4

Propositional Logic

OBJECTIVES

- Introduction
- Statement or Proposition
- Logical Connectives and Compound Statements
- Tautology, Contradiction and Contingency
- Logical Equivalence
- Predicates and Quantifiers
- Duality Law
- Normal Forms

4.1 Introduction

Logic is concerned with the methods of reasoning. It provides rules and techniques by which we can determine whether any particular argument is valid or not. Logical reasoning has its applications in the field of mathematics to prove theorems; in the field of computer science for design of computing machines, to artificial intelligence to programming languages, etc., in the field of natural and physical sciences to draw conclusions from experiments and in our day-to-day life to solve a multitude of problems.

4.2 Statement or Proposition

A *statement* (or *proposition*) is a declarative sentence that is either true or false, but not both. If the statement is true then we assign a value T to it and if it is false then we assign a value F to it. These values T and F are called the ***truth values*** of the statement.

Example 1. Which of the following sentences are statements ? Also find their truth values.

(i) Delhi is in India.

(ii) $3 + 5 = 8$

(iii) Do your homework.

(iv) What are you doing ?

(v) $x - 4 = 2$

(vi) Jaipur is a state.

Solution:

(i) and (ii) are statements that happen to be true, so their truth values are T.

(iii) is a command, not a statement.

(iv) is not a statement, it is a question.

(v) is a declarative sentence, but not a statement, since it is true or false depending on the value of x.

(vi) is a statement which is false, so its truth value is F.

4.2.1 Propositional Variables

In logic, it is required to draw conclusions from the given statements. Now, instead of writing the statements repeatedly, it is convenient to denote each of the statements by a unique variable, called *propositional variable*. These variables are usually denoted by an English alphabet $p, q, r, \ldots$, etc., and can be replaced by statements. For example, we can write,

p : Delhi is the capital of India; q : It is raining.

4.3 Logical Connectives and Compound Statements

Statements or propositional variables can be combined by means of logical connectives or operators to form a single statement called *compound statements* (or *compound propositions* or *molecular statement)*.

There are five logical connectives as shown in the table given below, which are frequently used for this purpose.

Table 4.1 Logical Connectives

Symbol	Connective	Name
$\sim$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	implies or if ... then	implication or conditional
$\leftrightarrow$	If and only if	equivalence or biconditional

Remark 1. The statement (or proposition) which does not contain any connective is called a ***prime statement*** or an ***atomic statement***.

4.3.1 Negation

If p denotes a statement, then the *negation* of p is the statement denoted by $\sim p$ (or $\neg p$)and read as 'it is not the case that p'. So, it follows that if p is true then $\sim p$ is false, and if p is false then $\sim p$ is true.

Example 2.

If p : Ramanujan was a great mathematician.

then $\sim p$: It is not the case that Ramanujan was a great mathematician.

or $\sim p$: Ramanujan was not a great mathematician.

The truth value of $\sim p$ is relative to that of p and can be expressed in a tabular form, known as **truth table**, as shown below :

Table 4.2 Truth table of $\sim p$

p	$\sim p$
T	F
F	T

Remark 2. Negation changes one statement into another, while other connectives combine two statements to form a third.

Remark 3. A truth table displays the relationships between the truth of propositions. It indicates the truth values of compound statements constructed from simpler statements.

4.3.2 Conjunction

If p and q are statements, then "p and q" is a compound statement, denoted as $p \wedge q$ and referred as the *conjunction* of p and q . The conjunction of p and q is true only when both p and q are true, otherwise, it is false.

Truth values of the statement $p \wedge q$ in terms of truth values of p and q are given in the truth table shown below

Table 4.3 Truth table of $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3. Form the conjunction of p and q for each of the following :

(i) p : It is snowing. q : I am cold.

(ii) $p : 6 < 7$. $q : -3 > -4$.

(iii) p : It is raining. $q : 1 > 4$.

Solution:

(i) $p \wedge q$: It is snowing and I am cold. Since here both p and q are true so the conjunction $p \wedge q$ is true.

(ii) $p \wedge q : 6 < 7$ and $-3 > -4$. Since here both p and q are true so the conjunction $p \wedge q$ is true.

(iii) $p \wedge q$: It is raining and $1 > 4$. Since here q is false so the conjunction $p \wedge q$ is false.

Remark 4. Example 3(iii) depicts that in logic, we may connect two totally unrelated statements by the connective *and*.

4.3.3 Disjunction

If p and q are statements, then "p or q" is a compound statement, denoted as $p \vee q$ and referred as the *disjunction* of p and q. The disjunction of p and q is true whenever at least one of the two statements is true, and it is false only when both p and q are false.

The truth table for disjunction of p and q can be constructed as shown below :

Table 4.4 Truth table of $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 4. Form the disjunction of p and q for each of the following :

(i) p : 4 is a positive integer. q : $\sqrt{5}$ is a rational number.

(ii) $p : 3 + 4 > 8$ q : Jaipur is the capital of Gujarat.

Solution:

(i) $p \vee q$: 4 is a positive integer or $\sqrt{5}$ is a rational number. Since here p is true, so the disjunction $p \vee q$ is true, even though q is false.

(ii) $p \vee q$: 3+4>8 or Jaipur is the capital of Gujarat. Here $p \vee q$ is false as both p and q are false.

Remark 5. Example 4(ii) depicts that in logic, we may connect two totally unrelated statements by the connective *or.*

Remark 6. In disjunction we have defined *or* in the *inclusive* sense, i.e., either p is true or q is true or both are true so this "or" could be known as *inclusive or.* But "or" can be used in the exclusive sense, also i.e., either p is true or q is true, but not both. In mathematics and computer science, we conventionally use the connective "or" in the inclusive manner.

Example 5. Make a truth table for each of the following

(i) $p \vee \sim q$ (ii) $(\sim p \wedge q) \vee p$

(iii) $(p \vee q) \vee \sim q$ (iv) $(p \vee q) \wedge r$

(v) $(\sim p \vee q) \wedge \sim r$ (vi) $\sim(\sim p)$.

Solution: (i)

Table 4.5 Truth table of $p \vee \sim q$

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(ii)

Table 4.6 Truth table of $(\sim p \wedge q) \vee p$

p	q	$\sim p$	$\sim p \wedge q$	$(\sim p \wedge q) \vee p$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

(iii)

Table 4.7 Truth table of $(p \vee q) \vee \sim q$

p	q	$p \vee q$	$\sim q$	$(p \vee q) \vee \sim q$
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(iv)

Table 4.8 Truth table of $(p \vee q) \wedge r$

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(v)

Table 4.9 Truth table of $(\sim p \vee q) \wedge \sim r$

p	q	r	~p	~p∨q	~r	(~p∨q)∧~r
T	T	T	F	T	F	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
F	T	T	T	T	F	F
T	F	F	F	F	T	F
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	T	T	T	T

(vi)

Table 4.10 Truth table of $\sim(\sim p)$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Here, we observe that double negation of any statement gives the original statement.

4.3.4 Conditional Statement (or Implication), Converse and Contrapositive

If p and q are statements, then "if p then q" is a compound statement, denoted as $p \rightarrow q$ and referred as a *conditional statement*, or *implication*. The implication $p \rightarrow q$ is false only when p is true and q is false; otherwise, it is always true. In this implication, p is called the ***hypothesis*** (or *antecedent* or *premise*) and q is called the ***conclusion*** (or *consequent*).

Now, before constructing the truth table for $p \rightarrow q$, let us first analyse the following implication:

$p \rightarrow q$: If you wash my car, then I will pay you Rs. 20.

Here p : If you wash may car,

and q : I will pay you Rs. 20.

Now we have following three cases:

Case 1: When both p and q are true

If you wash my car and if I pay you Rs. 25, then the implication is true, as I kept my promise.

Case 2: When p is true and q is false

If you wash my car and if I do not pay you Rs. 20, then the promise is violated and hence the implication is false.

Case 3: When p is false

If you do not wash my car, then I may give you Rs. 20 (being generous) or not, (so q may be true of false). In either case, my promise has not been tested and thus has not been violated. Therefore, the implication has not been proved false. If it is not false, it must be true. So we can say, if p is false $p \to q$ is always true by default.

In other words, if p is false the implication $p \to q$ is said to be **vacuously** true.

Hence, we have the following truth table:

Table 4.11 Truth table of $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 6. Write the implication $p \to q$ for each of the following :

(i) p : I am hungry. q : I will eat.

(ii) p : It is snowing. q : $3 + 8 = 11$

Solution:

(i) $p \to q$: If I am hungry, then I will eat.

(ii) $p \to q$: If it is snowing, then $3 + 8 = 11$.

Remark 7. Example 6(ii) depicts that in logic, the compound statement $p \to q$ does not say that p "caused" q in the usual sense. It only asserts that if p is true, then q will also be found to be true.

Remark 8. The statement $p \to q$ may be expressed as :

p implies q ; p is a sufficient condition for q ;

if p then q ; q is a necessary condition for p;

p only if q ; q follows from p;

q provided p ; q whenever p.

q, if p ; q is a consequence of p.

If $p \to q$ is an implication, then

the **converse** of $p \to q$ is the implication $q \to p$, the **contrapositive** of $p \to q$ is the implication $\sim q \to \sim p$,

and the **inverse** of $p \to q$ is the implication $\sim p \to \sim q$.

Table 4.12 Truth table of $q \to p$ (converse)

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Table 4.13 Truth table of $\sim q \to \sim p$ (contrapositive)

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Table 4.14 Truth table of $\sim p \to \sim q$ (inverse)

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \to \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Example 7. Find the converse, contrapositive and inverse of the following implications:

(i) "If today is Thursday, then I have a test today".

(ii) "If it is raining, then I get wet."

Solution:

(i) Let p : Today is Thursday, and q : I have a test today. Then

Converse is $q \to p$: If I have a test today, then today is Thursday.

Contrapositive is $\sim q \to \sim p$: If I do not have a test today, then today is not Thursday".

Inverse is $\sim p \to \sim q$: If today is not Thursday, then I do not have a test today.

(ii) Let p : It is raining ; and q : I get wet.

Then, the converse is $q \to p$: If I get wet, then it is raining.

The contrapositive is $\sim q \to \sim p$: If I do not get wet, then it is not raining.

The inverse is $\sim p \to \sim q$: If it is not raining, then I do not get wet.

4.3.5 Biconditional Statement

If p and q are statements, then "p if and only if q" is a compound statement, denoted as $p \leftrightarrow q$ and referred as a *biconditional statement* or *an equivalence*.

The equivalence $p \leftrightarrow q$ is true only when both p and q are true or when both p and q are false.

Table 4.15 Truth table of $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The equivalence $p \leftrightarrow q$ can also be defined as a conjunction of the implications $p \rightarrow q$ and $q \rightarrow p$.

Table 4.16 Truth table of $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example 8. Is the following equivalence a true statement ? **[RTU 2009]**

"$3>2$ if and only if $0<3-2$."

Solution: Let $p : 3 > 2$; $q : 0 < 3 - 2$

Since both p and q are true, so we conclude that $p \leftrightarrow q$ is true.

4.4 Tautology, Contradiction and Contingency

A compound statement that is always true for all possible truth values of its propositional variables, is called a *tautology* or *valid.* Obviously, its truth table contains only truth value T in the last column.

A compound statement that is always false, is called a *contradiction* or *absurdity*. Obviously, its truth table contains only value F in the last column.

A statement that is neither a tautology nor a contradiction is called a *contingency*. So, its truth table contains both T and F values at least once in its last column.

Example 9. Show that the statement $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tautology.

Solution:

Table 4.17 Truth table of $(p \to q) \leftrightarrow (\sim q \to \sim p)$

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since, the above table contains only T in the last column, so the given statement is a tautology.

Example 10. Show that the statement $p \wedge \sim p$ is a contradiction.

Solution: The truth table for the given statement is as follows :

Table 4.18 Truth table of $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Since the above table contains only F in the last column, so the given statement is a contradiction.

