Statistics 5350/7110
Forecasting

Lecture 19
Seasonal ARIMA Models

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Admin Issues

- Questions

- Assignments

- Quick review
  - Building ARIMA models in practice
  - Filling in some details: Lecture_18.Rmd
    - Similarity of the models for GDP
    - Model drift
    - Dealing with an outlier
  - IMA models as variations on exponential smoothing (next slide)

Review: IMA Model as Average

- Exponential smoothing
  - Equivalent to IMA(1,1) if treated as a forecasting procedure (i.e., with a time shift)
  - Forecasts are weighted average of past
    $$\hat{X}_{n+1|n} = \sum_{j=0} w_j X_{n-j} \quad \text{where} \quad \sum_{j=0} w_j = 1,\ w_j = (1-\lambda)\,\lambda^j$$
  - All ARIMA(0,1,q) models have this general form:
    One-step ahead prediction is weighted average of prior values
- Derivation
  - ARIMA(0,1,q) has the form
    $$(1-B)X_t = w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q} \quad \text{or} \quad (1-B)\,X_t = \theta(B)\,w_t$$
  - As an infinite AR model
    $$\underbrace{(1-B)\theta(B)^{-1}}_{\pi(B)}\ X_t = w_t \quad \text{where} \quad \pi(B) = 1 + \pi_1 B + \pi_2 B^2 + \cdots \text{ so that } \hat{X}_{n+1|n} = -\sum_{j=1} \pi_j X_{n+1-j}$$
  - Invertibility implies $\pi(1) = 0$ so that $\pi_1 + \pi_2 + \cdots = -1$.

Today's Topics

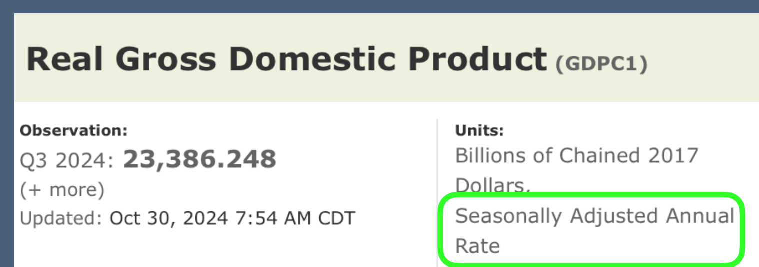
Text, §5.3

- Seasonal ARIMA models (SARIMA)
  - Parsimonious representation for efficiency
  - Same as ARIMA model with certain coefficients constrained to be zero
  - Not a new model, just a convenient way to express constraints

Explains the name of the "sarima" functions in R

- Pure and mixed types
  - Pure: Non-zero correlations at multiples of specific period
  - Mixed: Combine two types of dependence

- Relevance
  - Just about every macroeconomic time series you'll find has been seasonally adjusted.
  - SARIMA models allow you to incorporate that adjustment into YOUR model.



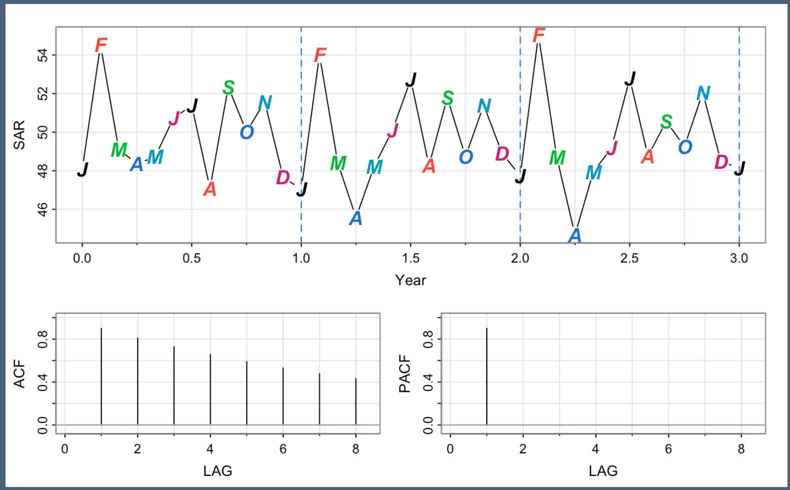
Seasonal ARIMA Models

Purely Seasonal ARIMA Model

- Nonzero autocorrelations at multiples of seasonal period
  - Quarterly data: autocorrelations at 4 months
  - Monthly data: autocorrelations at 12 months
- Example
  - Simulated for clarity (these are rare 😁)
  - Regular pattern from year to year:
    February routinely higher
    April routinely lower
  - Nonzero autocorrelations at seasonal spacing
- Model
  - Notation: monthly process (S=12) SARIMA$(1,0,0)_{12}$

    $X_t = \Phi X_{t-12} + w_t$ or $(1 - \Phi B^{12})X_t = w_t$
  - Equivalent to AR(12) model with all of the intervening coefficients $\phi_1, \phi_2, \ldots \phi_{11} = 0$.
  - Stationarity requires $|\Phi| < 1$ as in usual AR models

Example 5.11
Figure 5.9



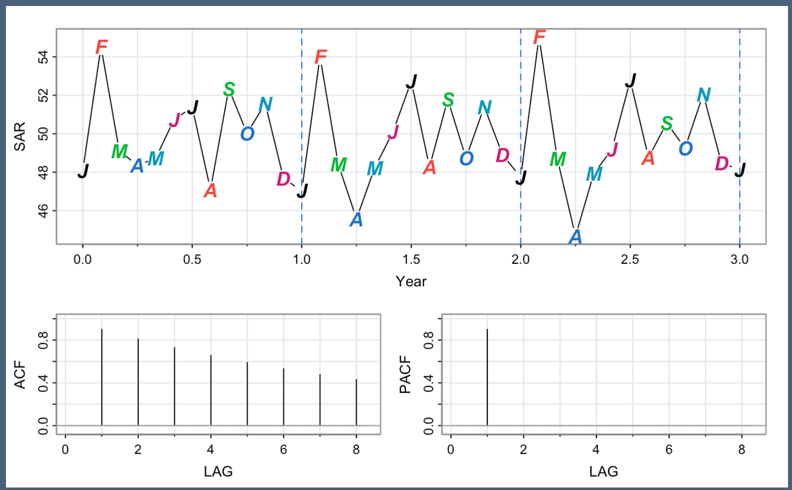
Purely Seasonal ARIMA Model

- Seasonal AR(1) model
  - Specification is $X_t = \Phi X_{t-12} + w_t$
- Patterns in ACF and PACF
  - Why are the intervening correlations zero?
  - Infinite MA representation

$$
\begin{aligned}
X_t &= \Phi X_{t-12} + w_t \\
&= \Phi(\Phi X_{t-24} + w_{t-12}) + w_t \\
&= w_t + \Phi\, w_{t-12} + \Phi^2(\Phi\, X_{t-36} + w_{t-24}) \\
&= w_t + \Phi\, w_{t-12} + \Phi^2 w_{t-24} + \Phi^3 X_{t-36} \\
&= \textstyle\sum_{j=0}^{\infty} \Phi^{12\,j} w_{t-12j}
\end{aligned}
$$

  - Nonzero ACF at spacing given by seasonal period
  - PACF: zero once we know value 12 months previous
  - Explains stationarity condition

Example 5.11
Figure 5.9



Mixed Seasonal ARIMA Model

Example 5.12

- Autocorrelations at many lags
  - Never so "pure" as in prior examples, with nothing but a few spikes
  - We observe estimates rather than true autocorrelations
- Example
  - Simulated for clarity (so we know the process ACF)
  - MA(1) combined with SAR(1)
  - Nonzero autocorrelations cluster near seasonal period
- Notation
  - Monthly process (S=12)

$$X_t = \Phi X_{t-12} + w_t + \theta w_{t-1}$$

  - Backshift polynomial form

$$(1 - \Phi B^{12}) X_t = \theta(B)\, w_t$$

  - Equivalent to an ARMA(12, 1) model with all of the intervening coefficients $\phi_1, \phi_2, \ldots \phi_{11} = 0$.

