ESE 5420 Homework 2

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Q1

a) Method of Moments

The probability density function (pdf) is:

$$f(x|\sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right)$$

First Moment:
The first moment $E[X]$ is:

$$E[X] = \int_{-\infty}^{\infty} x f(x|\sigma) dx = 0$$

For the Method of Moments, the first sample moment u_1 is:

$$u_1 = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Equating the theoretical first moment to the sample first moment:

$$E[X] = u_1$$

Since $E[X] = 0$, we have:

$$u_1 = 0$$

Second Moment:
The second moment $E[X^2]$ is:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x|\sigma) dx = 2\sigma^2$$

The second sample moment u_2 is:

$$u_2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

Equating the theoretical second moment to the sample second moment:

$$E[X^2] = u_2$$

Thus:

$$2\sigma^2 = u_2$$

Solving for σ:

$$\sigma = \frac{\sqrt{u_2}}{\sqrt{2}}$$

Thus, the Method of Moments estimate for σ is:

$$\hat{\sigma}_{\text{MoM}} = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}}{\sqrt{2}}$$

b) Maximum Likelihood Estimate (MLE)

The likelihood function is:

$$L(\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma} \exp\left(-\frac{|X_i|}{\sigma}\right)$$

The log-likelihood function is:

$$\log L(\sigma) = -n\log(2\sigma) - \frac{1}{\sigma}\sum_{i=1}^{n} |X_i|$$

First Derivative of the Log-Likelihood:
Taking the first derivative with respect to σ:

$$\frac{\partial}{\partial\sigma}\log L(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} |X_i|$$

Solve for σ:
Setting the derivative equal to zero:

$$-\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} |X_i| = 0$$

Multiply by σ^2:

$$-n\sigma + \sum_{i=1}^{n} |X_i| = 0$$

Solve for σ:

$$\hat{\sigma}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} |X_i|$$

c) Asymptotic Variance and Fisher Information

The second derivative of the log-likelihood is:

$$\frac{\partial^2}{\partial \sigma^2} \log L(\sigma) = \frac{n}{\sigma^2} - \frac{2}{\sigma^3} \sum_{i=1}^{n} |X_i|$$

The Fisher Information is the negative expected value of the second derivative:

$$I(\sigma) = -E\left[\frac{\partial^2}{\partial \sigma^2} \log L(\sigma)\right]$$

Since:

$$E\left[\sum_{i=1}^{n} |X_i|\right] = n\sigma$$

Substitute into the second derivative:

$$E\left[\frac{\partial^2}{\partial \sigma^2} \log L(\sigma)\right] = \frac{n}{\sigma^2} - \frac{2n\sigma}{\sigma^3} = -\frac{n}{\sigma^2}$$

Thus, the Fisher Information is:

$$I(\sigma) = \frac{n}{\sigma^2}$$

Asymptotic Variance:
The asymptotic variance of the MLE $\hat{\sigma}_{\text{MLE}}$ is:

$$\text{Var}(\hat{\sigma}_{\text{MLE}}) = \frac{1}{I(\sigma)} = \frac{\sigma^2}{n}$$

Q2

a) Likelihood Function

Let the probabilities of the outcomes be: - p_1 for the value 1, - p_2 for the value 2, - $1 - p_1 - p_2$ for the value 0.

Given the counts: - n_0 is the number of times 0 occurs, - n_1 is the number of times 1 occurs, - n_2 is the number of times 2 occurs,

the likelihood function is:

$$L(p_1, p_2) = (1 - p_1 - p_2)^{n_0} \cdot p_1^{n_1} \cdot p_2^{n_2}$$

where: - $n_0 = 19$, - $n_1 = 8$, - $n_2 = 3$.

The log-likelihood function is:

$$\log L(p_1, p_2) = n_0 \log(1 - p_1 - p_2) + n_1 \log(p_1) + n_2 \log(p_2)$$

b) MLE Estimates for p_1 and p_2

To find the Maximum Likelihood Estimates (MLE) for p_1 and p_2, we take the partial derivatives of the log-likelihood function with respect to p_1 and p_2.

