# 6

# Combinatorial Mathematics

**OBJECTIVES**

- Introduction
- Basic Counting Principles
- Permutation
- Combinations
- Binomial Theorem

## 6.1 Introduction

We count objects to solve many kinds of problems. In fact counting problems arise throughout in the field of mathematics as well as computer science. For example, we count – all possible outcomes and successful outcomes of an experiment to determine probabilities, the number of operations used by an algorithm to study it's time complexity, Internet protocol addresses to meet the demand etc. Hence in this chapter we will study counting techniques which are useful in the study of various concepts in mathematics and computer science, particularly permutations and combinations. More generally, counting techniques for ordered arrangements and unordered selections of distinct objects of a finite set will be introduced.

## 6.2 Basic Counting Principles

There are two basic principles of counting.

The Sum Rule : If a task can be done in m–ways and another task in n–ways then there are m + n ways of performing exactly one of these tasks.

The Product Rule : If a task can be done in m ways and another task in n ways then there are m × n ways in which both of these tasks can be performed.

These rules can be extended to any number of tasks.

**Example 1.** There are 18 mathematics majors and 325 computer science majors at a college.

(a) How many ways are there to pick two representatives, so that one is a mathematics major and other is a computer science major ?

(b) How many ways are there to pick one representative who is either a mathematics major or a computer science major ?

Solution. (a) Using product rule this can be done in $18 \times 325 = 5850$ ways.

(b) Using sum rule this can be done in $18 + 325 = 343$ ways.

**Example 2.** A student can choose a project from one of three lists, which contain 29, 12 and 5 possible projects, respectively. How many possible projects are there from which student can choose ?

Solution. By sum rule a student can choose from $29 + 12 + 5 = 46$ projects.

**Example 3.** A computer program consists of one letter followed by three digits. If repetition is allowed then in how many ways different label identifiers are possible ?

Solution. There are 26 English alphabet and 10 digits from 0 to 9. Hence total number of ways in which different label identifiers are possible is $26 \times 10 \times 10 \times 10 = 26{,}000$.

**Example 4.** A cricket stadium has five gates on the eastern boundary and three gates on the western boundary.

(a) In how many ways can a person enter through an east gate and leave by west gate ?

(b) In how many different ways in all can a person enter and get out through different gates ?

Solution. (a) As there are five ways of entering (east side) and three ways of leaving (west side), therefore the required number of ways $= 5 \times 3 = 15$.

(b) The total number of gates is 8. Hence a person can enter from any of the 8 gates and may leave from any of the seven gates (leaving the gate from which he entered).

$\therefore$ The required number of ways $= 8 \times 7 = 56$.

**Example 5.** What is the value of 'a' after the following code has been executed ?

```
    a : = 0
for j1 : = 1 to m1
for j2 : = 1 to m2

for jn : = 1 to mn
    a : = a + 1
```

Solution. Let $T_i$ be the task of traversing ith loop; then the number of ways to do task $T_i$, i = 1, 2, ..... n is $m_i$, since the $i^{th}$ loop is traversed once for each integer ji with $1 \le j_i \le m_i$. Hence by product rule, it is clear that nested loop is traversed $m_1m_2$ ..... $m_n$ times. Hence, the final value of a is $m_1m_2$ ..... $m_n$.

## 6.3 Permutation

A permutation of a set of distinct objects is an ordered arrangement of these objects. The total number of permutations of n–objects taken r at a time is known r–permutation and is denoted as $^nP_r$, $1 \le r \le n$.

**Theorem 1 :** If $1 \le r \le n$ then $^nP_r = \frac{n!}{(n-r)!}$

**Proof.** The total number of permutations of n–different objects taken 'r' (without repetition) at a time is equal to the number of ways in which 'r' places can be filled with n–different objects. Hence by product rule we have

$$^nP_r = n(n-1)(n-2) \ldots\ldots r \text{ factors}$$

$$= n(n-1)(n-2) \ldots\ldots (n-r+1)$$

$$= \frac{[n(n-1)\&(n-r+1)][(n-r)(n-r-1)\&3\cdot2\cdot1]}{(n-r)(n-r-1)\&3\cdot2\cdot1}$$

$$= \frac{n!}{(n-r)!} \qquad [\text{where } n! = n(n-1)! \text{ and } 0! = 1]$$

This can also be considered as the problem of filling r distinct boxes with n–distinct objects.

**Example 6.** A number consisting of 4 digits is to be formed taking digits out of digits 1 – 9. How many such numbers are possible ?

