3

Functions

OBJECTIVES

- Introduction
- Functions (or Mapping)
- Domain, Co–Domain and Range of a Function
- Special Functions
- Properties of Functions
- Diagrammatic Representation of Different Kinds of Mappings
- Inclusion Mapping
- Cardinally Equivalent Sets
- Inverse of a Function (Inverse Mapping)
- Cardinality of an Infinite Set
- Countable and Uncountable Sets
- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

3.1 Introduction

The concept of a function is so fundamental that it plays a vital role in every branch of mathematics, in particular discrete mathematics. It is used to define discrete structures such as sequences and strings. Even we are using the concept of a function, knowingly or unknowingly, in our day-to-day life for example, to compute electric or water bills. In this chapter, we discussed the basic concepts involving function needed in discrete mathematics.

3.2 Functions (or Mapping)

Let A and B be two given sets. Let there exists a rule denoted by f, which associate to each element of A, a unique element of B. Then f is called a function or mapping from A to B. It is denoted by the symbol

$$f : A \rightarrow B$$

which reads f is a function from A to B or f maps A to B.

3.3 f-image

Element b ($\in$ B) corresponding to any element a ($\in$ A) will be denoted by the symbol $f(a)$ and is called the f-image of a.

3.4 f-set

The set formed by all the f-image of the elements of A is called the image–set and is denoted by $f(A)$.

3.5 Function as a Set of Ordered Pairs

A function $f : A \rightarrow B$ can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and second element is the corresponding element of B.

As such a function $f : A \rightarrow B$ can be considered as a set of ordered pairs $\{a, f(a)\}$ where $a \in A$ and $f(a) \in B$ which is the f-image of a. Hence f is a subset of A × B.

3.6 Representation by a Diagram

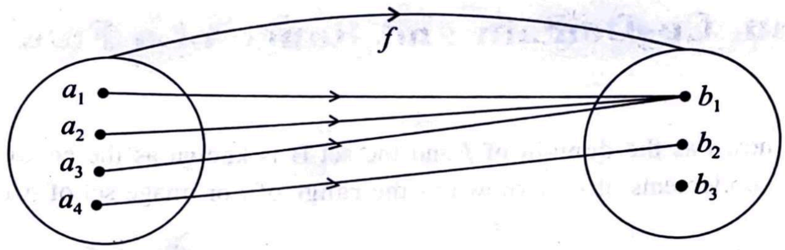
The mapping or function $f : A \rightarrow B$ is said to be well defined if

1. every element $a \in A$ has an image $f(a)$ in B,
2. an element $a \in A$ has only one image $f(a)$ in B.

But, it is possible that two or more elements $a \in A$ (say a_1, a_2, ...) may have the same image in B, i.e., $f(a_1) = f(a_2) = f(a_3) = ...$

The mapping or function $f : A \rightarrow B$ may be represented by a diagram.

Let the interior of the two areas represent the sets A and B.



The mapping or function $f : A \to B$ is represented by means of arcs joining the points representing the elements of A to the elements of B. Let $a_1, a_2, a_3, \ldots\ldots$ be the elements of A and $b_1, b_2, b_3, \ldots\ldots$ be the elements of B.

Every $a \in A$ is joined to some $b \in B$. Two or more points in A may be joined to the same point in B.

Remark 1. For mapping, two or more points of B cannot be joined to the same point of A.

Example 1. If A = {3, 4, 5}, B = {7, 8, 9}, and $f(3) = 7$, $f(4) = 8$, find out whether it defines a mapping.

Solution: Since the element $5 \in A$ does not have its image in B, it does not define a mapping.

Example 2. If A = {3, 4, 5}, B = {7, 8, 9}, $f(3) = 8$, $f(4) = 8$, $f(5) = 7$; $f(5) = 8$, find out whether it defines a mapping.

Solution: It does not define a mapping as the elements 7, 8 of B are the images of the same element 5 of A.

Remark 2.

(i) Every function is a relation but every relation is not necessarily a function.

(ii) $y = f(x)$ is analytic representation of a function and generally used in **Calculus.**

(iii) The geometrical representation of a function via $f : X \to Y$ is called mapping and generally used in **Algebra.**

(iv) f is a function iff f is a set of ordered pairs, no two of which have the same first coordinate.

Example 3. {(0, 1), (1, 1)} is a function but {(1, 0), (1, 1)} is not a function.

Example 4. {(1, 2), (2, 3), (3, 4)} is a function of {1, 2, 3} to {2, 3, 4}.

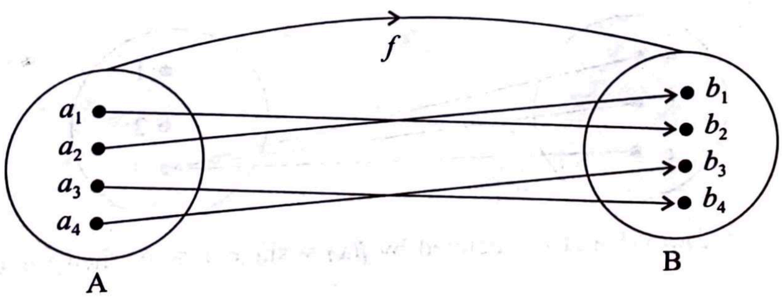
Example 5. {(1, 2), (1, 3), (3, 4)} is not a function as 1 occurs two times in the domain.

Example 6. Let $A = \{a_1, a_2, a_3, a_4\}$; $B = \{b_1, b_2, b_3, b_4\}$ be two sets and f-associates elements of A to elements of B as follows :

$$f(a_1) = b_2;\ f(a_2) = b_1;\ f(a_3) = b_4;\ f(a_4) = b_3.$$

Then f is a function (or mapping) from A to B, which can also be expressed as :

$$f = \{(a_1, b_2), (a_2, b_1), (a_3, b_4), (a_4, b_3)\}$$



3.7 Domain, Co–Domain and Range of a Function

If $f : A \to B$

then the set A is known as the **domain of** f and the set B is known as the **co–domain** of f. Moreover the set of all f–images of elements of A is known as the **range** of f or image set of A under f and is represented by $f(A)$.

Thus,

$$f(A) = \{f(a) \mid a \in A\}.$$

It should be observed that $f(A) \subset B$.

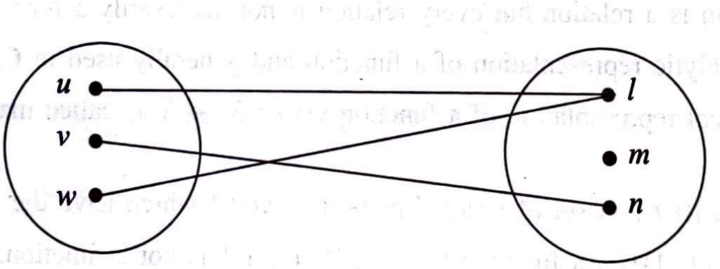
If the function is expressed as the set of ordered pairs then the domain of f will be the set of all first co–ordinates of elements of f and the range of f will be the set of the second co–ordinates of elements of f i.e.,

$$\text{domain of } f = \{x \mid (x, y) \in f\}$$

$$\text{range of } f = \{y \mid (x, y) \in f\}$$

Example 7. Let the function f be expressed as in the given diagram then

domain of $f = \{u, v, w\}$



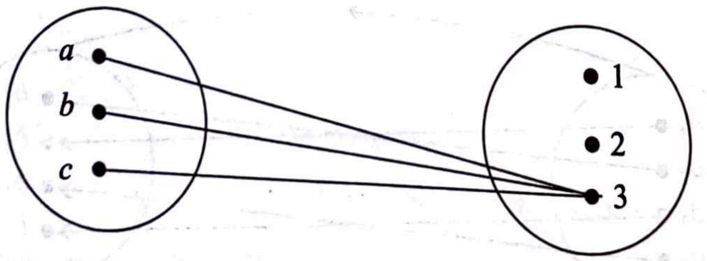
co–domain of $f = \{l, m, n\}$

range of $f = \{l, n\}$

3.8 Constant Function

The function $f : A \to B$ such that $f(a) = c;\ \forall\ a \in A$ and $c \in B$ is called a constant function.

In other words, $f : A \to B$ is a constant function if the range of f contains exactly one element. The adjoining figure, exhibits a constant mapping from a set $\{a, b, c\}$ into another set $\{1, 2, 3\}$.



Example 8. If $A = \{\pi, 2\pi, 3\pi\}$ and f is defined by $f(x) = \sin x,\ x \in A$, then f is constant [since $\sin x = 0$, $\forall\ x \in A$].

3.9 Identity Function

A function $f: A \to A$ is said to be an identity function of A if f associates every element of A to the element itself. Thus $f: A \to A$ is an identity function iff $f(x) = x, \forall x \in A$.

3.10 Equal Functions

Two functions f and g are said to be equal iff –

1. the domain of f = the domain of g
2. the co–domain of f = the co–domain of g and
3. $f(x) = g(x)$ for every x belonging to their common domain.

Example 9. Let A = {1, 2}, B = {3, 6} and

$$f: A \to B; f(x) = x^2 + 2$$

$$g: A \to B; g(x) = 3x$$

Then we find that f and g have the same domain and co–domain and also

$$\left.\begin{array}{l} f(1) = 3 = g(1) \\ f(2) = 6 = g(2) \end{array}\right\} \Rightarrow f = g$$

Hence the result.

Example 10. Let P be a computer program is which inputs and outputs are integers. The input output relation will be a function when for each input there is a unique output.

Solution: If each input, the answer be in the forms of yes or no, then it may be regarded as a function

$f: A \to (0, 1)$, defined by

$$f(a) = \begin{cases} 0, \text{ if the answer by the computer is No} \\ 1, \text{ if the answer is YES} \end{cases}$$

If a function be written symbolically as $y = f(x)$, x is called the independent variable assuming values from the set $x \in A$ = dom f and y, the dependent variable and $f(x) \in$ co–domain f. The dependent variable y is also called a function.

Example 11. Find the domains of the following functions :

(i) $f_1(x) = \dfrac{x}{x^2+1}$ (ii) $f_2(x) = \dfrac{x}{x-5}$

(iii) $f_3(x) = \sqrt{x-4}$ (iv) $f_4(x) = \log x$

(v) $f_5(x) = \dfrac{\sqrt{x+2}}{x-2}$ (vi) $f_6(x) = \sqrt{4-x^2}$

Solution: Assuming $f(x)$ to be a real number, domain will consists of those values of x for which:

(a) $f(x)$ does not tend to ∞,

(b) $f(x)$ does not become involving $\sqrt{-1}$

(c) $f(x)$ does not become indeterminate

(i) $f_1(x) = \dfrac{x}{x^2+1}$ is well defined $\forall\, x \in$ R. So dom. f_1 = set of real number R.

(ii) $f_2(x) = \dfrac{x}{x-5} \rightarrow \infty$ for $x = 5$, so domain = R – {5}

(iii) $f_3(x) = \sqrt{x-4}$ becomes imaginary when $x < 4$. So dom. = $\{4, \infty\}$.

(iv) $f_4(x) = \log x$, $\log x$ is real if x is positive $\log_a 0 \rightarrow \infty$, if $a > 1$ and $\log_a 0 \rightarrow -\infty$ if $a < 1$. So dom. = R^+

(v) $f_5(x) = \dfrac{\sqrt{x+2}}{x-2}$ becomes imaginary for $x + 2 < 0$ and tend to ∞ at $x = 2$, so domain = $(-2, \infty) - \{2\}$.

(vi) $f_6(x)$ is real if $-2 \le x \le 2$. So domain f_6 = interval $[-2, 2]$.

Example 12. Find the range of the functions :

(i) $y = f(x) = \dfrac{1}{x-5}$ (ii) $y = x^2$ (iii) $y = \dfrac{x}{1-x}$

(iv) $y = \dfrac{x}{1+x^2}$ (v) $y = \sqrt{9-x^2}$ (vi) $y = \dfrac{2+x}{2-x}$

Solution: First we should solve the given relation for x and then adopt the same criteria as we do for finding the domain

(i) From $y = \dfrac{1}{x-5}, x = \dfrac{1+5y}{y}$ when $y = 0$, $x \rightarrow \infty$. So range = $R - \{0\}$.

(ii) $y = x^2 \Rightarrow x = \pm\sqrt{y}$,, x is real when $y > 0$. So range = $(0, \infty)$

(iii) $y = \dfrac{x}{1-x} \Rightarrow x = \dfrac{y}{1+y}, x \rightarrow \infty$, for $y = -1$. So range = $R - \{-1\}$

(iv) $y = \dfrac{x}{1+x^2} \Rightarrow x = \dfrac{1 \pm \sqrt{1-4y^2}}{2y}$

x is real and finite when $-\dfrac{1}{2} \le y \le \dfrac{1}{2}$ and $y \varpi 0$. So range $f = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right] - \{0\}$.

(v) $y = \sqrt{9-x^2} \Rightarrow x = \pm\sqrt{9-y^2}$

x is real when $0 \le y \le 3$, negative values are left for y to be single valued as necessary for y to be a function, therefore, range = $[0, 3]$

(vi) $y = \dfrac{2+x}{2-x} \Rightarrow x = \dfrac{2y-2}{y+1}$, x is infinite when $y = -1$. So range = $R - \{-1\}$.

3.11 Sum and Product of Functions

Let f: A $\rightarrow$ R and g : B $\rightarrow$ R be two functions. Then their sum $f + g$ and product fg can be defined as below.

$$(f + g)(x) = f(x) + g(x) \qquad \text{[sum of } f \text{ and } g]$$

and
$$(fg)(x) = f(x).g(x) \qquad \text{[product of } f \text{ and } g]$$

Also
$$\text{dom}(f + g) = \text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g).$$

Example 13. Let $f(x) = x^2$ and $g(x) = \sqrt{x-1}$, where $dom(f) = (-\infty, \infty)$ and $\text{dom}(g) = [1, \infty)$. Then obtain the sum and product of f and g and also find their domains.

Solution:

Sum:
$$(f + g)(x) = f(x) + g(x)$$
$$= x^2 + \sqrt{x-1}$$

Product:
$$(fg)(x) = f(x).\ g(x)$$
$$= x^2 \sqrt{x-1}$$

Since dom $(f) \cap \text{dom}(g) = [1, \infty)$

So both $f(x)$ and $g(x)$ are defined only when $x \geq 1$.