Notation uses "capitalized" letters for the seasonal parameters

Mixed Seasonal ARIMA Model

Example 5.12

- Process
  - Monthly process (S=12) $X_t = \Phi X_{t-12} + w_t + \theta w_{t-1}$
- Autocorrelations
  - Variance is easy to find since the variables on the right side are uncorrelated

$$\text{Var}(X_t) = \gamma(0) = \Phi^2 \gamma(0) + (1+\theta^2)\sigma_w^2 \quad \Rightarrow \quad \gamma(0) = \frac{1+\theta^2}{1-\Phi^2}\,\sigma_w^2$$

  - Compute autocovariances as in Yule-Walker equations

$$\gamma(1) = \Phi\gamma(11) + \theta\,\sigma_w^2 \qquad \gamma(12) = \Phi\gamma(0) \qquad \gamma(11) = \gamma(13) = \Phi\gamma(1)$$

Hence

$$\gamma(1) = \Phi^2\gamma(1) + \theta\sigma_w^2 \quad \Rightarrow \quad \gamma(1) = \frac{\theta}{1-\Phi^2}\sigma_w^2 \quad \text{and} \quad \gamma(11) = \gamma(13) = \Phi\frac{\theta}{1-\Phi^2}\sigma_w^2$$

  - Autocorrelations

$$\rho(1) = \gamma(1)/\gamma(0) = \frac{\theta}{1+\theta^2} \quad \text{and} \quad \rho(11) = \rho(13) = \gamma(11)/\gamma(0) = \frac{\theta}{1+\theta^2}\Phi$$

In general

$$\rho(12h) = \Phi^h,\ \rho(12h \pm 1) = \frac{\theta}{1+\theta^2}\Phi^h, h = 1,2,\ldots$$

Mixed Seasonal ARIMA Model

- Understanding patterns in ACF
  - Model is

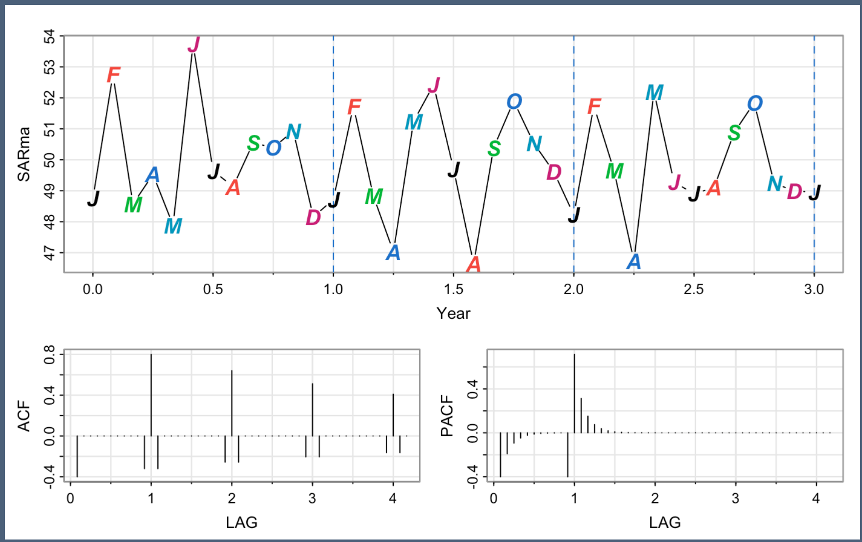
$$X_t = \Phi X_{t-12} + w_t + \theta w_{t-1}$$

- Patterns in ACF
  - Infinite MA representation sheds more light

$$\begin{aligned} X_t &= \Phi X_{t-12} + w_t + \theta w_{t-1} \\ &= w_t + \theta w_{t-1} + \Phi(\Phi X_{t-24} + w_{t-12} + \theta w_{t-13}) \\ &= w_t + \theta w_{t-1} + \Phi w_{t-12} + \Phi\, \theta w_{t-13} + \Phi^2\, X_{t-24} \end{aligned}$$

  - Nonzero weights at multiples of seasonal period and near the seasonal period
  - Nonzero autocorrelations symmetric around seasonal
- PACF
  - Conditional autocorrelations less obvious
  - Intuition from non-seasonal models

Example 5.12
Figure 5.10



Seasonal ARIMA, SARIMA

- Notation
  - Capital letters for the seasonal terms, with seasonal period S
  - Write model using backshift notation
  - SARIMA(p,0,q)(P,0,Q)$_S$

    $$(1 - \Phi_1 B^S - \cdots - \Phi_P B^{PS})(1 - \phi_1 B - \cdots - \phi_p B^p)X_t = (1 + \Theta_1 B^S + \cdots + \Theta_Q B^{QS})(1 + \theta_1 B + \cdots + \theta_q B^p)w_t$$

    or in more compact form

    $$\Phi(B^S)\,\phi(B)\,X_t = \Theta(B^S)\,\theta(B)\,w_t$$

- Constrained multiplicative structure
  - SARIMA(1,0,1)(1,0,1)$_4$ (seasonal period S=4)

    $$(1 - \Phi_1 B^4)(1 - \phi_1 B)X_t = (1 + \Theta_1 B^4)(1 + \theta_1 B)w_t$$

  - Equivalent to ARIMA(5,5) with constrained estimates (estimate 4 parameters, not 10)

    $$(1 - \phi_1 B - \Phi_1 B^4 + \phi_1 \Phi_1 B^5)\,X_t = (1 + \theta_1 B + \Theta_1 B^4 + \theta_1 \Theta_1 B^5)\,w_t$$