**Solution.** Here n = 9; r = 4

$\therefore$ Required solution = $^9P_4 = \frac{9!}{5!} = 9\cdot8\cdot7\cdot6 = 3024$

**Example 7.** Let a license plate consists of two letters followed by four digits. If the first letter on the plate is either a R or a P and the first digit is '7', the second letter is either a J or a I and the last digit is either 3 or 8 then find the different number of such license plates.

Solution. Given that the first letter can be R or P and the second letter can be J or I. So the number of permutations of two letters = 2 × 2 = 4.

Among the digits first digit is definitely 7; second and third can be chosen from the set (0, 1, 2, ....9) i.e., each place can choose digits in 10 ways and the fourth digit is either 3 or 8. Hence the number of permutations of the four digits are = 1 × 10 × 10 × 2 = 200. Hence by product rule the required number of permutations = 4 × 200 = 800.

**Example 8.** (a) Prove $^nP_{n-1} = {}^nP_n$.

(b) If $^{56}P_r + 6 : {}^{54}P_r + 3 = 30,800 : 1$, find r.

Solution. (a)

$$^nP_{n-1} = \frac{n!}{\{n-(n-1)\}!} = n!$$

$$^nP_n = \frac{n!}{(n-1)!} = n!$$

Hence $^nP_{n-1} = {}^nP_n$.

(b) $$^{56}P_{r+6} = \frac{56!}{(50-r)!},\ {}^{54}P_{r+3} = \frac{54!}{(51-r)!}$$

Given $$\frac{^{56}P_{r+6}}{^{54}P_{r+3}} = 30800$$

$\Rightarrow 56 \times 55 \times (51 - r) = 30800$

$\Rightarrow 51 - r = 10 \Rightarrow r = 41.$

**Example 9.** A sales woman has to visit eight different cities. She begins her trip from a specified city, but can visit the other seven cities in any order she wishes. How many possible orders can she use when visiting these cities ?

**Solution.** The number of possible paths

$= {}^7P_7 = 7! = 5040.$

### 6.3.1 Permutations with Repetition

Theorem 2. The number of permutations of n–objects, taken 'r' at a time, allowing repetition is nr (each such permutation is called a sequence of length 'r').

Proof. This can be considered equivalent to the problem of filling r distinct boxes with 'n' objects with repetition allowed. Hence there are n–ways to select an object for box I, again, as repetition is allowed there are n–ways to select an object for box II and so on. Hence there are n–ways to select an object for each of the r–boxes. Hence by product rule the number of such permutations

$= n \times n \ .....n(r \text{ times})$

$= n^r.$

Hence proved.

**Example 10.** A four digit number is formed with digits 1 – 9. How many such numbers can be formed if repetition is allowed ?

Solution. Here n = 9, r = 4.

$\therefore$ Required numbers $= 9^4 = 6561.$

**Example 11.** How many strings of six letters are there ?

**Solution.** There are 26 letters, therefore the required solution is $26^6$.

**Example 12.** How many permutations of the letters in the word COMPANY are possible,

(a) using all letters, without repetition

(b) using four letters, without repetition,

(c) using four letters with repetition ?

Solution. (a) $^7P^7 = 7! = 5040$

(b) $^7P_4 = \dfrac{7!}{(7-4)!} = 7 \times 6 \times 5 \times 4 = 840$

(c) $7^4 = 2401.$

### 6.3.2 Permutations of Objects not all Distinct

**Theorem 3.** The number of different permutations of n–objects out of which $n_1$ objects are of type 1 $n_2$ objects are of type 2 ...... and $n_k$ of type k, is

$$\frac{n!}{n_1!n_2! \ldots n_k!} \text{ where } n_1 + n_2 + \ldots + n_k = n.$$

**Proof.** Let the number of required permutations be T. Now consider one permutation '$p_1$' from the T permutations in which first all $n_1$ objects of type I are placed followed by all $n_2$ objects of type II followed by all objects of type III and so on last being nk objects of type k, where

$$n_1 + n_2 + \ldots\ldots + n_k = n$$

Number of permutations of $n_1$ objects taken all at a time $= {}^{n_1}P_{n_1} = n_1!$

Number of permutations of $n_2$ objects taken all at a time $= {}^{n_2}P_{n_2} = n_2!$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Number of permutations of $n_k$ objects taken all at a time $= {}^{n_k}P_{n_k} = n_k!$

Therefore, by product rule number of permutations $P_1 = n_1!n_2!\ldots\ldots n_k!$

But $p_1$ is just one kind of permutation of n element. Therefore, total number of permutations

$= T \times$ Number of permutations $p_1$

$= T \times n_1!\ n_2!\ \ldots\ n_k!$ .....(1)

But total number of permutations of n distinct objects $= {}^nP_n = n!$

$\therefore$ (1) $\Rightarrow n! = T \times n_1! \times n_2! \ldots n_k!$

$\Rightarrow$ Required number of permutations $= T = \dfrac{n!}{n_1!n_2!\cdots n_k!}$, $n_1 + n_2 + \ldots + n_k = n.$ Hence proved.

