ESE 542 - Homework #3

① (a) Given $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2 = 9)$ where $n=5$, $\bar{x}=12$, and $s^2=5$

$$\begin{cases} H_0: \mu = 10 \\ H_a: \mu \neq 10 \end{cases}$$

$\boxed{\text{Calculate Test Statistic}}$ (z-statistic since Normal Distrib. given and σ^2 known)

$$\underset{\sim N(0,1)}{TS} = \frac{\bar{x}-\mu_0}{SE(\bar{x})} = \frac{\bar{x}-\mu_0}{\sqrt{\frac{\sigma^2}{n}}} = \frac{12-10}{\sqrt{\frac{9}{5}}} = \frac{2\sqrt{5}}{3} \approx 1.491$$

$\boxed{\text{Calculate p-value}}$ (2-tailed test)

$$\begin{aligned} p &= Pr(Z < -TS) + Pr(Z > TS) && \text{for } Z \sim N(0,1) \\ &= 2 \cdot Pr(Z < -TS) && \text{by symmetry of Normal Distr.} \\ &= 2 \cdot \Phi(-TS) \\ &= 2 \cdot \Phi(-1.491)] \\ &= 0.136 \end{aligned}$$

$\boxed{\text{Determine Decision Rule for } \alpha = 0.05}$

We $\begin{cases} \text{Reject } H_0 \text{ when } p \leq \alpha \\ \text{Fail to Reject } H_0 \text{ when } p > \alpha \end{cases}$

Since $p = 0.1360 > 0.05$, we <u>fail to reject H_0</u> at the 5% signif. level.

(b) $\boxed{\text{Construct 95\% Confidence Interval for } \mu}$ (Acceptance/Non-Rejection Region)

<u>95% CI:</u> $\bar{x} - z_{\frac{\alpha}{2}} \cdot SE(\bar{x}) < \mu < \bar{x} + z_{\frac{\alpha}{2}} \cdot SE(\bar{x})$

$$\Rightarrow \bar{x} - z_{\frac{0.05}{2}} \cdot \sqrt{\frac{\sigma}{n}} < \mu < \bar{x} + z_{\frac{0.05}{2}} \cdot \sqrt{\frac{\sigma}{n}}$$

$$\Rightarrow 12 - 1.96\sqrt{\frac{9}{5}} < \mu < 12 + 1.96\sqrt{\frac{9}{5}}$$

$\therefore$ 95% CI is $[9.370, 14.630]$

Notice $\mu_0 = 10$ falls in 95% CI so consistent with failure to reject H_0!

★ Alternatively, we can compute p-value

$p = P(Z > TS) = P(Z > 4.341) = \Phi(-4.341) = 7.097 \times 10^{-6} \approx 0$

Since $p < 0.01$, reject H_0 at 1% level.

---

② For $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$ with $\sum_{i=1}^{n} X_i = 41$ and $n = 51$

(a) At $\alpha = 0.01$, want to test the hypotheses

$$\begin{cases} H_0: p = 0.5 \\ H_a: p > 0.5 \end{cases} \quad \text{(1-tailed test)}$$

From data, our estimate for p is $\hat{p} = \dfrac{\sum_{i=1}^{n} X_i}{n} = \dfrac{41}{51}$

Constructing a Test Statistic (Z-statistic since $n \geq 50$ so invoke CLT)

$$TS = \frac{\hat{p} - p_0}{SE(\hat{p})} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{\frac{41}{51} - 0.5}{\sqrt{\frac{0.5(1-0.5)}{n}}} \approx 4.341$$

$TS \xrightarrow[\text{by CLT}]{d} N(0,1)$

Next, obtain appropriate critical value. Since this is a 1-sided upper tail test,

Right Tail Critical Value $(z_\alpha) = \Phi^{-1}(1-\alpha)$

So at 1% significance ($\alpha = 0.01$),

$z_{0.01} = \Phi^{-1}(0.99) = 2.326$

Since $TS = 4.341 > 2.326 = z_{0.01}$

$\Rightarrow$ Reject H_0 at 1% significance Also see ★

$\therefore$ There is statistically significant evidence to conclude that more than 50% of all homes with dry wall have those problems.

