DISCRETE MATHEMATICS STRUCTURE

IV C.S.

SET THEORY, RELATION AND FUNCTION 1

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 Find the minimum number of students in a school to be sure that 5 of them are born in the same month. *[R.T.U. 2019]*

Ans. Use pigeonhole principle, first find boxes and objects. Suppose that for each month, we have a box that contains persons who was born in that month. The number of boxes is 12 and Let the number of objects is 60. By the generalized pigeonhole principle, at least one of these boxes contains at least 60/12= 5 persons. So, there must be at least 5 persons who were born in the same month.

Q.2 Find the generating function for the sequence $\{1, 1, 0, 0, 1, 1, 1, \ldots \infty\}$ *[R.T.U. 2019]*

Ans. $1 + 1.x + 0.x^2 + 0.x^3 + 1.x^4 + 1.x^5 + 1.x^6 + \ldots + x^9$

$$1 + x + x^4(1 + x + x^2) + \ldots$$

$$1 + x + x^4(1 - x)^{-1}$$

$$1 + x + \frac{x^4}{1-x}$$

Q.3 Find the domain of the following function :

$$f(x) = \sqrt{\log\left(\frac{5x - x^2}{4}\right)}$$ *[R.T.U. 2019]*

Ans. The function f(x) will defined

$$\log\left(\frac{5x - x^2}{4}\right) \geq 0$$

$$\frac{5x - x^2}{4} \geq e^0$$

$$\frac{5x - x^2}{4} \geq 1$$

$$5x - x^2 \geq 4$$

$$x^2 - 5x + 4 < 0$$

$$(x-1)(x-4) \leq 0$$

$$1 \leq x \leq 4$$

Thus, domain of $f(x) = [1, 4]$

Q.4 Prove that for any two sets A and B : $A - (A \cap B) = A - B$ *[R.T.U. 2019]*

Ans. Let $x \in A - (A \cap B)$

$\Rightarrow$ $x \in A$ or $x \notin (A \cap B)$

$\Rightarrow$ $x \in A$ or $\{x \notin A$ or $x \notin B\}$

$\Rightarrow$ $x \in A$ or $x \notin A$ or $x \in A$ or $x \notin B$

$\Rightarrow$ $x \in \phi$ or $x \in A - B$

$\Rightarrow$ $x \in A - B$

$A - (A \cap B) \leq A - B$ ...(1)

Let $x \in A - B$

$\Rightarrow$ $x \in \phi$ or $x \in A - B$

$x + A$ or $x \notin A$ or $x \in A$ or $x \notin B$

$x \in A$ or $x \notin A \cap B$

$x \in A - (A \cap B)$

$A - B \leq A - (A \cap B)$ ...(2)

From equation (1) and (2)

$A - (A \cap B) = A - B$

Q.5 *Define the Cross partition of a set.* *[R.T.U. 2017]*

Ans. Cross Partition of a set : Cross Partition of a set is defined on two partitions of the set as shown below

Let $[A_1, A_2 \ldots A_m]$ and $[B_1, B_2 \ldots B_n]$ be partitions of S. Then, the collection of sets $[A_i \cap B_j,\ i = 1, 1\ldots m,\ j = 1, 1\ldots n]$ is called the cross partition.

Empty set is not included in the cross partition.

Example : Let set = {1, 2, 3..., 8, 9}

We will define cross partition on set with partitions.

$$P_1 = \{\overset{A_1}{(1, 3, 5, 7, 9)}, \overset{A_2}{(2, 4, 6, 8)}\} \rightarrow A$$

$$P_2 = \{\underset{B_1}{(1, 2, 3, 4)}, \underset{B_2}{(5, 7)}, \underset{B_3}{(6, 8, 9)}\} \rightarrow B$$

$A_1 \cap B_1 = \{1, 3\}$ $\qquad$ $A_2 \cap B_1 = \{2, 4\}$

$A_1 \cap B_2 = \{5, 7\}$ $\qquad$ $A_2 \cap B_2 = \{5\}$

$A_1 \cap B_3 = \{9\}$ $\qquad$ $A_2 \cap B_3 = \{6, 8\}$

Cross Partition is {{1, 3}, {5, 7}, {9}, {2, 4}, {5}, {6, 8} }

Q.6 *Define the Duality.* *[R.T.U. 2017]*

Ans. Duality : Suppose that A be a statement dealing with the equality of sets expression. Every expression may, possibly involve one or more occurence of sets and their compliments, void set ϕ and universal set u and only the set operations union ($\cup$) and intersection ($\cap$). The dual of A, represented by A^d, is derived from A by replacing.

1. Every occurence of ϕ and u by u and ϕ respectively in A.
2. Every occurence of u and $\cap$ by $\cap$ and $\cup$ respectively in A.

Let $\quad A = (u \cap s) \cup (T \cap s) = S$

Then $\quad A^d = (\phi \cup s) \cap (T \cup s) = S$

Q.7 *Define the Floor function or greatest integer function.* *[R.T.U. 2017]*

Ans. Floor Functions : The floor is mathematical functions which convert arbitrary real numbers to close integers.

The floor function of a real number x, denoted by $\lfloor x \rfloor$ or floor (x), is a function that returns the highest integer less than or equal to x. Formally, for all real numbers x,

$$\lfloor x \rfloor = \max\{n \in Z : n \le x\}$$

For example,

Floor (2.9) = 2, floor (–2) = –2 and floor (–2.3) = –3.

For non-negative *x*, a more traditional name for floor(x) is the *integral part* or *integral* value of *x*. The function x–[x], also written as *x* mod 1 or {x}, is called the *fractional part of x*. Every fraction *x* can be written as a mixed number, the sum of an integer and a proper fraction. The floor function and fractional part functions extend this decomposition to all real values.

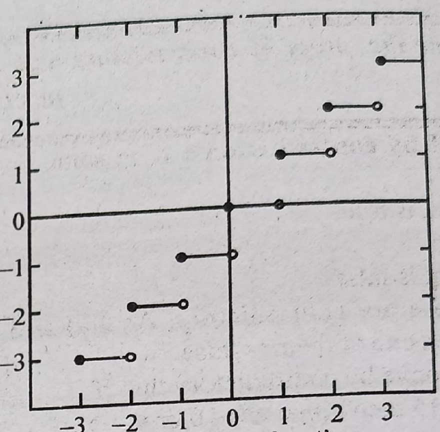


Fig. : The Floor function

Q.8 *Define the Bijection.* *[R.T.U. 2017]*

Ans. Bijection : A function that is both an injection and a surjection. In a bijection, each member of the range corresponds to an element of the domain i.e. mapped onto it and there is one-to-one correspondence between the members of the domain and the range.

Q.9 *Define countable and uncountable sets with example.* *[R.T.U. 2016]*

Ans. Uncountable Sets : An uncountable set is an infinite set that contains too many elements to be counted. The best known example of countable set is the set R of all real numbers.

Countable Sets : A countable set is either a finite set of natural numbers. The elements of a countable set can be counted. An example of countable set is as follows

$$A = \{1, 2, 3, 4\}$$

Q.10 *Define mod functions and div functions with example.* *[R.T.U. 2016]*

Ans. Mod functions and Div functions : The mod function f(x, y) = x mod y denotes the remainder when an integer x is divided by a positive integer y.

Div function

The **div function** g(x, y) = *x* div *y* denotes the quotient when *x* is divided by *y*. Programming languages often provide two such built-in operators, **mod** and **div;** in C++, the mod operator is denoted by the percent symbol %, and the div operator by the forward slash '/'.

For example, 23 mod 5 = 3, 18 mod 6 = 0, 23 div 5 = 4, and 5 div 6 = 0.

The mod function can determine the day of the week in n days from a given day.

Q.11 ***Define the proof by contradiction with example.*** *[R.T.U. 2016, 2013]*

Ans. Proof by contradiction : If in same case we have that

$\sim p \rightarrow q$, is true ... (A)

and also

$\sim p \rightarrow q$, is false ... (B)

But these are contradictory. As (A) and (B) are in contradiction one of them is false.

Ex. : Proove by contradictory that if

$x + y > 15$ then either $x > 10$ or $y > 5$

Sol. We assume the hypothesis $x + y > 15$. From here we must conclude that $x > 10$ or $y > 5$.

Assume to the contrary that

$x > 10$ or $y > 5$, is false

so $x \leq 10$ and $y \leq 5$

Adding both inequalities we get

$x + y \leq 10 + 5 = 15$

which contradicts the hypothesis

$x + y > 15$

From here we conclude that the assumption "$x \leq 10$ and $y \leq 5$" cannot be true,

So "$x > 10$" or "$y > 5$" must be true.

Q.12 ***Let f: R → R be a function defined as f(X) = 3X + 5 and g: R → R be another function defined as g (X) = X+4. Find $(gof)^{-1}$ and $f^{-1}og^{-1}$ and verify $(gof)^{-1} = f^{-1}og^{-1}$*** *[R.T.U. 2015]*

Ans. Given that,

$f(x) = 3x + 5$, and so, $x = (f - 5)/3$, i.e., $f^{-1} = (x - 5)/3$

$g(x) = x + 4$, and so, $x = (g - 4)$, i.e., $g^{-1} = x - 4$

Now,

$gof = g(f(x))$

$= g(3x + 5)$

$= (3x + 5) + 4$

$= 3x + 9$

Now, $x = ((gof) - 9)/3$

So, $(gof)^{-1} = (x - 9)/3$

Now,

$f^{-1}og^{-1} = f^{-1}(g^{-1}(x))$

$= f^{-1}(x - 4)$

$= ((x - 4) - 5)/3$

$= (x - 9)/3$

Hence, we see that $(gof)^{-1} = f^{-1}og^{-1}$

Q.13 ***Define the ceiling function with example.*** *[R.T.U. 2015]*

Ans. Ceiling Functions : The closely-related ceiling function, denoted by $\lceil x \rceil$ or ceil(x) or ceiling (x), is the function that returns the smallest integer nor less than *x* or formally,

$$\lceil x \rceil = \min \{n \in Z : x \leq n\}$$

For example, ceiling (2.3) = 3, ceiling (2) = 2 and ceiling (–2.3) = –2.

The names "floor" and "ceiling" and the corresponding notations were introduced by Kenneth E. Iverson in 1962.

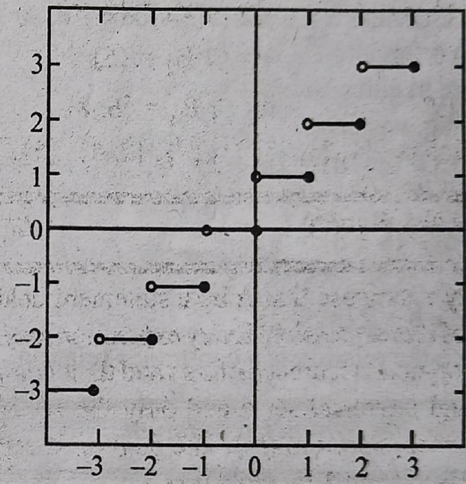


Fig. : The ceiling function

Q.14 ***Define the remainder function with example.*** *[R.T.U. 2015]*

Ans. Remainder Function : A remainder function (generally denoted by 'mod' or symbol '%') is one which given two numbers say, a,b returns the remainder when a is divided by b, i.e. "a mod b" is the remainder when a is divided by b. e.g.,

5 mod 2 = 1 (remainder when 5 is divided by 2).

11 mod 3 = 2 (remainder when 11 is divided by 3)

Q.15 ***Define the Reflexive relation.*** *[R.T.U. 2013]*

Ans. Reflexive Relation : A relation R on a non-void set A is known as reflexive relation if each member of A is R–related to itself, i.e. $x R x$ or $(x, x) \in R$, $\forall x \in A$.

A relation R on a set A is irreflexive if $(x,x) \notin R$, $\forall x \in A$.

Example 1. Let A be the set of all straight lines in a plane. The relation R in A defined by "*x* is parallel to *y*" is reflexive, since every straight line is parallel to itself.

Example 2. The relation "<" defined a set of real numbers is irreflexive because x is not less than x.

Example 3. If A be the set of men and R means "is husband of" then x R x is not true as man cannot be husband of himself. Thus, R is irreflexive.

Q.16 *Define the Congruency relation.* *[R.T.U. 2013]*

Ans. Congruency relation : An equivalence relation R on the semigroup (S, *) B called a congruence relation if

a R a' and b R b' imply (a * b) R (a' * b')

Example

Consider the semigroup (2, +) and the equivalence relation R on Z defined by

a R b if and only if a ≡ b (mod 2)

So

If a ≡ b (mod 2) and c ≡ d (mod 2)

then 2 divides a – b and 2 divide c – d, so

a – b = 2m and c – d = 2n

where m and n are in z.

Adding, we have

(a – b) + (c – d) = 2m + 2n

or

(a + c) – (b + d) = 2(m + n)

So

a + c ≡ b + d (mod 2)

Hence the relation is a congrance relation.

Q.17 *Define the Symmetric relation.* *[R.T.U. 2013]*

Ans. Symmetric relation : A relation R on a non void set A is known as symmetric relation if x R $y \Rightarrow y$ R x, i.e., whenever $(x, y) \in$ R then $(y, x) \in$ R.

Example 1 : Let A be the set of all straight lines in a plane. The relation R defined by "a is perpendicular to b" is symmetric relation because.

$a \perp b \Rightarrow b \perp a, a, b \in A$

Example 2 : If A be the set of members of a family and R means "is the brother of" and x R y means that x is brother of y then x R y does not imply y R x, as y may be the sister of x. Hence R is not symmetric.

Q.18 *Define the Asymmetric relation.* *[R.T.U. 2013]*

Ans. Asymmetric relation : A relation R on a set A is said to be asymmetric if (a, b) ∈ R ⇒ (b, a) ∉ R, a, b ∈ A.

Example : The relation $x < y$ is a symmetric as if $x < y$, then y is not less than x. The necessary and sufficient condition for a relation to be symmetric is $R = R^{-1}$

Q.19 *Define the Transitive relation.* *[R.T.U. 2013]*

Ans. Transitive relation : A relation on a set A is said to be transitive relation if for $x, y, z, \in$ A, x R y and y R $z \Rightarrow x$ R z

i.e. whenever $(x, y) \in$ R and $(y, z) \in$ R then $(x, y) \in$ R.

Example 1 : The relation ">" defined on the set of natural numbers N is transitive because for $x, y, z \in$ N,

If $x > y$ and $y > z$ then $x > z$

i.e. x R y and y R $z \Rightarrow x$ R z.

Example 2 : The relation R of parallelism in the set of straight lines in a plane is a transitive relation because.

$x \parallel y$ and $y \parallel z \Rightarrow x \parallel z$.

i.e. x R y and y R $z \Rightarrow x$ R z, $x, y, z \in$ A

The necessary and sufficient condition that a relation R be transitive is R operated on R ⊂ R.

Q.20 *Define the Anti symmetric relation.* *[R.T.U. 2013]*

Ans. Anti symmetric relation : A relation R on a set A is said to be antisymmetric if whenever $x \neq$ then either $x \not R\ y$ or $y \not R\ x$.

Or

A relation R on a set A is said to be antisymmetric

If x R y and y R $x \Rightarrow x = y$, $x\ y \in$ A.

