7

Recurrence Relations & Generating Functions

OBJECTIVES

- Introduction
- Linear Recurrence Relations with Constant Coefficients
- Discrete Numeric Functions
- Generating Functions
- Applications of Generating Functions

7.1 Introduction

Let $\{a_n\} = \{a_0, a_1, \ldots\ldots, a_r, \ldots\ldots\}$ be a sequence of real numbers. A relation that expresses an in terms of one or more of the previous terms i.e., $a_0, a_1, \ldots\ldots, a_{n-1}$, where n is a non–negative integer, is called a recurrence relation for the sequence $\{a_n\}$.

If a recurrence relation is satisfied by the terms of the sequence $\{a_n\}$ then an is called the solution of the recurrence relation. A recurrence relation is also known as a difference equation.

Example 1. $a_n = a_{n-1} - a_{n-2} + 2a_{n-3}$, $n \geq 3$ with initial conditions $a_0 = 1$, $a_1 = 3$, $a_2 = 5$ is a recurrence relation.

Example 2. The Fibonacci sequence i.e., 1, 1, 2, 3, 5, 8, ... can be defined by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 1$.

7.2 Order of a Recurrence Relation

The difference of the greatest and the smallest subscript appearing in the recurrence relation is called its order.

Example 3. The order of the recurrence relation appearing in Example 1 is $n - (n - 3) = 3$ and the order in Example 2 is $n - (n - 2) = 2$.

7.3 Linear Recurrence Relations with Constant Coefficients

A linear recurrence relation of order $k \in Z^+$ with constant coefficients is an equation of the form

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \ldots\ldots + c_ka_{n-k} = f(n) \qquad \ldots\ldots(1)$$

where $c_0, c_1, \ldots\ldots, c_k$ are constants such that $c_0 \neq 0$ and $c_k \neq 0$ and f(n) is a function of n alone.

If in relation (1), $f(n) = 0$ then it is said to be linear homogeneous recurrence relation of order k otherwise [i.e., when $f(n) \neq 0$] it is said to be linear non–homogeneous recurrence relation of order k.

Example 4. Classify the following recurrence relations :

(i) $a_n = a_{n-1} + a_{n-2}$ (ii) $3a_n + 5a_{n-1} = 2^n$

(iii) $a_n = 2a_{n-1} + a^2_{n-2}$ (iv) $a_n + n\,a_{n-1} = n^2 + n + 1$

Solution. (i) $a_n = a_{n-1} + a_{n-2}$ is a linear homogeneous recurrence relation of order 2 with constant coefficients.

(ii) $3a_n + 5a_{n-1} = 2^n$ is a linear non–homogeneous recurrence relation of order 1 with constant coefficients.

(iii) $a_n = 2a_{n-1} + a^2_{n-2}$ is a non–linear (pow . of all a_i's is not 1) homogeneous recurrence relation of order 2 with constant coefficients.

(iv) $a_n + n\,a_{n-1} = n^2 + n + 1$ is a linear non–homogeneous recurrence relation of order 1 with variable coefficients ($\because$ coefficients of all ai's are not constant).

7.4 Solution to Linear Homogeneous Recurrence Relations with Constant Coefficients

A linear homogeneous recurrence relation of order k with constant coefficients is of the form

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \ldots\ldots + c_ka_{n-k} = 0, \quad c_0 \neq 0,\ c_k \neq 0 \qquad \ldots\ldots(1)$$

Let the solution of (1) is of the form $a_n = r^n$, where r is a constant, then it must satisfy the given equation (1) i.e.,

$$c_0 r^n + c_1 r^{n-1} + c_2 r^{n-2} + \ldots\ldots + c_k r^{n-k} = 0$$

or, $c_0 r^k + c_1 r^{k-1} + c_2 r^{k-2} + \ldots\ldots + c_k = 0$ .....(2)

This equation (2) is called the characteristic equation for the relation (1) and the roots of this equation are called characteristic roots.

Since equation (2) is of k^{th} degree so it has k roots, namely, $r_1, r_2, \ldots\ldots, r_k$. The solution of relation (1) depends on the nature of these roots. For this we have to consider the following cases.

Case 1. When the roots are real and distinct

If $r_1, r_2, \ldots\ldots, r_k$ are real and distinct roots of equation (2) then the solution of relation (1) is given as

$$a_n = C_1 r_1^n + C_2 r_2^n + \ldots\ldots + C_k r_k^n.$$

Case 2. When the roots are real and equal

If two roots r_1 and r_2 are real and equal i.e., $r_1 = r_2 = r$(say) and the remaining roots are real and distinct then the solution of the relation (1) is given as

$$a_n = (C_1 + C_2 n)r^n + C_3 r_3^n + \ldots\ldots + C_k r_k^n.$$

Similarly, if three roots r_1, r_2 and r_3 are equal i.e., $r_1 = r_2 = r_3 = r$(say) and the remaining roots are real and distinct then the solution of the relation (1) is given as

$$a_n = (C_1 + C_2 n + C_3 n^2)r^n + C_4 r_4^n + \ldots\ldots + C_k r_k^n.$$

In general, if the root r1 is repeated k times, then the solution of the relation (1) is given as

$$a_n = (C_1 + C_2 n + C_2 n^2 + \ldots\ldots + C_k n^{k-1})\, r_1^n.$$

Case 3. When the roots are imaginary and distinct

If $\alpha \pm i\,\beta$ are two imaginary roots of the equation (2) then the corresponding part of the solution of the relation (1) is given as

$$a_n = (\alpha^2+\beta^2)^{\frac{n}{2}}(C_1 \cos n\theta + C_2 \cos n\theta); \theta = \tan^{-1}\frac{\beta}{\alpha}.$$

Case 4. When the roots are imaginary and repeated

If $\alpha \pm i\,\beta$ and $\alpha \pm i\,\beta$ are four roots of the equation (2) then the corresponding solution of the relation (1) is given as

$$a_n = (\alpha^2+\beta^2)^{\frac{n}{2}}\left[(C_1 + C_2 n)\cos n\theta + (C_3 + C_4 n)\sin n\theta\right]; \theta = \tan^{-1}\frac{\beta}{\alpha}$$

Remark 1. In all the above 4 cases $C_1, C_2, \ldots\ldots, C_k$ are the constants whose values are determined by the given initial conditions.

Example 5. Solve the recurrence relation an + an–1 – 6an–2 = 0

where $n \geq 2$ and $a_0 = 1$, $a_1 = 2$. [Raj. 2001]

$a_n = t^n$

Solution. The characteristic equation for the given relation is

$$r^2 + r - 6 = 0$$

$\Rightarrow r = 2, -3$ (real and distinct)

$\therefore$ The solution is

$a_n = C_1(2)^n + C_2(-3)^n$ [Case 1] .....(1)

Now, given that $a_0 = 1$ and $a_1 = 2$

put n = 0 in (1), we get

$a_0 = C_1 + C_2 \Rightarrow C_1 + C_2 = 1$ .....(2)

put n = 1 in (1), we get

$a_1 = 2C_1 - 3C_2 \Rightarrow 2C_1 - 3C_2 = 2$ .....(3)

On solving (2) and (3) we get

$C_1 = 1, C_2 = 0$

$\therefore$ The required solution is

$a_n = 2^n$.

Example 6. Solve the recurrence relation

$$a_n = a_{n-1} + a_{n-2}, n \geq 2$$

with initial conditions $a_0 = 0$ and $a_1 = 1$.

Solution. The characteristic equation for the given relation is

$$r^2 - r - = \Rightarrow r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ The solution is

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad .....(1)$$

Now, given that $a_0 = 0$ and $a_1 = 1$ so (1) gives

$a_0 = C_1 + C_2 \Rightarrow C_1 + C_2 = 0$ .....(2)

$$\text{and } a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad .....(3)$$

$$\Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right) - C_1\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad \text{[using (2)]}$$

$$\Rightarrow \frac{C_1}{2}[1+\sqrt{5}-1+\sqrt{5}] = 1 \Rightarrow \frac{C_1}{2}(2\sqrt{5}) = 1$$

$$\Rightarrow C_1 = \frac{1}{\sqrt{5}}$$

and then (2) $\Rightarrow C_2 = -C_1 = -\frac{1}{\sqrt{5}}$

$\therefore$ The required solution is

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example 7. Solve the recurrence relation

$$d_n = 2d_{n-1} - d_{n-2}$$

with initial conditions $d_1 = 1.5$ and $d_2 = 3$. [Raj. 2003]

Solution. The characteristic equation is

$$r^2 - 2r + 1 = 0 \Rightarrow r = 1, 1$$

$\therefore$ The solution is

$$d_n = (C1 + C_2 n)(1)^n = C_1 + C_2 n \quad .....(1)$$

Given that $d_1 = 1.5$ and $d_2 = 3$ so (1) gives

$$d_1 = C_1 + C_2 \Rightarrow C_1 + C_2 = 1.5 \quad .....(2)$$

and $d_2 = C_1 + 2C_2 \Rightarrow C_1 + 2C_2 = 3$ .....(3)

(3) – (2) $\Rightarrow C_2 = 1.5$

$\therefore$ (2) Þ $C_1 = 1.5 - C_2 = 1.5 - 1.5 = 0$

$\therefore$ The required solution is $d_n = 1.5n$.

Example 8. Solve $a_{n+2} = a_{n+1} + a_n$, $n \geq 0$ and $a_0 = 1$, $a_1 = 2$. [Raj. 2001, MREC 2001]

Solution. The characteristic equation is

$$r^2 - r - 1 \Rightarrow r = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ The solution is

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad .....(1)$$

Given that $a_0 = 1$ and $a_1 = 2$ so (1) gives

$$a_0 = C_1 + C_2 \Rightarrow C_1 + C_2 = 1 \quad .....(2)$$

and $a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$

$$\Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) \quad .....(3)$$

Now, $\left(\frac{1-\sqrt{5}}{2}\right) \times (2) - (3)$ gives

$$\left(\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right)C_1 = \frac{1-\sqrt{5}}{2} - 2$$

$$\Rightarrow -\sqrt{5}C_1 = -\frac{3+\sqrt{5}}{2} \Rightarrow C_1 = \frac{3+\sqrt{5}}{2\sqrt{5}} = \frac{1}{10}(5+3\sqrt{5})$$

$\therefore$ (2) $\Rightarrow C_2 = 1 - C_1 = 1 - \frac{1}{10}(5+3\sqrt{5}) = \frac{1}{10}(5-3\sqrt{5})$

$\therefore$ The required solution is

$$a_n = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example 9. Solve the recurrence relation

$$a_n = 5a_{n-1} + 6a_{n-2},\ n \geq 2$$

with $a_0 = 1$, a1 = 3. [Raj. 2000]

Solution. The characteristic equation is

$$r^2 - 5r - 6 = 0 \Rightarrow r = 6, -1$$

∴ The solution is

$$a_n = C_1 6^n + C_2(-1)^n \quad \text{.....(1)}$$

given $a_0 = 1 \Rightarrow 1 = C_1 + C_2$ .....(2)

and a1 = 3 Þ 3 = 6C1 – C2.....(3)

on solving (2) and (3), we get

$$C_1 = \frac{4}{7},\ C_2 = \frac{3}{7} \quad \text{.....(4)}$$

∴ The required solution is

$$an = \frac{4}{7}6^n + \frac{3}{7}(-1)^n \quad \text{[using (4) in (1)]}$$

Example 10. Solve the recurrence relation

$a_{n+2} + a_n = 0,\ n \geq 0$ with $a_0 = 0,\ a_1 = 3$. [Raj. 2000]

Solution. The characteristic equation is

$$r^2 + 1 = 0 \Rightarrow r = \pm i \quad \text{(Case 3)}$$

∴ The solution is

$$a_n = (1)^{n/2}(C_1 \cos n\theta + C_2 \sin n\theta)$$

where $\theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$$\therefore\ a_n = C_1 \cos\frac{n\pi}{2} + C_2 \sin\frac{n\pi}{2} \quad \text{.....(1)}$$

$$\left.\begin{array}{ll}\text{given} & a_0 = 0 \Rightarrow 0 = C_1 \Rightarrow C_1 = 0 \\ \text{aslo} & a_1 = 3 \Rightarrow 3 = C_2 \Rightarrow C_2 = 3\end{array}\right\} \quad \text{.....(2)}$$

using (2) in (1) we get the required solution as

$$a_n = 3\sin\frac{n\pi}{2}.$$

7.5 Solution to Linear Non-Homogeneous Recurrence Relations with Constant Coefficients

A linear non–homogeneous recurrence relation of order k with constant coefficients is of the form

$$c_0 a_n + c_1 a_{n-1} + \ldots\ldots + c_k a_{n-k} = F(n) \quad \text{.....(1)}$$

where $c_0, c_1, \ldots\ldots, c_k$ are real numbers such that $c_0 \neq 0$ and $c_k \neq 0$ and F(n) is a function of n alone. The complete solution of relation (1) is of the form

$$a_n = a_n^{(h)} + a_n^{(p)} \quad \text{.....(2)}$$

where $a_n^{(h)}$ is a solution of the following associated homogeneous recurrence relation

$$c_0a_n + c_1a_{n-1} + ... + c_ka_{n-k} = 0 \quad .....(3)$$

and $a_n^{(p)}$ is a particular solution of the relation (1).

There is no general method to find the particular solution of the recurrence relation (1). It depends on the nature (form) of F(n). Let, in relation (1)

$$F(n) = (b_tn^t + b_{t-1}n^{t-1} + ... + b_1n + b_0)\, s^n,$$

where $b_0, b_1, ..., b_t$ and s are real numbers. Now there are two possibilities for s, as given below.

Case 1. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation

In this case we assume that the form of a particular solution is

$$a_n^{(p)} = (p_tn^t + p_{t-1}n^{t-1} + ... + p_1n + p_0)\, s^n \quad .....(4)$$

Since $a_n^{(p)}$ is a particular solution of relation (1) so it must satisfy the relation (1). We substitute $a_n = a_n^{(p)}$ on the left hand side of (1) and then on comparing the coefficients on both sides we obtain the values of the constants $p_0, p_1, ..., p_t$ which on substitution in equation (4) give the desired particular solution.

Case 2. When s is a root of the characteristic equation of the associated linear homogeneous recurrence relation with multiplicity m

In this case we assume that the form of a particular solution is

$$a_n^{(p)} = n^m(p_tn^t + p_{t-1}n^{t-1} + ... + p_1n + p_0)\, s^n \quad .....(5)$$

where $p_0, p_1, ..., p_t$ are real numbers determined by the same procedure as given in Case 1 and m is any positive integer.