(b) $\boxed{\text{Lower Bound of a 1-sided 95\% Confidence Interval for } p}$

$p \in [LB, \infty) \Rightarrow p \geq LB$ where $LB = \hat{p} - z_\alpha \cdot SE(\hat{p})$

$$= \hat{p} - z_{0.01}\sqrt{\frac{p_0(1-p_0)}{n}}$$

$$= \frac{41}{51} - 2.326\sqrt{\frac{0.5(1-0.5)}{51}}$$

$$= 0.641$$

③ Given $n=100$ and $\hat{p}=0.2$ for parameter p.

(a) We want to test the hypotheses

$$\begin{cases} H_0: p=0.25 \\ H_a: p<0.25 \end{cases}$$ 1-sided lower tailed test

Constructing a Test Statistic (z-statistic since $n \geq 50$ so invoke CLT)

$$TS = \frac{\hat{p}-p_0}{SE(\hat{p})} = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.2-0.25}{\sqrt{\frac{0.25(1-0.25)}{100}}} \approx -1.155$$

$$TS \xrightarrow[\text{by CLT}]{d} N(0,1)$$

Next, obtain appropriate critical value. Since this is a 1-sided lower tail test,

Left Tail Critical Value $(-z_\alpha) = \Phi^{-1}(\alpha)$

- $\alpha=0.01$: $-z_{0.01} = \Phi^{-1}(0.01) = -2.326$

  Since $TS = -1.155 > -2.326 = -z_{0.01}$

  $\Rightarrow$ Fail to reject H_0 at 1% significance

  $\therefore$ There is insufficient evidence at 1% level. to conclude that proportion less than 0.25

- $\alpha=0.05$: $-z_{0.05} = \Phi^{-1}(0.05) = -1.645$

  Since $TS = -1.155 > -1.645 = -z_{0.05}$

  $\Rightarrow$ Fail to reject H_0 at 5% significance

  $\therefore$ There is insufficient evidence at 5% level. to conclude that proportion less than 0.25

Alternatively, we can compute p-value

$$p = P_r(Z < TS) = P_r(Z < -1.155) = \Phi(-1.155) = 0.124$$

Since $p = 0.124 > 0.05 > 0.01$, fail to reject at both 5% and 1% level.

[Question 3 continued]

(b) We want to test the hypotheses:

$$\begin{cases} H_0: p = 0.25 \\ H_a: p \neq 0.25 \end{cases}$$

$\boxed{\text{Calculate Test Statistic}}$ (Z-statistic since $n \geq 50$ so invoke CLT)

$$TS = \frac{\hat{p} - p_0}{SE(\hat{p})} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.2 - 0.25}{\sqrt{\frac{0.25(1-0.25)}{100}}} \approx -1.155 \quad \text{as in part (a)}$$

$$TS \xrightarrow[\text{by CLT}]{d} N(0,1)$$

$\boxed{\text{Calculate p-value}}$ (2-tailed test)

$$\begin{aligned} p &= Pr(Z < -TS) + Pr(Z > TS) && \text{for } Z \sim N(0,1) \\ &= 2 \cdot Pr(Z < -TS) && \text{by symmetry of Normal Distr.} \\ &= 2 \cdot \Phi(-TS) \\ &= 2 \cdot \Phi(-1.155) \\ &= 0.248 && \text{... twice that of the 1-tailed p-value} \end{aligned}$$

$\boxed{\text{Determine Decision Rule for } \alpha = 0.05 \text{ and } \alpha = 0.01}$

We $\begin{cases} \text{Reject } H_0 \text{ when } p \leq \alpha \\ \text{Fail to Reject } H_0 \text{ when } p > \alpha \end{cases}$

Since $p = 0.248 > 0.05 > 0.01$

we fail to reject H_0 at both 5% and 1% levels.