Thus dom $(f + g) = dom\ (fg) = [1, \infty)$.

3.12 Properties of Functions

In this section, we will study the following kinds of mapping.

1. Into Mapping
2. Onto Mapping or Surjective Mapping
3. One–one Mapping or Injective Mapping
4. Many–one Mapping
5. Bijection Mapping

1. **Into Mapping :** ~~If the mapping f : A $\rightarrow$~~ B, is such that at least one element of B is not the f–image of any element of A, then f is called an into mapping. In that case $f(A) \subset B$ and $f(A) \neq B$.

Example 14. Let f : R $\rightarrow$ R be defined by $f(x) = x^2$, $\forall\ x \in$ R

then f is an into function, because the negative numbers do not appear in the range, i.e., no negative number is the square of a real number.

2. **Onto Mapping or Surjective Mapping :** If the mapping f : A $\rightarrow$ B is such that each element of B is the f–image of at least one element of A, then f is called onto mapping. In that case, $f(A) = B$

Example 15. Let A = {1, 2, 3, 4} and B = {1, 4, 9, 16} and let $f(x) = x^2$, $\forall\ x \in$ A, then f : A $\rightarrow$ B is an onto function as every element of B has a pre–image in A under f.

3. **One–one Mapping or Injective Mapping :** A mapping f : A $\rightarrow$ B is said to be a one–one mapping if different elements of A have different f–images in B. Thus

$f : A \to B$ is one–one $\Leftrightarrow f(a) = f(b) \Rightarrow a = b, \forall\, a, b \in A$

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$.

Example 16. Let I be the set of integers and B the set of all even integers then the mapping $f : I \to B$, defined by $f(x) = 2x$, $x \in A$ is an into mapping which is also one–one.

Example 17. Let $f : Z \to Z$, $f(x) = -x$. Then f is an injection because for any $a, b \in Z$ $a \neq b \Rightarrow -a \neq -b \Rightarrow f(a) \neq f(b)$.

4. **Many–one Mapping :** A mapping $f : A \to B$ is said to be a many one mapping if two or more elements of A have the same f–image in B i.e., $a \neq b \Rightarrow f(a) = f(b)$.

Example 18. Let $f : Z \to Z$, $f(x) = |x|$. Then f is many one mapping because for every $a \in Z$, $a \neq 0$ and $a \neq -a \Rightarrow f(a) = f(-a)$.

5. **Bijection Mapping :** A mapping $f : A \to B$ is said to be a bijection if it is one–one as well as onto mapping. Thus

   $f : A \to B$ is bijection if $f(a) = f(b) \Rightarrow a = b$ and $f(A) = B$.

Remark 5. The bijection mappings are also termed as invertible, non–singular or biuniform mappings.

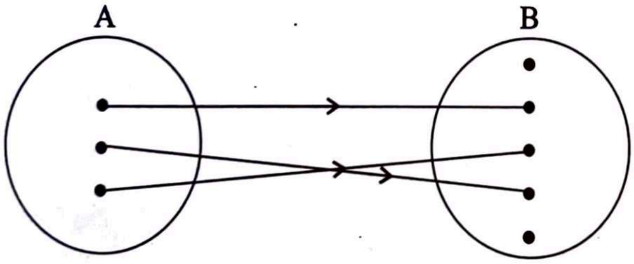
Example 19. The mapping $f : Z \to Z$, $f(x) = x + 1$ is a bijection because for any $x_1, x_2 \in Z$ $x_1 \neq x_2 \Rightarrow x_1 + 1 \neq x_2 + 1 \Rightarrow f(x_1) \neq f(x_2) \Rightarrow f$ is one–one, and $f(Z) = Z \Rightarrow f$ is onto.

Example 20. The mapping $f : Q \to Q$, $f(x) = 2x - 3$ is a bijection because for any $x_1, x_2 \in Q$.

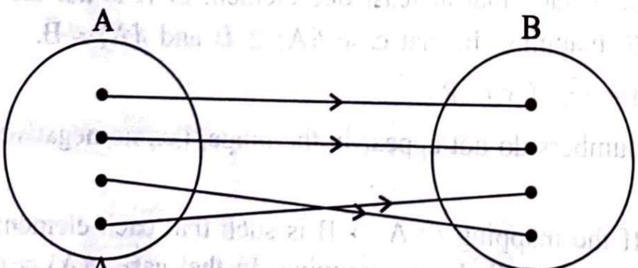
$x_1 \neq x_2 \Rightarrow 2x_1 - 3 \neq 2x_2 - 3 \Rightarrow f(x_1) \neq f(x_2) \Rightarrow$ f is one–one,

Also for each $x \in Q$ (co–domain) its pre–image $\frac{x+3}{2} \in Q$ (domain) which shows that f is onto.

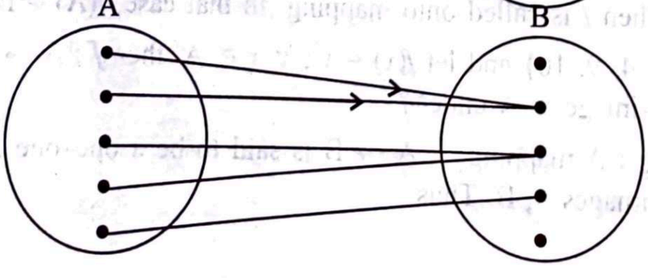
3.13 Diagrammatic Representation of Different Kinds of Mappings



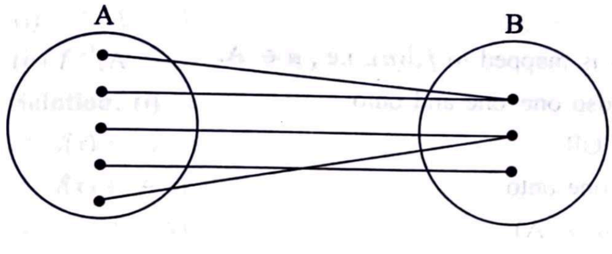
One–one into



One–one onto (bijection)



Many–one into



Many–one onto

3.14 Inclusion Mapping

If A ⊂ B, the mapping f : A → B, defined by $f(x) = x$, ∀ $x \in$ A is known as the **inclusion map.**

Example 21. Let A = {–3, –2, 1, 0, 1, 2, 3} and B = {.... 4, –3, –2, –1, 0, 1, 2, 3, 4, ...} and $f(-3) = -3, f(-2) = -2, f(-1) = -1, f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$, then it is an inclusion map of A to B because A ⊂ B and $f(x) = x$ ∀ $x \in$ A.

3.15 Cardinally Equivalent Sets

If there exists a map f : A → B which is one–one and onto, then the two sets A and B are said to be cardinally equivalent or equinumerous. It is denoted by A ~ B.

Remark 6. If any set A is equivalent to N, the set of natural numbers, then A is said to be denumerable set.

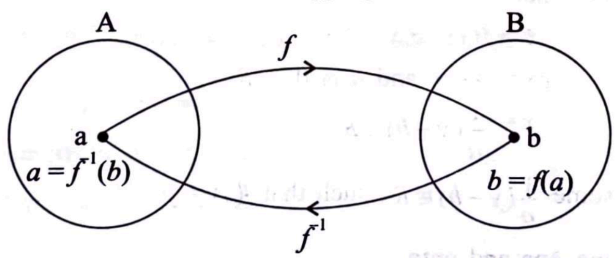
Example 22. Let A = {1, 2, 3}, B = {1, 4, 9} then f : A → B defined by $f(x) = x^2$, ∀ $x \in$ A is one–one and onto mapping therefore the set A and B are cardinally equivalent i.e., A ~ B.

Example 23. The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots.\right\}$ is denumerable set as it can be put in one–to–one correspondence with the set of natural numbers.

3.16 Inverse of a Function (Inverse Mapping)

If f : A → B be a one–one onto mapping, then the mapping f^{-1} : B → A which associates to each element $b \in$ B the unique element $a \in$ A, is called the inverse of the map f : A → B.

The following diagram will make the inverse mapping more clear.



Let f : A → B be a one–one and onto mapping. Then $a \in$ A is mapped onto $b \in$ B

∴ $b = f$–image of $a = f(a)$

Thus, $a = f^{-1}(b)$

Hence under the mapping $f^{-1} : B \to A$, $b \in B$ is mapped to $f^{-1}(a)$, i.e., $a \in A$.

Theorem 5. The inverse map $f^{-1} : B \to A$ is also one–one and onto

OR

If f is one–one onto map, then f^{-1} is also one–one onto

Proof. Let $f^{-1}(b_1) = a_1$, $(b_1 \in B, a_1 \in A)$

and $f^{-1}(b_2) = a_2$, $(b_2 \in B, a_2 \in A)$

$\therefore$ $f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow a_1 = a_2$

$\Rightarrow f(a_1) = f(a_2)$, f is one–one onto

$\Rightarrow b_1 = b_2$

Thus f^{-1} is one–one.

Again, since any element $a \in A$ is the f^{-1} image of the element $b \in B$, where $b = f(a)$, so the mapping is onto.

Hence the theorem.

Theorem 6. If $f : A \to B$ be one–one onto then the inverse map of f is unique.

Proof. Let $g : B \to A$ and $h : B \to A$ be two inverse mappings of f. We have to prove that $g = h$.

Let $b \in B$, $g(b) = a_1$ and $h(b) = a_2$. Since g and h are inverse mappings of f, therefore,

$g(b) = a_1 \Rightarrow f(a_1) = b$

and $h(b) = a_2 \Rightarrow f(a_2) = b$

But f is one–one mapping.

$\therefore$ $f(a_1) = b$

and $f(a_2) = b \Rightarrow a_1 = a_2$

$\Rightarrow$ $g(b) = h(b)$

Hence $g = h$.

Example 24. Let $f : R \to R$, defined by $f(x) = ax + b$, where $a, b, x \in R$ and $a \neq 0$. Show that f is invertible. **[RTU 2010]**

***f* is one–one :** For $x_1, x_2 \in R$, $f(x_1) = f(x_2)$

$\Rightarrow$ $ax_1 + b = ax_2 + b$

$\Rightarrow$ $ax_1 = ax_2$

$\Rightarrow$ $x_1 = x_2 \Rightarrow f$ is one–one

***f* in onto :** Let $y \in R$ such that

$y = f(x) \Rightarrow y = ax + b$

$\Rightarrow$ $ax = y - b$ and a ϖ $0 \in R$

$\Rightarrow$ $x = \frac{1}{a}(y-b) \in R$

$\therefore$ For each $y \in R$, $\exists$ some $\frac{1}{a}(y-b) \in R$, such that $f(x) = y$

$\therefore$ $f : R \to R$ is both one–one and onto.

Thus f is invertible.

Example 25. Let f be a function defined from the set X to the set Y and let A, B be the subsets of Y, then

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Solution: **(i)** Let $x \in f^{-1}(A \cup B)$

$\Rightarrow f(x) \in A \cup B$

$\Rightarrow f(x) \in A$ or $f(x) \in B$

$\Rightarrow x \in f^{-1}(A)$ or $x \in f^{-1}(B)$

$\Rightarrow x \in \{f^{-1}(A)\} \cup \{f^{-1}(B)\}$

$\therefore f^{-1}(A \cup B) \subseteq \{f^{-1}(A)\} \cup \{f^{-1}(B)\}$ .....(1)

Again let $y \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow y \in f^{-1}(A)$ or $y \in f^{-1}(B)$

$\Rightarrow f(y) \in A$ or $f(y) \in B$

$\Rightarrow f(y) \in A \cup B$

$\Rightarrow y \in f^{-1}(A \cup B)$

$\therefore f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

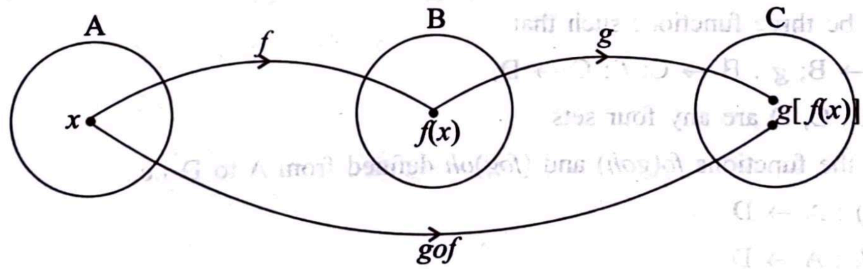
Likewise second result can be proved.

3.17 Product of Mappings or Composite of Functions

Let $f : A \to B$ and $g : B \to C$; then the composite of the function f and g denoted by gof (or gf) is a mapping

$gof : A \to C$ such that

$$(gof)\, x = g[f(x)],\ \forall\, x \in A$$



3.17.1 Properties of Composite of Functions

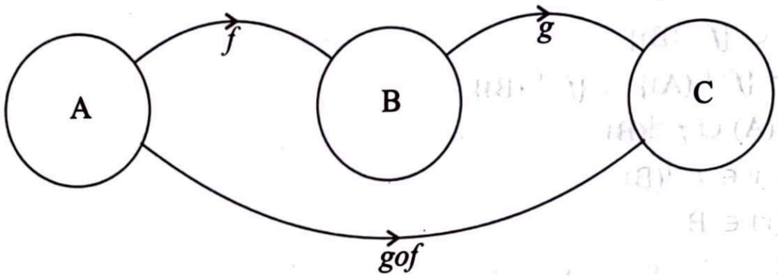
The following are the properties of composite of functions :

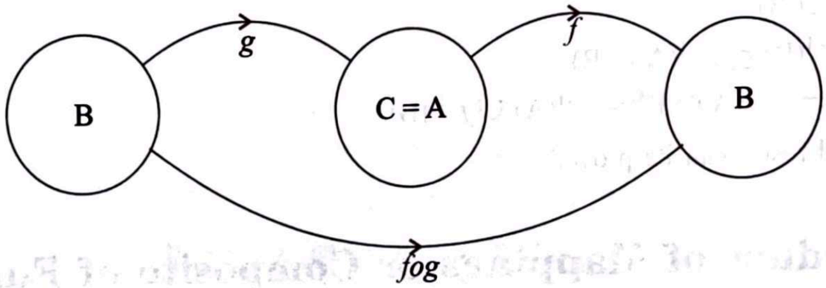
1. **The composite of functions is not commutative**
2. **The composite of functions is associative**
3. **The composite of two bijections is a bijection**

Proofs.

