6

Combinatorial Mathematics

OBJECTIVES

- Introduction
- Basic Counting Principles
- Permutation
- Combinations
- Binomial Theorem

6.1 Introduction

We count objects to solve many kinds of problems. In fact counting problems arise throughout in the field of mathematics as well as computer science. For example, we count – all possible outcomes and successful outcomes of an experiment to determine probabilities, the number of operations used by an algorithm to study it's time complexity, Internet protocol addresses to meet the demand etc. Hence in this chapter we will study counting techniques which are useful in the study of various concepts in mathematics and computer science, particularly permutations and combinations. More generally, counting techniques for ordered arrangements and unordered selections of distinct objects of a finite set will be introduced.

6.2 Basic Counting Principles

There are two basic principles of counting.

The Sum Rule : If a task can be done in m–ways and another task in n–ways then there are m + n ways of performing exactly one of these tasks.

The Product Rule : If a task can be done in m ways and another task in n ways then there are m × n ways in which both of these tasks can be performed.

These rules can be extended to any number of tasks.

Example 1. There are 18 mathematics majors and 325 computer science majors at a college.

(a) How many ways are there to pick two representatives, so that one is a mathematics major and other is a computer science major ?

(b) How many ways are there to pick one representative who is either a mathematics major or a computer science major ?

Solution. (a) Using product rule this can be done in $18 \times 325 = 5850$ ways.

(b) Using sum rule this can be done in $18 + 325 = 343$ ways.

Example 2. A student can choose a project from one of three lists, which contain 29, 12 and 5 possible projects, respectively. How many possible projects are there from which student can choose ?

Solution. By sum rule a student can choose from $29 + 12 + 5 = 46$ projects.

Example 3. A computer program consists of one letter followed by three digits. If repetition is allowed then in how many ways different label identifiers are possible ?

Solution. There are 26 English alphabet and 10 digits from 0 to 9. Hence total number of ways in which different label identifiers are possible is $26 \times 10 \times 10 \times 10 = 26{,}000$.

Example 4. A cricket stadium has five gates on the eastern boundary and three gates on the western boundary.

(a) In how many ways can a person enter through an east gate and leave by west gate ?

(b) In how many different ways in all can a person enter and get out through different gates ?

Solution. (a) As there are five ways of entering (east side) and three ways of leaving (west side), therefore the required number of ways $= 5 \times 3 = 15$.

(b) The total number of gates is 8. Hence a person can enter from any of the 8 gates and may leave from any of the seven gates (leaving the gate from which he entered).

$\therefore$ The required number of ways $= 8 \times 7 = 56$.

Example 5. What is the value of 'a' after the following code has been executed ?

a : = 0

for j_1 : = 1 to m_1

for j_2 : = 1 to m_2

for jn : = 1 to mn

a : = a + 1

Solution. Let T_i be the task of traversing ith loop; then the number of ways to do task T_i, i = 1, 2, ... n is m_i, since the i^{th} loop is traversed once for each integer ji with $1 \le j_i \le m_i$. Hence by product rule, it is clear that nested loop is traversed $m_1 m_2$ ..... m_n times. Hence, the final value of a is $m_1 m_2$ ..... m_n.

6.3 Permutation

A permutation of a set of distinct objects is an ordered arrangement of these objects. The total number of permutations of n–objects taken r at a time is known r–permutation and is denoted as ${}^nP_r$, $1 \le r \le n$.

Theorem 1 : If $1 \le r \le n$ then ${}^nP_r = \frac{n!}{(n-r)!}$

Proof. The total number of permutations of n–different objects taken 'r' (without repetition) at a time is equal to the number of ways in which 'r' places can be filled with n–different objects. Hence by product rule we have

$${}^nP_r = n(n-1)(n-2) \text{ ..... r factors}$$

$$= n(n-1)(n-2) \text{ ..... } (n-r+1)$$

$$= \frac{[n(n-1)\& (n-r+1)][(n-r)(n-r-1)\& 3\cdot 2\cdot 1]}{(n-r)(n-r-1)\& 3\cdot 2\cdot 1}$$

$$= \frac{n!}{(n-r)!} \qquad [\text{where } n! = n(n-1)! \text{ and } 0! = 1]$$

This can also be considered as the problem of filling r distinct boxes with n–distinct objects.

Example 6. A number consisting of 4 digits is to be formed taking digits out of digits 1 – 9. How many such numbers are possible ?

Solution. Here n = 9; r = 4

$\therefore$ Required solution = ${}^9P_4 = \frac{9!}{5!} = 9\cdot 8\cdot 7\cdot 6 = 3024$

Example 7. Let a license plate consists of two letters followed by four digits. If the first letter on the plate is either a R or a P and the first digit is '7', the second letter is either a J or a I and the last digit is either 3 or 8 then find the different number of such license plates.

Solution. Given that the first letter can be R or P and the second letter can be J or I. So the number of permutations of two letters = 2 × 2 = 4.

Among the digits first digit is definitely 7; second and third can be chosen from the set (0, 1, 2, ...9) i.e., each place can choose digits in 10 ways and the fourth digit is either 3 or 8. Hence the number of permutations of the four digits are = 1 × 10 × 10 × 2 = 200. Hence by product rule the required number of permutations = 4 × 200 = 800.

Example 8. (a) Prove ${}^nPn_{-1} = {}^nP_n$.

(b) If ${}^{56}P_r + 6 : {}^{54}P_r + 3 = 30,800 : 1$, find r. Property

Solution. (a)

$${}^nP_{n-1} = \frac{n!}{\{n-(n-1)\}!} = n!$$

$${}^nP_n = \frac{n!}{(n-1)!} = n!$$

Hence ${}^nP_{n-1} = {}^nP_n$.

(b) ${}^{56}P_{r+6} = \frac{56!}{(50-r)!}$, ${}^{54}P_{r+3} = \frac{54!}{(51-r)!}$

$$\text{Given } \frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = 30800$$

$$\Rightarrow 56 \times 55 \times (51 - r) = 30800$$

$$\Rightarrow 51 - r = 10 \Rightarrow r = 41.$$

Example 9. A sales woman has to visit eight different cities. She begins her trip from a specified city, but can visit the other seven cities in any order she wishes. How many possible orders can she use when visiting these cities ?

Solution. The number of possible paths

$= {}^7P_7 = 7! = 5040.$

6.3.1 Permutations with Repetition

Theorem 2. The number of permutations of n–objects, taken 'r' at a time, allowing repetition is nr (each such permutation is called a sequence of length 'r').