Example 1 : Let N be the set of natural numbers and let R be the relation defined by "a divides b", ∀ a, b, ∈ N. Then R is an antisymmetric relation as a divides b and b divides a ⇒ a = b.

Example 2 : Let P be a family of sets, then the relation R on P defined by "A is a subset of B" is anti-symmetric because A R B and B RA ⇒ A ⊂ B and B ⊂ A ⇒ A = B.

The necessary and sufficient condition that R is antisymmetric is

$R \cap R^{-1} = \phi$.

Q.21 *Show that in the power set P(A) = (Set of all subset of A), the relation of contained in defined as A1RA2 if A1 is a subset of A2, is a partial order relation.* *[R.T.U. 2013]*

Ans. Let P = P(A) be the power set of the set A. Recall that P(A) is defined to be the set {S | S ⊆ A}, that is the set of all subsets of A. For example P({ 1,2, 3}) is the set

{θ, {1}, {2}, {3}, {2,3}, {1,3}, {1,2}, {1,2,3}}

Let R = {(X, Y) ∈ P × P | X ⊆ Y}. That is R is a relation on P, and for X, Y ∈ P, XRY if and only if X ⊆ Y. Two elements X and Y of P are related by R if X is a subset of Y.

For any subset X of A note that XRX since $X \subseteq X$. Also θRX, and XRA, recall that $A \subseteq A$, so $A \in P$. For this relation it is true that if $X, Y \in P$ then $(X \cap Y)$ RX and $(X \cap Y)$ RY.

Also X R $(X \cup Y)$ and YR $(X \cup Y)$. That means for this relation, given X and Y there is always some U such that URX and URY and there is always some V such that XRV and YRV.

Q.22 ***Let $A = \{a, b, c, d, e\}$ and $B = \{c, e, f, h, k, m\}$ then prove if A and B are finite sets then $|A \cup B| = |A| + |B| - |A \cap B|$.*** ***[R.T.U. 2011]***

Ans. $A = \{a, b, c, d, e\}$

$n(A) = |A| = 5$

$B = \{c, e, f, h, k, m\}$

$n(B) = |B| = 6$

Now, $A \cup B = \{a, b, c, d, e, f, h, k, m\}$

$\therefore$ $|A \cup B| = 9$

and $A \cap B = \{c, e\}$

$\therefore$ $|A \cap B| = 2$

Thus, we have

$|A \cup B| = |A| + |B| - |A \cap B|$

$9 = 5 + 6 - 2$ **Hence Proved.**

PART-B

Q.23 ***Let $f : R \to R$ and $g : R \to R$ where R is the set of real numbers. Find gof and fog where $f(x) = x^2 - 2$ and $g(x) = x + 4$. State whether these functions are injective, surjective or bijective.*** ***[R.T.U. 2019]***

Ans. fog $= x^2 + 8x + 14$

gof $= x^2 + 2$

The minimum value taken up by fog $= x^2 + 8x + 14$ is 14. Similarly, the minimum value taken up by gof $= x^2 + 2$ is 2.

Hence the entire range is not mapped.

Minimum value for fog $= x^2 + 8x + 14$ is given for

$x = -4$ i.e. -2.

Q.24 ***Show that in the power set P(A) of all subsets of a set $A = \{a, b, c\}$, 'Set inclusion, $\subseteq$' is a partial order relation. Also draw the Hasse diagram for the POSET.*** ***[R.T.U. 2019]***

Ans. $A = \{a, b, c\}$

Power set of A

$P(A) = \{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$

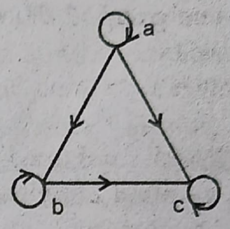
includes partial order relation

{<a, a>, <b, b>, <c, c>, <a, b>, <b, c>, <a, c>}

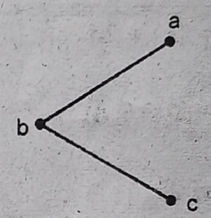
i.e. It satisfies (i) Reflexive (ii) Antisymmetric

(iii) Transitive properties

Hasse diagram of POSET



Digraph of partial ordered set



Hasse diagram

Q.25 ***Write the scope and objective of DMS in Computer Science?*** ***[R.T.U. 2019]***

Ans. Scope of D.M.S. : Though discrete mathematics has found application in almost every conceivable area of study. It is integral part of science course. It provides the mathematical foundation for many computer courses viz algorithms, database management, automata, compiler theory, operating system, computer language, to name a few with wrong mathematical foundation, these computer science subject become easy to understand.

Objectives : The objectives of this course is to provide the fundamental and concepts of Discrete Mathematical Structures with application of computer science including mathematical logic, Boolean Algebra, and its applications, switching circuits and logic gates. Groups and Trees, Important computer theorem with constructive proofs, real life problems and graphs, theoretic algorithms, to help the students to understand the computational and algorithmic assects of sets, relations, functions and algebric structure in field of computer science and its application.

Q.26 *(a) Show that the set of odd positive integers is a countable set.*

(b) A survey is taken on method of commuter travel. Each respondent is asked to check BUS, TRAIN or AUTOMOBILE as a major method of travelling to work. More than one answer is permitted. The results reported were as follows :

(i) 30 people checked BUS

(ii) 35 people checked TRAIN

(iii) 100 people checked AUTOMOBILE

(iv) 15 people checked BUS and TRAIN

(v) 15 people checked BUS and AUTOMOBILE

(vi) 20 people checked TRAIN and AUTOMOBILE

(vii) 5 people checked all three methods

How many respondents completed their surveys? **[R.T.U. 2017]**

Ans.(a) Let T be the set of odd positive integers and N be the set of natural numbers :

Consider a function

$f : N \to T$

such that $f(n) = 2n - 1, n \in N$

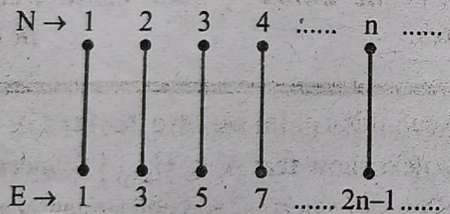


Fig. : Bisection between N and T

We need to show that f is bijection

One-One : Let $f(n_1) = f(n_2)$

$2n_1 - 1 = 2n_2 - 1 \quad 2n_1 = 2n_2$

$n_1 = n_2$

On-to : Let $t \in T$, then t is an odd positive integer

- $t + 1$ is on even positive integer.
- $\frac{t+1}{2}$ is a positive integer or $\frac{t+1}{2} \in N$.

Now for each $t \in T \; \exists \left(\frac{t+1}{2}\right) \in N$ such that

$$f\left(\frac{t+1}{2}\right) = 2\left(\frac{t+1}{2}\right) - 1 = (t+1) - 1 = t$$

So each element of T has its preimage in N. Thus f is onto.

Hence f is a bijection between N and T and then

$|N| = |T|$

Therefore, T is countable.

Ans.(b) Let A, B, C denote the set of respondent who checked Bus, Train and Automobile respectively.

So the number of respondent who checked Bus = $|A|$

The number of respondent who checked Train = $|B|$

The number of respondent who checked Automobile = $|C|$

The number of respondent who checked Bus and Train = $|A \cap B|$

The number of respondent who checked Train and Automobile = $|B \cap C|$

The number of respondent who checked Bus and Automobile = $|A \cap C|$

The number of respondent who checked all the three = $|A \cap B \cap C|$

In this question, it is given

$$|A| = 30, |B| = 35, |C| = 100$$
$$|A \cap B| = 15, |A \cap C| = 15,$$
$$|B \cap C| = 20, |A \cap B \cap C| = 5$$

It is required to find how many respondent completed the survey, i.e. $|A \cup B \cup C|$ is to be found out. According to the principle of inclusion and exclusion :

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$
$$= 30 + 35 + 100 - 15 - 15 - 20 + 5$$
$$|A \cap B \cap C| = 120$$

Ans.

Q.27 *If the set of integers I = {......., –3, –2, –1, 0, 1, 2, 3, ...} be partitioned by the equivalence relation aRb as a ≡ b (mod 3). Obtain the set I/R.* **[R.T.U. 2017]**

Ans. Since difference of any two elements in

$$P_1 = \{\ldots, -6, -3, 0, 3\ 6, 9, \ldots.\} = [0]$$

is a multiple of –3, p_1 is an equivalence class in I.

Next,

$$P_2 = \{\ldots.., -5, -2, 1, 4, 7, \ldots..\} = [1]$$

is a set where difference of any two elements is a multiple of 3. So all these elements are in relation. So P_2 is another equivalence class of I (or z).

and in $P_3 = \{\ldots, -4, -1, 2, 5, 8, \ldots\} = [2]$

The difference of any two elements is a multiple of 3. So all the elements of P_3 are in relation R. Hence P_3 is an equivalence class.

We have

$$P_1 \cup P_2 \cup P_3 = I$$

And $P_1 \cap P_2 = \phi, P_2 \cap P_3 = \phi, P_3 \cap P_1 = \phi$

So, P_1, P_2, P_3 are partitions of I induced by the relation R.

Hence the set $\frac{I}{R} = \{P_1, P_2, P_3\} = \{[0], [1], [2]\}$

Q.28 ***Let $A = Z$ the set of integers Relation R defined by A by aRb as 'a is congruent to b mod 2'. Show that R is an equivalence relations.*** ***[R.T.U. 2013, 12]***

OR

Define congruency relation in Modulo system. If $A = Z$ (the set of integers), Relation R defined in A set by aRb as "a is congruent to b mod 2", then prove that R is an equivalence relation. ***[R.T.U. 2017]***

OR

Let $A = Z$, the set of integers relation R define in A by aRb as "a congruent to b mod 2". Prove that R is an equivalence relation. ***[R.T.U. 2014]***

Ans. Here R is $a \equiv b \pmod 2$ *i.e.* $a - b$ is divisible by 2

or $a - b$ is multiple of 2.

(i) Reflexive : Let $a \in A$, then

$a - a = 0 = 0 \times (2)$ a multiple of 2

$\Rightarrow$ aRa

$\therefore$ R is reflexive.

(ii) Symmetric: Let $a, b \in A$, then

$aRb \Rightarrow a \equiv b \pmod 2 \Rightarrow a - b$ is divisible by 2

$\Rightarrow$ $a - b = 2k, k \in Z$

$\Rightarrow$ $b - a = -2k = 2(-k), -k \in Z$

$\Rightarrow$ $b \equiv a \pmod 2$

$\Rightarrow$ bRa

$\therefore$ R is symmetric.

(iii) Transitive: Let $a, b, c \in A$, then

$aRb \Rightarrow a - b = 2k_1;\ k_1 \in Z$ ....(i)

$bRc \Rightarrow b - c = 2k_2;\ k_2 \in Z$ ....(ii)

From eq. (i) + (ii)

$\Rightarrow$ $a - c = 2(k_1 + k_2);\ k_1 + k_2 \in Z$

$\Rightarrow$ $a \equiv c \pmod 2$

$\Rightarrow$ aRc

$\therefore$ R is transitive.

Hence, it is an equivalence relation.

Q.29 ***Define equivalence Relation. If R and S be two equivalence relations in a set A, then prove that $R \cap S$ is also an equivalence relation in A.*** ***[R.T.U. 2016]***

Ans. Equivalence relation : An equivalence relation is a binary relation which is reflexive relation, a symmetric relation and a transitive relation.

Now, to prove $R \cap S$ is an equivalence relation in A. Let $a \in A$ be an arbitary value. Since, R and S are equivalence relation on A, they are reflexive, so that $(a, a) \in R$ and $(a, a) \in S$. Hence, $(a, a) \in R \cap S$. Hence, $R \cap S$ is reflexive.

Now, Suppose $a, b \in A$ are such that $(a, b) \in R \cap S$. then, $(a, b) \in R$ and $(a, b) \in S$. Since R and S are symmetric, hence, $(b, a) \in R$ and $(b, a) \in S$. Hence, $(b, a) \in R \cap S$, so that $R \cap S$ is symmetric.

Suppose, $a, b, c \in A$ are such that $(a, b), (b, c) \in R \cap S$. Then, $(a, b), (b, c) \in R$ and $(a, b), (b, c) = S$. Since, R is transitive, $(a, b), (b, c) \in R$ implies that $(a, c) \in R$. Since, S is transitive, $(a, b), (b, c) \in S$ and hence, $(a, c) \in S$. Thus, $(a, c) \in R$ and $(a, c) \in S$, so that $(a, c) \in R \cap S$. Hence, $R \cap S$ is transitive.

Therefore, $R \cap S$ is an equivalence relation.

Q.30 ***An equivalence relation R on a set A decomposes A into equivalence classes which are either distinct or completely overlapping and the set A is the union of such distinct equivalence classes.*** ***[R.T.U. 2016]***

OR

Prove that an equivalence relation R on a set A decomposes A into equivalence classes which are either distinct or completely overlapping and the set A is the union of such distinct equivalence classes. ***[R.T.U. 2013, 09]***

Ans. By the definition of [a] we have that $[a] \subseteq A$. Hence $U_{a \in A}$ $[a] \subseteq A$. We next show that $A \subseteq U_{a \in A}$ [a]. Indeed let $a \in A$ since A is reflexive *i.e.* $a \in [a]$ and consequently $a \in U_{a \in A}$ [b]. Hence $A \subseteq U_{a \in A}$ [b]. It follows that $A = U_{a \in A}[a]$. This establishes (i) it remains to show that if $[a] \neq [b]$ then $[a] \cap [b] = \phi$ for $a, b \in A$. Suppose the contarary *i.e.*, suppose $[a] \cap [b] = \phi$. Then there is an element $c \in [a] \cap [b]$. This means that $c \in [a]$ and $c \in [b]$. Hence, aRc and bRc. Since R is symmetric and transitive then aRb. We will now prove that the conclusion aRb leads to $[a] = [b]$.

The proof is by double inclusions. Let $x \in [a]$. Thus, xRa, since aRb and R is transitive then xRb which means that $x \in [b]$. Thus, $[a] \subseteq [b]$. Now, interchange the letters *a* and *b* to show that $[b] \subseteq [a]$.

Hence $[a] = [b]$ which contradicts our assumption that $[a] \neq [b]$. Thus, A/R is a partition of A.

Let set [a] called the equivalence classes of A given by the relation R. The element a in [a] is called a representative of the equivalence class [a].