Example 11. Show that the statement $(p \to q) \wedge (p \vee q)$ is a contingency.

Solution: The truth table for the given statement can be constructed as follows :

Table 4.19 Truth table of $(p \to q) \wedge (p \vee q)$

p	q	$p \to q$	$p \vee q$	$(p \to q) \wedge (p \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Since the above table contains both T and F values in its last column, so the given statement is a contingency.

4.5 Logical Equivalence

Two compound statements p and q are said to be *logically equivalent* or simply *equivalent*, if $p \leftrightarrow q$ is a tautology. If p is equivalent to q then we write $p \equiv q$.

Example 12. Show that $(p \vee q)$ and $(q \vee p)$ are equivalent.

Solution: First, we construct the truth table for the statement $(p \vee q) \leftrightarrow (q \vee p)$ as follows:

Table 4.20 Truth table of $(p \vee q) \leftrightarrow (q \vee p)$

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Since, only T occurs in the last column so $(p \vee q) \leftrightarrow (q \vee p)$ is a tautology and hence $p \vee q \equiv q \vee p$.

Remark 9. Another way to determine whether two statements are equivalent is to construct a column for each statement and compare these columns, if they are identical then we say that the two statements are equivalent.

Example 13. Show that $p \to q \equiv (\sim p) \vee q$. **[RTU 2011, 2009]**

Solution:

Table 4.21 Truth table for $p \to q$ and $(\sim p) \vee q$

p	q	$p \to q$	$\sim p$	$(\sim p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns corresponding to $p \to q$ and $(\sim p) \vee q$ respectively, are identical so $p \to q \equiv (\sim p) \vee q$.

4.5.1 Operations For Propositions

The operations for propositions have the following properties :

(A) Commutative Properties

(i) $p \vee q \equiv q \vee p$

(ii) $p \wedge q \equiv q \wedge p$.

Proof :

(i) Refer Example 12.

(ii) The truth table for $p \wedge q$ and $q \wedge p$ can be constructed as below

Table 4.22 Truth table for $p \wedge q$ and $q \wedge p$

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Since the columns corresponding to $p \wedge q$ and $q \wedge p$ respectively, are identical so we conclude that $p \wedge q \equiv q \wedge p$.

Proved

(B) Associative Properties

(i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Proof : The truth table for the given propositions can be constructed as below :

Table 4.23 Truth table for $p \vee (q \vee r)$, $(p \vee q) \vee r$, $p \wedge (q \wedge r)$ and $(p \wedge q) \wedge r$

p (1)	q (2)	r (3)	$q \vee r$ (4)	$p \vee (q \vee r)$ (5)	$p \vee q$ (6)	$(p \vee q) \vee r$ (7)	$q \wedge r$ (8)	$p \wedge (q \wedge r)$ (9)	$p \wedge q$ (10)	$(p \wedge q) \wedge r$ (11)
T	T	T	T	T	T	T	T	T	T	T
T	T	T	T	T	T	T	F	F	T	F
T	F	F	T	T	T	T	F	F	F	F
F	T	T	T	T	T	T	T	F	F	F
T	F	F	F	T	T	T	F	F	F	F
F	T	F	T	T	T	T	F	F	F	F
F	F	T	T	T	F	T	F	F	F	F
F	F	F	F	F	F	F	F	F	F	F

(i) Since the columns (5) and (7) corresponding to $p \vee (q \vee r)$ and $(p \vee q) \vee r$ respectively, are identical so we have $p \vee (q \vee r) \equiv (p \vee q) \vee r$.

(ii) Since the columns (9) and (11), corresponding to $p \wedge (q \wedge r)$ and $(p \wedge q) \wedge r$ respectively, are identical so we have $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

(C) Distributive Properties

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ **[CE(RTU)-2009, 2007]**

Proof : (i) The truth table for the given propositions can be constructed as follows :

Table 4.24 Truth table for $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

p (1)	q (2)	r (3)	$q \wedge r$ (4)	$p \vee (q \wedge r)$ (5)	$p \vee q$ (6)	$p \vee r$ (7)	$(p \vee q) \wedge (p \vee r)$ (8)
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since the columns (5) and (8) are identical, so we have

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

(ii) Left as an exercise for the reader.

(D) Idempotent Properties

(i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$

Proof : The truth table is as follows :

Table 4.25 Truth table for p, $p \vee p$ and $p \wedge p$

p (1)	p (2)	$p \vee p$ (3)	$p \wedge p$ (4)
T	T	T	T
F	F	F	F

(i) Since columns (1) and (3) are identical, so we have $p \vee p \equiv p$

(ii) Since columns (1) and (4) are identical, so we have $p \wedge p \equiv p$.

(E) Properties of Negation

(i) $\sim(\sim p) \equiv p$ (Double negation law)

(ii) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$ (De Morgan's law)

(iii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$ (De Morgan's law)

Proof : The truth table for the given propositions is as follows :

Table 4.26 Truth table for $\sim(\sim p)$, $\sim(p \vee q)$, $(\sim p) \wedge (\sim q)$, $\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$

p (1)	q (2)	$\sim p$ (3)	$\sim(\sim p)$ (4)	$\sim q$ (5)	$p \vee q$ (6)	$\sim(p \vee q)$ (7)	$(\sim p) \wedge (\sim q)$ (8)	$p \wedge q$ (9)	$\sim(p \wedge q)$ (10)	$(\sim p) \vee (\sim q)$ (11)
T	T	F	T	F	T	F	F	T	F	F
T	F	F	T	T	T	F	F	F	T	T
F	T	T	F	F	T	F	F	F	T	T
F	F	T	F	T	F	T	T	F	T	T

(i) Since the columns (1) and (4) are identical, so we have $\sim(\sim p) \equiv p$.

(ii) Since the columns (7) and (8) are identical, so we have $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$.

(iii) Since the columns (10) and (11) are identical so we have $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$.

(F) Identity Laws

(i) $p \wedge \mathbf{T} \equiv p$ (ii) $p \vee \mathbf{F} \equiv p$,

where **T** denotes any proposition that is always true and **F** denotes any proposition that is always false.

Proof : The truth table is as shown below :

Table 4.27 Truth table for $p \wedge$T and $p \vee$F

p (1)	T (2)	$p \wedge$T (3)	F (4)	$p \vee$F (5)
T	T	T	F	T
F	T	F	F	F

(i) Since the columns (1) and (3) are identical, so $p \wedge \mathbf{T} \equiv p$.

(ii) Since the columns (1) and (5) are identical, so $p \vee \mathbf{F} \equiv p$.

(G) Domination Laws

(i) $p \vee T \equiv T$ (ii) $p \wedge F \equiv F$

Proof : The truth table is as shown below

Table 4.28 Truth table for $p \vee T$ and $p \wedge F$

p (1)	T (2)	F (3)	$p \vee T$ (4)	$p \wedge F$ (5)
T	T	F	T	F
F	T	F	T	F

(i) Since the columns (2) and (4) are identical, so $p \vee T \equiv T$.

(ii) Since the columns (3) and (5) are identical, so $p \wedge F \equiv F$.

(H) Absorption Laws

(i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

Proof : The truth table is as shown below

Table 4.29 Truth table for $p \vee (p \wedge q)$ and $p \wedge (p \vee q)$

p (1)	q (2)	$p \wedge q$ (3)	$p \vee q$ (4)	$p \vee (p \wedge q)$ (5)	$p \wedge (p \vee q)$ (6)
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	F	F	F

(i) Since the columns (1) and (5) are identical, so $p \vee (p \wedge q) \equiv p$.

(ii) Since the columns (1) and (6) are identical, so $p \wedge (p \vee q) \equiv p$.

The above logical equivalences can be used to form additional logical equivalences such as

$$p \vee (\sim p) \equiv T$$

$$p \wedge (\sim p) \equiv F$$

$$(p \rightarrow q) \equiv (\sim p \vee q), \quad \text{etc.,}$$

A statement in a compound statement can be replaced by its logical equivalent without affecting the truth value of the compound statement. This method is illustrated in the following examples.

Example 14. Show that $\sim(p \vee (\sim p \wedge q)) \equiv (\sim p) \wedge (\sim q)$ **[IT(RTU)-2008]**

Solution:

$\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim(\sim p \wedge q)$ [by De Morgan's law E(ii)]

$\equiv \sim p \wedge (\sim(\sim p) \vee (\sim q))$ [by De Morgan's law E(iii)]

$\equiv \sim p \wedge (p \vee \sim q)$ [by double negation, law E(i)]

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ [by distributive law C(ii)]

$\equiv F \vee (\sim p \wedge \sim q)$ [$\because \sim p \wedge p \equiv F$]

$\equiv (\sim p \wedge \sim q) \vee F$ [$\because p \vee q \equiv q \vee p$]

$\equiv \sim p \wedge \sim q$ [by identity law for F, F(ii)]

Example 15. Show that $(p\wedge q)\rightarrow(p\vee q)$ is a tautology.

Solution: To show that the given statement is a tautology it is sufficient to prove that it is logically equivalent to **T**.

$$
\begin{aligned}
(p\wedge q)\rightarrow(p\vee q) &\equiv \sim(p\wedge q)\vee(p\vee q) && [\square p\rightarrow q \equiv \sim p\vee q]\\
&\equiv (\sim p\vee\sim q)\vee(p\vee q) && [\text{by De Morgan's law}]\\
&\equiv ((\sim p\vee\sim q)\vee p)\vee q && [\text{by associativity}]\\
&\equiv (\sim p\vee(\sim q\vee p))\vee q && [\text{by associativity}]\\
&\equiv (\sim p\vee(p\vee\sim q))\vee q && [\square p\vee q \equiv q\vee p]\\
&\equiv (\sim p\vee p)\vee(\sim q\vee q) && \\
&\equiv \mathbf{T}\vee\mathbf{T} && [\square\sim p\vee p \equiv \mathbf{T}]\\
&\equiv \mathbf{T} && [\text{by domination law}]
\end{aligned}
$$

Remark 10. Ex. 14 and Ex. 15 can also be done by the method of truth table.

4.6 Predicates and Quantifiers

Consider the statement

"x is a positive integer."

This statement cannot has a truth value unless the value of the variable x is specified. The first part, i.e., the variable x, is called the subject of the statement, while the second part, i.e., "is a positive integer" - refers to a property that the subject of the statement can have, is called the ***predicate***. We can express the above statement by $P(x)$, where P denotes the predicate "is a positive integer" and x is the variable. The statement $P(x)$ is also called a **propositional function** because once a value has been assigned to x, it becomes a proposition and has a truth value. Propositional functions also occur in computer programs. The logic based upon the analysis of predicates in any statement is called ***predicate logic* or *first order logic***.

Example 16. Let $P(x)$ denote the statement "$x > 3$". What are the truth values of $P(2)$ and $P(4)$?

Solution: The statement $P(2)$ denotes $2>3$, which is false, while $P(4)$ denotes $4>3$, which is true.

Remark 11. There exist statements that involve two or more variables.

Example 17. Let $Q(x, y)$ be the statement "$x = y+3$". What are the truth values of the statements $Q(1, 2)$ and $Q(3,0)$?

Solution: To get $Q(1, 2)$, put $x = 1$ and $y = 2$ in $Q(x, y)$. We have the statement $Q(1, 2)$ as "$1 = 2+3$", which is false. The statement $Q(3, 0)$ it "$3 = 0+3$", which is true.

Example 18. Consider the statement "if $x > 0$ then $x := x + 1$.

Solution: When this statement is encountered in a program, the value of the variable x, which is "$x > 0$", at that point in the execution of the program is inserted into $P(x)$. If $P(x)$ is true for this value of x, the assignment statement $x := x + 1$ is executed, so the value of x is increased by 1. If $P(x)$ is false for this value of x, the assignment statement is not executed, so x is unchanged.

Quantifiers

Quantification is an another powerful technique to create a statement from a propositional function. There are two types of quantification, namely, universal quantification and existential quantification.