- Differencing

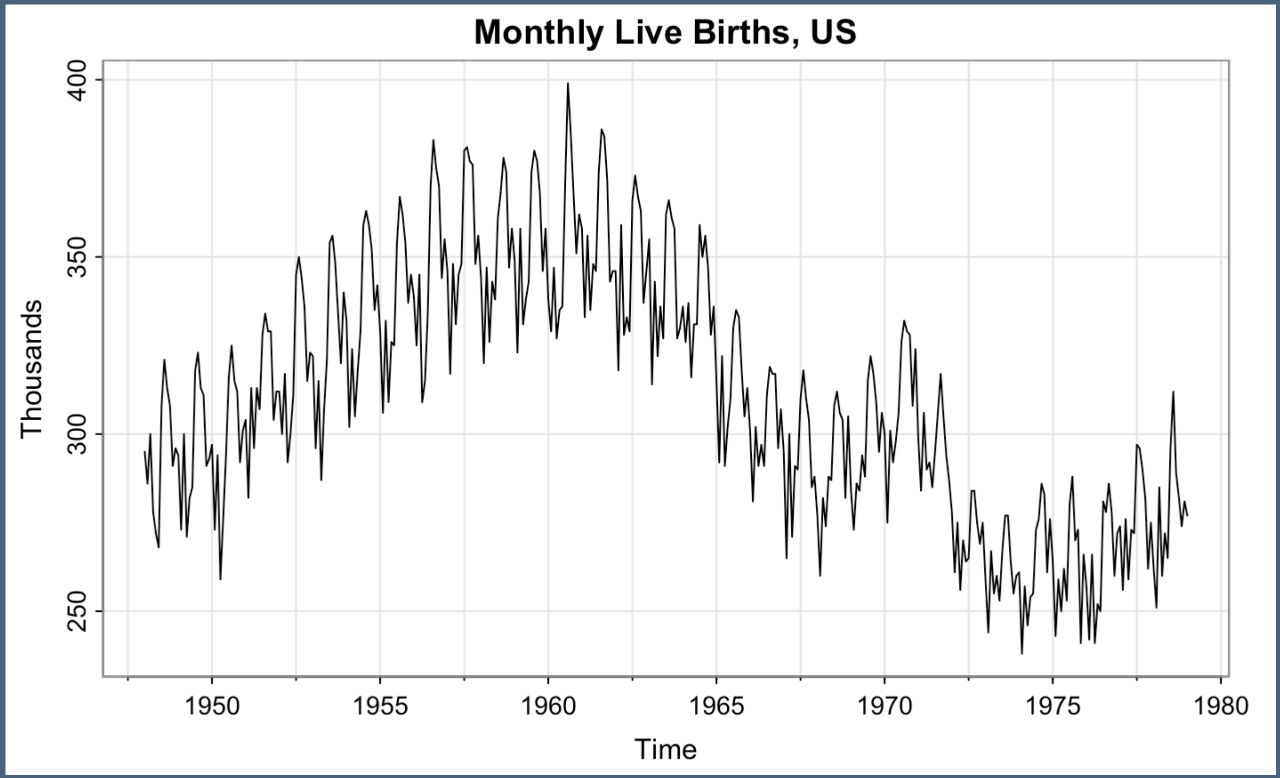
  Definition 5.13
  - SARIMA(p,d,q)(P,D,Q)$_S$ allows differencing $\Phi(B^S)\phi(B)\,(1 - B^S)^D(1 - B)^d\,X_t = \Theta(B^S)\theta(B)\,w_t$

Example: Births

Example 5.12

Example: Mixed Seasonal Model

- Birth series
  - Live births, monthly in the US from 1948 - 1979

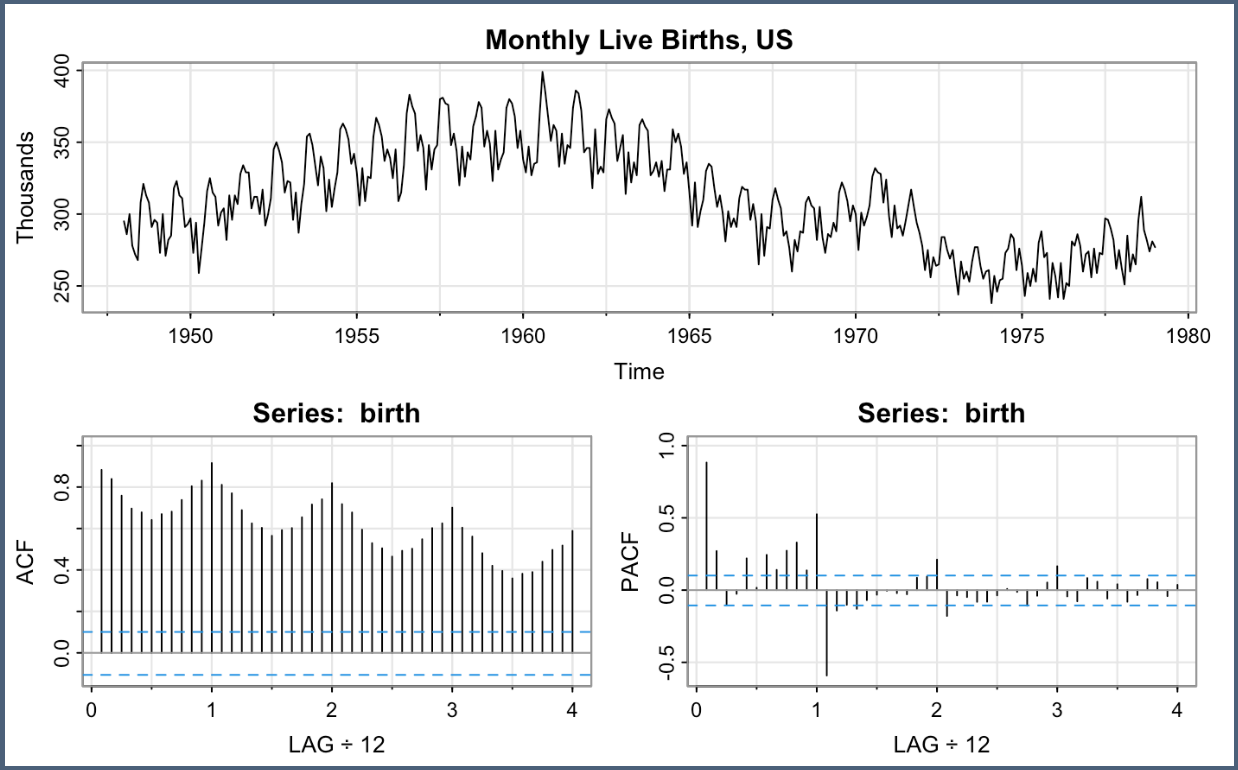


Post-war baby boom

How would you have modeled this process before this class?