(i) Let $f : A \to B$ and $g : B \to C$ be two functions. The function gof exists because the co–domain of f is the same set which is the domain of g. But the function fog cannot exists unless the range of g is the same as the domain of f i.e., unless $C = A$. As such we see that fog has no meaning if $A \varpi C$ but fog will be function from B to B if A = C. Because if A = C then

$$\left.\begin{array}{l} g: B \to C \Rightarrow g: B \to A \\ g: A \to B \Rightarrow f: A \to B \end{array}\right\} \Rightarrow fog: B \to B \text{ and } gof: A \to A$$





Again even when both these functions gof and fog exist they cannot be equal if A and B are distinct sets, which are their domains. However if A = B = C, then both gof and fog exist and both are from A to A, even then they may not be equal. Hence in general the composite of functions is not necessarily commutative.

(ii) If f, g, h are three functions such that $fo(goh)$ and $(fog)oh$ are defined, then

$fo(goh) = (fog)oh$

Let f, g, h be three functions such that

$h : A \to B;\ g : B \to C;\ f : C \to D,$

where A, B, C, D are any four sets.

Then both the functions $fo(goh)$ and $(fog)oh$ defined from A to D i.e.,

$fo(goh) : A \to D$

and, $(fog)oh : A \to D$

We have to prove that

$$[fo(goh)]\, x = [(fog)oh]\,(x)\ \forall\ x \in A$$

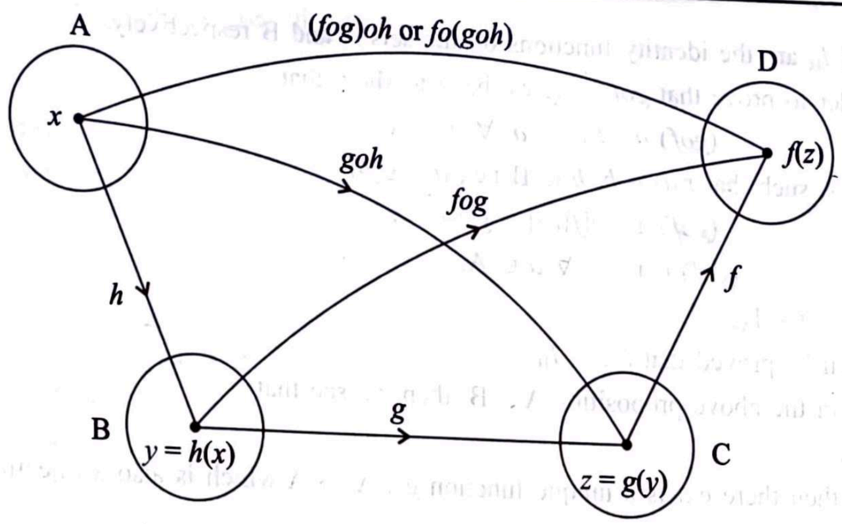
Let $x \in A$, $y \in B$, $z \in C$ such that $h(x) = y$ and $g(y) = z$.

Then $\quad [fo(goh)]\,(x) = f\,[(goh)\,(x)]$

$= f\,[g(h)(x)]$

$= f\,[g(y)] = f(z)$

$\therefore \quad [fo(goh)]\,(x) = f(z)$ .....(1)



Again $\quad [(fog)oh]\,(x) = (fog)\,[h(x)]$

$$= (fog)\,(y)$$

$$= f\,[g(y)] = f(z)$$

$\therefore \quad [(fog)oh]\,(x) = f(z)$ .....(2)

From (1) and (2), we find

$$[fo(goh)]\,(x) = [(fog)\,oh]\,(x) \qquad \forall\, x \in A$$

$\therefore \quad fo(goh) = (fog)\,oh$

(iii) If f and g are two bijections such that gof is defined then gof is also a bijection.

Let $f : A \to B$ and $g : B \to C$ be two bijections. Then gof exists such that $gof : A \to C$

We have to show that gof is one–one and onto.

One–one : let $a_1, a_2 \in A$ such that $(gof)\,a_1 = (gof)\,(a_2)$,

then $\quad (gof)\,(a_1) = (gof)\,(a_2)$

$\Rightarrow \quad g[f(a_1)] = g[f(a_2)]$

$\Rightarrow \quad f(a_1) = f(a_2)$ [∵ g is one–one]

$\Rightarrow \quad a_1 = a_2$ [∵ f is one–one]

$\therefore$ gof is also one–one function.

Onto : Let $c \in C$, then

$c \in C \Rightarrow \exists\, b \in B$ such that $g(b) = c$

and $b \in B \Rightarrow \exists\, a \in A$ such that $f(a) = b$.

Therefore, we see that

$c \in C \Rightarrow \exists\, a \in A$ such that

$$(gof)\,(a) = g[f(a)] = g(b) = c$$

i.e., every element of C is gof image of some element of A. As such gof is onto function.

Hence gof is a bijection.

Theorem 7. If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then

$$fog = I_B \text{ and } gof = I_A,$$

where I_A and I_B are the identity functions on the sets A and B respectively.

Proof. In order to prove that $gof = I_A$, we have to show that

$$(gof)\ a = I(a) = a,\ \forall\ a \in A$$

Now let $a \in A$ such that $f(a) = b,\ b \in B$ i.e., $a = g(b)$

Then $$(gof)\ a = g[f(a)] = g(b) = a$$

$\therefore$ $$(gof)\ (a) = a,\ \forall\ a \in A,$$

It proves that $gof = I_A$.

Similarly it can be proved that $fog = I_B$.

Corollary: If in the above proposition A = B, then we see that

$f : A \to A$

is a bijection, then there exists a unique function $g : A \to A$ which is also a bijection such that $fog = gof = I_A$.

Each of these bijections f and g are said to be inverse of each other.

Theorem 8. If f and g are two bijections $f : A \to B,\ g : B \to C$, then the inverse of gof exists and

$$(gof)^{-1} = f^{-1} \circ g^{-1}$$

Proof. Since $f : A \to B$ and $g : B \to C$ are two bijections, so by 2.18.1(iii)

$gof : A \to C$ is also bijection. As such gof has an inverse function $(gof)^{-1} : C \to A$. We have to prove that

$$(gof)^{-1} = f^{-1} \circ g^{-1}$$

Now let $a \in A,\ b \in B,\ c \in C$ such that $f(a) = b$ and $g(b) = c$

So $$(gof)(a) = g[f(a)] = g(b) = c,$$

Now $$f(a) = b \Rightarrow a = f^{-1}(b) \quad .....(i)$$

$$g(b) = c \Rightarrow b = g^{-1}(c) \quad .....(ii)$$

$$(gof)\ (a) = c \Rightarrow a = (gof)^{-1}(c) \quad .....(iii)$$

Also
$$\begin{aligned}(f^{-1} \circ g^{-1})\ (c) &= f^{-1}\ [g^{-1}\ (c)] && \text{[by dif.]}\\ &= f^{-1}(b) && \text{[by (ii)]}\\ &= a && \text{[by (i)]}\\ &= (gof)^{-1}\ c && \text{[by (iii)]}\end{aligned}$$

$\therefore$ $(gof)^{-1} = f^{-1} \circ g^{-1}$, which proves the theorem.

Example 26. Let f, g and h be mapping from N to N where N is the set of natural numbers so that

$$f(n) = n+1, g(n) = 2n, h(n) = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

(i) Show that f, g and h are functions.

(ii) Determine fof, fog, hog and $(fog)oh$, where o denotes composition of functions.

[RTU 2009, Raj. 2004]

Solution:

(i) To prove that given relation f is a function, it is sufficient to show that relation is defined for all values of domain and

$$f(x) \neq f(y) \in x \neq y,\ \forall\ x, y \in N.$$

***f* is a function**

Given $$f(n) = n + 1,\ n \in N$$

It is obvious that f is defined for all $n \in$ N.

Further, let $f(n_1) \neq f(n_2)$

$\Rightarrow$ $$n_1 + 1 \neq n_2 + 1 \Rightarrow n_1 \neq n_2$$

$\therefore$ f is a function

g is a function

Given $g(n) = 2n$

Since $g(n)$ exists for all $n \in$ N and also

$$g(n_1) \neq g(n_2)$$

$\Rightarrow$ $$2n_1 \neq 2n_2 \Rightarrow n_1 \neq n_2$$

$\therefore$ g is a function

h is a function

$$h(n) = \begin{cases} 0, n \text{ is even} \\ 1, n \text{ is odd} \end{cases}$$

Since, $0 \notin$ N so $h(n)$ does not exists for even n and then h is not a function. If we let that 0 includes in N then h is defined for all values of N.

Further, $h(n_1) \neq h(n_2)$

$\Rightarrow$ n_1 is even or n_2 is even but not both $\Rightarrow$ one of them is odd and the other is even.

$\Rightarrow$ $n_1 \neq n_2$

$\therefore$ h is a function

(ii) ***fof***

$fof(n) = f[f(n)] = f[n + 1] = (n + 1) + 1 = n + 2$

fog

$fog(n) = f[g(n)] = f[2n] = 2n + 1$

hog

$hog(n) = h[g(n)] = h[2n] = 0$

(fog)oh

$[(fog)oh]\ (n) = fog[h(n)]$

$$= \begin{cases} fog(0), n \text{ is even} \\ fog(1), n \text{ is odd} \end{cases}$$

$$= \begin{cases} f[g(0)], n \text{ is even} \\ f[g(1)], n \text{ is odd} \end{cases}$$

$$= \begin{cases} f(0), n \text{ is even} \\ f(2), n \text{ is odd} \end{cases}$$

$$= \begin{cases} 1, n \text{ is even} \\ 3, n \text{ is odd} \end{cases}$$

Example 27. Determine whether each of the following functions is a bijection (one–to–one and onto) from R to R

(i) $f(x) = -3x + 4$ (ii) $f(x) = -3x^2 + 7$

[RTU 2011]

(iii) $f(x)=\dfrac{x+1}{x+2}$ (iv) $f(x) = x^5 + 1$. [Raj. 2004]

Solution:

(i) $f(x) = -3x + 4$

One–one : Let $f(x_1) = f(x_2)$

$\Rightarrow -3x_1 + 4 = -3x_2 + 4 \Rightarrow -3x_1 = -3x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one–one.

Onto : Let $y = f(x) = -3x + 4 \Rightarrow x=\dfrac{4-y}{3}$

So for each $y \in$ R (co-domain) $\exists\, x=\dfrac{4-y}{3} \in$ R (Domain) such

that $f(x)=-3x+4=-3\left(\dfrac{4-y}{3}\right)+4=y$

Thus each element of co-domain has its pre–image in domain.

$\therefore$ f is onto

Hence f is a bijection.

(ii) $f(x) = -3x^2 + 7$

One–one : Since $1 \neq -1$ but $f(1) = 4$ and $f(-1) = 4$

$\Rightarrow$ two elements of domain have the same f–image

$\therefore$ f is not one–one.

Hence f is not a bijection.

(iii) $f(x)=\dfrac{x+1}{x+2}$

Since for $x = -2$, $f(x)$ tends to ∞, i.e, $f(x)$ is not defined for $x = -2$.

$\therefore$ f is not even a function.

Hence f cannot be a bijection.

(iv) $f(x) = x^5 + 1$

One–one : Let $f(x_1) = f(x_2)$

$\Rightarrow x_1^5 + 1 = x_2^5 + 1 \Rightarrow x_1^5 = x_2^5 \Rightarrow x_1 = x_2$

$\therefore$ f is one–one.

Onto : Let $y = x^5 + 1 \Rightarrow x = (y-1)^{1/5} \in R, \forall\, y \in R$

So for each $y \in R$ (co-domain) $\exists\, x = (y-1)^{1/5} \in R$ (domain) such that

$f(x) = x^5 + 1 = \{(y-1)^{1/5}\}^5 + 1 = y - 1 + 1 = y$

Thus each element of co-domain has its pre–image in domain.

$\therefore$ f is onto.

Hence, f is a bijection.

3.18 Recursive Function

If the definition of a function refers to itself then the function is said to be defined recursively, for example $\lfloor(n) = n\lfloor(n-1)$

If the recursively defined function has the following property :

(a) the value of the function at zero (base value) is specific.

(b) Each time the function does refer to itself, the argument of the function must be closure to a base value, then the recursive function is said to be well defined.

For example : (i) The factorial function n!

$$n!\begin{cases} = 1, \text{ if } n = 0 \\ = n(n-1)!, \text{ if } n > 0 \end{cases}$$

In both the property 0 is the base value and the function for other arbitrary value n is defined interms of values close to n.

i.e., 3! = 3(2)! = 3.2.1!

(ii) F_0, F_1, F_2 denotes the Fibonacci sequence such that

$F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n > 1$.

This function is also recursively well defined. Here the base values are 0 and 1. F_n is defined is terms of recurrence of F_{n-1} and F_{n-2}. F_n is defined in terms of smaller values of n which are closer to the base value.

3.19 Cardinality of an Infinite Set

We know that the cardinality of a finite set is the number of elements in the set. Now we extend this concept of cardinality to infinite sets by the following definition:

"Two set A and B have the same cardinality if and only if there is a one-to-one correspondence (bijection) from A to B."

The above definition can be used to split infinite sets into two disjoint classes, namely:

1. Infinite sets with the same cardinality as the set of natural numbers and
2. Infinite sets with different cardinality from that of the natural numbers.

Consider the set of natural numbers N = {1,2,3,.....}. If we add an element 0 to it, we get another infinite set W = {0,1,2,3,.....}.

We claim that the two sets N and W have the same cardinality. To prove this, define a mapping f : N → W such that $f(n) = n - 1$, $n \in$ N. Clearly, f is a bijection from N onto W. Thus |N| = |W|.