The *universal quantification* of a predicate P(x) is the statement "P(x) is true for all values of x in the universe of discourse".

The universe of discourse is the domain that specifies the possible values of the variable x.

The notation $\forall x$P(x) denotes the universal quantification of P(x). Here the symbol $\forall$ is called the **universal quantifier**. The statement $\forall x$P(x) can also be stated as

"for every xP(x)" or "for all xP(x)".

The *existential quantification* of a predicate $P(x)$ is the statement "There exists an element x in the universe of discourse for which $P(x)$ is true."

The notation $\exists x$P(x) denotes the existential quantification of P(x). Here the symbol $\exists$ is called the **existential quantifier**. The statement $\exists x$P(x) can also be stated as

"there is an x such that P(x)", "there is at least one x such that $P(x)$", "for some x P(x)", or "there exists an x such that $P(x)$".

Example 19. Express the statement "Every student in this class has studied calculus" as a universal quantification.

Solution: Let P(x) be the statement "x has studied calculus", and the universe of discourse consists of the students in this class. Then the given statement can be expressed as $\forall x$P(x).

Alternatively, this statement can also be expressed as

$\forall x(R(x) \rightarrow P(x))$

where R(x) is the statement "x is in this class", and the universe of discourse is the set of all students.

Example 20. What is the truth value of $\forall xP(x)$, where $P(x)$ is the statement "$x^2<10$" and the universe of discourse consists of the positive integers not exceeding 4 ?

Solution: Here universe of discourse U={1, 2, 3, 4}. Since P(4) states "$4^2<10$" which is false, so it follows that $\forall x$P(x) is false.

Example 21. What is the truth value of $\exists x$P(x), in Example 20 ?

Solution: Since universe of discourse U = {1, 2, 3, 4} and we observe that P(1), which states "$1^2<10$" is true so it follows that $\exists x$P(x) is true.

Example 22. What is the truth value of $\exists x$Q(x), where Q(x) is the statement "$x = x + 1$" and the universe of discourse is the set of real numbers ?

Solution: Since Q(x) is false for every real number x, so it follows that $\exists x$Q(x) is false.

Remark 12. The meaning of both types of quantifiers can be summarized as below.

Table 4.30 Table of Quantifiers for one variable

Statement	When True ?	When False ?
$\forall x$P(x)	P(x) is true for every x.	There is at least one x for which P(x) is false.
$\exists x$P(x)	There is atleast one x for which P(x) is true.	P(x) is false for every x.

Remark 13. Sometimes expressions involving quantifiers can be quite complicated so that their meaning cannot be understood. Translating the expression into English helps to understand its meaning.

Example 23. Translate the statement

$\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$,

into English, where C(x) is "x has a computer", F(x,y) is "x and y are friends", and the universe of discourse for both x and y is the set of all students in our school.

Solution: The statement says that, for every student x in our school x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, every student in our school has a computer or has a friend who has a computer.

Ans.

Example 24. Express the statement "Everyone has exactly one best friend" as a logical expression using quantifiers.

Solution: Let Q(x, y) denotes "y is the best friend of x". Now, the given statement means that for each person x there is another unique person y such that y is the best friend of x. It means if z is a person other than y, then z cannot be a best friend of x. Thus, we can translate the sentence as

$\forall x \exists y \forall z(Q(x, y) \wedge ((z \neq y) \rightarrow \sim Q(x, z)))$

4.6.1 Properties of Quantifiers

The negation of a quantified statement changes the quantifier and also negates the given statement as mentioned below :

(i) $\sim(\forall x P(x)) \equiv \exists x \sim P(x)$ (De Morgan's Law)

(ii) $\sim(\exists x P(x)) \equiv \forall x \sim P(x)$ (De Morgan's Law)

(iii) $\exists x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \exists x Q(x)$

(iv) $\exists x P(x) \rightarrow \forall x Q(x) \equiv \forall x(P(x) \rightarrow Q(x))$

(v) $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

(vi) $\sim(\exists x \sim P(x)) \equiv \forall x P(x)$

ILLUSTRATIVE EXAMPLES

Example 25. With the help of truth tables, prove that

(i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$

(ii) $p \rightarrow q \equiv \sim q \rightarrow \sim p$

(iii) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$ **[RTU 2010]**

Solution: Truth table for the given statements (i) and (ii) is as follows :

Table 4.31 Truth table for $\sim(p \rightarrow q)$, $p \wedge \sim q$, $\sim q \rightarrow \sim p$

p (1)	q (2)	$p \rightarrow q$ (3)	$\sim(p \rightarrow q)$ (4)	$\sim q$ (5)	$p \wedge \sim q$ (6)	$\sim p$ (7)	$\sim q \rightarrow \sim p$ (8)
T	T	T	F	F	F	F	T
T	F	F	T	T	T	F	F
F	T	T	F	F	F	T	T
F	F	T	F	T	F	T	T

(i) Since the columns (4) and (6) are identical, so $\sim(p\to q) \equiv p\wedge\sim q$.

(ii) Since the columns (3) and (8) are identical, so $p\to q \equiv \sim q\to\sim p$.

(iii) Truth table for the given statement is as follows :

Table 4.32 Truth table for $p\to(q\wedge r)$ and $(p\to q)\wedge(p\to r)$

p (1)	q (2)	r (3)	$q\wedge r$ (4)	$p\to(q\wedge r)$ (5)	$p\to q$ (6)	$p\to r$ (7)	$(p\to q)\wedge(p\to r)$ (8)
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
F	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns (5) and (8) are identical, so $p\to(q\wedge r) \equiv (p\to q)\wedge(p\to r)$.

Example 26. A firm of chartered accountants makes the following declaration : "An article clerk passing the final C.A. examination in the first attempt will be awarded a prize of Rs. 100. Five clerks A, B, C, D and E appeared for the first time from the firm and only A and B could pass. The firm awards prizes not only to them but to C and D also. Is this action logically justified? How should the statement be worded so that only A and B will be entitled for the prize ?

Solution: Let p : a clerk passes C.A. examination in the first attempt.

q : awarded a prize of Rs. 100.

Obviously, $p\to q$. But it does not exclude other because the statement be even true when p is false and q is true as shown in the table below :

Table 4.33 Truth table for $p\to q \equiv \sim p\vee q$

p (1)	q (2)	$p\to q$ (3)	$\sim p$ (4)	$\sim p\vee q$ (5)
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns (3) and (5) are identical, so $p\to q \equiv \sim p\vee q$.

Thus, it is logically justified to award the prizes to C and D.

Columns (1) and (2) indicate that only in one case when p is true, q is false, i.e. the prize is not given to those who have passed the exam in first attempt but given to those who have not passed. So, the statement should be worded as "only those who pass the examination in the first attempt should be awarded the prize, i.e. $p\leftrightarrow q$.

Example 27. Show that $p \wedge q \rightarrow p \vee q$ is a tautology. **[RTU 2011, 2010]**

Solution: The truth table for the given statement is as follows :

Table 4.34 Truth table for $p \wedge q \rightarrow p \vee q$

p	q	$p \wedge q$	$p \vee q$	$p \wedge q \rightarrow p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Since the given statement has all entries as T values in the last column of the table, so it is a tautology.

Example 28. Show that $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction or fallacy. **[RTU 2011, 2009]**

Solution: The truth table is as follows

Table 4.35 Truth table for $(p \wedge q) \wedge \sim(p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since the last column of the table, corresponds to the statement $(p \wedge q) \wedge \sim(p \vee q)$, contains only F as the truth value, so the given statement is a contradiction.

Example 29. Let p : It is cold.

q : It is raining.

Translate the following statements into English

(i) $\sim p$ (ii) $p \wedge q$

(iii) $p \vee q$ (iv) $q \vee \sim p$

Solution:

(i) $\sim p$: It is not cold.

(ii) $p \wedge q$: It is cold and raining.

(iii) $p \vee q$: It is cold or it is raining.

(iv) $q \vee \sim p$: It is raining or it is not cold.

Example 30.

Let p : Rahul reads Newsweek.

q : Rahul reads Time.

r : Rahul reads Fortune.

Write each of the following in symbolic form using the connectives $\wedge$, $\vee$, $\sim$.

(i) Rahul reads Newsweek or Time, but not Fortune.

(ii) Rahul reads Newsweek and Time, or he does not read Newsweek and Fortune.

(iii) It is not true that Rahul reads Newsweek but not Fortune.

(iv) It is not true that Rahul reads Fortune or Time but not Newsweek.

Solution:

(i) $(p \vee q) \wedge \sim r$ (ii) $(p \wedge q) \vee \sim (p \wedge r)$

(iii) $\sim (p \wedge \sim r)$ (iv) $\sim ((r \vee q) \wedge \sim p)$

Example 31. Rewrite the following statements without using the conditional.

(i) If it is cold, he wears a hat.

(ii) If productivity increases, then wages rise.

Solution: Since we know that "If p then q" is equivalent to "Not p or q", i.e. $p \rightarrow q \equiv \sim p \vee q$ (see Ex. 13). Hence

(i) Let p : It is cold.

q : He wears a hat.

so the statement can be rewritten as "It is not cold or he wears a hat".

(ii) Let p : Productivity increases.

q : Wages rise.

Then the statement can be rewritten as "Productivity does not increases or wages rise".

Example 32. Determine the contrapositive of each statement :

(i) If John is a poet, then he is poor.

(ii) Only if Mary studies will she pass the test.

Solution: The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

(i) Let p : John is a poet.

q : He is poor.

Hence the contrapositive of the given statement is:

If John is not poor, then he is not a poet.

(ii) The given statement is equivalent to "If Mary passes the test, then she studied".

Let p : Mary passes the test.

q : She studied.

Hence its contrapositive is:

If Mary does not study, then she will not pass the test.

Example 33. Write the negation of each of the following statements.

(i) He swims if and only if the water is warm.

(ii) If it snows, then they do not drive the car.

Solution:

(i) Since we have $\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q \equiv \sim p \leftrightarrow q$, hence the required negation is:

He swims if and only if the water is not warm.

OR

He does not swim if and only if the water is warm.

(ii) Since $\sim(p \rightarrow \sim q) \equiv p \wedge q$. Hence the required negation is

It snows and they drive the car.

Example 34. Over the universe of four-wheelers, let P(x) : x is a four wheeler; Q(x) : x is a car and R(x) : x is manufactured by Maruti Udyog Ltd. (MUL). Express the following statements using quantifiers.

(i) Every car is a four wheeler manufactured by MUL.

(ii) There are cars that are not manufactured by MUL.

(iii) Every four wheeler is a car.

Solution:

(i) $\forall x (P(x) \wedge Q(x)) \rightarrow R(x)$

(ii) $\exists x (Q(x) \wedge \sim R(x))$

(iii) $\forall x (P(x) \rightarrow Q(x))$

Example 35. Over the universe of animals, let

P(x) : x is a whale ; Q(x) : x is a fish : R(x) : x lives in water.

Translate the following into English

(i) $\exists x(\sim R(x))$

(ii) $\exists x(Q(x) \wedge \sim P(x))$

(iii) $\forall x(P(x) \wedge R(x)) \rightarrow Q(x)$

Solution:

(i) There exists an animal which does not lives in water.

(ii) There exists a fish that is not a whale.

(iii) Every whale that lives in the water, is a fish.

Example 36. Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using quantifiers.

Solution: Let the universe of discourse for the variable x be the set of students in this class.

Let M(x) : x has visited Mexico.

C(x) : x has visited Canada.

Then, the first statement can be written as $\exists x M(x)$.

The second given statement can be written as $\forall x(C(x) \vee M(x))$.

Example 37. Express the statement "If somebody is female and is a parent, then this person is someone's mother" as a logical expression.

Solution: Let F(x) : x is female ; P(x) : x is a parent and

M(x, y) : x is the mother of y.

Here the universe of discourse is the people of the world.

