2

Relations

OBJECTIVES

- Introduction
- Relations
- Graph of a Relation
- Binary Relation
- Properties of Relations
- Equivalence Relation
- Equivalence Class
- Composition of Relations
- Partial Order Relation & Hasse Diagram
- Lattices
- Job-Scheduling Problem

2.1 Introduction

Relationships between elements of sets occur in many contexts. Relations can be used to solve problems involving communications networks, project scheduling, etc.

2.2 Ordered Pairs

While listing the objects of a set we do not assign their position and order. But several times we need a pair of two objects a and b in which the position of these objects in also very important.

Let a and b be any two objects, a is assigned as the first position and b is assigned as the second position, then (a, b) is called the ordered pair of a and b.

Further we define

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d$$

Thus $\quad (a, b) \neq (b, a)$ unless $a = b$.

Example 1. Find the numbers x and y if the ordered pairs $(2x - 1, -5)$ and $(x, y + 1)$ are equal.

By definition, $(2x - 1, -5) = (x, y + 1)$

$\therefore \; 2x - 1 = x$ and $-5 = y + 1$

whence, we get $x = 1$ and $y = -6$.

2.3 Cartesian Product of Sets

The **cartesian product** of two non-empty sets A and B is the set of all ordered pairs (a, b), where $a \in A$, $b \in B$ and is denoted by $A \times B$.

Thus,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Obviously by definition, $A \times B \neq B \times A$ unless $A = B$.

When $B = A$, $A \times A$ is denoted by A^2.

Example 2. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$ then

$A \times B = \{(1, 3), (1, 4), (1,5), (1,6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

and

$B \times A = \{(3, 1), (3, 2), (3,3), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (6, 1), (6, 2), (6, 3)\}$.

Observe that $(1, 3) \in A \times B$ whereas $(1, 3) \notin B \times A$.

Hence $A \times B \neq B \times A$.

Example 3. If R is the set of all real numbers, then

$$R \times R = \{(x, y) \mid x, y \in R\}.$$

Thus, if (x, y) is thought of as a point in a plane (with reference to some fixed co–ordinate axis, of course), then $R \times R$ is the set of all points in the plane. $R \times R$ will usually be denoted by R^2.

Similarly, we define R^3 as follows :

$$R^3 = \{(x, y, z) \mid x, y, z \in R\}$$

and, in general

$$R^n = \{(x_1, x_2, \ldots\ldots, x_n) \mid x_i \in R, i = 1, 2, \ldots\ldots, n\}$$

Thus, R^3, may be thought of as the set of all points in 3 dimensional space, and R^n as the set of all points in n–dimensional space.

Theorem 1. For non–void sets A, B, C, D, prove

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$, **[Raj. 2003]**

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$, **[Raj. 2003]**

(c) $A \times (B - C) = (A \times B) - (A \times C)$

(d) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ **[Raj. 2003, 1999]**

Proof:

(a) Letting $A \times (B \cup C) = X$ and $(A \times B) \cup (A \times C) = Y$

we have to show that $X \subseteq Y$ and $Y \subseteq X$.

Let $(x, y) \in X$

$\Rightarrow (x, y) \in A \times (B \cup C)$

$\Rightarrow x \in A$ and $y \in B \cup C$

$\Rightarrow x \in A$ and $y \in B$ or C

Therefore, either $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$

$\Rightarrow (x, y) \in A \times B$ or $(x, y) \in A \times C$

so $(x, y) \in (A \times B) \cup (A \times C) = Y$

so $X \subseteq Y$ .....(1)

Next let $(x, y) \in Y$

$\Rightarrow (x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow (x, y) \in A \times B$ or $(x, y) \in (A \times C)$

$\Rightarrow (x \in A, y \in B)$ or $(x \in A, y \in C)$

It means x always $\in$ A and y may $\in$ B or C

so $x \in A$ and $y \in B \cup C$

i.e., $(x, y) \in A \times (B \cup C) = X$

Hence $Y \subseteq X$ .....(2)

(1) and (2) $\Rightarrow X = Y$ i.e. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) Letting $X = A \times (B \cap C)$ and $Y = (A \times B) \cap (A \times C)$

Suppose $(x, y) \in X$

$\Rightarrow (x, y) \in A \times (B \cap C)$

$\Rightarrow x \in A$ and ($y \in B$ and C both)

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in A \times C$

so $(x, y) \in (A \times B) \cap (A \times C) = Y$

Thus,

$X \subseteq Y$ .....(1)

Next let $(x, y) \in Y$

$\Rightarrow (x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Rightarrow (x \in A, y \in B)$ and $(x \in A, y \in C)$

$\Rightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow x \in A$ and $y \in B \cap C$

$\Rightarrow (x, y) \in A \times (B \cap C) = X$

$\Rightarrow Y \subseteq X$ .....(2)

Therefore (1) and (2) $\Rightarrow X = Y$

i.e. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(c) Letting $X = A \times (B - C)$ and $Y = (A \times B) - (A \times C)$

Let $(x, y) \in X$

$\Rightarrow (x, y) \in A \times (B - C)$

$\Rightarrow x \in A, y \in B - C$

$\Rightarrow x \in A$ and $y \in B$ but $y \notin C$

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in A, y \notin C)$

$\Rightarrow (x, y) \in (A \times B)$ but $(x, y) \notin (A \times C)$

$\Rightarrow (x, y) \in (A \times B) - (A \times C) = Y$

Therefore, $X \subseteq Y$ .....(1)

Net let $(x, y) \in Y$

$\Rightarrow (x, y) \in (A \times B) - (A \times C)$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$

$\Rightarrow (x \in A, y \in B)$ and (either $x \notin A$ or $y \notin C$)

$\Rightarrow x \in A, y \in B$ and $y \notin C$ $[\because x \in A]$

$\Rightarrow x \in A, y \in B - C$

$\Rightarrow (x, y) \in A \times (B - C) = X$

$\Rightarrow y \subseteq X$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow$ i.e. $A \times (B - C) = (A \times B) - (A \times C)$.

(d) Letting $X = (A \times B) \cap (C \times D)$

and $Y = (A \cap C) \times (B \cap D)$

Suppose $(x, y) \in X$

$\Rightarrow (x, y) \in (A \times B) \cap (C \times D)$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$

$\Rightarrow (x \in A, y \in B)$ and $(x \in C, y \in D)$

$\Rightarrow (x \in A$ and $x \in C)$ and $(y \in B$ and $y \in D)$

$\Rightarrow x \in A \cap C$ and $y \in B \cap D$

Thus, $(x, y) \in (A \cap C) \times (B \cap D) = Y$

so $X \subseteq Y$ .....(1)

Next let $(x, y) \in T$

$\Rightarrow (x, y) \in (A \cap C) \times (B \cap D)$

$\Rightarrow x \in A \cap C$ and $y \in B \cap D$

$\Rightarrow (x \in A$ and $x \in C)$ and $(y \in B$ and $y \in D)$

$\Rightarrow (x \in A, y \in B)$ and $(x \in C, y \in D)$

$\Rightarrow (x, y) \in A \times B$ and $(x, y) \in C \times D$

$\Rightarrow (x, y) \in (A \times B) \cap (C \times D) = X$

$\Rightarrow Y \subseteq X$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow X = Y$ i.e.,

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Theorem 2. For non–void sets A, B, C, D prove

(a) $A \subseteq B$ and $C \subseteq D \Rightarrow A \times C \subseteq B \times D$,

(b) If $A \times B = B \times A$, then $A = B$

(c) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

Proof: (a) Letting $A \subseteq B$ and $C \subseteq D$

Let $(x, y) \in A \times C \Rightarrow x \in A$ and $y \in C$

As $x \in A \Rightarrow x \in B$ $[\because A \subseteq B]$

$\Rightarrow y \in C \Rightarrow y \in D$ $[\because C \subseteq D]$

Thus, $(x, y) \in B \times D$

Hence $A \times C \subseteq B \times D$

(b) Given $A \times B = B \times A$

$\therefore$ $A \times B \subseteq B \times A$ and $B \times A \subseteq A \times B$

Let $(x, y) \in (A \times B) \Rightarrow (x, y) \in B \times A$ $[\because A \times B \subseteq B \times A]$

Thus $(x \in A, y \in B) \Rightarrow (x \in B, y \in A)$

so $x \in A \Rightarrow x \in B$ and $y \in B \Rightarrow y \in A$

$\Rightarrow A \subseteq B$ and $B \subseteq A$

Hence $A = B$.

(c) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap B \times A$

Let $(x, y) \in A \times A$

$\Rightarrow x \in A$ and $y \in A$

$\Rightarrow x \in B$ and $y \in B$ [$\because A \subseteq B$]

$\therefore\ x \in A, y \in B$ and $x \in B, y \in A$

so $(x, y) \in A \times B$ and $(x, y) \in B \times A$

Thus $(x, y) \in (A \times B) \cap (B \times A)$

Therefore, $A \times A \subseteq (A \times B) \cap (B \times A)$

Theorem 3. If A and B be two non–void sets with n elements in common, then prove $A \times B$ and $B \times A$ will have n^2 elements in common.

Proof: Let $S = A \cap B$ be the set of elements which are common in A and B.

Let number of elements $S = |S| = n$

then $|S \times X| = |S| \cdot |S| = n \cdot n = n^2$

Let $(x, y) \in S \times S \Rightarrow x \in S, y \in S$

but $S \subset A$ and $S \subset B$

so $x \in A$ and $y \in B$

Thus $(x, y) \in A \times B$

$\Rightarrow S \times S \subseteq A \times B$ .....(1)

Next $S \subseteq B$ and $S \subseteq A$

so $(x, y) \in S \times S \Rightarrow x \in S, y \in S$

$\Rightarrow x \in B, y \in A$ [$\because S \subseteq B, S \subseteq A$]

$\Rightarrow (x, y) \in B \times A$

$\therefore\ S \times S \subseteq B \times A$ .....(2)

Thus (1) and (2) $\Rightarrow S \times S \subseteq (A \times B) \cap (B \times A)$ .....(3)

Next let $(x, y) \in (A \times B) \cap (B \times A)$

$\Rightarrow (x, y) \in A \times B$ and $(x, y) \in B \times A$

$\Rightarrow (x \in A, y \in B)$ and $(x \in B, y \in A)$

$\Rightarrow (x \in A, x \in B)$ and $(y \in A, y \in B)$

$\Rightarrow x \in A \cap B$ and $y \in A \cap B$

$\Rightarrow (x, y) \in (A \cap B) \times (A \cap B)$

$\Rightarrow (x, y) \in S \times S$ [$\because S = A \cap B$]

$\Rightarrow (A \times B) \cap (B \times A) \subseteq S \times S$ .....(4)

$\therefore$ (3) and (4) $\Rightarrow S \times S = (A \times B) \cap (B \times A)$

So number of elements common in $A \times B$ and $B \times A = |(A \times B) \cap (B \times A)| = |S \times S|$

$$= |S| \cdot |S| = n \cdot n$$

$$= n^2$$

2.4 Relations

A relation is a set of ordered pairs.

Let A and B be two sets. A relation from A to B is a subset of A × B.

Symbolically, R is a relation from A to B iff $R \subseteq A \times B$.

If (x, y) be a member of a relation set R, we express it by writing x R y and say that 'x is related to y by the relation R.

Thus $(x, y) \in R \Leftrightarrow x\ R\ y$.

Example 4. If A = {2, 3, 5, 6} and R means "divide" then 2R2, 2R6, 3R3, 3R6, 5R5, 6R6 and as such relation set R = {(2, 2), (2, 6), (3, 3), (3, 6), (5, 5), (6, 6)}.

2.5 Domain And Range

Let R be a relation from A to B. Then the set of all first coordinates of the ordered pairs in relation set R is called the Domain of R and the set of all second members of ordered pair in R is called the range of R.

Thus, Domain R = $\{x \mid (x, y) \in R\}$

Range R = $\{y \mid (x, y) \in R\}$

Example 5. If A = {1, 2, 3, 4}; B = {3, 4, 5} and a relation R is defined from A to B by $x\ R\ y \Leftrightarrow x < y$, then

A × B = {(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5)}.

Now if R stands for "less than" then relation set R = {(1, 3), (1, 4), (1,5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

∴ Domain R = {1, 2, 3, 4}

and Range R = {3, 4, 5}

2.6 Relation in a Set

If A be a non–void set, then any subset of A × A defines a relation in A.

2.7 Empty Relation

If no element of A is in relation with any element of B, then the relation R = a Null set ϕ = { } = an empty set. It is known as **void** or **empty** relation.

2.8 Universal Relation

If A be a non–void set and $R : A \times A \rightarrow A$ such that

Domain R = Range R = A, the relation R is known as a **universal relation** in A.

2.9 Identity Relation

If a relation R be defined on a non–void set A such that

$$R = I = \{(a, a) \mid a \in A\},$$

then R is called an **identity relation** in A. It is denoted by $I(x) = x$.

2.10 Inverse Relation

Let R be a relation from a set A to a set B. The relation R^{-1} from B to A is said to be the **inverse relation** of R,

if $$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

Thus Domain of R^{-1} = Range of R

and Range of R^{-1} = Domain of R

Example 6. Let $A = \{l, m, n\}$, $B = \{a, b\}$ then $R = \{(l, a), (l, b), (m, a), (n, b)\}$ is a relation from A to B.

The relation $R^{-1} = \{(a, l), (b, l), (a, m), (b, n)\}$ from B to A is the inverse relation of R.

Remark 2. Every relation has an inverse relation.

2.11 R-Relative Set of an Element x

Let R be a relation from A to B and $x \in A$, then R relative set of x, denoted by $R(x)$, is the set of those elements of B which are related to that particular element x i.e.,

$$R(x) = \{y \in B \mid (x, y) \in R\}.$$

Example 7. Let $A = \{1, 2, 3, 4\}$, $B = \{r, s, t\}$

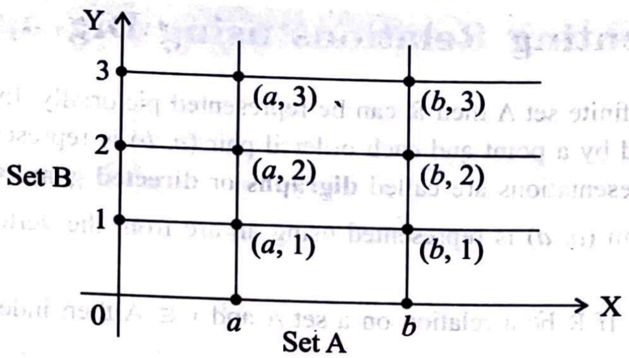
$$R = \{(1, s), (1, t), (2, 4), (2, s), (4, t)\}$$

$$R(1) = (s, t);\ R(2) = \{r, s\};\ R(4) = \{t\}.$$

2.12 Graph of a Relation

As a relation R from a set A to a set B is a subset of the product set $A \times B$, therefore R can be sketched on a co–ordinate diagram similar to that of the set $A \times B$.

Example 8. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$, then $R = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$ is a relation from A to B. Its graph can be drawn as shown in the following figure.



2.13 Adjacency Matrix of a Relation

Let elements of sets A and B be labelled as

$$A = \{a_1, a_2, \ldots, a_m\}$$

and

$$B = \{b_1, b_2, \ldots, b_n\}.$$

Suppose R be a relation from A to B. Then the relation R can be represented by an $m \times n$ matrix

$$M_R = [m_{ij}]_{m \times n},$$

where

$$m_{ij} = \begin{cases} 1, \text{ if } (a_i, b_j) \in R \\ 0, \text{ if } (a_i, b_j) \notin R \end{cases}$$

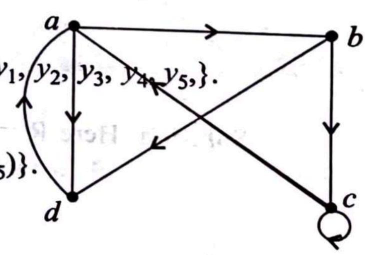
The matrix M_R is known as the matrix of relation R or adjacency matrix or Boolean matrix of R.