We can also show that the set N and the set of integers Z also have the same cardinality. For this, define a mapping f: N → Z

such that

$$f(n) = \begin{cases} -\dfrac{n}{2}, & n \text{ is even} \\ \dfrac{n-1}{2}, & n \text{ is odd}, \quad n \in N. \end{cases}$$

Clearly,

n	1	2	3	4	5	6	7	.....
$f(n)$	0	−1	1	−2	2	−3	3	.....

Thus, it is easily seen that f is a bijection from N onto Z. Therefore, $|N| = |Z|$.

In a certain sense, we can count the elements of N as 1,2,3,4,.... but this process continues forever, to count the whole set. Thus we call N a countably infinite set, and the same term is used for any set whose cordinality equals that of N.

3.20 Countable and Uncountable Sets

A set S is said to be *countable* if it is either finite or there exists a bijection between S and the set of natural numbers N (i.e. $|S| = |N|$).

If S is infinite and countable, then S is called *countably infinite*.

A set that is not countable is called *uncountable*.

Remark 7. If A and B are finite sets such that $A \subset B$ then $|A| < |B|$. This property need not be true in the case of infinite sets, for example, if E be the set of even positive integers then $E \subset N$ and also $|E| = |N|$.

Example 28. Show that the set of even positive integers is a countable set. **[RTU 2011]**

Solution: Let E be the set of even positive integers and N be the set of natural numbers.

Define a function $f: N \to E$

such that $f(n) = 2n$, $n \in N$.

Now, we will show that f is a bijection.

One-one: Let $f(n_1) = f(n_2)$

$\Rightarrow$
$$2n_1 = 2n_2 \Rightarrow n_1 = n_2$$

Thus f is one-one.

Onto: Let $b \in E$ then $\frac{b}{2} \in N$ ($\because$ b is an even positive integer) so for each $b \in E$ $\exists$ $\frac{b}{2} \in N$ such that

$$f\left(\frac{b}{2}\right) = 2\left(\frac{b}{2}\right) = b$$

Thus each element of E has its preimage in N. Therefore, f is onto.

Hence f is a bijection between N and E. Consequently, $|N| = |E|$.

So E is a countable set.

$$\begin{array}{cccccccc}
N \rightarrow & 1 & 2 & 3 & 4 & \cdots & n & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\
E \rightarrow & 2 & 4 & 6 & 8 & \cdots & 2n & \cdots
\end{array}$$

Figure (Bijection between N and E)

Example 29. Show that the set of odd positive integers is a countable set.

Solution: Let T be the set of odd positive integers and N be the set of natural numbers.

Consider a function

$$f: N \to T$$

such that
$$f(n) = 2n - 1,\ n \in N$$

$$\begin{array}{ccccccc} N \rightarrow 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ E \rightarrow 1 & 3 & 5 & 7 & \cdots & 2n-1 & \cdots \end{array}$$

Figure (Bijection between N and T)

We need to show that f is bijection.

One-one: Let $f(n_1) = f(n_2)$

$\Rightarrow$ $2n_1 - 1 = 2n_2 - 1 \Rightarrow 2n_1 = 2n_2$

$\Rightarrow$ $n_1 = n_2$

$\therefore$ f is one-one.

Onto: Let $t \in$ T, then t is an odd positive integer

$\Rightarrow$ $t + 1$ is an even positive integer

$\Rightarrow$ $\left(\frac{t+1}{2}\right)$ is a positive integer or $\left(\frac{t+1}{2}\right) \in N$

Now for each t $\in$ T $\exists$ $\left(\frac{t+1}{2}\right) \in N$ such that

$$f\left(\frac{t+1}{2}\right) = 2\left(\frac{t+1}{2}\right) - 1 = (t+1) - 1 = t$$

So each element of T has its preimage in N. Thus f is onto.

Hence f is a bijection between N and T and then |N| = |T|.

Therefore, T is countable.

Remark 8. An infinite set is countable if and only if it is possible to list the elements of the set in a sequence $a_1, a_2, a_3, \ldots a_n, \ldots$, where $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$.

Example 30. Show that the open interval (0, 1) is uncountable.

Solution: We shall prove it by contradiction method. Let the interval (0, 1) is countable. Then every real number between 0 and 1 can be listed as $a_1, a_2, a_3, \ldots$. Let the decimal representation of these real numbers be

$$\begin{aligned} a_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\ a_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\ a_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\ a_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{aligned}$$

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

[For e.g., if $a_1 = 0.25694102\ldots$, then $d_{11} = 2$, $d_{12} = 5$, $d_{13} = 6$, and so on.]

Now form a new real number $b = 0.b_1b_2b_3b_4\ldots$ as follows:

$$b_i = \begin{cases} 5, & \text{if } d_{ii} = 4 \\ 4, & \text{if } d_{ii} \neq 4 \end{cases}.$$

(For eg. if $a_1 = 0.25694102...$ and $a_2 = 0.44590137....$, $a_3 = 0.29118764....$, and so on. Then $b = 0.b_1b_2b_3b_4 .... = 0.454....$, where $b_1 = 4$ since $d_{11} \neq 4$, $b_2 = 5$ since $d_{22} = 4$, $d_3 = 4$ since $d_{33} \neq 4$, and so on.)

Clearly $0 < b < 1$ so b must be one of the number in the above list $a_1, a_2, a_3, a_4,...$

We observe that, $b_i \neq d_{ii}$ for every i (since if $d_{ii} = 4$ then $b_i = 5$ and if $d_{ii} \neq 4$ then $b_i = 4$).

So b cannot be in the list $a_1, a_2, a_3,....$

($\because$ $b_1 \neq d_{11} \Rightarrow b \neq a_1$; $b_2 \neq d_{22} \Rightarrow b \neq a_2$; $b_3 \neq d_{33} \Rightarrow b \neq a_3$ and so on)

This leads to a contradiction.

Thus our assumption that every real number between 0 and 1 can be listed, was wrong and hence the interval (0, 1) is uncountable.

Remark 9. Every subset of a countable set is also countable.

Example 31. Show that the set of real numbers is an uncountable set.

Solution : Suppose that the set R of real numbers is countable. Then every subset of R must be countable. In particular, the subset (0, 1) of R must be countable, but we have shown in the previous example (41), that (0, 1) is uncountable, and hence R cannot be countable.

Thus our assumption was wrong and hence R is uncountable.

3.21 The Pigeonhole Principle

Introduction : Suppose that a flock of pigeons flies into a set of pigeonholes to roost. The **pigeonhole principle** states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it. Of course, this principle applies to other objects besides pigeons and pigeonholes.

Theorem 9. The Pigeonhole Principle

Statement : If $R + 1$ or more objects are placed into R boxes, then there is atleast one box containing two or more of the objects.

Proof : Let none of the R boxes contains more than one object. Then the total number of objects would be at most R. This is a contradiction, since there are at least $R + 1$ objects. Thus there is atleast one box containing two or more of the objects.

Remark 10. A well–known proof technique in mathematics is the so called pigeonhole principle, also known as the **shoe box argument or Dirichlet drawer principle.**

Example 32. In any group of 27 English words, there must be at least two begin with the same letter, since there are 26 letters in the English alphabet.

Example 33. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if exam is graded on a scale from 0 to 100 points.

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Example 34. Prove that if any six numbers from the set [1, 2, 3, 4, 5, 6, 7, 8, 9] are chosen, then two of them will add to 10.

Solution: From the set we form different sets containing two numbers that add up to 10 as follows:

$S_1 = \{1, 9\}$, $S_2 = \{2, 8\}$, $S_3 = \{3, 7\}$, and $S_4 = \{4, 6\}$ and we left with a singleton set $S_5 = \{5\}$.

So, there are four such sets consisting of 8 numbers and one number 5 is left unused then 5 numbers are selected from S_1 to S_5, and one number must be selected from S_1 to S_4, therefore, two numbers will be chosen from any one of S_1 to S_4, thus the sum of there will be 10.

Example 35. Prove that if we allot 26 rooms to the students in a P.G. hostel from the room numbered between 1 and 50 inclusive, at least two are consecutively numbered.

Solution: Let $R_1, R_2, \ldots, R_i, \ldots, R_{26}$ .....(1)

be the chosen room numbers from the room numbers [1, 2, 3, ..., 50] where R_i indicate the i^{th} room number chosen from the room numbers [1, 2, 3, ..., 50].

The room numbers

$R_1 + 1, R_2 + 1, \ldots, R_{26} + 1$ .....(2)

together with room numbers of (1) which are 52 in numbers lie in the range [1, 2, 3, ..., 50, 51].

Hence 52 rooms are selected from the room numbers 1 to 51, therefore, by the pigeonhole principle at least one of the room number from (1) coincides with room number from (2) [since (1) are distinct and so also (2) are distinct]. Thus, for some i and j, we have

$$R_i = R_j + 1, i, j = 1, 2, \ldots, 26$$

and room number R_i follows R_j i.e., R_i and R_j are consecutively numbered.

Example 36. In a result list of 60 students, each marked "Pass" or "Fail". There are 35 students pass. Prove that there are at least two students pass in the list exactly nine students apart. [Considering that students at numbered at 2 and 11 or at numbered 50 and 59 satisfy the condition).

Solution: Let P_i denote the position of pass student at i^{th} numbered. We must prove that $P_i = P_j + 9$ for some i and j. Consider the positions,

$P_1, P_2, \ldots, P_{35}$ among 60 students position .....(1)

and $P_1 + 9, P_2 + 9, \ldots, P_{35} + 9$ .....(2)

The 70 numbers in (1) and (2) have possible values only from 1 to 60. Since (1) are distinct numbers so also (2). So by pigeonhole principle, two of the numbers from the sequence (1) and (2) [one from each] must coincide.

Thus, $P_i = P_j + 9$ for same i and j. Hence the result.

3.22 The Generalized Pigeonhole Principle

Theorem 10. Let n–pigeons are assigned R pigeonholes, then one of the pigeonholes must contain at least $\left\lfloor \frac{n-1}{R} \right\rfloor + 1$ pigeons, where $\left\lfloor \frac{n-1}{R} \right\rfloor$ is the floor of $\frac{n-1}{R}$ i.e., the greatest integer less than or equal to $\frac{n-1}{R}$.

Proof. We have n–pigeons and R–pigeonholes such that $n > R$. Assuming that each of R pigeonhole contains not more than $\left\lfloor \frac{n-1}{R} \right\rfloor$ pigeons, then total number of pigeons in the R pigeonholes must be less than or equal to

$\left\lceil \frac{n-1}{R} \right\rceil + 1$ pigeons, where $\left\lceil \frac{n-1}{R} \right\rceil$ is the floor of $\frac{n-1}{R}$.

$$R \times \left\lfloor \frac{n-1}{R} \right\rfloor \leq R \times \frac{n-1}{R} = n-1$$

but there are n–pigeons, so this contradicts our assumption that a pigeonhole contains not more than $\left\lfloor \frac{n-1}{R} \right\rfloor$ pigeons. So one of the pigeonholes must contain at least $\left\lfloor \frac{n-1}{R} \right\rfloor + 1$ pigeons.

Example 37. Let there are 5 separate departments in a departmental store and the total number of employee are 36. Show that one of the departments must have at least 8 employee.

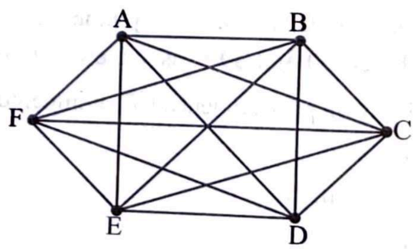
Solution: Let 36 employees are pigeons and 5 departments as pigeonholes, then according to the pigeonhole principle's, one of the department will have at least

$$\left[\frac{36-1}{5}\right] + 1 \text{ employees}$$

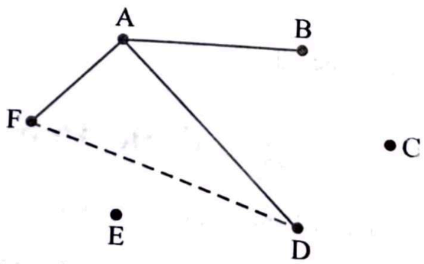
$$\Rightarrow \left[\frac{35}{5}\right] + 1 = 7 + 1 = 8.$$

Example 38. Let every pair of vertices of a hexagon is joined by a line segment, which is colored red or yellow. Prove that the line segments form at least one monochromatic triangle, i.e., a triangle with all its sides having the same color.

Solution: Consider a hexagon with vertices A, B, C, D, E and F.



Clearly, five line segments (pigeons) emanate from each vertex. Without loss of generality, consider the line segments at A.



Since there are exactly two colors (pigeonholes), so by the generalized pigeonhole principle, at least $\left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3$ of the line segments at A must have the same color, say, yellow. Let they are AB, AD and AF.

Case 1: Assume DF is colored yellow. Then ΔADF is monochromatic.

Case 2: Assume DF is not yellow. Then it is red. Now if BD is yellow, then ΔABD is monochromatic. If BD is not yellow, then consider BF. If BF is yellow, then ΔABF is monochromatic. If BF is red, then ΔBDF is a red triangle.

Hence the line segments form atleast one monochromatic triangle.

Example 39. If $(n+1)$ integers are selected from the set $\{1, 2, \ldots, 2n\}$, then show that one of them divides another integer that has been selected. **[RTU 2011]**

Solution: Let $x_1, x_2, \ldots, x_{n+1}$ are the integers selected from the set.

Now express each x_i, $1 \le i \le n+1$, as a product of a power of 2 and an odd integer, that is,

$$x_i = 2^{p_i} b_i$$

where $p_i \ge 0$ and b_i $(i = 1, 2, \ldots, n+1)$ are odd positive integers $\le 2n$.

Since there are exactly $\frac{2n}{2} = n$ odd positive integers $\le 2n$, so by the pigeonhole principle, two of the elements $b_1, b_2, \ldots, b_{n+1}$ must be equal, say $b_i = b_j$.

Then
$$x_j = 2^{p_j} b_j = 2^{p_j} b_i$$

Now
$$\frac{x_i}{x_j} = \frac{2^{p_i} b_i}{2^{p_j} b_i} = 2^{(p_i - p_j)}$$

or
$$x_i = 2^{p_i - p_j} x_j$$

If $p_i > p_j$, then clearly $x_j \mid x_i$ and if $p_i < p_j$ then $x_i \mid x_j$.