1. Differentiate with respect to p_1:

$$\frac{\partial}{\partial p_1} \log L(p_1, p_2) = \frac{-n_0}{1 - p_1 - p_2} + \frac{n_1}{p_1}$$

Set this equal to zero:

$$\frac{-n_0}{1 - p_1 - p_2} + \frac{n_1}{p_1} = 0$$

2. Differentiate with respect to p_2:

$$\frac{\partial}{\partial p_2} \log L(p_1, p_2) = \frac{-n_0}{1 - p_1 - p_2} + \frac{n_2}{p_2}$$

Set this equal to zero:

$$\frac{-n_0}{1 - p_1 - p_2} + \frac{n_2}{p_2} = 0$$

Solving the system of equations, we get:

$$p_1 = \frac{4}{15}, \quad p_2 = \frac{1}{10}$$

Thus, the MLE estimates for p_1 and p_2 are:

$$p_1 = 0.2667, \quad p_2 = 0.1$$

c) 95% Confidence Interval for p_1

The formula for the 95% confidence interval for p_1 is:

$$CI(95\%) = \hat{p_1} \pm Z_{\alpha/2} \cdot \frac{1}{\sqrt{n \cdot I(\hat{p_1})}}$$

Where:

$$I(p_1) = \frac{1 - p_2}{p_1(1 - p_1 - p_2)}$$

Substitute $p_1 = 0.2667$, $p_2 = 0.1$, and $n = 30$:

$$I(p_1) = \frac{1 - 0.1}{0.2667(1 - 0.2667 - 0.1)} = \frac{0.9}{0.2667 \times 0.6333} = 5.329$$

Now, calculate the standard error:

$$\text{Standard error} = \frac{1}{\sqrt{n \cdot I(p_1)}} = \frac{1}{\sqrt{30 \times 5.329}} = \frac{1}{\sqrt{159.87}} = 0.0790$$

$$CI(95\%) = 0.2667 \pm 1.96 \times 0.0790$$

$$CI(95\%) = 0.2667 \pm 0.1548$$

Thus, the confidence interval is:

$$CI(95\%) = (0.1117, 0.4217)$$

Q3

a) Method of Moments

The given probability density function is:

$$f(x|\theta) = e^{-(x-\theta)} \quad \text{for} \quad x \geq \theta$$

To find the first moment, we compute the expected value $E[X]$:

$$E[X] = \int_{\theta}^{\infty} x \cdot e^{-(x-\theta)} \, dx$$

Simplifying:

$$E[X] = \theta + 1$$

Equating to the sample mean $u_1 = \frac{1}{n} \sum_{i=1}^{n} X_i$, we have:

$$\hat{\theta}_{\text{MoM}} = u_1 - 1$$

b) Maximum Likelihood Estimate (MLE)

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} e^{-(X_i-\theta)} = e^{-\sum_{i=1}^{n}(X_i-\theta)}$$

Simplifying the log-likelihood:

$$\log L(\theta) = -\sum_{i=1}^{n}(X_i - \theta)$$

The likelihood is positive only if $\theta \leq X_i$ for all i, meaning:

$$\theta \leq \min(X_1, X_2, \dots, X_n)$$

Thus, the Maximum Likelihood Estimate for θ is:

$$\hat{\theta}_{\text{MLE}} = \min(X_1, X_2, \dots, X_n)$$

Q4

a) Calculating Unbiased Estimator

Given $X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda)$, we want to find an unbiased estimator of $\theta = e^{-\lambda}$.

The probability mass function of the Poisson distribution is:

$$P(X_i = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

For $k = 0$:

$$P(X_i = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = \theta$$

Define the indicator function:

$$I\{X_i = 0\} = \begin{cases} 1, & \text{if } X_i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

An estimator of θ is:

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} I\{X_i = 0\}$$

Verification of Unbiasedness:

$$\begin{aligned} E[\hat{\theta}] &= E\left[\frac{1}{n}\sum_{i=1}^{n} I\{X_i = 0\}\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E[I\{X_i = 0\}] \\ &= \frac{1}{n}\sum_{i=1}^{n} P(X_i = 0) \\ &= \frac{1}{n} \times n\theta \\ &= \theta \end{aligned}$$

Thus, $\hat{\theta}$ is an unbiased estimator of θ.

b) Computing Cramer Rao Lower Bound to understand the efficiency of computed Unbiased estimator

Since $I\{X_i = 0\}$ are independent Bernoulli random variables with success probability θ, their variance is:

$$\text{Var}(I\{X_i = 0\}) = \theta(1 - \theta)$$

The variance of $\hat{\theta}$ is:

$$\begin{aligned}\text{Var}(\hat{\theta}) &= \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} I\{X_i = 0\}\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(I\{X_i = 0\}) \\ &= \frac{n\theta(1-\theta)}{n^2} \\ &= \frac{\theta(1-\theta)}{n}\end{aligned}$$