Example 9. Let X = {1, 2, 3}, Y = {1, 2} and R = {(2, 1), (3, 1), (3, 2)} then M_R is a 3 × 2 matrix, whose elements m_{21}, m_{31}, m_{32} are each equal to unity and other elements are all zero,

$$\therefore \quad M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example 10. Find the relation R, when

$$\therefore \quad M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$



be a matrix of the relation R from X to Y, where X = $\{x_1, x_2, x_3\}$ and Y = $\{y_1, y_2, y_3, y_4, y_5\}$.

Solution. The relation is

$$R = \{(x_1, y_2), (x_2, y_1), (x_2, y_3), (x_2, y_4), (x_3, y_1), (x_3, y_3), (x_3, y_5)\}.$$

2.14 Representing Relations using Digraphs

If R be a relation on a finite set A then R can be represented pictorially. In this representation each element of the set is represented by a point and each ordered pair (a, b) is represented by an arc (or edge) directed from a to b. Such representations are called **digraphs** or **directed graphs.**

An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a loop.

Indegree of a vertex : If R be a relation on a set A and $v \in A$ then indegree of v is the number of edges directed towards v.

Outdegree of a vertex : If R be a relation on a set A and $v \in A$ then outdegree of v is the number of edges beginning from v i.e., going away from v.

Example 11. Let A = $\{a, b, c, d\}$ and

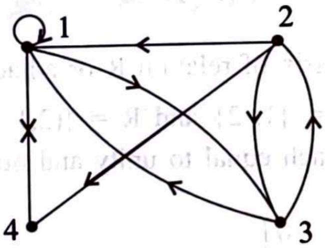
R = $\{(a, b), (a, d), (b, d), (c, a), (c, b), (c, c), (d, a)\}$

then digraph of R can be displayed as below

Here

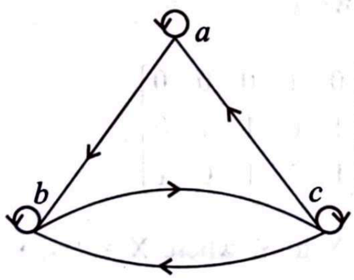
Vertices	:	a	b	c	d
Indegree	:	2	1	2	2
Outdegree	:	2	2	2	1

Example 12. Find the relation R, defined on the set A = {1, 2, 3, 4}, represented by the digraph



Here R = {(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}

Example 13. Let R be a relation on the set A = $\{a, b, c\}$. Find R and matrix representation M_R of R, whose digraph is given below



Solution. Here R = $\{(a, a), (a, b), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

$\therefore$ $$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

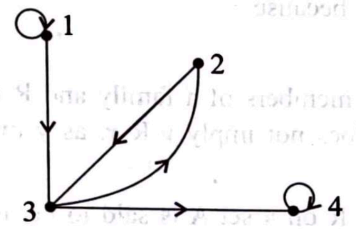
Example 14. If a relation R on the set A = {1, 2, 3, 4} has the matrix representation

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Draw the digraph of R.

Solution. The relation R = {(1, 1), (1, 3) (2, 3), (3, 2), (3, 4), (4, 4)}

∴ Digraph is as follows



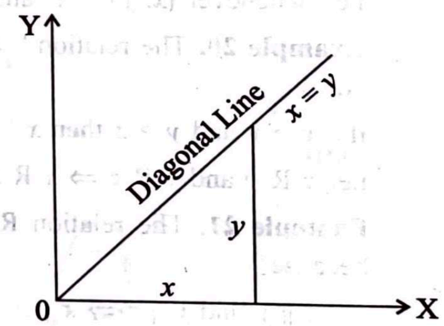
Vertices	:	1	2	3	4
Indegree	:	1	1	2	2
Outdegree	:	2	1	2	1

2.15 Binary Relation

A relation R between pairs of elements of a set A is called **binary relation.**

In the discussions here after the word relation will means binary relation.

Let A be any set and R be the set of all those pairs (x, y) of A × A for which $x = y$. Then the relation R is known as the **equality relation** in the set A. It is also known as the **diagram relation** in A and is denoted by Δ.



Thus $\Delta = \{(x, y) \mid x = y,\ x, y \in A\}$.

Remark 3. Any subset of A × A is called a binary relation on the set A.

2.16 Properties of Relations

1. Reflexive Relation

A relation R on a non–void set A is known as **reflexive relation** if each member of A is R–related to itself, i.e., $x \,R\, x$ or $(x, x) \in R,\ \forall\ x \in A$.

A relation R on a set A is irreflexive if $(x, x) \notin R, \forall\ x \in A$.

Example 15. Let A be the set of all straight lines in a plane. The relation R in A defined by "x is parallel to y" is reflexive, since every straight line is parallel to itself.

Example 16. The relation "<" defined on a set of real numbers is irreflexive because x is not less than x.

Example 17. If A be the set of men and R means "is husband of" then x R x is not true as man cannot be husband of himself. Thus, R is irreflexive.

Remark 4. The necessary and sufficient condition for a relation to be reflexive is $\Delta \subset R$.

2. Symmetric Relation

A relation R on a non–void set A is known as **symmetric relation** if $x \; R \; y \Rightarrow y \; R \; x$, i.e., whenever $(x, y) \in R$ then $(y, x) \in R$.

Example 18(a). Let A be the set of all straight lines in a plane. The relation R defined by "a is perpendicular to b" is symmetric relation because

$a \perp b \Rightarrow b \perp a$, $a, b \in A$.

Example 18(b). If A be the set of members of a family and R means "is the brother of" and x R y means that x is brother of y then x R y does not imply y R x, as y may be the sister of x. Hence R is not symmetric.

Asymmetric Relation : A relation R on a set A is said to be asymmetric if

$(a, b) \in R \Rightarrow (b, a) \notin R$, $a, b \in A$.

Example 19. The relation $x < y$ is a symmetric as if $x < y$, then y is not less than x.

Remark 5. The necessary and sufficient condition for a relation to be symmetric is $R = R^{-1}$.

3. Transitive Relation

A relation on a set A is said to be **transitive relation** if for $x, y, z \in A$,

$x \; R \; y$ and $y \; R \; z \Rightarrow x \; R \; z$

i.e., whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Example 20. The relation " > " defined on the set of natural numbers N is transitive because for $x, y, z \in$ N,

if $x > y$ and $y > z$ then $x > z$

i.e., $x \; R \; y$ and $y \; R \; z \Rightarrow x \; R \; z$.

Example 21. The relation R of parallelism in the set of straight lines in a plane is a transitive relation because

$x \parallel y$ and $y \parallel z \Rightarrow x \parallel z$

i.e., $x \; R \; y$ and $y \; R \; z \Rightarrow x \; R \; z$, $x, y, z \in A$

Remark 6. The necessary and sufficient condition that a relation R be transitive is R operated on $R \subset R$.

4. Anti Symmetric Relation

A relation R on a set A is said to be antisymmetric if whenever $x \neq y$ then either $x \not R \; y$ or $y \not R \; x$.

OR

A relation R on a set A is said to be antisymmetric

if $x \; R \; y$ and $y \; R \; x \Rightarrow x = y$, $\forall \; x, y \in A$.

Example 22. Let N be the set of natural numbers and let R be the relation defined by "a divides b", $\forall \; a, b \in$ N. Then R is an anti–symmetric relation as a divides b and b divides $a \Rightarrow a = b$.

Example 23. Let P be a family of sets, then the relation R on P defined by "A is a subset of B" is anti–symmetric because A R B and B R A $\Rightarrow$ A $\subseteq$ B and B $\subseteq$ A

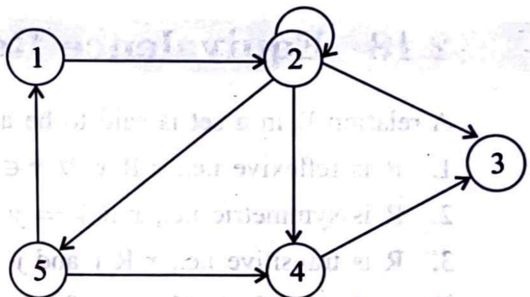
$\Rightarrow$ A = B.

2.17 Connectivity Relation

Let R be a relation on a set A. A path of length n in R from a to b is a finite sequence Π : $a, x_1, x_2, \ldots, x_{n-1}, b$, beginning with a and ending with b, such that

$$a \text{ R } x_1, x_1 \text{ R } x_2, \ldots\ldots, x_{n-1} \text{ R } b.$$

A path of length 'n' involves $(n + 1)$ elements of the set A, although they are not necessarily distinct. For example, in the digraph



Π_1 : 1, 2, 5, 4, 3 is a path of length 4 from vertex 1 to vertex 3.

Π_2 : 1, 2, 5, 1 is a path of length 3 from vertex 1 to itself.

A path whose first and the last vertex are same is called a cycle. In above example, Π_2 is a cycle of length 3.

If n is a fixed positive integer, we can define a relation R^n on A as a R^n b it implies that there is a path of length n from a to b in R.

We can also define a relation R^∞ on A as a R^∞ b it implies that there is some path in R from a to b, whose length depends on a and b. The relation R^∞ is called the connectivity relation for R. If A has n elements then $R^\infty = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

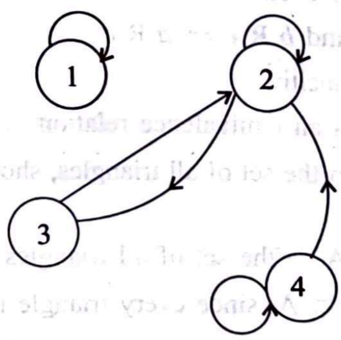
Example 24. Given A = {1, 2, 3, 4}, consider following relation in A.

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}.$$

Find (i) $R^2 = R \cdot R$.

(ii) R^∞ **[Raj. 2003]**

Solution. The digraph of R is as follows



Since 1 R 1 and 1 R 1 $\Rightarrow$ 1 R^2 1

2 R 2 and 2 R 2 $\Rightarrow$ 2 R^2 2

3 R 2 and 2 R 3 $\Rightarrow$ 3 R^2 3

4 R 4 and 4 R 2 $\Rightarrow$ 4 R^2 2

4 R 4 and 4 R 4 $\Rightarrow$ 4 R^2 4

4 R 2 and 2 R 3 $\Rightarrow$ 4 R^2 3

2 R 2 and 2 R 3 $\Rightarrow$ 2 R^2 3

3 R 2 and 2 R 2 $\Rightarrow$ 3 R^2 2

$\therefore$ R^2 = {(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 2), (4, 3)}

(ii) Since $R^{\infty} = R \cup R^2 \cup R^3 \cup R^4$

Now, $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$

$R^2 = \{(1, 1,), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 2), (4, 3)\}$

$R^3 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 2), (4, 3)\}$

$R^4 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 2), (4, 3)\}$

$\therefore$ $R^{\infty} = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$

2.18 Equivalence Relation

A relation R in a set is said to be an equivalence relation if

1. R is reflexive i.e., $x \, R \, x$, $\forall \, x \in R$,
2. R is symmetric i.e., $x \, R \, y \Rightarrow y \, R \, x$, $x, y \in R$, and
3. R is transitive i.e., $x \, R \, y$ and $y \, R \, z \Rightarrow x \, R \, z$, $x, y, z \in R$.

Example 25. If I be the set of integers and if R be defined over I by "$a \, R \, b$ iff $a - b$ is an even integer", where $a, b \in I$, then show that the relation R is an equivalence relation.

Solution.

(i) Reflexivity

Since $a - a = 0$ is even.

Thus $a \, R \, a$, $\forall \, a \in I$

Therefore, R is reflexive over I.

(ii) Symmetry

Since $a - b$ is even then $(b - a) = -(a - b)$ is also even and hence $a \, R \, b \Rightarrow b \, R \, a$. So R is symmetric

(iii) Transitivity

$a \, R \, b$ and $b \, R \, c$

$\Rightarrow a - b$ and $b - c$ are even

$\Rightarrow (a - b) + (b - c)$ is even

$\Rightarrow a - c$ is even

$\therefore a \, R \, b$ and $b \, R \, c \Rightarrow a \, R \, c$

So R is transitive

Hence R is an equivalence relation.

Example 26. In the set of all triangles, show that the relation of congruency (or similarity) is an equivalence relation.

Solution: Let A be the set of all triangles and R be the relation defined by $x \, R \, y$ iff $x \equiv y$, $x, y \in A$.

(i) $x \, R \, x$, $\forall \, x \in A$, since every triangle is congruent to itself. Thus, R is reflexive.

(ii) $x \, R \, y \Rightarrow y \, R \, x$, $x, y \in A$, because if x is congruent to y, then y is surely congruent to x. Thus R is symmetric.

(iii) $x \, R \, y$ and $y \, R \, z \Rightarrow x \, R \, z$, $\forall \, x, y, z \in A$ because if x is congruent to y and y is congruent to z, then x is also congruent to z. Thus R is transitive.

Hence it is an equivalence relation.

Example 27. Prove that in the set A = {1, 2, 3}, the relation

R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2), (3, 1), (1, 3)}

is an equivalence relation.

Solution:

(i) Reflexive

Since (1, 1), (2, 2), (3, 3) ∈ R ⇒ $a \, R \, a, \forall a \in A$.

∴ R is reflexive.

(ii) Symmetric

We have (1, 2) and (2, 1) ∈ R

(2, 3) and (3, 2) ∈ R

(1, 3) and (3, 1) ∈ R

∴ R is symmetric

(iii) Transitive

We have

when (1, 2), (2, 3) ∈ R ⇒ (1, 3) ∈ R

(2, 3), (3, 1) ∈ R ⇒ (2, 1) ∈ R

(3, 1), (1, 2) ∈ R ⇒ (3, 2) ∈ R

∴ R is transitive

Thus it is an equivalence relation.

Example 28. If R is an equivalence relation on a set A, then prove that R^{-1} is also an equivalence relation on A.

Solution: Since R is a relation on A

$\Rightarrow R \subset A \times A$

$\Rightarrow R^{-1} \subset A \times A$

$\Rightarrow R^{-1}$ is also a relation in A.

Now we have only to prove that R^{-1} is an equivalence relation.

(i) Reflexive : Let $a \in A$, then

$a \in A \Rightarrow (a, a) \in R$ [∵ R is reflexive]

$\Rightarrow (a, a) \in R^{-1}, \forall (a, a) \in A \times A$

∴ R^{-1} is reflexive.

(ii) Symmetric : Let $(a, b) \in R^{-1}$,

$(a, b) \in R^{-1} \Rightarrow (b, a) \in R$

$\Rightarrow (a, b) \in R$ [∵ R is symmetric]

$\Rightarrow (b, a) \in R^{-1}$

∴ R^{-1} is symmetric

(iii) Transitive : Let $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$, then

$(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$

$\Rightarrow (b, a) \in R$ and $(c, b) \in R$

$\Rightarrow (c, b) \in R$ and $(b, a) \in R$ [$\because$ R is transitive]

$\Rightarrow (c, a) \in R$

$\Rightarrow (a, c) \in R^{-1}$

$\therefore$ R^{-1} is transitive.

Hence R^{-1} is also an equivalence relation on A.

Example 29. Prove that a relation R in a set is symmetric iff $R = R^{-1}$.

Solution: First suppose that R is symmetric then we have to prove that $R = R^{-1}$

Let $(a, b) \in R$, then

$(a, b) \in R \Rightarrow (b, a) \in R$ [$\because$ R is symmetric]

$\Rightarrow (a, b) \in R^{-1}$ [by definition of inverse function]

$\therefore R \subseteq R^{-1}$ .....(1)

Also $(x, y) \in R^{-1} \Rightarrow (y, x) \in R$

$\Rightarrow (x, y) \in R$ [$\because$ R is symmetric]

$\therefore R^{-1} \subseteq R$ .....(2)

Thus from (1) and (2), we have $R = R^{-1}$.

Conversely: Suppose that $R = R^{-1}$, then we have to prove that R is symmetric relation.

Let $(a, b) \in R$, then

$(a, b) \in R \Rightarrow (b, a) \in R^{-1}$ [by dif. of inverse relation]

$\Rightarrow (b, a) \in R$ [$\because R = R^{-1}$]

R is a symmetric relation.