This can also be considered equivalent to distributing n–distinguishable objects into k distinguishable boxes so that $n_i$ objects are placed into box i, i = 1, 2, ...... k.

**Example 13.** Find the number of distinct permutations of the letters of the word ENGINEERING.

Solution. The given word consists of 11 letters out of which there are 3E's, 3N's, 2G's, 2I's and 1R.

$\therefore$ Required permutations $= \dfrac{11!}{3!3!2!2!1!} = 2,77,200.$

**Example 14.** How many permutations can be made with letters of the word CONSTITUTION ? Also find the number of permutations in which :

(a) two O's come together

(b) vowels occur together

(c) consonants and vowels occur alternatively

(d) two O's do not come together.

(e) letter N occurs both at the begining and at the end.

**Solution.** This word consists of 12 letters out of which there are 2O's, 2N's, 3T's, 2I's, 1C, 1S and 1U.

$\therefore$ All possible permutations $= \frac{12!}{2!2!3!2!} = 99{,}79{,}200.$

(a) Consider 2O's as one letter. Hence now we have 11 letters and there are 2N's, 3T's and 2I's.

$\therefore$ Required permutations $= \frac{11!}{2!3!2!} = 16{,}63{,}200.$

(b) There are seven consonants – C N N S T T T and five vowels O O I I U. Consider all five vowels as one letter. Hence we have 8 letters out of which there are 2N's and 3T's. So the number of permutations

$= \frac{8!}{2!3!} = 3360.$

Also the five vowels consist of 2O's, 2I's and 1U. Hence they can be permuted in $\frac{5!}{2!2!1!} = 30$ ways.

Hence the required number of total permutations = 3360 × 30 = 1,00,800.

(c) First we fix five vowels at alternate positions i.e., $2^{nd}$, $4^{th}$, $6^{th}$, $8^{th}$ and $10^{th}$ positions. This can be done in $\frac{5!}{2!2!} = 30$ ways. (5 vowels which have 2O's and 2I's). Again seven consonants can be put at the remaining seven places in $\frac{7!}{2!3!}$ different ways.

Hence required number of permutations $= \frac{5!}{2!2!} \times \frac{7!}{2!3!} = 12{,}600$.

(d) Required number of permutations = number of total permutations – number of permutations in which 2O's come together = 9979200 – 1663200 = 83,16,000.

(e) If we fix the letter N at the beginning and at the end. Then we have 10 letters which have 2O's, 3T's and 2I's.

$\therefore$ Required number of permutations $= \frac{10!}{2!2!3!} = 151200.$

**Example 15.** How many different arrangements of the letters in the word 'BOUGHT' can be formed if the vowels must be kept together. [Raj. 2000]

**Solution.** 'BOUGHT' consists of 6 letters out of which there are 2 vowels and 4 consonants. If the 2 vowels are kept together, then there remain five letters which can be arranged in 5! ways.

Also the two vowels can be arranged in 2! ways.

$\therefore$ The required number of different arrangements = 5! × 2! = 240.

**Example 16.** How many bytes contain :

(i) exactly two 1's ?

(ii) exactly four 1's ?

(iii) atleast six 1's ? [Raj. 2000]

**Solution.** A byte consists of 8 bits and a bit is either 0 or 1.

(i) If a byte has exactly two 1's and it contains exactly six 0's then number of such bytes

(ii) If it contains exactly four 1's then it has exactly four 0's.

then number of such bytes .

(iii) The number of bytes containing atleast six 1's = bytes containing six 1's + bytes containing seven 1's + bytes containing eight 1's = $\frac{8!}{6!2!}+\frac{8!}{7!1!}+\frac{8!}{8!0!}$

= 28 + 8 + 1 = 37.

**Example 17.** In how many ways can letters a, b, c, d, e, f be arranged so that the letter b is always to the immediate left of letter e ? [Raj. 1999]

Solution. Among six letters 'be' always forms a pair.

∴ We have to arrange five letters, which can be done in 5! = 120 ways.

### 6.3.3 Circular Permutations

The circular permutations are the permutations of the objects placed in a circle. As the objects are arranged in a circle, hence there is no starting and ending point. Here only the relative positions are important.

**Theorem 4.** The number of circular permutations of n–different objects in (n – 1)!.

Proof. Let us fix one object. The problem is now to arrange remaining (n – 1) objects which can be done in (n – 1)! ways.