Example 11. Solve the recurrence relation

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1.$$ [Raj. 2002]

Solution. The characteristic equation is

$$r^2 + 5r + 6 = 0 \Rightarrow r = -2, -3$$

$$\therefore\ a_n^{(h)} = C_1(-2)^n + C_2(-3)^n \quad .....(1)$$

Now, here $F(n) = (3n^2 - 2n + 1)\,(1)^n$

The particular solution of the given relation is of the form

$$a_n^{(p)} = (p_2n^2 + p_1n + p_0) \quad .....(2)$$

[Since here t = 2 (highest power of n) and s = 1 which is not a root of characteristic equation so we apply Case 1]

It must satisfy the given relation i.e.,

$$(p_2n^2 + p_1n + p_0) + 5\{p_2(n-1)^2 + p_1(n-1) + p_0\} + 6\{p_2(n-2)^2 + p_1(n-2) + p_0\} = 3n^2 - 2n + 1$$

or, $n^2(p_2 + 5p_2 + 6p_2) + n(p_1 - 10p_2 + 5p_1 - 24p_2 + 6p_1)$

$+ (p_0 + 5p_2 - 5p_1 + 5p_0 + 24p_2 - 12p_1 + 6p_0) = 3n^2 - 2n + 1$

or, $12p_2n^2 + (12p_1 - 34p_2)n + (12p_0 - 17p_1 + 29p_2) = 3n^2 - 2n + 1$

On comparing the coefficients of equal powers of n on both sides, we get

$$12p_2 = 3 \Rightarrow p_2 = \frac{1}{4} \quad .....(3),$$

$$12p_1 - 34p_2 = -2 \Rightarrow 12p_1 = 34\left(\frac{1}{4}\right) - 2 = \frac{13}{2}$$ [using (3)]

or $p_1 = \frac{13}{24}$ .....(4)

and $12p_0 - 17p_1 + 29p_2 = 1 \Rightarrow 12p_0 - 17\times\frac{13}{24} + 29\frac{1}{4} = 1$

or, $12p_0 = 1 + \frac{221}{24} - \frac{29}{4} = \frac{1}{24}(24 + 221 - 174) = \frac{71}{24}$

or, $p_0 = \frac{71}{288}$

$\therefore$ (2) $\Rightarrow a_n^{(p)} = \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$

Thus, the complete solution of the given relation is

$a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$$a_n = C_1(-2)^n + C_2(-3)^n + \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}.$$

Example 12. Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 42(4)^n$.

Solution. Here the characteristic equation is

$r^2 + 5r + 6 = 0 \Rightarrow r = -2, -3$

$\therefore$ $a_n^{(h)} = C_1(-2)^n + C_2(-3)^n$ .....(1)

Here $F(n) = 42(4)^n$ i.e.; $s = 4$ which is not a root of the characteristic equation so we assume that the particular solution be

$a_n^{(p)} = p_0(4)^n$ .....(2)

[by putting $t = 0$ in Case 1]

It must satisfy the given relation i.e.,

$$p_0(4)^n + 5p_0(4)^{n-1} + 6p_0(4)^{n-2} = 42(4)^n$$

or, $\left(p_0 + \frac{5}{4}p_0 + \frac{6}{16}p_0\right)(4)^n = 42(4)^n$

on comparing the coefficients on both sides, we get

$$p_0 + \frac{5}{4}p_0 + \frac{6}{16}p_0 = 42$$

or, $\left(1 + \frac{5}{4} + \frac{3}{8}\right)p_0 = 42 \Rightarrow \frac{21}{8}p_0 = 42$

or, $p_0 = 16$

$\therefore$ (2) $\Rightarrow a_n^{(p)} = 16(4)^n$

Hence the complete solution is

an $= a_n^{(h)} + a_n^{(p)}$ i.e.,

$a_n = C_1(-2)^n + C_2(-3)^n + 16(4)^n$.

Example 13. Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 1$. [MREC 2002]

Solution. The characteristic equation is

$$r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$$

$\therefore\ a_n^{(h)} = C_1(2)^n + C_2(3)^n$ .....(1)

Here $F(n) = 1$[i.e., here $t = 0$ and $s = 1$ which is not a root of C. equation]

Let the particular solution be

$a_n^{(p)} = p_0$ .....(2)

[in Case 1 put $t = 0$ and $s = 1$]

It must satisfy the given relation so

$$p_0 - 5p_0 + 6p_0 = 1 \Rightarrow 2p_0 = 1 \Rightarrow p_0 = \frac{1}{2}$$

$\therefore\ (2) \Rightarrow a_n^{(p)} = \frac{1}{2}$ .....(3)

Thus, the complete solution is

$a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$$a_n = C_1(2)^n + C_2(3)^n + \frac{1}{2}.$$

Example 14. Solve the recurrence relations

(a) $a_n + a_{n-1} = 3n\ 2^n$

(b) $a_n - 2a_{n-1} = 3(2)^n$.

Solution. (a) an + an–1 = 3n 2n

The characteristic equation is

$$r + 1 = 0 \Rightarrow r = -1$$

$\therefore\ a_n^{(h)} = C(-1)^n$ .....(1)

Here $F(n) = 3n\ 2^n$[i.e., here $t = 1$ and $s = 2$ which is not a root of C. equation]

Let the particular solution be

$a_n^{(p)} = (p_1 n + p_0)2^n$ .....(2)

[On putting $t = 1$ and $s = 2$ in Case 1]

It must satisfy the given relation so

$$(p_1 n + p_0)2^n + \{p_1(n-1) + p_0\}2^{n-1} = 3n2^n$$

or, $\left(p_1 + \frac{p_1}{2}\right)n2^n + \left(p_0 - \frac{p_1}{2} + \frac{p_0}{2}\right)2^n = 3n2^n$

or, $\left(\frac{3p_1}{2}\right)n2^n + \frac{1}{2}(3p_0 - p_1)2^n = 3n2n$

or, $\left(\frac{3p_1}{2}\right)n + \frac{1}{2}(3p_0 - p_1) = 3n$

on comparing the coefficients on both sides, we get

$$\frac{3p_1}{2} = 3 \Rightarrow p_1 = 2$$

and $\frac{1}{2}(3p_0 - p1) = 0 \Rightarrow p_0 = \frac{1}{3}p_1 \Rightarrow p_0 = \frac{2}{3}$

$\therefore$ (2) $\Rightarrow a_n^{(p)} = \left(2n+\frac{2}{3}\right)2^n$

Thus the complete solution is

$a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$a_n = C(-1)^n + \left(2n+\frac{2}{3}\right)2^n$

(b) $a_n - 2a_{n-1} = 3(2)^n$

The characteristic equation is

$r - 2 = 0 \Rightarrow r = 2$

$\therefore a_n^{(h)} = C(2)^n$ .....(1)

Here $F(n) = 3(2)^n$

[i.e., here $t = 0$ and $s = 2$ which is a root of C. equation with multiplicity 1]

Let the particular solution be

$a_n^{(p)} = np_0 2^n$ .....(2)

[by putting $t = 0$, $m = 1$ and $s = 2$ in Case 2]

It must satisfy the given relation so

$np_0 2^n - 2(n-1)p_0 2^{n-1} = 3(2)^n$

or, $(p_0 - p_0)n\, 2^n + p_0\, 2^n = 3(2)^n$

or, $p_0\, 2^n = 3(2)^n$

$\Rightarrow p_0 = 3$

$\therefore$ (2) $\Rightarrow a_n^{(p)} = 3n2^n$.

Thus the complete solution is .....(3)

$a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$a_n = C(2)^n + 3n(2)^n$.

Example 15. Solve the recurrence relation

$a_{n+2} - 4a_{n+1} + 4a_n = 2^n$. [MREC 2001]

Solution. The characteristic equation is

$r^2 - 4r + 4 = 0 \Rightarrow r = 2, 2$

$\therefore a_n^{(h)} = (C_1 + C_2 n)2^n$ .....(1)

Here $F(n) = 2^n$ i.e., $s = 2$ which is a root of the characteristic equation with multiplicity 2. Let the particular solution be

$a_n^{(p)} = n^2 p_0 2^n$ .....(2)

[by taking $m = 2$, $t = 0$ and $s = 2$ in Case 2]

It must satisfy the given relation so

$p_0(n+2)^2\, 2^{n+2} - 4p_0(n+1)^2\, 2^{n+1} + 4p_0 n^2 2^n = 2^n$

or, $4p_0(n^2 + 4n + 4) - 8p_0(n+1)^2 + 4p_0 n^2 = 1$

or, $n^2(4p_0 - 8p_0 + 4p_0) + n(16p_0 - 16p_0) + 16p_0 - 8p_0 = 1$

or, $8p_0 = 1 \Rightarrow p_0 = \frac{1}{8}$

$\therefore$ (2) $\Rightarrow a_n^{(p)} = \frac{1}{8}n^2 2^n$

Thus the complete solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \text{ i.e.,}$$

$$a_n = (C_1 + C_2 n)2^n + \frac{1}{8}n^2 2^n.$$

ILLUSTRATIVE EXAMPLES

Example 16. Solve the recurrence relation

$3a_{n+1} = 2a_n + a_{n-1}, n \geq 1$ with $a_0 = 7, a_1 = 3.$ [MREC 2000]

Solution. Given, $3a_{n+1} - 2a_n - a_{n-1} = 0$

The chracteristic equation is

$$3r^2 - 2r - 1 = 0 \Rightarrow r = 1, -\frac{1}{3}$$

$\therefore$ The solution is

$$a_n = C_1(1)^n + C_2\left(-\frac{1}{3}\right)^n = C_1 + C_2\left(-\frac{1}{3}\right)^n \quad .....(1)$$

given $a_0 = 7 \Rightarrow 7 = C_1 + C_2$ .....(2)

and $a_1 = 3 \Rightarrow 3 = C_1 - -\frac{1}{3}C_2$ .....(3)

(2) and (3) $\Rightarrow C_1 = 4, C_2 = 3$

$\therefore$ (1) $\Rightarrow a_n = 4+3\left(-\frac{1}{3}\right)^n.$

Example 17. Solve the recurrence relation

$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n, n \geq 0$ with $a_0 = 0, a_1 = 1, a_2 = 2.$ [Raj. 2000]

Solution. Given $2a_{n+3} - a_{n+2} - 2a_{n+1} - a_n = 0$

The characteristic equation is

$$2r^3 - r^2 - 2r + 1 = 0 \Rightarrow r = 1, -1, \frac{1}{2}$$

$\therefore$ The solution is

$$an = C_1(1)^n + C_2(-1)^n + C_3\left(\frac{1}{2}\right)^n = C_1 + C_2(-1)^n + C_3\left(\frac{1}{2}\right)^n \quad .....(1)$$

given $a_0 = 0 \Rightarrow C_1 + C_2 + C_3 = 0$ .....(2)

$a_1 = 1 \Rightarrow C_1 - C_2 + \frac{1}{2}C_3 = 1$ .....(3)

and $a_2 = 2 \Rightarrow C_1 + C_2 + \frac{1}{4}C_3 = 2$ .....(4)

on solving (2), (3) and (4) we get

$$C_1 = \frac{5}{2};\ C_2 = \frac{1}{6};\ C_3 = -\frac{8}{3}$$

$$\therefore\ (1) \Rightarrow a_n = \frac{5}{2}+\frac{1}{6}(-1)^n - \frac{8}{3}\left(\frac{1}{2}\right)^n.$$

Example 18. Solve the recurrence relation

$$a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n\ .$$

Solution. The characteristic equation is

$$r^2 - 4r + 4 = 0 \Rightarrow r = 2, 2$$

$$\therefore\ a_n^{(h)} = (C_1 + C_2 n)2^n$$

Here $F(n) = (n + 1)2^n$ i.e., $s = 2$ which is a root of the characteristic equation with multiplicity 2 so the particular solution is given as

$$a_n^{(p)} = n^2(p_1 n + p_0)2^n \qquad .....(1)$$

It must satisfy the given relation so

$$n^2(p_1 n + p_0)2^n - 4(n-1)^2\{p_1(n-1) + p_0\}2^{n-1} + 4(n-2)^2\{p_1(n-2) + p_0\}2^{n-2} = (n+1)2^n$$

or, $(p_1 n^3 + p_0 n^2) - 2(n^2 - 2n + 1)(p_1 n - p_1 + p_0) + (n^2 - 4n + 4)(p_1 n - 2p_1 + p_0) = n + 1$

or, $n^3(p_1 - 2p_1 + p_1) + n^2(p_0 + 2p_1 - 2p_0 + 4p_1 - 2p_1 - 4p_1 + p_0)$

$$+\, n(-4p_1 + 4p_0 - 2p_1 + 4p_1 + 8p_1 - 4p_0) + (2p_1 - 2p_0 - 8p_1 + 4p_0) = n + 1$$

or, $6p_1 n + 2p_0 - 6p_1 = n + 1$

On comparing the coefficients of both sides, we get

$$6p_1 = 1 \Rightarrow p_1 = 1/6$$

and

$$\Rightarrow p_0 = 1$$

$$\therefore\ (1) \Rightarrow a_n^{(p)} = n^2\left(\frac{1}{6}n+1\right)2^n = \frac{1}{6}(n+6)n^2 2^n$$

Thus the complete solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \text{ i.e.,}$$

$$a_n = (C_1+C_2 n)2^n + \frac{1}{6}(n+6)n^2 2^n.$$

Example 19. A bank pays 6% interest annually on savings, compounding the interest monthly. If Ram deposits Rs. 1000 on the first day of May, how much this deposit be worth a year later ?

Solution. Given that annually rate of interest is 6% so the monthly rate is $\frac{6}{12}$ = 0.5% Let P_n denote the amount of Ram's deposit after n months then $P_0 = 1000$ (given).

Also $P_{n+1} = P_n + \frac{P_n \times 0.5}{100}$ $\qquad$ ($\because$ one month interest $= \frac{PRT}{100} = \frac{P_n \times 0.5 \times 1}{100}$)

or, $P_{n+1} = 1.005\, P_n$ $\qquad .....(1)$

which is a recurrence relation whose solution can be obtained as follows.

The characteristic equation of relation (1) is

$r - 1.005 = 0 \Rightarrow r = 1.005$

So the solution is

$$P_n = C(1.005)^n \quad .....(2)$$

given that $P_0 = 1000$ so from (2), we have

$$1000 = C \Rightarrow C = 1000$$

$$\therefore \ (2) \Rightarrow P_n = 1000\ (1.005)^n \quad .....(3)$$

Thus, at the end of one year i.e., after 12 months the deposit of Ram is

$$P_{12} = 1000\ (1.005)^{12}$$

or, P_{12} = Rs. 1061.68.

Example 20. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die. Then find the population of rabbits after 10 months. [Raj. 2005]

Solution. Let f_n be the number of pairs of rabbits after n months.

Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	$1 = f_1$
2	0	1	$1 = f_2$
3	1	1	$2 = f_3$
4	1	2	$3 = f_4$
5	2	3	$5 = f_5$
6	3	5	$8 = f_6$
7	5	8	$13 = f_7$
...	...	...	

(Fibonacci sequence)

and so on.

From the above table we observe that

$$f_3 = f_2 + f_1;\ f_4 = f_3 + f_2;\ f_5 = f_4 + f_3$$

$$f_6 = f_5 + f_4\ ;\ f_7 = f_6 + f_5 \text{ and so on.}$$

Thus the recurrence relation for the given problem is given as

$$f_n = f_{n-1} + f_{n-2},\ n \geq 3 \quad .....(1)$$

with initial conditions $f_1 = 1$, $f_2 = 1$.