Hence R is a symmetric $\Leftrightarrow R = R^{-1}$.

Example 30. If R and S be two equivalence relations in a set A, then prove that $R \cap S$ is also an equivalence relation in A.

Solution: R and S are relations in A

$\Rightarrow R \subset A \times A$ and $S \subset A \times A$

$\Rightarrow R \cap S \subset A \times A$

$\Rightarrow R \cap S$ is also a relation in A.

Now we shall prove that it is an equivalence relation in A.

(i) Reflexive : Let $a \in A$, then

$a \in A \Rightarrow (a, a) \in R$ and $(a, a) \in S$ [$\because$ R and S are reflexive]

$\Rightarrow (a, a) \in R \cap S, \forall a \in A$

$\therefore R \cap S$ is reflexive.

(ii) Symmetric : Let $(a, b) \in R \cap S$, then

$(a, b) \in R \cap S \Rightarrow (a, b) \in R$ and $(a, b) \in S$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$ [$\because$ R and S are symmetric]

$\Rightarrow (b, a) \in R \cap S$

$\therefore$ R $\cap$ S is symmetric.

(iii) Transitive : Let $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$, then

$(a, b) \in R \cap S \Rightarrow (a, b) \in R$ and $(a, b) \in S$

$(b, c) \in R \cap S \Rightarrow (b, c) \in R$ and $(b, c) \in S$

But R and S are transitive so

$$\left.\begin{array}{l}(a,b)\in R \text{ and } (b,c)\in R \Rightarrow (a,c)\in R\\ (a,b)\in S \text{ and } (b,c)\in S \Rightarrow (a,c)\in S\end{array}\right\} \Rightarrow (a,c)\in R\cap S$$

Thus $(a, b) \in R \cap S$ and $(b, c) \in R \cap S \Rightarrow (a, c) \in R \cap S$.

$\therefore$ R $\cap$ S is transitive.

Hence R $\cap$ S is an equivalence relation in A.

Example 31. Union of two equivalence relations is not necessarily an equivalence relation.

Solution: It can be prove by giving a counter example.

Let set A= {1, 2, 3, 4}.

Suppose that R= {(1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3) (1, 2), (2, 1)}

and S= {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (2, 4), (3, 2), (4, 2)}

are two relations on A.

It can be easily verify that R and S are equivalence relations.

It is easy to check that both these relations are equivalences.

Now from the set R $\cup$ S = {(1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3), (1, 2), (2, 1), (2, 3), (3, 2), (2, 4), (4, 2)}

We notice here that the transitive property does not hold true in R $\cup$ S.

Here $(1, 2) \in R \cup S$, $(2, 3) \in R \cup S$ but $(1, 3) \notin R \cup S$

$\therefore$ R $\cup$ S is not transitive.

Hence it is not an equivalence relation.

Remark 7. If we let

R= {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)}

and S= {(1, 1), (2, 2), (3, 3), (3, 1), (1, 3)}

We see that R and S are equivalence relations and R $\cup$ S is also an equivalence relation.

Hence we conclude that R $\cup$ S may or is not necessarily equivalence.

Example 32. Let a relation R in the set of real numbers be defined as

$a \, R \, b$ iff $1 + ab > 0$

Show that this relation is reflexive and symmetric but not transitive.

Solution. (i) Reflexive : Let A be the set of real numbers and $a \in A$, then

$a \in A \Rightarrow 1 + a \times a = 1 + a^2 > 0$ [$\because$ square of a real number $\geq 0 \; \forall \; (a, a) \in A \times A$]

$\Rightarrow (a, a) \in R$ $\therefore$ R is reflexive.

(ii) Symmetric : Let $(a, b) \in R$, then

$$(a, b) \in R \Rightarrow 1 + ab > 0$$
$$\Rightarrow 1 + ba > 0 \qquad [\because ab = ba, \forall a, b \in A]$$
$$\Rightarrow (b, a) \in R$$

$\therefore$ R is symmetric.

(iii) Transitive : R is not transitive because we find that

$$\left(1, \frac{1}{2}\right) \in R \text{ and } \left(\frac{1}{2}, -1\right) \in R \text{ but } (1, -1) \notin R$$

since $1 + 1 \times (-1) = 0$ (not +ve)

Hence R is reflexive and symmetric but not transitive.

Example 33. Consider a set A = {a, b, c, d, e, f} and a relation R defined on A given by R = {(a, a), (a, b), (b, a), (b, b), (c, c), (d, d), (d, e), (d, f), (e, d), (e, e), (e, f), (f, d), (f, e), (f, f)}

Write the matrix representation M_R of the relation and hence prove that it is an equivalence relation.

Solution: The set A = {a, b, c, d, e, f}.

The matrix representation of the given relation R is

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(i) **Reflexive :** The diagonal elements of the matrix M_R are non–zero it implies that every element of the set A is related to itself hence the relation is reflexive.

(ii) **Symmetric :** The matrix M_R and its transpose $(M_R)^T$ are identical hence the relation is symmetric.

(iii) **Transitive :** The matrix $(M_R)^2$ is given by

$$M_R \cdot M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$= M_R$ (keeping in view 1 + 1 =1)

So we notice that $M_R M_R = M_R$.

Thus relation is transitive.

Hence the relation is an equivalence relation.

2.19 Congruency Relation Modulo System

It is of common experience that railway time–table is fixed with the provision of 24 hours in a day and night. When we say that train is arriving at 14 hours it means that the train will arrive at 2 p.m. according to our watch. Thus, all the timing starting from 12 to 23 hours correspond to one of 0, 1, 2, 3, ..., 11, O'clock as indicated in routine watches. In other words all integers from 12 to 23 are equivalent to one or the other of the integers 0, 1, 2, ..., 11 in a system of integers upto 12. In saying like this the integers in question are divided into 12 classes. Now, we shall extend this idea of all the integers whether positive or negative, by dividing them into 12 classes. Then every integer whatsoever will be equivalent to one or the other of the integers 0, 1, 2, ..., 11. For example

$$-2 \equiv 10, -20 \equiv 4, 32 \equiv 8 \text{ etc..}$$

To indicate the division into 12 classes, we write modulo 12 in bracket on the extreme right hand side of these statement. Symbolically, it is written as mod.

In this way the integers could be divided into 3 classes or 6 classes or m (m being a positive integer) classes and we write mod 3 or mod 6 or mod m. This system of representing integers is known as **modulo system.**

Two integers a and b are said to be congruent modulo m if $a-b$ is divisible by m and we denoted it as $a \equiv b \pmod{m}$, $a, b, m \in Z$, where $m \geq 2$.

The integer m is known as the **modulus** of the **congruence relation.**

Thus,

$13 \equiv 1 \pmod{12}$, as $13 - 1 = 12$ which is a multiple of 12

$27 \equiv 3 \pmod{12}$, as $27 - 3 = 24 = 2.12$

$25 \equiv 0 \pmod{5}$, as $25 - 0 = 25 = 5.5$

$3m + 2 \equiv 2 \pmod{m}$, as $3m + 2 - 2 = 3m$.

Remark 8. Obviously, if some member is to be expressed as congruence mod m, then that number should be divided by m and then remainder is the desired number.

Thus, in a congruence modulo m, the whole set of integers can be denoted by the numbers 0, 1, 2, 3, ..., $m - 1$ because remainder (when quotient is m) cannot be m or greater than m.

The set of all these numbers which are equivalent to 0 will be represented by $\{0\}$. Similarly others by $\{1\}, \{2\}, \{3\}, ..., \{m-1\}$.

The sets $\{0\}, \{1\}, \{2\}, \{3\}, ..., \{m-1\}$ are known as **residue classes.**

Let us take $m = 5$, then the residue classes are

$$\{0\} = \{...., -15, -10, -5, 0, 5, 10, 15, ...\}$$
$$\{1\} = \{....., -14, -9, -4, 1, 6, 11, 16, ...\}$$
$$\{2\} = \{....., -13, -8, -3, 2, 7, 12, 17, ...\}$$
$$\{3\} = \{....., -12, -7, -2, 3, 8, 13, 18, ...\}$$
$$\{4\} = \{....., -11, -6, -1, 4, 9, 14, 19, ...\}$$

If $a \equiv b \pmod{m}$ then a and b have the same remainder when divided by m.

2.20 Addition Modulo m

If a and b are any two integers and m is a fixed positive integer then for mod m,

$$a + b \equiv r \pmod{m},\ 0 \le r < m$$

where r is the least non–negative remainder obtained on dividing $a + b$ by m, $a + b$ being the ordinary sum of a and b. Symbolically, $a +_m b = r,\ 0 \le r < m$.

Example 34. $2 + 3 = 5,\ 2 + 4 = 0,\ 2 + 5 = 1 \pmod 6$, actually $2 + 4 = 6$, but when we divide it by 6 we get 0 as remainder, similarly when we divide $2 + 5 = 7$ by 6, we get 1 as remainder.

2.21 Multiplication Modulo m

If a and b are an integers and m is a fixed positive integer then

$$a \times b \equiv r \pmod{m},\ 0 \le r < m$$

where r is the least non–negative remainder when $a \times b$, the ordinary multiplication of a and b is divided by m. Symbolically, $a \times_m b = r,\ 0 \le r < m$.

Example 35. The multiplication of 4 and 3 modulo 5 is

$4 \times 3 = 12 \equiv 2 \pmod 5$ i.e., $4 \times_5 3 = 2$

likewise, $4 \times 2 = 8 \equiv 1 \pmod 7$ i.e., $4 \times_7 2 = 1$

It can be easily verified that

If $a \equiv b \pmod m$ then

$a \times c \equiv b \times c \pmod m$

Remark 9. All the laws of ordinary addition and multiplication hold good for addition and multiplication modulo m.

Example 36. Show that in the set of integers I = {...., – 2, – 1, 0, 1, 2,....}, the relation a R $b \Rightarrow a$ is congruent to $b \pmod n$, $n \in$ N, means that $a - b$ = multiple of $n = n\,k,\ k \in$ I is an equivalence relation.

[RTU 2011]

Solution: (i) Reflexivity : Let $a \in$ I, then

$a - a = 0 \cdot n$ = multiple of n

$\Rightarrow a\,\mathrm{R}\,a \quad \therefore$ R is reflexive

(ii) Symmetric : Let $a, b \in$ I, then

$a\,\mathrm{R}\,b \Rightarrow a \equiv b \pmod n \Rightarrow a - b = nk\ \ k \in \mathrm{I}$

$\Rightarrow b - a = n(-k),\ -k \in \mathrm{I}$

$\Rightarrow b \equiv a \pmod n$

$= b\,\mathrm{R}\,a \qquad \therefore$ R is symmetric

(iii) Transitive : Let a R b and b R c, then $a\,\mathrm{R}\,b \Rightarrow a - b = nk_1$ .....(1)

and $b\,\mathrm{R}\,c \Rightarrow b - c = nk_2$ .....(2)

$\therefore$ (1) + (2) $\Rightarrow a - b + b - c = nk_1 + nk_2$

$\Rightarrow a - c = n(k_1 + k_2),\ k_1 + k_2 \in \mathrm{I}$

$\Rightarrow a \equiv c \pmod n$

$\Rightarrow a \text{ R } c$, so R is transitive

Hence it is an equivalence relation.

Example 37. Let A = Z, the set of integers relation R defined in A by a R b as "a is congruent to b mod 2". Prove that R is an equivalence relation.

[RTU 2010, Raj. 2003, 2001]

Solution: The relation R is defined on the set of integers A by a R b as $a - b$ is divisible by 2 (which is a is congruent to b mod 2).

(i) Reflexive : Let $a \in$ A, then

$$a - a = 0 = 0 \times 2 \quad \Rightarrow (a - a) \text{ is divisible by } 2$$

$$\Rightarrow a \text{ R } a, \ \forall \ a \in \text{A}$$

$\therefore$ R is reflexive,.

(ii) Symmetric : Let a R b, then

$$a \text{ R } b \Rightarrow a \equiv b \pmod{n} \quad \Rightarrow a - b \text{ is divisible by } 2$$

$$\Rightarrow a - b = 2p, \ p \in \text{Z}$$

$$\Rightarrow b - a = 2(-p), \ -p \in \text{Z}$$

$$\Rightarrow b \equiv a \pmod 2$$

$$\Rightarrow b \text{ R } a$$

$\therefore$ R is symmetric.

(iii) Transitive : Let a R b and b R c, then

$$a \text{ R } b \Rightarrow a - b = 2p, \ p \in \text{Z} \quad \text{.....(1)}$$

$$\text{and } b \text{ R } c \Rightarrow b - c = 2q, \ q \in \text{Z} \quad \text{.....(2)}$$

$$\therefore (1) + (2) \Rightarrow a - b + b - c = 2(p + q)$$

$$\Rightarrow a - c = 2(p + q), \ p + q \in \text{Z}$$

$$\Rightarrow a \equiv c \pmod 2$$

$$\Rightarrow a \text{ R } c$$

$\therefore$ R is transitive

Hence it is an equivalence relation.

2.22 Equivalence Classes

If X be a set, R be an equivalence relation on X and $x \in$ X, then the set of all those members $y \in$ X, for which x R y, is called equivalence class of x and is denoted by

$$[x] = \{y \mid x, y \in \text{X}, x \text{ R } y\} = [y]$$

$$y \in [x] \Leftrightarrow x \text{ R } y$$

Thus, an equivalence class is a subset of the given set, such that any two elements of it are equivalent to each other.

Remark 10. Some authors write $\frac{x}{R}$ for the equivalence class x with respect to R. The set of all equivalence classes will be denoted by $\frac{X}{R}$ and read as "X modulo R" or "X mod R". This is called the quotient set of X by R. Thus, the set of all disjoint equivalence classes defined by an equivalence relation R over a set S, is called the quotient set of S relative to R.

Example 38. Let X be the set of all triangles in a plane, and the equivalence relation R is for similarity. If $x, y \in X$, then by $[x]$ we mean the class or set of all those triangles which are similar to the triangle x and by $[y]$ we mean the class or set of all those triangle which are similar to the triangle y.

All the equivalence classes form a subset of P(X), the assertion can be proved by noting that a subset A of P(X) is an equivalence class if and only if for any $x \in A$, $[x] = A$, which is the quotient set of x modulo.

2.23 Properties of Equivalence Classes

Theorem 4 : Let A be a non–empty set and let R be an equivalence relation in A. Let x and y be arbitrary elements in A.

Then

(i) $x \in [x]$

(ii) If $y \in [x]$, then $[y] = [x]$,

(iii) $[x] = [y] \Leftrightarrow (x, y) \in R$ i.e., iff $x \text{ R } y$,

(iv) Either $[x] = [y]$ or $[x] \cap [y] = \phi$ i.e., two equivalence classes are either disjoint or identical.

Proof.

(i) Since R is reflexive, $x \text{ R } x, \forall x \in x$

But $[x] = \{a \mid a \in A \text{ and } a \text{ R } x\}$

Thus, $x \text{ R } x \Rightarrow x \in [x]$.

(ii) $y \in [x] \Rightarrow y \text{ R } x, \forall x, y \in A$

Let a be any arbitrary element of $[y]$, then $a \in [y] \Rightarrow a \text{ R } y$.

But R is transitive, hence $a \text{ R } y$ and $y \text{ R } x$

$\Rightarrow a \text{ R } x \Rightarrow a \in [x]$

Thus, $a \in [y] \Rightarrow a \in [x]$

$\therefore [y] \subseteq [x]$ .....(1)

Again let b be any arbitrary element of $[x]$, then

$b \in [x] \Rightarrow b \text{ R } x$.

Since R is symmetric, $y \text{ R } x \Rightarrow x \text{ R } y$

Now $b \text{ R } x$ and $x \text{ R } y \Rightarrow b \text{ R } y$ [$\because$ R is transitive]

$\Rightarrow b \in [y]$

Thus, $b \in [x] \Rightarrow b \in [y]$

$\therefore [x] \subseteq [y]$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow [x] = [y]$.

(iii) Let $[x] = [y]$, then we have to prove $x \text{ R } y$ first.