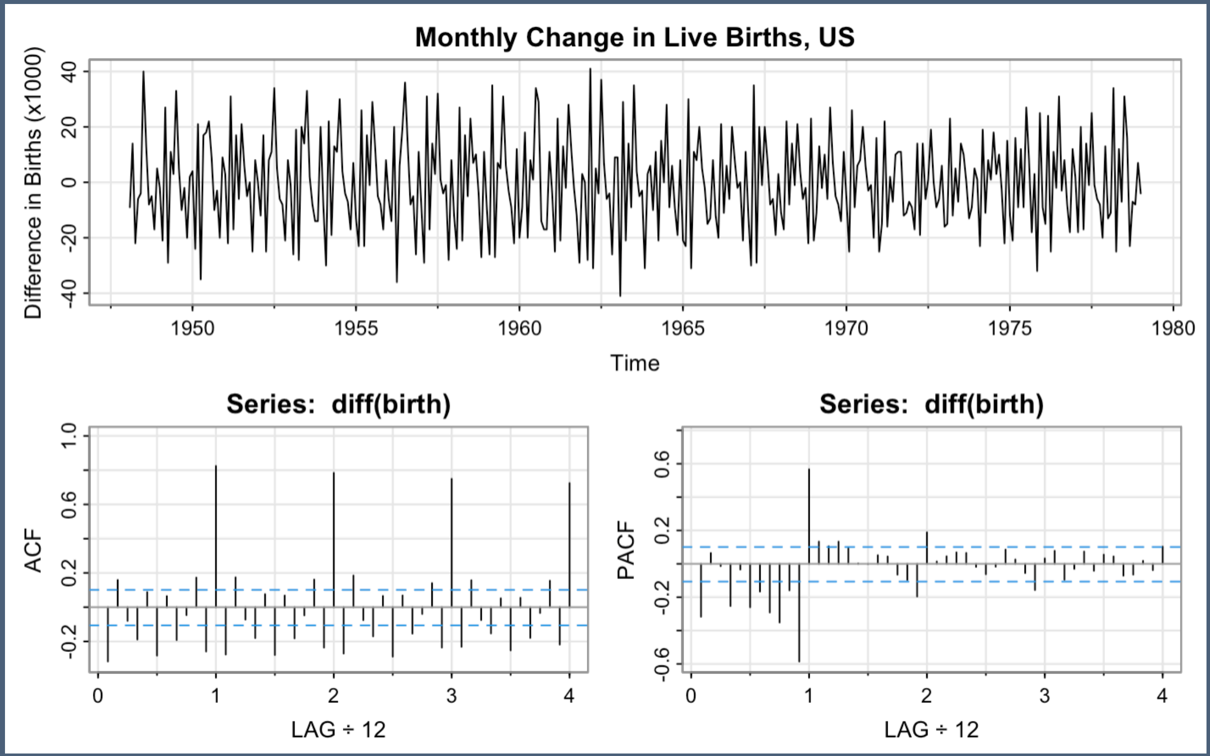
Example: Mixed Seasonal

- Birth series
  - Persistent autocorrelations suggest better to model differences (non-stationary)



Example: Mixed Seasonal

- Birth series
  - Month-to-month differences in the live births, monthly in the US from 1949 - 1979

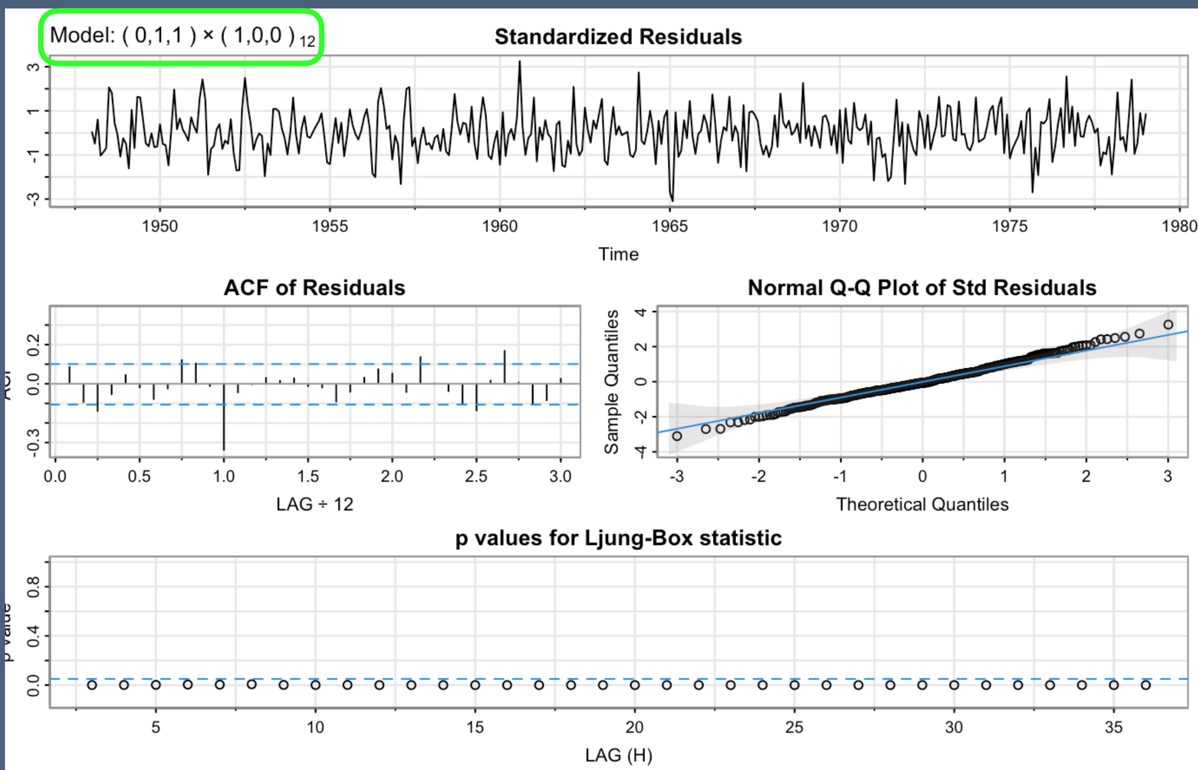


Example: Mixed Seasonal

- Parameters are significant, but substantial autocorrelation remains
  - Autocorrelation in residuals at lag 12

Interpreting the process label in the figure

$$(1-\Phi B^{12})\,\nabla X_t = (1-\theta B)w_t$$



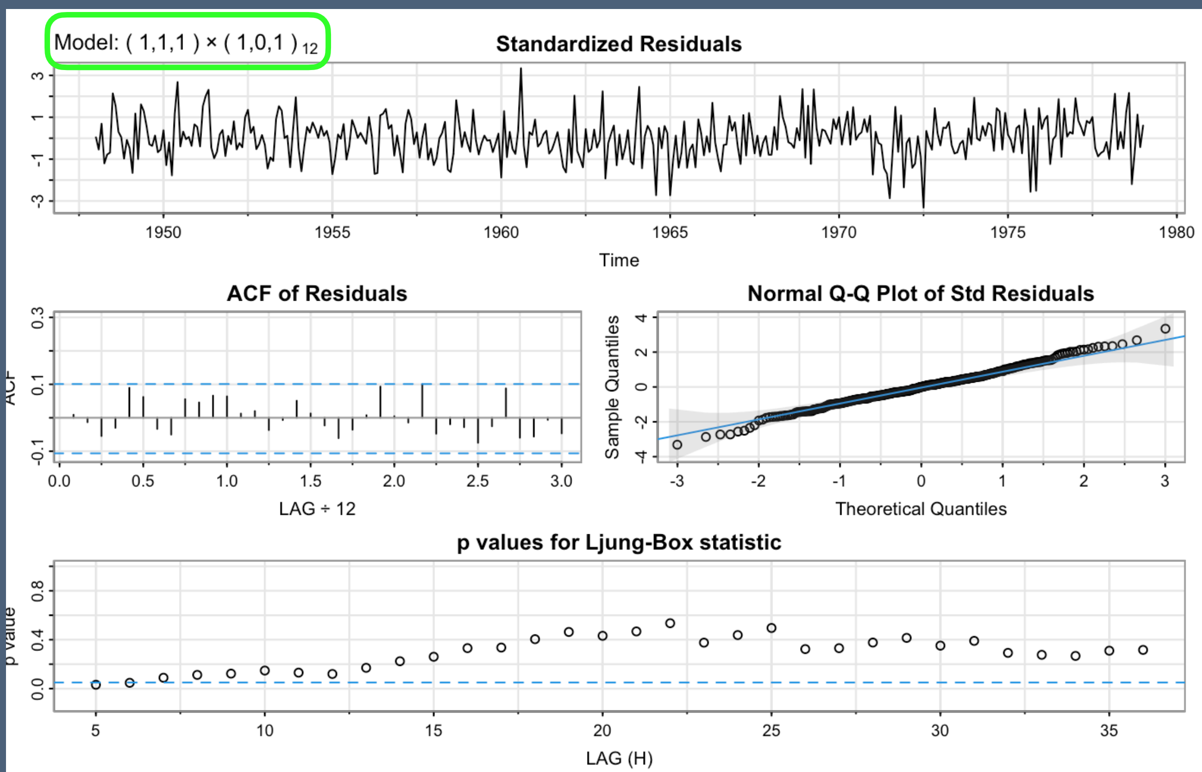
Coefficients:

	Estimate	SE	t.value	p.value
ma1	-0.5036	0.0578	-8.7122	0.0000
sar1	0.8697	0.0239	36.3372	0.0000
xmean	0.0665	1.3542	0.0491	0.9609

Example: Mixed Seasonal

- Expand model
  - ARMA(1,1) plus seasonal ARMA(1,1)
  - Seasonal AR coefficient near boundary of stationarity

$$(1-\phi B)(1-\Phi B^{12})\nabla X_t = (1-\theta B)(1-\Theta B^{12})w_t$$



Coefficients:

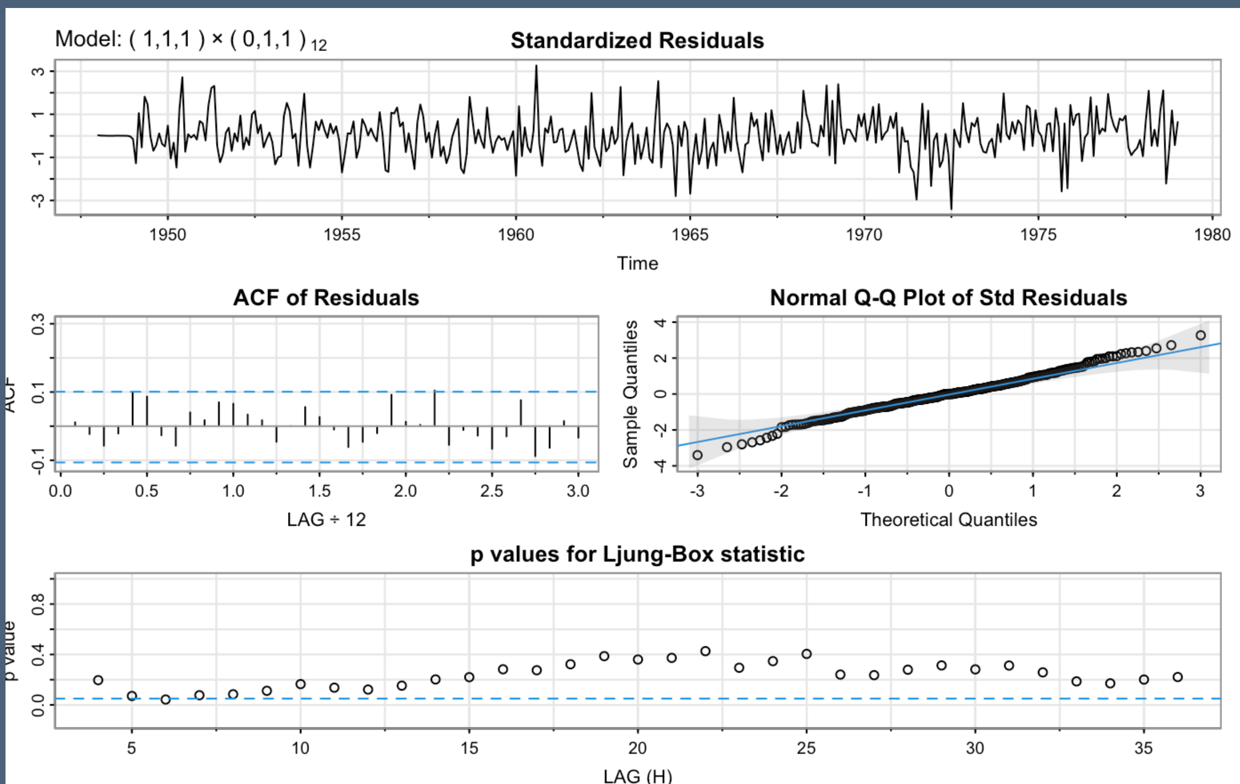
	Estimate	SE	t.value	p.value
ar1	0.3144	0.0857	3.6686	0.0003
ma1	-0.7095	0.0597	-11.8795	0.0000
sar1	0.9953	0.0028	351.8633	0.0000
sma1	-0.7926	0.0461	-17.1982	0.0000
constant	0.1574	1.7213	0.0915	0.9272

Example: Mixed Seasonal

- Seasonal difference
  - Difference at seasonal spacing rather than fit the AR term at seasonal spacing

$$(1-\phi B)(1-B^{12})\nabla X_t = (1-\theta B)(1-\Theta B^{12})w_t$$

$$(1-\phi B)\nabla_{12}\nabla X_t = (1-\theta B)(1-\Theta B^{12})w_t$$

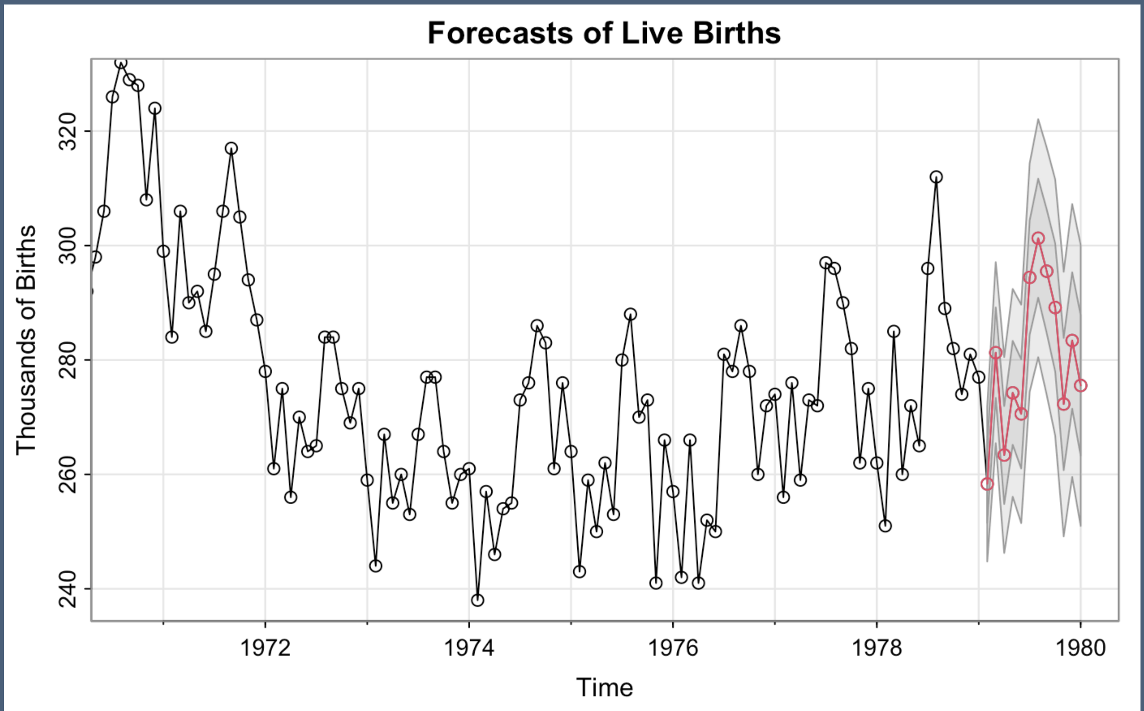



```
Coefficients:
     Estimate     SE  t.value p.value
ar1    0.3038 0.0865   3.5104   5e-04
ma1   -0.7006 0.0604 -11.5984   0e+00
sma1  -0.8000 0.0441 -18.1302   0e+00
```