Hence, one of the selected integers must divides the another integer that has been selected.

Example 40. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Solution: The sequence 8,11,9,1,4,6,12,10,5,7 contains 10 terms $(10 = 3^2+1)$. There are four increasing subsequences of length 4, namely, 1,4,6,12 ; 1,4,6,7 ; 1,4,6,10 and 1,4,5,7. There is also a decreasing subsequence of length 4, namely, 11,9,6,5.

Let $x_1, x_2, \ldots\ldots x_{n^2+1}$ be a sequence of n^2+1 distinct real numbers. Associate an ordered pair with each term of the sequence, namely, associate (i_k, d_k) to the term x_k, where i_k is the length of the longest increasing subsequence starting at x_k and d_k is the length of the longest decreasing subsequence starting at x_k.

We shall solve this problem by contradiction method.

Suppose that there are no increasing or decreasing subsequences of length $n+1$. Then i_k and d_k are both positive integers less than or equal to n, for $k = 1, \ldots, n^2+1$. Hence, by the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these n^2+1 ordered pairs are equal. In other words, there exist terms x_s and x_t, $s < t$. Such that $i_s = i_t$ and $d_s = d_t$. We will show that this is impossible. Since the terms of the sequence are distinct, therefore either $x_s < x_t$ or $x_s > x_t$. If $x_s < x_t$, then, since $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at x_s, by taking x_s followed by an increasing subsequence of length i_t beginning at x_t. This is a contradiction to our assumption $i_s = i_t$. Similarly, if $x_s >$

x_r, it can be shown that d_s must be greater than d_r, which is a contradiction.

Hence there must exist a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Example 41. Assume that in a group of 6 people, each pair of individuals consists of two friends or two enemies (or strangers). Show that there are either 3 mutual friends or 3 mutual enemies (or strangers) in the group.

Solution: Let A be one of the 6 people. Of the 5 other people in the group, there are either 3 or more who are friends of A, or 3 or more who are enemies (or strangers) of A. This follows from the generalized pigeonhole principle, since when 5 objects are divided into two sets, one of the sets has at least $\left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3$ elements. In the former case, suppose that B, C and D are friends of A. If any two of these 3 individuals are friends, then these 2 and A form a group of 3 mutual friends. Otherwise, B, C and D form a set of 3 mutual enemies (or strangers).

The proof in the latter case, when there are 3 or more enemies (or strangers) of A, can be established in a similar manner.

Thus in the group, there are either 3 mutual friends or 3 mutual enemies (or strangers).

Example 42. An inventory consists of a list of 80 items, each marked 'available' or 'unavailable'. There are 45 available items. Show that there are at least two available items in the list exactly nine items apart.

Solution: Let e_i denotes the position of the i^{th} available item. We need to show that $e_i - e_j = 9$ for some i and j.

Consider the numbers

$$e_1, e_2, \ldots, e_{45}, \text{ where } 1 \leq e_j \leq 80,$$

and

$$e_1 + 9, e_2 + 9, \ldots, e_{45} + 9, \text{ where } 10 \leq e_j + 9 \leq 89.$$

Since $e_1, e_2, \ldots, e_{45}, e_1 + 9, e_2 + 9, \ldots, e_{45} + 9$ are 90 numbers having possible values only from 1 to 89. Hence, by the pigeonhole principle, two of the numbers must coincide. Since no two of $e_1, e_2, \ldots, e_{45}$ or no two of $e_1 + 9, e_2 + 9, \ldots, e_{45} + 9$ are equal, thus some e_i $(1 \leq i \leq 45)$ is equal to some $e_j + 9$ $(1 \leq j \leq 45)$, i.e.,

$$e_i = e_j + 9$$

or

$$e_i - e_j = 9, \text{ for some } i \text{ and } j.$$

Example 43. During a month with 30 days a baseball team plays at least 1 game a day, but not more than 45 games altogether. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let e_j be the number of games played on or before the j^{th} day of the month. Then $e_1, e_2, \ldots, e_{30}$ is an increasing sequence of distinct positive integers, where $1 \leq e_j \leq 45$. Also, $e_1 + 14, e_2 + 14, \ldots, e_{30} + 14$ is also an increasing sequence of distinct positive integers with $15 \leq e_j + 14 \leq 59$.

Since $e_1, e_2, \ldots, e_{30}, e_1 + 14, e_2 + 14, \ldots, e_{30} + 14$ are 60 positive integers less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Since $e_1, e_2, \ldots, e_{30}$ are all distinct and $e_1 + 14, e_2 + 14, \ldots, e_{30} + 14$ are all distinct, so there must be indices i and j with $e_i = e_j + 14$.

Thus $$e_i - e_j = 14$$

$\Rightarrow$ (number of games played on or before i^{th} day) – (number of games played on or before j^{th} day) = 14

or, number of games played from $(j+1)^{th}$ day to i^{th} day = 14

Hence exactly 14 games were played from day $(j+1)$ to day i.

EXERCISE 3.1

Q.1 Evaluate each, where n is an integer

(i) $\left\lfloor n+\frac{1}{2} \right\rfloor$

(ii) $\left\lceil n+\frac{1}{2} \right\rceil$

Ans. (i) n

(ii) $n+1$

Q.2 Find the range of $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor,\ x \in R$

Ans. $R = \{-1, 0\}$.

Q.3 Find the number of positive integers ≤ 3076 and divisible by

(i) 3 or 4

(ii) 3, 5 or 6

Ans. (i) 1538

(ii) 1435.

Q.4 Compute the number of leap years after 1600 and not beyond each of the following years.

(i) 2000

(ii) 3076

Ans. (i) 97

(ii) 358.

Q.5 If today is Monday, then find the day of the week after 234 days from today.

Ans. Thursday.

Q.6 Is the function $f : \mathrm{N} \to \mathrm{N}$, $f(x) = 2x + 3$ surjective ?

Ans. No

Q.7 Are the following sets of ordered pairs functions ? If so, examine whether the mapping is surjective or injective

(a) $\{(x, y) \mid x$ is a person, y is the mother of $x\}$.

(b) $\{(a, b)\ \} \mid$ a is a person, b is an ancestor of $a\}$.

Ans. (a) Neither injective nor surjective (b) Not a function

Q.8 Find gof and fog when $f : R \to R$ and $g : R \to R$ defined by

(i) $f(x) = x^2 - 1$ and $g(x) = 1+\frac{1}{1-x}$

(ii) $f(x) = 3x + 4$ and $g(x) = \frac{1}{3}(x-4)$

Also find the value of (gof) (1).

Ans. (i) $(gof)x = \frac{3-x^2}{2-x^2}, (fog)x = \frac{3-2x}{(1-x)^2}, gof(1) = 2$

(ii) $(gof)\ x = x$, $(fog)\ (x) = x$, $(gof)\ (1) = 1$

Q.9 A chess player wants to prepare for a championship match by playing some practice game in 77 days. She wants to play at least one game a day but no more than 132 games altogether. Show that there is a period of consecutive days within which she plays exactly 21 games. **[MNIT 2003]**

Q.10 Six positive integers are selected. Show that at least two of them will have the same remainder when divided by five.

Q.11 There are six matching pairs of gloves. Show that any set of seven gloves will contain a matching pair.

Q.12 The sum of nine integers in the range 1–25 is 83. Show that one of them must be at least 10.

Q.13 The total cost of 13 refrigerator at a department store is Rs. 1,23,050. Show that one refrigerator must cost at least Rs. 9,466.

Q.14 Show that if five points are chosen inside a unit square, then the distance between at least two of them is no more than $\sqrt{2}/2$.

Q.15 If 10 points are selected inside an equilateral triangle of unit side, then at least two of them are no more that $\frac{1}{3}$ of a unit apart.

Q.16 Suppose there are three men and five women at a party. Show that if these people are lined up in a row, at least two women will be next to each other.

Q.17 Let P, Q and R be sets. Let Δ denote the symmetric difference operator defined as $P \Delta Q = (P \cup Q) - (P \cap Q)$. Using Venn diagrams, determine which of the following is/are TRUE.

I. $P \Delta (Q \cap R) = (P \Delta Q) \cap (P \Delta R)$

II. $P \cap (Q \Delta R) = (P \cap Q) \Delta (P \cap R)$

[GATE 2006]

(a) I only
(b) II only
(c) Neither I and II
(d) Both I and II

Q.18 The number of functions from an m elements set to an n element set is **[GATE 1998]**

(a) $m + n$
(b) m^n
(c) n^m
(d) $m * n$

Q.19 Let X,Y, Z be sets of sizes x,y,z respectively. Let $W = X \times Y$ and E be the set of all subsets of W. The number of function from Z to E is **[GATE 2006]**

(a) Z^{2xy}
(b) $Z \times 2^{xy}$
(c) Z^{2x+y}
(d) 2^{xyz}

Q.20 Let A and B be sets with cardinalities m and n respectively. The number of one-one mappings (injections) from A to B, when $m < n$, is : **[GATE 1993]**

(a) m^n
(b) ${}^nP_m$

(c) ${}^{m}C_{n}$ (d) ${}^{n}C_{m}$

(e) ${}^{m}P_{n}$

Q.21 Let f: B → C and g: A → B be two function let h = fog. Given that h is an onto function which one of the following is TRUE? **[GATE 2005]**

(a) f and g should both be onto functions

(b) f should be onto but g not be onto

(c) g should be onto but f not be onto

(d) g and f should both not be onto

Q.22 How many onto (or subjective) functions are there from an n-element ($n \geq 2$) set to a 2-element set? **[GATE 2012]**

(a) 2^n

(b) $2^n - 1$

(c) $2^n - 2$

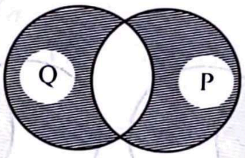
(d) $2(2^n - 2)$

Answer Key

17	d	18	c	19	d	20	b	21	b
22	c								

Sol.17 Both I and II.

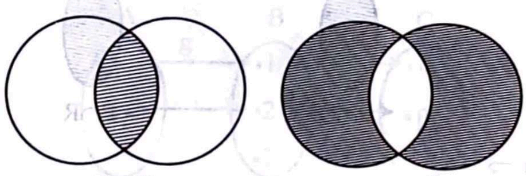
Given that PΔQ = (P ∩ Q) – (P ∩ Q)



(PΔQ)

(i) PΔ(Q ∩ Q) = (PΔQ) ∩ (PΔR)

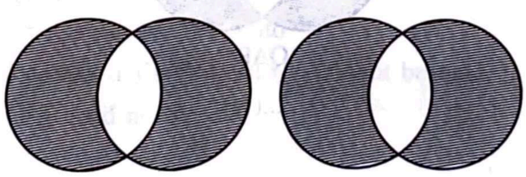
PΔ(Q ∩ R) ⇒



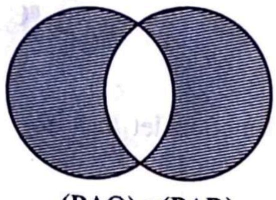
(Q∩P) PΔ(Q∩R)

.....(i)

(PΔQ) ∩ (PΔR)



PΔQ PΔR



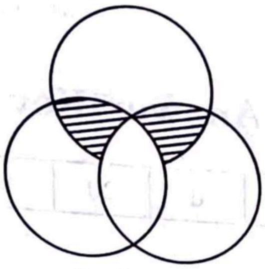
$(P \Delta Q) \cap (P \Delta R)$

(ii) is same as (i)

$\therefore$ (I) is true.

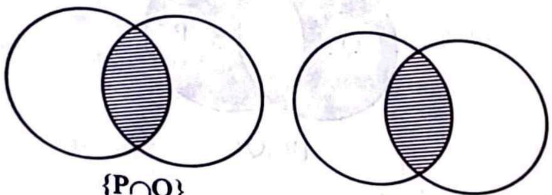
(iii) $P \cap (Q \Delta R) = (P \cap Q) \Delta (P \cap R)$

$P \cap (Q \Delta R) \Rightarrow$



$P \cap (Q \Delta R)$

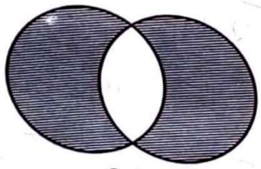
$(P \cap Q) \Delta (P \cap R)$ .....(iii)



$\{P \cap Q\}$ $P \cap R$

$P \cap Q$ $P \cap R$

$(P \cap Q) \Delta (P \cap R) \Rightarrow$



$Q \Delta R$

.....(iv)

(iv) is same as (iii)

$\therefore$ (II) is also true.

Sol.18 Any one element of the m–element set may be associated with any one element of the n–element set in exactly n ways.

$\Rightarrow$ By MP, all m elements of the first set may be associated with those of the n–element set is $n \times n \times \ldots\ldots m$ times $= n^m$ ways. Hence n^m functions are possible.

Sol.19 Given $|x| = x$, $|y| = y$ and $|z| = z$

$W = x \times y$

so $|W| = x\,y$

$|E| = 2^{|W|} = 2^{xy}$

so the number of function for $Z + E = |E|^{|Z|} = (2^{xy})^z = 2^{xyz}$

Sol.20 Any element of A may be associated with any element of B in n ways. Hence, by multiplication principle, all m elements of A may be associated in $n \times n \times n \ldots\ldots m$ times $= n^m$ ways. Hence n^m functions are possible from A to B.

Out of these, exactly ${}^nP_m$ function are injective.

There are no surjective possible from A to B.

For example,

$A = \{a, b\}$, $B = \{1, 2, 3\}$, $m = 2$, $n = 3$

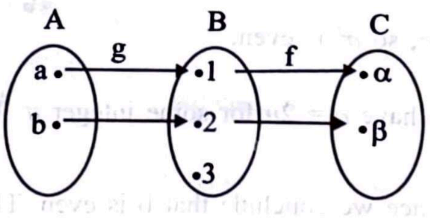
There are $3^2 = 9$ function from A to B. They are

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} a & b \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} a & b \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} a & b \\ 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} a & b \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} a & b \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} a & b \\ 3 & 1 \end{pmatrix}$$

out of these are injective, i.e. ${}^3P_2$ and no surjective.