As R is reflexive, $x \text{ R } x$ and therefore

$x \text{ R } x \Rightarrow x \in [x]$

$\Rightarrow x \in [y]$ [$\because [x] = [y]$]

$\Rightarrow x \text{ R } y$

Thus, $[x] = [y] \Rightarrow x \text{ R } y$.

Conversely : Suppose $x \text{ R } y$, then we have to prove $[x] = [y]$.

Let a be any arbitrary element of $[x]$, then $a \text{ R } x$ but it is given that $x \text{ R } y$, therefore,

$a \text{ R } x$ and $x \text{ R } y \Rightarrow a \text{ R } y$ [$\because$ R is transitive]

$\Rightarrow a \in [y]$

Hence $a \in [x] \Rightarrow a \in [y]$ i.e., $[x] \subseteq [y]$ .....(1)

Again let b be any arbitrary element of $[y]$, then

$b \in [y] \Rightarrow b \text{ R } y$

Since R is symmetric, $x \text{ R } y \Rightarrow y \text{ R } x$

$\therefore$ $b \text{ R } y$ and $y \text{ R } x \Rightarrow b \text{ R } x$ [$\because$ R is transitive]

$\Rightarrow b \in [x]$

$\therefore$ $b \in [y] \Rightarrow b \in [x]$ i.e., $[y] \subseteq [x]$ .....(2)

Hence (1) and (2) $\Rightarrow [x] = [y]$

Finally $[x] = [y] \Rightarrow x \text{ R } y$ and $x \text{ R } y \Rightarrow [x] = [y]$

$\therefore$ $[x] = [y]$ iff $x \text{ R } y$.

(iv) If $[x] \cap [y] = \phi$ then the result is obvious. Hence let us take $[x] \cap [y]$ non–empty and we have to prove $[x] = [y]$.

$[x] \cap [y] \neq \phi \Rightarrow \exists\, x \in A$ s.t. $x \in [x] \cap [y]$.

Now $a \in [x] \cap [y] \Rightarrow a \in [x]$ and $a \in [y]$

$\Rightarrow a \text{ R } x$ and $a \text{ R } y$

$\Rightarrow x \text{ R } a$ and $a \text{ R } y$ [as R is symmetric, $a \text{ R } x \Rightarrow x \text{ R } a$]

$\Rightarrow x \text{ R } y$ [$\because$ R is transitive]

$\Rightarrow [x] = [y]$ [by (3)]

Thus $[x] \cap [y] \neq \phi \Rightarrow [x] = [y]$

or $[x] \neq [y] \Rightarrow [x] \cap [y] = \phi$.

2.24 Partition of a Set

Let X be a non–void set. A set P = {A, B, C, ...,} of non–void subsets of X is called a partition of X if

1. $X = A \cup B \cup C \cup$ ...., i.e., the set X is the union of the sets in P, and
2. the intersection of every pair of distinct subsets of $X \in P$ is the null set i.e., if A and $B \in P$ then either $A = B$ or $A \cap B = \phi$.

Example 39. Consider the set X = {1, 2, 3, ..., 9, 10} and its subsets

$B_1 = \{1, 3\}, B_2 = \{7, 8, 10\}, B_3 = \{2, 5, 6\}, B_4 = \{4, 9\}$

The set $P = \{B_1, B_2, B_3, B_4\}$ is such that

(i) B_1, B_2, B_3, B_4 are all non–void subsets of X,

(ii) $B_1 \cup B_2 \cup B_3 \cup B_4 = X$

(iii) For any disjoint sets $B_i, B_j, B_i \cap B_j = \phi$

Hence the set $\{B_1, B_2, B_3, B_4\}$ is a partition of X.

Theorem 5 : An equivalence relation defined in set S decomposes the set into disjoint classes.

Proof. Let an equivalence relation R be defined in a set S. Let $a \in S$ and T be a subset of S consisting of all those elements which are equivalent to a i.e.,

$$T = \{x \mid x \in S \text{ and } a \, R \, x\}.$$

Then $a \in T$, for $a \, R \, a$ (R is reflexive). Any two elements of T are equivalent to each other, for if $x, y \in T$, then $x \, R \, a$ and $y \, R \, a$.

Again $x \, R \, a, y \, R \, a \Rightarrow x \, R \, a, a \, R \, y$ [R is symmetric]

$\Rightarrow x \, R \, y$

Thus T is an equivalence class.

Let T_1 be another equivalence class i.e., $T_1 = \{x \mid x \in S \text{ and } x \, R \, b\}$,

where b is not equivalent to a. Then the classes T and T_1 must be disjoint. For if they have a common element s, $s \, R \, a$ and $s \, R \, b$, so that $b \, R \, a$ which is contrary to our hypotheses.

The set S can now be decomposed into equivalence classes T, T_1, T_2, ... such that every element of S belongs to one of these classes. Since these classes are mutually disjoint, we obtain the required partition of S.

Theorem 6 : An equivalence relation R in a non–empty set S determines a partition of S.

Proof. Let R be an equivalence relation in S. Let A be the set of equivalence classes of S w.r.t. R i.e. $A = \{[a] : a \in S\}$ where $[a] = \{x : x \in S \text{ and } a \, R \, x\}$.

Now, R is an equivalence relation. Therefore $\forall \, a \in S$, we have $a \, R \, a$ (by reflexivity). Hence $a \in [a] \Rightarrow [a] \neq \phi$.

Further every element 'a' of S is an element of the equivalence class $[a]$ in A. From this we conclude that $S = \bigcup_{a \in S} [a]$.

Finally, if $[a]$ and $[b]$ are two equivalence classes then either $[a] = [b]$ or $[a] \cap [b] = \phi$.

Hence, A is a partition of S.

Example 40. Let A = {1, 2, 3, 4, 5, 6} and let R be the equivalence relation on A defined by

R = {(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6),
(4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)}

Find the partition of A induced by R i.e., find the equivalence classes. **[Raj. 2003]**

Solution: The equivalence class [1] is the set of elements of A related to 1 and these are 1 and 5, so

$$[1] = \{1, 5\}, \text{ similarly other classes are}$$
$$[2] = \{2, 3, 6\}$$
$$[3] = [2] = \{2, 3, 6\}$$
$$[4] = \{4\}$$
$$[5] = [1]$$
$$[6] = [2]$$

Hence the partition of A induced by R is

$$\{[1], [2], [4]\} = \{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$$

Example 41. Consider A = {1, 2, 3, 4} and let R be an equivalence relation

R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)}. Find $\frac{A}{R}$.

Solution. The equivalence classes are

$$[1] = \{1, 2\}$$
$$[3] = \{3, 4\}$$

So $$\frac{A}{R} = \{\{1, 2\}, \{3, 4\}\}$$

Quotient set of A by R : The partition P of A obtained by an equivalence relation R on A is called quotient set of A and this partition is represented by $\frac{A}{R} = \{[a] \mid a \in A\}$.

2.25 Composition of Relations

Let R be a relation from A to B and S be a relation from B to C then R and S give rise to a relation SoR from A to C, called composition relation.

Thus, SoR = {(*a*, *c*) | (*a*, *b*) ∈ R and (*b*, *c*) ∈ S }

Example: Let A = {1, 2, 3, 4} and R = {(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)} & S = {(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)} are relations on A.

Then, SoR = {(1, 3), (1, 4), (1, 1), (2, 1), (3, 3)}.

2.26 Number of Partitions of a Finite Set

Suppose X be a finite set of size *n*. Then the number of partitions of X (or the number of equivalence relations on X) is given by $\sum_{r=1}^{n} S(n, r)$, where S(*n*, *r*) denotes a sterling number of the second kind, defined as

$$S(n, 1) = 1 = S(n, n)$$

and $S(n, r) = S(n-1, r-1) + r\, S(n-1, r)$,

where $1 < r < n$.

Example 42. Compute the number of partitions of a set with

(i) Four elements (ii) Five elements

Solution:

(i) Here $n = 4$

Thus the required number of partitions $= \sum_{r=1}^{4} S(4, r)$

$$= S(4, 1) + S(4, 2) + S(4, 3) + S(4, 4) \quad .....(1)$$

Now $S(4, 1) = 1 = S(4, 4)$ .....(2)

$S(4, 2) = S(3, 1) + 2S(3, 2)$ .....(3)

$S(4, 3) = S(3, 2) + 3S(3, 3)$ .....(4)

Since $S(3, 2) = S(2, 1) + 2S(2, 2)$ .....(5)

Now, $S(2, 1) = 1 = S(2, 2)$ [$\because S(n, 1) = 1 = S(n, n)$]

Thus equation (5) gives

$S(3, 2) = 1 + 2(1) = 3$ .....(6)

$\therefore$ (3) $\Rightarrow$ $S(4, 2) = 1 + 2(3)$ [using (6) and $S(3, 1) = 1$]

$= 7$

(4) $\Rightarrow$ $S(4, 3) = 3 + 3(1) = 6$

Hence the required number of partitions

$= 1 + 7 + 6 + 1$

$= 15$

(ii) Here $n = 5$

Thus the required number of partitions

$$= \sum_{r=1}^{5} S(5, r)$$

$$= S(5, 1) + S(5, 2) + S(5, 3) + S(5, 4) + S(5, 5) \quad .....(1)$$

Now, $S(5, 1) = 1 = S(5, 5)$

Further, $S(5, 2) = S(4, 1) + 2S(4, 2)$

$= 1 + 2(7)$ [from part (i)]

$= 15$

$S(5, 3) = S(4, 2) + 3S(4, 3)$

$= 7 + 3(6)$ [from part (i)]

$= 25$

$S(5, 4) = S(4, 3) + 4S(4, 4)$

$= 6 + 4(1)$ [from part (i)]

$= 10$

Hence from (1), the required number of partitions

$= 1 + 15 + 25 + 10 + 1 = 52.$

ILLUSTRATIVE EXAMPLES

Example 43. Prove that the relations R_1 and R_2 in the set N × N, where N is the set of natural numbers, defined as follows :

(i) $(a, b)\ R_1\ (c, d) \Leftrightarrow a + d = b + c,$

(ii) $(a, b)\ R_2\ (c, d) \Leftrightarrow a{\cdot}d = b{\cdot}c$

[Raj. 2006, 2003]

are equivalence relations for $a, b, c, d \in N$.

Solution: (i) Reflexive : Let $(a, b) \in N \times N$, then $(a, b) \in N \times N \Rightarrow a, b \in N$

$\Rightarrow a + b = b + a$

$\Rightarrow (a, b)\ R_1\ (a, b)\ \forall\ (a, b) \in N \times N.$

$\therefore$ R_1 is reflexive

Symmetric : Let $(a, b)\ R_1\ (c, d)$, then

$(a, b)\ R_1\ (c, d) \Rightarrow a + d = b + c$

$\Rightarrow b + c = a + d$

$\Rightarrow c + b = d + a$

$\Rightarrow (c, d)\ R_1\ (a, b)$

$\therefore$ R_1 is symmetric.

Transitive : Let $(a, b)\ R_1\ (c, d)$ and $(c, d)\ R_1\ (e, f)$, then

$(a, b)\ R_1\ (c, d) \Rightarrow a + d = b + c$ .....(1)

and $(c, d)\ R_1\ (e, f) \Rightarrow c + f = d + e$ .....(2)

$\therefore$ (1) + (2) $\Rightarrow (a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow a + f = b + e$

$\Rightarrow (a, b)\ R_1\ (e, f)$

$\therefore$ R_1 is transitive.

Hence R_1 is an equivalence relation.

(ii) Reflexive : Let $(a, b) \in N \times N$, then

$(a, b) \in N \times N \Rightarrow a, b \in N \Rightarrow ab \in N$

$\Rightarrow ab = ba$

$\Rightarrow (a, b)\ R_2\ (a, b)$

$\therefore$ R_2 is reflexive

Symmetric : Let $(a, b)\ R_2\ (c, d)$, then

$(a, b)\ R_2\ (c, d) \Rightarrow ad = bc$

$\Rightarrow bc = ad$

$\Rightarrow cb = da$

$\Rightarrow (c, d)\ R_2\ (a, b)$

$\therefore$ R_2 is symmetric

Transitive : Let $(a, b)\ R_2\ (c, d)$ and $(c, d)\ R_2\ (e, f)$, then

$(a, b)\ R_2\ (c, d) \Rightarrow ad = bc$ .....(1)

and $(c, d)\ R_2\ (e, f) \Rightarrow cf = de$ .....(2)

$\therefore$ $(1) \times (2) \Rightarrow (ad)\ (cf) = (bc)\ (de)$

$\Rightarrow af = be$

$\Rightarrow (a, b)\ R_2\ (e, f)$

$\therefore$ R_2 is transitive

Hence R_2 is an equivalence relation.

Example 44. Let A = {1, 2, 3, 4} and consider the partition P = {{1, 2, 3}, {4}} of A. Obtain the equivalence relation R on A determined by P. **[RTU 2009, Raj. 2001]**

Solution: Let P = {{1, 2, 3}, {4}} be a partition A/R of the set A induced by the relation R, so the sets {1, 2, 3} and {4} are equivalence classes. In each class every element is related to every other element in the same class and only to these elements. Thus the elements of R related to block {1, 2, 3} are (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) and the elements R related to block {4} is (4, 4).

$\therefore$ R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)}.

Example 45. In the set of integers Z = A, a relation R is defined by a R b as $a \equiv b \pmod 4$, show that R is an equivalence relation. Also determine A/R.

Solution: Here $a\ R\ b \Rightarrow a \equiv b \pmod 4 \Rightarrow a - b$ is divisible by 4.

(i) Reflexive Let $a \in$ A, then

$a - a = 0 = 0 \times 4 \Rightarrow (a - a)$ is divisible by 4

$\Rightarrow a \equiv b \pmod 4$

$\Rightarrow a\ R\ a, \forall a \in A$

$\therefore$ R is reflexive

(ii) Symmetric Let a R b, then

$a\ R\ b \Rightarrow a \equiv b \pmod 4 \Rightarrow a - b$ is divisible by 4

$\Rightarrow a - b = 4 \times p \quad [p \in Z]$

$\Rightarrow b - a = 4 \times (-p)\ [-p \in Z]$

$\Rightarrow b - a$ is divisible by 4

$\Rightarrow b \equiv a \pmod 4$

$\Rightarrow b\ R\ a$

$\therefore$ R is symmetric

(iii) Transitive Let a R b and b R c, then

a R b ⇒ $(a - b)$ is divisible by 4

$\Rightarrow a - b = 4p, \quad p \in Z$ .....(1)

and b R c ⇒ $(b - c)$ is divisible by 4

$\Rightarrow b - c = 4q, \quad q \in Z$ .....(2)

∴ (1) + (2) ⇒ $(a - b) + (b - c) = 4p + 4q$

$\Rightarrow a - c = 4(p + q), p + q \in Z$

⇒ $(a - c)$ is divisible by 4

$\Rightarrow a \equiv c \pmod 4$

⇒ a R c

∴ R is transitive

So R is an equivalence relation.

Now we construct the equivalence classes of A = Z = {...., – 2, – 1, 0, 1, 2, ...} as follows:

$$[0] = [-8, \ -4, 0, 4, 8, ...]$$
$$[1] = [-7, -3, 1, 5, 9, ...]$$
$$[2] = [-6, -2, 2, 6, 10, ...]$$
$$[3] = [-5, -1, 3, 7, 11, ...]$$
$$[4] = [-8, -4, 0, 4, 8, ...] = [0]$$

Hence $\dfrac{A}{R} = \{\{0\}, \{1\}, \{2\}, \{3\}\}.$

Example 46. In the set of integers Z = A, a relation R is defined by a R b as $a \equiv b \pmod 3$, show that R is an equivalence relation. Also determine the equivalence classes. **[IT(RTU)-2008]**

Solution: Given that $x \, Ry \Leftrightarrow x{-}y$ is divisible by 3; $x, y \in N$.

Reflexive: Since $x{-}x = 0, \ \forall \ x \in N$

and 0 is divisible by 3

$\Rightarrow x \, R \, x, \ \forall \ x \in$ N

∴ R is reflexive.