Thus, the statement can be written as

$$\forall x[(F(x) \wedge P(x)) \rightarrow \exists y M(x, y)]$$

Example 38. Let $P(x, y) : x+y = y+x$. What is the truth value of the quantification $\forall x \forall y\ P(x, y)$?

Solution: The quantification $\forall x\ \forall y\ P(x, y)$ denotes the statement "For every real number x and for every real number y, it is true that $x+y = y+x$."

Since $P(x, y)$ is true for all real numbers x and y, hence the statement $\forall x\ \forall y\ P(x, y)$ is true.

Example 39. Let $P(x, y) : x+y = 0$. What are the truth values of the quantifications $\exists y\ \forall x P(x, y)$ and $\forall x\ \exists y\ P(x, y)$?

Solution: The quantification $\exists y\ \forall x\ P(x, y)$, denotes the statement "There is a real number y such that for every real number x, $x + y = 0$."

Since there is no real number y satisfying $x + y = 0$ for all real numbers x, hence the statement $\exists y\ \forall x P(x, y)$ is false.

The quantification $\forall x\ \exists y P(x, y)$ denotes the statement - "For every real number x there is a real number y such that $x + y = 0$."

Since for each x, there is a $y = -x$ such that $x + y = 0$.

Hence, the statement $\forall x\ \exists y P(x, y)$ is true.

Example 40. Without using truth table, show that for any two statements p and q, $p \vee q \equiv \sim(\sim p \wedge \sim q)$.

Solution: Here we have to use biconditional technique which states that, if the truth value of p is equal to the truth value of q for every set of truth values assigned to propositional variables, then p and q are said to be equivalent and we write $p \equiv q$.

Let the truth value of $\sim(\sim p \wedge \sim q)$ be T then by definition of negation, the truth value of $(\sim p \wedge \sim q)$ is F. Further, by definition of conjunction, the truth value of $\sim p$ or $\sim q$ is F.

Then by definition of negation, the truth value of p or q is T. Hence, the truth value of $p \vee q$ is T.

Similarly, let the truth value of $\sim(\sim p \wedge \sim q)$ be F. By negation, the truth value of $\sim p \wedge \sim q$ is T. Further, by conjunction, the truth values of both $\sim p$ and $\sim q$ are T. Then by negation it follows that the truth values of both p and q are F. Hence the truth value of $p \vee q$ is F.

Thus, we have $p \vee q \equiv \sim(\sim p \wedge \sim q)$.

Example 41. Establish the following logical equivalences, where A is a proposition not involving any quantifiers.

(i) $(\forall x P(x)) \wedge A \equiv \forall x (P(x) \wedge A)$

(ii) $(\exists x P(x)) \wedge A \equiv \exists x (P(x) \wedge A)$

Solution:

(i) If A is true, then both sides are logically equivalent to $\forall x P(x)$.

If A is false, the left-hand side is clearly false and for every x, $P(x) \wedge A$ is false, so the right-hand side is also false. Hence, the two sides are logically equivalent.

(ii) If A is true, then both sides are logically equivalent to $\exists x P(x)$

If A is false, LHS is clearly false, and for every x, $P(x) \wedge A$ is also false, so RHS is also false. Hence, the two sides are logically equivalent.

Example 42. Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x (P(x) \wedge Q(x))$ are not logically equivalent.

Solution: Assume that $P(x)$: x is positive and $Q(x)$: x is negative; with universe of discourse the set of integers. Then

$\exists x P(x) \wedge \exists x Q(x)$ is true but $\exists x(P(x) \wedge Q(x))$ is false.

Example 43. Show that $\forall x P(x) \wedge \exists x Q(x)$ and $\forall x \exists y(P(x) \wedge Q(y))$ are equivalent.

Solution: Let $\forall x P(x) \wedge \exists x Q(x)$ is true then $P(x)$ is true for every x and there is an element y for which $Q(y)$ is true. Since $P(x) \wedge Q(y)$ is true for every x and there is a y for which $Q(y)$ is true, so $\forall x \exists y(P(x) \wedge Q(y))$ is true.

Conversely, let $\forall x \exists y(P(x) \wedge Q(y))$ is true. Let x be an element in the universe of discourse. There is a y such that $Q(y)$ is true, so $\exists x Q(x)$ is true. Since $\forall x P(x)$ is also true, it follows that $\forall x P(x) \wedge \exists x Q(x)$ is true.

Example 44. Show that the propositional formula

$(p \wedge q) \wedge (r \wedge s) \rightarrow p$ for any propositions p, q, r, s is a tautology.

[CE(RTU)-2007]

Solution: $(p \wedge q) \wedge (r \wedge s) \rightarrow p \equiv \sim[(p \wedge q) \wedge (r \wedge s)] \vee p$ $\quad [\because p \rightarrow q \equiv \sim p \vee q]$

$\equiv [\sim(p \wedge q) \vee \sim(r \wedge s)] \vee p$

$\equiv [(\sim p \vee \sim q) \vee (\sim r \vee \sim s)] \vee p$

$\equiv \sim p \vee \sim q \vee \sim r \vee \sim s \vee p$

$\equiv T \vee \sim q \vee \sim r \vee \sim s$ $\quad (\because p \vee \sim p \equiv T)$

$\equiv T$ $\quad (\because T \vee p \equiv T)$

$\Rightarrow (p \wedge q) \wedge (r \wedge s) \rightarrow p$ is a tautology.

Example 45. Write contrapositive, converse and inverse of the statement

"The home team wins whenever it is raining".

Also construct the truth table for each statement. **[RTU 2010, 2008]**

Solution: The given statement can be rewritten as

"If it is raining then the home team wins."

Let p : It is raining.

q : The home team wins.

Then the statement is of the form

$$p \rightarrow q.$$

(a) Its contrapositive is $\sim q \rightarrow \sim p$, i.e.,

"If the home team does not win then it is not raining."

(b) Its converse is $q \rightarrow p$, i.e.,

"If the home team wins then it is raining."

(c) Its inverse is $\sim p \rightarrow \sim q$, i.e.

"If it is not raining then the home team does not win."

The truth table for the above statements is as follows :

p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Example 46. What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers? **[IT(RTU)-2008]**

Solution: Let P(x) : $x^2 \geq x$.

If the domain consists of all real numbers then $\forall x$ P(x) is false as P : $\left|\frac{1}{2}\right| : \frac{1}{4} \geq \frac{1}{2}$ is false. Thus the truth value is F.

If the domain consists of all integers then the quantification $\forall x$P(x) is true. Thus the truth value is T.

Example 47. Show that the statement $p \wedge q$ is a contingency where

p : John is a bachelor, q : Smith is married. **[IT(RTU)-2009]**

Solution: The truth table for the given statement $p \wedge q$ is as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Since the above table contains both T and F values in its last column (corresponds to the given statement), so the given statement is a contingency.

EXERCISE 4.1

Q.1 Define a statement and its truth values.

Q.2 What are logical connectives ? Explain the following terms

(i) Negation

(ii) Conjunction

(iii) Disjunction.

Q.3 Define conditional and biconditional statements. Explain the following terms by giving suitable example

(i) Converse

(ii) Contrapositive

(iii) Inverse.

Q.4 Explain the terms tautology and contradiction in the context of truth tables.

Q.5 Define universal and existential quantifiers.

Q.6 Write each of the following in terms of p, q, r, and logical connectives, where p : Today is Monday; q : The grass is wet ; and r : The dish ran away with the spoon.

(i) Today is Monday and the dish did not run away with the spoon.

(ii) Either the grass is wet or today is Monday

(iii) Today is not Monday and the grass is dry.

(iv) The dish ran away with the spoon, but the grass is wet.

Ans. (i) $p\wedge\sim r$ (ii) $q\vee p$

(iii) $\sim p\wedge\sim q$ (iv) $r\wedge q$

Q.7 Translate the following in English

(i) $\sim r\wedge q$ (ii) $\sim q\vee p$

(iii) $\sim(p\vee q)$ (iv) $p\vee\sim r$,

where p, q and r are the statements as defined in Q.6.

Ans. (i) The dish did not run away with the spoon and the grass is wet.

(ii) The grass is dry or today is Monday.

(iii) It is not true that today is Monday or the grass is wet.

(iv) Today is Monday or the dish did not run away with the spoon.

Q.8 Let P(x) : x is even ; Q(x) : x is a prime number ; and R(x, y) : $x + y$ is even, and the universe of discourse for x and y is the set of integers. Then, translate the following in English.

(i) $\forall x\ \exists y\ R(x, y)$

(ii) $\exists x\ \forall y\ R(x, y)$

(iii) $\sim(\exists x\ P(x))$

(iv) $\sim(\forall x\ Q(x))$.

Ans. (i) For every integer x there exists an integer y such that $x + y$ is even.

(ii) There is some x for all y such that $x + y$ is even.

(iii) It is not true that there is an x such that x is even.

(iv) It is not true that every x is a prime number.

Q.9. Show that each of the following is a tautology :

(i) $(p\wedge q)\rightarrow p$ (ii) $(p\wedge q)\rightarrow q$

(iii) $p\rightarrow(p\vee q)$ (iv) $q\rightarrow(p\vee q)$

(v) $\sim p\rightarrow(p\rightarrow q)$ (vi) $\sim(p\rightarrow q)\rightarrow p$

(vii) $(p\wedge(p\rightarrow q))\rightarrow q$ (viii) $(\sim p\wedge(p\vee q))\rightarrow q$

(ix) $(\sim q\wedge(p\rightarrow q))\rightarrow\sim p$ (x) $((p\rightarrow q)\wedge(q\rightarrow r))\rightarrow(p\rightarrow r)$.

Q.10 Let p : I am awake ; q : I work hard ; r : I dream of home. Write the following in terms of p, q, r and logical connectives.

(i) I am awake implies that I work hard.

(ii) I dream of home only if I am awake.

(iii) Working hard is sufficient for me to be awake.

(iv) Being awake is necessary for me not to dream of home.

Ans. (i) $p \to q$ (ii) $r \to p$

(iii) $q \to p$ (iv) $\sim r \to p$.

Q.11 Let p : I will study discrete structures ; q : I will go to a movie ; r : I am in a good mood. Then, translate the following in English.

(i) $((\sim p) \wedge q) \to r$ (ii) $r \to (p \vee q)$

(iii) $(\sim r) \to ((\sim q) \vee p)$ (iv) $(q \wedge (\sim p)) \leftrightarrow r$.

Ans. (i) If I don't study discrete structures and I go to a movie, then I am in a good mood.

(ii) If I am in a good mood, then I will study discrete structures or I will go to a movie.

(iii) If I am not in a good mood, then I will not go to a movie or I will study discrete structures.

(iv) I will go to a movie or I will not study discrete structures if and only if I am in a good mood.

Q.12 If $p \to q$ is false, can you determine the truth value of $(\sim(p \wedge q)) \to q$? Explain, your answer.

Ans. Yes. If $p \to q$ is false, then p is true and q is false. Thus $p \wedge q$ is false, $\sim(p \wedge q)$ is true, and $(\sim(p \wedge q)) \to q$ is false.

Q.13 State the converse of each of the following :

(i) If 2+2=4, then I am not the Queen of England.

(ii) If I am late, then I did not take the train to work.

(iii) If I have enough money, then I will buy a car and I will buy a house.

Ans. (i) If I am not the Queen of England, then 2+2=4.

(ii) If I did not take the train to work, then I an late.

(iii) If I buy a car and I buy a house, then I have enough money.

Q.14 By means of truth table, prove the following :

(i) $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$

(ii) $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$

(iii) $\sim(p \leftrightarrow q) \equiv \sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$.

Q.15 Show that the statements $\sim\exists x \forall y P(x, y)$ and $\forall x \exists y \sim P(x, y)$ have the same truth value.

Q.16 Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ have the same truth value.

Q.17 Establish the following logical equivalences, where A is a proposition not involving any quantifiers.