Sol.21 Consider the arrow diagram shown below:



$h(a)$ $\quad f.g(a) = \alpha$

$h(b)$ $\quad f.\ g(b) = \beta$

Here f is onto but g is not onto, yet h is onto

As can be seen from diagram if f is not onto, h cannot be onto.

$\therefore$ f should be onto, but g need not be onto .

$\therefore$ Answer is (b).

Sol.22 Each function can be repres entered by a binary word e.g., for 3 elements, there is a function that maps

(1, 2, 3) to (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1) . . .

The number of function like this is 2^n since, for each element you make an independent choice bit 2, multiplying all these choices gives 2^n.

We only want surjective function, so we can't have (0, 0, 0) or (1, 1, 1) because range = co-domain.

We need to take these away from our sets of functions.

So our final answer is 2^n-2.

3.23 Theorem Proving Techniques

In this section we will discuss the most common techniques for proving implications or theorems.

3.24 Proof by Contradiction

This method of proof exploits the fact [derived from De Morgan's laws and $p \to q \equiv \sim p \vee q$] that $p \to q$ is true iff $p \wedge (\sim q)$ is false. Thus, this proof is constructed as follows :

1. Assume $p \wedge (\sim q)$ is true.
2. Discover on the basis of that assumption some conclusion that is identically false or violates some other fact already established in the frame of reference.
3. Then the contradiction discovered in step (b) leads us to conclude that the assumption in step (a) was false and therefore $(\sim p) \vee q$ is true or $p \to q$ ($\because$ $p \to q \equiv (\sim p) \vee q$) is true.

Example 44. Prove that there is no rational number $\frac{a}{b}$ whose square is 2. In other words, show that $\sqrt{2}$ is irrational. **[RTU 2011, 2009]**

Solution: Let p : $\sqrt{2}$ is irrational.

Suppose that $\sim p$ is true. Then $\sqrt{2}$ is rational. We will show that this leads to a contradiction. Now, $\sqrt{2}$ is rational follows that there exist integers a and b such that $\frac{a}{b} = \sqrt{2}$, where a and b have no common factors so $\frac{a}{b}$ is in its lowest-term form.

Since $\frac{a}{b} = \sqrt{2}$, it gives $a^2 = 2b^2$, so a^2 is even.

It implies that a is even, so we have $a = 2n$ for some integer n. Thus

$$(2n)^2 = 2b^2 \quad \text{so } b^2 = 2n^2.$$

It means that b^2 is even and hence we conclude that b is even. Thus, both a and b are even and therefore have a common factor of 2. This is a contradiction, since we have shown that $\sim p$ implies both r and $\sim r$ where r is the statement that a and b are integers with no common factors. Hence $\sim p$ is false, so that p : "$\sqrt{2}$ is irrational" is true.

3.25 Principle of Mathematical Induction

In mathematics, as in science there are two main aspects of inquiry whereby we can discover new results : deductive and inductive. As we have seen that the deductive aspect involves accepting certain statements as premises and axioms and then deducing other statements on the basis of valid inferences. The inductive aspect, on the other hand, is concerned with the search for facts by observation and experimentation - we arrive at a conjecture for a general rule by inductive reasoning. Frequently we may arrive at a conjecture that we believe to be true for all positive integers n. But then before we can put any confidence in our conjecture we need to verify the truth of the conjecture. *Mathematical induction* is a technique for proving conjectures of this kind. In other words, mathematical induction is used to prove propositions of the form $\forall n P(n)$, where the universe of discourse is the set of positive integers.

A proof using principle of mathematical induction that $P(n)$ is true for every positive integer n, consists of two steps.

1. **Basis Step:** The proposition $P(1)$ is shown to be true.
2. **Inductive Step:** For all $k \geq 1$, $P(k)$ implies $P(k+1)$.

Here, the proposition $P(n)$ for a fixed positive integer n is called the *inductive hypothesis.* When we complete the above steps of a proof by mathematical induction, we have proved that $P(n)$ is true for all positive integers n, that is, we have shown that $\forall n P(n)$ is true.

This technique can be stated as a rule of inference as

$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$.

Remark 6. In a proof by mathematical induction it is not assumed that $P(n)$ is true for all positive integers. It is only shown that if it is assumed that $P(n)$ is true, then $P(n+1)$ is also true. Thus, it is not a case of begging the question, or circular reasoning.

Remark 7. If the Steps 1 and 2 are replaces by $P(n_0)$ is true, and for all $k \geq n_0$, $P(k)$ implies $P(k+1)$, respectively, then we can prove $P(n)$ holds for all $n \geq n_0$, and the starting point n_0, or *basis of induction,* may be any integer-positive, negative, or zero.

Example 45. Show by mathematical induction that, $\forall n \geq 1$, **[RTU 2010]**

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

Solution: Let $P(n): 1+2+3+\ldots+n=\frac{n(n+1)}{2}$

Basis Step : $P(1)$ is the statement $1=\frac{1(1+1)}{2}$, which is true.

Inductive Step : We must show that for $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ must also be true. In other words, if $P(k): 1+2+\ldots+k=\frac{k(k+1)}{2}$ holds then we have to show that

$P(k+1): 1+2+3+\ldots+(k+1)=\frac{(k+1)(k+2)}{2}$ also holds.

The left hand side of $P(k+1)$ can be written as

$1+2+3+\ldots+k+(k+1)=(1+2+3+\ldots+k)+(k+1)=\frac{k(k+1)}{2}+(k+1)$ [using $P(k)$]

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2} = \text{right hand side of P(k+1)}$$

$\therefore$ By the principle of mathematical induction, it follows that P(n) is true for all n ≥ 1.

Example 46. Let $A_1, A_2, A_3, \ldots, A_n$ be any n sets, then show by mathematical induction that

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}.$$

Solution: Let P(n) : The equality holds for any n sets, i.e.

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}$$

Basis Step : P(1) is the statement $\overline{A_1} = \overline{A_1}$, which is true.

Inductive Step : Let P(k) be true, then we have to prove that P(k+1) is also true.

Now, if P(k) is true then

$$\overline{\left(\bigcup_{i=1}^{k} A_i\right)} = \bigcap_{i=1}^{k} \overline{A_i} \qquad \text{.....(1)}$$

Further, LHS of P(k+1) is

$$\overline{\left(\bigcup_{i=1}^{k+1} A_i\right)} = \overline{(A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1})}$$

$$= \overline{(A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1}}$$

(by associativity of ∪)

$$= \overline{(A_1 \cup A_2 \cup \ldots \cup A_k)} \cap \overline{A}_{k+1}$$

(by De Morgan's law)

$$= \overline{\left(\bigcup_{i=1}^{k} A_i\right)} \cap \overline{A}_{k+1} = \bigcap_{i=1}^{k} \overline{A}_i \cap \overline{A}_{k+1} \qquad \text{(using (1))}$$

$$= \bigcap_{i=1}^{k+1} \overline{A}_i \text{ RHS of P}(k+1)$$

Thus the implication P(k) → P(k+1) is a tautology, and hence by the principle of mathematical induction, P(n) is true for all $n \geq 1$.

3.26 Second Principle of Mathematical Induction (or Complete Induction)

It is another form of mathematical induction which is often useful in proofs. In this form, we use the same basis step as set in mathematical induction but we use a different inductive step. We assume that P(k) is true for $k = 1, \ldots, n$ and show that P(n+1) must also be true based on this assumption.

To show that P(n) is true for all positive integers n, this method requires the following two steps.

1. **Basis Step:** The proposition P(1) is shown to be true.
2. **Inductive Step:** It is shown that $[P(1) \wedge P(2) \wedge \ldots \wedge P(n)] \rightarrow P(n+1)$ is true for every positive integer n.

Example 47. Prove that for each positive integer n, the n^{th} Fibonacci number F_n is less than $(7/4)^n$.

Solution: Let P(n) : $F_n < (7/4)^n$.

Basis Step : P(1) is true, since $F_1 = 1 < 7/4$.

Inductive Step : Let P(k) is true for all $1 \le k \le n$, where $n \ge 2$, that is, suppose $F_k < (7/4)^k$ for each $1 \le k \le n$. Then we need to show that $F_{n+1} < (7/4)^{n+1}$ on the basis of the second principle of mathematical induction. Since $n \ge 2$, $n-1$ is a positive integer and thus $F_n < (7/4)^n$ and $F_{n-1} < (7/4)^{n-1}$.

Hence

$$F_{n+1} = F_n + F_{n-1} < \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^{n-1} = \left(\frac{7}{4}\right)^{n-1}\left(\frac{7}{4}+1\right)$$

$$= \left(\frac{7}{4}\right)^{n-1}\left(\frac{11}{4}\right) < \left(\frac{7}{4}\right)^{n-1}\left(\frac{7}{4}\right)^2 \qquad \left(\because \frac{11}{4} = \frac{44}{16} < \left(\frac{7}{4}\right)^2 = \frac{49}{16}\right)$$

$$= (7/4)^{n+1}.$$

Thus P(n + 1) is true. Hence by second principle of mathematical induction P(n) holds for each positive integer n.

ILLUSTRATIVE EXAMPLES

Example 48. Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd".

Solution: Let p : $3n+2$ is odd ; and q : n is odd.

Now to prove the implication $p \rightarrow q$ by contradiction method it suffices to show that $p \wedge \sim q$ is false. Let $p \wedge \sim q$ is true, i.e. 3n+2 is odd and n is even. Now n is even implies $n = 2k$ for some integer k. Then $3n+2 = 3(2k + 2) = 6k + 2 = 2(3k + 1)$, which shows that $3n + 2$ is even. This contradicts the assumption that $3n + 2$ is odd, i.e. $p \wedge \sim q$ is true . So $p \wedge \sim q$ must be false and then $p \rightarrow q$, i.e. "if $3n+2$ is odd, then n is odd" is true.

Example 49. Prove the theorem "The integer n is odd if and only if n^2 is odd". **[RTU 2009]**

Solution: Let p : n is odd ; and q : n^2 is odd.

Then the theorem has the form $p \leftrightarrow q$. To prove the theorem, we need to show that $p \rightarrow q$ and $q \rightarrow p$ are true.

We have already shown (in Ex. 5) that $p \rightarrow q$ is true. We will use an indirect proof (method of contraposition) to show that $q \rightarrow p$ is true. Suppose that its conclusion, i.e. n is odd, is false then we have n is even. Then $n = 2k$ for some integer k. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, so n^2 is even.

Thus $\sim p \rightarrow \sim q$ is true so this completes the indirect proof of $q \rightarrow p$.

Hence given theorem is true.

Example 50. Show by mathematical induction, that, any finite, nonempty set is countable; that is, it can be arranged in a list.

Solution: Let P(n) : If A is any set with $|A|=n \geq 1$, then A is countable.

Basis Step : P(1) is the statement that, if A is any set with $|A|=1$, say $A=\{x\}$, then A is countable. Since x forms a sequence all by itself whose set is A, so P(1) is true.

Inductive Step : Let, if A is any set with k elements, then A is countable. Now select any set B with $k+1$ elements and choose any element x in B. Since $B-\{x\}$ is a set with k elements thus by the induction hypothesis there is a sequence $x_1, x_2, \ldots, x_k$ with $B-\{x\}$ as its corresponding set. Then the sequence $x_1, x_2, \ldots, x_k, x$ has B as the corresponding set, so B is countable. Since B is any arbitrary set with $k+1$ elements, P(k + 1) is true if P(k) is true. Thus, by the principle of mathematical induction P(n) is true for all $n \geq 1$.

Example 51. Show that for all $n \geq 1$, $n! \geq 2^{n-1}$.

Solution: Let P(n) : $n! \geq 2^{n-1}$

Basis Step: P(1) is the statement $1! \geq 2^0$ implies $1 \geq 1$, which is true.

Inductive Step: Let P(k) be true, i.e. $k! \geq 2^{k-1}$ for some $k \geq 1$. Then to complete the inductive step, we need to show that the implication $P(k) \rightarrow P(k+1)$ is true. Now, P(k+1) is the statement $(k+1)! \geq 2^{(k+1)-1}$, the left hand side of P(k+1) is

$$\begin{aligned}
(k+1)! &= (k+1)k! \\
&\geq (k+1)2^{k-1} && \text{[using } P(k)\text{]} \\
&\geq 2 \times 2^{k-1} && [\because k+1 \geq 2, \forall k \geq 1] \\
&\geq 2^{(k+1)-1} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Thus, P(k+1) is true. Hence by the principle of mathematical induction, it follows that P(n) is true for all $n \geq 1$.

Example 52. Prove by contradiction that in a room of 13 people, 2 or more people have their birthdays in the same month.

Solution: Let p : A room has 13 people ; and q : 2 or more people have their birthdays in the same month. Then we need to prove that $p \rightarrow q$ is true, but by contradiction we prove instead $(p \wedge \sim q)$ is false. So assuming that the room has 13 people and no pair of people have their birthdays in the same month. But then since each person is born in some month, and since we are assuming that no two people were born in the same month, there must be 13 months representing the birth months of the people in the room. This contradicts the fact that there are only 12 months. So $(p \wedge \sim q)$ is false, and then $p \rightarrow q$ is true.

Example 53. Suppose that the 10 integers 1, 2, . . . , 10 are randomly positioned around a circular wheel. Show that the sum of some set of 3 consecutively positioned numbers is at least 17.

Solution: Let us give a proof by contradiction. Let X_i represents the integer positioned at position i on the wheel. Then we have to show :

Either $X_1 + X_2 + X_3 \geq 17,$

$X_2 + X_3 + X_4 \geq 17$

.

.

.

or $X_{10} + X_1 + X_2 \geq 17.$

If, on the contrary, we assume that the conclusion is false, then by De Morgan's laws we assume the conjunction of the following statements :

$X_1 + X_2 + X_3 < 17,$

$X_2 + X_3 + X_4 < 17$

.

.

.

or $X_{10} + X_1 + X_2 < 17$

The above statements can also be written as

$X_1 + X_2 + X_3 \leq 16,$

$X_2 + X_3 + X_4 \leq 16,$

.

.