∴ Number of circular permutations = (n – 1)!. **Hence proved.**

In the circular sequencing of, beads in a necklace, flowers in a garland etc. the clockwise and anticlockwise arrangements are not distinct. Hence here the number of circular permutations of 'n' different objects $= \frac{1}{2}(n-1)!$.

**Example 18.** In how many ways can 5 programmers and 3 software engineers sit around a table so that no two software engineers are together.

**Solution.** The programmers are arranged round the circular table in (5 – 1)! = 4! ways. As no two software engineers should sit together, hence they are made to sit in between the programmes. Now three places between 5 programmers can be filled in 3! ways.

∴ Required number of permutations = 4! × 3! = 144 Ans.

## 6.4 Combinations

A combination is a collection of objects where order does not matter. If there are n–objects then the number of objects taken r at a time is denoted by $^nC_r$

The difference between permutation and combination can be seen by taking an example of choosing any two digits from set {1, 2, 3, 4}.

| | Permutation ($^nP_r$) | ($^nC_r$) Combination |
|---|---|---|
| | 12, 21 | 12 |
| | 13, 31 | 13 |
| | 14, 41 | 14 |
| | 23, 32 | 23 |
| | 24, 42, | 24 |
| | 34, 43 | 34 |
| **Total** | $^4P_2 = 12$ | $^4C_2 = 6$ |

**Theorem 5.** If $0 \le r \le n$ then, the number of r combinations of a set with n elements is given by

$$^nC_r = \frac{n!}{r!(n-r)!}.$$

**Proof.** The 'r' permutations of the set with n–elements is given by P(n, r) where ordering of elements is considered. The ordering can be done in P(r, r) = r! ways. As the order does not matter in combination, hence, P(n, r) = C(n, r) × P(r, r)

$$\Rightarrow C(n,r) = \frac{P(n,r)}{P(n,r)} = \frac{n!}{(n-r)!r!}.$$ **Hence proved.**

In general when order matters, we count the number of sequences or permutations; when order does not matter, we count the number of subsets or combinations.

## 6.4.1 Some Important Results

(1) $^nC_0 = {}^nC_n = 1$

(2) $^nC_{n-1} = n$

(3) $^nC_r = \frac{n}{r} \times {}^{n-1}C_{r-1}$

(4) $^nP_r = r! \times {}^nC_r$ i.e, $\frac{^nP_r}{^nC_r} = r$

(5) $^nC_1 = n$; $^nC_2 = \frac{n(n-1)}{2!}$; $^nC_3 = \frac{n(n-1)(n-2)}{3!}$ and so on.

(6) $^nC_r = {}^nC_{n-2}$

**Example 19.** If n and k be positive integers with $n \ge k$. Then prove that

C(n + 1, k) = C (n, k – 1) + C(n, k) (Pascal's identity). [MREC 2000, Raj. 2002, 2003]

**Solution.** $C(n, k-1) = \frac{n!}{[n-k-1!]!(k-1)!} = \frac{n!}{[n-k+1!]!(k-1)!}$

$$C(n, k) = \frac{n!}{(n-k)!k!}$$

$$\therefore \text{ R.H.S.} = C(n, k-1) + C(n, k) = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!}$$

$$= \frac{n!}{(n-k+1)(n-k)!(k-1)!} + \frac{n!}{(n-k)!k(k-1)!}$$

$$= \frac{n!}{(n-k)!(k-1)!}\left[\frac{1}{n-k+1}+\frac{1}{k}\right] = \frac{n!}{(n-k)!(k-1)!}\cdot\frac{n+1}{k(n-k+1)}$$

$$= \frac{(n+1!)}{(n-k+1)!k!} = \frac{(n+1)!}{[(n+1)-k]!k!} = C(n+1,k)$$

$= \text{L.H.S.}$

Hence $C(n + 1, k) = C(n, k - 1) + C(n, k)$ **Proved.**

**Example 20.** How many diagonals has a regular polygon with n–sides? Which of them has the same number of diagonals as sides? [Raj. 2002, 2005]

**Solution.** The regular polygon with n–sides has n–vertices. Any two vertices determine either a side or a diagonal. Now, from n–vertices, two vertices can be chosen in nC2 ways.

$\therefore\ {}^nC_2 = \text{sides} + \text{diagonals}$

or $\dfrac{n(n-1)}{2} = \text{sides} + \text{diagonals}$

$\therefore$ Diagonals $= \dfrac{n(n-1)}{2} - n$ (as there are n–sides)

$= \dfrac{n(n-3)}{2}$ **Ans.**

Further, when diagonals = sides

$\Rightarrow \dfrac{n(n-3)}{2} = n \Rightarrow n = 0, 5$

as $n \neq 0 \Rightarrow n = 5$.