`sarima` does not provide constant/mean when both seasonal and regular differencing

Example: Forecasts of Live Births

- Non-trivial extrapolation
  - Contrast to simple forecast evolution with stationary and ARIMA models



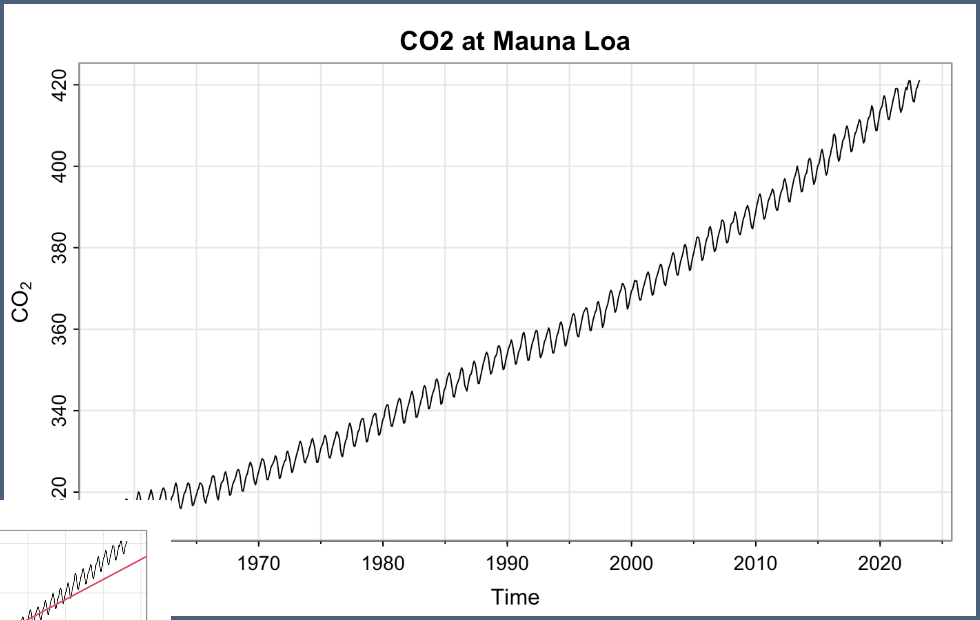
	Prediction	SE
Feb 1979	258.3	6.8
Mar 1979	281.3	7.9
Apr 1979	263.4	8.6
May 1979	274.3	9.1
Jun 1979	270.6	9.5
Jul 1979	294.4	10.0
Aug 1979	301.3	10.4
Sep 1979	295.5	10.8
Oct 1979	289.2	11.2
Nov 1979	272.3	11.6
Dec 1979	283.4	11.9
Jan 1980	275.5	12.3

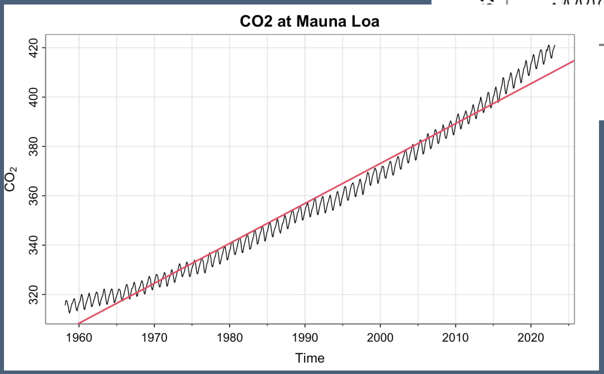
Example: CO_2 Levels

CO2 Levels

Example 5.15

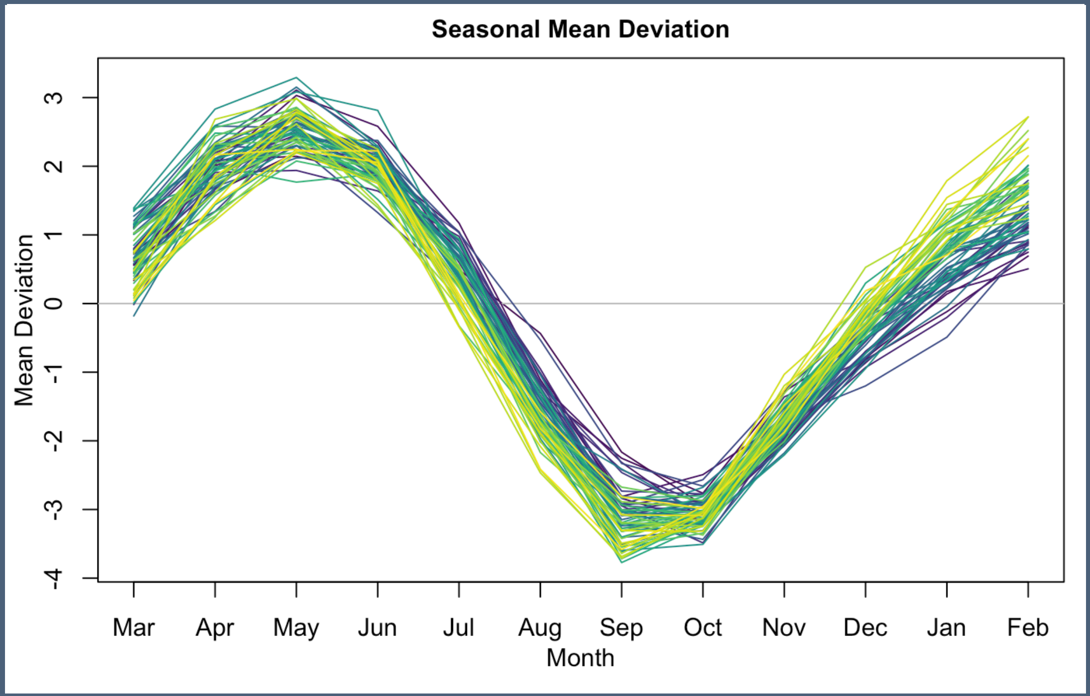
- CO2 measured at Mauna Loa Observatory
  - Monthly, from March 1958 through March 2023
  - Definition
    Dry mole fraction defined as the number of molecules of carbon dioxide divided by the number of molecules of dry air multiplied by one million (ppm)
- Discussion
  - Regular oscillation
  - Faster growth in later data
  - Certainly non stationary