Compute the CRLB.
The Fisher information for one observation is:

$$I(\lambda) = \frac{1}{\lambda}$$

For n observations:

$$I_n(\lambda) = n \times I(\lambda) = \frac{n}{\lambda}$$

Compute the derivative of θ with respect to λ:

$$\frac{\partial\theta}{\partial\lambda} = \frac{\partial}{\partial\lambda}e^{-\lambda} = -e^{-\lambda} = -\theta$$

The CRLB is:

$$\begin{aligned}\text{Var}(\hat{\theta}) &\geq \frac{\left(\frac{\partial\theta}{\partial\lambda}\right)^2}{I_n(\lambda)} \\ &= \frac{(-\theta)^2}{\frac{n}{\lambda}} \\ &= \frac{\theta^2\lambda}{n}\end{aligned}$$

Since $\lambda = -\ln\theta$ (because $\theta = e^{-\lambda}$):

$$\text{Var}(\hat{\theta}) \geq \frac{\theta^2(-\ln\theta)}{n}$$

Compare the variance of $\hat{\theta}$ with the CRLB:

$$\begin{aligned}\frac{\text{Var}(\hat{\theta})}{\text{CRLB}} &= \frac{\frac{\theta(1-\theta)}{n}}{\frac{\theta^2(-\ln\theta)}{n}} \\ &= \frac{\theta(1-\theta)}{\theta^2(-\ln\theta)} \\ &= \frac{1-\theta}{\theta(-\ln\theta)}\end{aligned}$$

Since $0 < \theta < 1$ and $-\ln\theta > 0$, the ratio is greater than or equal to 1. Thus, the variance of $\hat{\theta}$ exceeds the CRLB indicating the unbiased estimator is not efficient.

Q5

a) Estimating the Mean μ and a 95% Confidence Interval

The mean is calculated as:

$$\mu = \frac{1 \cdot 2600 + 2 \cdot 5200 + 3 \cdot 2200}{10000} = 1.96$$

Variance σ^2 is:

$$\sigma^2 = (1 - 1.96)^2 \cdot \frac{2600}{10000} + (2 - 1.96)^2 \cdot \frac{5200}{10000} + (3 - 1.96)^2 \cdot \frac{2200}{10000} = 0.4784$$

$$\sigma = \sqrt{0.4784} \approx 0.6917$$

$$SE = \frac{0.6917}{\sqrt{10000}} = 0.006917$$

$$ME = 1.96 \times 0.006917 = 0.01355$$

$$\text{CI} = 1.96 \pm 0.01355 = [1.9465, 1.9735]$$

b) Estimating p_1 and Checking if the Estimator is Unbiased

The estimate of p_1 is:

$$\hat{p_1} = \frac{2600}{10000} = 0.26$$

The estimator $\hat{p_1} = \frac{1}{n} \sum_{i=1}^{n} 1\{X_i = 1\}$ is an unbiased estimator for p_1. This is because $E[\hat{p_1}] = E\left(\frac{1}{n} \sum_{i=1}^{n} 1\{X_i = 1\}\right)$, which, by the linearity of expectation, simplifies to $\frac{1}{n} \sum_{i=1}^{n} E[1\{X_i = 1\}] = p_1$. Hence, the expected value of $\hat{p_1}$ is equal to the true parameter p_1, proving that the estimator is unbiased.

c) Method of Moments Estimation of p_1 and p_2

1. First moment (mean):

$$E[X] = p_1 + 2p_2 + 3(1 - p_1 - p_2) = 3 - 2p_1 - p_2$$

Equate with the sample mean:

$$1.96 = 3 - 2p_1 - p_2$$

$$2p_1 + p_2 = 1.04$$

2. Second moment (variance):

$$E[X^2] = p_1 + 4p_2 + 9(1 - p_1 - p_2) = 9 - 8p_1 - 5p_2$$

Equate with sample variance:

$$0.4784 = 9 - 8p_1 - 5p_2 - 3.8416 = 4.68 = 8p_1 + 5p_2$$

3. Solving the system of equations:

$$2p_1 + p_2 = 1.04, \quad 8p_1 + 5p_2 = 4.68$$

Multiply the first equation by 4:

$$8p_1 + 4p_2 = 4.16$$

Subtract from the second equation:

$$p_2 = 0.52$$

Substitute into the first equation:

$$2p_1 + 0.52 = 1.04 \quad \Rightarrow \quad p_1 = 0.26$$

d) Maximum Likelihood Estimation for p_2 and a 95% Confidence Interval

1. Likelihood function:

$$L(p_2) = \left(\frac{1}{4}\right)^{n_1} \cdot p_2^{n_2} \cdot \left(\frac{3}{4} - p_2\right)^{n_3}$$