.

or $X_{10} + X_1 + X_2 \leq 16$

Now adding up all these inequalities, we get

$3(X_1 + X_2 + \ldots + X_{10}) \leq (10)(16) = 160$

But $(X_1 + X_2 + \ldots + X_{10})$ is just the sum of first 10 positive integers, which is equal to 55. Therefore, the last inequality becomes

$$3(55) \leq 160$$

or

$$165 \leq 160$$

This is clearly a contradiction and hence the proof.

Example 54. If 41 balls are chosen from a collection of red, white, blue, garnet, and gold colored balls, then show that there are at least 12 red, 15 white, 4 blue, 10 garnet, or 4 gold balls chosen.

Solution: Let X_1, X_2, X_3, X_4, X_5 represent respectively, the number of red, white, blue, garnet, and gold balls chosen. We have to show that either $X_1 \geq 12$, $X_2 \geq 15$, $X_3 \geq 4$, $X_4 \geq 10$, $X_5 \geq 4$. Suppose, on the contrary, that $X_1 \leq 11$, $X_2 \leq 14$, $X_3 \leq 3$, $X_4 \leq 9$ or $X_5 \leq 3$. Then $X_1 + X_2 + \ldots X_5 \leq 11 + 14 + 3 + 9 + 3 = 40$. But it is given that the sum $X_1 + X_2 + \ldots X_5 = 41$, since this is that total number of balls chosen. Thus, we have arrived at the contradiction $41 \leq 40$ and, as a result the conclusion is verified.

Example 55. Find and prove a formula for the sum of the first n cubes, that is, $1^3 + 2^3 + \ldots + n^3$.

Solution: Let $S(n) = 1 + 2 + \ldots + n$, and $T(n) = 1^3 + 2^3 + \ldots + n^3$ then we can construct the following table:

n	1	2	3	4	5	6
S(n)	1	3	6	10	15	21
T(n)	$1=1^2$	$9=3^2$	$36=6^2$	$100=10^2$	$225=15^2$	$441=21^2$

One might observe the following pattern : $T(1) = [S(1)]^2$, $T(2) = [S(2)]^2$, $T(3) = [S(3)]^2$, $T(4) = [S(4)]^2$, $T(5) = [S(5)]^2$, and $T(6) = [S(6)]^2$.

We conjecture then that $T(n) = [S(n)]^2 = [n(n+1)/2]^2$ since we have already proved that $S(n) = n(n+1)/2$ (in Ex. 17). Let us verify this formula by mathematical induction.

Let $P(n): 1^3 + 2^3 + \ldots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Basis Step: Since P(1) is the statement $1^3 = \left[\dfrac{1(1+1)}{2}\right]^2$, which is true.

Inductive Step: Let $P(k)$ is true, i.e. $1^3 + 2^3 + \ldots + k^3 = [k(k+1)/2]^2$, then we need to show that $P(k+1)$ is true, that, is, $1^3 + 2^3 + \ldots + (k+1)^3 = [(k+1)(k+2)/2]^2$

LHS of $P(k+1)$ is

$$= 1^3 + 2^3 + \ldots + (k+1)^3$$

$$= 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \qquad \text{[using } P(k)\text{]}$$

$$= (k+1)^2\left[\left(\frac{k}{2}\right)^2 + (k+1)\right]$$

$$= (k+1)^2\left[\frac{k^2}{4} + k + 1\right] = (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$$

$$= (k+1)^2\left(\frac{k+2}{2}\right)^2 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$= \text{RHS of } P(k+1)$$

Hence, $P(k + 1)$ is true and thus by the principle of mathematical induction $P(n)$ is true for all positive integers n.

Example 56. Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer

n.

Solution: Let P(n) : $6^{n+2} + 7^{2n+1}$ is divisible by 43. **[IT(RTU)-2008]**

Basis Step: Since P(1) is $6^{1+2} + 7^{2+1} = 559 = 43(13)$, which is clearly divisible by 43, so P(1) is true.

Inductive Step: Let P(k) is true, i.e., $6^{k+2} + 7^{2k+1}$ is divisible by 43 then we need to show that P(k+1) is true, i.e. $6^{k+3} + 7^{2k+3}$ is divisible by 43.

Now observe:

$$6^{k+3} + 7^{2k+3} = 6^{k+2} \cdot 6 + 7^{2k+1} \cdot 49 = 6^{k+2} \cdot 6 + 7^{2k+1}\ (6+43)$$

$$= 6(6^{k+2} + 7^{2k+1}) + 43(7^{2k+1})$$

$$= 6(43x) + 43(7^{2k+1})$$ [Since P(k) suggests that $6^{k+2} + 7^{2k+1} = 43x$ for some integer x]

$$= 43(6x + 7^{2k+1})$$

which is clearly divisible by 43. Thus P(k+1) is true and hence by mathematical induction P(n) holds for each positive integer n.

Example 57. Show that for each positive integer n, there are more than n prime integers.

Solution: Let P(n) : There are more than n prime integers.

Basis Step: P(1) is the statement that there are more than 1 prime integers, which is true since 2 and 3 are prime.

Inductive Step: Let P(k) is true, then there exists $k + 1$ distinct prime integers, say, $a_1, a_2, \ldots a_{k+1}$. Form the integer.

$$N = a_1\, a_2 \ldots a_{k+1} + 1 = \prod_{i=1}^{k+1} a_i + 1$$

Now N is not divisible by any of the primes a_i. But N is either a prime or is divisible by a new prime a_{k+2}. In either case there are more than k+1 primes. Thus P(k+1) is true. Hence by the principle of mathematical induction P(n) holds for each positive integer n.

Example 58. Suppose the postal department prints only 5- and 9-rupees stamps. Prove that it is possible to make any postage of n-rupees using only 5- and 9-rupees stamps for $n \geq 35$.

Solution: Let P(n) : Any postage of n-rupees can be done by using only 5- and 9-rupees stamps for each $n \geq 35$.

Basis Step: Here minimum value of n is 35 and P(35) says that postage of 35 rupees can be done using only 5- and 9-rupees stamps, which is true since it can be done with seven 5-rupees stamps.

Inductive Step: Let P(k) is true, that is, k-rupees postage can be done with 5- and 9-rupees stamps where $n \geq 35$.

Now consider postage of (k+1) rupees. There are two possibilities :

(a) The k-rupees postage is made with only 5-rupees stamps, or

(b) There is at least one 9-rupee stamp involved in the makeup of k-rupees postage.

In case (a), the number of 5-rupee stamps is atleast seven since $n \geq 35$. Thus, we can replace those seven 5-rupee stamps by four 9-rupee stamps and make a postage of $(k+1)$ rupees.

In case (b), the k-rupees postage includes at least one 9-rupees stamp. Therefore, if we replace that one 9-rupee stamp by two 5-rupee stamps we can make a postage of $(k+1)$ rupees.

Thus, in either case we have shown that $(k+1)$ rupees postage can be made with only 5- and 9-rupees stamps. So, $P(k+1)$ is true, and then by mathematical induction $P(n)$ holds for each $n \geq 35$.

Example 59. Prove that for all integers $n \geq 4$, $3^n > n^3$.

Solution: Let $P(n) : 3^n > n^3$, where $n \geq 4$.

Basis Step: $P(n)$ is true since $3^4 = 81 > 4^3 = 64$.

Inductive Step: Let $P(k)$ is true for all $k \geq 4$, that is, $3^k > k^3$. Then we need to show that $P(k+1)$ is true, i.e. $3^{k+1} > (k+1)^3$.

Let us rewrite
$$\begin{aligned}(k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ &= k^3\left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)\end{aligned}$$

Since $3^k > k^3$ (using $P(k)$), we would be done if we could also prove that $3 > 1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}$ for $k \geq 4$. Observe that the function $f(k) = 1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}$ decreases as k increases, so that $f(k)$ is largest when k is smallest. In other words, $f(4)$ is the largest value of $f(k)$, where $k \geq 4$. Since

$$f(4) = 1+\frac{3}{4}+\frac{3}{4^2}+\frac{1}{4^3} = \frac{125}{64}$$

is obviously less than three, we have for any integer $k \geq 4$, $3 > 1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}$

Thus combining the two facts:

$3 > 1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}$ and $3^k > k^3$ for $k \geq 4$, we can multiply and get

$$3^{k+1} > k^3\left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$$

or $3^{k+1} > (k+1)^3$

So $P(k+1)$ is true and then by mathematical induction $P(n)$ is true for all integers $n \geq 4$, i.e. $3^n > n^3$.

Example 60. Prove that if F_n is the n^{th} Fibonacci number, then

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \text{ for all integers } n \geq 0.$$

Solution: Let $P(n): F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

Basis Step: P(0) is true, since $F_0 = 1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right]$ Similarly, P(1) is true, since

$$F_1 = 1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right].$$

Inductive Step: Let P(*k*) is true for all $0 \le k \le n$, that is assume

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right], \ 0 \le k \le n$$

We need to show that

$$F_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]$$

Let

$$a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2}$$

Then since we know that $F_{n+1} = F_n + F_{n-1}$ and

$$F_n = \frac{1}{\sqrt{5}}\left[a^{n+1} - b^{n+1}\right]$$

and

$$F_{n-1} = \frac{1}{\sqrt{5}}\left[a^n - b^n\right]$$

So, it follows that

$$F_{n+1} = \frac{1}{\sqrt{5}}\left[a^{n+1} - b^{n+1}\right] + \frac{1}{\sqrt{5}}\left[a^n - b^n\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(a^{n+1} - b^{n+1}\right) + \left(a^n - b^n\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[a^n(a+1) - b^n(b+1)\right]$$

Now, $a+1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2}$ and $a^2 = \frac{3+\sqrt{5}}{2}$ so that $a + 1 = a^2$. Similarly, $b + 1 = b^2$. Therefore,

$$F_{n+1} = \frac{1}{\sqrt{5}}\left[a^n \cdot a^2 - b^n \cdot b^2\right] = \frac{1}{\sqrt{5}}\left[a^{n+2} - b^{n+2}\right]$$

Hence the proof.

Example 61. Prove that the sum $1^2+2^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

Solution: Let $P(n):1^2+2^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Basis Step: P(1) is $1^2=\dfrac{1(1+1)(2+1)}{6}$, which is true.

Inductive Step: Let P(k) is true, that is

$$1^2+2^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$$

We need to show that P(k+1) is true, i.e.

$$1^2+2^2+\ldots+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$$

Now LHS of P(k+1) is

$$= 1^2 + 2^2 + \ldots + (k+1)^2$$
$$= 1^2 + 2^2 + \ldots + k^2 + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \qquad \text{[using P(}k\text{)]}$$
$$= \frac{(k+1)}{6}[k(2k+1)+6(k+1)]=\frac{(k+1)}{6}\left[2k^2+7k+6\right]$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}=\text{RHS of P}(k+1)$$

Thus P(k+1) is true, hence by mathematical induction P(n) holds for every integer $n \geq 1$.

Example 62. Let $A=\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}$ be a matrix. Prove by mathematical induction that $A^n=\begin{bmatrix}1 & an\\ 0 & 1\end{bmatrix}$, n ≥ 1.

Solution: Let $P(n): A^n=\begin{bmatrix}1 & an\\ 0 & 1\end{bmatrix}$

Basis Step: P(1) is true, since $A^1=\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}$

Inductive Step: Let P(k) is true, that is,

$$A^k = \begin{bmatrix}1 & ak\\ 0 & 1\end{bmatrix}$$

We need to show that P(k+1) is true, i.e.,

$$A^{k+1} = \begin{bmatrix}1 & a(k+1)\\ 0 & 1\end{bmatrix}$$

Now, LHS of P(k+1) is = A^{k+1}

$$= A^k \cdot A = \begin{bmatrix} 1 & ak \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$ [using P(k) and A]

$$= \begin{bmatrix} 1 & a+ak \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a(k+1) \\ 0 & 1 \end{bmatrix}$$

= RHS of P(k+1)

Thus P(k+1) is true. Hence, by mathematical induction P(n) holds for each positive integer n.

Example 63. Prove that $1 + 3 + 5 + \ldots + (2n-1) = n^2$, for every positive integer n.

Solution: Let P(n) : $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

Basis Step: P(1) is true, since $1 = 1^2$.

Inductive Step: Let P(k) is true, i.e. $1 + 3 + 5 + \ldots + (2k-1) = (k)^2$

We need to show that P(k+1) is true, i.e., $1 + 3 + 5 + \ldots + (2k+1) \equiv (k+1)^2$

Now, LHS of P(k+1) is

$$= 1+3+5+\ldots+(2k-1)+(2k+1)$$

$$= k^2 + (2k + 1)$$ [using P(k)]

$$= (k+1)^2$$

= RHS of P(k+1)

Thus P(k+1) is true. Hence by mathematical induction P(n) holds for every positive integer n.

Example 64. Show that for any positive integer $n > 1$,

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n}$$

Solution: Let $P(n): \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n}$

Basis Step: $P(2)$ is true, since $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}} = 1+\frac{1}{\sqrt{2}} > \sqrt{2}$

Inductive Step: Let P(k) is true, i.e., $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}} > \sqrt{k}$

Then we need to show that P(k+1) is true, i.e., $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k+1}} > \sqrt{(k+1)}$

Now LHS of P(k+1) is

$$= \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$$

$$> \sqrt{k}+\frac{1}{\sqrt{k+1}}$$ [using P(k)]

$$= \sqrt{k+1}\left[\frac{\sqrt{k}\sqrt{k+1}+1}{k+1}\right]$$

$$> \sqrt{k+1}\left[\frac{\sqrt{k}\sqrt{k+1}}{k+1}\right] \qquad \left(\because \sqrt{k+1} > \sqrt{k}\right)$$

$$= \sqrt{k+1}\left[\frac{k+1}{k+1}\right] = \sqrt{k+1}$$

$$= \text{RHS of } P(k+1)$$

Thus P(k+1) is true. Hence by mathematical induction P(n) holds for every integer n>1.

Example 65. Show that for any integer $n \geq 1$, $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

Solution: Let P(n) : $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Basis Step : P(1) is $11^{1+2} + 12^{2+1} = 11^3 + 12^3 = 3059 = 133(23)$, which is clearly divisible by 133.