$\therefore$ Pentagon is the only regular polygon having same number of diagonals as the sides.

**Example 21.** Suppose that a valid computer password consists of four characters, the first of which is a letter, chosen from the set {A, B, C, D, E, F} and the remaining three characters are letters chosen from the English alphabet or digits chosen from the set T = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, how many different passwords are there? [Raj. 2001]

**Solution.** An English alphabet can be chosen in ${}^{26}C_1 = 26$ ways. A digit from the given set can be chosen in ${}^{10}C_1 = 10$ ways. A letter from the given set can be chosen in ${}^6C_1 = 6$ ways.

Hence by sum rule, a character may be an English alphabet or a digit chosen from given set in (26 + 10) = 36 ways

$\therefore$ By product rule, the number of different passwords

$= 6 \times 36 \times 36 \times 36 = 2{,}79{,}936$.

**Example 22.** How many different seven person committee can be formed containing three female members from an available set of 20 females and four males from an available set of 30 male members.

**Solution.** Three females can be chosen from an available set of 20 in ${}^{20}C_3$ ways. Four male members can be chosen from an available set of 30 males in ${}^{30}C_4$ ways.

$\therefore$ Seven person committee can be formed in required manner in $= {}^{20}C_3 \times {}^{30}C_4$ ways

$=$ 31,241,700 ways.

**Example 23.** How many ways are there to pick a five person basket ball team from 12 possible players ? How many selections include the weakest and the strongest players? **[MREC 2000]**

**Solution**. Required number of ways $= {}^{12}C_5 = 792$.

(11 –(3 + 2 +1) are to be chosen from the set of three elements (repetition allowed). Hence required number of ways = number of solutions = C(3 + 5 – 1), 5) = C(7, 5) = 21.

**Example 31.** In how many ways can a prize winner choose three CD's from top ten list if repeats are allowed ?

**Solution.** Required number of ways = C(10 + 3 – 1, 3) = C(12, 3) = 220.

### 6.4.4 Combinations of Objects not all Different

**Theorem 8.** The total number of combinations which can be made of n–different objects taken some or all at a time is ${}^nC_1 + {}^nC_2 + {}^nC_3 + ..... + {}^nC_n = 2^n - 1$

**Proof.** The total number of combinations of n–different objects

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + ..... + {}^nC_n \quad .....(1)$$

Since $(x + y)^n = {}^nC_0\, x^ny^0 + {}^nC_1\, x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + ..... + {}^nC_n\, x^0y^n$

taking x = y = 1, we get

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ..... + {}^nC_n \quad .....(2)$$

Since, we must select atleast one object at a time so the required number of combinations $= {}^nC_1 + {}^nC_2 + ..... + {}^nC_n$

$\therefore$ (2) $\Rightarrow {}^nC_1 + {}^nC_2 + ..... + {}^nC_n = 2^n - 1$ $\quad (\because {}^nC_0 = 1)$ **Hence proved.**

**Example 32.** Determine all possible ways in which a student may attempt one or more questions from 6 given questions having an alternative.

**Solution.** There are three ways in which each of the 6 questions may be attempted – the question itself, it's alternative, or none of them.

$\therefore$ 6 questions may be attempted in $3^6$ ways.

But as this also includes the case when no question (i.e. all six questions are not done) is attempted. Hence required number of ways = $3^6 - 1 = 729 - 1 = 728$.

**Example 33.** In order to pass B.E. III semester examination minimum marks have to be secured in each of the six subjects. In how many cases can a student fail ?

**Solution.** There are two possibilities a student fails or passes in each of the six subjects. Thus, possibility of fail or pass in all the six subjects can be considered in $2^6$ ways.

But, this number includes the case in which student passes in all six subjects.

$\therefore$ Required number of ways = $2^6 - 1 = 63$.

**Example 34.** In how many ways can a teacher choose one or more students from six eligible students.

Solution. There are $2^6 = 64$ subsets of the set consisting of the 6 students. Also as the empty set must be deleted since one or more student is to be chosen, therefore required number of ways

$$= 2^6 - 1 = 64 - 1 = 63.$$

**Example 35.** In how many ways three or more persons be selected from 12 persons.

**Solution.** Number of ways of choosing one or more of 12 persons = $2^{12} - 1 = 4095$. Number of ways of choosing one or two of the 12 persons = ${}^{12}C_1 + {}^{12}C_2 = 12 + 66 = 78$. Hence number of ways of choosing three or more persons out of 12 = 4095 – 78 = 4017.

## ILLUSTRATIVE EXAMPLES

**Example 36.** A computer science professor has seven different programming books on a bookshelf. Three of the books deal with FORTRAN, the other four are concerned with C. In how many ways can the professor arrange these books on the shelf :–

(i) if there are no restrictions.