Symmetric : Let $x \, R \, y \Rightarrow x{-}y$ is divisible by 3

⇒ $-(x{-}y)$ is also divisible by 3

⇒ $y{-}x$ is divisible by 3

$\Rightarrow y \, R \, x$

∴ R is symmetric.

Transitive : Let $x \, R \, y$ and $y \, R \, z$ then

$x{-}y = 3n_1$ .....(i)

and $y{-}z = 3n_2$ .....(ii)

$n_1, n_2 \in$ Z

(i) + (ii) $\Rightarrow x - z = 3(n_1 + n_2) = 3n, \ n \in z$

$\Rightarrow (x - z)$ is divisible by 3

$\Rightarrow x\ R\ z$

$\therefore$ R is transitive.

Hence R is an equivalence relation.

Further $N = \{1, 2, 3, 4, ...\}$

Now $[1] = \{1, 4, 7, 10, ...\}$

$[2] = \{2, 5, 8, 11, ...\}$

$[3] = \{3, 6, 9, 12, ...\}$

$[4] = \{1, 4, 7, 10, ...\} = [1]$

So N has only three distinct equivalence classes namely,

$[1], [2]$ and $[3]$,

such that $[1] \cup [2] \cup [3] = N$.

Example 47. Prove that the relation R_1 defined by $a\ R\ b$ iff $ab > +1$ is not an equivalence relation, on the set of real numbers.

Solution: For any a, relation R_1 can be reflexive if $a \cdot a > +1$ is true but this is not true for any $a \in R$. In view of this the relation is not reflexive. Hence the given relation R_1 is not an equivalence relation.

Example 48. If $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3)\}$ be an equivalence relation in $A = \{1, 2, 3, 4\}$, obtain the partitions of A.

Solution: We have $(1, 2) \in R \Rightarrow 1$ and 2 should be in same partition.

Similarly as $(3, 4) \in R \Rightarrow 3$ and 4 belongs to another partition as $(2, 3) \notin R$ so 2 and 3 must be in different partitions. So the partitions are

$$P_1 = \{1, 2\}, P_2 = \{3, 4\}$$

Set of all partitions of A by the relation R is defined by $\frac{A}{R}$.

Hence $\frac{A}{R} = \{\{1, 2\}, \{3, 4\}\}$

Example 49. If the set of integers I(or Z) $= \{......, -3, -2, -1, 0, 1, 2, 3, ...\}$ be partitioned by the equivalence relation $a\ R\ b$ as $a \equiv b \pmod 3$. Obtain the set $\frac{I}{R}$.

Solution: Since difference of any two elements in

$$P_1 = \{... -6, -3, 0, 3, 6, 9, ...\} = [0]$$

is a multiple of -3, P_1 is an equivalence class in I.

Next, $P_2 = \{... -5, -2, 1, 4, 7, ...\} = [1]$

is a set where difference of any two elements is a multiple of 3. So all these elements are in relation. So P_2 is another equivalence class of I(or Z).

and in $P_3 = \{... -4, -1, 2, 5, 8, ...\} = [2]$

The difference of any two elements is a multiple of 3 so all the elements of P_3 are in relation R. Hence P_3 is an equivalence class, we have

$$P_1 \cup P_2 \cup P_3 = I$$

and $P_1 \cap P_2 = \phi$, $P_2 \cap P_3 = \phi$, $P_3 \cap P_1 = \phi$

so P_1, P_2, P_3 are partitions of I induced by the relation R.

Hence the set $\frac{I}{R} = \{P_1, P_2, P_3\}, = \{[0], [1], [2]\}$.

Example 50. Let $A = \{1, 2, 3, 4, 5\}$ and $P_1 = \{1, 2\}$, $P_2 = \{3\}$, $P_3 = \{4, 5\}$ be three partitions of A. Obtain an equivalence relation R pertaining to the partition.

Solution: Let R be an equivalence relation on R. Now all elements of P_1 are in relation R.

$\Rightarrow \{(1, 1), (2, 2), (1, 2), (2, 1)\} \subseteq R$

All elements of P_2 and P_3 are to be in R, so we must have

$\{(3, 3)\} \subseteq R$

and $\{(4, 4), (5, 5), (4, 5), (5, 4)\} \subseteq R$

Thus $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (4, 5), (5, 4), (5, 5)\}$.

Example 51. Suppose that R and S are reflexive relations on a set A. Prove or disprove each of the following statements.

(i) $R \cup S$ is reflexive **[Raj. 2005, 2004]**

(ii) $R \cap S$ is reflexive **[Raj. 2005, 2004]**

(iii) $R \oplus S$ is irreflexive **[Raj. 2005, 2004]**

(iv) $R - S$ is irreflexive **[Raj. 2005, 2004]**

(v) S o R is reflexive (composition of S & R) **[Raj. 2005]**

Solution: Given R and S are reflexive, i.e.,

$(a, a) \in R, \forall a \in A$

and $(a, a) \in S, \forall a \in A$.

(i) $R \cup S$ is reflexive

Since, $(a, a) \in R \Rightarrow (a, a) \in R \cup S, \forall a \in A$

$\therefore$ $R \cup S$ is reflexive.

(ii) $R \cap S$ is reflexive

Since, $(a, a) \in R$ and $(a, a) \in S, \forall a \in A$

$\Rightarrow (a, a) \in R \cap S, \forall a \in A$

$\therefore$ $R \cap S$ is reflexive

(iii) $R \oplus S$ is irreflexive

Since, $R \oplus S = (R - S) \cup (S - R)$

Now, $(a, a) \in R$ and $(a, a) \in S, \forall a \in A$

$\Rightarrow (a, a) \notin R - S$ and $(a, a) \notin S - R$

$\Rightarrow (a, a) \notin (R - S) \cup (S - R), \forall a \in A$

$\Rightarrow (a, a) \notin R \oplus S, \forall a \in A$

$\therefore$ $R \oplus S$ is irreflexive

(iv) R − S is irreflexive

Since, $(a, a) \in R$ and $(a, a) \in S, \forall a \in A$

$\Rightarrow (a, a) \notin R - S, \forall a \in A$

$\therefore$ R − S is irreflexive

(v) S o R is reflexive

Since, $R : A \to A$ and $S : A \to A$

so $SoR : A \to A$

Now, $$SoR(a) = S[R(a)], \forall a \in A$$
$$= S[a] = a, \forall a \in A$$

$\Rightarrow (a, a) \in SoR, \forall a \in A$

$\therefore$ S o R is reflexive

Example 52. Suppose that A is a non–empty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) where $f(x) = f(y)$. Show that R is an equivalence relation on A. **[Raj. 2004]**

Solution: Given $(x, y) \in R \Leftrightarrow f(x) = f(y)$.

Reflexive : Since $f(x) = f(x), \forall x \in A$

$\Rightarrow (x, x) \in R, \forall x \in A$

$\therefore$ R is reflexive.

Symmetric : Let $(x, y) \in R, \quad x, y \in A$

$\Rightarrow f(x) = f(y)$

$\Rightarrow f(y) = f(x)$

$\Rightarrow (y, x) \in R$

$\therefore$ R is symmetric

Transitive : Let $(x, y) \in R, (y, z) \in R$

$\Rightarrow f(x) = f(y)$ and $f(y) = f(z)$

$\Rightarrow f(x) = f(y) = f(z)$

$\Rightarrow f(x) = f(z)$

$\Rightarrow (x, z) \in R$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

Example 53. Show that the relation R, consisting of all pairs (x, y) where x and y are bits strings of length 3 or more that agree in their first three (3) bits, is an equivalence relation on the set of all bit strings of length three or more. **[Raj. 2004]**

Solution: Let A be the set of all bits strings of length 3 or more.

Given $(x, y) \in R \Leftrightarrow x$ and y agree in their first 3 bits.

Reflexive : Since it is obvious that x agree with x in first 3 bits

$\Rightarrow (x, x) \in R, \forall x \in A$

$\therefore$ R is reflexive

Symmetric : Let $(x, y) \in R$

$\Rightarrow$ x and y agree in their first 3 bits

$\Rightarrow$ y and x agree in their first 3 bits

$\Rightarrow$ $(y, x) \in R$

$\therefore$ R is symmetric.

Transitive : Let $(x, y) \in R$, $(y, z) \in R$

$\Rightarrow$ x and y agree in their first 3 bits and y and z agree in their first 3 bits

$\Rightarrow$ x, y and z agree in their first 3 bits

$\Rightarrow$ x and z agree in their first 3 bits

$\Rightarrow$ $(x, z) \in R$

$\therefore$ R is transitive

Hence R is an equivalence relation.7

EXERCISE 2.1

Q.1 Prove that if $A \subset B$ and $C \subset D$ then $A \times C \subset B \times D$.

Q.2 If A = {1, 2}, B = {2, 3}, C = {3, 7},

Find:

(i) $(A \times B) \cup (A \times C)$

(ii) $(A \times B) \cap (A \times C)$

Ans. (i) {(1, 2), (1, 3), (1, 7), (2, 2), (2, 3), (2, 7)}

(ii) {(1, 3), (2, 3)}

Q.3 Let A be the set of first ten natural numbers (from 1 to 10). Let a relation R on A be defined by $x \text{ R } y \Leftrightarrow x + 2y = 10$. Express:

(i) domains of R and R^{-1}

(ii) ranges of R and R^{-1}

Ans. R = {(2, 4), (4, 3), (6, 2), (8, 1)}, R^{-1} = {(4, 2), (3, 4), (2, 6), (1, 8)

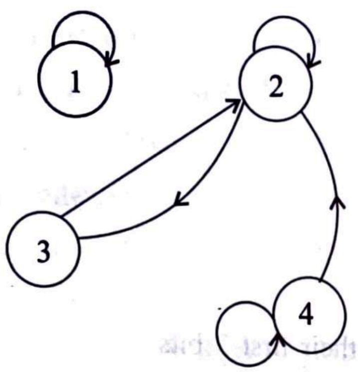
domain of R = range of R^{-1} = {2, 4, 6, 8} and range of R = domain of R^{-1} = {1, 2, 3, 4}].

Q.4 Define equivalence relation. Let A = {1, 2, 3}. Show that the relation R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)} is an equivalence relation. **[MREC 2002]**

Q.5 Prove that the relation a R b such that $a - b$ in an even integer defined on the set I (or Z) of integers is an equivalence relation.

Q.6 Draw the digraph and matrix M_R of the relation R defined on A = {1, 2, 3, 4} by R = {(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)}

Ans.



$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Q.7 Show that the relation of parallelism of two lines drawn in a plane is an equivalence relation.

Q.8 In the set of integers prove that "congruence modulo 5 is an equivalence relation.

Q.9 Obtain the distinct equivalence classes of the relation R "congruence modulo 5" in the set I.

Ans. $\frac{I}{R}$ = {Cl(0), Cl(1), Cl(2), Cl(3), Cl(4)}.

Q.10. Determine the equivalence relation R determined by the partitioning A = $P_1 \cup P_2$ where P_1 = {1, 2}, P_2 = {3, 4, 5}.

Ans. R = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (3, 4), (4, 3), (4, 5), (5, 4), (3, 5), (5, 3)}

Q.11 If A = {1, 2, 3, 4, 5} and R is an equivalence relation on A that induced the partition = {{1, 2}, {3, 4}, {5}}, find R.

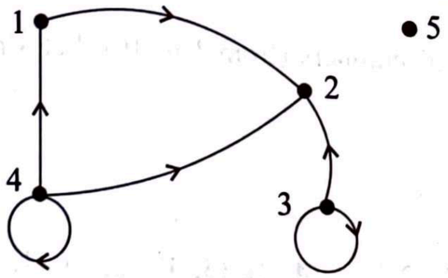
Ans. R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)

Q.12 If A ⊆ B, prove that A × C ⊆ B × C.

Q.13 Show that the identity relation I(x) = x, ∀ x ∈ A is a subset of every equivalence relation on A.

Q.14 If R and S be relations on A and R be anti–symmetric, prove that R ∩ S is also anti–symmetric.

Q.15 If R be a relation in the set A = {1, 2, 3, 4, 5} represented by digraph, find R as a set of ordered pairs



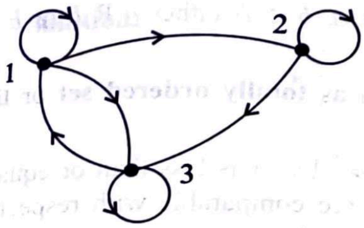
Ans. R = {(3, 3), (4, 4), (1, 2), (3, 2), (4, 2), (4, 1)}

Q.16 Represent the relation

R = {(1, 2), (2, 3), (3, 1), (1, 1), (2, 2), (3, 3), (1, 3)}

in the set A = {1, 2, 3} by (i) a digraph (ii) by adjacency matrix

Ans.



$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Q.17. Give an example of a relation which is

(a) transitive but not symmetric (b) symmetric but not transitive. **[Raj. 1998]**

Q.18. Let $B = B_1 \cup B_2 \cup \ldots\ldots \cup B_k$ be a partition of B into k–parts. Prove that a relation R defined on B by x R y iff $x, y \in B_i$ for some i, is an equivalence relation. **[Raj. 1997 , 1996]**

Q.19. Show that the relation R on a set A is transitive iff $R^n \subseteq R$, for $n = 1, 2, 3, \ldots$, where $R^n = R \cdot R \cdot R \ldots$ R (n–times).

2.27 Partial Order Relation

A relation R defined on a set A is said to be a **partial order relation** in A if

1. R is reflexive
2. R is anti–symmetric
3. R is transitive.

A set A together with a defined partial order relation R in A is known as **partially ordered set or poset** and is denoted by (A, R).

Example 54. The relation of "set inclusion" in a family of sets is a partial order relation, since this relation is reflexive, anti–symmetric and transitive.

Example 55. Let R be a relation in the set N of natural numbers defined by "x divides y". Then N is a partially ordered set with the relation R.

2.28 Comparability

The elements a and b of a poset $(A, \leq)$ are said to be comparable if either $a \leq b$ or $b \leq a$ and a and b are incomparable if neither $a \leq b$ nor $b \leq a$.

The word 'partial' is used to describe partial orderings since pair of elements may be incomparable.

Example 56. In the poset $(Z^+, |)$, the elements 3 and 6 are comparable, since $3|6$. Also, the elements 2 and 5 are incomparable since neither $2|5$ nor $5|2$.

2.29 Total Order Relation or Linear Order Relation

A relation R defined on a set A, is said to be a **total order relation** in A if

1. R is a partial order relation in A and

2. every pair of elements of A is comparable with respect to R i.e., $\forall\ a, b \in$ A either a R b or b R a or $a = b$.

A set A together with a defined total order relation R in A, is known as **totally ordered set** or **linearly ordered set.**

Example 57. Let R be a relation in the set N of natural numbers defined by "x is less than or equal to y". Then R is a total order relation in N; since any two natural numbers are comparable with respect to this relation i.e., for any two numbers $a, b \in$ N either $a < b$ or $b < a$ or $a = b$.

Example 58. Let R be a relation in the set A = {1, 2, 3, 4, 5, 6} defined by "x divides y". Then R is not a total order relation in A; since 2 and 3 are not comparable.

2.30 Digraph of a POSET

The digraph of partial order relation on a finite set can be denoted in a simpler manner than these of general relations.

We have the following observations.

1. The partial order relation is reflexive so every vertex of the poset has a cycle of length one.
2. The partial order is anti–symmetric so if there is an edge from a to b then there is no edge from b to a, if $b \neq a$.
3. If there is an edge from a to b and also an edge from b to c then by transitivity there will be an edge from a to c.
4. In view of (b) and (c) we can say that the digraph of a partial order relation has no cycle of length more than one.

2.31 Hasse Diagram

A partial ordering $\leq$ on a set P can be represented by means of a diagram known as a **Hasse diagram** or a partially ordered set diagram of (P, $\leq$).