Question 4

Set-Up

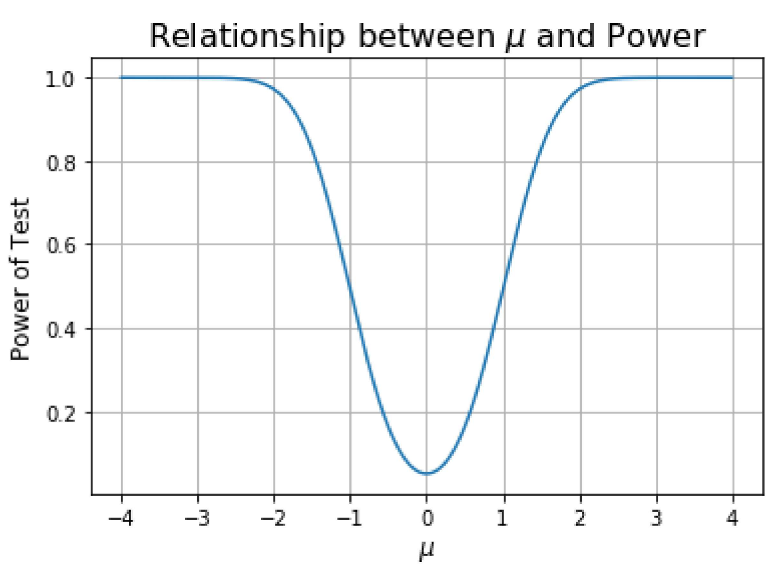
$$\beta = \Pr(\text{Type II Error}) = \Pr(\text{Fail to Reject } H_0 \mid H_0 \text{ is false})$$
$$= \Phi\left(\frac{\mu_0 - \mu + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{\mu_0 - \mu - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}}\right)$$

In [1]:
```
# Imports
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
%matplotlib inline
```

In [23]:
```
# Plotting Helper Function
def plot_power_effect(x_list, power_list, x_label, title):

    # Plotting mechanics
    plt.plot(x_list, power_list)
    plt.grid()
    plt.title(title, size = 16)
    plt.xlabel(x_label, size = 12)
    plt.ylabel('Power of Test', size = 12)
```

4(a): μ on Power of Test

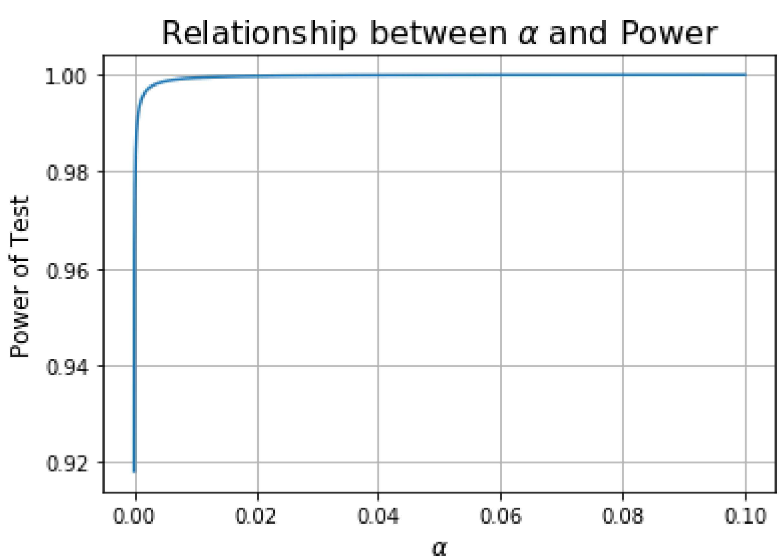
In [3]:
```
# Initialized values
n = 15
alpha = 0.05
mu_0 = 0
variance = 4

# Calculated values
SE = np.sqrt(variance/n)
crit_value = norm.ppf(1 - alpha/2)

# Set range of x-axis
mu_list = np.arange(-4, 4.01, 0.01)

# Calculate beta values
beta_list = norm.cdf((mu_0 - mu_list + crit_value*SE)/SE) \
- norm.cdf((mu_0 - mu_list - crit_value*SE)/SE)

# Calculate power values
power_list = 1 - beta_list

# Plot
plot_power_effect(mu_list, power_list, '$\\mu$', 'Relationship between $\\mu$
```



4(b): α on Power of Test

In [4]:

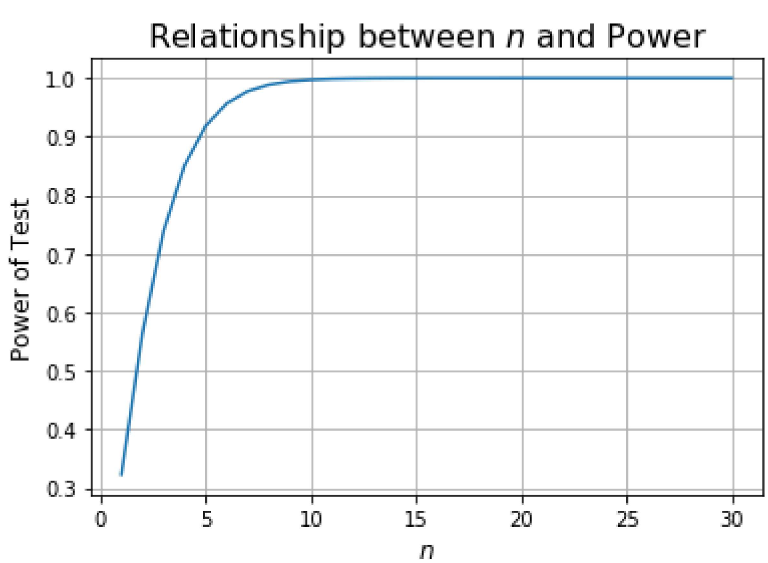
```
# Additional initialized values
mu = 3

# Set range of alphas
alpha_list = np.arange(0.00001, 0.10001, 0.00001)

# Calculated values
SE = np.sqrt(variance/n)
crit_value = norm.ppf(1 - alpha_list/2)

# Calculate beta values
beta_list = norm.cdf((mu_0 - mu + crit_value*SE)/SE) - norm.cdf((mu_0 - mu -

# Calculate power values
power_list = 1 - beta_list

# Plot
plot_power_effect(alpha_list, power_list, '$\\alpha$', 'Relationship between
```



4(c): n on Power of Test

In [9]:

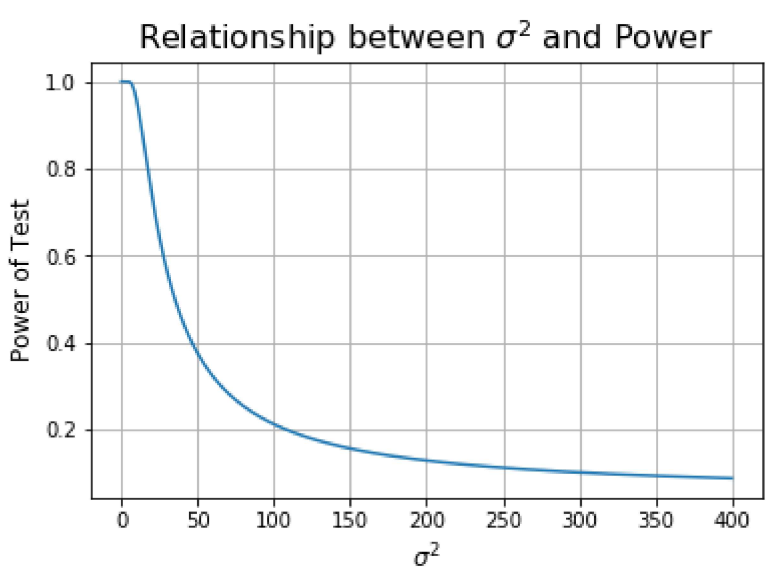
```
# Reset initialized values
alpha = 0.05
crit_value = norm.ppf(1 - alpha/2)

# Set range of n
n_list = np.arange(1, 31)

# Calculated values
SE = np.sqrt(variance/n_list)

# Calculate beta values
beta_list = norm.cdf((mu_0 - mu + crit_value*SE)/SE) \
- norm.cdf((mu_0 - mu - crit_value*SE)/SE)

# Calculate power values
power_list = 1 - beta_list

# Plot
plot_power_effect(n_list, power_list, '$n$', 'Relationship between $n$ and Po
```