Proof. This can be considered equivalent to the problem of filling r distinct boxes with 'n' objects with repetition allowed. Hence there are n–ways to select an object for box I, again, as repetition is allowed there are n–ways to select an object for box II and so on. Hence there are n–ways to select an object for each of the r–boxes. Hence by product rule the number of such permutations

$= n \times n \;.....n(r \text{ times})$

$= n^r$

Hence proved.

Example 10. A four digit number is formed with digits 1 – 9. How many such numbers can be formed if repetition is allowed ?

Solution. Here $n = 9$, $r = 4$.

$\therefore$ Required numbers $= 9^4 = 6561$.

Example 11. How many strings of six letters are there ?

Solution. There are 26 letters, therefore the required solution is 26^6.

Example 12. How many permutations of the letters in the word COMPANY are possible,

(a) using all letters, without repetition

(b) using four letters, without repetition,

(c) using four letters with repetition ?

Solution. (a) $^7P^7 = 7! = 5040$

(b) $^7P_4 = \dfrac{7!}{(7-4)!} = 7\times6\times5\times4 = 840$

(c) $7^4 = 2401$.

6.3.2 Permutations of Objects not all Distinct

Theorem 3. The number of different permutations of n–objects out of which n_1 objects are of type 1 n_2 objects are of type 2 ...... and n_k of type k, is

$$\frac{n!}{n_1!n_2!\ldots n_k!} \text{ where } n_1 + n_2 + \ldots + n_k = n.$$

Proof. Let the number of required permutations be T. Now consider one permutation 'p_1' from the T permutations in which first all n_1 objects of type I are placed followed by all n_2 objects of type II followed by all objects of type III and so on last being nk objects of type k, where

$n_1 + n_2 + \ldots + n_k = n$

Number of permutations of n_1 objects taken all at a time $= {}^{n_1}P_{n_1} = n_1!$

Number of permutations of n_2 objects taken all at a time $= {}^{n_2}P_{n_2} = n_2!$

. . . .

. . . .

Number of permutations of n_k objects taken all at a time $= {}^{n_k}P_{n_k} = n_k!$

Therefore, by product rule number of permutations $P_1 = n_1!n_2!\ldots\ldots n_k!$

But p_1 is just one kind of permutation of n element. Therefore, total number of permutations

$= T \times$ Number of permutations p_1

$= T \times n_1!\ n_2!\ \ldots\ n_k!$ .....(1)

But total number of permutations of n distinct objects $= {}^nP_n = n!$

$\therefore$ (1) $\Rightarrow n! = T \times n_1! \times n_2! \ldots n_k!$

$\Rightarrow$ Required number of permutations $= T = \dfrac{n!}{n_1!n_2!\cdots n_k!}$, $n_1 + n_2 + \ldots + n_k = n$. Hence proved.

This can also be considered equivalent to distributing n–distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, i = 1, 2, ... k.

Example 13. Find the number of distinct permutations of the letters of the word ENGINEERING.

Solution. The given word consists of 11 letters out of which there are 3E's, 3N's, 2G's, 2I's and 1R.

$\therefore$ Required permutations $= \dfrac{11!}{3!3!2!2!1!} = 2,77,200.$

Example 14. How many permutations can be made with letters of the word CONSTITUTION ? Also find the number of permutations in which :

(a) two O's come together

(b) vowels occur together

(c) consonants and vowels occur alternatively

(d) two O's do not come together.

(e) letter N occurs both at the begining and at the end.

Solution. This word consists of 12 letters out of which there are 2O's, 2N's, 3T's, 2I's, 1C, 1S and 1U.

$\therefore$ All possible permutations $= \dfrac{12!}{2!2!3!2!} = 99{,}79{,}200.$

(a) Consider 2O's as one letter. Hence now we have 11 letters and there are 2N's, 3T's and 2I's.

$\therefore$ Required permutations $= \dfrac{11!}{2!3!2!} = 16{,}63{,}200.$

(b) There are seven consonants – C N N S T T T and five vowels O O I I U. Consider all five vowels as one letter. Hence we have 8 letters out of which there are 2N's and 3T's. So the number of permutations

$= \dfrac{8!}{2!3!} = 3360.$

Also the five vowels consist of 2O's, 2I's and 1U. Hence they can be permuted in $\dfrac{5!}{2!2!1!} = 30$ ways. Hence the required number of total permutations $= 3360 \times 30 = 1{,}00{,}800.$

(c) First we fix five vowels at alternate positions i.e., 2nd, 4th, 6th, 8th and 10th positions. This can be done in $\dfrac{5!}{2!2!} = 30$ ways. (5 vowels which have 2O's and 2I's). Again seven consonants can be put at the remaining seven places in $\dfrac{7!}{2!3!}$ different ways.

Hence required number of permutations $= \dfrac{5!}{2!2!} \times \dfrac{7!}{2!3!} = 12{,}600$.

(d) Required number of permutations = number of total permutations – number of permutations in which 2O's come together = 9979200 – 1663200 = 83,16,000.

(e) If we fix the letter N at the beginning and at the end. Then we have 10 letters which have 2O's, 3T's and 2I's.

$\therefore$ Required number of permutations $= \dfrac{10!}{2!2!3!} = 151200.$

Example 15. How many different arrangements of the letters in the word 'BOUGHT' can be formed if the vowels must be kept together. [Raj. 2000]

$\boxed{OU}$ 5

Solution. 'BOUGHT' consists of 6 letters out of which there are 2 vowels and 4 consonants. If the 2 vowels are kept together, then there remain five letters which can be arranged in 5! ways.

Also the two vowels can be arranged in 2! ways.

$5! \times 2! = 240$

$\therefore$ The required number of different arrangements $= 5! \times 2! = 240.$

Example 16. How many bytes contain :

(i) exactly two 1's ?

(ii) exactly four 1's ?

(iii) atleast six 1's ? [Raj. 2000]

Solution. A byte consists of 8 bits and a bit is either 0 or 1.

(i) If a byte has exactly two 1's and it contains exactly six 0's then number of such bytes

(ii) If it contains exactly four 1's then it has exactly four 0's.

then number of such bytes .

(iii) The number of bytes containing atleast six 1's = bytes containing six 1's + bytes containing seven 1's + bytes containing eight 1's = $\frac{8!}{6!2!}+\frac{8!}{7!1!}+\frac{8!}{8!0!}$

$= 28 + 8 + 1 = 37.$

Example 17. In how many ways can letters a, b, c, d, e, f be arranged so that the letter b is always to the immediate left of letter e ? [Raj. 1999]

Solution. Among six letters 'be' always forms a pair.