(i) $(\forall x P(x)) \vee A \equiv \forall x (P(x) \vee A)$

(ii) $(\exists x P(x)) \vee A \equiv \exists x (P(x) \vee A)$

Q.18 Show that $\forall x P(x) \vee \exists x Q(x)$ and $\forall x \exists y (P(x) \vee Q(y))$ are equivalent.

Q.19 The notation $\exists! x P(x)$ means "There exists a unique x such that P(x) is true."

If the universe of discourse is the set of integers, what are the truth values of the following?

(i) $\exists! x (x > 1)$ (ii) $\exists! x (x^2 = 1)$

(iii) $\exists!x(x + 3 = 2x)$ (iv) $\exists!x(x = x + 1)$

Ans. (i) False (ii) False

(iii) True (iv) False

Q.20 What are the truth values of the following statements ?

(i) $\exists!x P(x) \rightarrow \exists x P(x)$

(ii) $\forall x P(x) \rightarrow \exists!x P(x)$

(iii) $\exists!x{\sim}P(x) \rightarrow {\sim}\forall x P(x)$

Here $\exists!x P(x)$ means "There exists a unique x such that $P(x)$ is true."

Ans. (i) True

(ii) False, unless the universe of discourse is a singleton set

(iii) True

4.7 Normal Forms

We observed in the previous sections that the construction of truth tables may not be practical, even with the help of a computer if the number of variables, involved in a given statement becomes large. Therefore, we consider other method known as reduction to normal form. In this method, we use the word "product" in place of "conjunction" and "sum" in place of "disjunction".

4.7.1 Some Basic Terms Related to Normal Form

1. **Elementary Product :** A product of the variables and their negation is called an elementary product.

Example 48. Let p and q be any two atomic variables. Then p, q, $p \wedge q$, $p \wedge {\sim}q$, ${\sim}p \wedge q$, ${\sim}p \wedge {\sim}q$, ${\sim}q \wedge q$, ${\sim}p \wedge p$, $p \wedge q \wedge {\sim}q$ and ${\sim}p \wedge p \wedge q$ are elementary products.

2. **Elementary Sum :** A sum of the variables and their negation is called an elementary sum.

Example 49. If p and q are any two atomic variables, then p, q, $p \vee q$, $p \vee {\sim}q$, ${\sim}p \vee q$, ${\sim}p \vee {\sim}q$, ${\sim}q \vee q$, ${\sim}p \vee p$, $p \vee q \vee {\sim}q$ and ${\sim}p \vee p \vee q$ are elementary sums.

3. **Factor :** A factor of the given elementary sum or product, is a part of it and is itself an elementary sum or product.

Example 50. If p and q are any two atomic variables, then ${\sim}q$, p , q, $p \wedge {\sim}q$, ${\sim}q \wedge p$ and ${\sim}q \wedge q$ are the factors of ${\sim}q \wedge p \wedge q$.

4. **Minterms :** Let p and q be two propositional variables. All possible formulas which consist of product of p or its negation and product of q or its negation, but should not contain both the variable and its negation in any one of the formula are called *minterms* of p and q.

Example 51. For two variables p and q, there are $2^2 = 4$ minterms, namely, $p \wedge q$, $p \wedge {\sim}q$, ${\sim}p \wedge q$ and ${\sim}p \wedge {\sim}q$.

Remark 14. Each minterm has the truth value T for exactly one combination of the truth values of the variables p and q.

Remark 15. If there are n variables then number of minterms $= 2^n$. Out of these minterms no two are equivalent.

5. **Maxterm :** For given variables, the *maxterm* consists of sums (disjunctions) in which each variable or its negation, but not both, appears only once. Thus the maxterms are the duals of minterms.

Example 52. For two variables p and q, there are $2^2=4$ maxterms, namely, $p\vee q$, $p\vee\sim q$, $\sim p\vee q$ and $\sim p\vee\sim q$.

Remark 16. Each maxterm has the truth value F for exactly one combination of the truth values of the variables.

4.7.2 Disjunctive Normal Form (DNF)

A statement which consists of a sum of elementary products of propositional variables and is equivalent to the given compound statement, is called a *disjunctive normal form* of the given statement. This form is not unique for the given statement.

For example, the statement $p\vee(q\wedge r)$, which is already in the disjunctive normal form, is equivalent to $(p\wedge p)\vee(p\wedge q)\vee(p\wedge r)\vee(q\wedge r)$.

We have the following method to obtain disjunctive normal form of a given statement.

If the statement contains the connectives $\rightarrow$ and $\leftrightarrow$, then we obtain an equivalent formula in which $\rightarrow$ and $\leftrightarrow$ disappear. That is an equivalent formula can be obtained if $\rightarrow$ and $\leftrightarrow$ are replaced by $\wedge$, $\vee$ and $\sim$. Finally, we do manipulation to get an equivalent form which is the sum of elementary product terms.

Example 53. Obtain DNF of the statement $\sim(p\vee q)\leftrightarrow p\wedge q$.

Solution: $\sim(p\vee q)\leftrightarrow(p\wedge q)\equiv(\sim(p\vee q)\rightarrow(p\wedge q))\wedge((p\wedge q)\rightarrow\sim(p\vee q))$ [using $p\leftrightarrow q\equiv(p\rightarrow q)\wedge(q\rightarrow p)$]

$\equiv((p\vee q)\vee(p\wedge q))\wedge(\sim(p\wedge q)\vee\sim(p\vee q))$[using $p\rightarrow q\equiv\sim p\vee q$]

$\equiv[((p\vee q)\vee(p\wedge q))\wedge\sim(p\wedge q)]\vee[((p\vee q)\vee(p\wedge q))\wedge\sim(p\vee q)]$

$\equiv((p\vee q)\wedge\sim(p\wedge q))\vee((p\wedge q)\wedge\sim(p\vee q))$ [using $p\wedge\sim p\equiv$ F]

$\equiv((p\vee q)\wedge(\sim p\vee\sim q))\vee((p\wedge q)\wedge(\sim p\wedge\sim q))$

$\equiv((p\vee q)\wedge(\sim p\vee\sim q))\vee(p\wedge q\wedge\sim p\wedge\sim q)$ [using $p\vee$F $\equiv p$]

$\equiv(p\vee q)\wedge(\sim p\vee\sim q)\equiv(p\wedge(\sim p\vee\sim q))\vee(q\wedge(\sim p\vee\sim q))$

$\equiv(p\wedge\sim p)\vee(p\wedge\sim q)\vee(q\wedge\sim p)\vee(q\wedge\sim q)$

Example 54. Obtain DNF of the statement $p\wedge(p\rightarrow q)$.

Solution: $p\wedge(p\rightarrow q)\equiv p\wedge(\sim p\vee q)$

$\equiv(p\wedge\sim p)\vee(p\wedge q)$

which is the required DNF.

4.7.3 Conjunctive Normal Form (CNF)

[RTU 2010]

A statement which consists of a product of elementary sums of propositional variables and is equivalent to the given statement, is called a *conjunctive normal form* of the given statement. This form is not unique for the given statement.

The method for obtaining conjunctive normal form of a given statement is similar to that of disjunctive normal forms.

Example 55. Obtain CNF of $p\wedge(p\rightarrow q)$.

Solution:

$$p\wedge(p\rightarrow q) \equiv p\wedge(\sim p\vee q)$$
$$\equiv (p\vee(q\wedge\sim q))\wedge(\sim p\vee q)$$
$$\equiv (p\vee q)\wedge(p\vee\sim q)\wedge(\sim p\vee q)$$

which is the required CNF.

Example 56. Obtain CNF of $\sim(p\vee q) \leftrightarrow (p\wedge q)$.

Solution:

$$\sim(p\vee q)\leftrightarrow(p\wedge q) \equiv (\sim(p\vee q)\rightarrow(p\wedge q))\wedge((p\wedge q)\rightarrow\sim(p\vee q))$$
$$\equiv ((p\vee q)\vee(p\wedge q))\wedge(\sim(p\wedge q)\vee(\sim p\wedge\sim q))$$
$$\equiv ((p\vee q\vee p)\wedge(p\vee q\vee q))\wedge((\sim p\vee\sim q)\vee(\sim p\wedge\sim q))$$
$$\equiv (p\vee q\vee p)\wedge(p\vee q\vee q)\wedge(\sim p\vee\sim q\vee\sim p)\wedge(\sim p\vee\sim q\vee\sim q)$$

which is the required CNF.

4.7.4 Principal Disjunctive Normal Form (PDNF)

For a given formula, an equivalent formula consisting of disjunctions of minterms only is known as its *principal disjunctive normal form* or *sum-of-products canonical form.*

We assert that if the truth table of any statement containing only the variables p and q, is known then we can easily obtain an equivalent statement which consists of a sum (disjunction) of some of the minterms. This can be done in the following way :

For every truth value T in the truth table of the given statement, choose the minterm which also has the value T for the same combination of the truth values of p and q. The sum of these minterms will then be equivalent to the given statement and this sum denotes the principal disjunctive normal form of the given statement.

Example 57. Obtain the principal disjunctive normal forms of each of the following :

(i) $p\rightarrow q$ (ii) $p\vee q$ (iii) $\sim(p\wedge q)$.

Solution: The truth table for the given statements is shown below.

Table 4.36 Truth table for $p\rightarrow q$, $p\vee q$ and $\sim(p\wedge q)$

p	q	$p\vee q$	$p\wedge q$	$\sim(p\wedge q)$	$p\rightarrow q$
T	T	T	T	F	T
T	F	T	F	T	F
F	T	T	F	T	T
F	F	F	F	T	T

Then,

(i) $p\rightarrow q \equiv (p\wedge q)\vee(\sim p\wedge q)\vee(\sim p\wedge\sim q)$ which is the required PDNF.

(ii) $p\vee q \equiv (p\wedge q)\vee(p\wedge\sim q)\vee(\sim p\wedge q)$ which is the required PDNF.

(iii) $\sim(p\wedge q) \equiv (p\wedge\sim q)\vee(\sim p\wedge q)\vee(\sim p\wedge\sim q)$ which is the required PDNF.

4.7.4.1 Alternative Method to Obtain PDNF

In order to obtain the principal disjunctive normal form of a given statement without help of truth table, first replace the biconditionals and conditional by their equivalent formulas containing only the connectives $\wedge$, $\vee$ and $\sim$. Next, the negations are applied to the variables by using De Morgan's laws followed by distributive laws. Any elementary product which is a contradiction (identically false) such as $p\wedge\sim p$, is dropped. Minterms are obtained in the disjunctions by introducing the missing factors. Identical minterms, if appearing in the disjunctions, must be deleted.

Example 58. Obtain PDNF of $p\rightarrow((p\rightarrow q)\wedge\sim(\sim q\vee\sim p))$. **[RTU 2010]**

Solution: Since $p\rightarrow q \equiv \sim p\vee q$

$\therefore$

$$p\rightarrow((p\rightarrow q)\wedge\sim(\sim q\vee\sim p))$$

$$\equiv \sim p\vee((\sim p\vee q)\wedge(q\wedge p))$$

$$\equiv \sim p\vee(\sim p\wedge(q\wedge p))\vee(q\wedge(q\wedge p)) \quad \text{[by distributivity]}$$

$$\equiv \sim p\vee(q\wedge p) \quad (\because \sim p\wedge p \text{ is identically false})]$$

$$\equiv (\sim p\wedge(q\vee\sim q))\vee(q\wedge p) \quad (\text{introducing } q\vee\sim q)$$

$$\equiv (\sim p\wedge q)\vee(\sim p\wedge\sim q)\vee(q\wedge p) \text{ which is the required PDNF.}$$

4.7.5 Principal Conjunctive Normal Form (PCNF)

For a given statement, an equivalent statement consisting of conjunctions of the maxterms only is known as its *principal conjunctive normal form* or *product-of-sums canonical form. The* method for obtaining the principal conjunctive normal form of a given statement using truth table is as follows:

For every truth value F in the truth table of the given statement, select the maxterm which also has the value F for the same combination of the truth values of the variables involved in the statement. The product (conjunction) of these maxterms will then be equivalent to the given formula and is also the required principal conjunctive normal form for the given statement.