4(d): σ^2 on Power of Test

In [21]:

```
# Reset initialized values
n = 15

# Set range of sigma^2
var_list = np.arange(0.01, 400, 0.01)

# Calculated values
SE = np.sqrt(var_list/n)

# Calculate beta values
beta_list = norm.cdf((mu_0 - mu + crit_value*SE)/SE) \
- norm.cdf((mu_0 - mu - crit_value*SE)/SE)

# Calculate power values
power_list = 1 - beta_list

# Plot
plot_power_effect(var_list, power_list, '$\\sigma^2$', 'Relationship between
```



4(e): Comparisons and Interpretations

(a): μ *and Power*

When the true mean (μ) is **further from the hypothesized value under H_0 (μ_0)**, i.e. $\mu \to -\infty$ (for $\mu < 0$) or $\mu \to +\infty$ (for $\mu > 0$), we are **more likely to reject** the null hypothesis. Thus, in the event that H_0 is false, we are more likely to reject H_0 in such a case as well. As a result, **power of the test increases**.

(b): α *and Power*

As α **increases**, the probability of a Type I error increases so it becomes **easier to reject H_0**. Thus, in the event that H_0 is false, we are more likely to reject H_0 in such a case as well. As a result, **power of the test increases**.

(c): n *and Power*

As sample size **increases**, we **need a *smaller* effect size** to reject H_0 so it becomes **easier to reject H_0**. Thus, in the event that H_0 is false, we are more likely to reject H_0 in such a case as well. As a result, **power of the test increases**.

(d): σ^2 *and Power*

As population variance **increases**, we **need a *larger* effect size** to reject H_0 so it becomes **harder to reject H_0**. Thus, in the event that H_0 is false, we are less likely to reject H_0 in such a case as well. As a result, **power of the test decreases.**

⑤ Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$

We want to conduct a Likelihood Ratio Test for

$$\begin{cases} H_0: \lambda = \lambda_0 \\ H_a: \lambda = \lambda_1 \end{cases} \quad \text{such that } \lambda_1 > \lambda_0$$

<u>Likelihood Function</u>: $\mathcal{L}(\lambda) = \prod_{i=1}^{n} \dfrac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \dfrac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{\prod_{i=1}^n x_i!}$

<u>Likelihood Ratio</u>: (Test Statistic) $\dfrac{\mathcal{L}(\lambda_0)}{\mathcal{L}(\lambda_1)} = \dfrac{\lambda_0^{\sum_{i=1}^n x_i} e^{-n\lambda_0}}{\lambda_1^{\sum_{i=1}^n x_i} e^{-n\lambda_1}} = \left(\dfrac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^n x_i} e^{n(\lambda_1 - \lambda_0)}$

<u>Compare</u>: $\left(\dfrac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^n x_i} e^{n(\lambda_1 - \lambda_0)} \underset{\text{Reject } H_0}{\overset{\text{Fail to reject } H_0}{\gtrless}} k$

$$\Rightarrow \ln\left[\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^n x_i} e^{n(\lambda_1 - \lambda_0)}\right] \gtrless \ln(k)$$

$$\Rightarrow \sum_{i=1}^n x_i \underbrace{\left[\ln(\lambda_0) - \ln(\lambda_1)\right]}_{<0:\ \text{division by this reverses signs}} + n\lambda_1 - n\lambda_0 \gtrless \ln(k)$$

$$\Rightarrow \sum_{i=1}^n x_i \lessgtr \frac{\ln(k) - n\lambda_1 + n\lambda_0}{\ln(\lambda_0) - \ln(\lambda_1)} = k'$$

Denote $S = \sum_{i=1}^n x_i$ and observe $S \sim \text{Poisson}(n\lambda)$ since sum of $X_i \sim \text{Poisson}(\lambda)$ is also Poisson Distributed with $E(S) = E\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n E(x_i) = \sum_{i=1}^n \lambda = n\lambda$

Then $\alpha = \Pr(\text{Reject } H_0 \mid H_0 \text{ True})$

$= \Pr(S > k' \mid X_i \sim \text{Poisson}(\lambda_0))$

$= 1 - \Pr(S \leq k' \mid X_i \sim \text{Poisson}(\lambda_0))$

$= 1 - F(k')$ where $F(\cdot)$ = CDF of $\text{Poisson}(n\lambda_0)$

$\therefore k' = F^{-1}(1-\alpha)$