∴ We have to arrange five letters, which can be done in 5! = 120 ways.

6.3.3 Circular Permutations

The circular permutations are the permutations of the objects placed in a circle. As the objects are arranged in a circle, hence there is no starting and ending point. Here only the relative positions are important.

Theorem 4. The number of circular permutations of n–different objects in (n – 1)!.

Proof. Let us fix one object. The problem is now to arrange remaining (n – 1) objects which can be done in (n – 1)! ways.

∴ Number of circular permutations = (n – 1)!. **Hence proved.**

In the circular sequencing of, beads in a necklace, flowers in a garland etc. the clockwise and anticlockwise arrangements are not distinct. Hence here the number of circular permutations of 'n' different objects $= \frac{1}{2}(n-1)!$.

Example 18. In how many ways can 5 programmers and 3 software engineers sit around a table so that no two software engineers are together.

Solution. The programmers are arranged round the circular table in (5 – 1)! = 4! ways. As no two software engineers should sit together, hence they are made to sit in between the programmes. Now three places between 5 programmers can be filled in 3! ways.

∴ Required number of permutations = 4! × 3! = 144 Ans.

6.4 Combinations

A combination is a collection of objects where order does not matter. If there are n–objects then the number of objects taken r at a time is denoted by ^nC_r.

The difference between permutation and combination can be seen by taking an example of choosing any two digits from set {1, 2, 3, 4}.

	Permutation (^nP_r)	(^nC_r) Combination
	12, 21	12
	13, 31	13
	14, 41	14
	23, 32	23
	24, 42,	24
	34, 43	34
Total	$^4P_2 = 12$	$^4C_2 = 6$

Theorem 5. If $0 \le r \le n$ then, the number of r combinations of a set with n elements is given by $^nC_r = \dfrac{n!}{r!(n-r)!}$.

Proof. The 'r' permutations of the set with n–elements is given by P(n, r) where ordering of elements is considered. The ordering can be done in P(r, r) = r! ways. As the order does not matter in combination, hence, P(n, r) = C(n, r) × P(r, r)

$$\Rightarrow C(n,r) = \frac{P(n,r)}{P(n,r)} = \frac{n!}{(n-r)!r!}.$$

Hence proved.

In general when order matters, we count the number of sequences or permutations; when order does not matter, we count the number of subsets or combinations.

6.4.1 Some Important Results

(1) $^nC_0 = {^nC_n} = 1$

(2) $^nC_{n-1} = n$

(3) $^nC_r = \dfrac{n}{r} \times {^{n-1}C_{r-1}}$

(4) $^nP_r = r! \times {^nC_r}$ i.e, $\dfrac{^nP_r}{^nC_r} = r!$

(5) $^nC_1 = n$; $^nC_2 = \dfrac{n(n-1)}{2!}$; $^nC_3 = \dfrac{n(n-1)(n-2)}{3!}$ and so on.

(6) $^nC_r = {^nC_{n-r}}$

Example 19. If n and k be positive integers with $n \ge k$. Then prove that

C(n + 1, k) = C (n, k – 1) + C(n, k) (Pascal's identity). [MREC 2000, Raj. 2002, 2003]

Solution. $C(n, k-1) = \dfrac{n!}{[n-k-1!]!(k-1)!} = \dfrac{n!}{[n-k+1!]!(k-1)!}$

$$C(n, k) = \frac{n!}{(n-k)!k!}$$

$$\therefore \text{R.H.S.} = C(n, k-1) + C(n, k) = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!}$$

$$= \frac{n!}{(n-k+1)(n-k)!(k-1)!} + \frac{n!}{(n-k)!k(k-1)!}$$

$$= \frac{n!}{(n-k)!(k-1)!}\left[\frac{1}{n-k+1}+\frac{1}{k}\right] = \frac{n!}{(n-k)!(k-1)!}\cdot\frac{n+1}{k(n-k+1)}$$

$$= \frac{(n+1!)}{(n-k+1)!k!} = \frac{(n+1)!}{[(n+1)-k]!k!} = C(n+1,k)$$

$=$ L.H.S.

Hence $C(n + 1, k) = C(n, k - 1) + C(n, k)$ **Proved.**

Example 20. How many diagonals has a regular polygon with n–sides? Which of them has the same number of diagonals as sides? [Raj. 2002, 2005]

Solution. The regular polygon with n–sides has n–vertices. Any two vertices determine either a side or a diagonal. Now, from n–vertices, two vertices can be chosen in nC2 ways.

$\therefore$ ${}^nC_2$ = sides + diagonals

or $\frac{n(n-1)}{2}$ = sides + diagonals

$\therefore$ Diagonals $= \frac{n(n-1)}{2} - n$ (as there are n–sides)

$= \frac{n(n-3)}{2}$ **Ans.**

Further, when diagonals = sides

$\Rightarrow \frac{n(n-3)}{2} = n \Rightarrow n = 0,5$

as $n \neq 0 \Rightarrow n = 5$.

$\therefore$ Pentagon is the only regular polygon having same number of diagonals as the sides.

Example 21. Suppose that a valid computer password consists of four characters, the first of which is a letter, chosen from the set {A, B, C, D, E, F} and the remaining three characters are letters chosen from the English alphabet or digits chosen from the set T = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, how many different passwords are there? [Raj. 2001]

Solution. An English alphabet can be chosen in ${}^{26}C_1 = 26$ ways. A digit from the given set can be chosen in ${}^{10}C_1 = 10$ ways. A letter from the given set can be chosen in ${}^6C_1 = 6$ ways.

Hence by sum rule, a character may be an English alphabet or a digit chosen from given set in $(26 + 10) = 36$ ways

$\therefore$ By product rule, the number of different passwords

$= 6 \times 36 \times 36 \times 36 = 2,79,936$.

Example 22. How many different seven person committee can be formed containing three female members from an available set of 20 females and four males from an available set of 30 male members.

Solution. Three females can be chosen from an available set of 20 in ${}^{20}C_3$ ways. Four male members can be chosen from an available set of 30 males in ${}^{30}C_4$ ways.

$\therefore$ Seven person committee can be formed in required manner in $= {}^{20}C_3 \times {}^{30}C_4$ ways

$=$ 31,241,700 ways.