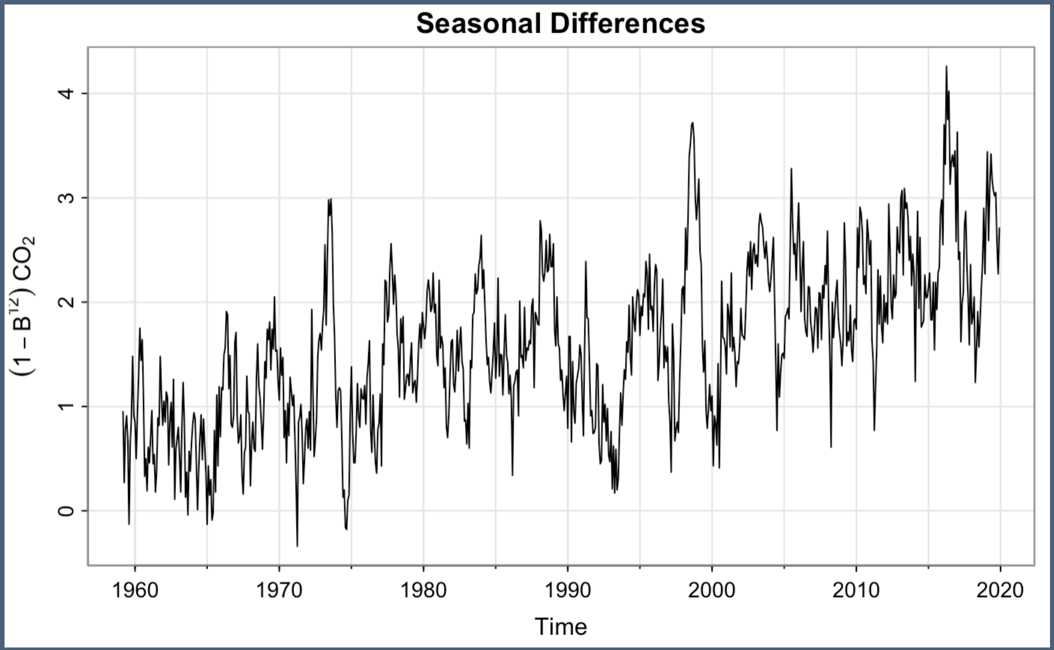
Persistent Seasonality

- Regular pattern
  - Periodic plot
  - Center values for 12 months
  - Plot versus month.
- Colors
  - Dintinguish curves over time
  - Viridis palette
    dark purple -> yellow
- Interpretation
  - Deeper Sep-Oct dip in recent years



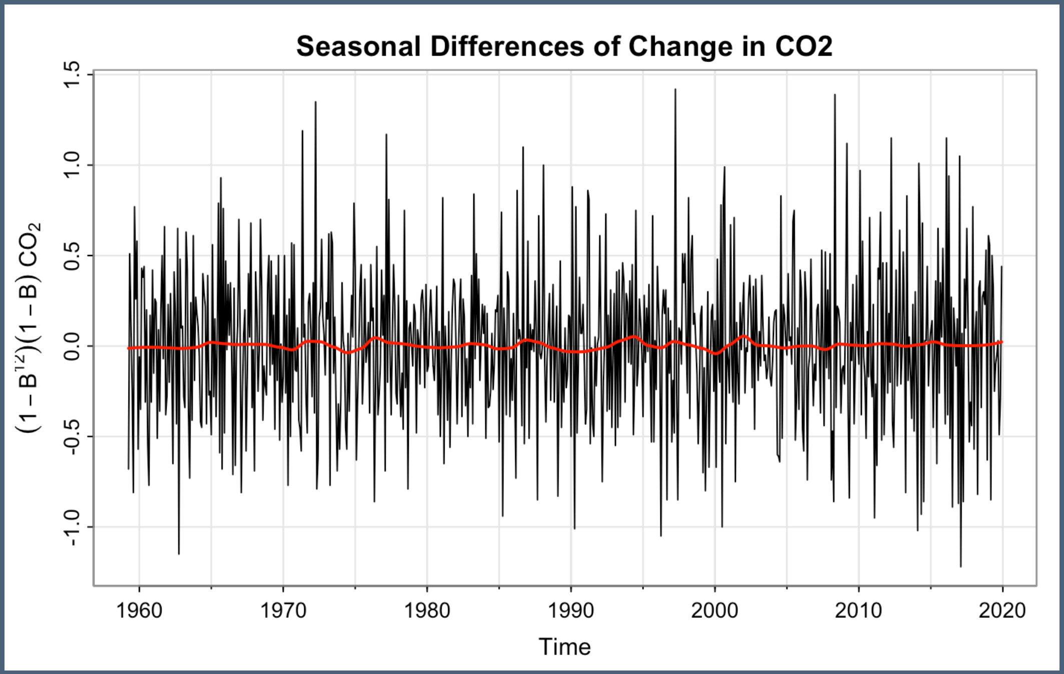
Modeling: Seasonal Differences

- Seasonal differences
  - Plot shows $(1-B^{12}) X_t$
  - Evident upward trend consistent with growth rate in initial sequence plot.



Modeling: Seasonal Differences

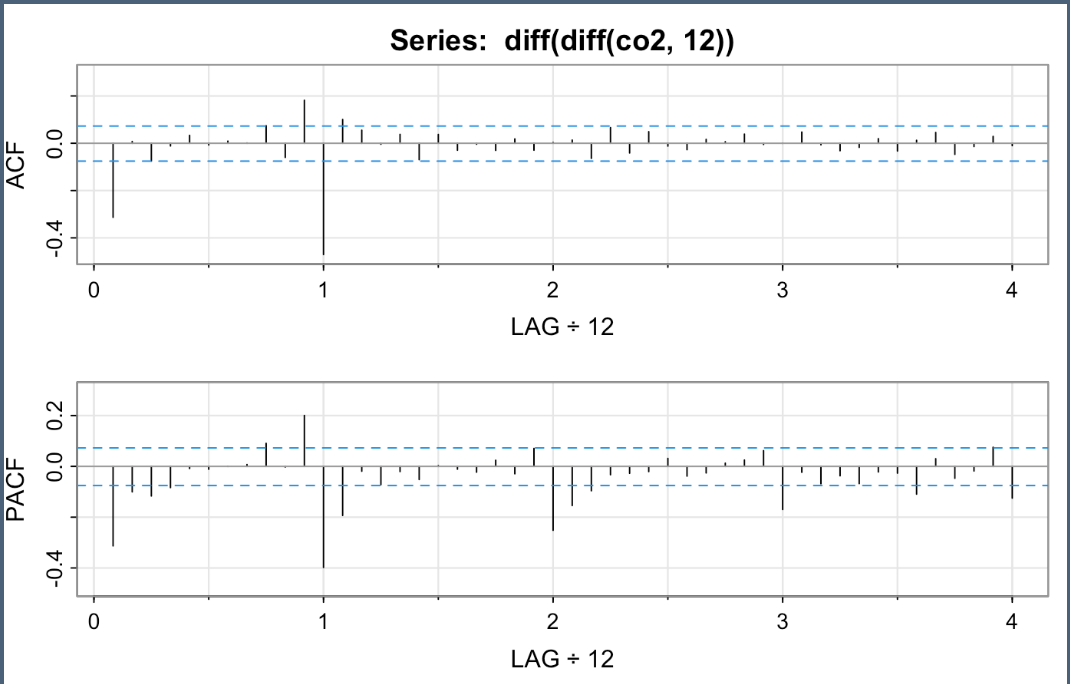
- Difference the seasonal differences
  - Plot shows $(1\text{-}B)(1\text{-}B^{12})\ X_t$
  - This second round of differencing takes care of the changing slope noticed in the original sequence plot
- Loess smooth
  - Visually confirm absence of a trend
  - Important since `sarima` will not fit a constant term in presence of two layers of differencing.