Now the characteristic equation for relation (1) is

$$r^2 - r - 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ The solution is given as

$$f_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad .....(2)$$

given $f_1 = 1$

$$\therefore \ (2) \Rightarrow 1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\text{or, } C_1\left(\frac{1+\sqrt{5}}{2}\right)+C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \qquad \text{.....(3)}$$

given $f_2 = 1$

$$\therefore\ (2) \Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right)^2+C_2\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 \qquad \text{.....(4)}$$

$$\text{Now, } \left(\frac{1-\sqrt{5}}{2}\right)\times(3)-(4)$$

$$\Rightarrow C_1\left[-1-\frac{3+\sqrt{5}}{2}\right] = -\frac{(1+\sqrt{5})}{2}$$

$$\Rightarrow -\sqrt{5}(1+\sqrt{5})C_1 = -(1+\sqrt{5}) \Rightarrow C_1 = \frac{1}{\sqrt{5}} \qquad \text{.....(5)}$$

using (5) in (3), we get

$$C_2 = -\frac{1}{\sqrt{5}} \qquad \text{.....(6)}$$

$$\therefore\ (2) \Rightarrow f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad \text{.....(7)}$$

Now, the number of pairs of rabbits after 10 months can be obtained by putting $n = 10$ in relation (7) i.e.,

$$f_{10} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{10} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{10}$$

$$= \frac{1}{2^{10}\sqrt{5}}\left[(1+\sqrt{5})^{10}-(1-\sqrt{5})^{10}\right]$$

$$= \frac{1}{2^{10}\sqrt{5}}\left[(62976+28160\sqrt{5})-(62976-28160\sqrt{5})\right]$$

$$= \frac{1}{2^{10}\sqrt{5}}(56320\sqrt{5}) = \frac{56320}{2^{10}} = \frac{56320}{1024}$$

$\Rightarrow f_{10} = 55$

$\Rightarrow$ after 10 months there are 55 pairs of rabbits

$\Rightarrow$ population of rabbits after 10 months $= 2 \times 55 = 110$.

Example 21. (Tower of Hanoi) Let there are three pegs mounted on a board together with n disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom. Find the number of moves required to have all the disks on the second peg in order of size, with the largest on the bottom if there is a condition that disks to be moved one at a time from one peg to another and a disk is never placed on top of a smaller disk. Also there is a third peg on which disks can be placed temporarily. Also find the number of moves if there are 6 disks.

Solution. Let f_n denote the number of moves required to solve the given problem with n disks. Start with n disks on peg 1. Now following the given condition we can transfer the top $(n - 1)$ disks to peg 3 using f_{n-1} moves, keeping the largest disk fixed during these moves. Then we use one move for transferring the largest disk to the peg 2. Further, to transfer the $(n - 1)$ disks on peg 3 to peg 2 we require f_{n-1} additional moves, putting them on top of the largest disk which stays fixed on the bottom of peg 2. Thus the total moves required to solve the problem = $f_{n-1} + 1 + f_{n-1} = 2f_{n-1} + 1$. So the recurrence relation for the given problem can be established as

$f_n = 2f_{n-1} + 1, n \geq 1$

and $f_1 = 1$ (since one disk needed one move to transferred from peg 1 to peg 2)

or, $f_n - 2f_{n-1} = 1, n \geq 1$ .....(1)

with $f_1 = 1$.

The characteristic equation can be given as

$r - 2 = 0 \Rightarrow r = 2$

$\therefore\ f_n^{(h)} = C(2)^n$ .....(2)

Here F(n) = 1 so the form of particular solution is

$f_n^{(p)} = p_0$ .....(3)

It must satisfy the relation (1) so

$p_0 - 2p_0 = 1 \Rightarrow p_0 = -1$

$\therefore\ (3) \Rightarrow f_n^{(p)} = -1$

Thus the complete solution is

$f_n = f_n^{(h)} + f_n^{(p)}$ i.e.,

$f_n = C(2)^n - 1$ .....(4)

given $f_1 = 1$

$(4) \Rightarrow 1 = 2C - 1 \Rightarrow C = 1$

$\therefore\ (4) \Rightarrow f_n = 2^n - 1.$

Further, if there are 6 disks then number of moves required to solve the problem is given as

$f_6 = 2^6 - 1 = 63.$

Example 22. Solve the recurrence relation

$a_n - 5a_{n-1} + 6a_{n-2} = 4^n, n \geq 2.$ [Raj. 2003]

Solution. The characteristic equation is

$r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$

$\therefore\ a_n^{(h)} = C_1(2)^n + C_2(3)^n$ .....(1)

Here F(n) = 4^n so the form of particular solution is

$a_n^{(p)} = p_0 4^n$ .....(2)

It must satisfy the given relation so

$p_0 4^n - 5p_0\, 4^{n-1} + 6p_0\, 4^{n-2} = 4^n$

or, $\left(p_0 - \frac{5}{4}p_0 + \frac{6}{16}p_0\right)4^n = 4^N$

or, $\frac{1}{8}p_0 4^n = 4^n \Rightarrow \frac{1}{8}p_0 = 1 \Rightarrow p_0 = 8$

$\therefore\ (2) \Rightarrow a_n^{(p)} = 8(4)^n$

Thus the complete solution is

$a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$a_n = C_1(2)^n + C_2(3)^n + 8(4)^n$.

Example 23. Let $A_1, A_2, \ldots\ldots, A_n$ be an array of real numbers which is to be sorted into ascending order. One of the techniques for doing so is called bubble sort. In this, comparisons are made and the smallest number is chosen. Find the recurrence relation and then solve it. Also find out after how many comparisons will the 10 numbers be arranged in ascending order.

Solution. Let an be the number of comparisons required for sorting n numbers. To sort out the smallest number out of n numbers we require $(n - 1)$ comparisons. The remaining $(n - 1)$ numbers then require a_{n-1} comparisons to be sorted completely. So the recurrence relation can be written as

$$a_n = (n - 1) + a_{n-1},\ n \geq 1 \text{ and } a_0 = 1$$

$$\text{or, } a_n - a_{n-1} = n - 1 \qquad \text{.....(1)}$$

Now, the characteristic equation is

$$r - 1 = 0 \Rightarrow r = 1$$

$$\therefore\ a_n^{(h)} = C(1)^n = C \qquad \text{.....(2)}$$

Here $F(n) = n - 1 = (n - 1)\,(1)^n$ i.e., here $s = 1$ which is a root of the characteristic equation with multiplicity 1 so the particular solution is of the form

$$a_n^{(p)} = n(p_1 n + p_0) \qquad \text{.....(3)}$$

It must satisfy the relation (1) so

$$n(p_1 n + p_0) - (n - 1)\{p_1(n - 1) + p_0\} = n - 1$$

$$\text{or, } (p_1 - p_1)\,n^2 + (p_0 + p_1 - p_0 + p_1)n + (-p_1 + p_0) = n - 1$$

$$\text{or, } 2p_1 n + (p_0 - p_1) = n - 1$$

equating the coefficients on both sides, we get

$$2p_1 = 1 \Rightarrow p_1 = \frac{1}{2}$$

$$\text{and } p_0 - p_1 = -1 \Rightarrow p_0 = -\frac{1}{2}$$

$$\therefore\ (3) \Rightarrow a_n^{(p)} = n\left(\frac{1}{2}n - \frac{1}{2}\right) = \frac{1}{2}n(n-1) \qquad \text{.....(4)}$$

Thus the complete solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \text{ i.e.,}$$

$$a_n = C + \frac{1}{2}n(n - 1) \qquad \text{.....(5)}$$

given $a_0 = 0$ so $(5) \Rightarrow C = 0$

$$\therefore\ (5) \Rightarrow a_n = \frac{1}{2}n(n - 1). \qquad \text{.....(6)}$$

Further, if there are 10 numbers then the number of comparisons required to arrange them in ascending order can be obtained by putting $n = 10$ in relation (6) i.e.,

$$a_{10} = \frac{1}{2}10(10 - 1) = 45.$$

Example 24. If there are n people $(n \geq 2)$ at a party and each of these people shakes hands (exactly one time) with all of the other people then find the number of total handshakes.

Solution. Let a_n be the total number of handshakes of n people. Since a person shakes hand with remaining (n − 1) people i.e., for one person the number of handshakes is (n − 1). So if we left this person then there are (n − 1) people and for them the total number of handshakes is a_{n-1}. So we have −

$$a_n = (n-1) + a_{n-1},\ n \geq 2 \text{ and } a_2 = 1$$

or, $a_n - a_{n-1} = n - 1$ .....(1)

The characteristic equation is

$$r - 1 = 0 \Rightarrow r = 1$$

$\therefore\ a_n^{(h)} = C(1)^n$ .....(2)

Here F(n) = (n − 1) = (n − 1) $(1)^n$ i.e., s = 1 which is a root of characteristic equation with multiplicity 1 so the form of particular solution is

$a_n^{(p)} = n(p_1 n + p_0)$ .....(3)

It must satisfy the relation (1) so

$$n(p_1 n + p_0) - (n-1)\{p_1(n-1) + p_0\} = n - 1$$

or, $n_2(p_1 - p_1) + n(p_0 - p_0 + 2p_1) + p_0 - p_1 = n - 1$

or, $2p_1 n + (p_0 - p_1) = n - 1$

equating the coefficients on both sides, we get

$$2p_1 = 1 \Rightarrow p_1 = \frac{1}{2}$$

and $p_0 - p_1 = -1 \Rightarrow p_0 = -\frac{1}{2}$

$\therefore\ (3) \Rightarrow a_n^{(p)} = \frac{1}{2}n(n-1)$ .....(4)

Thus the complete solution is

$a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$an = C(1)n + \frac{1}{2}n(n-1)$ ....(5)

given that $a_2 = 1$

$\Rightarrow\ 1 = C(1)2 + \frac{1}{2}2(2-1) \Rightarrow 1 = C + 1$

$\Rightarrow\ C = 0$

$\therefore\ (5) \Rightarrow a_n = \frac{1}{2}n(n-1).$

Example 25. Solve the following recurrence relations :

(i) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$
with $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$. [Raj. 2004]

(ii) $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$. [Raj. 2004]

(iii) $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$, $a_1 = 7$. [Raj. 2005]

Solution. (i) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

The characteristic equation is

$r^3 + 3r^2 + 3r + 1 = 0 \Rightarrow (r + 1)^3 = 0 \Rightarrow r = -1, -1, -1$

Thus the solution is

$a_n = (C_1 + C_2 n + C_3 n^2)(-1)^n$ .....(1)

given $a_0 = 1$

$\Rightarrow C_1 = 1$ .....(2)

given $a_1 = -2$

$\Rightarrow (C_1 + C_2 + C_3)(-1) = -2$

$\Rightarrow C_1 + C_2 + C_3 = 2 \Rightarrow C_2 + C_3 = 1$ .....(3)

given $a_2 = -1$

$\Rightarrow (C_1 + 2C_2 + 4C_3)(-1)^2 = -1$

$\Rightarrow C_1 + 2C_2 + 4C_3 = -1$

$\Rightarrow 2C_2 + 4C_3 = -2 \Rightarrow C_2 + 2C_3 = -1$ .....(4)

$(4) - (3) \Rightarrow C3 = -2$ .....(5)

$(3) \Rightarrow C2 = 3$ .....(6)

using (2), (5), (6) in (1), we get

$a_n = (1 + 3n - 2n^2)(-1)^n$.

(ii) $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.

The characteristic equation is

$r^2 - 6r + 9 = 0 \Rightarrow (r - 3)^2 = 0 \Rightarrow r = 3,3$

Thus the solution is

$a_n = (C_1 + C_2 n)\, 3^n$ .....(1)

given $a_0 = 1$

$\Rightarrow C_1 = 1$ .....(2)

given $a_1 = 6$

$\Rightarrow 3(C_1 + C_2) = 6 \Rightarrow C_1 + C_2 = 2 \Rightarrow C_2 = 1$ .....(3)

using (2) and (3) in (1), we get

$a_n = (1 + n)\, 3^n$.

(iii) $a_n = a_{n-1} + 2a_{n-2}$

The characteristic equation is

$r^2 - r - 2 = 0 \Rightarrow (r - 2)(r + 1) = 0 \Rightarrow r = -1, 2$

Thus the solution is

$a_n = C_1(-1)^n + C_2(2)^n$

given $a_0 = 2$

$\therefore (1) \Rightarrow 2 = C_1 + C_2$ .....(1)

given $a_1 = 7$

$\therefore (1) \Rightarrow 7 = -C_1 + 2C_2 \Rightarrow C_1 = 2C_2 - 7 = 2(2 - C_1) - 7$ .....(2)

[using (2)]

$\Rightarrow 3C_1 = -3 \Rightarrow C_1 = -1$

$\therefore$ (2) $\Rightarrow C_2 = 3$

Thus, (1) $\Rightarrow a_n = -(-1)^n + 3(2)^n = (-1)^{n+1} + 3(2)^n$.

Example 26. A coin is tossed n times. Find the number of sequences of outcomes in which head never appear on successive tosses.

Solution. If we toss a coin n times then there are 2^n possible sequences of outcomes. Let an denote the number of sequences in which heads never appear on successive tosses.

Now to each sequence of (n – 1) heads and tails in which there are no successive heads, we can append a tail to get a sequence of n heads and tails in which there are no successive heads. Similarly to each sequence of (n – 2) heads and tails in which there are no successive heads, we can append a tail and then a head to get a sequence of n heads and tails in which there are no successive heads. Now, these sequences exhaust all sequences of n heads and tails in which there are no consecutive heads. Thus the recurrence relation for the given problem is

$$a_n = a_{n-1} + a_{n-2} \text{ with } a_1 = 2 \text{ and } a_2 = 3,\ n \geq 3.$$

The characteristic equation is

$$r^2 - r - 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{5}}{2}$$

Thus the solution is

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{....(1)}$$

given $a_1 = 2$

$$\Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 2$$

$$\text{or, } C_1(1+\sqrt{5}) + C_2(1-\sqrt{5}) = 4 \quad \text{.....(2)}$$

given $a_2 = 3$

$$\Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + C_2\left(\frac{1-\sqrt{5}}{2}\right)^2 = 3$$

$$\text{or, } C_1(1+\sqrt{5})^2 + C_2(1-\sqrt{5})^2 = 12 \quad \text{.....(3)}$$

Now, $(1-\sqrt{5}) \times (2) - (3)$ gives

$$C_1\left[-4-(1+\sqrt{5})^2\right] = 4(1-\sqrt{5}) - 12$$

$$\text{or, } C_1(-2\sqrt{5}-10) = -4(1-\sqrt{5})$$

$$\Rightarrow C_1 = \frac{4(\sqrt{5}+2)}{2(\sqrt{5}+5)} \times \left(\frac{\sqrt{5}-5}{\sqrt{5}-5}\right) = \frac{2(-5-3\sqrt{5})}{(-20)}$$

$$\text{or, } C_1 = \frac{5+3\sqrt{5}}{10} \quad \text{.....(4)}$$

$$\therefore (2) \Rightarrow C_2 = \frac{5-3\sqrt{5}}{10} \quad \text{.....(5)}$$

$$\therefore\ (1) \Rightarrow a_n = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n,\ n \geq 1$$

Example 27. Ram invest Rs. 100 at 6% interest compounded quarterly. How many months must he wait for his money to double ?

Solution. Since the interest is compounded quarterly so here rate $= \frac{6}{4} = \frac{3}{2}\%$ and time = 4n where n be the number of years required to get the money double.