Example 23. How many ways are there to pick a five person basket ball team from 12 possible players ? How many selections include the weakest and the strongest players? **[MREC 2000]**

Solution. Required number of ways $= {}^{12}C_5 = 792$.

Among 12 possible players 1 is weakest and 1 is strongest. If these two are always selected then we just have to select 3 persons out of 10.

$\therefore$ Required number of ways for it = ${}^{10}C_3 = 120$.

Example 24. Five cards are drawn from a standard deck of 52 cards. How many ways are there when this selection results in (a) no hearts? (b) atleast one heart ?

Solution. (a) In this case all 5 cards are selected from 39 cards (52 – 13), which can be done in ${}^{39}C_5$ = 5,75,757 ways.

(b) Required number of ways = number of ways in which five card can be chosen from 52 cards – number of ways in which no heart is chosen

= ${}^{52}C_5 - {}^{39}C_5$ = 25, 98, 960 – 5,75,757

= 20,23,203.

6.4.2 The Binomial Theorem

Theorem 6. Let x and y be variables and let n be a positive integer. Then

$$(x+y)^n = \sum_{i=0}^{n} C(n,i)x^{n-i}y^i = {}^nC_0x^n + {}^nC_1x^{n-1}y + \cdots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

Proof. The terms in the sum are of the form $x^{n-i}y^i$, which are obtained by choosing all (n – i) x's from the sum, so that other i–terms are automatically y's. Hence coefficient of

x^{n-i} = number of such terms = C(n, n – i) = C(n, i) $[\because {}^nC_r = {}^nC_{n-r}]$

Example 25. Let n be a positive integer. Then $\sum_{k=0}^{n} C(n,k) = 2^n$.

Solution. Binomial theorem gives $\sum_{k=0}^{n} C(n,k)x^{n-k}y^k = (x+y)^n$

Let x = y = 1

$$\Rightarrow \sum_{k=0}^{n} C(n,k) = 2^n.$$

Hence proved.

Example 26. Let m, n and r be non–negative integers with $r \le m, n$.

Then $C(m+n, r) = \sum_{k=0}^{r} C(m,r-k)C(n,k)$ [Raj. 2004]

Proof. This problem can be treated as there are m elements in one set and n elements in the second set. Then the total number of ways to select r elements out of total elements (m + n) is C(m + n, r). Another way to select r elements from (m + n) elements is to select k elements from the first set and the remaining (r – k) elements from the second set, where $0 \le k \le r$, $k \in Z$. Then by product rule, this can be done in C(m, k) C(n, r – k) ways. Hence, the total number of ways to select r elements from the total (m + n) elements is given as

$$C(m+n, r) = \sum_{k=0}^{r} C(m,k)C(n,r-k)$$

or, $C(m + n, r) = \sum_{k=0}^{r} C(m, r-k)C(n,k)$ (on interchanging m and n as it does not affect the equality).

Hence proved.

Example 27. Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k C(n,k) = 0$ **[Raj. 2003]**

Solution. Binomial theorem gives

$$\sum_{k=0}^{n} C(n,k)x^{n-k}y^k = (x + y)^n$$

Let x = 1, y = – 1

$$\Rightarrow \sum_{k=0}^{n} (-1)^k C(n,k) = (1 - 1)^n = 0$$

Proved.

6.4.3 Combinations with Repetition

Theorem 7. The number of combinations from a set with n–elements taken r at a time when repetition is allowed is $C(n + r - 1, r) = \frac{(n+r-1)!}{r!(n-1)!}$. $(^{n+r-1}C_r)$ r = similar things

Proof. The problem of r–combination of a set with n–elements when repetition is allowed can be represented by considering a box with n–compartments, hence (n – 1) partitions, each can hold maximum of r–objects. The number of ways to select r–objects corresponds to the number of ways of arranging (n – 1) partitions and r–objects which is equal to the number of ways to select the positions of r–objects from [(n – 1) + r] positions. This is equal to unordered selections of r–objects from a set of n + r – 1 objects.

Consequently required combinations = C(n + r – 1, r). **Hence proved.**

Example 28. A cookie shop has four different kinds of cookies. In how many different ways can six cookies be chosen? Consider that only the type of cookie, and not the individual cookie or the order in which they are chosen is important.

Solution. The required number of ways is equal to the number of 6–combinations of a set with 4–elements (i.e. repetitions allowed) = C (4 + 6 – 1, 6) = C(9, 6) = 84.

Example 29. How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 8$ have, where x_1, x_2, x_3 and x_4 are non–negative integers ?

Solution. This problem can be viewed as the number of ways of selecting objects from a set with four elements, so that x_1 objects are of type one, x_2 objects of type two, x_3 objects of type three and x_4 objects of type four are chosen.

∴ It is a problem of finding number of 8–combinations from a set with four elements (repetition allowed).

∴ Required number of such combinations

= number of solutions

= C(4 + 8 – 1, 8) = C(11, 8) = 165.

Example 30. How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are non–negative integers and $x_1 \geq 1$, $x_2 \geq 2$ and $x_3 \geq 3$.

Solution. The problem corresponds to selection of 11 items, in which there is atleast one item of type one, two items of type two and three items of type three, from a set with three elements, with x_1 items of type one, x_2 items of type two and x_3 items of type three.

∴ Choose one item of type one, two items of type two, three items of type three. Remaining five

(11 −(3 + 2 +1)) are to be chosen from the set of three elements (repetition allowed). Hence required number of ways = number of solutions = C(3 + 5 − 1), 5) = C(7, 5) = 21.

Example 31. In how many ways can a prize winner choose three CD's from top ten list if repeats are allowed ?

Solution. Required number of ways = C(10 + 3 − 1, 3) = C(12, 3) = 220.

6.4.4 Combinations of Objects not all Different

Theorem 8. The total number of combinations which can be made of n-different objects taken some or all at a time is ${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$

Proof. The total number of combinations of n-different objects

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n \quad .....(1)$$

Since $(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + ... + {}^nC_n\, x^0 y^n$

taking x = y = 1, we get

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n \quad .....(2)$$

Since, we must select atleast one object at a time so the required number of combinations $= {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$\therefore$ (2) $\Rightarrow {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$ $\quad (\because {}^nC_0 = 1)$ **Hence proved.**

Example 32. Determine all possible ways in which a student may attempt one or more questions from 6 given questions having an alternative.

Solution. There are three ways in which each of the 6 questions may be attempted – the question itself, it's alternative, or none of them.