Q.31 (i) ***Prove, for finite sets A and B;***

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

(ii) In a class of 50 students, 15 play Tennis, 20 play Cricket and 20 play Hockey, 3 play Tennis and Cricket, 6 play Cricket and Hockey, and 5 play Tennis and Hockey, 7 play no game at all. How many play Cricket, Tennis and Hockey? ***[R.T.U. 2014]***

Ans.(i) We know that

$$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B \quad ...(1)$$

and $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint therefore,

$$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A) \quad ...(2)$$

Further $A = (A - B) \cup (A \cap B)$

and $(A - B) \cap (A \cap B) = \phi$

so $n(A) = n(A - B) + n(A \cap B) \quad ...(3)$

Similarly

$$n(B) = n(A \cap B) + n(B - A) \quad ...(4)$$

Adding (3) and (4), we have

$$n(A) + n(B) = \{n(A - B) + n(A \cap B) + n(B - A)\} + n(A \cap B)$$
$$= n(A \cup B) + n(A \cap B) \text{[Using (2)]}$$

Thus

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Ans.(ii) Let A = Student play Tennis

B = Student play Cricket

C = Student play Hockey

so $n(A) = 15; n(B) = 20; n(c) = 20;$

$N(A \cap B) = 3; n(B \cap C) = 6; n(A \cap C) = 5$

$\because$ 50 students in the class in which 7 play no game at all so

$$n(A \cup B \cup C) = 50 - 7 = 43$$

Now number of students those play Cricket, Hockey and Tennis is

$$N(A \cap B \cap C) = n(A \cup B \cup C) - n(A) - n(B) - n(C) + n(A \cap B) + n(B \cap C) + n(C \cap A)$$
$$= 43 - 15 - 20 - 20 + 3 + 6 + 5$$
$$= 57 - 55$$
$$= 2$$

Q.32 (i) ***If $f: A \to B$ be one-one onto then the inverse map of f is unique. Prove it.***

(ii) Show that set of even positive integers is a countable set. ***[R.T.U. 2014]***

Ans.(i) If possible, let g and h be inverses of f. Then by the definition of inverse

$$g.f = f.g = I \quad ... (1)$$

and $h.f = f.h = I \quad ... (2)$

where I is an identity function. Now

$$h = h.I = h.(f.g) \quad \text{(Using (1))}$$
$$= (h.f).g \quad \text{(Associativity Rule)}$$
$$= I.g \quad \text{(Using (2))}$$
$$= g$$

So $h = g$

Hence the result.

Ans. (ii) Proof : Let's count the elements in the set of even positive integers. We do that by comparing the even numbers with the counting numbers:

1 2 3 4 5 6 7 8 9 10 11 . . . (counting numbers)

2 4 6 8 10 12 14 16 18 20 22 . . . (even numbers)

This correspondence goes on forever. And we can find no even number that does not match up with a counting number. We can only deduce that the two sets have the same number of elements. This may seem counter-intuitive, but it is straightforward. The set of even positive integers has the same number of elements as the set of counting numbers. They are both infinite sets of the same size. The set of all integers (positive and negative) matches up the same way. So does the set of all integers ending in 000. These are all **countable sets**, because they have the same number of elements as the counting numbers.

Q.33 ***Compute the number of partitions of a set with four elements.*** ***[R.T.U. 2014]***

Ans. Here $n = 4$

$$\text{Thus the number of partitions} = \sum_{r=1}^{4} S(4, r)$$

$$= S(4, 1) + S(4, 2) + S(4, 3) + S(4, 4) \quad ... (1)$$

Now $S(4, 1) = 1 = S(4, 4) \quad ... (2)$

$S(4, 2) = S(3, 1) + 2S(3, 2) \quad ... (3)$

$S(4, 3) = S(3, 2) + 3S(3, 3) \quad ... (4)$

Since $S(3, 2) = S(2, 1) + 2S(2, 2) \quad ... (5)$

Now $S(2, 1) = S(2, 2) = 1$

$[\because S(n, 1) = 1 = S(n, n)]$

Thus eq.(5) gives

$$S(3, 2) = 1 + 2(1) = 3$$

From (3)

$$S(4, 2) = 1 + 2(3) = 7$$

From (4)

$$S(4, 3) = 3 + 3(1) = 6$$

Hence the number of partition $= 1 + 7 + 6 + 1 = 15$

Q.34 ***Out of 250 failed students, 128 fails in Maths, 87 in Physics and 134 in Aggregate, 31 failed in Maths and Physics, 54 failed in Aggregate and in Maths, 30 failed in Aggregate and in Physics. Find how many candidates failed.***
(i) All three subjects
(ii) In Maths not in Physics
(iii) In Aggregate but not in Maths
(iv) In Physics but not in Aggregate or Maths
(v) In the Aggregate or in Maths, but not in Physics.
[R.T.U. 2013]

Ans. Let M, P and A be the sets of students who failed in Maths, Physics and Aggregate, respectively.

Given

$|M| = 128$

$|P| = 87$

$|A| = 134$

$|M \cap P| = 31$

$|A \cap M| = 54$

$|A \cap P| = 30$

We know that

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

(i) All three subjects

$|M \cup P \cup A| = |M| + |P| + |A| - |M \cap P| - |P \cap A| - |A \cap M| + |M \cap P \cap A|$

$250 = 128 + 87 + 134 - 31 - 30 - 54 + |M \cap P \cap A|$

$250 = 349 - 115 + |M \cap P \cap A|$

$250 - 234 = |M \cap P \cap A|$

$16 = |M \cap P \cap A|$

Candidates failed in all three subjects = 16.

(ii) In Maths not in Physics

After seeing the following fig.

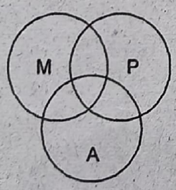


Fig.

Candidates failed in Maths not in Physics

$= 128 - 31 = 97$

(iii) In Aggregate but not in Maths

$= 134 - 54 = 80$

(iv) In Physics but not in Aggreegate or Maths

$= 87 - 30 - 31 + 16 = 42$

(v) In the aggregate or in Maths, but not in Physics.

$= 128 + 134 - 31 - 30 + 16 = 217$

Q.35 ***(a) In the Survey of 60 people it was found that 25 read News week, 26 read Time and 26 read the magazine Fortune. Also 9 read both News weak and Fortune, 11 read News week and Time and 8 read both time and fortune. If 8 read none of the three magazines, determine the number of people who read exactly one magazine.***
[R.T.U 2012, Raj. Univ. 2007, 2001]

(b) Let $f : A \to B$ and $g : B \to C$ be two invertible then
(i) gof is invertible and
(ii) $(gof)^{-1} = f^{-1} \circ g^{-1}$ ***[R.T.U. 2012]***

Ans.(a) Using the given data and the number of elements in various parts of the sets, we have the equation

$C = 8$ ....(1)

Those who read News week only = x

Those who read Fortune only = y

Those who read Time only = z

$(x + y + z) + (p + q + r) + \lambda = 60 - 8 = 52$ ....(2)

$x + (p + q) + \lambda = 25$ ....(3)

$y + (q + r) + \lambda = 26$ ....(4)

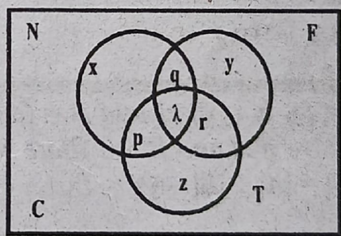
$z + (r + p) + \lambda = 26$ ....(5)

$q + \lambda = 9$ ....(6)

$p + \lambda = 11$ ....(7)

$r + \lambda = 8$ ....(8)

$n(u) = 60$



Adding eq. (6), (7), (8),

$(p + q + r) + 3\lambda = 28$ ....(9)

Adding eq. (3), (4), (5)

$(x + y + z) + 2(p + q + r) + 3\lambda = 77$ ....(10)

Let $x + y + z = a$, $p + q + r = b$

then eq. (2) gives

$a + b + \lambda = 52$ ....(11)

Eq.(9) gives $b + 3\lambda = 28$ ....(12)

and eq. (10) gives $a + 2b + 3\lambda = 77$ ....(13)

Doing eq.(11) + (12) − (13) we have

$\lambda = 52 + 28 - 77 = 3$ ....(14)

Hence from eq. (12)

$$b = 28 - 3 \times 3 = 19 \quad ....(15)$$

From eq. (11), $a = 52 - b - \lambda$

$$= 52 - 19 - 3 = 30 \quad ....(16)$$

Those who read only one magzine $= x + y + z = 30$

Ans.(b) Since f and g are one-to-one functions, therefore we have

$$f(x_1) = f(x_2) \quad \Rightarrow x_1 = x_2 \text{ for all } x_1, x_2 \in R$$

$$g(y_1) = g(y_2) \quad \Rightarrow y_1 = y_2 \text{ for all } y_1, y_2 \in R$$

$$(gof)(x_1) = (gof)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$$

$$\Rightarrow f(x_1) = f(x_2)$$

$$\Rightarrow x_1 = x_2$$

Also as f and g are both onto functions, so for each $z \in C$, there exists $y \in B$ such that $g(y) = z$ and $f(x) = y$ for each $x \in A$. Thus $z = g(y) = g(f(x)) = (g \circ f)(x)$. Hence each element $z \in C$ has preimage under $g \circ f$ and therefore $g \circ f$ is an onto function. Since it has been proved that $g \circ f$ is one-one onto function, therefore $(g \circ f)$ exists. This implies that if $g \circ f : A \to C$, then $(g \circ f)^{-1} : C \to A$ or $g^{-1} o f^{-1}$: $C \to A$.

Now $(g \circ f)^{-1}(z) = x \Leftrightarrow (g \circ f)(x) = z$

$$\Leftrightarrow g(f(x)) = z \text{ or } g(y) = z$$

$$\Leftrightarrow y = g^{-1}(z)$$

$$\Leftrightarrow f^{-1}(y) = f^{-1}(g^{-1}(z)) = (f^{-1} o g^{-1})(z)$$

$$\Leftrightarrow x = (f^{-1} o g^{-1})(z).$$

Hence $(g o f)^{-1}(z) = (f^{-1} o g^{-1})(z)$

or $(g o f)^{-1} = (f^{-1} o g^{-1})$.

Q.36 ***(a)*** ***Let $A = \{a, b, c, d, e\}$ and $B = \{c, e, f, h, k, m\}$ then prove if A and B are finite sets then $|A \cup B| = |A| + |B| - |A \cap B|$.***

[R.T.U. 2011]

(b) ***Let 100 of 120 students of mathematics of a college take at least one of the languages Hindi, English and German. Also suppose 65 study Hindi, 45 study English, and 42 study German. If 20 study Hindi and English, 25 study English and German and 15 study Hindi and German. Find the number of students who study all the three languages.***

[R.T.U. 2011, 2009]

Ans.(a) $A = \{a, b, c, d, e\}$

$n(A) = |A| = 5$

$B = \{c, e, f, h, k, m\}$

$n(B) = |B| = 6$

Now, $A \cup B = \{a, b, c, d, e, f, h, k, m\}$

$\therefore$ $|A \cup B| = 9$

and $A \cap B = \{c, e\}$

$\therefore$ $|A \cap B| = 2$

Thus, we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$9 = 5 + 6 - 2$$

Hence Proved.

Ans.(b) Let H, E and G denote the set of students who study Hindi, English and German respectively.

Then,

$n(H \cup E \cup G) = 100$

$n(H) = 65$, $n(E) = 45$, $n(G) = 42$,

$n(H \cap E) = 20$, $n(E \cap G) = 25$, $n(H \cap G) = 15$

We have to find $n(H \cap E \cap G)$

By the principle of inclusion – exclusion we have

$$n(H \cup E \cup G) = n(H) + n(E) + n(G) - n(H \cap E)$$

$$= - n(E \cap G) - n(H \cap G) + n(H \cap E \cap G)$$

$$100 = 65 + 45 + 42 - 20 - 15 - 25 + n(H \cap E \cap G)$$

$$n(H \cap E \cap G) = 160 - 152 = 8$$

Thus, 8 students study all three languages.

Q.37 ***Determine whether each of the following function is bijection (one to one and one to many) from R to R.***

(i) $f(x) = -3x + 4$

(ii) $f(x) = -3x^2 + 7$

(iii) $f(x) = x^5 + 1$ ***[R.T.U. 2011, MREC 2002]***

Ans. (i) $f(x) = -3x + 4$ to prove one–one for two input if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ for all value of x_1 and x_2 in domain so it is one–one

To prove onto for all values of x in R it will give all the values of R in result so it is onto.

(ii) $f(x) = -3x^2 + 7$

To prove one–one if $x_1 \neq x_2$ then $f(x_1) = f(x_2)$ so this not one–one such as for 1 and –1 it will give result 4. So this is not bijection function.

(iii) $f(x) = x^5 + 1$

To prove one–one if $x_1 \neq x_2$ then if $f(x_1) \neq f(x_2)$ for all values of x_1 and x_2 so this is one–one

To prove onto, for all values for x in R, it will give all values in R. So this is a bijection function.

Q.38 *(a) Show that in the set of integers $I = \{..., -2, -1, 0, 1, 2, ...\}$, the relation $aRb \Rightarrow a \equiv b \pmod{n}$, $n \in N$, is an equivalence relation.*

(b) Show that an equivalence relation defined in a set A decomposes the set into disjoint classes.

[R.T.U. 2011, 2009]

Ans. (a) The relation is $a \equiv b \pmod{n}$ *i.e.* a is congruent to b (mod n) meaning that

$a - b$ = multiple of $n = np$, $p \in I$

For equivalence relation, we show that given relation is reflexive, symmetric and transitive.

(i) Reflexive: Let $a \in I$, then

$a - a = 0 = 0 \cdot n$ = multiple of n

$\Rightarrow aRa$

$\therefore$ R is reflexive.

(ii) Symmetric: Let $a, b \in I$, then

$aRb \Rightarrow a \equiv b \pmod{n} \Rightarrow a - b = np, p \in I$

$\Rightarrow b - a = -np = n(-p), -p \in I$

$\Rightarrow b \equiv a \pmod{n} = bRa$

$\therefore$ R is symmetric.

(iii) Transitive: Let $a, b, c \in I$, and

$aRb \Rightarrow a - b = n\,p_1$ ....(i)

$bRc \Rightarrow b - c = n\,p_2$ ....(ii)

(i) + (ii) $\Rightarrow a - c = n(p_1 + p_2), p_1 + p_2 \in I$

$\Rightarrow a \equiv c \pmod{n}$

$\Rightarrow aRc$,

$\therefore$ R is transitive.

Hence, it is an equivalence relation.

Ans.(b) Let an equivalence relation R be defined in A. Let $a \in A$ and T be a subset of A consisting of all those elements which are equivalent to a *i.e.*,

$T = \{x \mid x \in A \text{ and } a\,R\,x\}$.

Then $a \in T$, for $a\,R\,a$ (R is reflexive). Any two elements of T are equivalent to each other, for if $x, y \in T$, then $x\,R\,a$ and $y\,R\,a$.

Again $x\,R\,a$, $y R\,a \Rightarrow x\,R\,a$, $a\,R\,y$ [R is symmetric]

$\Rightarrow x\,R\,y$

Hence T is an equivalence class.

suppose T_1 be another equivalence class *i.e.*,

$T_1 = \{x \mid x \in A \text{ and } x\,R\,b\}$,

where b is not equivalent to a. Then the classes T and T_1 must be disjoint. For if they have a common element A, $A\,R\,a$ and $s\,R\,b$, so that $b\,R\,a$ which is contrary to our hypotheses.

Now the set A can be decomposed into equivalence classes T, T_1, T_2, ... such that every element of A belongs to one of these classes. Since these classes are mutually disjoint, we obtain the required partition of A.

Q.39 *(a) Let S be any non empty set. Show that $(P(S), \subseteq)$ is a partially ordered set for the relation inclusion.*

(b) Let R be the relation defined on a set of natural numbers N s.t. for $x, y \in N$, $xRy \Leftrightarrow x - y$ is divisible by 3, then show that R is an equivalence relation on N. Find equivalence classes also.

[R.T.U. 2008]

Ans.(a) Let A_1, A_2, A_3 be three subsets of S so they are elements of P(S).

Relation is reflexive : $A_1 \subseteq A_1$ is true for any set S, so the 'set inclusion' relation is reflexive.