Inductive Step : Let P(k) is true, i.e., $11^{k+2} + 12^{2k+1}$ is divisible by 133 then we have $11^{k+2} + 12^{2k+1} = 133\text{m}$, for some integer m. We need to show that P(k+1) is true, i.e., $11^{k+3} + 12^{2k+3}$ is divisible by 133.

Now, $$11^{k+3} + 12^{2k+3} = 11^{k+2} \cdot 11 + 12^{2k+1} \cdot 12^2 = 11^{k+2} \cdot 11 + 12^{2k+1} \cdot 144$$

$$= 11^{k+2} \cdot 11 + 12^{2k+1} \cdot (11+133)$$

$$= 11(11^{k+2} + 12^{2k+1}) + 133(12^{2k+1})$$

$$= 11(133m) + 133(12^{2k+1}) \qquad \text{[using P(}k\text{)]}$$

$$= 133(11m + 12^{2k+1})$$

which is clearly divisible by 133, since $11m + 12^{2k+1}$ is an integer. Thus P(k+1) is true.

Hence by mathematical induction, P(n) is true for any integer $n \geq 1$.

Example 66. Show that any integer composed of 3^n identical digits is divisible by 3^n.

Solution: Let P(n) : Any integer composed of 3^n identical digits is divisible by 3^n.

Basis Step: P(1) is true, since any 3-digit integer with three identical digits is divisible by 3.

Inductive Step: Let P(k) is true, i.e., any integer composed of 3^k identical digits is divisible by 3^k. Then we need to show that P(k+1) is true.

Let x be an integer composed of 3^{k+1} identical digits. Then x can be written as

$$x = y \times z$$

where y is an integer composed of 3^k identical digits and

$$z = 10^{2\cdot 3^k} + 10^{3^k} + 1 = 1\underbrace{000\ldots0}_{(3^k-1)0's}1\underbrace{000\ldots0}_{(3^k-1)0's}1$$

Since we assume that y is divisible by 3^k, and z is clearly divisible by 3, we conclude that x is divisible by 3^{k+1}. Thus P(k+1) is true. Hence by mathematical induction P(n) holds for any integer $n \geq 1$.

Example 67. Show that $2^n > n^3$ for $n \geq 10$.

Solution: Let P(n) : $2^n > n^3$, $n \geq 10$

Basis Step : P(10) is $2^{10} = 1024 > 10^3$, which is true.

Inductive Step : Let P(k) is true, i.e., $2^k > k^3$. Then we need to show that P(k+1) is true, i.e., $2^{k+1} > (k+1)^3$.

Now, LHS of P(k+1) is

$$= 2^{k+1}$$

$$= 2\cdot 2^k > \left(1+\frac{1}{10}\right)^3 \cdot 2^k \geq \left(1+\frac{1}{k}\right)^3 \cdot 2^k$$

$$> \left(1+\frac{1}{k}\right)^3 \cdot k^3$$ **[using P(*k*)]**

$$= (k+1)^3$$

= RHS of P(k+1).

Thus P(k+1) is true.

Hence by mathematical induction P(n) is true for all $n \geq 10$.

Example 68. Show that $n < 2^n$, for every integer $n > 0$. **[RTU 2009]**

Solution: Let P(n) : $n < 2^n$.

Basis Step : P(1) is $1 < 2^1 = 2$, which is true.

Inductive Step : Let P(k) holds, that is, $k < 2^k$. We need to show that P(k+1) holds, i.e., $(k+1) < 2^{k+1}$.

Now LHS of P(k+1) is

$$= k+1$$

$$< 2^k + 1$$ **[using P(*k*)]**

$$< 2^k + 2^k \qquad \left(\because 2^k \geq 1,\ \forall\, k \geq 0\right)$$

$$= 2\cdot 2^k = 2^{k+1}$$

= RHS of P(k+1)

So, P(k+1) is true. Hence by mathematical induction P(n) is true for all positive integral values of n.

Example 69. Show that $2^n < n!$ for $n \geq 4$.

Solution: Let P(n) : $2^n < n!$

Basis Step: P(n) is $2^4 = 16 < 4! = 24$, which is true.

Inductive Step: Let P(k) holds for any $k > 4$, i.e., $2^k < k!$. We have to show that P(k+1) holds, i.e., $2^{k+1} < (k+1)!$

Now LHS of P(k+1) is

$$= 2^{k+1}$$

$$= 2\cdot 2^k < 2(k!)$$

$$< (k+1)(k!) = (k+1)!$$

Thus P(k+1) holds.

Hence by mathematical induction, P(n) holds for all $n \geq 4$.

Example 70. Show that $n^3 + 2n$ is divisible by 3 for $n \geq 0$.

Solution: Let P(n) : $n^3 + 2n$ is divisible by 3.

Basis Step : P(0) is, 0 is divisible by 3, which is true.

Inductive Step : Let P(k) is true, that is, k^3+2k is divisible by 3, then $k^3+2k = 3m$ for some integer m. We need to show that P(k+1) holds, i.e., $(k+1)^3 + 2(k+1)$ is divisible by 3.

Now $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$

$$= k^3 + 3k^2 + 5k + 3 = (k^3 + 2k) + 3(k^2 + k + 1)$$

$$= 3m + 3(k^2 + k + 1) \qquad \text{[using P}(k)]$$

$$= 3(m + k^2 + k + 1)$$

which is clearly divisible by 3. Thus P(k+1) holds. Hence by mathematical induction P(n) is true for all $n \geq 0$.

Proved

Example 71. Show that $B \cup \left(\bigcap_{i=1}^{n} A_i \right) = \bigcap_{i=1}^{n} (B \cup A_i)$, for $n \geq 2$.

Solution: Let $P(n): B \cup \left(\bigcap_{i=1}^{n} A_i \right) = \bigcap_{i=1}^{n} (B \cup A_i)$

Basis Step: P(2) is $B \cup \left(\bigcap_{i=1}^{2} A_i \right) = \bigcap_{i=1}^{2} (B \cup A_i)$, implies $B \cup (A_1 \cap A_2) = (B \cup A_1) \cap (B \cup A_2)$, which follows from the distributive law of union and intersection.

Inductive Step: Let P(k) holds for any k, that is,

$$B \cup \left(\bigcap_{i=1}^{k} A_i \right) = \bigcap_{i=1}^{k} (B \cup A_i)$$

Then we have to show that P(k+1) holds, i.e.,

$$B \cup \left(\bigcap_{i=1}^{k+1} A_i \right) = \bigcap_{i=1}^{k+1} (B \cup A_i)$$

Now LHS of P(k+1) is

$$= B \cup \left(\bigcap_{i=1}^{k+1} A_i \right)$$

$$= B \cup \left(\bigcap_{i=1}^{k} A_i \cap A_{k+1} \right)$$

$$= \left[B \cup \left(\bigcap_{i=1}^{k} A_i \right) \right] \cap [B \cup A_{k+1}]$$

$$= \left(\bigcap_{i=1}^{k} B \cup A_i \right) \cap (B \cup A_{k+1}) \qquad \text{[using P}(k)]$$

$$= \bigcap_{i=1}^{k+1} (B \cup A_i)$$

$$= \text{RHS of P}(k+1)$$

Thus P(k+1) holds.

Hence P(n) holds for every integer $n \geq 2$. **Proved**

Example 72. Show that $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$, for all $n \in N$.

Solution: Let $P(n): \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Basis Step: P(1) is $\frac{1}{1.3}=\frac{1}{2+1}$ which is true.

Inductive Step: Let P(k) is true, that is,

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Then we need to show that P(k+1) holds,

i.e.,
$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Now LHS of P(k+1) is

$$=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \qquad \text{[using P}(k)]$$

$$=\frac{k}{(2k+1)}\left[\frac{(2k+3)k+1}{2k+3}\right]=\frac{1}{(2k+1)}\left(\frac{2k^2+3k+1}{2k+3}\right)$$

$$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

= RHS of P(k+1)

Thus P(k+1) holds. Hence by induction P(n) holds for all $n \geq 2$.

Example 73. Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps. Prove this by using-

(i) principle of mathematical induction. **[CE(RTU)-2007]**

(ii) principle of complete induction. **[IT(RTU)-2008]**

Solution:

(i) Let P(n) : Postage of n cents can be formed using 4-cent and 5-cent stamps.

Basis Step: P(12) is true since postage of 12 cents can be formed using three 4-cent stamps.

Inductive Step: We have to show that P(k) → P(k + 1).

Let P(k) be true i.e., postage of k cents can be formed using just 4-cent and 5-cent stamps. If at least one 4-cent stamp was used then on replacing it with a 5-cent stamp we get the postage of k+1 cents. If no 4-cent stamps were used then it is obvious that atleast three 5-cent stamps were used to form postage of k cents as $k \geq 12$. So, replace three 5-cent stamps with four 4-cent stamps to form postage of $k + 1$ cents. This completes the proof of $P(k) \rightarrow P(k + 1)$.

Thus by the principle of mathematical induction P(n) holds for every natural number $n \geq 12$.

(ii) To prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps, we will show that postage of 12, 13, 14 and 15 cents can be formed and then show that postage of $n + 1$ cents for $n \geq 15$ can be formed from postage of n–3 cents.

Basic Step: we can form postage of-

(i) 12 cents using three 4-cent stamps i.e.,

$$12 = 4 \times 3 + 5 \times 0$$

(ii) 13 cents using two 4-cent stamps and one 5-cent stamp i.e.,

$$13 = 4 \times 2 + 5 \times 1$$

(iii) 14 cents using one 4-cent stamp and two 5-cent stamps i.e.,

$$14 = 4 \times 1 + 5 \times 2$$

(iv) 15 cents using three 5-cents stamps i.e.,

$$15 = 4 \times 0 + 5 \times 3$$

So all the above postages have the form $4x + 5y$, where x and y are non negative integers.

Inductive Step: Let $n \geq 15$ and also assume that we can form the postage of k cents, where $12 \leq k \leq n$, using just 4-cent and 5-cent stamps.

Now we have to show that postage of (n+1) cents can also be formed using only 4-cent and 5-cent stamps.

Since $\quad k = n - 3 < n$

So by above assumption the postage of $n - 3$ cents can be formed using just 4-cent and 5-cent stamps, i.e.,

$$n-3 = 4x + 5y; \; x, y \in W = \{0, 1, 2, ...\}$$

or $\quad (n-3)+4 = (4x + 5y)+4$

or $\quad n + 1 = 4(x + 1) + 5y$

$\Rightarrow$ postage of n+1 cents can be formed by using the stamps that form postage of n–3 cents together with one 4-cent stamp. This completes the inductive step, as well as the proof by the principle of complete induction.

EXERCISE 3.2

Q.1 Prove by contradiction : If the product of a certain 2-digit decimal integer n by 5 is a 2-digit number, then the tens digit of n is 1.

Q.2 If x and y are each integers > 2, prove by contradiction that $xy>x+y$.

Q.3 (a) Prove by contradiction : There do not exist 3 consecutive integers such that the cube of the largest is equal to the sum of the cubes of the 2 other integers.

(b) Prove by contradiction : If n has the form $4k+3$, where k is an integer, then the equation $x^2 + y^2 = n$ has no integral solutions for x and y.

Q.4 Suppose that a man hiked 6 miles the first hour and 4 miles the twelfth hour and hiked a total of 71 miles in 12 hours. Prove that he must have hiked at least 12 miles within a certain period of two consecutive hours. (Hint : prove by contradiction)

Q.5 Use mathematical induction to show that

$$1^2 + 3^2 + 5^2 + \ldots +(2n-1)^2 = n(2n-1)(2n+1)/3,\ n \geq 1.$$

Q.6 For each integer $n \geq 5$, show that $2^n > n^2$.

Q.7 For each integer $n \geq 1$, show that the nth Fibonacci number F_n is less than $\left(\frac{13}{8}\right)^n$.

Q.8 Prove by the principle of mathematical induction that

$$1+3+6+\ldots+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6},\ n\geq 1$$

Q.9 Prove by the principle of mathematical induction

$$1\cdot 2 + 2\cdot 2^2 + \ldots + n\cdot 2^n = (n-1)2^{n+1} + 2,\ \ n \geq 1$$

Q.10 Prove by the principle of mathematical induction that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1},\ n\geq 1$$

Q.11 Prove by the principle of mathematical induction the inequality

$$(a+1)^n \geq 1 + na,\ \ \text{for}\ \ a > -1;\ \ n = 2, 3, 4, \ldots$$

Q.12 Using the principle of mathematical induction, prove that

$$1\cdot 3+2\cdot 4+3\cdot 5+\ldots+n(n+2)=\frac{n}{6}(n+1)(2n+7), n\geq 1$$

Q.13 Prove by the principle of mathematical induction that

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}, n\geq 1$$

Q.14 By the principle of induction, show that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14, for all positive integral value of n including zero.

Q.15 (a) If nth term of A.P. is $a+(n-1)d$, then show by the principle of mathematical induction that the sum of n terms of A.P. is $\frac{n}{2}\{2a+(n-1)d\}$

That is, by the principle of mathematical induction, prove that

$$a+(a+d)+(a+2d)+\ldots+\{a+(n-1)d\}=\frac{n}{2}\{2a+(n-1)d\},n\geq 1$$

(b) Prove by the principle of mathematical induction the result

$$a+ar+ar^2+\ldots+ar^{n-1}=a\cdot\frac{r^n-1}{r-1}, \text{ if } r\neq 1, n\geq 1$$

Q.16 Prove by the principle of mathematical induction that $10^n+3\cdot 4^{n+2}+5$ is divisible by 9.

Q.17 Show by mathematical induction that $n(n^2-1)$ is divisible by 24; n is positive odd integer.

Q.18 Show that $1^2-2^2+3^2+\ldots+(-1)^{n+1}n^2=\frac{(-1)^{n+1}n(n+1)}{2}$.

Q.19 Show that x^n-1 is divisible by $x-1$ for all positive integral value of n.

Q.20 Show that $10^{2n-1}+1$ is divisible by 11 for each natural number n.

Q.21 Show that $\frac{1}{5}n^5+\frac{1}{3}n^3+\frac{7}{15}n$ is a natural number for all $n\in$ N.

Q.22 Prove that every amount of postage of 12 rupees or more can be formed using just 4 rupees and 5 rupees stamps.

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