Example 59. The truth table for a statement S is given in the table shown below. Determine its principal conjunctive normal form.

Table 4.37 Truth table for S

p	q	r	S
T	T	T	$\boxed{\text{F}}$
T	T	F	$\boxed{\text{F}}$
T	F	T	T
F	T	T	T
T	F	F	$\boxed{\text{F}}$
F	T	F	T
F	F	T	$\boxed{\text{F}}$
F	F	F	T

Solution: From the table, the maxterms correspond to the F values of S are

$(\sim p\vee\sim q\vee\sim r)$; $(\sim p\vee\sim q\vee r)$; $(\sim p\vee q\vee r)$; and $(p\vee q\vee\sim r)$

The maxterms are written down by including the variable if its truth value is F and its negation if the value is T.

Hence, the required PCNF of S is

$$\equiv(\sim p\vee\sim q\vee\sim r)\wedge(\sim p\vee\sim q\vee r)\wedge(\sim p\vee q\vee r)\wedge(p\vee q\vee\sim r)$$

4.7.5.1 Alternative Method To Obtain PCNF

The method for finding the principal conjunctive normal form of a given statement without using truth table, is similar to the one described previously for the principal disjunctive normal form.

Example 60. Obtain PCNF of the statement S given by $(\sim p\rightarrow r)\wedge(q\leftrightarrow p)$.

[IT(RTU) 2009]

Solution:
$$\begin{aligned}(\sim p\rightarrow r)\wedge(q\leftrightarrow p) &\equiv (p\vee r)\wedge((q\rightarrow p)\wedge(p\rightarrow q))\\
&\equiv (p\vee r)\wedge((\sim q\vee p)\wedge(\sim p\vee q))\\
&\equiv ((p\vee r)\vee(q\wedge\sim q))\wedge((\sim q\vee p)\vee(r\wedge\sim r))\wedge((\sim p\vee q)\vee(r\wedge\sim r))\\
&\equiv (p\vee q\vee r)\wedge(p\vee\sim q\vee r)\wedge(p\vee\sim q\vee r)\wedge(p\vee\sim q\vee\sim r)\wedge(\sim p\vee q\vee r)\wedge(\sim p\vee q\vee\sim r)\\
&\equiv (p\vee q\vee r)\wedge(p\vee\sim q\vee r)\wedge(p\vee\sim q\vee\sim r)\wedge(\sim p\vee q\vee r)\wedge(\sim p\vee q\vee\sim r)\end{aligned}$$

which is the required PCNF. $\vee\wedge$

4.7.6 To Obtain PCNF From PDNF and Vice-Versa

If the principal disjunctive (or conjunctive) normal form of a given statement S, containing n variables, is known then the principal disjunctive (or conjunctive) normal form of ~S will consist of the disjunction (or conjunction) of the remaining minterms (or maxterms) which are not present in the principal disjunctive (or conjunctive) normal form of S. Since $S\equiv\sim(\sim S)$, so we can obtain the principal conjunctive (or disjunctive) normal form of S by applying De Morgan's laws to the principal disjunctive (or conjunctive) normal form of ~S.

Example 61. Find PCNF of a statement S whose PDNF is

$(p\wedge q\wedge r)\vee(p\wedge q\wedge\sim r)\vee(\sim p\wedge q\wedge r)\vee(p\wedge\sim q\wedge r)$.

Solution: First we obtain the principal disjunctive normal form of ~S, which is the sum (disjunction) of those minterms which are not present in the given PDNF of S. Hence the PDNF of ~S is

$(\sim p\wedge\sim q\wedge\sim r)\vee(\sim p\wedge\sim q\wedge r)\vee(p\wedge\sim q\wedge\sim r)\vee(\sim p\wedge q\wedge\sim r)$

Thus, the PCNF of $S\equiv\sim$[PDNF of $(\sim S)$], i.e.

$$\equiv\sim((\sim p\wedge\sim q\wedge\sim r)\vee(\sim p\wedge\sim q\wedge r)\vee(p\wedge\sim q\wedge\sim r)\vee(\sim p\wedge q\wedge\sim r))$$
$$\equiv(p\vee q\vee r)\wedge(p\vee q\vee\sim r)\wedge(\sim p\vee q\vee r)\wedge(p\vee\sim q\vee r)$$

Example 62. Find PDNF of a statement S whose PCNF is

$(p\vee q\vee r)\wedge(p\vee\sim q\vee r)\wedge(p\vee\sim q\vee\sim r)\wedge(\sim p\vee q\vee r)\wedge(\sim p\vee q\vee\sim r)$.

Solution: First we find the PCNF of ~S, which is the product (conjunction) of those maxterms which do not appear in the given PCNF of S.

Hence the PCNF of ~S is $(p\vee q\vee\sim r)\wedge(\sim p\vee\sim q\vee r)\wedge(\sim p\vee\sim q\vee\sim r)$

Thus, the PDNF of $S \equiv \sim$[PCNF of $(\sim S)$], i.e.

$$\equiv \sim((p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee \sim r))$$

$$\equiv (\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r).$$

Example 63. Write the expression

$$f(p, q, r) \equiv (p \wedge q) \vee (p \wedge r) \vee (q' \wedge r)$$

in DNF and CNF. **[CE(RTU)-2007]**

Solution: Given expression is

$$f(p, q, r) \equiv (p \wedge q) \vee (p \wedge r) \vee (q' \wedge r)$$

The expression is already in disjunctive normal form.

Conjunctive Normal form

$$(p \wedge q) \vee (p \wedge r) \vee (q' \wedge r) \equiv ([p \vee (p \wedge r)]) \wedge [q \vee (p \wedge r)]) \vee (q' \wedge r)$$

$$\equiv [(p \vee p) \wedge (p \vee r) \wedge (q \vee p) \wedge (q \vee r)] \vee (q' \wedge r)$$

$$\equiv (p \vee p \vee q') \wedge (p \vee p \vee r) \wedge (p \vee r \vee q') \wedge (p \vee r \vee r)$$

$$\wedge (q \vee p \vee q') \wedge (q \vee p \vee r) \wedge (q \vee r \vee q') \wedge (q \vee r \vee r)$$

$$\equiv (p \vee q') \wedge (p \vee r) \wedge (p \vee q' \vee r) \wedge (p \vee r) \wedge (p \vee q \vee q') \wedge (p \vee q \vee r) \wedge (q \vee r \vee q') \wedge (q \vee r)$$

$$\equiv (p \vee q') \wedge (p \vee r) \wedge (p \vee q' \vee r) \wedge (p \vee q \vee q') \wedge (p \vee q \vee r) \wedge (q \vee r \vee q') \wedge (q \vee r)$$

which is the required CNF.

EXERCISE 4.2

Q.4 Show that from

(a) $\exists x(F(x) \wedge S(x) \rightarrow \forall y(M(y) \rightarrow W(y))$

(b) $\exists y(M(y) \wedge \sim W(y))$

the conclusion $\forall x(F(x) \rightarrow \sim S(x))$ follows.

Q.5 Prove that $\exists x(P(x) \wedge Q(x)) \Rightarrow \exists x P(x) \wedge \exists x Q(x)$.

Q.8 Which of the following propositions is a tautology? **[GATE 1997]**

(a) $(p \vee q) \rightarrow p$

(b) $p \vee (q \rightarrow p)$

(c) $p \vee (p \rightarrow q)$

(d) $p \rightarrow (p \rightarrow q)$

Q.9 If the proposition $\sim p \Rightarrow q$ is true, then the truth value of the proposition $\sim p \vee (p \Rightarrow q)$, where $\sim$ is negation, '$\vee$' is inclusive or and $\Rightarrow$ is implication, is **[GATE 1995]**

(a) true

(b) multiple-valued

(c) false

(d) cannot be determined

Q.10 The proposition $p \wedge (\sim p \vee q)$ is **[GATE 1993]**

(a) A tautology

(b) A contradiction

(c) Logical equivalence to $p \wedge q$

(d) Logical equivalence to $p \vee q$

Q.11 Which of the following is/are tautology ? [GATE 1992]

(a) $a \vee b \to b \wedge c$

(b) $a \wedge b \to b \vee c$

(c) $a \vee b \to (b \to c)$

(d) $a \to b \to (b \to c)$

Q.12 Which one of the following is the most appropriate logical formula to represent the statement ?

"Gold and silver ornaments are precious".

The following notations are used :

G(x) : x is a gold ornament.

S(x) : x is a silver ornament.

P(x) : x is precious [GATE 2009]

(a) $\forall x(P(x) \to (G(x) \wedge S(x)))$

(b) $\forall x((G(x) \wedge S(x)) \to P(x))$

(c) $\exists x((G(x) \wedge S(x)) \to P(x))$

(d) $\forall x((G(x) \vee S(x)) \to P(x))$

Q.13 Which of the following tuple relational calculus expression(s) is/are equivalent to $\forall t \in r(P(t))$

I. $\neg \exists t \in r(P(t))$

II. $\exists t \notin r(P(t))$

III. $\neg \exists t \notin r(\neg P(t))$

IV. $\exists t \notin r(\neg P(t))$ [GATE 2008]

(a) I only

(b) II only

(c) III only

(d) III and IV only

Q.14 Let fsa and pda be two predicates such that fsa(x) means x is a finite state automation and pda(y) means, that y is a push down automation.

Let equivalent be another predicate such that equivalent (a,b) means a and b are equivalent. Which of the following first order logic statement represent the following:

Each finite state automation has an equivalent pushdown automation. [GATE 2008]

(a) $(\forall x \text{ fsa}(x)) \Rightarrow (\exists y \text{ pda}(y) \wedge \text{equivalent}(x,y))$

(b) $\sim\forall y(\exists x \text{ fsa}(x) \Rightarrow (\exists y \text{ pda}(y) \wedge \text{equivalent}(x,y))$

(c) $\forall x \exists y (\text{fsa}(x) \wedge \text{pda}(y) \wedge \text{equivalent}(x,y))$

(d) $\forall x\, \exists y$ (fsa(x) $\wedge$ pda(x) $\wedge$ equivalent(x,y))

Q.15 Let Graph(x) be a predicate which denotes that x is a graph. Let connected(x) be a predicate which denotes that x is connected. Which of the following first order logic sentences DOES NOT represent the statement. "Not every graph is connected"? **[GATE 2007]**

(a) $\neg\forall x(\text{Graph}(x) \Rightarrow \text{Connected}(x))$

(b) $\exists x(\text{Graph}(x) \wedge \neg\text{Connected}(x))$

(c) $\neg\forall x(\neg\text{Graph}(x) \vee \text{Connected}(x))$

(d) $\forall x(\text{Graph}(x)) \Rightarrow \neg\text{Connected}(x))$

Q.16 Which one of the first order predicate calculus statements given below correctly expresses the following English statement?