Relation is anti-symmetric : $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$ both be true only when $A_1 = A_2$ so the relation is anti-symmetric.

Relation is transitive : Whenever $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$, both are true then $A_1 \subseteq A_3$ is also true so relation is transitive also. So relation is partial order relation.

Hence $\{P(S), \subseteq\}$ is a partially ordered set for the relation inclusion.

Ans.(b) Here R is $x - y$ is divisible by 3 or $x - y$ is multiple of 3.

Reflexive : Let $x \in N$

$x - x = 0 \times 3$ is a multiple of 3.

$\Rightarrow$ $x\,R\,x\ \forall x \in N$

$\therefore$ R is reflexive.

Symmetric : Let $x, y \in N$

then $x\,R\,y \Rightarrow x \equiv y \pmod{3}$

$\Rightarrow$ $x - y$ is divisible by 3.

$\Rightarrow x - y = 3k$

$\Rightarrow y - x = -3k$

$y \equiv x \pmod{3}$

$\Rightarrow y\,R\,x$

$\therefore$ R is symmetric.

Transitive: Let $x, y, z \in N$

$x\,R\,y \Rightarrow x - y = 3\,k_1$

$y\,R\,z \Rightarrow y - z = 3k_2$

$x - z = 3(k_1 + k_2)$

$x \equiv z \pmod{3}$

$x\,R\,z$

$\therefore$ R is transitive.

Hence it is an equivalence relation.

Q.40 ***If $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$ be functions on R, the set of the real numbers then find :***

(i) gof (ii) fog (iii) fof (iv) gofg (v) foh (vi) hog (vii) hof (viii) (foh)og **[Raj. Univ. 2006]**

Ans. Given that $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$

(i) gof(x) $= g[f(x)] = g[x+2] = x+2-2 = x$

(ii) fog(x) $= f[g(x)] = f[x-2] = x-2+2 = x$

(iii) fof(x) $= f[f(x)] = f[x+2] = x+2+2 = x+4$

(iv) gog(x) $= g[g(x)] = g[(x-2)] = x-2-2 = x-4$

(v) foh(x) $= f[h(x)] = f[3x] = 3x+2$

(vi) hog(x) $= h[g(x)] = h[x-2] = 3(x-2) = 3x-6$

(vii) hof(x) $= h[f(x)] = h[x+2] = 3(x+2) = 3x+6$

(viii) (foh)og(x) $= foh[g(x)] = foh[x-2]$
$= f[3(x-2)] = f(3x-6) = 3x-6+2 = 3x-4$

PART-C

Q.41 ***Let $R = \{(1, 2), (2, 3), (3, 1)\}$ and $A = \{1, 2, 3\}$. Find reflexive, symmetric and transitive closure of R using-***

(a) Composition of relation R

(b) Composition of matrix relation R

(c) Graphical representation of R **[R.T.U. 2019]**

Ans.(a) The reflexive closure of R, denoted by $R^{(r)}$, is given by

$R^{(r)} = R \cup I_A = \{(1, 2),(2, 3),(3, 1)\} \cup \{(1, 1),(2, 2),(3, 3)\}$
$= \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}$

The symmetric closure of R, denoted by $R^{(S)}$, is given by

$R^{(S)} = R \cup R^{-1} = \{(1, 2), (2, 3), (3, 1)\} \cup \{(2, 1), (3, 2), (1, 3)\}$
$= \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$

Now, $RoR = \{(1, 2), (2, 3), (3, 1)\} \text{ o } \{(1, 2), (2, 3), (3, 1)\}$

$R^2 = \{(1, 3), (2, 1), (3, 2)\}$

$R^3 = R^2oR = \{(1, 3), (2, 1), (3, 2)\} \text{ o } \{(1, 2), (2, 3), (3, 1)\}$
$= \{(1, 1), (2, 2), (3, 3)\}$

$R^4 = R^3oR = \{(1, 1), (2, 2), (3, 3)\} \text{ o } \{(1, 2), (2, 3), (3, 1)\}$
$= \{(1, 2), (2, 3), (3, 1)\} = R$

Thus, $R^5 = R^4oR = RoR = R^2$, $R^6 = R^5oR = RoR = R^2oR = R^3$ and so on.

Hence, the transitive closure of R, denoted by R^*, is given by

$R^* = R \cup R^2 \cup R^3 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(b) Let M be the relation matrix of R. Then

$$\therefore \quad M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The symmetric closure matrix of R, denoted by $M_R^{(S)}$ is given by

$$M_R^{(S)} = M_R \vee M_R^T$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

so that $R^{(S)} = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$

The reflexive closure matrix of R, denoted by $M_R^{(r)}$, is given by

$$M_R^{(r)} = M_R \vee I_3$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

so that $R^{(r)} = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}$

Now $M_R^2 = M_R \cdot M_R$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_R^3 = M_R^2 \cdot M_R$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transitive closure relation matrix of R, denoted by, M_R^* is given by

$$M_R^* = M_R \vee M_R^2 \vee M_R^3$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

so that R* = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

(c) The graphical representation of R is shown in Fig.1

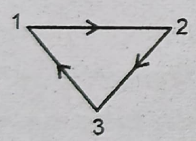


Fig. 1

To find out the reflexive closure representation of R we add all the arrows from points to themselves which is shown in Fig. 2.

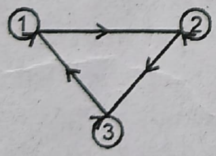


Fig. 2

To find the symmetric closure representation of R, we add missing reverses of all arrows in graphical representation of R. This is shown in Fig. 3.

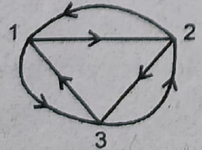


Fig. 3

To find transitive arrow, we add arrow 1 to 1 since $1 \to 2 \to 3 \to 1$. Similarly, 2 to 2 and 3 to 3. Again we add arrow 1 to 3, since $1 \to 2 \to 3$. Similarly, 2 to 1 and 3 to 2. This is shown in Fig. 4.

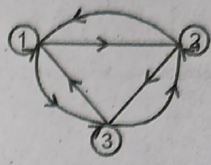


Fig. 4

Q.42 ***What is the Pigeonhole principle and the Extended Pigeonhole principle also prove both.*** ***[R.T.U. 2012]***

OR

State and prove the generalized pigeonhole principle. ***[R.T.U. 2017]***

OR

Explain pigeonhole and extended pigeonhole principle with example. ***[R.T.U. 2010]***

OR

State and prove the Pigeonhole and Generalized Pigeonhole Principles. ***[R.T.U. 2014]***

Ans. Pigeonhole Principle

If n pigeonholes are occupied by n pigeons and m > n then at least one pigeonhole is occupied by more than one pigeon.

The pigeon hole principle is nothing more than the obvious remark : if you have fewer pigeon holes than pigeons and you put every pigeon in a pigeon hole, then there must result at least one pigeon hole with more than one pigeon. It is surprising how useful this can be as a proof strategy.

The pigeonhole principle, also known as *Dirichlet's* box (or drawer) principle, states that, given two natural numbers *n* and *m* with n > m, if *n* items are put into m pigeonholes, then at least one pigeonhole must contain more than one item. Another way of stating this would be that *m* holes can hold at most *m* objects with one object to a hole; adding another object will force one to reuse one of the holes, provided that m is finite. More formally, the theorem states that there does not exist an injective function on finite sets whose co-domain is smaller than its domain.

The pigeonhole principle is an example of a **counting argument** which can be applied to many formal problems, including ones involving infinite sets that cannot be put into one-to-one correspondence.

Generalizations of the Pigeonhole Principle

A generalized version of this principle states that, if *n* discrete objects are to be allocated to *m* containers, then at least one container must hold no fewer than [n/m] objects, where [x] is the ceiling function, denoting the smallest integer larger than or equal to x.

A probabilistic generalization of the pigeonhole principle states that if n pigeons are randomly put into m pigeonholes with uniform probability 1/m, then at least one pigeonhole will hold more than one pigeon with probability

$$1-\frac{m!}{(m-n)!m^n}=1-\frac{(m)_n}{m^n}\left(\because (m)_n=\frac{\sqrt{m+1}}{\sqrt{m-n+1}}\right)$$

where $(m)_n$ is falling factorial, for n = 0 and for n = 1 (and m > 0), that probability is zero; in other words, if there is just one pigeon, there cannot be conflict. For n > m (more pigeons than pigeonholes) it is one, in which case it coincides with the ordinary pigeonhole principle. But even if the number of pigeons does not exceed the number of pigeonholes ($n \le m$), due to the random nature of the assignment of pigeons to pigeonholes there is often a substantial chance that clashes will occur. For example, if 2 pigeons are randomly assigned to 4 pigeonholes, there is a 25% chance that at least one pigeonhole will hold more than one pigeon; for 5 pigeons and 10 holes, that probability is 69.76%; and for 10 pigeons and 20 holes it is about 93.45%. This problem is treated at much greater length at birthday paradox.

Pigeonhole Principle : Simple Form

Theorem : If n + 1 objects are put into n boxes, then at least one box contains two or more objects.

Proof : Suppose none of the n boxes contains more than one object. Then the total number of objects would be at most n. This is a contradiction, since there are at least n + 1 objects.

Example : There are n married couples. How many of the 2n people must be selected in order to guarantee that one has selected a married couple?

Q.43 ***Prove by mathematical induction that $3^n > n^3$ for all integers $n \ge 4$.*** ***[R.T.U. 2017]***

Ans. Let $P(n) = 3^n > n^3$ where $n \ge 4$.

Basic Step : P(n) is true since $3^4 = 81 > 4^3 = 64$

Inductive step : Let P(k) is true for all $k \ge 4$, i.e. $3^k > k^3$.

Then we need to show that P(k+1) is true, i.e. $3^{k+1} > (k+1)^3$

Let us rewrite

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

$$= k^3\left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$$

Since $3^k > k^3$ (using P(k)), we would be done if we could also prove that $3 > \left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$ for $k \ge 4$.

Observe that the function $f(k) = 1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}$ decreases as k increases, so that f(k) is largest when k is smallest. In other words, f(4) is the largest value of f(k), where $k \ge 4$.

Since

$$f(4) = 1+\frac{3}{4}+\frac{3}{4^2}+\frac{1}{4^3}$$

$$= \frac{125}{64}$$

is obviously less then 3. We have, for any integer $k \ge 4$,

$$3 > \left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$$

Thus combining the two facts :

$3 > \left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$ and $3^k > k^3$ for $k \ge 4$, we can multiply and get

$$3^{k+1} > k^3\left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$$

or $\quad 3^{k+1} > (k+1)^3$

So P(k+1) is true and then by mathematical induction P(n) is true for all integers $n \ge 4$, i.e. $3^n > n^3$.

Q.44 ***Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.*** ***[R.T.U. 2016]***

OR

Define principle of mathematical induction and hence prove that $6^{(n+2)} + 7^{(2n+1)}$ is divisible by 43 for each positive integer n. ***[R.T.U. 2008]***

Ans. Principle of Mathematical Induction : Let p(n) be some statement involving a positive integer n. Suppose it is required to show that p(n) is true for integer n greater than some fixed integer n_0. The method adopted here to prove is called the *method of mathematical induction*. This method consists of the following three steps :

Basis Step : First it is verified whether p(n) is true for certain number. Generally we take n_0 to have the value one. If p (n) is not true for n = 1 then the least value of n found for which it is true.

Hypothesis : Then it is assumed that p (n) is true for n = k.

Inductive Step : Taking p(n) to be true for n = k, it is proved that p(n) is true also for the next value n = (k+1).

Since it has been found to be true for $n = n_0$, so it is true for $n = n_0 + 1$. When it is true for $n = n_0 + 1$, it is true for the next value $n = n_0 + 1 + 1$.

Arranging in this way, it is concluded that p(n) is true for all positive integer values of $n \ge n_0$. The above method of proving a proposition p(n) involving a positive integer n is called the *Method of Mathematical Induction* or the *Principle of Mathematical Induction.*

Proof

Let $p(n): 43$ divides $6^{n+2}+7^{2n+1}$

Basis Step : Let n = 1, then p(1) is true

$$= 6^{1+2}+7^{2+1}=6^3+7^3=216+343$$

$= 559$, which is divisible by 43.

Inductive Hypothesis : Let p(k) be true for $k \geq 1$.

$6^{k+2}+7^{2k+1}$ is divisible by 43 (say 43λ).

Inductive Step : We shall show that

$p(k+1): 6^{k+3}+7^{2k+3}$ is divisible by 43, is true whenever p(k) is true.

$$6^{k+3}+7^{2k+3}=6^{k+2}.6^1+7^{2k+1}.7^2$$

$$=6^{k+2}.6+7^{2k+1}.(43+6)$$

$$=43.7^{2k+1}+6\left(6^{k+2}+7^{2k+1}\right)$$

$$=43.7^{2k+1}+6.43\lambda=43\,(7^{2k+1}+6\lambda)$$

$\Rightarrow$ $6^{k+3}+7^{2k+3}$ is divisible by 43.

Now

$\Rightarrow$ p (k + 1) is true.

Hence by principle of mathematical induction 43 divides $6^{(n+2)}+7^{(2n+1)}$.

Q.45 *(a) Prove that $A-B = A \cap B' = B' \cap A'$*

(b) Consider the following collection of subsets $\{A_1, A_2, A_3\}$ of a set $A = \{1,2,3,4,5,6,7,8,9,10\}$

[a] [{1,6,9}, {2,3,8}, {4,5,7,10}]

[b] {1}, {2,4,8}, {5,7,9}] and

[c] [{1,5}, {2,3,8}, {4,5,6,7,9,10}]

Determine which one is a partition of a set A

(c) Let f, g, h be mapping from N to N when N is the set of naturals such that f(n) = n + 1,

$$g(n)=2n,\ h(n)=\begin{cases}0, n \text{ is even}\\ 1, n \text{ is odd}\end{cases}$$

(i) Show that f, g and h are functions

(ii) Determine fof, fog, hog and (fog) oh

Where 'o' stands for composition of functions

[R.T.U. 2016]

Ans.(a) Prove that $A-B = A \cap B' = B' - A'$

Let $x \in A-B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \notin A'$ and $x \in B'$

$\Leftrightarrow x \in B'$ and $x \notin A'$

$\Leftrightarrow x \in B'-A'$

Thus $A-B = B'-A'$

Again

$x \in A-B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \in B'$

$\Leftrightarrow x \in A \cap B'$

thus $A-B = A \cap B'$

from (1) and (2)

$$A-B = A \cap B' = B'-A'$$

Ans.(b) $A=\{1,2,3,4,5,6,7,8,9,10\}$

A family of sets A_1, A_2, A_3 is called partition of A if,

(i) $A_j \cap A_k = \phi$ for any k # j

(ii) Their union completely covers the space, $S = A_1 \cup A_2 \ldots A_n$

[a] $A_1=\{1,6,9\}$

$A_2=\{2,3,8\}$

$A_3=\{4,5,7,10\}$

$A_1 \cap A_2 = \{\phi\}$

$A_1 \cap A_3 = \phi$

$A_2 \cap A_3 = \phi$

Also, $A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,6,7,8,9,10\} = A$

$\therefore A_1, A_2, A_3$ is partition of A.

