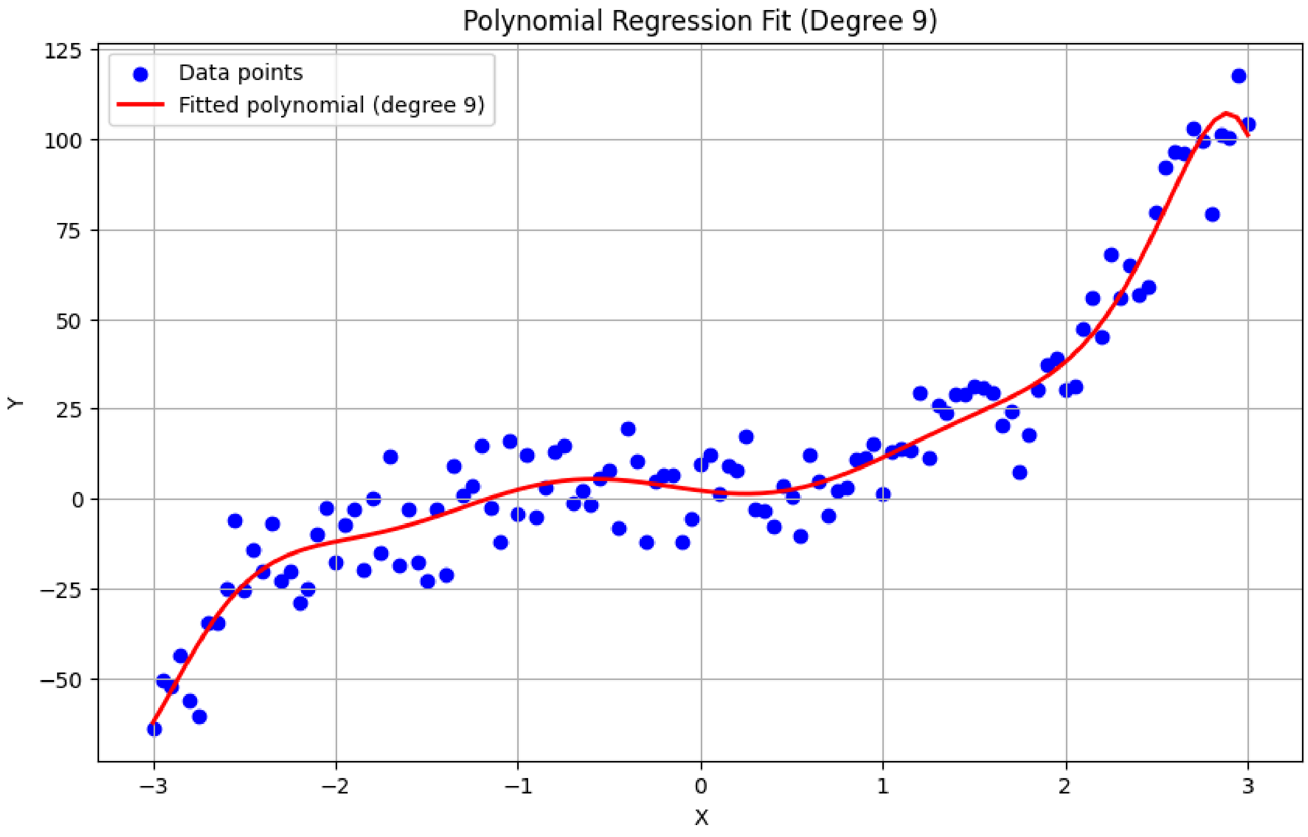
```
best_degree = degrees[np.argmin(cv_errors)]
print(f"The best polynomial degree based on {k} cross-validation is: {best_degree}")
```

```
The best polynomial degree based on 5 cross-validation is: 9
```

3(b)