$\therefore$ 6 questions may be attempted in 3^6 ways.

But as this also includes the case when no question (i.e. all six questions are not done) is attempted. Hence required number of ways = $3^6 - 1 = 729 - 1 = 728$.

Example 33. In order to pass B.E. III semester examination minimum marks have to be secured in each of the six subjects. In how many cases can a student fail ?

Solution. There are two possibilities a student fails or passes in each of the six subjects. Thus, possibility of fail or pass in all the six subjects can be considered in 2^6 ways.

But, this number includes the case in which student passes in all six subjects.

$\therefore$ Required number of ways = $2^6 - 1 = 63$.

Example 34. In how many ways can a teacher choose one or more students from six eligible students.

Solution. There are $2^6 = 64$ subsets of the set consisting of the 6 students. Also as the empty set must be deleted since one or more student is to be chosen, therefore required number of ways

$$= 2^6 - 1 = 64 - 1 = 63.$$

Example 35. In how many ways three or more persons be selected from 12 persons.

Solution. Number of ways of choosing one or more of 12 persons = $2^{12} - 1 = 4095$. Number of ways of choosing one or two of the 12 persons = ${}^{12}C_1 + {}^{12}C_2 = 12 + 66 = 78$. Hence number of ways of choosing three or more persons out of 12 = 4095 − 78 = 4017.

ILLUSTRATIVE EXAMPLES

Example 36. A computer science professor has seven different programming books on a bookshelf. Three of the books deal with FORTRAN, the other four are concerned with C. In how many ways can the professor arrange these books on the shelf :–

(i) if there are no restrictions.

(ii) if the language should alternate.

(iii) if all the FORTRAN books must be next to each other.

(iv) if all FORTRAN books must be next to each other and all C books must be next to each other.

[MREC 2000]

Solution. (i) Number of required ways = 7! = 5040.

(ii) The number of ways to arrange seven different books when languages are alternate is C F C F C F C = 4! × 3! = 24 × 6 = 144.

(iii) Here we have 5 books in total, which can be arranged in 5! ways. Further the three books of FORTRAN can be arranged in 3! ways

∴ Required number of ways = 5! × 3! = 720.

(iv) In this case we have only two books which can be arranged in 2! ways. Also the three FORTRAN books can be arranged in 3! ways and four C books can be arranged in 4! ways

∴ Required number of ways = 3! × 4! × 2 = 288.

Example 37. In how many ways can the letters in MISSISSIPPI be arranged so that (i) two P's are to be next to each other, (ii) two P's are seperated. **[Raj. 1999]**

Solution. Total number of ways in which letters of the given words can be arranged = $\frac{11!}{4!4!2!}$ = 34650.

(i) Total arrangements in which two P's always come together.

(ii) Required number of ways, when two P's don't come together = 34650 – 6300 = 28350.

Example 38. Five boys and five girls are to be seated in a row. In how many ways can they be seated if

(i) All boys must be seated in five left most seats.

(ii) John (one boy out of five) and Mary (one girl out of five) must be seated together. **[Raj. 1999]**

Solution. (i) They should be seated in the following pattern BBBBBGGGGG. The five boys can be seated in 5! ways within themselves and the five girls can be seated in 5! ways within themselves.

∴ Required number of ways = 5! × 5! = 14400.

(ii) Here we have to arrange 4 Boys, 4 Girls and 1 Pair (John and Mary). But John and Mary may sit together in 2! ways.

∴ Required number of ways = 9! × 2! = 725760.

Example 39. Show that for all integers n, r > 0 if n + 1 > r, then

$$P(n+1, r) = \frac{n+1}{(n+1-r)}P(n,r)$$

Solution. $P(n+1, r) = \frac{(n+1)!}{(n+1-r)!}$

$$= \frac{(n+1)n!}{(n+1-r)(n-r)!} = \frac{n+1}{(n+1-r)}\ {}^{n}P_r$$ Hence proved.

Example 40. If n is a positive integer and n > 1, prove that ${}^{n}C_2 + {}^{n-1}C_2$ is a perfect square.

Solution. Since ${}^{n}C_2 + {}^{n-1}C_2 = \frac{n!}{2!(n-2)!} + \frac{(n-1)!}{2!(n-3)!}$

$$= \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2}$$

$$= \frac{2n^2-4n+2}{2} = n^2-2n+1 = (n-1)^2$$ **Hence proved.**

Example 41. A student is to answer seven out of 10 questions in an examination. In how many ways can he make his selection if :

(a) there are no restriction

(b) he must answer the first two questions

(c) he must answer atleast four of the first six questions.

Solution. (a) Required number of ways = ${}^{10}C_7 = {}^{10}C_3 = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 120$.

(b) As it is compulsory to answer first two questions, hence now five questions are to be answered out of eight so required number of ways = ${}^{8}C_5 = 56$.

(c) Required number of ways = choose four from first six and three from last four

or choose five from first six and two from last four

or choose six from first six and one from last four

$= {}^{6}C_4 \times {}^{4}C_3 + {}^{6}C_5 \times {}^{4}C_2 + {}^{6}C_6 \times {}^{4}C_1 = 60 + 36 + 4 = 100.$

Example 42. Find the number of subsets of a set 'X' containing n elements.

Solution. The number of subsets of a set 'X' with r < n elements is given by ${}^{n}C_r$

$\therefore$ Altogether we have ${}^{n}C_0 + {}^{n}C_1 + ... + {}^{n}C_n$ subsets of X. .....(1)

Also by Binomial theorem : $(x+y)^n = \sum_{k=0}^{n} {}^{n}C_k x^{n-l} y^k$

Take x = 1 = y

$$\Rightarrow \quad (1+1)^n = \sum_{k=0}^{n} {}^{n}C_k = {}^{n}C_0 + {}^{n}C_1 + \& \, {}^{n}C_n$$

$$\Rightarrow \quad {}^{n}C_0 + {}^{n}C_1 + \ldots + {}^{n}C_N = 2^n$$

$\therefore$ By (1), we have 2^n subsets of X.