[b] $A_1=\{1\}$

$A_2=\{2,4,8\}$

$A_3=\{5,7,9\}$

There, $A_1 \cap A_2 = \phi$

$A_2 \cap A_3 = \phi$

$A_1 \cap A_3 = \phi$

Now, $A_1 \cup A_2 \cup A_3 = \{1,2,4,5,7,8,9\} \neq A$

$\therefore A_1, A_2, A_3$ is not the partition of A.

[c] $A_1=\{1,5\}$,

$A_2=\{2,3,8\}$,

$A_3=\{4,5,6,7,9,10\}$

Now, $A_1 \cap A_2 = \phi$

$A_2 \cap A_3 = \phi$

$$A_1 \cap A_2 = \{5\}$$

$\Rightarrow$ A_1, A_2 and A_3 is not a partition of A.

Ans.(c) $f(n) = n+1,$

$$g(n) = 2n,\ h(n) = \begin{cases} 0, \text{ n is ever} \\ 1, \text{ n is odd} \end{cases}$$

(i) As we know that a relation is a function if it is defined for all values of domain and $f(x) \neq f(y) = x \neq y, \forall x, y \in N$.

f is a function

Here $f(n) = n+1, n \in N$.

Also, f is defined for all $n \in N$ let $f(n_1) \neq f(n_2)$

$$n_1+1 \neq n_2+1 \Rightarrow n_1 \neq n_2$$

$\therefore$ f is a function.

g is a function

Here, $g(n) = 2n$

$\because$ g(n) exists for all $n \in N$ and

$$g(n_1) \neq g(n_2)$$
$$g(n_1) \neq g(n_2) = 2n_1 \neq 2n_2$$
$$\Rightarrow n_1 \neq n_2$$

$\therefore$ g is a function.

h is a function

$$h(n) = \begin{cases} 0, \text{ n is ever} \\ 1, \text{ n is odd} \end{cases}$$

$\because$ $0 \notin N$, so h(n) doesn't exist for even n and then h is not a function.

(ii)
$$fof = (fof)(n) = f(f(n))$$
$$= f(n+1)$$
$$= n+1+1$$
$$= n+2$$
$$(fog)(n) = f(g(n)) = f(2n)$$
$$= 2n+1$$
$$(hog)(n) = h(g(n))$$
$$= h(2n)$$
$$= 0$$
$$(fog)oh = (fog)(h(n))$$

$$= \begin{cases} fog(0), \text{ n is even} \\ fog(1), \text{ n is odd} \end{cases} = \begin{cases} f[g(1)], \text{ n is even} \\ f[g(1)], \text{ n is odd} \end{cases}$$

$$= \begin{cases} f(0), \text{ n is even} \\ f(2), \text{ n is odd} \end{cases} = \begin{cases} 1, \text{ n is even} \\ 3, \text{ n is odd} \end{cases}$$

Q.46 *(a) State and prove the Principle of Inclusion and Exclusion for three sets A,B and C.*

(b) There are 250 students in a computer Institute of these 180 have taken a course in Pascal, 150 have taken a course in C++, 120 have taken a course in Java. Further 80 have taken Pascal and C++, 60 have taken C++ and Java, 40 have taken Pascal and Java and 35 have taken all 3 courses. So find-

(i) How many students have not taken any course?

(ii) How many study at least one of the languages?

(iii) How many students study only Java?

(iv) How many students study Pascal and C++ but not Java?

(c) Let A = {1,1,1,2,2,3,4,4} and B = {1,2,4,4,5,5,5}. Find $A \cup B$, $A \cap B$, $A - B$ and A+B. **[R.T.U. 2015]**

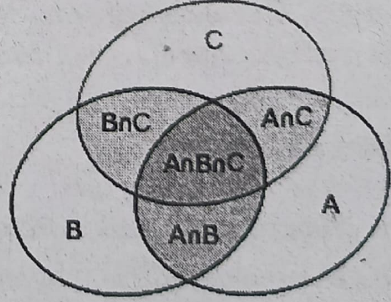
Ans.(a) In combinatorics (combinatorial mathematics), the inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite or more sets.

For three sets A, B, C, the inclusion–exclusion principle is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Where, A, B, C are three finite sets.

Proof : Using Venn diagram



Now, we can also prove it otherwise :

Lets say we have three sets A, B and C.

A big hint is to prove the result for three sets, A, B, C, given the result for two sets.

$|A \cup B| = |A| + |B| - |A \cap B|$

We start by writing the three-fold union as the union of two sets:

$|A \cup B \cup C| = |(A \cup B) \cup C|$

and so we can use the two-set result on $A \cup B$ and C, so

$|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$

Now, we can use the two-set result on $|A \cup B|$ to get

$|A \cup B \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$

$= |A| + |B| - |A \cup B| + |C| - |(A \cup B) \cap C|$

$= |A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C|$

Now, use the distributive property on the last term:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$

$- |(A \cap B)| \cup (B \cap C|$

and use the two-set property yet again on the last term, with sets $A \cap C$ and $B \cap C$ to get

$|(A \cap C) \cup (B \cap C)| = |(A \cap C)| + |(B \cap C)|$

$- |(A \cap C) \cap (B \cap C)|$

$= |(A \cap C)| + |(B \cap C)| - |A \cap B \cap C|$

Finally, we substitute this into our big expression (remembering that it was negated) to get

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$

$- |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Ans.(b) Given that,

Total (N) = 250

Pascal(P) = 180

C++ (C) = 150

Java (J) = 120

PC = 80

CJ = 60

PJ = 40

PCJ = 35

No. of students who have taken atleast one course

$= P \to C \to J$

$= P + C + J - PC - CJ - PJ + PCJ$

$= 180 + 150 + 120 - 80 - 60 - 40 + 35$

$= 305$

Looks like the total no. of students given is wrong. Most probably it will be N = 350. In that case,

(i) How many students have not taken any course?

= Total – (No.of students who have taken atleast one course)

$= 350 - 305$

$= 45$

(ii) How many study atleast one of the languages?

= no. of students who have taken atleast one course

$= 305$

(iii) How many study only java?

$= J - PJ - CJ + PCJ$

$= 120 - 60 - 40 + 35$

$= 55$

(iv) How many study Pascal and C++ but not Java?

$= PC - PCJ$

$= 80 - 35$

$= 45$

Ans.(c) Given that,

$A = \{1,1,1,2,2,3,4,4\}$

$B = \{1,2,4,4,5,5,5\}$

Since, we are performing set operations on A and B, then A and B are sets and therefore,

$A = \{1,1,1,2,2,3,4,4\} = \{1,2,3,4\}$

$B = \{1,2,4,4,5,5,5\} = \{1,2,4,5\}$

Now,

(i) $A \cup B = \{1,2,3,4,5\}$

(ii) $A \cap B = \{1,2,4\}$

(iii) $A - B = A - (A \cap B)$

$= \{1,2,3,4\} - \{1,2,4\}$

$= \{3\}$

(iv) $A + B = A \cup B$ {'+' is nothing but union of sets}

$= \{1,2,3,4,5\}$

Q.47 ***(a) Let R be a relation defined on a set of orderded pairs of positive integers such that for all $(x,y),(u,v) \in Z^+ \times Z^+$, $(x,y)\ R\ (u, v)$ if and only if $\frac{u}{x} = \frac{v}{y}$. Determine whether R is an equivalence relation.***

(b) Let A = { 1,2,3,4} and R = {(a,b):a+b>4} be a relation on A. Draw the graph of the relation R.

(c) Let R be an equivalence relation on a set of positive integers defined by x R y if and only if $x \equiv y(mod 3)$. Then, find the equivalence class of 2 and also find the partition generated by the equivalence relation. ***[R.T.U. 2015]***

Ans.(a) For a relation to be equivalent, it needs to be reflexive, symmetric and transitive.

Reflexive: A relation is said to be reflexive if it contains (a,a) for all possible a

R is reflexive because (x,y)R(u, v) is contained in R for all combinations of (x,y) because $\frac{u}{x} = \frac{v}{y} = 1$.

Symmetry: A relation is said to be symmetric if for every (a,b) pair it contains, it also contains a corresponding (b,a) pair.

R is symmetric because if a pair ((x,y),(u,v)) satisfies the relation (u/x = v/y) then it also satisfies the relation (x/u = y/v) and hence pair ((u,v),(x,y)) is also in the relation.

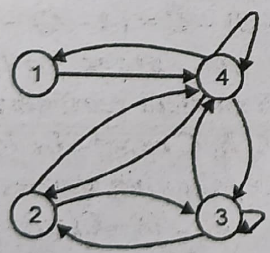
Transitive: A relation is said to be transitive if for every (a,b) and (b,c) pair it contains, it also contains an (a,c) pair.

R is transitive because for all pairs ((x,y),(u,v))&((u,v),(w,z)) it contains, it satisfies the relation x/w = y/z by the mathematical property a= b & b=c implies a = c.

Since, R is reflexive, symmetric and transitive, R is an equivalent relation.

Ans.(b) A = {1, 2, 3, 4} R = {c, b) | a + b > 4}. Draw graph of R.

R contains {(1, 4), (2, 3), (2, 4), (3, 3), (3, 4), (3, 2), (4, 1), (4, 2), (4, 3), (4, 4)}



Graph of R has an edge between vertex i & j if (i, j) ∈ R.

Ans.(c) Here equivalence relation is

$x \equiv y \pmod 3$

Which means on dividing y by 3 the remainder should be equal to reminder when x is divided by 3.

Relation is defmed on set of positive integers.

$x \equiv y \pmod 3$

is read "x is congruent to y modulo 3d"

Here (4, 7) will belong to the relation as dividing 4 by 3 gives remainder 1 & dividing 7 by 3 also gives 1.

R = {(1, 4), (4, 1), (2, 5), (4, 7), (7, 4) ...; (x, y)|x ≡ y mod 3}

Equivalence Class of 2

It is the class of (x, y) from the relation of such y values which relates to x = 2.

As (2, 2), (2, 5), (2, 8), (2, 11) ∈ R

Equivalence Class of 2, [|Z|] = {2, 5, 8, 11, 14, 17, ...}

For final partition by equivalence relation will find all classes,

[|0|] = {3, 6, 9, 12, 15, ...}

[|1|] = {1, 4, 7, 10, 13, 16, ...}

[|2|] = {2, 5, 8, 11, 14, 17, ...}

[|3|] = {3, 6, 9, 12, 15, ...} → similiar to [|0|]

So no further partioning by class. Partition by equivalence relation can be written as

[{1, 4, 7, 10, 13, 16, ...}, {2, 5, 8, 11, 14, 17, ...}, {3, 6, 9, 12, 15, ...}]

Q.48 *(a) If A, B, C be finite sets then*

$n(A \cup B \cup B) = n(A) + n(B) + n(C) - n(A \cap B) - (B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

(b) Participation in sports is compulsory in a school. In a class of 80 students, 60 play football 40 play basket ball. Find

(i) How many play both the games

(ii) Play football only **[R.T.U. 2013]**

Ans. (a) Suppose that $A \cup B = X$

Then $n(X) = n(A) + n(B) - n(A \cap B)$ ...(1)

Also $n(A \cup B \cup C) = n(X \cup C)$

$= n(X) + n(C) - n(X \cap C)$

$= n(X) + n(C) - n[(A \cup B) \cap C]$

$= n(X) + n(C) - n[(A \cap C) \cup (B \cap C)]$

$= n(X) + n(C) - \{n(A \cap C) + n(B \cap C) - n(A \cap C \cap B)\}$ ...(2)

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A)$

$+ n(A \cap B \cap C)$

Ans. (b) Let F and B are the costs of students who play football and basket ball, respectively,

Given

$|F| = 60$

$|B| = 40$

We know that

(i) $|A \cup B| = |A| + |B| - |A \cap B|$

$$80 = 60 + 40 - |A \cap B|.$$

$20 = |A \cap B|$

So 20 students play both the games.

(ii) Play football only

$\because$ given that 60 play football and as we calculated that 20 students play both games so, students who play football only are,

$= 60 - 20 = 40$ students.

Q.49 *(a) Let R be an equivalence relation an A, and let p be the collection of all distinct equivalence classes [a] for a $\in$ A. Then show that P is a partition of A and R is the equivalence relation determine by P.*

(b) In the set of natural number N = {1, 2, ...} show that the relation defined as a R b$\Leftrightarrow$ a = b^k for a, b, R $\in$ N is a partial order relation.

[R.T.U. 2012; Raj. Univ. 2006, 2002, MNIT 2003]

Ans. (a) Theorem : Let X be a non-empty set. Then every partition of X induces an equivalence relation on X, and every equivalence relation induces a partition of X.

Proof : Let R be an equivalence relation on X, Let P be the collection of distinct equivalence classes of X wrt R :

$$P = \{(x)\} : x \in X.$$

It's clear that the union of subsets [x] of X in P is all of X as x ranges over X. If $z \in [x] \cap [y]$ then $(x, z), (y, z) \in R$, which, by symmetry and transitivity gives $(x, y) \in R$, so $[x] = [y]$. It follows that P is a partition of X.

On the other hand if P is a partition of X, define a relation R on X by $(x, y) \in R$ if x and y belong to the same subset of X contained in P. Clearly R is reflexive since the union of the subsets of X in P is X. R is clearly symmetric, and finally if $x, y \in P$ and $y, z \in P$ and $x, z \in P$, so R is transitive.

Since R is reflexive, symmetric and transitive, it follows that R is an equivalence relation. Note the the equivalence classes are exactly the members of P.

Ans. (b) The relation R defined on set of natural numbers is

$$a R b \Leftrightarrow a = b^k \qquad \forall a, b, k \in N$$

(i) To show R is reflexive

$\forall a \in N$ we have $a = a^1, 1 \in N$

$\therefore R$ is reflexive

(ii) To show R is antisymmetric we have to prove that if aRb then bRA when a = b

Suppose a R b and bRa .

$\Rightarrow a = b^{k_1}$ and $b = a^{k_2}$, $k_1 k_2 \in N$

$\therefore a = b^{k_1} = \left(a^{k_2}\right)^{k_1} = a^{k_1 k_2} \quad \therefore k_1 = k_2 = 1$

Since $k_1, k_2 \in N$, we must have $k_1 = k_2 = 1$ thus $a = b$

$\therefore$ R is antisymmetric.

(iii) To show R is transitive

Let aRb and bRc then

$a = b^{k_1}$ and $b = c^{k_2} \quad k_1, k_2 \in N$

$\because a = b^{k_1} = c^{k_1 k_2} \quad$ as $k_1 k_2 \in N$

$\Rightarrow a = c^k \qquad \{k = k_1 k_2 \in N\}$

$\Rightarrow a R c \therefore$ R is transitive

Since R is reflexive , antisymmetirc and transitive .

Hence R is a partial order relation.

Q.50 *(a) Let A, B and C are arbitrary sets, show that :*

(i) (A – B) – C = A – (B ∪ C)

(ii) (A – B) – C = (A – C) – B

(iii) (A – B) – C = (A – C) – (B – C)

Do not prove with examples. Give mathematical reasoning.

(b) Let A, B, C be sets. Under what conditions each of the following statements are true ?