(ii) if the language should alternate.

(iii) if all the FORTRAN books must be next to each other.

(iv) if all FORTRAN books must be next to each other and all C books must be next to each other. **[MREC 2000]**

**Solution.** (i) Number of required ways = 7! = 5040.

(ii) The number of ways to arrange seven different books when languages are alternate is C F C F C F C

= 4! × 3! = 24 × 6 = 144.

(iii) Here we have 5 books in total, which can be arranged in 5! ways. Further the three books of FORTRAN can be arranged in 3! ways

∴ Required number of ways = 5! × 3! = 720.

(iv) In this case we have only two books which can be arranged in 2! ways. Also the three FORTRAN books can be arranged in 3! ways and four C books can be arranged in 4! ways

∴ Required number of ways = 3! × 4! × 2 = 288.

**Example 37.** In how many ways can the letters in MISSISSIPPI be arranged so that (i) two P's are to be next to each other, (ii) two P's are seperated. **[Raj. 1999]**

**Solution.** Total number of ways in which letters of the given words can be arranged = $\frac{11!}{4!4!2!}$ = 34650.

(i) Total arrangements in which two P's always come together.

(ii) Required number of ways, when two P's don't come together = 34650 – 6300 = 28350.

**Example 38.** Five boys and five girls are to be seated in a row. In how many ways can they be seated if

(i) All boys must be seated in five left most seats.

(ii) John (one boy out of five) and Mary (one girl out of five) must be seated together. **[Raj. 1999]**

Solution. (i) They should be seated in the following pattern BBBBBGGGGG. The five boys can be seated in 5! ways within themselves and the five girls can be seated in 5! ways within themselves.

∴ Required number of ways = 5! × 5! = 14400.

(ii) Here we have to arrange 4 Boys, 4 Girls and 1 Pair (John and Mary). But John and Mary may sit together in 2! ways.

∴ Required number of ways = 9! × 2! = 725760.

**Example 39.** Show that for all integers n, r > 0 if n + 1 > r, then

$$P(n + 1, r) = \frac{n+1}{(n+1-r)} P(n,r)$$

**Solution.** $P(n + 1, r) = \frac{(n+1)!}{(n+1-r)!}$

(a) Given $2n = \frac{n(n-3)}{2} \Rightarrow 4n = n^2 - 3n$

$\Rightarrow (n^2 - 7n) = 0 \Rightarrow n = 0, 7$

$n \neq 0 \Rightarrow n = 7.$

(b) Given $3n = \frac{n(n-3)}{2} \Rightarrow n^2 - 9n = 0 \Rightarrow n = 9.$

**Example 50.** How many ways can you choose three of seven fiction books and two of six non fiction books to take with you on your vacation ?

Solution. Required number of ways $= {}^7C_3 \times {}^6C_2$

$$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} \times \frac{6 \cdot 5}{2} = 35 \times 15$$

$$= 525.$$

**Example 51.** Into how many parts are the diagonals of a convex octagon decomposed, given that no three of these diagonals are concurrent except at a vertex.

**Solution.** For a convex n–gon, number of diagonals $= \frac{n(n-3)}{2}$.

Also four vertices will result in exactly one intersection between two diagonals.

$\therefore$ Number of such intersections $= {}^nC_4$.

Also as the intersection point divides each diagonal into 2 line–segments, hence the number of required line segments $= 2\,{}^nC_4 + \frac{n(n-3)}{2}$.

For an octagon n = 8

$\therefore$ Number of required line segments

$$= 2 \times {}^8C_4 + \frac{8(8-3)}{2} = 140 + 20 = 160.$$

**Example 52.** How many solutions are there to the equation

$x_1 + x_2 + x_3 + x_4 = 17$

where $x_1, x_2, x_3$ and $x_4$ are non–negative integers?

**Solution.** The problem consists of selection of 17 items where $x_1$ items corresponds to type I, $x_2$ items corresponds to type II, $x_3$ items corresponds to type III and $x_4$ items corresponds to type IV.

$\therefore$ Required number of solutions $= C(4 + 17 - 1, 17) = {}^{20}C_{17} = {}^{20}C_3 = 1140.$

**Example 53.** A password consists of two alphabets of English followed by 2 digits. How many passwords exists when upper case and lower case letters are treated (i) equivalent (ii) different. [Raj 2003]

Solution. (i) There are 26 English alphabets and 10 digits.

$\therefore$ Required number of passwords $= 26 \times 26 \times 10 \times 10 = 67600.$

(ii) Now there are 26 + 26 = 52 English alphabets and 10 digits.