Let P_n denote the amount after n years then $P_n = P_{n-1}\left(1+\frac{R}{100}\right)^T = P_{n-1}\left(1+\frac{3}{200}\right)^4$ where P_{n-1} is the amount after (n – 1) years which becomes the principal for nth year.

So we have the recurrence relation as

$$P_n = P_{n-1}\left(\frac{203}{200}\right)^4,\ n \geq 1 \text{ and } P_0 = 100.$$

or, $P_n = P_{n-1}(1.015)^4 = 1.061\ P_{n-1}$

or, $P_n - 1.061\ P_{n-1} = 0$

$\therefore$ The characteristic equation is

$r - 1.061 = 0 \Rightarrow r = 1.061$

Thus the solution is

$P_n = C(1.061)^n$ .....(1)

given $P_0 = 100$

$\therefore$ (1) $\Rightarrow$ C = 100

Thus, $P_n = 100\ (1.061)^n$ .....(2)

Now we want that money (i.e., Rs. 100) must be double i.e., $P_n = 200$.

$\therefore$ (2) $\Rightarrow 200 = 100\ (1.061)^n$

or, $(1.061)^n = 2$

or, $n \log (1.061) = \log 2 = 0.3010$

or, $n(0.0257) = 0.3010$

or, $n = \frac{0.3010}{0.0257} = 11.712$

$\Rightarrow$ he must wait 11.712 years for his money to double.

$\Rightarrow$ number of months, he must wait for his money to double = 11.712 × 12 = 140.54

$\because$ 141 months.

Example 28. Suppose that a person deposits Rs. 10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years ?

Solution. Let P_n denote the amount in the account after n years. Now this amount is equal to the sum of the amount after (n – 1) years and the interest of the n^{th} year.

Since the interest for 1 year on amount $P_{n-1} = P_{n-1} \times \frac{11}{100} = (0.11)P_{n-1}$. Thus the recurrence relation for the given problem is

$P_n = P_{n-1} + (0.11)\ P_{n-1}$

or, $P_n - (1.11)\ P_{n-1} = 0$ with $P_0 = 10{,}000$ and $n \geq 1$ .....(1)

Now the characteristic equation for the relation (1) is

$$r - 1.11 = 0 \Rightarrow r = 1.11$$

Thus the solution is

$$P_n = C(1.11)^n \quad .....(2)$$

given $P_0 = 10{,}000$

$\therefore$ (2) $\Rightarrow C = 10{,}000$

Thus, $P_n = 10000(1.11)^n$ .....(3)

Hence the amount after 30 years (i.e., n = 30) the account have is P_{30} i.e.,

$$P_{30} = 10000(1.11)^{30} = \text{Rs. } 228{,}922.97.$$

Example 29. A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For example, 12044097 is valid while 10230520 is not valid. Find the number of valid n–digit codewords.

Solution. Let a_n denote the number of valid n–digit codewords. Now there are two ways to form a valid string with n–digits from a string with (n – 1) digits.

First, if a string with (n – 1) digits is valid then a valid string of n digits can be obtained by appending a digit other than 0 to the string with (n – 1) digits. Since this appending can be done in 9 ways (appending 1 or 2 or 3 or .... or 9) so in this case a valid string with n digits can be formed in $9a_{n-1}$ ways.

Second, if a string with (n – 1) digits is not valid then a valid string of n digits can be obtained by appending a 0 to the string with (n – 1) digits.

The number of ways that this can be done is equal to the number of invalid (n – 1) digit strings. Since there are 10^{n-1} strings of length (n – 1) out of which a_{n-1} are valid so in this case a valid string with n digits can be formed in $(10^{n-1} - a_{n-1})$ ways.

As all valid strings of length n are obtained in one of the above two ways, so the recurrence relation for the problem can be given as

$$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$$

or, $a_n - 8a_{n-1} = 10_{n-1}$, $n \geq 1$ with $a_1 = 9$ .....(1)

(Here $a_1 = 9$ because there are 10 one–digit strings out of which only one i.e., 0, is not valid).

The characteristic equation for the relation (1) is

$$r - 8 = 0 \Rightarrow r = 8$$

$\therefore\ a_n^{(h)} = C(8)^n$ .....(2)

Here $F(n) = 10^{n-1} = \frac{1}{10}(10)^n$ so the form of particular solution is

$a_n^{(p)} = p_0(10)^n$ .....(3)

It must satisfy the relation (1) so

$$p_0 10^n - 8p_0(10)^{n-1} = 10^{n-1}$$

or, $p_0\left(1-\frac{8}{10}\right)10^n = 10^{n-1} \Rightarrow 2p_0 10^{n-1} = 10^{n-1} \Rightarrow 2p_0 = 1$

or, $p_0 = \frac{1}{2}$

$\therefore$ (3) $\Rightarrow$ an(p) $= \frac{1}{2}10^n$

Thus the complete solution is

$a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$a_n = C8^n + \frac{1}{2}10^n$ .....(4)

given $a_1 = 9$

$\therefore$ (4) $\Rightarrow 9 = 8C + \frac{1}{2}(10) \Rightarrow 8C = 9 - 5 = 4$

$\Rightarrow C = \frac{1}{2}$

$\therefore$ (4) $\Rightarrow a_n = \frac{1}{2}(8^n + 10^n)$.

Example 30. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month. Find an explicit formula for the number of cars produced in the first n months by this factory. Also find how many cars are produced in the first year ?

Solution. Let an denote the number of cars produced in the first n months. Since in the nth month n cars are made so a_n can be expressed as a sum of the number of cars produced in the first (n – 1) months and the number of cars produced in the nth month i.e.,

$a_n = a_{n-1} + n, n \geq 1$ with $a_1 = 1$

The characteristic equation for the relation (1) is .....(1)

$r - 1 = 0 \Rightarrow r = 1$

$\therefore a_n^{(h)} = C(1)^N = C$ .....(2)

Here $F(n) = n(1)^n$ i.e., s = 1 which is a root of characteristic equation with multiplicity 1 so the form of particular solution is

$a_n^{(p)} = n(p_1 n + p_0)$

It must satisfy the relation (1) so .....(3)

$n(p_1 n + p_0) - (n - 1)\{p_1(n - 1) + p_0\} = n$

or, $n^2(p_1 - p_1) + n(p_0 + p_1 - p_0 + p_1) + p_0 - p_1 = n$

or, $2p_1 n + p_0 - p_1 = n$

equating the coefficients on both sides, we get $2p_1 = 1 \Rightarrow p_1 = \frac{1}{2}$

and $p_0 - p_1 = 0 \Rightarrow p_0 = p_1 = \frac{1}{2}$

$\therefore$ (3) $\Rightarrow a_n^{(p)} = \frac{n}{2}(n+1)$

Thus the complete solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \text{ i.e.,}$$

$$a_n = C + \frac{n}{2}(n+1) \qquad .....(4)$$

given $a_1 = 1$

$$\therefore \ (4) \Rightarrow 1 = C + \frac{1}{2}(1+1) \Rightarrow C = 0$$

Thus, $a_n = \frac{n}{2}(n+1)$ .....(5)

which is the required explicit formula for the number of cars produced in the first n months. Further, the number of cars produced in the first year (i.e., 12 months) is

$$a_{12} = \frac{12}{2}(12+1) = 78.$$

Example 31. Find the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time. Also find that how many ways can this person climb a flight of 8 stairs ?

Solution. Let a_n denote the number of ways the person can climb n stairs. Now there are two possibilities for the first stride.

First, if the person take one stair then there are $(n - 1)$ stairs left to climb and for that there are an–1 number of ways to climb.

Second, if the person take two stairs in first stride then there are $(n - 2)$ stairs left to climb and for that there are a_{n-2} number of ways to climb.

The above two cases are exhaustive and mutually exclusive so that total number of ways to climb n stairs is $a_{n-1} + a_{n-2}$ i.e.,

$$a_n = a_{n-1} + a_{n-2}, \ n \geq 3 \text{ with } a_1 = 1 \text{ and } a_2 = 2.$$

The characteristic equation for the above relation is

$$r^2 - r - 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{5}}{2}$$

Thus the solution is

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad .....(1)$$

given $a_1 = 1$

$$\therefore \ (1) \Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \qquad .....(2)$$

given $a_2 = 2$

$$\therefore \ (1) \Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + C_2\left(\frac{1-\sqrt{5}}{2}\right)^2 \qquad .....(3)$$

On solving (2) and (3) we get

$$C_1 = \frac{5+\sqrt{5}}{10} \text{ and } C_2 = \frac{5-\sqrt{5}}{10}$$

$$\therefore\ a_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \quad .....(4)$$

If there are 8 stairs to climb then the number of ways this person can climb the flight is

$$a_8 = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^8 + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^8$$

$$= \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{6016+2688\sqrt{5}}{256}\right) + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{6061-2688\sqrt{5}}{256}\right)$$

$$= \frac{1}{256}(6016+2688) = 34$$

$\Rightarrow a_8 = 34.$

Example 32. Find the number of binary sequences of length n, in which the pattern 101 occurs at the end.

Solution. Let a_n denote the number of binary sequences of length n having the pattern 101 at the end. Since the last three places of the sequence are fixed so the remaining (n – 3) places can be filled up in 2^{n-3} ways. So there are 2^{n-3} binary sequences of length n that end up with 101. Now not all of these sequences will have the pattern 101 at the end. For example, the sequence 0 1 0 1 0 1 of length 6 ends with 101 but the pattern 101 already occurs at the 4th digit so the 4th digit cannot be counted in a later occurrence of the pattern. On the other hand, the sequence 0 1 0 1 0 1 0 1 of length 9 ends with 101 and also has a pattern 101 at the end (also at 4th digit). It must be clear now that a sequence of length n ending with 101 will have the pattern 101 at the nth digit if such pattern does not occurs at the (n – 2)th digit. Since the number of binary sequences of length n, having the pattern 101 at the (n – 2)th digit, is a_{n-2}. So we have the relation.

$$a_n = 2^{n-3} - a_{n-2},\ n \geq 3 \text{ with } a_1 = 0,\ a_2 = 0.$$

$$\text{or, } a_n + a_{n-2} = \frac{1}{8}2^n \quad .....(1)$$

The characteristic equation is

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

$$\therefore\ a_n^{(h)} = C_1\cos n\theta + C_2\sin n\theta$$

$$\text{where } \theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\therefore\ a_n^{(h)} = C_1\cos\frac{n\pi}{2} + C_2\sin\frac{n\pi}{2} \quad .....(2)$$

Here $F(n) = \frac{1}{8}2^n$ so the form of particular solution is

$$a_n^{(p)} = p_0 2^n \quad .....(3)$$

It must satisfy the relation (1) so,

$$p_0 2^n + p_0 2^{n-2} = \frac{1}{8}2^n$$

$$\Rightarrow p_0 + \frac{p_0}{4} = \frac{1}{8} \Rightarrow 5p_0 = \frac{1}{2} \Rightarrow p_0 = \frac{1}{10}$$

$$(3) \Rightarrow a_n^{(p)} = \frac{1}{10}2^n \quad .....(4)$$

Thus the complete solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \text{ i.e.,}$$

$$a_n = C_1 \cos\frac{n\pi}{2} + C_2 \sin\frac{n\pi}{2} + \frac{1}{10}2^n \quad .....(5)$$

given $a_1 = 0$

$$\therefore \ (5) \Rightarrow C_2 + \frac{2}{10} = 0 \Rightarrow C_2 = -\frac{1}{5} \quad .....(6)$$

given $a_2 = 0$

$$\therefore \ (5) \Rightarrow -C_1 + \frac{4}{10} = 0 \Rightarrow C_1 = \frac{2}{5} \quad .....(7)$$

using (6) and (7) in (5), we get

$$a_n = \frac{2}{5}\cos\frac{n\pi}{2} - \frac{1}{5}\sin\frac{n\pi}{2} + \frac{1}{10}2^n$$

$$\text{or, } a_n = \frac{1}{5}\left(2\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2} + 2^{n-1}\right).$$

Example 33. Evaluate the determinant of the n × n matrix A_n which has all 1's on its principal diagonal and the immediate subdiagonals and 0's everywhere else.

Solution. Let an denote the determinant of the matrix A_n i.e.,

$$a_n = \det(A_n)$$

where

$$a_n = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}_{n\times n}$$

Now on expanding det(An) along its first row, we get

$$a_n = \det(A_n) = \det(A_{n-1}) - \det(B_{n-1}) \quad .....(1)$$

where B_{n-1} is the (n – 1) × (n – 1) matrix given as

$$B_{n-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

On expanding the $\det(B_{n-1})$ along its first column, we get

$$\det(B_{n-1}) = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{bmatrix}_{(n-2)\times(n-2)}$$

$= \det(An-2)$

$\therefore$ (1) $\Rightarrow a_n = \det(A_{n-1}) - \det(A_{n-2}) = a_{n-1} - a_{n-2}$

or, $a_n - a_{n-1} + a_{n-2} = 0$, $n \geq 2$ with $a_1 = 1$, $a_2 = 0$ .....(1)

The characteristic equation for the relation (1) is

$$r^2 - r + 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Thus the solution is

$$a_n = C_1 \cos n\theta + C_2 \sin n\theta$$

where $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{2}$

$$\therefore a_n = C_1 \cos\frac{n\pi}{3} + C_2 \sin\frac{n\pi}{3} \quad .....(2)$$

given $a_1 = 1$

$$\therefore (2) \Rightarrow C_1 \cos\frac{\pi}{3} + C_2 \sin\frac{\pi}{3} = 1$$

or, $C_1 + C_2\sqrt{3} = 2$ .....(3)

given $a_2 = 0$

$$\therefore (2) \Rightarrow C_1 \cos\frac{2\pi}{3} + C_2 \sin\frac{2\pi}{3} = 0$$

$\Rightarrow C_1 - C_2\sqrt{3} = 0$ .....(4)

Now (3) + (4) $\Rightarrow C_1 = 1$

$$\Rightarrow C_2 = \frac{1}{\sqrt{3}}$$

Thus, $a_n = \cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3}$ .....(5)

which is the required determinant of An.