Example 43. Prove ${}^{2n}C_n + {}^{2n}C_{n-1} = \frac{1}{2}\left({}^{2n+2}C_{n+1}\right)$

Solution. ${}^{2n}C_n + {}^{2n}C_{n-1} = \frac{(2n)!}{n!n!} + \frac{(2n)!}{(n-1)!(n+1)!}$

$$= \frac{(2n)!}{n(n-1)!n!} + \frac{(2n)!}{(n-1)!(n+1)n!}$$

$$= \frac{(2n)!}{(n-1)!n!}\left(\frac{1}{n}+\frac{1}{n+1}\right) = \frac{(2n)!}{(n-1)!n!}\frac{(2n+1)}{n(n+1)}$$

$$= \frac{(2n+2)!}{2(n-1)n!(n+1)!} = \frac{1}{2}\frac{(2n+2)!}{(n+1)!(n+1)!}$$

$$= \frac{1}{2}\,{}^{2n+2}C_{n+1}$$ **Hence proved.**

Example 44. Find the number 'n' of ways that a judge can award first, second and third places in a contest with eighteen contestants.

Solution. Required number of ways = n = ${}^{18}P_3$ = 4896

Example 45. On the annual function of the college there are 10 prizes to be distributed one for each of the ten activities. In how many ways the prizes can be distributed among the four classes I year, II year, III year and IV year. If

(a) one or more prizes can be distributed to any one class,

(b) each of the four classes get at least one prize,

(c) each of the four classes get atleast one prize and the IV year class gets atleast five prizes.

Solution. (a) Required number of ways = C(4 + 10 –1, 10) = ${}^{13}C_{10}$ = 286.

(b) Here each class gets atleast one prize. Hence now six prizes are to be distributed among four classes. Required number of ways = C(4 + 6 – 1, 6) = ${}^{9}C_6$ = 84.

(c) Here each class gets atleast one prize and the IV year class gets atleast five prizes. Therefore now eight prizes are already fixed and two prizes are to be distributed among four classes.

∴ Required number of ways = C(4 + 2 – 1, 2) = ${}^{5}C_2$ = 10.

Example 46. How many different signals, each consisting of 10 flags hung in a vertical line, can be formed from a set of 5 blue, 3 green and 2 white, indistinguishable flags.

Solution. Required number of ways $\frac{10!}{5!3!2!} = 2520.$

Example 47. Sixteen books are placed one over the other on 4 tables. 7 books are of Mathematics on first table, 4 are of Computer on second table, 3 are of English on third table and remaining 2 are of Hindi on fourth table. The Librarian wants to put numbers on each book taking one by one (taking uppermost book on each table). In how many ways can he lift the books?

Solution. The required number of ways $\frac{16!}{7!4!3!2!} = 144,14,400$.

Example 48. On their home from college 8 students stop at a restaurant where each of them has one of the following; a cup of tea; a cup of coffee and a cold drink. How many different purchases are possible?

Solution. Here three things are to distributed among eight people with repetition allowed.

∴ Number of different purchases = C(3 + 8 – 1, 8) = ${}^{10}C_8$ = 45.

Example 49. Find the regular polygon which has

(a) twice as many diagonals as sides.

(b) three times as many diagonals as sides.

Solution. Let the regular polygon has n sides then number of diagonals = ${}^{n}C_{2} - n = \frac{n(n-3)}{2}$.

(a) Given $2n = \frac{n(n-3)}{2} \Rightarrow 4n = n^2 - 3n$

$\Rightarrow (n^2 - 7n) = 0 \Rightarrow n = 0, 7$

$n \neq 0 \Rightarrow n = 7.$

(b) Given $3n = \frac{n(n-3)}{2} \Rightarrow n^2 - 9n = 0 \Rightarrow n = 9.$

Example 50. How many ways can you choose three of seven fiction books and two of six non fiction books to take with you on your vacation ?

Solution. Required number of ways $= {}^7C_3 \times {}^6C_2$

$$= \frac{7\cdot 6\cdot 5}{3\cdot 2} \times \frac{6\cdot 5}{2} = 35 \times 15$$

$$= 525.$$

Example 51. Into how many parts are the diagonals of a convex octagon decomposed, given that no three of these diagonals are concurrent except at a vertex.

Solution. For a convex n–gon, number of diagonals $= \frac{n(n-3)}{2}$.

Also four vertices will result in exactly one intersection between two diagonals.

$\therefore$ Number of such intersections $= {}^nC_4$.

Also as the intersection point divides each diagonal into 2 line–segments, hence the number of required line segments $= 2\,{}^nC_4 + \frac{n(n-3)}{2}$.

For an octagon $n = 8$

$\therefore$ Number of required line segments

$$= 2 \times {}^8C_4 + \frac{8(8-3)}{2} = 140 + 20 = 160.$$

Example 52. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where x_1, x_2, x_3 and x_4 are non–negative integers?

Solution. The problem consists of selection of 17 items where x_1 items corresponds to type I, x_2 items corresponds to type II, x_3 items corresponds to type III and x_4 items corresponds to type IV.

$\therefore$ Required number of solutions $= C(4 + 17 - 1, 17) = {}^{20}C_{17} = {}^{20}C_3 = 1140.$

Example 53. A password consists of two alphabets of English followed by 2 digits. How many passwords exists when upper case and lower case letters are treated (i) equivalent (ii) different. [Raj 2003]

Solution. (i) There are 26 English alphabets and 10 digits.

$\therefore$ Required number of passwords $= 26 \times 26 \times 10 \times 10 = 67600.$

(ii) Now there are $26 + 26 = 52$ English alphabets and 10 digits.

$\therefore$ Required number of passwords $= 52 \times 52 \times 10 \times 10 = 270400.$

Example 54. Prove that $\sum_{k=0}^{r} C(n+k,k) = C(n + r + 1, r).$ [Raj. 2003]

Solution. For r = 0, the given relation is true. Let it be true for r = p i.e.,

$$\sum_{k=0}^{p} C(n+k,k) = C(n+p+1,\ p) \qquad \text{.....(1)}$$

Now, consider

$$\sum_{k=0}^{p+1} C(n+k,k) = \sum_{k=0}^{p} C(n+k,k) + C(n+p+1,\ p+1)$$

$$= C(n+p+1,\ p) + C(n+p+1,\ p+1) \quad \text{[using (1)]}$$

$$= \frac{(n+p+1)!}{(n+1)!p!} + \frac{(n+p+1)!}{n!(p+1)!}$$

$$= \frac{(n+p+1)!}{n!p!}\left[\frac{(n+p+2)}{(n+1)(p+1)}\right]$$

$$= \frac{(n+p+2)!}{(n+1)!(p+1)!} = C\left(n+\overline{p+1}+1, p+1\right)$$

It shows that the relation is true for r = p + 1. Thus by induction the result holds for all r i.e.,

$$\sum_{k=0}^{r} C(n+k,k) = C(n+r+1,\ r)$$

Proved.