Log-likelihood:

$$\log L(p_2) = n_1 \log\left(\frac{1}{4}\right) + n_2 \log(p_2) + n_3 \log\left(\frac{3}{4} - p_2\right)$$

Differentiate:

$$\frac{d}{dp_2} \log L(p_2) = \frac{n_2}{p_2} - \frac{n_3}{\frac{3}{4} - p_2}$$

Set to 0 and solve:

$$\frac{5200}{p_2} = \frac{2200}{\frac{3}{4} - p_2}$$

Solve for p_2:

$$p_2 = 0.527$$

2. Standard error: The Fisher Information $I(p_2)$ is given by:

$$I(p_2) = E\left[\left(\frac{d}{dp_2} \log L(p_2)\right)^2\right]$$

After simplification, the Fisher Information for p_2 becomes:

$$I(p_2) = \frac{3}{p_2(3 - 4p_2)}$$

Substitute $p_2 = 0.527$ into the Fisher Information expression:

$$I(0.527) = \frac{3}{0.527(3 - 4 \cdot 0.527)} \approx 6.381$$

Variance of $\hat{p}_2$:
The variance of $\hat{p}_2$ is given by:

$$\mathrm{Var}(\hat{p}_2) = \frac{1}{n \cdot I(p_2)}$$

For the sample size $n = 10,000$, we compute:

$$\mathrm{Var}(\hat{p}_2) = \frac{1}{10,000 \times 6.381} \approx 0.00001567$$

Standard Error (SE) of $\hat{p}_2$:
The standard error is the square root of the variance:

$$SE(\hat{p}_2) = \sqrt{\mathrm{Var}(\hat{p}_2)} = \sqrt{0.00001567} \approx 0.00396$$

95% Confidence Interval for p_2:
The 95% confidence interval is calculated as:

$$CI(95\%) = \hat{p}_2 \pm Z \cdot SE$$

Using $Z = 1.96$ for a 95% confidence level:

$$CI(95\%) = 0.527 \pm 1.96 \times 0.00396$$

This gives:

$$CI(95\%) = 0.527 \pm 0.00776$$

Thus, the 95% confidence interval for p_2 is:

$$CI(95\%) = (0.51924, 0.53476)$$

Q6.

a) Calculating Mean and Variance Manually and Comparing it with Numpy Functions

```
import pandas as pd
data = pd.read_csv('./data_HW2.csv', header=None)
data.columns = ['observation']
```

```
sample_mean = sum(data['observation'])/len(data)
sample_mean
```

$$\text{Sample Mean} = 4.234069250505728$$

```
sum_sq = 0
for i in data['observation']:
    sum_sq += (i - sample_mean) ** 2

sample_variance = sum_sq/(len(data) - 1)
sample_variance
```

The calculated sample variance is:

$$\text{Sample Variance} = 18.258337155141206$$

```
import numpy as np

numpy_sample_mean = np.mean(data['observation'])
numpy_sample_mean
```

The result from NumPy is:

$$\text{NumPy Sample Mean} = 4.234069250505729$$

```
numpy_sample_variance = np.var(data['observation'])
numpy_sample_variance
```

The population variance (with `ddof=0`) is:

$$\text{NumPy Variance (Population)} = 18.185303806520647$$

To calculate the sample variance, we must set `ddof=1`, which ensures that we divide by `n-1` instead of `n`, as is standard for sample variance:

```
numpy_variance = np.var(data['observation'], ddof=1)
numpy_variance
```

This yields the sample variance, which matches our manually computed result:

$$\text{NumPy Variance (Sample)} = 18.25833715514121$$

In NumPy, the `var()` function calculates the variance. By default, it uses `ddof=0`, meaning it computes the **population variance**, dividing by `n`. To calculate the **sample variance**, we set `ddof=1`, which ensures that we divide by `n-1`. This adjustment is necessary to get an unbiased estimator of the variance in the case of a sample rather than the entire population.