$\therefore$ Required number of passwords $= 52 \times 52 \times 10 \times 10 = 270400.$

**Example 54.** Prove that $\sum_{k=0}^{r} C(n+k,k) = C(n + r + 1, r).$ [Raj. 2003]

**Solution.** For r = 0, the given relation is true. Let it be true for r = p i.e.,

$$\sum_{k=0}^{p} C(n+k,k) = C(n + p + 1, p) \quad .....(1)$$

Now, consider

$$\sum_{k=0}^{p+1} C(n+k,k) = \sum_{k=0}^{p} C(n+k,k) + C(n + p + 1, p + 1)$$

$$= C(n + p + 1, p) + C(n + p + 1, p + 1) \quad \text{[using (1)]}$$

$$= \frac{(n+p+1)!}{(n+1)!p!} + \frac{(n+p+1)!}{n!(p+1)!}$$

$$= \frac{(n+p+1)!}{n!p!}\left[\frac{(n+p+2)}{(n+1)(p+1)}\right]$$

$$= \frac{(n+p+2)!}{(n+1)!(p+1)!} = C(n+\overline{p+1}+1, p+1)$$

It shows that the relation is true for r = p + 1. Thus by induction the result holds for all r i.e.,

$$\sum_{k=0}^{r} C(n+k,k) = C(n + r + 1, r)$$ **Proved.**

**Example 55.** Prove that $^{n}P_{r} = {}^{n-1}P_{r} + r\,{}^{n-1}P_{r-1}$.

**Solution.** R.H.S. $= {}^{n-1}P_{r} + r\,{}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[1 + \frac{r}{n-r}\right]$$

$$= \frac{(n-1)!}{(n-r-1)!}\left(\frac{n}{n-r}\right)$$

$$= \frac{n!}{(n-r)!}$$

$$= {}^{n}P_{r}$$ **Proved.**

**Example 56.** Prove that C(2n, 2) = 2C(n, 2) + $n^2$.

**Solution.** LHS = C(2n, 2)

$$= \frac{(2n)!}{(2n-2)!2!} = \frac{2n(2n-1)(2n-2)}{2(2n-2)!}$$

$$= n(2n - 1)$$

$$= n(n - 1) + n^2$$

$$= 2\left[\frac{1}{2}n(n-1)\right] + n^2$$

$$= 2C(n, 2) + n^2$$

= R.H.S. **Proved.**

**Example 57.** How many bit strings of length 8 do not contain '6' consecutive 0's ?

**Solution.** Total bit strings of length 8 = $2^8$.

Number of bit strings containing six consecutive 0's = 2 × 2 × 2 = 8.

∴ Required number of bit strings that do not contain 6 cosecutive 0's = $2^8 - 8 = 248$.

**Example 58.** Show that if n and k are positive integers, then $C(n + 1, k) = \frac{(n+1)C(n,k-1)}{k}$.

Proof. L.H.S. $= C(n + 1, k)$

$$= {}^{n+1}C_k = \frac{(n+1)!}{k!(n+1-k)!} \quad .....(1)$$

R.H.S. $= \frac{(n+1)}{k}C(n,k-1)$

$$= \frac{(n+1)}{k} \times \frac{n!}{(k-1)!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} \quad .....(2)$$

∴ (1) and (2) ⇒ L.H.S. = R.H.S. Hence proved.

**Example 59.** How many license plates consisting of three letters followed by three digits contain no letter or digits twice?

**Solution.** There are 26 letters and 10 digits.

∴ Required number of plates

= 26 × 25 × 24 × 10 × 9 × 8

= 11,232,000.

**Example 60.** How many strings are there of lowercase letters of lengths four or less ?

Solution. There are 26 lower case letters.

∴ Required number of strings

$= 26^4 + 26^3 + 26^2 + 26^1 + (26)^0$ (counting the empty string)

= 4,569,76 + 17,576 + 676 + 26 + 1

= 4,75,255.

**Example 61.** Determine the number of ways in which we can make up strings of four distinct letters followed by three distinct digits. [Raj. 2003]

**Solution.** There are 26 letters out of which 4 can be chosen in ${}^{26}C_4$ ways. Also there are 10 digits out of which three can be chosen in ${}^{10}C_3$ ways.

∴ Required number of ways = ${}^{26}C_4 \times 4! \times {}^{10}C_3 \times 3!$.

**Example 62.** How many different bit strings can be formed using six 1's and eight 0's.

Solution. Total length of bit string = 6 + 8 = 14.

Number of different strings = ${}^{14}C_8 = {}^{14}C_6 = 3003$.

**Example 63.** Five fair coins are tossed and the results are recorded:

(i) How many different sequences of heads and tails are possible?

(ii) How many of the sequences recorded have exactly one head?