(i) (A–B) ∪ (A–C) = A (ii) (A–B) ∪ (A–C) = ϕ

(iii) (A – B) ⊕ (A – C) = ϕ

Justify your answer. *[Raj. Univ. 2005, 2003, 1998, 1995]*

Ans. (a) (i) Let $x \in (A - B) - C$

$\Rightarrow x \in (A - B)$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow x \in A$ and $x \notin B$ and $x \notin C$

$\Rightarrow x \in A - (B \cup C)$

Hence $(A - B) - C \subseteq A - (B \cup C)$ .....(1)

Again let $x \in A - (B \cup C)$

$\Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow$ $x \in A$ and ($x \notin B$ and $x \notin C$)

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$\Rightarrow$ $x \in (A - B)$ and $x \notin C$

$\Rightarrow$ $x \in (A - B) - C$

Hence $A - (B \cup C) \subseteq (A - B) - C$ .....(2)

$\therefore$ from eq. (1) and (2)

$\Rightarrow (A - B) - C = A - (B \cup C)$ **Hence Proved.**

(ii) Let $x \in (A-B)-C$

$\Rightarrow$ $x \in (A - B)$ and $x \notin C$

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$\Rightarrow$ ($x \in A$ and $x \notin C$) and $x \notin B$

$\Rightarrow$ $x \in (A-C)$ and $x \notin B$

$\Rightarrow$ $x \in (A-C)-B$

Hence $(A-B)-C \subseteq (A-C)-B$ .....(1)

Again Let $x \in (A-C)-B$

$\Rightarrow$ $x \in (A-C)$ and $x \notin B$

$\Rightarrow$ ($x \in A$ and $x \notin C$) and $x \notin B$

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$\Rightarrow$ $x \in (A-B)$ and $x \notin C \Rightarrow$ $x \in (A-B)-C$

Hence $(A-C)-B \subseteq (A-B)-C$ .....(2)

$\therefore$ from (1) and (2)

$(A - B) - C = (A - C) - B$ **Hence Proved.**

(iii) Let $x \in (A-B)-C$

$\Rightarrow$ $x \in (A-B)$ and $x \notin C$

($x \in A$ and $x \notin B$) and $x \notin C$

$\Rightarrow$ ($x \in A$ and $x \notin C$) and $x \notin B$

$\Rightarrow$ $x \in (A-C)$ and $x \notin (B-C)$

$\Rightarrow$ $x \in (A-C)-(B-C)$

Hence $(A - B) - C \subseteq (A - C) - (B - C)$ .....(1)

Again, let $x \in (A - C) - (B - C)$

$\Rightarrow$ $x \in (A-C)$ and $x \notin (B-C)$

$\Rightarrow$ ($x \in A$ and $x \notin C$) and $x \notin (B-C)$

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$[\because x \notin C, x \notin (B-C) \Rightarrow x \notin B]$

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$\Rightarrow$ $x \in (A-B)$ and $x \notin C$

$\Rightarrow$ $x \in (A-B)-C$

$\Rightarrow$ $(A-C)-(B-C) \subseteq (A-B)-C$ .....(2)

$\therefore$ From eq. (1) and eq. (2)

$(A - B) - C = (A - C) - (B - C)$ **Hence Proved.**

Ans.(b) (i) $(A - B) \cup (A - C) = A$

It is true if B or C is null set or .....(1)

A and B are disjoint sets or ....(2)

A and C are disjoint sets ....(3)

Since if eq. (1) is true then $B = \phi$ or $C = \phi$

$\Rightarrow A - B = A$ or $A - C = A$

and then $(A - B) \cup (A - C) = A$.

If eq. (2) is true then

$A - B = A$

$\Rightarrow (A - B) \cup (A - C) = A \cup (A - C) = A$

If eq.(3) is true then $A - C = A$

$\Rightarrow (A - B) \cup (A - C) = (A - B) \cup A = A$

(ii) $(A - B) \cup (A - C) = \phi$

It is true if eq. (1) $A = B = C$ or eq. (2) A is a null set *i.e.* $A = \phi$

As if eq. (1) is true *i.e.* $A = B = C$ then $A - B = \phi$ and $A - C = \phi$ so it follows that

$(A - B) \cup (A - C) = \phi \cup \phi = \phi$

If eq. (2) is true *i.e.* $A = \phi$ then $A - B = \phi - B = \phi$

and $A - C = \phi - C = \phi$ so it follows that

$(A - B) \cup (A - C) = \phi \cup \phi = \phi$

(iii) $(A - B) \oplus (A - C) = \phi$

It is true if

(1) $A \subseteq B$ and $A \subseteq C$ or

(2) $A \subseteq B$ and $A \subseteq C$ but $B = C$

As, if eq. (1) is true *i.e.* $A \subseteq B$ and $A \subseteq C = A - B = \phi$ and $A - C = \phi$

So it follows that $(A - B) \oplus (A - C) = \phi \oplus \phi$

$= (\phi \cup \phi) - (\phi \cap \phi) = \phi - \phi = \phi$

If eq. (2) is true *i.e.*

$A \notin B$, $A \notin C$ but $B = C$

then it follows that

$(A - B) \oplus (A - C) = (A - B) \oplus (A - B)$

$[(A-B) \cup (A-B)] - [(A-B) \cap (A-B)] = (A-B) - (A-B) = \phi$

□□□

PROPOSITIONAL LOGIC 2

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 *Define finite state machines.* ***[R.T.U. 2019]***

Ans. Finite automation as a machine equipped with an input tape. The machine works on a discrete time scale. At every point of time the machine is in one of its states, then it reads the next letter on the tape (the letter under the reading head), or maybe nothing (in the first variation), and then, according to the transition function (depending on the actual state and the letter being read, if any) it goes to a/the next state. It may happen in some variation that there is no transitions defined for the actual state and letter, then the machine gets stuck and cannot continue its run.

Q.2 *In how many ways can a team of 11 cricketers be chosen for 6 bowlers, 4 wicket keepers and 11 batsmen to give majority of batsmen so that at least 4 bowlers are there and 1 wicketkeeper?* ***[R.T.U. 2019]***

Ans. 1 wicketkeeper can be selected in C(4, 1) ways

4 bowlers chosen = C(6, 4)

Remaining 6 batsmen = C(11, 6)

Total possibilities = C(4, 1) * C(6, 4) * C(11, 6) = 27720

The batsmen has to be majority. So the split cannot be 1 WC, 5 Bowlers, 5 Batsmen. It can only be 1WC, 4 bowlers and 6 batsmen.

Q.3 *Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.* ***[R.T.U. 2017]***

Ans. $(p \wedge q) \rightarrow (p \vee q)$

First, we construct the truth table

p	q	$p \wedge q$	$p \vee q$	$p \wedge q \rightarrow p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Since in the last column, all are true *i.e.* T therefore $p \wedge q \rightarrow p \vee q$ is a tautology.

Q.4 *Find PCNF of a statement S whose PDNF is $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$.* ***[R.T.U. 2017]***

Ans. First we obtain PDNF of ~s, which is the sum (disjunction) of those minterms which are not present in the given PDNF of s. Hence the PDNF of ~s is

$$(\sim P \wedge \sim q \wedge \sim r) \vee (\sim P \wedge \sim q \wedge r) \vee (P \wedge \sim q \wedge \sim r) \vee (\sim P \wedge q \wedge \sim r)$$

Thus, the PCNF of $S \equiv$ [PDNF of (~s)], i.e.

$$\equiv \sim ((\sim P \wedge \sim q \wedge \sim r) \vee (\sim P \wedge \sim q \wedge r) \vee (P \wedge \sim q \wedge \sim r) \vee (\sim P \wedge q \wedge \sim r))$$

$$\equiv (P \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim P \vee q \vee r) \wedge (P \vee \sim q \vee r)$$

Q.5 *Explain the following for propositions with example:*

(i) Logical Equivalence

(ii) Tautological Implication

(iii) Normal Forms ***[R.T.U. 2015]***

Ans.(i) Logical Equivalence : Any two propositions for wihich the truth table is same are said to be LOGICALLY EQUIVALENT.

Ex. $p \rightarrow q$ & $\neg p \vee q$

p	q	$p \to q$	$\neg q$	$(\neg p \vee q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	T	F	T

(ii) Tautological Implication : Compound statements which are always true regardless of the truth or false of component statements are called tautologies. Obviously, the truth table of a tautology will contain only T entries in the last column.

Example : The statement $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(iii) Normal Forms : A literal L is either on after P or its negation ($\neg P$). A clause D is a disjunction literals. A formula C is in NORMAL FORM if it is a conjunction of clause.

$L \to P | \neg P$
$D \to L | L \vee D$
$C \to D | D \wedge C$

Example : (a) $(\neg q \vee p \vee r) \wedge (\neg p \vee r) \wedge q$
(a) $(p \vee r) \wedge (\neg p \vee r) \wedge (p \vee \neg r)$

Q.6 ***Over the universe of animals, let***
P(x) : x is a whale ; Q(x) : x is a fish
R(x) : x lives in water.
Translate the following into English
$\exists x (\sim R(x))$
$\exists x (Q(x) \wedge \sim P(x))$
$\forall x (P(x) \wedge R(x)) \to Q(x)$ *[R.T.U. 2014]*

Ans. $\exists x(\sim R(x))$: There exists an animal which does not lives in water.

$\exists x(Q(x) \wedge \sim P(x))$: There exists a fish that is not a whale.

$\exists x(P(x) \wedge R(x)) \to Q(x)$: Every whale that lives in the water, is a fish.

Q.7 ***Consider the following :***
p : It is hot today
q : The temperature is 35°C
Write in simple sentence the meaning of the following :
(i) $p \vee q$ ***(ii)*** $\sim(p \vee q)$ ***(iii)*** $\sim(p \wedge q)$
(iv) $\sim p \wedge \sim q$ *[R.T.U. 2013]*

Ans. Given

p : It is hot today

q : The temperature is 35°C

(i) $p \vee q$

Either it is hot today or the temperature is 35° C

(ii) $\sim(p \vee q)$

Neither it is hot today nor the temperature is 35° C

(iii) $\sim(p \wedge q)$

It is not hot today and the temperature is not 35° C

(iv) $\sim p \vee \sim q$

It is not hot today and the temperature is not 35° C

PART-B

Q.8 ***(a) Show that*** $\sim(p \vee (\sim p \wedge q)) \equiv (\sim p) \wedge (\sim q)$ ***(without truth table)***

(b) Write contrapositive converse and inverse of the statement "The home team wins whenever it is raining". Also construct the truth table for each statement. *[R.T.U. 2019]*

Ans.(a) $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim(\sim p \wedge \sim q)$ [De Morgan's Law]

$\equiv \sim p \wedge (\sim\sim p \vee \sim q)$ [De Morgan's Law]

$\equiv \sim p \wedge (p \vee \sim q)$ [Double Negation Law]

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ [Distributive Law]

$\equiv F \vee (\sim p \wedge \sim q)$ [$\because \sim p \wedge p \equiv F$]

$\equiv \sim p \wedge \sim p$

Ans.(b) The given proposition is in the form "q whenever p" such that,

q (Conclusion) : The home team wins.

p (hypothesis) : It is raining.

Converse : $q \to p$ is "If the home team wins then it is raining",

Inverse : $\neg p \to \neg q$ is "If it is not raining then the home team does not win".

Contra positive: $\neg q \to \neg p$ is "If the home team does not win then it is not raining".

p	q	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Q.9 Draw the transition diagram of the finite state machine M (I, S, O, S_0, f, g), where I = {a, b}, S = {S_0, S_1}, O = {0, 1} and the transition table is as follows-

I / S	f: a	f: b	g: a	g: b
S_0	S_1	S_0	0	1
S_1	S_0	S_1	1	0

Also, find the output string for the input b b a a.

[R.T.U. 2019]

Ans. FSM, M (I, S, O, S_0, f , g)

I = {a, b}

S = {S_0, S_1}

O = {0, 1}

$A \to I$

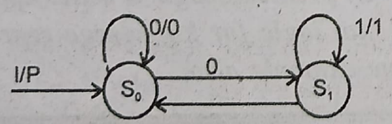
$z \to O$

1. A finite set I of alphabet
2. A finite set S of internal state
3. A finite state Z of O/P symbol
4. An initial state S_0 in S
5. A next state function f from S × I into S.
6. An op state functions of from S × I into Z.

Transition of function $f(f : S \times I \Rightarrow P(S))$

S_0 - initial state

F - finite state



Transitional diagram (Result)

Output string for bbaa is 1101.

Q.10 In a test 70% of the candidate passed in Science, 65% in Mathematics, 27% failed in both Science and Mathematics and 124 passed in both the subjects. Find the total number of candidates for the test.

[R.T.U. 2019]

Ans. 70% passed in science, 65% passed in mathematics, 27% failed in both.

$100 - 27 = 73\%$

$n(S) = 70\%, n(M) = 65\%, n(S \cap M) = 73\%$

$n(S \cup M) = n(S) + n(M) - n(S \cap M)$

$= 70 + 65 - 73 = 62\%$

$\because$ 62 passed in both subjects then total no of students = 100

$\because$ 124 passed in both subjects then total no of students

$\therefore \frac{100}{62} \times 124 = 200.$

Q.11 Obtain the Principal disjunctive normal forms of $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$. ***[R.T.U. 2019]***

Ans.

p	q	r	p∧q≡a	~p	~p∧r≡b	q∧r≡c	a∨b∨c
T	T	T	T	F	F	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	F	F
T	F	F	F	F	F	F	F
F	T	T	F	T	T	T	T
F	T	F	F	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	F	F	F

PDNF of $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

$= ((p \wedge q) \wedge (r \vee \sim r)) \vee ((\sim p \wedge r) \wedge (q \vee \sim q)) \vee ((q \wedge r) \wedge (p \vee \sim p))$

$= (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$

$= (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim p)$

(Deletion of identical min terms)

Q.12 (a) Define Tautology, contradiction and contingency. Determine the contrapositive of each statement:

(i) If John is a poet, then he is poor.

(ii) Only if Mary studies will she pass the test.

(b) Determine the validity of the argument :

All men are fallible

All kings are Men

Therefore, all kings are fallible. ***[R.T.U. 2017]***

Ans.(a) Tautology : A compound statement that is always true for all possible truth values of its propositional variables, is called a tautology. Obviously, its truth table contains only truth value T in the last column.

Example : *Refer to Q.5(ii).*

Contradiction : A compound statement that is always false, is called a contradiction. Obviously its truth table contains only false value in the last column.

Contingency : A statement that is neither a tautology nor a contradiction is called contingency. So its truth table contains both T and F values at least once in its last column.

The contrapositive of $p \to q$ is $\sim q \to \sim p$.

(i) Let p : John is a poet

q : He is poor.

Hence the contrapositive of the given statement is :

If join is not poor, then he is not a poet.

(ii) The given statement is equivalent to "If mary passes the test, then she studied".

Let p : Mary passes the test

q : she studied.

Hence its contrapositive is :

If Mary does not study, then she will not pass the test.

Ans.(b) Let

M(x) : x is a Man

K (x) : x is a king

F (x) : x is fallible

Then the argument takes the form

$$\forall x[M(x) \to F(x)]$$

$$\frac{\forall x[k(x) \to M(x)]}{\therefore \forall x[k(x) \to F(x)]}$$

A formal proof is as follows:

S.No.	Step	Reason
1.	$\forall x\,[M(x) \to F(x)]$	Premise 1
2.	$M(c) \to F(c)$	Step 1 and universal instantiation
3.	$\forall x[k(x) \to M(x)]$	Premise 2
4.	$K(c) \to M(c)$	Step 3 and universal instantiation
5.	$K(c) \to F(c)$	Hypothetical syllogism using step 2 and 4
6.	$\forall x[K(x) \to F(x)]$	Step 5 and universal generalization

Hence the argument is valid.