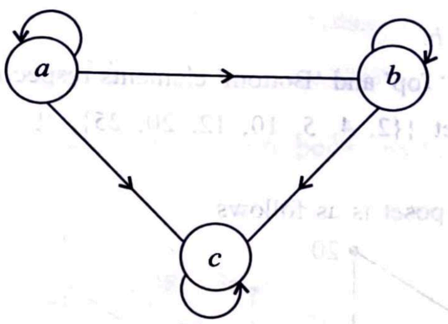
In the digraph of the relation of partial order, we go through the following in order to make it simple.

1. Delete all loops from the vertices since in partial order relation loop at every vertex is obvious.
2. Delete all edges that must be present because of the transitivity.
3. Draw the digraph of all partial order with all edges pointing upward in order to remove arrows from the edges.
4. Denote vertices by dots in place of circles.

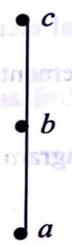
Hence the digraph so obtained is known as Hasse–diagram of the partial order relation.

Example 59. Draw the general diagram and Hasse diagram of the poset (A, R), where A = $\{a, b, c\}$ and R = $\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$

Solution:



The general digraph



Hasse digraph

In a poset some elements are of special importance for many of the properties and applications which will be used throughout our discussion, some of them are defined as given below

(i) Upper Bound

Let (A, $\leq$) be a poset and a, $b \in$ A. Then an element $c \in$ A is called **upper bound** of a, b if $a \leq c$ and $b \leq c$.

(ii) Least Upper Bound (LUB)

An element $l \in$ A is called **least upper bound** of a and b in A iff.

(I) $a \leq l$ and $b \leq l$, i.e., l is the upper bound of a and b,

(II) If there exists an element $l' \in$ A s.t. $a \leq l'$ and $b \leq l'$ then $l \leq l'$, i.e., if l is another upper bound of a and b then l' is also the upper bound of l.

(iii) Lower Bound

Let (A, $\leq$) be a poset and a, $b \in$ A. Then an element $d \in$ A is called **lower bound** of a, b if $d \leq a$ and $d \leq b$.

(iv) Greatest Lower Bound (GLB) or Infimum

An element $g \in$ A is called **greatest lower bound** of a and b in A iff

(I) $g \leq a$ and $g \leq b$, i.e., g is the lower bound of a and b,

(II) If there exists an element $g' \in$ A s.t. $g' \leq a$ and $g' \leq b$ then $g' \leq g$, i.e., if g' is another lower bound of a and b then g' is also the lower bound of g.

(v) Maximal and Minimal elements

Maximal Element : An element a in the poset (A, $\leq$) is said to be maximal if there is no element $b \in$ A such that $a < b$.

Minimal Element : An element a in the poset (A, $\leq$) is said to be minimal if there is no element $b \in$ A such that $b < a$.

(vi) Greatest and Least element

Greatest element : An element a of the poset (A, $\leq$) is said to be greatest if $b \leq a$, $\forall$ $b \in$ A. The greatest element is unique, when it exists.

Least element : An element a of the poset (A, $\leq$) is said to be least if $a \leq b$, $\forall$ $b \in$ A. The least element is unique, when it exists.

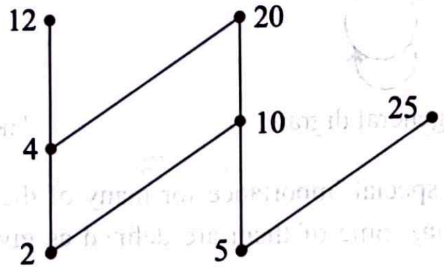
Remark 11.

(i) Here $a < b$ denotes that $a \leq b$ but $a \neq b$.

(ii) Maximal and minimal elements are the 'Top' and 'Bottom' elements respectively, in the Hasse diagram.

Example 60. Which elements of the poset $\{\{2, 4, 5, 10, 12, 20, 25\}, /\}$ are maximal and which are minimal? **[Raj. 2004, 2003]**

Solution: The Hasse diagram of the given poset is as follows



$\therefore$ Maximal elements are $\rightarrow$ 12, 20, 25 (Top elements) and minimal elements are $\rightarrow$ 2, 5 (Bottom elements).

Thus, maximal and minimal elements may not be unique.

Remark 12.

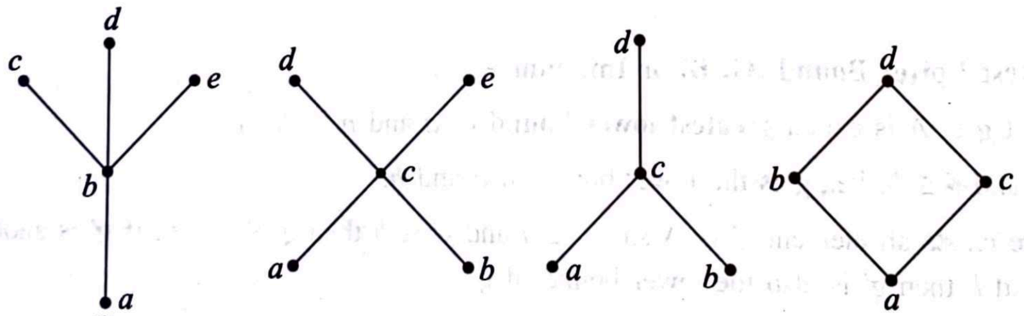
(i) If in the poset (A, $\leq$), maximal element is unique then it is also the greatest element of the poset.

(ii) If in the poset (A, $\leq$), minimal element is unique then it is also the least element of the poset (A, $\leq$).

(iii) If maximal element is not unique then the greatest element does into exists.

(iv) If minimal element is not unique then the least element does not exists.

Example 61. Find the greatest and least element in the following Hasse diagrams –



(i) (ii) (iii) (iv)

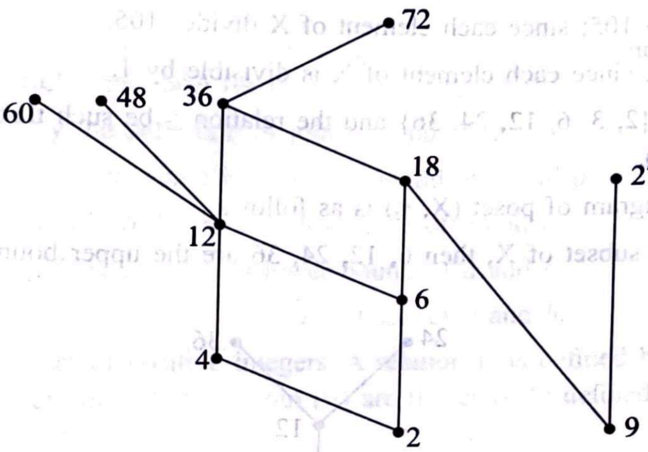
Solution:

Fig.	(i)	(ii)	(iii)	(iv)
Greatest element	does not exists	does not exists	d	d
Least element	a	does not exists	does not exists	a

Example 62. Answer the following questions concerning the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

(a) Find the maximal and minimal elements.
(b) Find the greatest and least element, if exists.
(c) Find lub of {2, 9}, if exists.
(d) Find glb of {60, 72}, if it exists. **[Raj. 2007, 2003]**

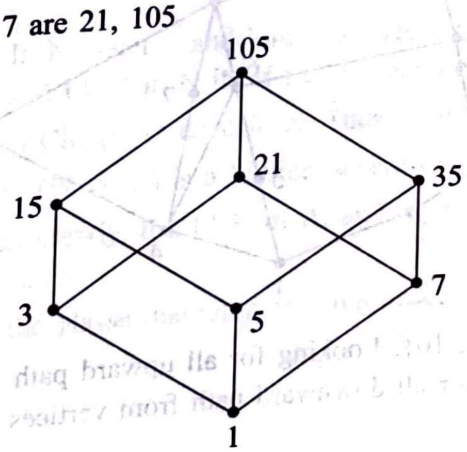
Solution: The Hasse diagram of the given poset can be constructed as follows



(i) Maximal elements (top elements) are 27, 48, 60 and 72.
Minimal elements (bottom elements) are 2 and 9.
(ii) Since maximal and minimal elements are not unique so greatest and least element does not exists.
(iii) Upper bounds of {2, 9} are 18, 36 and 72 and the least upper bound is 18.
(iv) Lower bounds of {60, 72} are 12, 6, 4 and 2 and the greatest lower bound is 12.

Example 63. Let X = {1, 3, 5, 7, 15, 21, 35, 105} and R be the relation '|' (divides) on the set X then X is the poset. Determine the following
(i) LUB of 3 and 7
(ii) GLB of 15 and 35
(iii) Greatest and least element of X.

Solution: The Hasse diagram of the given poset is as follows
(i) From the diagram, upper bounds of 3 are 3, 15, 21, 105 and upper bounds of 7 are 7, 21, 35, 105
∴ upper bounds of 3 and 7 are 21, 105



Since 21|105 (21 divides 105) $\Rightarrow$ 21 is LUB of 3 and 7.

(ii) From the diagram, lower bounds of 15 are, 1, 3, 5, 15

and lower bounds of 35 are 1, 5, 7, 35

$\therefore$ lower bounds of 15 and 35 are 1 and 5

Since 1|5 $\Rightarrow$ 5 is GLB of 15 and 35.

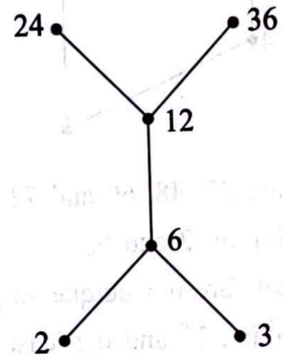
(iii) Greatest element $\rightarrow$ 105; since each element of X divides 105.

Least element $\rightarrow$ 1; since each element of X is divisible by 1.

Example 64. Let X = {2, 3, 6, 12, 24, 36} and the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the Hasse digram of (X, $\leq$).

Solution: The Hasse digram of poset (X, $\leq$) is as follows

Let Y = {2, 3 6} be a subset of X, then 6, 12, 24, 36 are the upper bounds of Y and then 6 is the least upper bound.

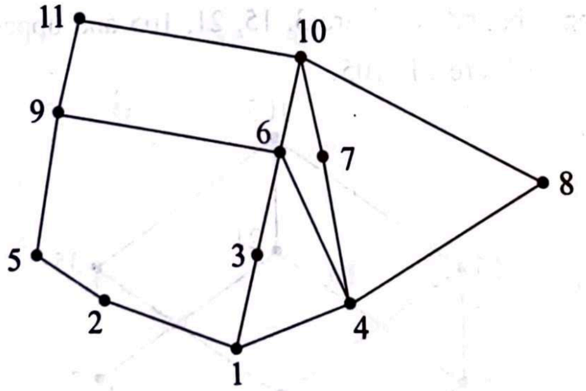


There is no lower bound of Y. consequently has no greatest lower bound.

If we take C = {6, 12}, a subset of X then 12, 24 are the upper bounds of C and least upper bound is 12 also, 2, 3 and 6 are lower bounds of C and 6 is the greatest lower bound.

Example 65. Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} be the poset whose Hasse diagram is given and Let B = {6, 7, 10} $\subset$ A. Find LUB and GLB of B, provided they exists.

Solution:



Considering the set B = {6, 7, 10}. Looking for all upward path form vertices 6, 7,10 we find that LUB of B = 10. Likewise looking for all downward path from vertices 6, 7, 10, we have GLB of B = 4.

2.32 Lattices

A lattice is a partially ordered set (L, ≤) in which every pair of elements $\{a, b\} \in L$ has a greatest lower bound and a least upper bound.

Lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

LUB of $\{a, b\} \in L$ is denoted by $a \vee b$ and is known as the join of a and b and GLB of $\{a, b\} \in L$ is denoted by $a \wedge b$ and is called the meet (or product) of a and b.

Remark 11. It is obvious by the definition of join (∨) and meet (∧) :

(i) $a \leq a \vee b$ and $b \leq a \vee b \Rightarrow a \vee b$ is an upper bound of a and b.

(ii) If $a \leq c$ and $b \leq c$, then $a \vee b \leq c \Rightarrow a \vee b$ is LUB of a and b.

(iii) $a \wedge b \leq a$ and $a \wedge b \leq b \Rightarrow a \wedge b$ is a lower bound of a and b.

(iv) If $c \leq a$ and $c \leq b$, then $c \leq a \wedge b \Rightarrow a \wedge b$ is GLB of a and b.

Example 66. Let N be the set of positive integers. A relation R is defined by x R y iff x divides y. Prove that (N, R) is a Lattice where meet (∧) and join (∨) are respectively defined by $x \wedge y = \text{HCF}(x, y)$ and $x \vee y = \text{LCM}(x, y)$.

Solution. Firstly, we have to prove that (N, R) is a poset :

(i) Reflexive : Since each element of N is divisible by itself so R is reflexive.

(ii) Antisymmetric : Let x R y and y R x both are true.

Now, $x R y \Rightarrow x$ divides $y \Rightarrow y = p x, p \in N$

Also, $y R x \Rightarrow y$ divides $x \Rightarrow x = qy, q \in N$

Thus we have, $y = p q y \Rightarrow p q = 1$

$\Rightarrow p = 1 = q$

$\therefore x = y$

hence R is antisymmetric.

(iii) Transitive: Let x R y and y R z are true, then $y = p x$ for some $p \in N$ and $z = q y$, for some $q \in N$

$\Rightarrow z = p q x$

$\Rightarrow x$ divides $z \Rightarrow x R z$

So R is a partial order relation. In N, meet ∧ and join ∨ are HCF and LCM repectively and every pair of positive integers has their HCF and LCM in N itself i.e., for each pair of elements $x, y \in N$,

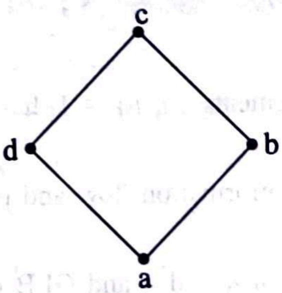
$x \wedge y = \text{HCF}(x, y)$ and $x \vee y = \text{LCM}(x, y)$ exist in N. Thus, N is a lattice.

Example 67. Show that the poset (P(A), ⊆) is a Lattice, where P(A) is the power set of a set A.

Solution: Let $A_i, A_j \in P(A)$ then clearly, the LUB of A_i and A_j is $A_i \cup A_j$ and the GLB of A_i and A_j is $A_i \cap A_j$. Hence (P(A), ⊆) is a Lattice.

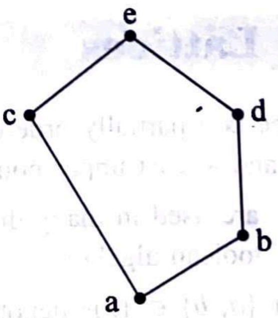
Example 68. The following are the Hasse diagrams of some posets, explain which are Lattices?

Solution.

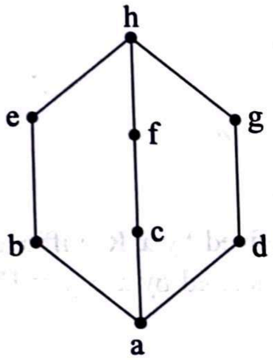


(i)

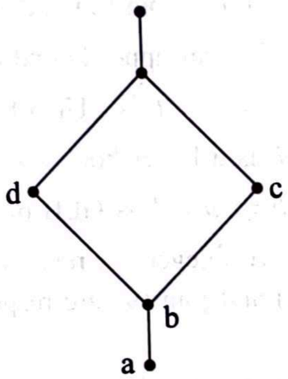
(ii)



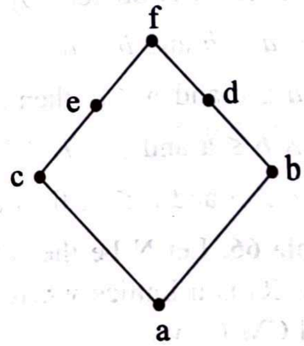
(iii)



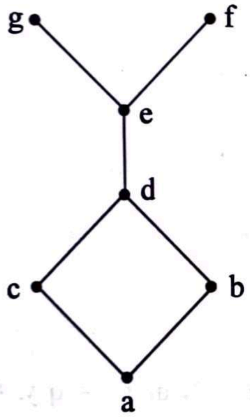
(iv)



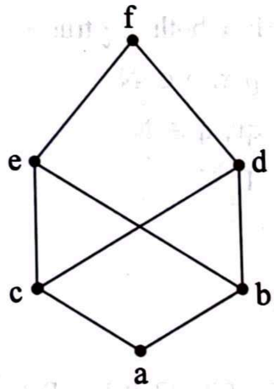
(v)



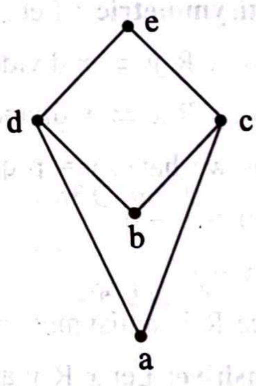
(vi)



(vii)



(viii)



(ix)

Solution.