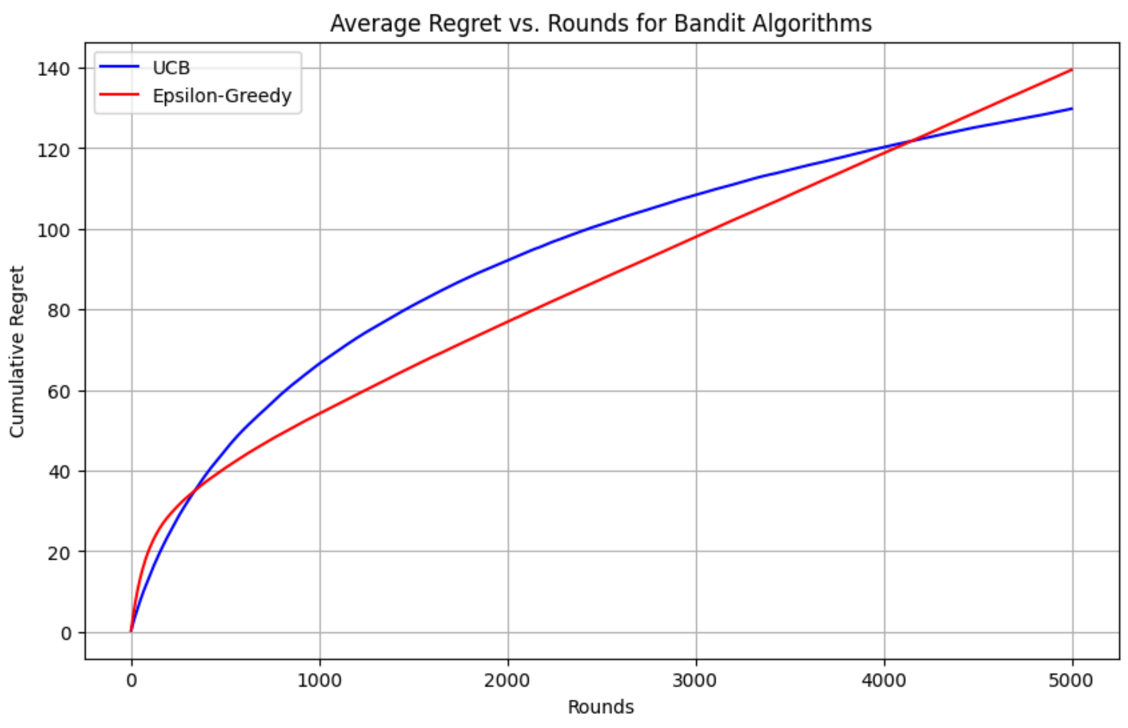
```
best_degree = degrees[np.argmin(cv_errors)]

poly = PolynomialFeatures(best_degree)
X_poly = poly.fit_transform(X)
model = LinearRegression()
model.fit(X_poly, y)

print(f"Polynomial coefficients for degree {best_degree}:")
print(model.coef_)

X_curve = np.linspace(X.min(), X.max(), 100).reshape(-1, 1)
X_curve_poly = poly.transform(X_curve)
y_curve = model.predict(X_curve_poly)

plt.figure(figsize=(10, 6))
plt.scatter(X, y, color='blue', label='Data points')
plt.plot(X_curve, y_curve, color='red', linewidth=2, label=f'Fitted polynomial (degree {best_degre
plt.xlabel('X')
plt.ylabel('Y')
plt.title(f'Polynomial Regression Fit (Degree {best_degree})')
plt.legend()
plt.grid(True)
plt.show()
```

```
Polynomial coefficients for degree 9:
[ 0.         -6.32550987  7.72580525 15.2293662  -3.78550145 -5.20498237
  0.85899842  0.80888092 -0.05668653 -0.04139501]
```



Q5

```
import numpy as np
import matplotlib.pyplot as plt


arm_probs = [0.02, 0.1, 0.25, 0.4]
num_arms = len(arm_probs)


T = 5000
num_trials = 1000

def run_ucb(num_rounds, arm_probs):
    num_arms = len(arm_probs)
    Q = np.zeros(num_arms)
    N = np.zeros(num_arms)
    regret = np.zeros(num_rounds)

    for t in range(num_rounds):
        if t < num_arms:
            arm = t
        else:
            ucb_values = Q + np.sqrt(2 * np.log(t + 1) / (N + 1e-5))
            arm = np.argmax(ucb_values)

        reward = np.random.binomial(1, arm_probs[arm])
        N[arm] += 1
        Q[arm] += (reward - Q[arm]) / N[arm]
        regret[t] = np.max(arm_probs) - arm_probs[arm]

    return np.cumsum(regret)


def run_epsilon_greedy(num_rounds, arm_probs, epsilon=0.1):
    num_arms = len(arm_probs)
    Q = np.zeros(num_arms)
    N = np.zeros(num_arms)
    regret = np.zeros(num_rounds)

    for t in range(num_rounds):
        if np.random.rand() < epsilon:
            arm = np.random.choice(num_arms)
        else:
            arm = np.argmax(Q)

        reward = np.random.binomial(1, arm_probs[arm])
        N[arm] += 1
        Q[arm] += (reward - Q[arm]) / N[arm]
        regret[t] = np.max(arm_probs) - arm_probs[arm]

    return np.cumsum(regret)


ucb_regret = np.zeros(T)
egreedy_regret = np.zeros(T)

for _ in range(num_trials):
    ucb_regret += run_ucb(T, arm_probs)
    egreedy_regret += run_epsilon_greedy(T, arm_probs)

ucb_regret /= num_trials
egreedy_regret /= num_trials


plt.figure(figsize=(10, 6))
plt.plot(ucb_regret, label='UCB', color='blue')
plt.plot(egreedy_regret, label='Epsilon-Greedy', color='red')
plt.xlabel('Rounds')
plt.ylabel('Cumulative Regret')
plt.title('Average Regret vs. Rounds for Bandit Algorithms')
plt.legend()
plt.grid()
plt.show()
```


Average Regret vs. Rounds for Bandit Algorithms
UCB
Epsilon-Greedy
Cumulative Regret
Rounds
0
20
40
60
80
100
120
140
0
1000
2000
3000
4000
5000