EXERCISE 7.1

1. Solve the following recurrence relations:

(i) $a_{n+2} - 4a_{n+1} + 4a_n = 0,\ n \geq 0$ and $a_0 = 1,\ a_1 = 3.$ [Raj. 2005]

(ii) $a_n + 2a_{n-1} + 2a_{n-2} = 0,\ n \geq 2$ and $a_0 = 1,\ a_1 = 3.$

(iii) $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$

(iv) $4a_n - 20a_{n-1} + 17a_{n-2} - 4a_{n-3} = 0$

(v) $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0,\ n \geq 3$ and $a_0 = 1,\ a_1 = 4,\ a_2 = 8.$

[Ans. (i) $a_n = (n + 2)\ 2^{n-1}$

(ii) $a_n = 2^{\frac{n}{2}}\left(\cos\frac{3n\pi}{4} + 4\sin\frac{3n\pi}{4}\right)$

(iii) $a_n = (C_1 + C_2 n + C_3 n^2)\ (-2)^n$

(iv) $a_n = C_1 (4)^n + (C_2 + C_3 n)\left(\frac{1}{2}\right)^n$

(v) $a_n = 5(2)^n + 3n(2)^n - 4(3)_n$]

2. Solve the following recurrence relations :

(i) $a_n - a_{n-1} = 3(n - 1),\ n \geq 1$ and $a_0 = 2$

(ii) $a_n - 7a_{n-1} + 10a_{n-2} = 7(3)^n,\ n \geq 2$

(iii) $a_n - 4a_{n-1} + 4a_{n-2} = 2^n,\ n \geq 2$

(iv) $a_n - 3a_{n-1} = 5(3)^n,\ n \geq 1,\ a_0 = 2$

(v) $a_n - 3a_{n-1} = 5(7)^n,\ n \geq 1,\ a_0 = 2$

(vi) $a_n - 2a_{n-1} = 1,\ n \geq 2$ and $a_1 = 6$

[Ans. (i) $a_n = 2 + \frac{3}{2}n(n-1), n \geq 0$

(ii) $a_n = C_1 2^n + C_2 5^n - \frac{63}{2} 3^n$

(iii) $a_n = (C_1 + C_2 n)2^n + n^2 2^{n-1}$

(iv) $a_n = (2 + 5n)3^n$

(v) $a_n = \frac{5}{7}(7)^{n+1} - \frac{1}{4}(3)^{n+3},\ n \geq 0$

(vi) $a_n = 7(2)^{n-1} - 1$]

3. Solve the relation $a_n = n\ a_{n-1},\ n \geq 1$ and $a_0 = 1.$

[Ans. $a_n = n!$]

4. For the sequence 0, 2, 6, 12, 20, ... form the recurrence relation and then solve it.

[Ans. $a_n - a_{n-1} = 2n$ and $a_n = n^2 + n$]

5. Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not contain two consecutive 0s. How many such bit strings are there of length 5 ?

[Ans. $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 2$ and $a_2 = 3,\ a_5 = 13$]

6. Suppose that the number of bacteria in a colony triples every hour. Find the number of bacteria after n hours have elapsed. Also find that how many bacteria will be in a colony if 100 bacteria are used to begin the colony.

[Ans. $a_n = 3a_{n-1}$, 59,04,900]

7. Find the sum of the first n–positive integers with the help of recurrence relation.

[Hint : $a_n = a_{n-1} + n$, $n \geq 2$ with $a_1 = 1$]

[Ans. an = $\frac{n(n+1)}{2}$]

8. A deposit of Rs. 100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

(a) Find a recurrence relation for $\{P_n\}$, where Pn is the amount in the account at the end of n years if no money is ever withdrawn.

(b) How much is in the account after n years if no money has been withdrawn ?

[Ans. (a) $P_n = 1.2\ P_{n-1} + 0.45\ P_{n-2}$, $P_0 = 100{,}000$ and $P_1 = 120{,}000$

(b) $P_n = \frac{250000}{3}\left(\frac{3}{2}\right)^n + \frac{50000}{3}\left(\frac{-3}{10}\right)^n$]

9. A young pair of rabbits (one of each sex) is placed on an island. They produces two new pairs of rabbits at the age of 1 month and six new pairs of rabbits at the age of 2 months and every month afterward. Determine the number of pairs of rabbits on the island after n months, assuming that none of the rabbits ever die or leave the island.

[Hint : $f_n = 3f_{n-1} + 4f_{n-2}$, $n \geq 2$ with $f_0 = 1$ and $f_1 = 3$]

[Ans. $f_n = \frac{1}{5}\{4^{n+1} + (-1)^n\}$]

10. If S = {1, 2, 3, ..., n}, $n \geq 0$ and S = f for n = 0 then find the number of subsets of S that contain no consecutive integers.

[Hint : $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$ with $a_0 = 1$ and $a_1 = 2$]

[Ans. $a_n = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$]

7.6 Discrete Numeric Functions

As we defined earlier that a function is a binary relation under which each element of domain is uniquely associated with some element of codomain. The collection of these elements of codomain, to which the elements of domain are associated, is called the range of the function. If we made a class of functions whose domain is the set of natural numbers and whose range is the set of real numbers, then these functions are called discrete numeric functions or, simply, numeric functions.

Numeric functions are denoted by bold face lower case letters. Like a numeric function 'a' is denoted by a and it can be written as

$$\mathbf{a} = (a_0, a_1, a_2, \ldots\ldots, a_r, \ldots\ldots)$$

where $a_0, a_1, a_2, \ldots\ldots, a_r, \ldots\ldots$ are the values of the function at 0, 1, 2, ...r, ...

Example 34. Let we deposit Rs 100 in a savings account at the rate of 7% per year, compounded annually. Now at the end of the Ist year, the amount becomes Rs 107, at the end of the 2^{nd} year the amount becomes Rs. 114.49, at the end of 3^{rd} year, the amount becomes Rs. 122.50 and so on. The amount in the account at the end of each year can be expressed by a numeric function a which can be written as

(100, 107, 114.49, 122.50, ...)

or as

$$a = 100(1.07)^r.$$

7.7 Generating Functions

Generating functions are used to represent numeric functions in series form.

Let $a = (a_0, a_1, \ldots\ldots, a_r, \ldots.)$ be a numeric function or be the sequence of real numbers. Then the generating function for this sequence can be written in series form as

$$A(x) = a_0 + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots = \sum_{r=0}^{\infty} a_rx^r$$

The expression A(x) is called a formal power series and the formal power series is called a generating function for the sequence $a = (a_0, a_1, a_2, \ldots\ldots a_r, \ldots.)$.

Example 35. Find the generating function for the sequence 1, 1, 1, ...

Solution. The generating function for the given sequence is

$$G(x) = 1 + a_1x + a_2x^2 + \ldots\ldots$$

$$= 1 + x + x^2 + \ldots\ldots [\because \text{ Here } a_0 = a_1 = \ldots\ldots a_r = \ldots\ldots = 1)$$

$$= \frac{1}{1-x} \quad [\text{for } |x|<1]$$

7.7.1 Properties of Generating Functions

Let be $A(x) = \sum_{k=0}^{\infty} a_kx^k$ and $B(x) = \sum_{k=0}^{\infty} b_kx^k$ the generating functions of the numeric functions

$a = (a_0, a_1, \ldots\ldots a_k, \ldots.)$ and

$b = (b_0, b_1, \ldots\ldots b_k, \ldots.)$ respectively.

Then

(i) $A(x) = B(x)$ if $a_n = b_n$ for each $n \geq 0$.

(ii) $cA(x) = \sum_{k=0}^{\infty} (ca_k)x^k$,where c is any scalar number

(iii) $A(x) + B(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k$

(iv) $A(x)B(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_jb_{k-j} \right) x^k$

Example 36. Express $F(x) = \frac{1}{(1-x)^2}$ in summation form.

Solution. Let $f(x) = \frac{1}{1-x}$ and $g(x) = \frac{1}{1-x}$ then

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k \qquad \text{.....(1)}$$

$$\text{and } g(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k \qquad \text{.....(2)}$$

$\therefore$ By property (iv)

$$f(x) = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$$

Since (1) $\Rightarrow a_k = 1 \ \forall$ j, and (2) $\Rightarrow$ bk = 1, $\forall$ k.

$$\therefore \quad \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} 1\right)x^k = \sum_{k=0}^{\infty}(k+1)x^k.$$

To apply generating functions to solve a number of counting problems we require the Binomial theorem for non–positive integral exponents. Before that, we define extended Binomial coefficients.

7.7.2 Extended Binomial Coefficients

Let p be a real number and k be a non–negative integer then the extended Binomial coefficient ${}^{p}C_k$ is defined as

$${}^{p}C_k = \begin{cases} \frac{p(p-1)\ldots(p-k+1)}{k!}, & k > 0 \\ 1, & k = 0 \end{cases}$$

If p is a negative integer then the extended Binomial coefficient can be described in terms of an ordinary Binomial coefficient as given below

$${}^{-n}C_k = \frac{-n(-n-1)\ldots(-n-k+1)}{k!}; \ p = -n, \text{ n is a positive integer}$$

$$= (-1)^k \frac{n(n+1)\ldots(n+k-1)}{k!}$$

$$= (-1)^k \frac{(n+k-1)(n+k-2)\ldots(n+1)n}{k!} \cdot \frac{(n-1)\ldots 3\cdot 2\cdot 1}{(n-1)\ldots 3\cdot 2\cdot 1}$$

$$= (-1)^k \frac{(n+k-1)!}{k!(n-1)!}$$

$$= (-1)^k \, {}^{n+k-1}C_k.$$

Example 37. Evaluate (i) ${}^{-2}C_3$ and (ii) $\left(\frac{3}{2}\right)_{C_3}$ .....(1)

Solution. (i) We have,

$${}^{-n}C_k = (-1)^k \, {}^{n+k-1}C_k$$

Here n = 2, k = 3

$\therefore \quad {}^{-2}C_3 = (-1)^3 \, {}^{2+3-1}C_3 = {}^{4}C_3 = 4.$

(ii) We have

$$ {}^{p}C_k = \frac{p(p-1)\ldots(p-k+1)}{k!}, \; k > 0$$

Here $p = \frac{3}{2}$, k = 3 > 0

$$\therefore \quad {}^{\left(\frac{3}{2}\right)}C_3 = \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!} = \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{6}$$

$$= -\frac{1}{16}.$$

Remark 2. If p is a positive integer then the extended Binomial coefficient reduces to the ordinary Binomial coefficient and in that case ${}^{p}C_k = 0$ for k > p.

Theorem 1. Extended Binomial Theorem

Let x be a real number such that |x| < 1 and also let p be a real number then

$$(1+x)^p = \sum_{k=0}^{\infty} {}^{p}C_k x^k$$

Remark 3. If p is a positive integer then the above theorem 1 reduces to the ordinary Binomial theorem and in that case ${}^{p}C_k = 0$ for k > p.

Example 38. Evaluate the generating function for $(1 + x)^{-n}$ and $(1 - x)^{-n}$ where n is a positive integer.

Solution. By Theorem 1,

$$(1+x)^{-n} = \sum_{k=0}^{\infty} {}^{-n}C_k x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \, {}^{n+k-1}C_k x^k \qquad \text{[using (1) of 7.7.1]}$$

Now replacing x by – x, we get

$$(1-x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \, {}^{n+k-1}C_k (-x)^k$$

$$= \sum_{k=0}^{\infty} {}^{n+k-1}C_k x^k$$

7.7.3 Exponential Generating Function

Since, we have

$$(1+x)^n = {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \ldots + {}^{n}C_r x^r + \ldots + {}^{n}C_n x^n$$

Also, $${}^{n}C_r = \frac{n!}{(n-r)!\,r!} = \frac{1}{r!}\,{}^{n}P_r$$

So, $(1+x)^n = {}^nP_0 + {}^nP_1 x + {}^nP_2 \frac{x^2}{2!} + \ldots {}^nP_r \frac{x^r}{r!} + \ldots + {}^nP_n \frac{x^n}{n!}$ .....(1)

It is clear from (1) that the coefficient of $\frac{x^r}{r!}$ gives the sequence ${}^nP_r$

Now, let

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a^3 \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \quad \text{.....(2)}$$

then f(x) is called the exponential generating function for the sequence $a_0, a_1, a_2, \ldots, a_r \ldots$.

7.8 Applications of Generating Functions

Generating functions can be used to solve many counting problems. For this, we need the coefficient of xk in the various generating functions. Some useful generating functions and corresponding coefficient of xk are given in the following table.

Table – 1

S.No.	Generating Functions G(x)	Coefficient of x^k
1.	$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$	${}^nC_k$
2.	$(1+ax)^n = \sum_{k=0}^{n} a^k\, {}^nC_k x^k$	$a^k\, {}^nC_k$
3.	$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$	1
4.	$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$	a^k
5.	$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} {}^{-n}C_k (-a^k) x^k$	${}^{-n}C_k(-a)^k = a^k\, {}^{n+k-1}C_k$
6.	$\frac{1}{(1+x)} = \sum_{k=0}^{\infty} {}^{-1}C_k x^k$ $= \sum_{k=0}^{\infty} (-1)^k\, {}^kC_k x^k$ $= \sum_{k=0}^{\infty} (-1)^k x^k$	$(-1)^k$

7.	$\frac{1}{(1+ax)^n} = \sum_{k=0}^{\infty} {}^{-n}C_k a^k x^k$ $= \sum_{k=0}^{\infty} (-1)^k\, {}^{n+k-1}C_k a^k x^k$	$(-1)^k a^k\, {}^{n+k-1}C_k$
8.	$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$	$\frac{1}{k!}$

7.9 Solving Recurrence Relations by the Method of Generating Functions

The solution of a recurrence relation can be determined by finding the associated generating function.

Example 39. Solve the recurrence relation

$a_n = 3a_{n-1}$, n = 1, 2, 3, ... with $a_0 = 2$

by the method of generating functions.

Solution. Let the generating function for the sequence $\{a_n\}$ is G(x) then

$$G(x) = \sum_{n=1}^{\infty} a_n x^n \qquad .....(1)$$

Given $a_n = 3a_{n-1}$ .....(2)

Now multiply the equation (2) by x^n and taking summation of both sides from n = 1 to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n$$

or $a_1x + a_2x^2 + a_3x^3 + ... = 3[a_0x + a_1x^2 + a_2x^3 + ...]$ .....(3)

Since $G(x) = \sum_{n=1}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \ldots$

$\therefore$ (3) $\Rightarrow G(x) - a_0 = 3x\,[a_0 + a_1x + a_2x^2 + ...] = 3x\,G(x)$

$\Rightarrow (1 - 3x)\,G(x) = a_0$

$\Rightarrow G(x) = \frac{2}{1-3x}$ $\quad (\because a_0 = 2)$

or, $G(x) = 2\sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} (2\cdot 3^n)x^n$ .....(4)

On comparing (1) and (4), we get the required solution

$a_n = 2\cdot 3^n$.