Q.13 *(a) Prove that $(\sim p \wedge q) \to [\sim(q \to p)]$ is a tautology with constructing truth table.*

(b) Obtain the principle disjunctive normal form of $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$ by constructing truth table. *[R.T.U. 2016]*

Ans.(a) To prove : $(\sim p \wedge q) \to [\sim(q \to p)]$

Truth table is as follows:

p	q	~p	~p ∧ q	~p→(p→q)
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

Since all the values are true, it is a tautology.

Ans.(b) *Refer to Q.11.*

Q.14 *Translate each of these statements into logical expressions using predicates, quantifiers and logical connectives.*

(i) No one is perfect

(ii) Not everyone is perfect

(iii) All your friends are perfect

(iv) One of your friend is perfect

(v) Everyone is your friend and perfect

(vi) Not every body is your friend or someone is not perfect. *[R.T.U. 2016]*

Ans. Let p(x): x is perfect

q(x) : x is your friend

And the universe of discourse be the set of all people. Then,

(i) $\forall x \sim p(x)$

(ii) $\sim(\forall x(p(x)))$

(iii) $\forall x(q(x) \to p(x))$

(iv) $\exists x(p(x) \wedge q(x))$

(v) $\forall x(p(x) \wedge q(x))$

(vi) $(\forall x \sim q(x)) v (\exists x \sim p(x))$

Q.15 *Explain why predicate logic is better approach than propositional logic for knowledge representation? Give some example also.* *[R.T.U. 2016]*

Ans. (i) Predicate logic also known as predicate calculus, and quantification theory- is a collection of formal systems used in mathematics, philosophy, linguistics, and computer science. Predicate logic admits quantified variables over non-logical objects and allows the use of sentences that contain such variables, so that rather than just propositions such as Socrates is a man one can have expressions in the form X is a man where X is a variable. This distinguishes it from propositional logic, which does not allow quantifiers.

(ii) Though propositional logic is one of the simplest languages that demonstrates all the important points, it has a very limited ontology, making only the commitment that, the world consists of facts. This made it difficult to represent even something as simple as the wumpus world. Predicate logic makes a stronger set of ontological commitments. The main one is that the world consists of objects, that is, things with individual identities and properties that distinguish them from other objects.

(iii) First-order logic can also express facts about all of the objects in the universe. This, together with the implication connective from propositional logic, enables one to represent : general laws or rules.

(iv) Firstorder logic is universal in the sense that it can express anything that can be programmed.

(v) It is by far the most studied and best understood scheme yet devised.

Q.16 ***Find the DNF of following:***

(i) $P \rightarrow ((p \rightarrow Q) \wedge \sim (\sim P \vee \sim P))$

(ii) $\sim (P \rightarrow (Q \wedge R)).$ ***[R.T.U. 2015]***

Ans. (i) A : P ® $((P \rightarrow Q) \wedge \neg(\neg P \vee \neg P))$

A DNF is an OR of AND

we know that $a \rightarrow b \equiv \neg a \vee b$

So, $A \equiv \neg P \vee ((P \rightarrow Q) \wedge \neg(\neg P \vee \neg P)$

we also know, $a \vee a \equiv a$ & $\neg(\neg a) = a$

So $A \equiv \neg P V ((P \rightarrow Q) \wedge P)$

$A \equiv \neg P \vee ((\neg P \vee Q) \wedge P)$

$A \equiv \neg P \vee ((\neg P \wedge P) \wedge (Q \wedge P))$

$A \equiv \neg P \vee T \vee (Q \wedge P)$ (To true)

($\therefore a \wedge \neg a$ is a tautology)

$A \equiv \neg(P \rightarrow (Q \wedge p)) \rightarrow$ DNF

(ii) $B \equiv \neg(P \rightarrow (Q \wedge R))$

$B \equiv \neg(\neg P \vee (Q \wedge R))$

$\neg(\neg P \vee Q) \wedge (\neg P \vee R)$

By de Morgan's Law $\neg(a \wedge b) = (\neg a \vee \neg b)$

$B \equiv (\neg(\neg P \vee Q)) \vee (\neg(\neg P \vee R))$

$B \equiv (P \wedge \neg Q) \vee (P \wedge \neg R)$

Q.17 ***Without constructing the truth table, show that*** $(\sim p \wedge (P \vee Q)) \rightarrow Q$ ***is a tautology.*** ***[R.T.U. 2015]***

Ans. We have,

$(\sim P \wedge (P \vee Q)) \rightarrow Q$

$\equiv \sim(\sim P \wedge (P \vee Q)) \vee Q$

$\{$using, $A \rightarrow B \equiv \sim A \vee B\}$

$\equiv (P \vee (\sim(P \vee Q))) \vee Q$

$\equiv (P \vee (\sim P \wedge \sim Q)) \vee Q$

$\equiv ((P \vee \sim P) \wedge (P \vee \sim Q)) \vee Q$

$\equiv (TRUE \wedge (P \vee \sim Q)) \vee Q$

$\equiv (P \vee \sim Q) \vee Q$

$\equiv P \vee \sim Q \vee Q$

$\equiv$ TRUE

Hence, it is a tautology.

Q.18 ***State the difference between deterministic and non deterministic finite automata.*** ***[R.T.U. 2015, 2014]***

Ans. Deterministic and Non-deterministic Finite Acceptors

S. No.	Deterministic Finite Acceptors	Non- deterministic Finite Acceptors
1.	For every symbol of the alphabet, there is only one state transition.	We do not need to specify how does the NFA react according to some symbol.
2.	Cannot use empty string transition.	Can use empty string transition.
3.	Can be understood as one machine.	Can be understood as multiple little machines computing at the same time.
4.	It will reject the string if it end at other than accepting state.	If all of the branches of NFA dies or rejects the string, we can say that NFA reject the string.
5.	The transition function is single valued.	The transition function is multi-valued.
6.	Checking membership is easy.	Checking membership is difficult.
7.	Construction is difficult.	Construction is easy.
8.	Space required is more.	Space required is comparatively less.
9.	Backtracking is allowed.	Not possible in every case.
10.	Can be constructed for every input and output.	Cannot be constructed for every input and output.
11.	There can be more than one final state.	There can only be one final state.

Q.19 ***Prove*** $pV (q \Lambda r) \equiv (p V q) \Lambda (p V r).$ ***[R.T.U. 2014]***

Ans. $pV (q \Lambda r) \equiv (p V q) \Lambda (p V r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since the column (5) and (8) are identical.

So we have $pV (q \Lambda r) \equiv (p V q) \Lambda (p V r)$.

Q.20 ***Show that propositional formula $(p \wedge q) \wedge (r \wedge s) \Rightarrow P$ for any propositions p,q,r,s is a tautology.*** *[R.T.U. 2013, 07]*

Ans. Given $(p \wedge q) \wedge (r \wedge s) \Rightarrow P$

p	q	r	s	$(p\wedge q)$	$(r\wedge s)$	$(p\wedge q)\wedge(r\wedge s)$	$(p\wedge q)\wedge(r\wedge s)\to p$
T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	F	T	F	T
F	T	T	T	F	T	F	T
T	T	F	F	T	F	F	T
T	F	F	T	F	F	F	T
F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	T
T	F	T	T	F	T	F	T
F	T	F	F	F	F	F	T
T	F	F	F	F	F	F	T
F	F	F	T	F	F	F	T
F	F	T	F	F	F	F	T
F	T	F	F	F	F	F	T
F	F	F	F	F	F	F	T

$\because (p\wedge q)\wedge(r\wedge s)\to p$ is always true

Hence proved that the given propositional formula is a tautology.

Q.21 ***Using propositional Logic, prove the validity of the argument. $[(p\vee \sim q)] \Rightarrow r \wedge (r \Rightarrow s) \wedge p \Rightarrow s$*** *[R.T.U. 2012]*

Ans. Suppose

p : I will study Physics

q : I will study Maths.

r : I will study Accounts

s : I will study Computer.

$(p \vee \sim q)$: I will study Physics or I will not study Maths.

$(p \vee \sim q) \to r$: If I will study Physics or I will not study Maths, then I will study Accounts.

$r \to s$: If I will study Accounts, then I will study Computer.

$p \to s$: If I will study Physics, then I will study Computer.

From the last two arguments we can see that Computer will be studied if either Physics or Accounts is studied. From the second and third argument we can see that

$(p \vee \sim q) \to s$

Thus, if Physics is studied or Maths is not studied, then Computer is studied is valid because if Physics is studied then by fourth statement Computer is studied and if Maths is not studied then either Physics or Accounts is studied which leads to study of Computer.

Q.22 *(a)* ***Show that $p \to q \equiv (\sim p) \vee q$***

(b) ***Show that $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.*** *[R.T.U. 2011]*

Ans. (a)

p	q	$\sim p$	$p \to q$	$(\sim p)\vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Thus, from truth table we have proved that $p \to q \equiv (\sim p) \vee q$.

Ans. (b)

p	q	$p\wedge q$	$p\vee q$	$\sim(p\vee q)$	$(p\wedge q)\wedge\sim(p\vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since in last column all the false result show that $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

Q.23 *(a)* ***Obtain principal conjunctive normal form of S where $S \equiv (\sim p \to r) \wedge (q \leftrightarrow p)$.***

(b) ***Check the validity of following argument***

$$\begin{array}{l} p \vee q \\ p \to \sim q \\ p \to r \\ \hline r \end{array}$$

[R.T.U. 2011, 2009]

Ans.(a) For obtaining the PCNF of S, we prepare the truth table

p	q	r	$\sim p$	$\sim p \to r$	$p \leftrightarrow q$	$(\sim p \to r)\wedge(p\leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
$\boxed{T}$	$\boxed{F}$	$\boxed{T}$	F	T	F	$\boxed{F}$
$\boxed{T}$	$\boxed{F}$	$\boxed{F}$	F	T	F	$\boxed{F}$
$\boxed{F}$	$\boxed{T}$	$\boxed{T}$	T	T	F	$\boxed{F}$
$\boxed{F}$	$\boxed{T}$	$\boxed{F}$	T	F	F	$\boxed{F}$
F	F	T	T	T	T	T
$\boxed{F}$	$\boxed{F}$	$\boxed{F}$	T	F	T	$\boxed{F}$

Here F appears in last column in III, IV, V, VI and VIII row. The required principal conjunctive normal form (PCNF) is

$(\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee r)$.

Ans.(b) Constructing truth table to verify the given argument is valid or not as :

Row No.	p	q	r	~q	p_1 p ∨ q	p_2 p → ~ q	p_3 p → r	Q r
0	T	T	T	F	T	F	T	T
1	T	T	F	F	T	F	F	F
2	T	F	T	T	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$
3	T	F	F	T	T	T	F	F
4	F	T	T	F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$
5	F	T	F	F	$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	$\boxed{F}$
6	F	F	T	T	F	T	T	T
7	F	F	F	T	F	T	T	F

Since in 5th row all the three premises namely $p \vee q$, $p \to \sim q$ and $p \to r$ are true (T) but the corresponding truth value of the conclusion is false (F) so invalid.

Q.24 ***Define the Contradiction.*** ***[R.T.U. 2011]***

Ans. Contradiction : If a compound statement is false for all value assignments for its component statements, then it is called a contradiction, *i.e.* a compound statement is said to be contradicting its truth value if it is false (F) for all its entries in the truth table. If a statement is a contradiction then its negation will be a tautology.

Example : The statement $p \wedge \sim p$ is a contradiction. The truth table for the given statement is as follows :

p	~p	p ∧ ~p
T	F	F
F	T	F

Q.25 ***Prove that the following implications are tautologies :***

(i) $p \wedge q \to p \vee q$

(ii) $\sim p \to (p \to q)$ ***[R.T.U. 2011]***

Ans.(i) $p \wedge q \to p \vee q$ **:** *Refer to Q.3.*

(ii) $\sim p \to (p \to q)$

p	q	~p	p → q	~p → (p → q)
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Since in the last column all entries are true.

So $\sim p \to (p \to q)$ is a tautology.

Q.26 ***Prove the following equivalences :***

(i) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

(ii) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$ ***[R.T.U. 2010]***

Ans.(i) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

p	q	r	p → q	p → r	p ∧ r	(p → q) ∧ (p → r)	p → (q ∧ r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

From the 7 and 8 columns we see that

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ **Hence proved**

(ii) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

p	q	r	p → r	q → r	(p ∨ r)	(p → r) ∧ (q → r)	(p ∨ q) → r
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	T	T	T	T	T
T	F	F	F	T	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	T

From the last two columns we see that

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$ **Hence proved**

PART-C

Q.27 ***Define fallacy and prove the following-***

$(p \wedge q) \vee \sim (p \wedge q)$ *is ∧ fallacy* ***[R.T.U. 2019]***

Ans.Fallacy : A compound preposition, that is always false for any assignment of false value to the prepositional variable is called fallacy.

p	q	p ∧ q	~ (p ∧ q)	(p ∧ q) ∨ ~ (p ∧ q)
T	T	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

$(p \wedge q) \vee \sim (p \wedge q)$ fallacy.

Q.28 Define tautology and prove the following-
$\{(p \to q) \wedge p\} \to q$ is tautology ***[R.T.U. 2019]***

Ans. Tautology : A compound preposition that is always true for any assigned of the truth values to the prepositional variable is called Tautology

p	q	$p \to q$	$(p \to q) \wedge p$	$(p \to q) \wedge p \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$\{(p \to q) \wedge p\} \to q$ Tautology

Q.29 Describe the block diagram of a finite automaton. Consider the transition system given below :

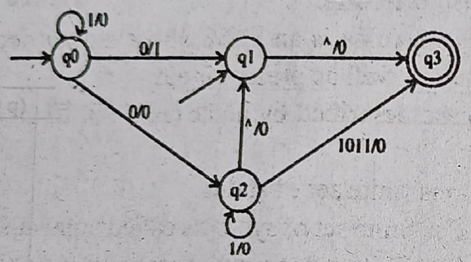


Fig.

Determine the initial states, the final state and the acceptability of 101011 and 111010.

[R.T.U. 2016]

Ans. Finite Automaton : A finite automaton can be represented by a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

(i) Q is a finite non-empty set of states.

(ii) Σ is a finite non-empty set of inputs called input alphabets.

(iii) δ is a function which maps $Q \times \Sigma$ into Q and is usually called direct transition function. This is the function which describes the changes of states during the transition. This mapping is usually represented by a transition table or a transition diagram.

(iv) $q_0 \in Q$ is the initial state.

(v) $F \subseteq Q$ is the set of final states. It is assumed here that there may be more than one final state.

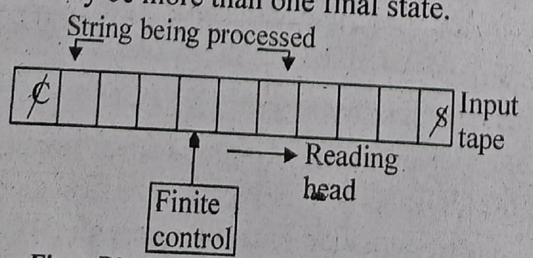


Fig. : Block diagram of finite automaton

(vi) Input Tape : The input tape is divided into squares, each square containing a single symbol from the input alphabet Σ. The end square of the tape contain end-markers ¢ at the left end and $ at the right end. Absence of end-markers indicates that the tape is of infinite length. The left-to-right sequence of symbol between the end markers is the input string to be processed.