Diagram (i), (ii), (iii), (iv), (v) and (vi) are Lattices. While (vii), (viii) and (ix) are not Lattices.

Diagram (vii) is not a lattice as LUB of the subset {f, g} i.e., f ∨ g does not exists.

Diagram (viii) is not a lattice because d, e, f are the upper bounds of the subset {b, c} but no LUB.

Also (ix) does not represent a lattice as it has no GLB of the subset {c, d}.

2.33 Properties of Lattices

Theorem 7. If (L, ≤) be a Lattice and $a, b \in$ L, then

(i) $a \vee b = b$ if and only if $a \leq b$.

(ii) $a \wedge b = a$ if and only if $a \leq b$.

(iii) $a \wedge b = a$ if and only if $a \vee b = b$.

Proof: (i) Let $a \vee b = b$. Since $a \leq a \vee b = b \Rightarrow a \leq b$.

Conversely, let $a \leq b$ and also we know that $b \leq b$. so b is an upper bound of a and b and hence $a \vee b \leq b$. Since $a \vee b$ is an upper bound of b, i.e. $b \leq a \vee b$, thus $a \vee b = b$.

(ii) The proof can be established on the similar lines of proof (i).

(iii) It follows from parts (i) and (ii).

Theorem 8. If (L, ≤) is a lattice with two binary operations ∨ and ∧, then for elements $a, b, c \in$ L.

(i) Absorption Law

$a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

(ii) Associative Law

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

$a \vee (b \vee c) = (a \vee b) \vee c$

(iii) Idempotent Law

$a \wedge a = a$

$a \vee a = a$

(iv) Commutative Law

$a \wedge b = b \wedge a$

$a \vee b = b \vee a$

Proof: (i) since ≤ is reflexive so $a \leq a$ and also $a \leq a \vee b$.

So we have

$a \leq a \wedge (a \vee b)$ .....(1)

Again by the definition of GLB, we have

$a \wedge (a \vee b) \leq a$ .....(2)

(1) and (2) $\Rightarrow a \wedge (a \vee b) = a$

Likewise, $a \leq a$ and also $a \wedge b \leq a$

Thus by the definition of LUB .....(3)

$a \vee (a \wedge b) \leq a$ .....(4)

Also $a \leq a \vee (a \wedge b)$

(3) and (4) $\Rightarrow a = a \vee (a \wedge b)$

(ii) By definition of GLB, we have .....(5)

$a \wedge (b \wedge c) \leq b \wedge c \leq c$

Also, $a \wedge (b \wedge c) \leq b \wedge c \leq b$ and $a \wedge (b \wedge c) \leq a$

Now by definition of GUB,

$a \wedge (b \wedge c) \leq a \wedge b$ .....(6)

(5) and (6) (by definition of GUB)

$\Rightarrow a \wedge (b \wedge c) \leq (a \wedge b) \wedge c$ ...(7)

Again $(a \wedge b) \wedge c = c \wedge (a \wedge b)$

$\leq (c \wedge a) \wedge b$ [by (7)]

$= b \wedge (c \wedge a)$

$\leq (b \wedge c) \wedge a$ [by (7)]

$= a \wedge (b \wedge c)$

or $(a \wedge b) \wedge c \leq a \wedge (b \wedge c)$ .....(8)

(7) and (8) $\Rightarrow a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Likewise, we can prove that $a \vee (b \vee c) = (a \vee b) \vee c$

(iii) Since $a \leq a$ and $a \leq a \Rightarrow a$ is the LB of $\{a,a\}$

$\Rightarrow a \leq a \wedge a$ ...(9)

Also, $a \wedge a$ is the GLB of a and a

$\Rightarrow a \wedge a \leq a$ ...(10)

$\therefore$ From (9) and (10), $a \wedge a = a$

Similarly, we can show that $a \vee a = a$.

(iv) Since $a \vee b = \text{LUB } \{a, b\}$

$= \text{LUB } \{b, a\} = b \vee a$

$\therefore a \vee b = b \vee a$

Also, $a \wedge b = \text{GLB } \{a, b\}$

$= \text{GLB } \{b, a\}$

$= b \wedge a$

$\therefore a \wedge b = b \wedge a.$

Theorem 9. If (L, $\leq$) is a lattice with binary operations $\vee$ and $\wedge$, then for any $a, b, c, d \in$ L

(i) $a \leq b$ and $c \leq d \Rightarrow a \wedge c \leq b \wedge d$;

(ii) $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$.

Proof: Let (L, $\leq$) be a lattice and for any $a, b, c, d \in$ L

(i) Let $a \leq b$ and $c \leq d$

since $a \wedge c \leq a$ and $a \wedge c \leq c$

Thus by transitivity,

$a \wedge c \leq a$ and $a \leq b \Rightarrow a \wedge c \leq b$

and $a \wedge c \leq c$ and $c \leq d \Rightarrow a \wedge c \leq d$

Again $a \wedge c \leq b$ and $a \wedge c \leq d$

$\Rightarrow a \wedge c$, is the LB of b and d

$\Rightarrow a \wedge c \leq b \wedge d.$

(ii) Suppose that $a \leq b$ and $c \leq d$

Now because $b \leq b \vee d$ and $d \leq b \vee d$

Therefore, by transitivity, $a \leq b$ and $b \leq b \vee d \Rightarrow a \leq b \vee d$

and $c \leq d$ and $d \leq b \vee d \Rightarrow c \leq b \vee d$

Now $a \leq b \vee d$ and $c \leq b \vee d \Rightarrow b \vee d$ is UB of a and c

$\Rightarrow a \vee c \leq b \vee d.$

Theorem 10. If (L, ≤) is a lattice with binary operations $\vee$ and $\wedge$, then for any $a, b, c \in$ L,

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Proof: Let (L, ≤) is a lattice and $a, b, c \in$ L.

(i) $a \wedge b \leq a$ and $a \wedge b \leq b \vee c$

$\therefore a \wedge b \leq a \wedge (b \vee c)$ .....(1)

Likewise, $a \wedge c \leq a$ and $a \wedge c \leq c \leq b \vee c$

$\therefore a \wedge c \leq a \wedge (b \vee c)$ .....(2)

(1) and (2) $\Rightarrow a \wedge (b \vee c)$ is the UB of $a \wedge b$ and $a \wedge c$.

Thus $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

(ii) $a \leq a \vee b$ and $b \wedge c \leq b \leq a \vee b$

$\Rightarrow a \vee b$ is the UB of a and $b \wedge c$

$\therefore a \vee (b \wedge c) \leq a \vee b$ .....(3)

Likewise, $a \leq a \vee c$ and $b \wedge c \leq c \leq a \vee c$

$\Rightarrow a \vee (b \wedge c) \leq a \vee c$ .....(4)

(3) and (4) $\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Theorem 11. The dual of a lattice is also a lattice.

Proof: Let (L, R) be a lattice, where R is a partial order relation on a non–void set L, i.e., (L, R) is a poset. If R^{-1} is the inverse of R, then R^{-1} is also a partial order relation. Consequently (L, R^{-1}) is a poset.

Now we will prove that for any $a, b \in$ L, $a \vee b$ and $a \wedge b$ are the GLB and LUB respectively of the subset {a, b} with regard to R^{-1}.

Since $a, b \in$ L [(L, R) is a lattice]

$\therefore$ With regard to R, $a \vee b$ = LUB $\{a, b\}$

$\therefore a$ R $(a \vee b)$ and b R $(a \vee b)$

$\Rightarrow (a \vee b)$ R^{-1} a and $(a \vee b)$ R^{-1} b

$\Rightarrow a \vee b$ is the LB of a and b with regard to R^{-1}

Now, if c is the another LB of a and b with regard to R^{-1}, then

c R^{-1} a and c R^{-1} $b \Rightarrow a$ R c and b R c

$\Rightarrow$ c is the UB of a and b with regard to R

$\Rightarrow$ $(a \vee b)$ R c [$a \vee b$ = LUB $\{a, b\}$

$\Rightarrow$ c R^{-1} $(a \vee b)$

$\therefore$ $(a \vee b)$ is the GLB of a and b with regard to R^{-1}

so in (L, R^{-1}) $a \vee b$ = GLB $\{a, b\}$

Likewise, it can be prove that

In (L, R^{-1}), $a \wedge b$ = LUB $\{a, b\}$

Hence (L, R^{-1}) is a lattice.

2.34 Bounded Lattices

Let (L, $\leq$) is a lattice. An element I will be used to denote the upper bound (UB) of the set L(i.e., for each $a \in$ L, we have $a \leq I$). Obviously I is unique in the lattice, if it exists. Similarly 0, to denote lower bound (LB) of the set L (i.e., for any $a \in$ L, we have $0 \leq a$). In other words, the greatest element in (L, $\leq$), if exists, is denoted as I and the least element in (L, $\leq$), if exists, is denoted as 0.

A lattice which has both elements 0 and I is called a bounded lattice.

Example 69. The lattice (P(X), $\subseteq$), where X is a finite set, is bounded.

Its UB is X and LB is ϕ

i.e. element I = X

and element $0 = \phi$.

Example 70. The lattice (Z^+, |) is not bounded since it has a least element, i.e. the integer 1, but does not possesses a greatest element.

Example 71. The lattice (Z, $\leq$), where $\leq$ is the relation of 'less than or equal', is not bounded since it has neither a greatest element nor a least element.

Example 72. If L = $\{a_1, a_2, a_3, \ldots a_n\}$ be a finite lattice then L is bounded as its greatest element $(I) = a_1 \vee a_2 \vee \ldots \vee a_n$ and its least element $(0) = a_1 \wedge a_2 \wedge \ldots \wedge a_n$.

Theorem 12. Let (L, $\leq$) be a lattice with 0 and I as lower and upper bounds, then $\forall\ a \in$ L

(i) $a \vee I = I$ and $a \wedge I = a$

(ii) $a \vee 0 = a$ and $a \wedge 0 = 0$

Proof: (i) Let $a \in$ L, as I is the UB of L

$\therefore$ $a \vee I \leq I$ .....(1)

Again $a \vee I$ is the LUB of a and I

$\therefore$ $I \leq a \vee I$ .....(2)

(1) and (2) $\Rightarrow$ $a \vee I = I$

Now as $a \wedge I$ is the GLB of a and I

$\therefore$ $a \wedge I \leq a$ .....(3)

Again $a \leq a$ and $a \leq I$ and $a \wedge I$ is the GLB

$\therefore$ $a \leq a \wedge I$ .....(4)

(3) and (4) $\Rightarrow a \wedge I = a$

(ii) Let $a \in$ L, then $0 \leq a$ and $a \leq a$

$\therefore\ a \vee 0 \leq a$ ......(5)

Since $a \vee 0$ is the LUB of a and 0

$\therefore\ a \leq a \vee 0$ .....(6)

(5) and (6) $\Rightarrow a \vee 0 = a$

Further, as $a \wedge 0$ is the GLB of a and 0

$\therefore\ a \wedge 0 \leq 0$ ....(7)

Again $0 \leq a$ and $0 \leq 0 \Rightarrow 0 \leq a \wedge 0$ ......(8)

(7) and (8) $\Rightarrow a \wedge 0 = 0$

2.35 Distributive lattices

A lattice (L, $\wedge$, $\vee$) is called a distributive lattice if for any $a, b, c \in$ L,

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ [Distributive over each other]

Example 73. The lattice (P(S), $\subseteq$) is distributive, for a set S. The join and meet operation in P(S) are $\cup$ and $\cap$ respectively. Since from set theory, we know that for A, B, C $\in$ P(S)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

2.36 Complement of an element of a lattice

Let (L, $\leq$) be a bounded lattice, whose greatest and least elements are I and 0 respectively. Then an element $a' \in$ L is called a complement of an element $a \in$ L if

$a \vee a' = I$ and $a \wedge a' = 0$. Also $0' = I$ and $I' = 0$.

Theorem 13. Let (L, $\leq$) be a bounded distributive lattice, if an element $a \in$ L has a complement then it is unique.

Proof: Let (L, $\leq$) be a distributive bounded lattice having I and 0 as its greatest and least elements respectively. Let b and c are the different complement of an element $a \in$ L, then

$a \vee b = I$ and $a \wedge b = 0$

with $a \vee c = I$ and $a \wedge c = 0$

Since L is distributive

Thus, $b = b \vee 0 = b \vee (a \wedge c) = (b \vee a) \wedge (b \vee c)$

$= I \wedge (b \vee c) = b \vee c$ .....(1)

and $c = c \vee 0 = c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b)$

$= I \wedge (c \vee b) = c \vee b = b \vee c$ .....(2)

(1) and (2) $\Rightarrow b = c$

Hence the complement of $a \in L$ is unique.

2.37 Complemented lattice

A bounded lattice $(L, \wedge, \vee, 0, I)$ is said to be complemented if every element of L has at least one complement.

Example 74. The lattice $(P(X), \subseteq)$ is a complemented lattice in which the complement of any subset A of X is the set X – A

$$A \vee (X - A) = A \cup (X - A) = X \;(= I)$$

$$\text{and } A \wedge (X - A) = A \cap (X - A) = \phi \;(= 0)$$

Theorem 14. If $(L, \leq)$ is a complemented distributive lattice, then $\forall\, a, b \in L$

(i) $(a \vee b)' = a' \wedge b'$

(ii) $(a \wedge b)' = a' \vee b'$ [De–Morgan's laws]

Proofs: (i) $\forall\, a, b \in L$

$$(a \vee b) \vee (a' \wedge b') = [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b'] \quad \text{[L is distributive]}$$

$$= [(a \vee a') \vee b] \wedge [a \vee (b \vee b')] \quad \text{[by associativity and commutativity]}$$

$$= [(I \vee b) \wedge [a \vee I] = I \wedge I = I \quad .....(1)$$

and $(a \vee b) \wedge (a' \wedge b') = [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]$

$$= [(a \wedge a') \wedge b'] \vee [a' \wedge (b \wedge b')]$$

$$= [0 \wedge b'] \vee [a' \wedge 0]$$

$$= 0 \wedge 0 = 0 \quad .....(2)$$

Obviously from (1) and (2) we have, $(a \vee b)' = a' \wedge b'$

(ii) $(a \wedge b) \vee (a' \vee b') = [a \vee (a' \vee b') \wedge [b \vee (a' \vee b')]$

$$= [(a \vee a'] \vee b'] \wedge [a' \vee (b \vee b')]$$

$$= [I \vee b'] \wedge [a' \vee I]$$

$$= I \wedge I = I \quad .....(3)$$

and $(a \wedge b) \wedge (a' \vee b') = [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b']$

$$= [(a \wedge a') \wedge b] \vee [a \wedge (b \wedge b')]$$

$$= [0 \wedge b] \vee [a \wedge 0]$$

$$= 0 \vee 0 = 0 \quad .....(4)$$

(3) and (4) $\Rightarrow (a \wedge b)' = a' \vee b'$.

Theorem 15. The dual of a complemented lattice is also a complemented lattice.

Proof: Let (L, R) be a complemented lattice where the greatest and the least elements are I and 0 respectively. Then we know that (L, R^{-1}) is also a lattice.

I is the greatest element of the lattice (L, R)

$\therefore\ a \text{ R } I, \forall a \in \text{L}$

$\Rightarrow I \text{ R}^{-1} a, \forall a \in \text{L}$

$\Rightarrow I$ is the least element of the lattice (L, R^{-1})

Likewise, 0 is the least element of (L, R)

$\therefore\ 0 \text{ R } a, \forall a \in \text{L} \Rightarrow a \text{ R}^{-1} 0, \forall a \in \text{L}$

$\Rightarrow 0$ is the greatest element of the lattice (L, R^{-1}).