"Tigers and lions attack if they are hungry or threatened." **[GATE 2006]**

(a) $\forall x$ [(tiger(x) $\wedge$ lion(x)) $\rightarrow$ {(hungry(x) $\vee$ threatened(x)) $\rightarrow$ attacks(x)}]

(b) $\forall x$ [(tiger(x) $\vee$ lion(x)) $\rightarrow$ {(hungry(x) $\vee$ threatened(x)) $\wedge$ attacks(x)}]

(c) $\forall x$ [(tiger(x) $\vee$ lion(x)) $\rightarrow$ {(attacks(x) $\rightarrow$ (hungry(x) $\rightarrow$ threatened(x)}]

(d) $\forall x$ [(tiger(x) $\vee$ lion(x)) $\rightarrow$ {(hungry(x) $\vee$ threatened(x)) $\rightarrow$ attacks(x)}]

Q.17 Consider the following propositional statements:

P1 : $((A \wedge B) \rightarrow C) \equiv ((A \rightarrow C) \wedge (B \rightarrow C))$

P2 : $((A \vee B) \rightarrow C) \equiv ((A \rightarrow C) \vee (B \rightarrow C))$

Which of the following is true? **[GATE 2006]**

(a) P1 is tautology, but not P2

(b) P2 is a tautology, but not P1

(c) P1 and P2 are both tautologies

(d) Both P1 and P2 are not tautologies

Q.18 What is the first order predicate calculus statement equivalent to the following? 'Every teacher is liked by some student' **[GATE 2005]**

(a) $\forall(x)$ {teacher(x) $\rightarrow \exists(y)$ [student(y) $\rightarrow$ likes(y,x)]}

(b) $\forall(x)$ {teacher(x) $\rightarrow \exists(y)$ [student(y) $\wedge$ likes (y,x)]}

(c) $\exists(y)\, \forall(x)$ [teacher(x) $\rightarrow$ [student(y) $\wedge$ likes(y,x)]]

(d) $\forall(x)$ [teacher(x) $\wedge \exists(y)$ [student(y) $\rightarrow$ likes(y,x)]]

Q.19 Let P(x) and Q(x) be arbitrary predicates. Which of the following statements is always TRUE? **[GATE 2005]**

(a) $((\forall x\, P(x) \vee Q(x))) \Rightarrow ((\forall x\, P(x)) \vee (\forall x\, Q(x)))$

(b) $(\forall x\, (P(x) \Rightarrow Q(x))) \Rightarrow ((\forall x\, P(x)) \Rightarrow (\forall x\, Q(x)))$

(c) $((\forall x\, (P(x)) \Rightarrow (\forall x\, Q(x))) \Rightarrow (\forall x\, (P(x) \Rightarrow Q(x)))$

(d) $((\forall x\ (P(x)) \Leftrightarrow (\forall x\ Q(x)))) \Rightarrow (\forall x\ (P(x) \Leftrightarrow Q(x)))$

Q.20 Identify the correct translation into logical notation of the following assertion :

Some boys in the class are taller than all the girls.

Note: taller (x, y) is true if x is taller than y. **[GATE 2004]**

(a) $(\exists x)\ (\text{boy}(x) \rightarrow (\forall y)\ (\text{girl}(y) \wedge \text{taller}(x,y)))$

(b) $(\exists x)\ (\text{boy}(x) \wedge (\forall y)\ (\text{girl}(y) \wedge \text{taller}(x,y)))$

(c) $(\exists x)\ (\text{boy}(x) \rightarrow (\forall y)\ (\text{girl}(y) \rightarrow \text{taller}(x,y)))$

(d) $(\exists x)\ (\text{boy}(x) \wedge (\forall y)\ (\text{girl}(y) \wedge \text{taller}(x,y)))$

Q.21 Which of the following is a valid first order formula? (Here α and β are first order formulae with x as their only free variable) **[GATE 2003]**

(a) $((\forall x)\ [\alpha] \Rightarrow (\forall x)\ [\beta]) \Rightarrow (\forall x)\ [\alpha \Rightarrow \beta]$

(b) $(\forall x)\ [\alpha] \Rightarrow (\exists x)\ [\alpha \wedge \beta]$

(c) $(\forall x)\ [\alpha \vee \beta] \Rightarrow (\exists x)\ [\alpha]) \Rightarrow (\forall x)\ [\alpha]$

(d) $(\forall x)\ [\alpha \Rightarrow \beta] \Rightarrow ((\forall x)\ [\alpha]) \Rightarrow (\forall x)\ [\beta])$

Q.22 "If X then Y unless Z" is represented by which of the following formulas in propositional logic? ("$\neg$" is negation, "$\wedge$" is conjunction, and "$\rightarrow$" is implication) **[GATE 2002]**

(a) $(X \wedge \neg Z) \rightarrow Y$

(b) $(X \wedge Y) \rightarrow \neg Z$

(c) $X \rightarrow (Y \wedge \neg Z)$

(d) $(X \rightarrow Y) \wedge \neg Z$

Q.23 Let a, b, c, d be propositions. Assume that the equivalences $a \leftrightarrow (b \vee \neg b)$ and $b \leftrightarrow c$ hold. Then the truth value of the formula $(a \wedge b) \rightarrow ((a \wedge c) \vee d)$ is always **[GATE 2000]**

(a) True

(b) False

(c) Same as the truth value of b

(d) Same as the truth value of d

Q.24 Which of the following predicate calculus statements is/are valid ? **[GATE 1992]**

(a) $(\forall x)\ P(x) \vee (\forall x)\ Q(x) \rightarrow (\forall x)\ \{P(x) \vee Q(x)\}$

(b) $(\exists x)\ P(x) \wedge (\exists x)\ Q(x) \rightarrow (\exists x)\ \{P(x) \wedge Q(x)\}$

(c) $(\forall x)\ \{P(x) \vee Q(x)\} \rightarrow (\forall x)\ P(x) \vee (\forall x)\ Q(x)$

(d) $(\exists x)\ \{P(x) \vee Q(x)\} \rightarrow \sim(\forall x)\ P(x) \vee (\exists x)\ Q(x)$

Q.25 If F1, F2 and F3 are propositional formulae such that

F1 $\wedge$ F2 ---> F3 and F1 $\wedge$ F2 ---> $\sim$ F3 **[GATE 1991]**

are both tautologies, then which of the following is TRUE ?

(a) Both F1 and F2 are tautologies

(b) The conjunction F1 $\wedge$ F2 is not satisfiable

(c) Neither is tautologies

(d) Neither is satisfiable

Q.26 Which of the following well–formed formulas are equivalent? **[GATE 1988]**

(a) $P \to Q$

(b) $\sim P \to Q$

(c) $\sim P \lor Q$

(d) $\sim Q \to P$

Q.27 P and Q are two propositions. Which of the following logical expressions are equivalent?

I. $P \lor \sim Q$

II. $\sim(\sim P \land Q)$

III. $(P \land Q) \lor (P \land \sim Q) \lor (\sim P \land \sim Q)$

IV. $(P \land Q) \lor (P \land \sim Q) \lor (\sim P \land Q)$ **[GATE 2008]**

(a) Only I and II

(b) Only I, II and III

(c) Only I, II and IV

(d) All of I, II, III and IV

Q.28 A logical binary relation □, is defined as follows:

A	B	A □ B
True	True	True
True	False	True
False	True	False
False	False	True

Let ~ be the unary negation (NOT) operator, with higher precedence then □, Which one of the following is equivalent to $A \land B$? **[GATE 2006]**

(a) (~A □ B)

(b) ~(A □ ~B)

(c) ~(~A □ ~B)

(d) ~(~A □ B)

Q.29 Let P, Q and R be tree atomic prepositional assertions, Let X denote $(P \lor Q) \to R$ and Y denote $(P \to R) \lor (Q \to R)$. Which one of the following is a tautology? **[GATE 2005]**

(a) $X \equiv Y$

(b) $X \to Y$

(c) $Y \to X$

(d) $\neg Y \to X$

Q.30 The following prepositional statement is $(P \to (Q \lor R)) \to ((P \land Q) \to R)$ **[GATE 2004]**

(a) satisfiable but not valid

(b) valid

(c) a contradiction

(d) none of the above

Q.31 Consider the following logic program P

$$A(x) \leftarrow B(x, y), C(y)$$
$$\leftarrow B(x, x)$$

Which of the following first order sentences is equivalent to P? **[GATE 2003]**

(a) $(\forall x)\ [(\exists y)\ [B(x, y) \wedge C(y)] \Rightarrow A(x)] \wedge \neg(\exists x)\ [B(xx)]$

(b) $(\forall x)\ [(\forall y)\ [B(x, y) \wedge C(y)] \Rightarrow A(x)] \wedge \neg(\exists x)\ [B(xx)]$

(c) $(\forall x)\ [(\exists y)\ [B(x, y) \wedge C(y)] \Rightarrow A(x)] \vee \neg(\exists x)\ [B(xx)]$

(d) $(\forall x)\ [(\forall y)\ [B(x, y) \wedge C(y)] \Rightarrow A(x)] \wedge\ (\exists x)\ [B(xx)]$

Q.32 The following resolution rule is used in logic programming :

Derive clause $(P \vee Q)$ from clauses $(P \vee R)$, $(Q \vee \neg R)$

Which of the following statements related to this rule is FALSE? **[GATE 2003]**

(a) $((P \vee R) \wedge (Q \vee \neg R)) \Rightarrow (P \vee Q)$ is logically valid.

(b) $(P \vee Q) \Rightarrow ((P \vee R) \wedge (Q \vee \neg R))$ is logically valid.

(c) $(P \vee Q)$ is satisfiable if and only if $(P \vee R) \wedge (Q \vee \neg R)$ is satisfiable.

(d) $(P \vee R) \Rightarrow$ FALSE if and only if both P and Q are unsatisfiable.

Q.33 Suppose the predicate $F(x, y, t)$ is used to represent the statement that person x can fool person y at time t. which one of the statements below expresses best the meaning of the formula $\forall x \exists y \exists t(\neg F(x, y, t))$? **[GATE 2003]**

(a) Everyone can fool some person at some time

(b) No one can fool everyone all the time

(c) Everyone cannot fool some person all the time

(d) No one can fool some person at some time

Q.34 Which one of the following options is CORRECT given three positive integers x, y and z, and a predicate
$P(x) = \neg(x = \ ) \wedge \forall y(\exists z(x = y \times z) \Rightarrow (y = x) \vee (y = 1))$ **[GATE 2011]**

(a) $P(x)$ being true means that x is a prime number

(b) $P(x)$ being true means that x is a number other than 1

(c) $P(x)$ is always true irrespective of the value of x

(d) $P(x)$ being true means that x has exactly two factors other than 1 and x

Q.35 Consider the following logical inferences.

I_1 : If it rains then the cricket match will not be played.
The cricket match was played.
Inference: There was no rain.

I_2 : If it rains then the cricket match will not be played.
It did not rain.
Inference: The cricket match was played.

Which of the following is TRUE? **[GATE 2012]**

(a) Both I_1 and I_2 are correct inferences

(b) I_1 is correct but I_2 is not a correct inference

(c) I_1 is not correct but I_2 is a correct inference

(d) Both I_1 and I_2 are not correct inferences

Q.36 What is the correct translation of the following statement into mathematical logic?

"Some real numbers are rational" **[GATE 2012]**

(a) $\exists x\,(\text{real}(x) \wedge \text{rational}(x))$

(b) $\forall x\,(\text{real}(x) \rightarrow \text{rational}(x))$

(c) $\exists x\,(\text{real}(x) \wedge \text{rational}(x))$

(d) $\exists x\,(\text{rational}(x) \rightarrow \text{real}(x))$

Q.37 What is the logical translation of the following statement?

"None of my friends are perfect" **[GATE 2013]**

(a) $\exists x(F(x) \wedge \rightarrow P(x))$

(b) $\exists x(\neg F(x) \wedge \rightarrow P(x))$

(c) $\exists x(\neg F(x) \wedge \neg P(x))$

(d) $\neg \exists x(F(x) \wedge P(x))$

Q.38 Which one of the following is NOT logically equivalent to $\neg\exists x(\forall y(\alpha) \wedge \forall z(\beta))$? **[GATE 2013]**

(a) $\forall x(\exists z(\neg\beta) \rightarrow \forall y)(\alpha))$

(b) $\forall x(\forall z(\beta) \rightarrow \exists y)(\neg\alpha))$

(c) $\forall x(\forall y(\alpha) \rightarrow \exists z)(\neg\beta))$

(d) $\forall x(\exists y(\neg\alpha) \rightarrow \exists z)(\neg\beta))$

ANSWER KEY

8.	c	9.	b	10.	c	11.	b	12.	d
13.	c	14.	a	15.	b	16.	d	17.	d
18.	b	19.	a	20.	d	21.	d	22.	d
23.	a	24.	a, d	25.	b	26.	a, c	27.	b
28.	d	29.	b	30.	a	31.	c	32.	b
33.	b	34.	a	35.	b	36.	c	37.	d
38.	a, d								

Sol.8

p	q	$p \rightarrow q$	$p \vee (p \rightarrow q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

$\therefore\ p \vee (p \rightarrow q)$ is a tautology. Others are not tautology.