Thus:

- **Population Variance** (default `ddof=0`): Divides by `n`, the total number of observations.
- **Sample Variance** (`ddof=1`): Divides by `n-1`, correcting for the bias when estimating population variance from a sample.

b) Calculating a 90% Confidence Interval for the Population Mean

```
from scipy import stats

pop_mean = np.mean(data['observation'])

std_dev = 4
confidence_interval = .90
z_score = stats.norm.ppf(1 - (1 - confidence_interval) / 2)

error = z_score * std_dev / np.sqrt(len(data))

lower_interval = pop_mean -  error
higher_interval = pop_mean + error

lower_interval, higher_interval
```

The calculated 90% confidence interval is:

$$(3.817950540205283, 4.650187960806175)$$

Q7

a) Simulate Means for different data points and understand the Variance relationship)

```
choice = [10, 100, 1000]

means_10 = []
means_100 = []
means_1000 = []

for i in choice:
    means = []
    for _ in range(2000):
        sample = np.random.randn(i)
        sample_mean = np.mean(sample)
        means.append(sample_mean)
```

```
        if i == 10:
            means_10 = means
        elif i == 100:
            means_100 = means
        elif i == 1000:
            means_1000 = means


import matplotlib.pyplot as plt
plt.hist(means_10, bins=30, color='blue', edgecolor='black', alpha=0.7)
plt.title('Histogram of Sample Means (n=10)')
plt.xlabel('Sample Mean')
plt.ylabel('Frequency')
plt.show()

plt.hist(means_100, bins=30, color='blue', edgecolor='black', alpha=0.7)
plt.title('Histogram of Sample Means (n=100)')
plt.xlabel('Sample Mean')
plt.ylabel('Frequency')
plt.xlim(-1, 1)
plt.show()

plt.hist(means_1000, bins=30, color='blue', edgecolor='black', alpha=0.7)
plt.title('Histogram of Sample Means (n=1000)')
plt.xlabel('Sample Mean')
plt.ylabel('Frequency')
plt.xlim(-1, 1)
plt.show()
```

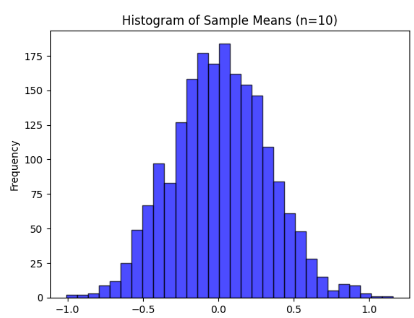

Figure 1: Histogram of Sample Means for $n = 10$

As n increases, the variance of the sample mean X_n decreases. This is evident from the histograms:

- For $n = 10$, the histogram is wider and more spread out, indicating a

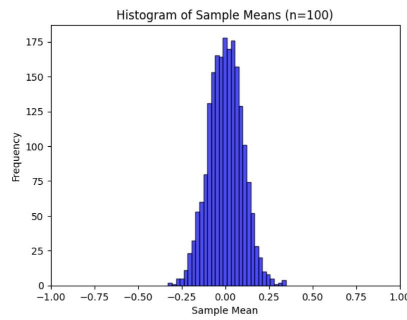

Figure 2: Histogram of Sample Means for $n = 100$

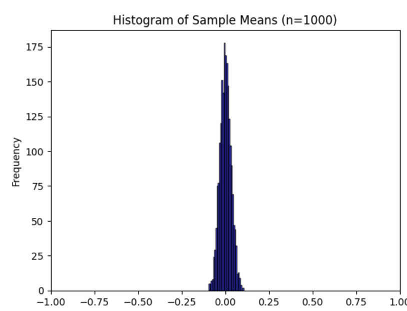

Figure 3: Histogram of Sample Means for $n = 1000$

larger variance. This shows that the values of X_n have more variability for smaller n.

- For $n = 100$, the histogram becomes narrower, reflecting a smaller variance. As the number of observations increases, the distribution of the sample means becomes more concentrated around the true mean.
- For $n = 1000$, the histogram is even more tightly concentrated around the mean, showing that the variance is significantly smaller.