ILLUSTRATIVE EXAMPLES

Example 40. Determine the coefficient of x^{15} in each of the following :

(i) $x^3(1 - 2x)^{10}$

(ii) $\dfrac{(x^3 - 5x)}{(1-x)^3}$

(iii) $\dfrac{(1+x)^4}{(1-x)^4}$ [Raj. 1999, 2001, MREC 2000, 2001]

Solution. (i) Coefficient of x^{15} in $x^3(1 - 2x)^{10}$ = coefficient of x^{12} in $(1 - 2x)^{10}$

Since, $(1 - ax)^n = \sum_{k=0}^{n} {}^nC_k (-1)^k x^k$

Here a = 2, n = 10 and k = 12

$\therefore$ Required coefficient $= (-2)^{12}\ {}^{10}C_{12}$

$= 0$ ($\because$ here $n < k$)

(ii) Since $\dfrac{(x^3 - 5x)}{(1-x)^3} = x^3(1-x)^{-3} - 5x(1-x)^{-3}$

so coefficient of x^{15} in $\dfrac{x^3 - 5x}{(1-x)^3}$

= coefficient of x^{15} in $x^3(1 - x)^{-3}$ + coefficient of x^{15} in $\{- 5x\ (1 - x)^{-3}\}$

= coefficient of x^{12} in $(1 - x)^{-3}$ – 5 [coefficient of x^{14} in $(1 - x)^{-3}$] .....(1)

Now, coefficient of x^{12} in $(1 - x)^{-3} = {}^{3+12-1}C_{12} = {}^{14}C_{12} = {}^{14}C_2$

and coefficient of x^{14} in $(1 - x)^{-3} = {}^{3+14-1}C_{14} = {}^{16}C_{14} = {}^{16}C_2$

[Refer S. No. 5 of Table – 1 with n = 3]

$\therefore$ (1) $\Rightarrow$ coefficient of x^{15} in $\dfrac{x^3 - 5x}{(1-x)^3}$

$= {}^{14}C_2 - 5.{}^{16}C_2 = 91 - 600$

$= - 509.$

(iii) $\dfrac{(1+x)^4}{(1-x)^4} = (1+x)^4(1-x)^{-4}$

$= ({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)(1 - x)^4$

$= (1 + 4x + 6x^2 + 4x^3 + x^4)\ (1 - x)^{-4}$

So, coefficient of x^{15} in $\dfrac{(1+x)^4}{(1-x)^4}$

= coefficient of x^{15} in $(1 + 4x + 6x^2 + 4x^3 + x^4) \times (1 - x)^{-4}$

= coefficient of x^{15} in $(1 - x)^{-4}$ + coefficient of x^{14} in $4(1 - x)^{-4}$

$+$ coefficient of x^{13} in $6(1-x)^{-4}$ + coefficient of x^{12} in $4(1-x)^{-4}$

$+$ coefficient of x11 in $(1-x)^{-4}$

$= {}^{4+15-1}C_{15} + 4\,{}^{4+14-1}C_{14} + 6\,{}^{4+13-1}C_{13} + 4\,{}^{4+12-1}C_{12} + {}^{4+11-1}C_{11}$

$= {}^{18}C_{15} + 4\,{}^{17}C_{14} + 6\,{}^{16}C_{13} + 4\,{}^{15}C_{12} + {}^{14}C_{11}$

$= {}^{18}C_{3} + 5\,{}^{17}C_{3} + 6\,{}^{16}C_{3} + 4\,{}^{15}C_{3} + {}^{14}C_{3}$

$= 816 + 3400 + 3360 + 1820 + 364$

$= 9760.$

Example 41. Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$. [Raj. 2000]

Solution. Since $\dfrac{1}{(x-3)(x-2)^2} = \dfrac{1}{x-3} - \dfrac{1}{x-2} - \dfrac{1}{(x-2)^2}$

$$= -\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{x}{2}\right)^{-1} - \frac{1}{4}\left(1-\frac{x}{2}\right)^{-2}$$

So, coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$

$$= \text{coefficient of } x^8 \text{ in } \left[-\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1}\right]$$

$$+ \text{ coefficient of } x^8 \text{ in } \left[\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}\right] + \text{ coefficient of } x^8 \text{ in } \left[-\frac{1}{4}\left(1-\frac{x}{2}\right)^{-2}\right]$$

$$= -\frac{1}{3}\,{}^{1+8-1}C_8\left(\frac{1}{3}\right)^8 + \frac{1}{2}\,{}^{1+8-1}C_8\left(\frac{1}{2}\right)^8 - \frac{1}{4}\,{}^{2+8-1}C_8\left(\frac{1}{2}\right)^8$$

$$= -\left(\frac{1}{3}\right)^9 + \left(\frac{1}{2}\right)^9 - \frac{9}{2}\left(\frac{1}{2}\right)^9 = -\frac{1}{3^9} - \frac{7}{2}\left(\frac{1}{2}\right)^9$$

$$= -\left[\frac{1}{3^9} + \frac{7}{2^{10}}\right].$$

Example 42. Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 5$, where $0 \le x_i \le 2$, $i = 1, 2, 3, 4$. [Raj. 2005]

Solution. Since each x_i can take value 0, 1, or 2 so the generating function for each $x_i = x^0 + x^1 + x^2 = 1 + x + x^2$. As this generating function is common for all x1, x2, x3 and x4 so the number of integer solutions with given constraints

$=$ coefficient of x^5 in the expansion of $(1 + x + x^2)(1 + x + x^2)(1 + x + x^2)(1 + x + x^2)$

$=$ coefficient of x^5 in $(1 + 2x + 3x^2 + 2x^3 + x^4)^2$

$=$ coefficient of x^5 in $(1 + 4x + 10x^2 + 16x^3 + 15x^4 + 16x^5 + 10x^6 + 4x^7 + x^8)$

$= 16$

Thus there are 16 solutions to the given equation.

Example 43. Find the number of solutions of the equation $e_1 + e_2 + e_3 = 17$, where e_1, e_2 and e_3 are non–negative integers with

$2 \le$ e1 ≤ 5, $3 \le$ e2 ≤ 6, and $4 \le$ e3 ≤ 7. [Raj. 2005]

Solution. Since $2 \le e_1 \le 5 \Rightarrow e_1$ can take value 2, 3, 4 or 5. So the generating function corresponds to e_1 is $(x_2 + x_3 + x_4 + x_5)$

Similarly, generating functions corresponds to e_2 and e_3 are

$(x^3 + x^4 + x^5 + x^6)$ and $(x^4 + x^5 + x^6 + x^7)$ respectively.

Thus the number of solutions of the given equation

$=$ coefficient of x^{17} in the expansion of

$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$

$=$ coefficient of x^{17} in $x^9(1 + x + x^2 + x^3)(1 + x + x^2 + x^3)(1 + x + x^2 + x^3)$

$=$ coefficient of x^8 in $(1 + x + x^2 + x^3)^3$

Since $1 + x + x^2 + x^3 = \dfrac{x^4 - 1}{x - 1}$ (sum of Geometric series), $|x| < 1$

$= -(x^4 - 1)(1 - x)^{-1}$

Thus coefficient of x^8 in $(1 + x + x^2 + x^3)^3$

$=$ coefficient of x^8 in $[-(x^4 - 1)^3 (1 - x)^{-3}]$

$=$ coefficient of x^8 in $[-(x^{12} - 1 - 3x^8 + 3x^4)(1 - x)^{-3}]$

$=$ coefficient of x^8 in $(1 - x)^{-3}$ + coefficient of x^0 in $3(1 - x)^{-3}$ – coefficient of x^4 in $3(1 - x)^{-3}$

$= (-1)^8(-1)^8\ {}^{3+8-1}C_8 + 3(-1)^0(-1)^0\ {}^{3+0-1}C_0 - 3(-1)^4(-1)^4\ {}^{3+4-1}C_4$

$= 45 + 3 - 45$

$= 3$

Hence the required number of solutions = 3.

Example 44. In how many different ways can 8 identical cookies be distributed among 3 distinct children if each child receives at least 2 cookies and no more than 4 cookies ? [Raj 2005]

Solution. Since each child receives at least 2 cookies and maximum 4 cookies so the factor corresponding to each child is $(x^2 + x^3 + x^4)$.

As there are 3 children so the generating function is $(x^2 + x^3 + x^4)^3$.

Also here exponents are the number of cookies that child receives. As there are total 8 cookies so the number of ways to distribute them

$=$ coefficient of x^8 in $(x^2 + x^3 + x^4)^3$

$=$ coefficient of x^8 in $x^6(1 + x + x^2)^3$

$=$ coefficient of x^2 in $(1 + x + x^2)^3$

$=$ coefficient of x^2 in $\left(\dfrac{x^3 - 1}{x - 1}\right)^3$

$=$ coefficient of x^2 in $[-(x^3 - 1)^3 (1 - x)^{-3}]$

Since $(x^3 - 1)^3 (1 - x)^{-3} = (x^9 - 1 - 3x^6 + 3x^3)(1 - x)^{-3}$

so coefficient of x^2 in $[-(x^3 - 1)^3 (1 - x)^{-3}]$ = coefficient of x2 in $(1 - x)^{-3}$

$= (-1)^2(-1)^2\ {}^{3+2-1}C_2 = 6$

Thus the required number of ways = 6.

Example 45. Use generating function to show that

$$\sum_{r=0}^{n}\left({}^{n}C_r\right)^2 = {}^{2n}C_n \text{ where n is a positive integer.}$$

Solution. Since $(1 + x)^{2n} = \sum_{k=0}^{2n} {}^{2n}C_k x^k$

$\Rightarrow$ coefficient of x^n in $(1 + x)^{2n}$ is ${}^{2n}C_n$ .....(1)

Also $(1 + x)^{2n} = [(1 + x)^n]^2$

$= [{}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n]^2$

$= ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n)({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n)$

$\Rightarrow$ coefficient of $x_n = {}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + ... + {}^nC_n\,{}^nC_0$

$= {}^nC_0\,{}^nC_0 + {}^nC_1\,{}^nC_1 + ... + {}^nC_n\,{}^nC_n$

$= ({}^nC_0)^2 + ({}^nC_1)^2 + ... + ({}^nC_n)^2$ [using ${}^nC_r = {}^nC_{n-r}$]

$= \sum_{r=0}^{n}\left({}^{n}C_r\right)^2$ .....(2)

Since (1) and (2) both are the coefficients of x^n in the expansion of $(1 + x)^{2n}$ so they must be equal i.e.,

$$\sum_{r=0}^{n}\left({}^{n}C_r\right)^2 = {}^{2n}C_n$$

Hence the result.

Example 46. Find the generating function for the finite sequence 2, 2, 2, 2, 2, 2.

Solution. Since $G(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

Here $a_0 = 2 = a_1 = a_2 = a_3 = a_4 = a_5$

$G(x) = 2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5$

$= 2(1 + x + x^2 + x^3 + x^4 + x^5)$

$= 2\left(\frac{x^6-1}{x-1}\right)$ [sum of G.P. $= \frac{r^n-1}{r-1}$]

which is the required generating function.

Example 47. Using generating function, evaluate the sum $0^2 + 1^2 + 2^2 + ... + k^2$.

Solution. First we find the generating function for the sequence $(0^2, 1^2, 2^2, 3^2, ..., k^2, ...)$.

Since, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^k + \ldots$

on differentiating it w.r.t x, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + kx^{k-1} + \ldots$$

$$\Rightarrow \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots + kx^k + \ldots \quad .....(1)$$

On differentiating it, we get

$$\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = 1^2 + 2^2x + 3^2x^2 + \ldots + k^2x^{k-1} + \ldots$$

or, $x\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = 0^2 + 1^2.x + 2^2.x^2 + 3^2.x^3 + \ldots + k^2.x^k + \ldots$

or, $x\left[\frac{(1+x)}{(1-x)^3}\right] = 0^2 + 1^2.x + 2^2.x^2 + 3^2.x^3 + \ldots + k^2.x^k + \ldots$

$\Rightarrow$ Generating function for the sequence $(0^2, 1^2, 2^2, 3^2, \ldots\ldots, k^2, \ldots\ldots)$ is

$$\frac{x(1+x)}{(1-x)^3} \quad \ldots\ldots(2)$$

Now, $\frac{x(1+x)}{(1-x)^3}\cdot\frac{1}{1-x} = (0^2 + 1^2.x + 2^2.x^2 + 3^2.x^3 + \ldots\ldots + k^2.x^k + \ldots.)(1 + x + x^2 + x^3 + \ldots\ldots x^k + \ldots.)$

$$\Rightarrow \frac{x(1+x)}{(1-x)^4} = 0^2 + (0^2 + 1^2)x + (0^2 + 1^2 + 2^2)\,x^2 + (0^2 + 1^2 + 2^2 + 3^2)$$

$$x^3 + \ldots\ldots + (0^2 + 1^2 + 2^2 + \ldots\ldots + k^2)\,x^k + \ldots\ldots \quad \ldots\ldots(3)$$

$\Rightarrow \frac{x(1+x)}{(1-x)^4}$ is a generating function for the sequence

$(0^2, 0^2 + 1^2, 0^2 + 1^2 + 2^2, \ldots., 0^2 + 1^2 + 2^2 + \ldots.. + k^2, \ldots.)$

Now the coefficient of xk in the expansion of

$\frac{x(1+x)}{(1-x)^4}$ = coefficient of x^k in $x(1-x)^{-4}$ + coefficient of x^k in $x^2(1-x)^{-4}$

= coefficient of x^{k-1} in $(1-x)^{-4}$ + coefficient of x^{k-2} in $(1-x)^{-4}$

$= (-1)^{k-1}(-1)^{k-1}\,{}^{4+k-1-1}C_{k-1} + (-1)^{k-2}(-1)^{k-2}\,{}^{4+k-2-1}C_{k-2}$

$= {}^{k+2}C_{k-1} + {}^{k+1}C_{k-2} = {}^{k+2}C_3 + {}^{k+1}C_3$

$= \frac{(k+2)(k+1)k}{3\cdot2\cdot1} + \frac{(k+1)k(k-1)}{3\cdot2\cdot1} = \frac{k(k+1)(k+2+k-1)}{6} = \frac{k(k+1)(2k+1)}{6} \quad \ldots\ldots(4)$

Now from (3), coefficient of $x^k = 0^2 + 1^2 + 2^2 + \ldots.. + k^2 \quad \ldots\ldots(5)$

Since (4) and (5) both represent the coefficient of xk in the expansion of $\frac{x(1+x)}{(1-x)^4}$ so they must be equal i.e.,

$$0^2 + 1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Example 48. Using generating function show that

$${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$$

Solution. Since $(1 + x)^n = (1 + x)^{n-1}(1 + x) = (1 + x)^{n-1} + x(1 + x)^{n-1}$

Now, coefficient of xr in LHS = coefficient of xr in RHS

$\Rightarrow$ coefficient of xr in $(1 + x)^n$ = coefficient of x^r in $(1 + x)^{n-1}$ + coefficient of x^r in $x(+ x)^{n-1}$

$\Rightarrow {}^nC_r = {}^{n-1}C_r$ + coefficient of x^{r-1} in $(1 + x)^{n-1}$

Hence Proved.

Example 49. Find the sequence corresponding to generating function $f(x) = (1 + x)^{-n}$, where n is a positive integer.

Solution. By extended Binomial theorem, we have

$$(1 + x)^p = \sum_{k=0}^{\infty} {}^pC_k x^k \text{, p be a real number}$$

$$= {}^pC_0 + {}^pC_1x + {}^pC_2x^2 + \ldots + {}^pC_rx^r + \ldots$$

replace p by – n, we get

$$(1 + x)^{-n} = {}^{-n}C_0 + {}^{-n}C_1x + {}^{-n}C_2x^2 + \ldots + {}^{-n}C_rx^r + \ldots$$

$\Rightarrow$ $(1 + x)^{-n}$ is a generating function for the sequence

$${}^{-n}C_0, {}^{-n}C_1, {}^{-n}C_2, \ldots, {}^{-n}C_r, \ldots$$

Example 50. Use generating functions to find the number of r–combinations of a set with n elements.

Solution. Let ar be the required number of r–combinations of a set with n elements and G(x) be the generating function for $\{a_r\}$ i.e.,

$$G(x) = \sum_{r=0}^{n} a_r x^r \quad \text{.....(1)}$$

Now each element can be selected in 2 ways i.e., 0 times or 1 times so the factor corresponding to each element is 1 + x.

Since there are n elements and each of these elements contributes the factor (1 + x) to the generating function G(x). Hence,

$$G(x) = (1 + x)^n$$

$$= \sum_{r=0}^{n} {}^nC_r x^r \quad \text{.....(2)}$$

Comparing (1) and (2), we get

$$a_r = {}^nC_r$$

$\Rightarrow$ number of r combinations of a set with n elements

$$= {}^nC_r$$

$$= \frac{n!}{(n-r)!r!}$$

Example 51. In how many ways can we select, with repetition allowed, r objects from n distinct objects?