Example 55. Prove that ${}^{n}P_r = {}^{n-1}P_r + r\ {}^{n-1}P_{r-1}$.

Solution. R.H.S. $= {}^{n-1}P_r + r\ {}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[1+\frac{r}{n-r}\right]$$

$$= \frac{(n-1)!}{(n-r-1)!}\left(\frac{n}{n-r}\right)$$

$$= \frac{n!}{(n-r)!}$$

$$= {}^{n}P_r$$

Proved.

Example 56. Prove that $C(2n, 2) = 2C(n, 2) + n^2$.

Solution. LHS $= C(2n, 2)$

$$= \frac{(2n)!}{(2n-2)!2!} = \frac{2n(2n-1)(2n-2)}{2(2n-2)!}$$

$$= n(2n-1)$$

$$= n(n-1) + n^2$$

$$= 2\left[\frac{1}{2}n(n-1)\right]+n^2$$

$$= 2C(n, 2) + n^2$$

= R.H.S.

Proved.

Example 57. How many bit strings of length 8 do not contain '6' consecutive 0's ?

Solution. Total bit strings of length 8 = 2^8.

Number of bit strings containing six consecutive 0's = 2 × 2 × 2 = 8.

∴ Required number of bit strings that do not contain 6 cosecutive 0's = $2^8 - 8 = 248$.

Example 58. Show that if n and k are positive integers, then $C(n+1, k) = \frac{(n+1)C(n,k-1)}{k}$.

Proof. L.H.S. $= C(n+1, k)$

$$= {}^{n+1}C_k = \frac{(n+1)!}{k!(n+1-k)!}$$

R.H.S. $= \frac{(n+1)}{k}C(n,k-1)$

$$= \frac{(n+1)}{k} \times \frac{n!}{(k-1)!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!}$$

∴ (1) and (2) ⇒ L.H.S. = R.H.S.

Hence proved.

Example 59. How many license plates consisting of three letters followed by three digits contain no letters or digits twice?

Solution. There are 26 letters and 10 digits.

∴ Required number of plates

= 26 × 25 × 24 × 10 × 9 × 8

= 11,232,000.

Example 60. How many strings are there of lowercase letters of lengths four or less ?

Solution. There are 26 lower case letters.

∴ Required number of strings

$= 26^4 + 26^3 + 26^2 + 26^1 + (26)^0$ (counting the empty string)

= 4,569,76 + 17,576 + 676 + 26 + 1

= 4,75,255.

Example 61. Determine the number of ways in which we can make up strings of four distinct letters followed by three distinct digits. [Raj. 2003]

Solution. There are 26 letters out of which 4 can be chosen in ${}^{26}C_4$ ways. Also there are 10 digits out of which three can be chosen in ${}^{10}C_3$ ways.

∴ Required number of ways = ${}^{26}C_4 \times 4! \times {}^{10}C_3 \times 3!$.

Example 62. How many different bit strings can be formed using six 1's and eight 0's.

Solution. Total length of bit string = 6 + 8 = 14.

Number of different strings = ${}^{14}C_8 = {}^{14}C_6 = 3003$.

Example 63. Five fair coins are tossed and the results are recorded:

(i) How many different sequences of heads and tails are possible?

(ii) How many of the sequences recorded have exactly one head?

(iii) How many of the sequences have exactly three heads recorded? [Raj. 2004]

Solution. Tossing of each coin results in two outcomes head or a tail.

(i) ∴ Requirednumber of sequences = 2 × 2 × 2 × 2 × 2 = 32.

(ii) Now there is one possible outcome for 1 coin i.e., Head and one possible outcome for all the other four coins i.e., Tail. But this one coin may be chosen from five coins in ${}^5C_1$ ways.

$\therefore$ Required number of sequences = ${}^5C_1 \times 1 = 5$.

(iii) Three coins showing Head can be chosen among five coins in ${}^5C_3$ ways.

$\therefore$ Required number of sequences = ${}^5C_3 = 10$.

Example 64. Determine the number of ways to place 2k + 1 indistinguishable balls in three distinct boxes so that any two boxes together will contain more balls than other one.

Solution. The total number of ways in which (2k + 1) objects can be placed in three distinct boxes

$$= C(3 + (2k + 1) - 1, 2k + 1)$$

$$= {}^{2k+3}C_{2k+1} = \frac{(2k+3)(2k+2)}{2}.$$

Now consider the case when number of balls in two boxes is less than the number of balls in third box. This means that objects or greater can be place in this third box and remaining (2k + 1) – (k + 1) = k or less will be placed in the two boxes.

Number of such ways = $C(3 + k - 1, k) = {}^{k+2}C_k = \dfrac{(k+2)(k+1)}{2}$.

But this third box can again be chosen in three ways, out of the three boxes. Hence total number of ways in which balls in two boxes is less than number of balls in third box

$$= 3\times C(3 + k - 1, k) = \frac{3}{2}(k+2)(k+1) \quad .....(2)$$

$\therefore$ (1) and (2) give the required number of ways

$$= \frac{(2k+3)(2k+2)}{2} - \frac{3}{2}(k+2)(k+1)$$

$$= \frac{(k+1)}{2}[2(2k+3)-3(k+2)] = \frac{k(k+1)}{2} \quad \textbf{Ans.}$$

Example 65. If out of p + q + r things p be alike, q be alike and rest are different, then find the total number of selections in which atleast one thing is selected.

Solution. The number of selections from p alike things = p + 1 as we may choose 0 or 1 or 2 or 3 or ..... or p things from the given p things.

Similarly, the number of selections from q alike things = q + 1, and the number of selection from r different things = 2^r

$\therefore$ The total number of selections = $(p + 1)(q + 1)\, 2^r - 1$.

(as the case, when all things are left must be excluded)

EXERCISE 6

1. How many seven digits telephone numbers are possible, if
   (a) only odd digits may be used.
   (b) the number must be a multiple of 100 ?
   (c) the first three digits are 481 ?
   [Ans. (a) 5^7 (b) 10^5 (c) 10^4]

2. A bookshelf is used to display six new books. Suppose that there are eight computer science books and six French books from which to choose. If we decide to show four computer science books and two French books and we are required to keep the books of each subject together, how many different displays are possible?

[Ans. 67,2000]

3. (a) Find the number of distinguishable permutations of the letters in ASSOCIATIVE.

(b) Find the number of distinguishable permutations of the letters in REQUIREMENTS.