This behavior aligns with the theoretical understanding of variance in random variables. The variance of X_n increases with n, and the sample mean variance decreases. This is confirmed by the progressively narrower histograms as n increases.

b) Simulating the Random Walk and Plotting Mean and Variance

```
X = []

for i in range(1, 51):
    positions = []
    for _ in range(1000):
        x_o = 0
        for _ in range(i):
            ch = np.random.choice(['H', 'T'])
            if ch == 'H':
                x_o += 1
            else:
                x_o -= 1
        positions.append(x_o)
    X.append(positions)

data = []
for i in range(len(X)):
    mean = np.mean(X[i])
    variance = np.var(X[i])
    data.append({'n': i+1, 'mean': mean, 'variance': variance})

df = pd.DataFrame(data)

import matplotlib.pyplot as plt

plt.plot(df['n'], df['mean'])
plt.title('Plot of Mean of Xn vs n')
plt.xlabel('n')
plt.ylabel('Mean of Xn')
```

```
plt.savefig('mean_plot.png')  # Save the figure

plt.plot(df['n'], df['variance'])
plt.title('Plot of Variance of Xn vs n')
plt.xlabel('n')
plt.ylabel('Variance of Xn')
plt.savefig('variance_plot.png')  # Save the figure
```

The two figures below show the plots for the mean and variance of X_n as a function of n.

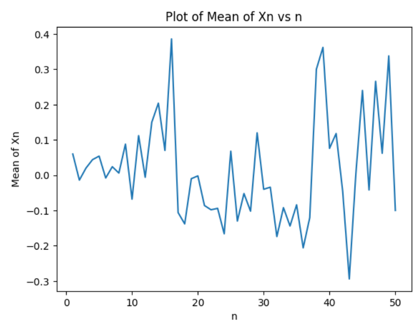

Figure 4: Plot of Mean of X_n vs n

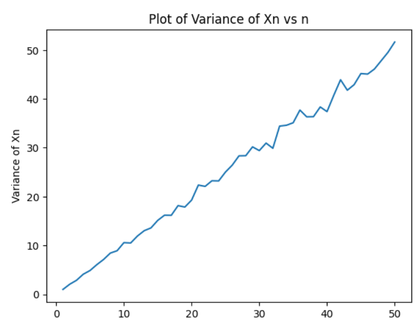

Figure 5: Plot of Variance of X_n vs n

c) Compute $E(X_n)$ and Var(X_n) theoretically and compare them with your plots

Theoretical Computation of $E(X_n)$

$$E(S_i) = 1 \times P(S_i = 1) + (-1) \times P(S_i = -1) = 1 \times 0.5 + (-1) \times 0.5 = 0$$

Now, the position after n steps is given by:

$$X_n = \sum_{i=1}^{n} S_i$$

Using the linearity of expectation, we can compute the expected value of X_n as:

$$E(X_n) = E\left(\sum_{i=1}^{n} S_i\right) = \sum_{i=1}^{n} E(S_i) = 0$$

Thus, the theoretical expected value of X_n is:

$$E(X_n) = 0$$

This theoretical result aligns with our plot of the empirical mean of X_n, which fluctuates around 0 as expected.

Theoretical Computation of Var(X_n)

$$\mathrm{Var}(X_n) = \mathrm{Var}\left(\sum_{i=1}^{n} S_i\right) = \sum_{i=1}^{n} \mathrm{Var}(S_i)$$

Each step S_i is either +1 or -1 with equal probability, i.e., $P(S_i = 1) = 0.5$ and $P(S_i = -1) = 0.5$.

Now, we calculate $E(S_i^2)$:

$$E(S_i^2) = 1^2 \times P(S_i = 1) + (-1)^2 \times P(S_i = -1) = 1 \times 0.5 + 1 \times 0.5 = 1$$

Since $E(S_i) = 0$, the variance of S_i is:

$$\mathrm{Var}(S_i) = E(S_i^2) - E(S_i)^2 = 1 - 0^2 = 1$$

Thus, the variance of X_n becomes:

$$\mathrm{Var}(X_n) = n \times 1 = n$$

Therefore, the theoretical variance grows linearly with n, i.e.,

$$\mathrm{Var}(X_n) = n$$

Comparison with Empirical Results

Comparing the theoretical results with the empirical plots:

- For $E(X_n)$, the theoretical expected value is 0, which matches the empirical plot where the mean of X_n fluctuates around 0.

- For $\text{Var}(X_n)$, the theoretical variance increases linearly with n. This is consistent with the empirical plot, where the variance of X_n grows approximately as a straight line starting from the origin, confirming the relationship $\text{Var}(X_n) = n$. There may be slight fluctuations due to random noise in the simulations, but the overall trend aligns with the plot.