Modeling: ACF and PACF

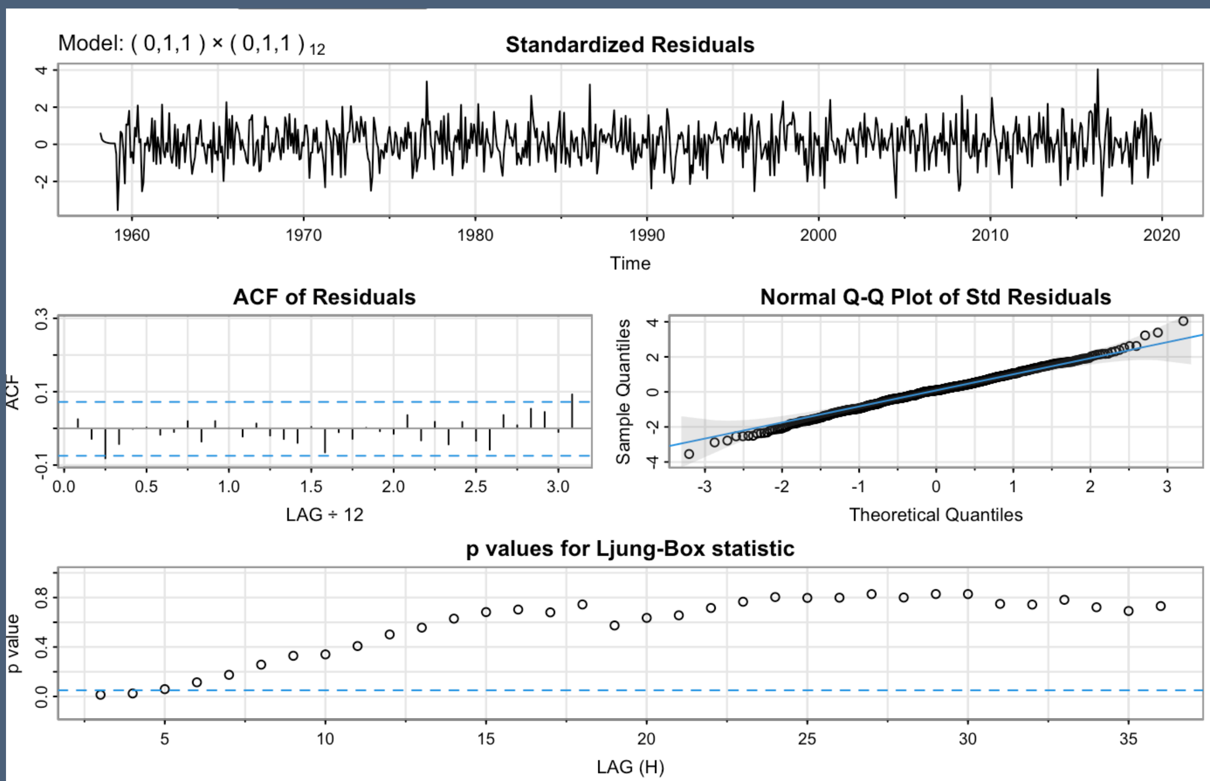
- Model selection
  - Plot shows ACF and PACF of (1-B)(1-B^{12}) X_t
  - Could use a "seasonal version" of the fit_arma_models function to help choosing p and q.
- Comments
  - ACF has large terms at lags 1 and 12
  - PACF shows gradual decay at seasonal spacing and perhaps at the origin.
- Initial choice of model?
  - SARIMA(0,1,1)(0,1,1)$_{12}$

$$(1-B^{12})(1-B)X_t = (1-\theta B)(1-\Theta B^{12})w_t$$



Modeling: Estimated Model

- Fitted model
  - Text adds another parameter, but doesn't add much.

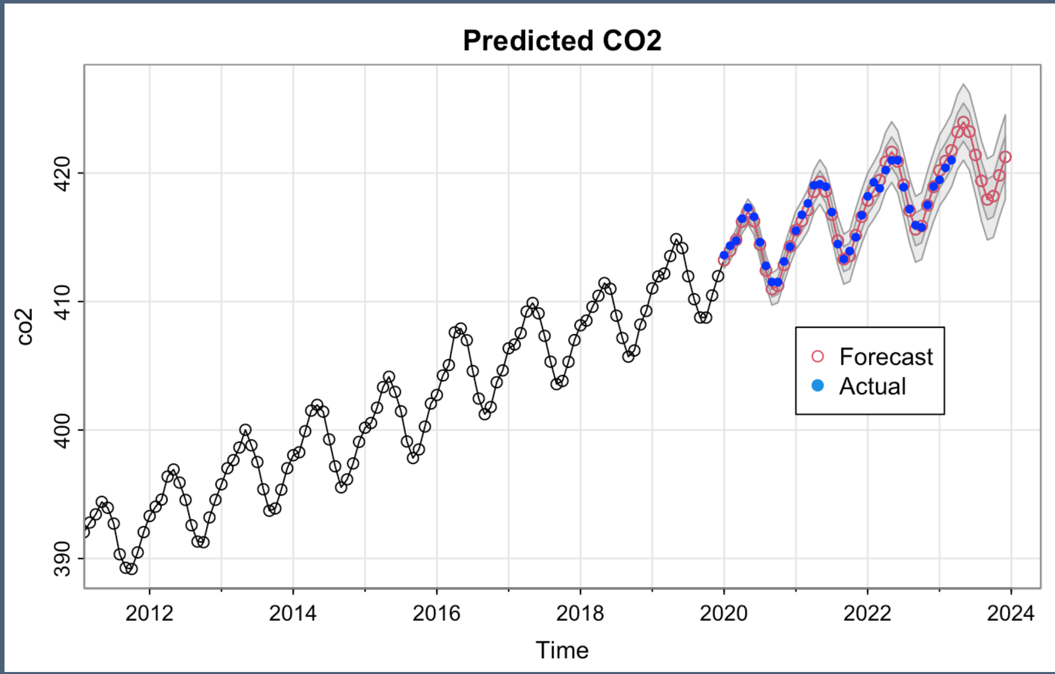



```
Coefficients:
     Estimate     SE  t.value p.value
ma1   -0.3803 0.0383  -9.9407       0
sma1  -0.8629 0.0190 -45.4596       0

sigma^2 estimated as 0.09633493 on 727
```

Predictions

- Compare to actual values
  - Use data from 2020-2023 to evaluate model
  - Note the narrow width of the prediction intervals



Prediction intervals for 2020

	Lower CI	Actual	Upper CI
Jan	412.6	413.6	413.8
Feb	413.2	414.3	414.7
Mar	414.0	414.7	415.6
Apr	415.3	416.4	417.1
May	416.0	417.3	418.0
Jun	415.2	416.6	417.3
Jul	413.3	414.6	415.6
Aug	411.2	412.8	413.6
Sep	409.7	411.5	412.2
Oct	409.9	411.5	412.5
Nov	411.5	413.1	414.2
Dec	412.9	414.3	415.7

What's next?

- More examples
- Comparison to regression seasonal models
  - Rigid vs fluctuating seasonal patterns
  - Incorporating calendar effects