[Ans. (a) 4,989,600, (b) 39,916,800]

4. In how many different ways 3 rings of a lock can be combined when each ring has 10 digits (0 to 9) ? If the lock opens with only one combination of three digits, how many unsuccessful events are possible ?

[Ans. 1000, 999]

5. How many different words containing all the letters of the word TRIANGLE can be formed so that :

(a) Consonants are never seperated.

(b) Consonants never come together.

(c) Vowels occupy odd places ?

[Ans. (a) 720 (b) 39,600 (c) 460]

6. Find the value of n if,

(i) $^{n-1}P_5 : {}^{n-1}P_n = 14 : 1$

(ii) $^{16}P_{x-1} : {}^{15}P_{x-2} = 30 : 11$

(iii) $^7P_x = 2 \cdot {}^7P_{x-2}$

[Ans. (i) 9, (ii) 7, (iii) 7, 10]

7. How many numbers greater than a million can be formed with the digits 1, 7, 10, 7, 3, 7 ?

[Ans. 360]

8. The CEO of 18 software companies meet to discuss the problems of software industry in India. In how many ways can they sit themselves around the table so that CEO of GE, HCL and Infosys, choose to sit together ?

[Ans. 15! × 3!]

9. (a) How many 6-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6 (without repetition) ? How many of these are divisible by 5 ?

(b) How many numbers greater than 7000 can be formed with the digits 3, 5, 7, 8, 9 (without repetition).

[Ans. (a) 720, 120 (b) 72]

10. A debating team consists of 3 boys and 2 girls. Find the number of ways they can sit in a row if :

(a) the boys and girls are each to sit together.

(b) just the girls are to sit together.

[Ans. (a) 24 (b) 48]

11. Ten students attended a counselling for taking admission in an Engg. college. Each of them has to choose one of the following stream : Mechanical, Electrical, Electronics and Computer. How many different choices are possible for all ten students ?

[Ans. 13C10]

12. If $a_k = \sum_{k=0}^{n} \frac{1}{^nC_k}$ then prove that $\sum_{k=0}^{n} \frac{1}{^nC_k} = \frac{1}{2} na_n$. (MREC 2008)

13. (a) A palindrome is a string whose reversal is identical to the string. How many bit strings of length 'n' are palindromes ?

[Ans. (a) $2^{n/2}$ if n is even, $2^{(n+1)/2}$ if n is odd]

(b) How many palindromes of length n can be formed from an alphabet of k letters.

[Ans. $k^{n/2}$ if n is even, $k^{(n-1)/2}$ if n is odd]

14. A person has three sons. He owns 101 shares of a company. He wants to give these to his sons so that no son should have more shares than the combined total of the other two. In how many ways can he do so?

[Ans. $^{103}C_2 - 3 \times {}^{52}C_2$)]

15. Prove that :

(i) $^nC_r = {}^nC_{n-r}$

(ii) $^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$.

(iii) $^{n+1}C_{r+1} = {}^nC_{r+1} + {}^{n-1}C_r + {}^{n-1}C_{r-1}$.

16. How many bit strings of length 10 have (a) exactly three 0's (b) atleast seven 1's (c) atleast three 1's ?

[Ans. (a) 120, (b) 176, (c) 968]

17. How many different 8–bit strings are there that begin and end with 1 ?

[Ans. 26 = 64]

18. How many bit strings of length 6 contains atleast four 1's ?

[Ans. 22]

19. Prove that $\sum_{k=1}^{n} kC(n,k)$.

20. How many bit strings contain exactly eight 0's and ten 1's. If every 0 must be immediately followed by a 1.

[Ans. 45]

21. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

where xi, i = 1, 2, 3, 4, 5 is a non–negative integer such that

(a) $x1 \geq 1$

(b) $xi \geq 2$ for i = 1, 2, 3, 4, 5

(c) $0 \leq x_1 \leq 10$

(d) $0 \leq x_1 \leq 3$, $1 \leq x_2 \leq 4$ and $x_3 \geq 15$.

[Ans. (a) 10, 626 (b) 1,365 (c) 11,649 (d) 106]

22. How many distinct permutations of the letters in the words

(i) BANANA (B) MISSISSIPPI are there ? [Raj. 2003]

[Ans. (i) 60 (ii) 34650]

23. At a certain college, the housing office has decided to appoint, for each floor, one male and one female residential advisor. How many different pairs of advisors can be selected for a seven–storey building from 12 male candidates and 15 female candidates ?

[Ans. 5,096,520]

24. In how many ways can we select objects from a collection of size 2n that consists of n distinct and n identical objects ?

[Ans. 2n]

25. Find the sum of all numbers greater than 10000 formed by using the digits 1, 3, 5, 7, 9, no digi... repeated in any number.

[Ans. 6666600]

26. In how many ways 10 programmers can sit on a round table to discuss the project so that the projec... and a particular programmer always sit together ?

[Ans. $8! \times 2! = 80640$]

27. How many 8–bit strings are there that end with 0111 ?

[Ans. 24]

28. Determine the number of triangles that are formed by selecting points from a set of 15 points out of ... 8 are collinear.

[Ans. ${}^{15}C_3 - {}^{8}C_3 = 854$]

29. If n fair coins are tossed and the results recorded, how many

(a) record sequences are possible ?

(b) sequences contain exactly three tails, assuming $n > 3$?

(c) sequences contain exactly k–heads, assuming $n > k$?

[Ans. (a) 2^n (b) ${}^{n}C_3$ (c) ${}^{n}C_k$]

30. An urn contain 15 balls, 8 of which are red and 7 are black. In how many ways can 5 balls be cho... that

(a) all 5 are red ?

(b) all 5 are black ?

(c) 2 are red and 3 are black ?

(d) 3 are red and 2 are black ?

[Ans. (a) 56 (b) 21 (c) 980 (d) 1176]

31. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign sc... to the problems if the sum of scores is 100 and each question is worth at least 5 points.

[Ans. 59C50]

32. How many ways are there to distribute five indistinguishable objects into three indistinguishable box...

[Ans. 5]

33. Find the number of triangles that can be formed by the vertices of an octagon. Also find the numb... triangles formed by the vertices of the octagon if its sides are not the sides of any triangle.

[Ans. 56, 16]

34. How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11$$

where x_1, x_2 and x_3 are non–negative integers ?

[Hint : Introduce an auxiliary variable x4 such that $x_1 + x_2 + x_3 + x_4 = 11$]

[Ans. 364]

□□