Sol.9 First of all, note that $p \Rightarrow q$ and $\sim p \vee q$ are logically equivalent.

Hence $\sim p \Rightarrow q \equiv \sim(\sim p) \vee q \equiv p \vee q$

$\sim p \vee (p \Rightarrow q) \equiv \sim p \vee (\sim p \vee q) \equiv \sim p \vee q$

It is given that $p \vee q$ is true.

$\Rightarrow p \vee q$ takes on the values $T \vee T$, $T \vee F$, $F \vee T$.

$\Rightarrow \sim p \vee q$ takes on the values $F \vee T$, $F \vee F$, $T \vee T$. i.e. T, F, T.

$\Rightarrow$ the truth value of $\sim p \vee q$ is multiple–valued.

Sol.10 $p \wedge (\sim p \vee q)$

$\Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$

$\Leftrightarrow F \vee (p \wedge q)$

$\Leftrightarrow p \wedge q$

Sol.11

a	b	c	$a \vee b$	$a \wedge b$	$b \vee c$	$b \wedge c$	$b \rightarrow c$	$a \rightarrow b$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	T
T	F	T	T	F	T	F	T	F
T	F	F	T	F	F	F	T	F
F	T	T	T	F	T	T	T	T
F	T	F	T	F	T	F	F	T
F	F	T	F	F	T	F	T	T
F	F	F	F	F	F	F	T	T

If follows that $a \vee b \rightarrow b \wedge c$ is not tautology, since $T \rightarrow F$ is F.

$a \wedge b \rightarrow b \vee c$ is a tautology since $F \rightarrow T$ is T and $F \rightarrow F$ is T.

$a \vee b \rightarrow (b \rightarrow c)$ is not a tautology since $T \rightarrow F$ is F.

$a \rightarrow b \rightarrow (b \rightarrow c)$ is not a tautology since $T \rightarrow F$ is F.

Sol.12 The correct translation of "Gold and silver ornaments are precious" is choice (d)

$\forall x\ ((G(x) \vee S(x)) \rightarrow P(x))$

which is read as "If an ornament is gold or silver, then it is precious". Now since a given ornament cannot be both gold and silver at the same time.

choice (b) $\forall x\ ((G(x) \wedge S(x)) \rightarrow P(x))$ is incorrect.

Sol.14 "For x which is an fsa, there exists a y which is a pda and which is equivalent to x."

$(\forall x\ \text{fsa}(x)) \Rightarrow (\exists y\ \text{pda}(y) \wedge \text{equivalent}(x,y))$ is the logical representation.

Sol.15 The statement "Not every graph is connected" is same as "There exists some graph which is not connected" which is same as $\exists x(\text{Graph}(x) \wedge \text{Connected}(x))$ which is choice (b).

Sol.16 The given statement should be read as "If an animal is a tiger or a lion, then (if the animal is hungry or threatened, then it will attack).

Therefore the correct translation is

$\forall x$ [(tiger(x) $\vee$ lion(x)) $\rightarrow$ {(hungry(x) $\vee$ threatened (x)) $\rightarrow$ attacks(x)}]

which is choice (d).

Sol.17

A	B	C	A ∨ B	A ∧ B	(A ∨ B) → C (3)	(A ∧ B) → C (1)
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	F	T
F	T	T	T	F	T	T
F	T	F	T	F	F	T
F	F	T	F	F	T	T
F	F	F	F	F	T	T

CA → C	B → C	(A → C) ∧ (B → C) (2)	(A → C) ∨ (B → C) (4)
T	T	T	T
F	F	F	F
T	T	T	T
F	T	F	T
T	T	T	T
T	F	F	T
T	T	T	T
T	T	T	T

Since column (1) and (2) are not identical so P_1 is not a tautology.

Similarly, column (3) and (4) are not identical so P_2 is not a tautology.

Sol.18 "Every teacher is liked by some student", then logical expression is (b); where likes (y,x) means y likes x : such that y represent the student and x represent the teacher.

Sol.20 The statement is "some boys in the class are taller than all the girls". So the notation for the given statement is

($\exists x$) (boy(x) $\wedge$ ($\forall y$) (girl(y) $\rightarrow$ taller(x,y))).

Sol.21 (b) and (c) are certainly false. (a) doesn't follow logically.

(d) is the only valid formula.

Sol.23 b $\vee$ ~b = T (Tautology)

a $\leftrightarrow$ (b $\vee$ ~b) $\equiv$ a $\leftrightarrow$ T

$\Rightarrow$ a is a tautology.

b $\leftrightarrow$ c

$\Rightarrow$ b and c have same truth valued.

Let $b = T$

$\Rightarrow c = T$

$(a \wedge b) \rightarrow ((a \wedge c) \vee d) \equiv (T \wedge T) \rightarrow ((T \wedge T) \vee d) \equiv T \rightarrow T \vee d$

$\Rightarrow$ Even if d is T or F, $T \rightarrow T \vee d \equiv T \rightarrow T \equiv T$.

Hence (d) is false.

Let $b = F$

$\Rightarrow c = F$

$(a \wedge b) \rightarrow ((a \wedge c) \vee d)$

$\equiv (T \wedge F) \rightarrow ((T \wedge F) \vee d) \equiv F \rightarrow (F \vee d)$

Let $d = F$

$\equiv F \rightarrow (F \vee F) \equiv F \rightarrow F \equiv T$

Let $d = T$

$\equiv F \rightarrow (F \vee T) \equiv F \rightarrow T \equiv T$

Hence (c) is false and ($\wedge$) is true.

$\therefore$ (a).

Sol.24 (a), (d)

(a) and (d) are valid and (b) and (c) are invalid. If we want to prove $p \rightarrow q$; it is enough to prove if p = T then q = T.

Sol.25 $F_1 \wedge F_2 \rightarrow F_3$ is a valid argument since it is a tautology.

$F_1 \wedge F_2 \rightarrow \sim F_3$ is also a valid argument since it is a tautology.

But this itself is a contradiction.

Hence it follows that the conjuction $F_1 \wedge F_2$ is not satisfiable.

Sol.26 Two statement patterns are said to be logically equivalent if they have the same truth values corresponding to the truth values of their components.

Hence $P \rightarrow Q$ and $\sim P \vee Q$ are logically equivalent as can be seen from the following table:

P	Q	P→Q	~P	~P∨Q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence (a) and (c) are equivalent.

Sol.27 (i) $P \vee \sim Q \equiv p + q'$

(ii) $\sim(\sim P \wedge Q) \equiv (p' q)' \equiv p + q'$

(iii) $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

$$\equiv pq + pq' + p'q'$$

$$\equiv p(q + q') + p'q' \equiv p + p'q'$$

$$\equiv (p + p')(p + q') \equiv p + q'$$

(iv) $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

$$\equiv pq + pq' + p'q \equiv p(q + q') + p'q$$

$$\equiv p + p'q \equiv (p + p')(p + q) \equiv p + q$$

Clearly (i), (ii) and (iii) are equivalent. Correct choice is (b).

Sol.28 By using min terms we can define

$$(A\ B) = AB + AB' + A'B' = A + A'B'$$

$$= (A + A') \cdot (A + B') = A + B'$$

(a) $\sim A$ □ $B = A'$ □ $B = A' + B'$

(b) $\sim (A$ □ $\sim B) = (A$ □ $B')' = (A + (B')')'$

$$= (A + B)' = A'B'$$

(c) $\sim(\sim A$ □ $\sim B) = (A'$ □ $B')' = (A' + (B')')'$

$$= (A' + B)' = AB'$$

(d) $\sim(\sim A$ □ $B) = (A'$ □ $B)' = (A' + B')'$

$$= (A' + B)' = A \cdot B = A \wedge B$$

$\therefore$ only choice (d) $\equiv A \wedge B$.

Sol.29 $X : (P \vee Q) \to R$

$Y : (P \to R) \vee (Q \to R)$

$X : P + Q \to R \equiv (P + Q)' + R \equiv P'Q' + R$

$Y : (P' + R) + (Q' + R) \equiv P' + Q' + R$

clearly $X \neq Y$

consider $X \to Y$ $\equiv (P'Q' + R) \to (P' + Q' + R) \equiv (P'Q' + R)' + P' + Q' + R$

$$\equiv (P'Q')'.R' + P' + Q' + R$$

$$\equiv (P + Q).R' + R' + Q' + R \equiv PR' + QR' + R' + Q' + R$$

$$\equiv (PR' + R) + (QR' + Q') + P'$$

$$\equiv (P + R)(R' + R) + (Q + Q') \times (R' + Q') + P'$$

$$\equiv (P + R) + (R' + Q') + P'$$

$$\equiv P + P' + R + R' + Q' \equiv 1 + 1 + Q' \equiv 1$$

$X \to Y$ is a tautology.

Sol.30 $(P \to (Q \vee R) \to ((P \wedge Q) \to R)$

$$\equiv (P \to Q+R) \to (PQ \to R) \equiv [P' \to Q+R] \to [(PQ)' \to R]$$
$$\equiv [P' + Q + R] \to [P' + Q' + R] \equiv (P' + Q + R)' + P' + Q' + R \equiv PQ'R' + P' + Q' + R$$
$$\equiv Q' + Q'PR' + P' + R \equiv Q' + P' + R \quad \text{(by absorption law)}$$

Which is a contingency (i.e. satisfiable but not valid.)

Sol.31 $p \Rightarrow q \equiv \sim p \vee q$

$B(x, x) \to [B(x, y), C(y) \to A(x)] \equiv \neg B(x, x) \vee [B(x, y) \wedge C(y) \to A(x)]$

Which is same as choice (c).

Sol.32 Derive clause $P \vee Q$ from clauses $P \vee R$, $Q \vee \neg R$ means that $(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q$

$\therefore$ (a) is true.

Since, $x \Rightarrow y$ does not imply that $y \Rightarrow x$

$\therefore$ $P \vee Q \to (P \vee R) \wedge (Q \vee \neg R)$

$\therefore$ may or may not be true. Hence (b) is not true.

Sol.33 $\because \forall x \exists y \exists t (\neg F(x, y, t))$

$\Rightarrow \neg \exists x \forall y \forall t (F(x, y, t))$

Meaning there does not exist a person who can fool everyone everytime/no one can fool everyone everytime.

Sol.34 $P(x) = \neg(x = 1) \wedge \forall y(\exists z(x = y \times z)$

$\Rightarrow (y = x) \vee (y = 1))$

$\neg(x = 1)$ means $x \neq 1$

and if x can be written as product of y and z implies either z is 1 or y is 1 so it means it has no factors other than 1 or itself.

So $P(x)$ being true means that x is a prime number.

Sol.36 "Some real numbers are rational"

or

There exist real number that are rational.

Therefore '$\exists$' quantifier is used. So, $\exists x$(real $(x) \wedge$ rational(x)) is correct one.

Sol.37 **Given:** Statement, "None of my friends are perfect".

To find: It's logical translation.

Analysis: The statement means that of all the friends that I have, none of them are perfect.

Hence, if we look at only friends and if any of them being perfect. We get logically

$\exists x(F(x) \wedge P(x))$

This means there is a x who is friend of mine and is also perfect.

Now, we need to negate the statement since none are perfect.

$\therefore$ $\neg \exists x(F(x) \wedge P(x))$

which is option (d).

In english, this statement is written as there is no x such that F(x) and P(x) is true simultaneously.

Sol.38 Consider a universe of one element.

The given statement is $!\alpha V!\beta$

Choice (A) is $\beta V!\alpha$,

Choice (B) is $!\beta V!\alpha$,

Choice (C) is $!\alpha V!\beta$,

Choice (D) is $\alpha V!\beta$

The examiner has allowed two correct answers in advertently.

Answer is (a) or (d) depending on the examinery choice.

❑❑❑