Solution. Let ar denote the number of ways to select r objects from n distinct objects when repetition is allowed and let G(x) be the generating function for the sequence $\{a_r\}$ i.e.,

$$G(x) = \sum_{r=0}^{\infty} a_r x^r \quad \text{.....(1)}$$

Since each element can be selected 0 times, 1 times, 2 times, ... so the factor corresponding to each element is $(1 + x + x^2 + \ldots..)$.

As there are n elements and each of these n elements contributes the factor (1 + x + x2 + ...) to the generating function G(x). Hence,

$$G(x) = (1 + x + x^2 + \ldots.)n \quad \left[\because 1 + x + x^2 + \ldots = \frac{1}{1-x}, |x| < 1 \text{ (sum of infinite G.P.)}\right]$$

$$= \frac{1}{(1-x)^n}$$

$$= \sum_{r=0}^{\infty} {}^{-n}C_r (-x)^r = \sum_{r=0}^{\infty} (-1)^r \, {}^{-n}C_r x^r \qquad .....(2)$$

on comparing (1) and (2), we have

$a_r = (-1)^r \, {}^{-n}C_r$

$= (-1)^r (-1)^r \, {}^{n+r-1}C_r$ (by extended Binomial coefficient)

$= {}^{n+r-1}C_r$

Thus, number of required ways $= {}^{n+r-1}C_r$

Example 52. Find the generating function for the sequence 0, 2, 6, 12, 20, 30, 42, ...

Solution. Here

$a_0 = 0 = 0^2 + 0, a_1 = 2 = 1^2 + 1, a_2 = 6 = 2^2 + 2, a_3 = 12 = 3^2 + 3, a_4 = 20 = 4^2 + 4, a_5 = 30 = 5^2 + 5, a_6 = 42 = 6^2 + 6$ and so on.

In general, we have

$a_n = n^2 + n, n \geq 0$

Since the generating function for the sequence n^2 i.e.,

$0^2, 1^2, 2^2, 3^2, ..., k^2, ...$

is $\dfrac{x(1+x)}{(1-x)^3}$.

Also, the generating function for the sequence n i.e.,

0, 1, 2, 3, ..., k, ...

is $\dfrac{x}{(1-x)^2}$.

So the generating function for the sequence a_n

= generating function for the sequence n^2 + generating function for the sequence n

$$= \frac{x(1+x)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(1+x)+x(1-x)}{(1-x)^3}$$

$$= \frac{2x}{(1-x)^3}.$$

Example 53. Find the number of ways in which 2t + 1 marbles can be distributed among 3 distinct boxes so that no box will contain more than t marbles.

Solution. Since each box can contain 0 marble, 1 marble, 2 marbles, ..., t marbles, so the factor corresponding to each box is $(1 + x + x^2 + ... + x^t)$. As there are 3 boxes and each box contributes the factor $(1 + x + x^2 + ... + x^t)$ so the generating function can be formed as

$$G(x) = (1 + x + x^2 + ... + x^t)^3$$

$$= \left(\frac{1-x^{t+1}}{1-x}\right)^3 = \left(1-x^{t+1}\right)^3 (1-x)^{-3}$$

$$= \left(1-3x^{t+1}+3x^{2t+2}-x^{3t+3}\right)^3 (1-x)^{-3}$$

So the required number of ways to put 2t + 1 marbles in 3 boxes = coefficient of x^{2t+1} in the expansion of G(x)

$= $ coefficient of x^{2t+1} in $(1 - 3x^{t+1} + 3x^{2t+2} - x^{3t+3})(1-x)^{-3}$

$= $ coefficient of x^{2t+1} in $(1-x)^{-3}$ + coefficient of x^{2t+1} in $(-3x^{t+1})(1-x)^{-3}$

$= $ coefficient of x^{2t+1} in $(1-x)^{-3}$ + coefficient of x^{t} in $-3(1-x)^{-3}$

$$= {}^{3+2t+1-1}C_{2t+1} - 3\,{}^{3+T-1}C_t$$

$$= {}^{2t+3}C_{2t+1} - 3\,{}^{t+2}C_t = {}^{2t+3}C_2 - 3\,{}^{t+2}C_2$$

$$= \frac{(2t+3)(2t+2)}{2\cdot 1} - 3\cdot\frac{(t+2)(t+1)}{2\cdot 1}$$

$$= \left(\frac{t+1}{2}\right)[2(2t+3) - 3(t+2)] = \frac{1}{2}(t+1)(t)$$

$$= \frac{1}{2}t(t+1).$$

Example 54. Show that the generating function for a, where $a_r = {}^{2r}C_r$ is $(1-4x)^{-1/2}$.

Solution. Since the sequence is $a_0, a_1, a_2, a_3, \ldots$

i.e., ${}^0C_0, {}^2C_1, {}^4C_2, {}^6C_3, \ldots$

or 1, 2, 6, 20, ...

So the corresponding generating function is

$G(x) = 1 + 2x + 6x^2 + 20x^3 + \ldots$

$$= 1 + \left(-\frac{1}{2}\right)(-4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-4x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(-4x)^3 + \ldots$$

$$= \left(-\tfrac{1}{2}\right)_{C_0} + \left(-\tfrac{1}{2}\right)_{C_1}(-4x) + \left(-\tfrac{1}{2}\right)_{C_2}(-4x)^2 + \left(-\tfrac{1}{2}\right)_{C_3}(-4x)^3 + + \ldots$$

(by extended Binomial coefficients)

(refer Example 49)

$$= (1-4x)^{-1/2}$$

$$= \frac{1}{(1-4x)^{1/2}}$$ Proved.

Example 55. Use generating functions to find the number of ways to select k objects of n different kinds if we must select at least one object of each kind.

Solution. Since atleast one object of each kind must be selected so each of the n kinds of objects contributes the factor $(x + x^2 + x^3 + \ldots)$ to the generating function G(x) i.e.,

$G(x) = (x + x^2 + x^3 + \ldots)^n$

$$= x^n(1 + x + x^2 + \ldots)^n = \frac{x^n}{(1-x)^n}$$

Now the number of ways to select k objects of n different kinds such that atleast one object of each kind must be selected

$= $ coefficient of x^k in the expansion of G(x).

$= $ coefficient of x^k in $x^n(1-x)^{-n}$

$= $ coefficient of x^{k-n} in $(1-x)^{-n}$

$= {}^{n+k-n-1}C_{k-n}$

$= {}^{k-1}C_{k-n}$.

Example 56. Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 20$ with $-3 \le x_1$, $-3 \le x_2$, $-5 \le x_3 \le 5$ and $0 \le x_4$.

Solution. Given $-3 \le x_1$, $-3 \le x_2$, $-5 \le x_3 \le 5$, $0 \le x_4$

$\Rightarrow 0 \le x_1 + 3,\ 0 \le x_2 + 3,\ 0 \le x_3 + 5 \le 10,\ 0 \le x_4$

$\Rightarrow 0 \le y_1,\ 0 \le y_2,\ 0 \le y_3 \le 10,\ 0 \le y_4$

where $y_1 = x_1 + 3$, $y_2 = x_2 + 3$, $y_3 = x_3 + 5$, $y_4 = x_5$

$\Rightarrow y_1 + y_2 + y_3 + y_4 = x_1 + x_2 + x_3 + x_4 + 11 = 31$ [given $x_1 + x_2 + x_3 + x_4 = 20$]

Thus the number of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

with $-3 \le x_1$, $-3 \le x_2$, $-5 \le x_3 \le 5$ and $0 \le x_4$ is equal to the number of integer solutions to the equation

$$y_1 + y_2 + y_3 + y_4 = 31 \qquad .....(1)$$

with $0 \le y_1$, $0 \le y_2$, $0 \le y_3 \le 10$, $0 \le y_4$

Now the factor corresponding to each y_1, y_2 and y_4 is $(1 + x + x^2 + ...)$ and the factor corresponding to y_3 is $(1 + x + x^2 + ... + x^{10})$

Thus the generating function for the problem is

$G(x) = (1 + x + x^2 + ...)^3 (1 + x + x^2 + ... + x^{10})$

$= \dfrac{1}{(1-x)^3} \cdot \dfrac{1-x^{11}}{1-x} = \dfrac{1-x^{11}}{(1-x)^4}$

Therefore, the required number of solutions to the equation (1) = coefficient of x^{31} in the expansion of G(x)

$=$ coefficient of x^{31} in $(1 - x^{11})(1 - x)^{-4}$

$=$ coefficient of x^{31} in $(1 - x)^{-4}$ – coefficient of x^{20} in $(1 - x)^{-4}$

$= {}^{34}C_3 - {}^{23}C_3 = 5984 - 1771$

$=$ 4,213.

Hence there are 4,213 integer solutions to the given equation.

Example 57. In how many ways can a police captain distribute 24 rifle shells to 4 police officers so that each police officer gets at least 3 shells but not more than 8 ?

Solution. Each police officer contributes the factor $(x^3 + x^4 + x^5 + x^6 + x^7 + x^8)$ to the generating function G(x) i.e.,

$G(x) = (x^3 + x^4 + x^5 + ... + x^8)^4$

$= x^{12}(1 + x + x^2 + ... + x^5)^4$

$= x^{12}\left(\dfrac{1-x^6}{1-x}\right)^4$

Thus the required number of ways

$=$ coefficient of x^{24} in the expansion of G(x)

$=$ coefficient of x^{24} in $x^{12}(1 - x^6)^4 (1 - x)^{-4}$

$=$ coefficient of x^{12} in $(1-x^6)^4(1-x)^{-4}$

$=$ coefficient of x^{12} in $(1-4x^6+6x^{12}-4x^{18}+x^{24})(1-x)^{-4}$

$=$ coefficient of x^{12} in $(1-x)^{-4}$ + coefficient of x^6 in $-4(1-x)^{-4}$ + coefficient of x^0 in $6(1-x)^{-4}$

$= {}^{4+12-1}C_{12} - 4\ {}^{4+6-1}C_6 + 6 = {}^{15}C_{12} - 4\ {}^{9}C_6 + 6$

$= {}^{15}C_3 - 4\ {}^{9}C_3 + 6 = 455 - 336 + 6$

$=$ 125.

Example 58. In how many ways can 3000 identical envelopes be distributed, in packet of 25, among 4 groups so that each group gets at least 150, but not more than 1000, of the envelopes.

Solution. Here 3000 envelopes = 120 packets (each having 25). Now each group gets at least 150 envelopes i.e. 6 packets but not more than 1000 envelops i.e. 40 packets.

So each group contributes the factor

$$(x^6 + x^7 + \ldots\ldots + x^{40})$$

to the generating function G(x) i.e.,

$G(x) = (x^6 + x^7 + \ldots\ldots + x^{40})^4$

$= x^{24}(1 + x + x^2 + \ldots\ldots + x^{34})^4$

$$= x^{24}\left(\frac{1-x^{35}}{1-x}\right)^4 = x^{24}(1-x^{35})^4(1-x)^{-4}$$

$= x^{24}(1-4x^{35}+6x^{70}-4x^{105}+x^{140})(1-x)^{-4}$

Since the number of packets to be distributed is 120 so the required number of ways to distribute them

$=$ coefficient of x^{120} in the expansion of G(x)

$=$ coefficient of x^{120} in $x^{24}(1-4x^{35}+6x^{70}-4x^{105}+x^{140})(1-x)^{-4}$

$=$ coefficient of x^{96} in $(1-x)^{-4}$ + coefficient of x^{61} in $(-4)(1-x)^{-4}$ + coefficient of x^{26} in $6(1-x)^{-4}$

$= {}^{4+96-1}C_{96} - 4\ {}^{4+61-1}C_{61} + 6\ {}^{4+26-1}C_{26}$

$= {}^{99}C_{96} - 4\ {}^{64}C_{61} + 6\ {}^{29}C_{26} = {}^{99}C_3 - 4\ {}^{64}C_3 + 6\ {}^{29}C_3.$

$=$ 12117.

Example 59. Solve the recurrence relation using generating functions,

$$a_n = 3a_{n-1} - 2a_{n-2},\ n > 2$$ **[MNIT 2003]**

with $a_1 = 5$, $a_2 = 3$.

Solution. Let the generating function for the sequence $\{a_n\}$ is G(x) i.e.,

.....(1)

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad .....(2)$$

or, $G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$ .....(3)

Given $a_n = 3a_{n-1} - 2a_{n-2}$

multiply (3) by x^n and taking summation over n = 3 to ∞ on both sides, we get

$$\sum_{n=3}^{\infty} a_n x^n = 3\sum_{n=3}^{\infty} a_{n-1}x^n - 2\sum_{n=3}^{\infty} a_{n-2}x^n$$

or, $a_3x^3 + a_4x^4 + \ldots\ldots = 3[a_2x^3 + a_3x^4 + \ldots\ldots] - 2[a_1x^3 + a_2x^4 + \ldots\ldots]$

or, $[G(x) - a_0 - a_1x - a_2x^2] = 3x(a_2x^2 + a_3x^3 + ...) - 2x^2(a_1x + a_2x^2 + ...]$

$= \quad 3x\,[G(x) - a_0 - a_1x] - 2x^2[G(x) - a_0]$ .....(4)

Now from (3), $a_2 = 3a_1 - 2a_0 \Rightarrow 2a_0 = 3a_1 - a_2 = 15 - 3 = 12$

$\Rightarrow a_0 \;= 6$

$\therefore$ (4) $\Rightarrow G(x) - 6 - 5x - 3x^2 = 3x[G(x) - 6 - 5x] - 2x^2\,[G(x) - 6]$

or, $(1 - 3x + 2x^2)\,G(x) = -\,18x - 15x^2 + 12x^2 + 6 + 5x + 3x^2$

$= 6 - 13\,x$

$$\text{or, } G(x) = \frac{6-13x}{(1-3x+2x^2)} = \frac{6-13x}{(x-1)(2x-1)}$$

$$= \frac{1}{2x-1} - \frac{7}{x-1} = \frac{7}{1-x} - \frac{1}{1-2x} = 7\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}(2x)^n$$

$$= \sum_{n=0}^{\infty}(7-2n)x^n \qquad .....(5)$$

On comparing (1) and (5), we get the required solution as

$a_n = 7 - 2^n$.

Example 60. How many four element subsets of S = {1, 2, 3, ..., 15} contain no consecutive integers ?

Solution. Let P = {r, s, t, u} be any arbitrary required subset such that $1 < r < s < t < u < 15$.

Let $a = r - 1$, $b = s - r$, $c = t - s$, $d = u - t$ and $e = 15 - u$

Then clearly, $a + b + c + d + e = 14$ ......(1)

with $a \geq 0$, $b \geq 2$, $c \geq 2$, $d \geq 2$ and $e \geq 0$ .....(2)

Now, the number of four element subsets of the set S containing no consecutive integers

= number of integer solutions to the equation (1) with constraints (2).