(iii) How many of the sequences have exactly three heads recorded? [Raj. 2004]

**Solution.** Tossing of each coin results in two outcomes head or a tail.

(i) ∴ Requirednumber of sequences = 2 × 2 × 2 × 2 × 2 = 32.

(ii) Now there is one possible outcome for 1 coin i.e., Head and one possible outcome for all the other four coins i.e., Tail. But this one coin may be chosen from five coins in ${}^5C_1$ ways.

∴ Required number of sequences = ${}^5C_1 \times 1 = 5$.

(iii) Three coins showing Head can be chosen among five coins in ${}^5C_3$ ways.

∴ Required number of sequences = ${}^5C_3 = 10$.

**Example 64.** Determine the number of ways to place 2k + 1 indistinguishable balls in three distinct boxes so that any two boxes together will contain more balls than other one.

**Solution.** The total number of ways in which (2k + 1) objects can be placed in three distinct boxes

$$= C(3 + (2k + 1) - 1, 2k + 1)$$

$$= {}^{2k+3}C_{2k+1} = \frac{(2k+3)(2k+2)}{2}.$$

Now consider the case when number of balls in two boxes is less than the number of balls in third box. This means that objects or greater can be place in this third box and remaining (2k + 1) – (k + 1) = k or less will be placed in the two boxes.

Number of such ways $= C(3 + k - 1, k) = {}^{k+2}C_k = \frac{(k+2)(k+1)}{2}$.

But this third box can again be chosen in three ways, out of the three boxes. Hence total number of ways in which balls in two boxes is less than number of balls in third box

$$= 3 \times C(3 + k - 1, k) = \frac{3}{2}(k+2)(k+1) \qquad .....(2)$$

∴ (1) and (2) give the required number of ways

$$= \frac{(2k+3)(2k+2)}{2} - \frac{3}{2}(k+2)(k+1)$$

$$= \frac{(k+1)}{2}[2(2k+3) - 3(k+2)] = \frac{k(k+1)}{2} \qquad \textbf{Ans.}$$

**Example 65.** If out of p + q + r things p be alike, q be alike and rest are different, then find the total number of selections in which atleast one thing is selected.

**Solution.** The number of selections from p alike things = p + 1 as we may choose 0 or 1 or 2 or 3 or ..... or p things from the given p things.

Similarly, the number of selections from q alike things = q + 1, and the number of selection from r different things = $2^r$

∴ The total number of selections = $(p + 1)(q + 1)\, 2^r - 1$.

(as the case, when all things are left must be excluded)

## EXERCISE 6

1. How many seven digits telephone numbers are possible, if
   (a) only odd digits may be used.
   (b) the number must be a multiple of 100 ?
   (c) the first three digits are 481 ?
   [Ans. (a) $5^7$ (b) $10^5$ (c) $10^4$]

[Ans. 2n]

25. Find the sum of all numbers greater than 10000 formed by using the digits 1, 3, 5, 7, 9, no digit being repeated in any number.

[Ans. 6666600]

26. In how many ways 10 programmers can sit on a round table to discuss the project so that the project leader and a particular programmer always sit together ?

[Ans. $8! \times 2! = 80640$]

27. How many 8–bit strings are there that end with 0111 ?

[Ans. 24]

28. Determine the number of triangles that are formed by selecting points from a set of 15 points out of which 8 are collinear.

[Ans. ${}^{15}C_3 - {}^{8}C_3 = 854$]

29. If n fair coins are tossed and the results recorded, how many

(a) record sequences are possible ?

(b) sequences contain exactly three tails, assuming $n > 3$ ?

(c) sequences contain exactly k–heads, assuming $n > k$ ?

[Ans. (a) $2^n$ (b) ${}^{n}C_3$ (c) ${}^{n}C_k$]

30. An urn contain 15 balls, 8 of which are red and 7 are black. In how many ways can 5 balls be chosen so that

(a) all 5 are red ?

(b) all 5 are black ?

(c) 2 are red and 3 are black ?

(d) 3 are red and 2 are black ?

[Ans. (a) 56 (b) 21 (c) 980 (d) 1176]

31. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of scores is 100 and each question is worth at least 5 points.

[Ans. 59C50]

32. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes ?

[Ans. 5]

33. Find the number of triangles that can be formed by the vertices of an octagon. Also find the number of triangles formed by the vertices of the octagon if its sides are not the sides of any triangle.

[Ans. 56, 16]

34. How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11$$

where $x_1$, $x_2$ and $x_3$ are non–negative integers ?

[Hint : Introduce an auxiliary variable x4 such that $x_1 + x_2 + x_3 + x_4 = 11$]

[Ans. 364]

□□□