Thus, (L, R^{-1}) is a bounded lattice.

Since (L, R) is complemented so for every $a \in \text{L} \; \exists \; a' \in \text{L}$ such that $a \wedge a' = 0$ and $a \vee a' = I$.

Thus, In (L, R), $0 = \text{GLB} \{a, a'\}$

and $I = \text{LUB} \{a, a'\}$

$\Rightarrow 0 \text{ R } a, 0 \text{ R } a'$ and $a \text{ R } I, a' \text{ R } I$

$\Rightarrow a \text{ R}^{-1} 0, a' \text{ R}^{-1} 0$ and $I \text{ R}^{-1} a, 1 \text{ R}^{-1} a'$

$\Rightarrow$ In (L, R^{-1}), 0 is the UB of $\{a, a'\}$ and I is the LB of $\{a, a'\}$.

Now, if K be another UB of a and a' in (L, R^{-1}), then

$a \text{ R}^{-1} \text{K}$ and $a' \text{ R}^{-1} \text{K}$

$\Rightarrow \text{K R } a$ and $\text{K R } a'$

$\Rightarrow$ K is LB of a and a'

$\Rightarrow \text{K R } 0$ [$\because 0$ is the GLB of a and a' in (L, R)]

$\Rightarrow 0 \text{ R}^{-1} \text{K}$

$\Rightarrow 0$ is the LUB of a and a' in (L, R^{-1})

Thus, $0 = a \vee a'$ in (L, R^{-1}) .....(1)

Likewise, it can be proved that

$a \wedge a' = I$ in (L, R^{-1}) .....(2)

(1) and (2) $\Rightarrow a$ is the complement of a' in (L, R^{-1})

Hence (L, R^{-1}) is complemented.

2.38 Job-Scheduling Problem

Let $\text{T} = \{\text{T}_1, \text{T}_2, \ldots, \text{T}_m\}$ denotes a set of tasks to be executed on n identical processors, say $\text{P}_1, \text{P}_2, \ldots, \text{P}_n$ in a multiprocessor computing system. The problem is to identify the schedule of these tasks on the given system. Suppose that the execution of a task requires one and only one processor. Since the processors are identical so the task can be performed on any one of the given processors. Here we also define a partial order relation '$\leq$' on T such that $\text{T}_i \leq \text{T}_j$ ($\text{T}_i \neq \text{T}_j$) if and only if the process of T_j cannot begin until the task T_i has been completed. Now for scheduling the tasks $\text{T}_1, \text{T}_2, \ldots, \text{T}_m$ on the computing system, we need to specify for each task T_i - the time interval in which it will be executed and the processor P_r on which the execution will happens. This schedule can be described by a diagram, called timing diagram.

2.38.1 Idle Period

A time interval in which no task is executed on a certain processor, called the idle period of that processor and is denoted by ϕ.

2.38.2 Total Elapsed Time

It is the total time requires to complete the execution of all tasks as per the schedule. In scheduling problems, we need to identity the schedule for which the total elapsed time is minimum.

Although there is no known method to identify the best scheduling, but to obtain a good scheduling (may not be the best one) never leave a processor intentionally idle.

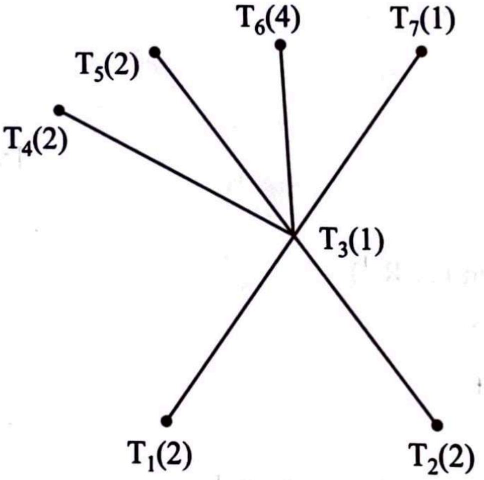
Remark 16. Let t denotes the total elapsed time and T_0 denotes the minimum possible total elapsed time then $\frac{t}{t_0} \leq 2 - \frac{1}{n}$, where n is the number of processors.

Example 75: Consider a set of tasks T = {T_1, T_2, ..., T_7}, a set of processors P = {P_1, P_2, P_3} and a partial order relation '≤' on T having the pairs (T_1, T_3), (T_2, T_3), (T_3, T_4), (T_3, T_5), (T_3, T_6), (T_3, T_7) as its elements other than the pairs due to reflexivity and transitivity and execution time for T_1 is 2, T_2 is 2, T_3 is 1, T_4 is 2, T_5 is 2, T_6 is 4 and T_7 is 1.

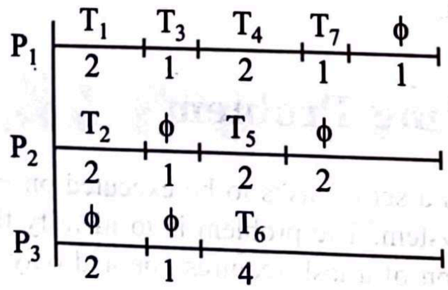
(i) Describe the problem graphically.

(ii) Draw the timing diagram.

Solution: (i) The problem can be described as per the following Hasse diagram:



(ii) The timing diagram can be constructed as below:



∵ Total elapsed time = 7 units.

ILLUSTRATIVE EXAMPLES

Example 76. For the set A = {1, 2, 3}, write two partial order relations.

Solution: (i) $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

(ii) $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$

Example 77. In the set of natural numbers N = {1, 2, 3, ...}, prove that the relation R defined as

$a \text{ R } b \Leftrightarrow a = b^K, \forall\ a, b, K \in \text{N}$

is a partial order relation.

[MNIT 2003, Raj. 2002]

Solution:

(i) R is **reflexive**

For any $a \in$ N, we have $a = a^1$ $\quad (K = 1)$

$\therefore$ R is reflexive

(ii) R is **anti–symmetric**

If a R b and b R a both exist, then

$a = b^{K_1}$ and $b = a^{K_2}$

So, we have

$$a = \left(a^{K_2}\right)^{K_1} = a^{K_1 K_2}$$

$\therefore$ $K_1 K_2 = 1$ as $K_1 K_2 \in$ N, we must have

$K_1 = K_2 = 1$

So that $a = b^{K_1} = b$

Hence a R b and b R a both exist for $a = b$

$\therefore$ R is anti–symmetric

(iii) R is **transitive**

Let a R b and b R c be true, then

$a = b^{K_1}$ and $b = a^{K_2}$,

$K_1, K_2 \in$ N

$\therefore a = (c^{K_2})^{K_1} = c^{K_1 K_2}$ as $K_1 K_2 \in$ N

$\Rightarrow a = c^K, K = K_1 K_2 \in$ N

Thus, a R c exists so R is transitive.

Hence R is a partial order relation on N.

Example 78. Show that the power set P(A) of a non–void set is not a totally ordered set for the relation 'set inclusion'.

Solution: Let P(A) be a power set of A. Now consider a relation R as "set inclusion" in P(A), then R is a partial order relation.

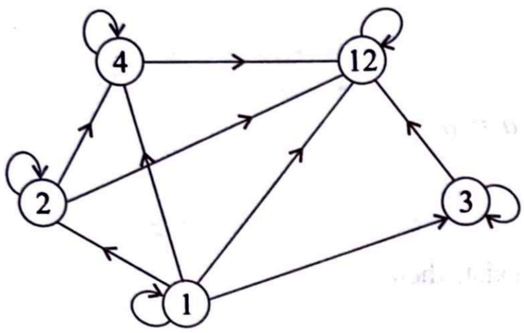
But it is not totally ordered. Because for the set A = {1, 2, 3, 4},

Let A_1 = {1, 2}, A_2 = {2, 3} be its two subsets, we have neither A_1 is contained in A_2(nor $A_1 = A_2$) nor A_2 in A_1.

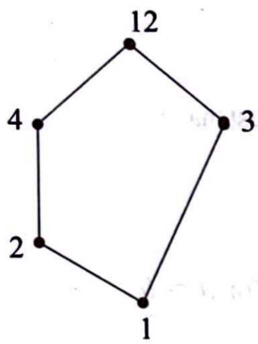
Thus, we notice here that the elements A_1, A_2 of the power set P(A) are not comparable by the relation of contained in thought it is a partial order relation, the set P(A) is not a totally ordered set.

Example 79. Let A = {1, 2, 3, 4, 12}. Consider the partial order of divisibility on A i.e., $a \leq b$ if a divides b. Draw the digraph and Hasse diagram of the poset (A, ≤).

Solution: Digraph



Hasse–diagram



Example 80. Show that the set of real numbers is a totally ordered set for the relation '≤'.

Solution: The set of real numbers R consists of all real numbers in which every pair of numbers (a, b), satisfy

(i) Either $a < b$ (ii) or $a > b$ (iii) or $a = b$

Thus, R is a totally ordered set.

Example 81. Prove that in the power set P(A) the relation of "contained in" defined as A_1 R A_2 if $A_1 \subseteq A_2$ is a partial order relation. **[IT(RTU)-2008]**

Solution:

(i) For any subset A_i of A, $A_i \in$ P(A), $A_i \subseteq A_i$ is true.

$\therefore$ R is **reflexive**

(ii) For any two subsets A_i and A_j of A,

and

$$\left.\begin{aligned} A_i &\subseteq A_j \\ A_j &\subseteq A_i \end{aligned}\right\} \Rightarrow A_i = A_j$$

$\therefore$ R is **antisymmetric**

(iii) For any three subsets

A_i, A_j and A_k of A,

if $A_i \subseteq A_j$ and $A_j \subseteq A_k$

then $A_i \subseteq A_k$ is true.

Thus $A_i \mathrel{R} A_j$ and $A_j \mathrel{R} A_k \Rightarrow A_i \mathrel{R} A_k$

$\therefore$ R is **transitive**

Hence R is a partial order relation.

Example 82. Prove that the relation R given by

$a \mathrel{R} b \Leftrightarrow a \leq b$,

is a partial order relation in the set of real numbers.

Solution:

(i) R is reflexive

As $a \leq a$ is true so $(a, a) \in R$, for every real number a.

(ii) R is anti–symmetric

If $a \mathrel{R} b$ and $b \mathrel{R} a$ be both true, then we have $a \leq b$ and $b \leq a$ only when $a = b$.

(iii) R is transitive

If $a \leq b$ and $b \leq c \Rightarrow a \leq c$

Hence, it is a partial order relation.

Example 83. Show that (Z^+, divisibility) is a poset. **[RTU 2011, 2009, 2007]**

Solution: Reflexive: Given $x \mathrel{R} y \Leftrightarrow x$ divides y

since x divides x, $\forall x \in Z^+$

$\Rightarrow x \mathrel{R} x, \qquad \forall x \in Z^+$

$\therefore$ R is reflexive.

Antisymmetric: Let $x \mathrel{R} y$ and $y \mathrel{R} x$ then

x divides y and y divides x

which is only possible when $x = y$

$\therefore$ R is antisymmetric.

Transitive: Let $x \mathrel{R} y$ and $y \mathrel{R} Z$

$\Rightarrow \quad x = k_1 y$ and $y = k_2 z$, $k_1, k_2 \in Z$

$\Rightarrow \quad x = k_1(k_2 z) = (k_1 k_2)z$

or $\quad x = kz$, $k = k_1 k_2 \in Z$

$\Rightarrow \quad x$ divides z

$\Rightarrow \quad x \mathrel{R} z$

$\therefore \quad$ R is transitive.

Thus R is a partial order relation on Z^+, i.e., (Z^+,divisibility) is a poset.

EXERCISE 2.2

Q.1 Define poset. Answer the following questions concerning the poset ({3, 5, 9, 15, 24, 45}, /).

(a) Find the maximal and minimal elements.

(b) Find the greatest and the least element, if exists.

(c) Find lub of {3, 5}, if exists.

(d) Find glb of {15, 45}, if it exists.

Ans. (a) maximal → 24, 45 minimal → 3, 5; (b) not exist (c) 15 (d) 15

Q.2 If poset is ({1}, {2}, {4}, {1, 2}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}), ⊆).

(a) Find maximal and minimal elements.

(b) Is there a greatest and a least element.

(c) Find lub of {{2}, {4}}, if it exists.

(d) Find glb of {{1, 3, 4}, {2, 3, 4}}, if it exists.

Ans. (a) Maximal → {1, 2}, {1, 3, 4}, {2, 3, 4}, minimal → {1}, {2}, {4}

(b) not exist (c) {2, 4} (d) {3, 4}

Q.3 The binary relation R = {(1, 1), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)} on the set A = {1, 2, 3, 4} is **[GATE 1998]**

(a) reflexive, symmetric and transitive

(b) neither reflexive, nor irreflexive but transitive

(c) irreflexive, symmetric and transitive

(d) irreflexive and antisymmetric

Q.4 Let R be a symmetric and transitive relation on a Set A. Then **[GATE 1995]**

(a) R is reflexive and hence an equivalence relation

(b) R is reflexive and hence a partial order

(c) R is reflexive and hence not an equivalence relation

(d) None of the above

Q.5 Let X = {2, 3, 6, 12, 24}, let ≤ be the partial order defined by X ≤ Y iff *x*/*y*. The number of edge as in the Hasse diagram of (X, ≤) is **[GATE 1996]**

(a) 3

(b) 4

(c) 9

(d) None of the above

Q.6 Let A be a finite set of size *n*. The number of elements in the power set A × A is **[GATE 1993]**

(a) 2^{2n}

(b) 2^{n2}

(c) $(2^n)^2$

(d) $(2^2)^n$

Q.7 The less-than relation, <, on real is : **[GATE 1993]**

(a) a partial ordering since it is asymmetric and reflexive

(b) a partial ordering since it is antisymmetric and reflexive

(c) not a partial ordering because it is not asymmetric and not reflexive

(d) not a partial ordering because it is not antisymmetric and reflexive

(e) none of the above

Q.8 Determine whether the following posets are lattices.

(a) $(\{1, 3, 6, 9, 12\}, /)$

(b) $(\{1, 5, 25, 125\}, /)$

(c) $(Z, \geq)$

(d) $(P(S), \subseteq)$, where P(S) is the power set of a set S.

ANSWER KEY

3.	b	4.	d	5.	b	6.	b	7.	e

Sol.3 Let R be the relation on set A.

Reflexive Relation:

If$(x, x) \in R, \forall x \in A$

i.e. $x \, R \, x, \forall x \in A$

Irreflexive Relation:

$x \not{R} \, x, \forall x \in A$

More $R = \{(1, 1), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$

$A = \{1, 2, 3, 4\}$

Now $(4, 4) \notin R$.

$\therefore$ R is not reflexive.

$\therefore$ Option(a) eliminates.

Also, $\because$ (1, 1), (2, 2), (3, 3) are elements of R.

$\therefore$ R is also not irreflexive.

$\therefore$ Option (c) and (d) eliminates.

Hence (b) is the answer.

Sol.4 A relation that is symmetric and transitive need not be reflexive.

For example,

$A = \{a, b, c\}$ and consider

$R = \{(a, a), (b, b), (a, b), (b, a)\}$

Certainly R is symmetric and transitive but not reflexive.

Since $(c, c) \notin R$.

Sol.5 The Hasse diagram of $(X, \leq)$ is shown below:

It follows that the number of edges is 4.

Sol.6 The power set $A \times A$ has cardinality n^2.

$\Rightarrow$ The cardinality of $P(A \times A)$ is 2^{n^2}.

(Note that if cardinality of A be n, then the cardinality of P(A) is 2^n. P(A) = set of all subsets of A including the empty set ϕ and the set A itself.)

Sol.7 A relation R on a non-empty set A is said to be a partial order if it is reflexive, antisymmetric and transitive. The relation '<' is anti–symmetric and transitive, but not reflexive.

❑❑❑