Now, the generating function corresponding to constraints (2) is

$G(x) = (1 + x + x^2 + ...)^2\,(x^2 + x^3 + x^4 + ...)^3$

$= \quad x^6(1 + x + x^2 + ...)^5 = x^6\left(\dfrac{1}{1-x}\right)^5$

$= \quad x^6(1 - x)^{-5}$

Thus, the number of integer solutions to equation (1) = coefficient of x^{14} in the expansion of G(x)

= coefficient of x^{14} in x6 $(1 - x)^{-5}$

= coefficient of x^8 in $(1 - x)^{-5}$

$= \; {}^{5+8-1}C_8 = {}^{12}C_8 = {}^{12}C_4$

$= \; \dfrac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495.$

Hence, there are 495 four element subsets of S that contain no consecutive integers.

Example 61. Find the generating function for the number of ways an advertising agent can purchase n minutes of air time sold for commercials in blocks of 20, 40 or 60 seconds.

Solution. Let 20 seconds represent one time unit. Then the required number of ways is the number of integer solutions to the equation

$$a + 2b + 3c = n, \qquad a \geq 0, \; b \geq 0, c \geq 0.$$

Now the corresponding generating function is

$$G(x) = (1 + x + x^2 + ...)\,(1 + x^2 + x^4 + ...)\,(1 + x^3 + x^6 + ...)$$

Here 1st factor $(1 + x + x^2 + ...)$ corresponds to the variable a, 2nd factor $(1 + x^2 + x^4 + ...)$ corresponds to the variable 2b i.e. multiple of 2 and 3rd factor $(1 + x^3 + x^6 +......)$ corresponds to the variable 3c i.e. multiple of 3.

Also, $G(x) = \dfrac{1}{(1-x)} \cdot \dfrac{1}{(1-x^2)} \cdot \dfrac{1}{(1-x^3)}$

So the required number of ways = coefficient of x^n in the expansion of G(x)

$$= \text{coefficient of } x^n \text{ in } \frac{1}{(1-x)} \cdot \frac{1}{(1-x^2)} \cdot \frac{1}{(1-x^3)}.$$

Example 62. In how many ways can four of the alphabets in BETTER be arranged ?

Solution. Here we apply exponential generating function.

The factor corresponding to B is $(1 + x)$, the factor corresponding to E is $\left(1+x+\dfrac{x^2}{2!}\right)$, the factor corresponding to T is $\left(1+x+\dfrac{x^2}{2!}\right)$ and the factor corresponding to R is $(1 + x)$.

Thus the exponential generating function is

$$G(x) = (1+x)^2\left(1+x+\frac{x^2}{2!}\right)^2$$

Now the required number of ways

$$= \text{coefficient of } \frac{x^4}{4!} \text{ in the expansion of } G(x)$$

$$= \text{coefficient of } \frac{x^4}{4!} \text{ in } (1+x)^2\left(1+x+\frac{x^2}{2!}\right)^2$$

$$= \text{coefficient of } \frac{x^4}{4!} \text{ in } (x^2+2x+1)\left(\frac{x^4}{4}+x^3+2x^2+2x+1\right)$$

Since coefficient of x^4 in $(x^2+2x+1)\left(\dfrac{x^4}{4}+x^3+2x^2+2x+1\right)$

$$= 2+2+\frac{1}{4}=\frac{17}{4}$$

$$\Rightarrow \text{coefficient of } \frac{x^4}{4!} = 4! \times \frac{17}{4} = 102$$

$\therefore$ The required number of ways = 102.

Example 63. A ship carries 48 flags, 12 each of the colours red, white, blue and black, 12 of these flags are placed on a vertical pole in order to communicate a signal to other ships.

(i) How many of these signals use an even number of blue flags and an odd number of black flags?

(ii) How many of these signals have at least 3 white flags or no white flags?

Solution. (i) The factor corresponding to each red and white flags is

$$\left(1+x+\frac{x^2}{2!}+\ldots\right)$$

The factor corresponding to blue flags is

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)$$

The factor corresponding to black flags is

$$\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right)$$

Thus the corresponding generating function is given as

$$G(x) = \left(1+x+\frac{x^2}{2!}+\ldots\right)^2\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right)$$

Since, $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots$

$$e^{-1} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\ldots$$

$$\Rightarrow \frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$$

and $\frac{e^x-e^{-x}}{2} = 1+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$

$$\therefore\ (G)x = e^{2x}\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)=\frac{1}{4}\left(e^{4x}-1\right)$$

$$= \frac{1}{4}\left[\left\{1+(4x)+\frac{(4x)^2}{2!}+\frac{(4x)^3}{3!}+\ldots\right\}-1\right]$$

$$= \frac{1}{4}\left[\left\{(4x)+\frac{(4x)^2}{2!}+\frac{(4x)^3}{3!}+\ldots\right\}\right]$$

$$= \frac{1}{4}\sum_{r=1}^{\infty}\frac{(4x)^r}{r!}$$

Now, the required number of signals = coefficient of $\frac{x^{12}}{12!}$ in G(x)

$= \text{ coefficient of } \frac{1}{4}(4)^{12} \text{ in } \frac{1}{4}\sum_{r=1}^{\infty}\frac{(4x)^r}{r!}$

$= \frac{1}{4}(4)^{12}$ [On taking r = 12]

$= 4^{11}$.

(ii) In this case the corresponding generating function is

$$G(x) = \left(1+x+\frac{x^2}{2!}+\ldots\right)^3\left(1+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)$$

$$= e^{3x}\left(e^x - x - \frac{x^2}{2!}\right) = e^{4x} - xe^{3x} - \frac{x^2}{2}e^{3x}$$

$$= \sum_{r=0}^{\infty}\frac{(4x)^r}{r!} - x\sum_{r=0}^{\infty}\frac{(3x)^r}{r!} - \frac{x^2}{2}\sum_{r=0}^{\infty}\frac{(3x)^r}{r!}$$

Now, the required number of signals

$$= \text{coefficient of } \frac{x^{12}}{12!} \text{ in } G(x)$$

$$= \text{coefficient of } \frac{x^{12}}{12!} \text{ in } \left[\sum_{r=0}^{\infty}\frac{(4x)^r}{r!} - x\sum_{r=0}^{\infty}\frac{(3x)^r}{r!} - \frac{x^2}{2}\sum_{r=0}^{\infty}\frac{(3x)^r}{r!}\right]$$

$$= 4^{12} - 3^{11}\times 12 - \frac{1}{2}\times 12\times 11\times 3^{10} = 4^{12} - 12\times 3^{11} - 66\times 3^{10}$$

$$= 4^{12} - 102 \times 3^{10} = 16777216 - 6022998$$

$$= 10754218.$$

Example 64. A company hires 25 new employees. Give the exponential generating function for the number of ways to assign these people to the 4 subdivisions so that each subdivision will get at least 3 new employee but no more than 10 new employee.

Solution. The corresponding generating function is

$$G(x) = \left(\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^{10}}{10!}\right)^4$$

Thus the required number of ways = coefficient of $\frac{x^{25}}{25!}$ in G(x)

$$= \text{coefficient of } \frac{x^{25}}{25!} \text{ in } \left(\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^{10}}{10!}\right)^4$$

Example 65. Find the number of ways in which an arrangement of 4 letters can be made from the letters of the word PROPORTION.

Solution. The corresponding generating function is

$$G(x) = (1+x)^3\left(1+x+\frac{x^2}{2!}\right)^2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)$$

Thus the required number of ways

$$= \text{coefficient of } \frac{x^4}{4!} \text{ in } G(x)$$

Since, $G(x) = (x^3+3x^2+3x+1)\left(\frac{x^4}{4}+x^3+2x^2+2x+1\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)$

$\therefore$ coefficient of x^4 in G(x)

$$= \text{coefficient of } x^4 \text{ in } \left[\left(2x^4+x^3+6x^4+6x^3+3x^2+3x^4+6x^3+6x^2+3x+\frac{x^4}{4}+x^3+2x^2+2x+1\right)\times\left(1+x+\frac{x2}{2}+\frac{x^3}{6}\right)\right]$$

$= \text{coefficient of } x^4 \text{ in } \left(\frac{45}{4}x^4+14x^3+11x^2+5x+1\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)$

$= \frac{45}{2}+14+\frac{11}{2}+\frac{5}{6}=\frac{379}{12}=\frac{758}{4!}$

Hence, the required number of ways

$= \text{coefficient of } \frac{x^4}{4!} \text{ in } G(x)$

$= 758.$

EXERCISE 7.2

1. Find a closed form for the generating function for each of the following sequences.
   (a) 0, 0, 0, 1, 1, 1, 1, 1, 1, ...
   (b) 2, 4, 8, 16, 32, 64, 128, 256, ...
   (c) $^7C_0$, $^7C_1$, $^7C_2$, ..., $^7C_7$, 0, 0, 0
   (d) $^8C_1$, $2\,^8C_2$, $3\,^8C_3$, ..., $8\,^8C_8$
   (e) 0, 0, 1, a, a^2, a^3, ... $(a \neq 0)$
   (f) 0, 1, 2, 3, 4, ...

   [Ans. (a) $\frac{x^3}{1-x}$ (b) $\frac{2}{1-2x}$ (c) $(1+x)^7$ (d) $8(1+x)^7$ (e) $\frac{x^2}{1-ax}$ (f) $\frac{x}{(1-x)^2}$]
2. Find the coefficient of
   (a) x^{10} in $(1+x^5+x^{10}+x^{15}+\ldots)^3$
   (b) x^{10} in $(x^2+x^4+x^6+\ldots)(x^3+x^6+x^9+\ldots)(x^4+x^8+x^{12}+\ldots)$
   (c) x^{10} in $\frac{1}{(1+2x)^4}$
   (d) x^{10} in $\frac{x^4}{(1-3x)^3}$
   (e) x^{15} in $(x^2+x^3+x^4+\ldots)^4$
   (f) x^5 in $(1-2x)^{-7}$
   (g) x^7 in $(1+x+x^2+\ldots)^{15}$
   (h) x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$

   [Ans. (a) 6 (b) 0 (c) 292,864 (d) 20,412 (e) 120 (f) 14784 (g) 116,280 (h) 381]
3. Find the number of integer solutions for the following equations.
   (i) $x_1+x_2+x_3+x_4=20$, $0 \le x_i \le 7\ \forall\ i = 1, 2, 3, 4$.
   (ii) $x_1+x_2+x_3+x_4+x_5=30$, with $2 \le x_1 \le 4$, $3 \le x_i \le 8$, $i = 2, 3, 4, 5$.

   [Ans. (i) 161 (ii) 246]
4. In how many ways can 10 identical balloons be distributed to 4 children if each child receives at least two balloons ?

[Ans. 10]

5. In how many ways 25 identical toys can be distributed among 4 children ?

[Ans. 3276]

6. Solve the following recurrence relations using generating functions.

(i) $a_n - 3a_{n-1} = n, n \geq 1, a_0 = 1$

(ii) $a_{n+1} - a_n = n^2, n \geq 0, a_0 = 1$

(iii) $a_n - 3a_{n-1} = 5^{n-1}, n \geq 1, a_0 = 1$

(iv) $a_{n+2} - 2a_{n+1} + a_n = 2^n, n \geq 0, a_0 = 1, a_1 = 2$

(v) $a_{n+2} - 5a_{n+1} + 6a_n = 2, n \geq 0, a_0 = 3, a_1 = 7$

(vi) $a_n = 8a_{n-1} + 10_{n-1}, n \geq 1, a_1 = 9$

(vii) $a_n = 3a_{n-1} + 2, n \geq 1, a_0 = 1$

(viii) $a_n = 4a_{n-1} - 4a_{n-2} + n^2, n \geq 2, a_0 = 2, a_1 = 5$

(ix) $a_n = 5a_{n-1} - 6a_{n-2}, n \geq 2, a_0 = 6, a_1 = 30$

(x) $a_n = a_{n-1} + a_{n-2}, n \geq 2, a_0 = 0, a_1 = 1$

[Ans. (i) $a_n = \frac{7}{3}3^n - \frac{1}{2}n - \frac{3}{4}$ (ii) $an = 1 + \frac{n(n-1)(2n-1)}{6}$

(iii) $a_n = \frac{1}{2}(3^n + 5^n)$ (iv) $a_n = 2^n$

(v) $a_n = 2 \cdot 3^n + 1$ (vi) $a_n = \frac{1}{2}(8^n + 10^n)$

(vii) $a_n = 2 \cdot 3^n - 1$ (viii) $a_n = n^2 + 8n + 20 + (6n - 18)\, 2^n$

(ix) $a_n = 18 \cdot 3^n - 12 \cdot 2^n$

(x) $a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

7. In how many ways can a person select n marbles from a large supply of blue, white and green marbles if the selection must include an even number of blue one ?

[Ans. $\frac{1}{8}\{1+(-1)^n\} + \frac{1}{4}\,{}^{n+1}C_n + \frac{1}{2}\,{}^{n+2}C_n$]

8. Determine the sum of the following

(i) ${}^nC_1 + 2\,{}^nC_2 + \ldots + n\,{}^nC_n$

(ii) ${}^nC_0 + 2\,{}^nC_1 + 2^2\,{}^nC_2 + \ldots + 2^n\,{}^nC_n$

[Ans. (i) 0 (ii) 3^n]

9. A company hires 11 new employees, each of whom is to be assigned to one of the four subdivisions so that each subdivision will get at least one new employee. In how many ways can these assignment be made ?

[Ans. Coefficient of $\frac{x^{11}}{(11)!}$ in $\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^4$]

10. Find the exponential generating function for the number of ways to arrange n letters (n > 0), selected from each of the following words

(i) MISSISSIPPI (ii) HAWAII

[Ans. (i) $(1+x)\left(1+x+\frac{x^2}{2!}\right)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)^2$ (ii) $(1+x)^2\left(1+x+\frac{x^2}{2!}\right)^2$]

11. Find the generating function for the number of ways an advertising agent can purchase n minutes of air time sold for commericals in blocks of 30, 60 or 120 seconds.

$$\left[\text{Ans. Coefficient of } x^n \text{ in } \frac{1}{(1-x)}\cdot\frac{1}{(1-x^2)}\cdot\frac{1}{(1-x^4)}\right]$$

12. How many ways can the four of the letters in ENGINE be arranged ?

$$\left[\text{Ans. Coefficient of } \frac{x^4}{(4)!} \text{ in } (1+x)^2\left(1+x+\frac{x^2}{2!}\right)^2\right]$$

13. If, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ show that $\sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1} a_i\right)x^n = \frac{xf(x)}{1-x}$.

14. Find the generating function for the sequence $a_0, a_1, a_2, \ldots$ where an $= \sum_{i=0}^{n}\left(\frac{1}{i!}\right)$, $n \in N$.

$$\left[\text{Ans. } \frac{e^x}{1-x}\right]$$

15. How many ways Rs. 1200 can be distributed among 3 persons Ajay, Amit and Rakesh so that Ajay gets atleast 400, and Amit and Rakesh gets at least 200 each but Rakesh gets not more than 500 ? The distribution is in the multiple of Rs. 100.

[Ans. 14]

16. If there are 30 red, 30 blue, 30 green and 30 white shirts then in how many ways a person can select 30 shirts so that he has any number of white, green or blue shirts and at least 6 red ones.

[Ans. Coefficient of x^{30} in $(1 + x + x^2 + \ldots.. x^{30})^3 (x^6 + x^7 + \ldots.. + x^{30}) = 2925$]

17. Find a formula to express $1^3 + 2^3 + \ldots. + n^3$ as a function of n.

□□□