(vii) Reading Head : The head examines only one square at a time and can move one square either to the left or to the right. For further analysis, we restrict the movement of R-head only to the right side.

(viii) Finite Control : The input to the finite control will be usually : symbol under the R-head, say a, or the present state of the machine, say q, to give the following outputs : (a) A motion of R-head along the tape to the next square (In some a null move *i.e.* R-head remaining to the same square is permitted); (b) The next state of a finite state machine given by δ (q, a).

Initial states are q_0 and q_1.

Final state is q_3.

For checking the string acceptability, we start the string from initial state. If we reach the final state after completing the string, then we say that this string is accepted by transition system or not.

For string 101011, the path value is $q_0\ q_0\ q_2\ q_3$. Since q_3 is the final state so this string is accepted by above transition system.

For string 111010, there is no path value. So this string is not accepted by the above transition system.

Q.30 (a) Determine whether the conclusion C follows logically from the premises H_1, H_2 and H_3.

$$H_1 : P \vee Q$$
$$H_2 : P \to R$$
$$\frac{H_3 : \sim Q \vee S}{C : S \vee R}$$

(b) Explain the followings:

(i) Argument

(ii) Predicates

(iii) Quantifiers ***[R.T.U. 2015]***

Ans. (a) We know that,

$$A \to B \equiv \sim A \vee B$$

Now we have,

$$\begin{array}{ll} P \vee Q & \{\text{from H1}\} \\ \hline \sim P \rightarrow Q & \\ Q \rightarrow S & \{\text{From H3}\} \\ \hline \sim P \rightarrow S & \equiv P \vee S \equiv \sim S \rightarrow P \end{array}$$

Hence,

$$\begin{array}{ll} \sim S \rightarrow R & \\ P \rightarrow R & \{\text{from H2}\} \\ \hline \sim S \rightarrow R & \equiv R \vee S \end{array}$$

Hence, yes, the conclusion C follows logically from the premises H1,H2 and H3.

Ans.(b) (i) An arangument is a statement which gives some facts and then claims something. The facts are called a premise & the claim is CONCLUSION. A sound argument is an argument which is valid and which has true premises. A valid argument is an argument in which if the premises are true then conclusion must be true.

(ii) Predicates : A function P which can take two values, TRUE & FALSE

$P : X \rightarrow$ {True, False} (X is any set)

(iii) Quantifiers

The way in which sentence can be turned into a proposition is quantification. A *quantifier* is an operator that limits the variables of a proposition. The two quantifiers, are the universal and existential quantifiers are as follows :

(1) The Universal Quantifier : Suppose that P(x) is a propositional function with domain D. The universal quantification of P(x) is the proposition that asserts that P(x) is true for all values of x in the universe of discourse D *i.e.* 'P(x) is true for all values of x in D' written '$\forall x P(x)$' and read 'for all x P(x)' or 'for every x P(x)'. The symbol $\forall$ is read as 'for all' or 'for every'.

(2) The Existential Quantifier : With existential quantification, we form a proposition that is true if and only if P(x) is true for at least one value in the universe of discourse.

The existential quantification of P(x) is the proposition. 'There exists an element x in domain D such that P(x) is true' denoted by '$\exists x P(x)$' read as 'There exists x such that P(x)' or 'for some x P(x)' or 'There is at least one x such that P(x).' The symbol $\exists$ is read as 'there exists'.

Properties of Quantifiers

The negation of a quantified statement changes the quantifier and also negates the given statement as mentioned below :

(i) $\sim\{\forall x P(x)\} \equiv \exists x \sim P(x)$ [De Morgan's law]

(ii) $\sim(\exists x P(x)) \equiv \forall x \sim P(x)$ [De Morgan's law]

(iii) $\exists x P(x) \rightarrow Q(x) \equiv \forall x P(x) \rightarrow \exists x Q(x)$

(iv) $\exists x P(x) \rightarrow \forall x Q(x) \equiv \forall x (P(x) \rightarrow Q(x))$

(v) $\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

(vi) $\sim(\exists x \sim P(x)) \equiv \forall x P(x)$.

Q.31 ***Discuss Mealy & Moore machines.***

[R.T.U. 2015]

Ans. Finite automata may have outputs corresponding to each transition. There are two types of finite state machine that generate output

(i) Mealy Machine

(ii) Moore Machine

Mealy Machine:

Mealy Machine is an FSM whose output depends on present state as well as present input

It can be described by a six tuple (Q, Σ, O, δ, X, g_0) where :

- Q is finite set of states
- Σ is finite set of symbols called input alphabets
- O is finite set of symbols called output alphabets
- δ is input transition function where $\delta : Q \times \Sigma \rightarrow Q$
- X is output transition function where $X : Q - O$
- g_0 is initial state from where any input is processed (g_0, $\in$Q). State diagram of Mealy machine is shown below :

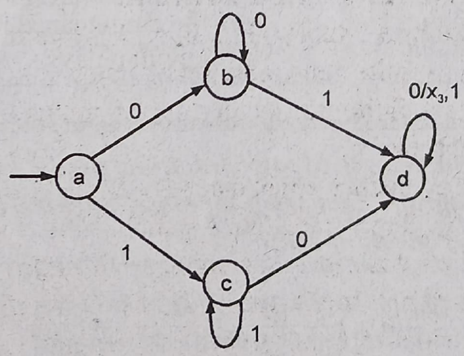


Fig.

Moore Machine:

Moore Machine is an FSM whose outputs depend on only present state. It can be described by a 6 tuple (Q, Σ, O, δ, X, g_0) where :

- Q is a finite set of states
- Σ is a finite set of symbols called input alphabets
- O is a finite set of symbols called output alphabets
- δ is input transition function where $\delta : Q \times \Sigma \rightarrow Q$
- X is the output transition function where $X : Q \times \Sigma \rightarrow 0$

- g_0 is initial state from where any input is processed ($g_0 \in Q$). State diagram is as shown below :

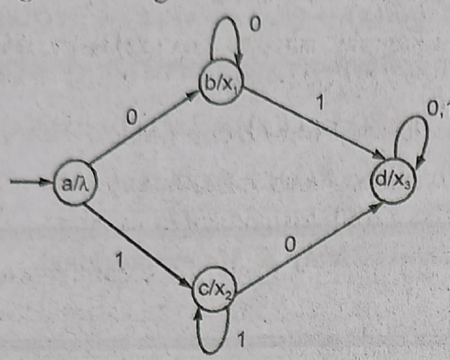


Fig.

Mealy Machine v/s Moore Machine

S. No.	Mealy Machine	Moore Machine
(i)	Output depends both upon present state and present input.	Output depends only upon present state.
(ii)	Generally, it has fewer states than Moore machine.	Generally, it has more states than Mealy machine
(iii)	Output changes at clock edges.	Input change can cause change in output change as soon as logic is done.
(iv)	Mealy machines react faster to inputs.	In moore machines, more logic is needed to decode the output since it has more circuit delays.

Q.32 ***Consider a Mealy machine given by transition diagram. Construct a moore machine equivalent to this mealy machine.***

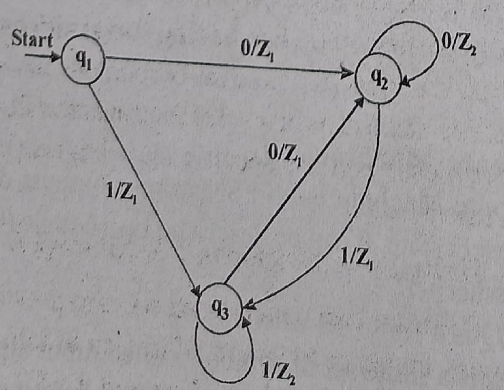


Fig.

[R.T.U. 2013, 2009]

Ans. First of all we have to convert the transition diagram into transition table as follows :

Present state	Next state			
	Input 0	Output	Input 1	Output
$\rightarrow q_1$	q_2	Z_1	q_3	Z_1
q_2	q_2	Z_2	q_3	Z_1
q_3	q_2	Z_1	q_3	Z_2

Step 1 : We look in the next state column for states q_1, q_2 and q_3.

q_1 is associated with no output, q_2 and q_3 are associated with two different outputs *i.e.* Z_1 and Z_2.

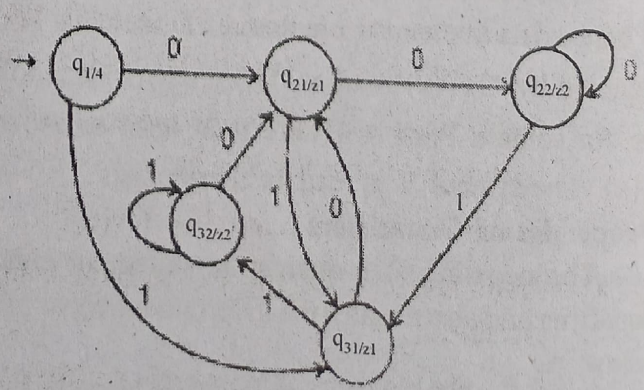
Step 2 : We don't split q_1 now we split q_2 and q_3 into q_{21}, q_{22} and q_{31}, q_{32} respectively. After splitting these state, the transition table we get is as follows :

Present state	Next state			
	Input 0	Output	Input 1	Output
$\rightarrow q_1$	q_{21}	Z_1	q_{31}	Z_1
q_{21}	q_{22}	Z_2	q_{31}	Z_1
q_{22}	q_{22}	Z_2	q_{31}	Z_1
q_{31}	q_{21}	Z_1	q_{32}	Z_2
q_{32}	q_{21}	Z_1	q_{32}	Z_2

Step 3 : Now we rearrange the transition table in such a way that each state in present state column is associated with a single output.

Present state	Next state		Output
	Input 0	Input 1	
$\rightarrow q_1$	q_{21}	q_{31}	0
q_{21}	q_{22}	q_{31}	Z_1
q_{22}	q_{22}	q_{31}	Z_2
q_{31}	q_{21}	q_{32}	Z_1
q_{32}	q_{21}	q_{32}	Z_2

The transition diagram for Moore Machine.



This is the required solution.

Q.33 *Write a note on :*

(a) First Order Logic

(b) Finite state machine as language recognizer

Ans. (a) First Order Logic :

- For some applications, propositional logic is not expressive enough but First-order logic is more expressive: allows representing more complex facts and making more sophisticated inferences.

  For instance, consider the statement "Anyone who drives fast gets a speeding ticket"
- From this, we should be able to conclude "If Joe drives fast, he will get a speeding ticket"
- Similarly, we should be able to conclude "If Rachel drives fast, she will get a speeding ticket" and so on.
- But Propositional Logic does not allow inferences like that because we cannot talk about concepts like "everyone", "someone" etc.
- First-order logic (predicate logic) allows making such kinds of Inferences.

Building Blocks of First-order Logic

- The building blocks of propositional logic were propositions.
- In first order logic, there are three kinds of basic building blocks, constants, variables, predicates.
- **Constants :** refer to specific objects (in a universe of discourse)

  Examples : George, 6, Austin CS311,.....
- **Variables :** range over objects (in a universe of discourse)

  Example : x,y,z, ...
- In universe of discourse is cities in Texas, x can represent Houston, Austin, Dallas, San Antonio....
- **Predicates :** describe properties of objects or relationships between objects.

  Example : ishappy, better than, loves, >...
- Predicates can be applied to both constants and variables.

  Example : ishappy (George), better than (x,y) loves (George, Rachel), x > 3, ...
- A predicate P(x) is true or false depending on whether property P holds for x.

  Example : ishappy (George) is true if George is happy, but false otherwise.

Semantics of First-Order Logic

- In propositional logic, the truth value of formula depends on a truth assignment to variables.
- In FOL, truth value of a formula depends interpretation of predicate symbols and variable over some domain D.
- Consider a FOL formula –P(x)
- A possible interpretation:

  $D = (*, o)$, $P(*) = \text{true}$, $P(o) = \text{false}$, $x = *$
- Under this interpretation, what's value of –P(x)?
- What about of x = o?.

Ans.(b) Finite state machine as language recognizer :

Let A =(Q, T, a_0, δ, F). It is a finite automation (recognizer), where Q is the finite set of (inner) states, T is the input (or tape) alphabet, $a_0 \in Q$ is the initial state, $F \subseteq Q$ in the set of final (or accepting) states and δ is the transition function as follows.

- $\delta : Q \times (T \cup \{\lambda\}) \rightarrow 2^Q$ (for nondeterministic finite automata with allowed λ-transitions):
- $\delta : Q \times T \rightarrow 2^Q$ (for nondeterministic finite automata without λ-transitions):
- $\delta : Q \times T \rightarrow Q$ (for deterministic finite automata, λ can be partially defined):
- $\delta : Q \times T \rightarrow Q$ (for completely defined deterministic finite automata (it is not allowed that δ is partial function, it must be completely defined.))

One can observe, that the second variation is a special case of the first one (not having λ-transitions. The third variation is a special case of the second one having sets with at most one element as images of the transition function, while the fourth case is more specific allowing sets exactly with one element.

Finite State Machines : *Refer to Q.1.*

There are two widely used ways to present automata; by Cayley tables or by graph. When an automaton is given by a Cayley table, then the 0^{th} line and the 0^{th} column of the table are reserved for the states and for the alphabet, respectively (and it is marked in the 0^{th} element of the 0^{th}

row). In some cases it is more convenient to put the states in the 0^{th} row, while in some cases it is a better choice to put the alphabet there. We will look at both possibilities. The initial state should be the first among the states (it is advisable to mark it by a $\rightarrow$ sign also. The final states should also be marked, they should be circled. The transition function is written into the table: the elements of the set $\delta(q,a)$ are written (if any) in the field of the column and row marked by the state q and by the letter a. In the case when λ-transitions are also allowed, then the 0^{th} row of the column (that contains the symbols of the alphabet) should be extended by the empty word (λ)also. Then λ-transitions can also be indicated in the table.

Automata can also be defined in a graphical way: let the vertices (nodes, that are drawn as circles in this case) of a graph represent the states of the automaton (we may write the names of the states into the circles). The initial state is marked by an arrow going to it not from a node. The accepting states are marked by double circles. The labeled arcs (edges) of the graph represent the transitions of the automaton. If $p \in \delta(q, a)$ for some $p,q \in Q$, $a \in T \cup \{\lambda\}$, then there is an edge from the circle representing state q to the circle representing state p and this edge is labeled by a. (Note that our graph concept is wider here than the usual digraph concept, since it allow multiple edges connecting two states, in most cases these multiple edges are drawn as normal edges having more than 1 labels.)

In this way, implicitly, the alphabet is also given by the graph (only those letters are used in the automaton which appear as labels on the edges).

In order to provide even more clarification, we present an example. We describe the same nondeterministic both by a table and by a graph.

TQ	$\rightarrow q_0$	q_1	$\subset q_2 \supset$	$\subset q_3 \supset$
a	q_1	q_1	q_2, q_3	-
b	q_0	q_0	-	q2
c	q_0	q_2	-	q_1, q_2, q_3

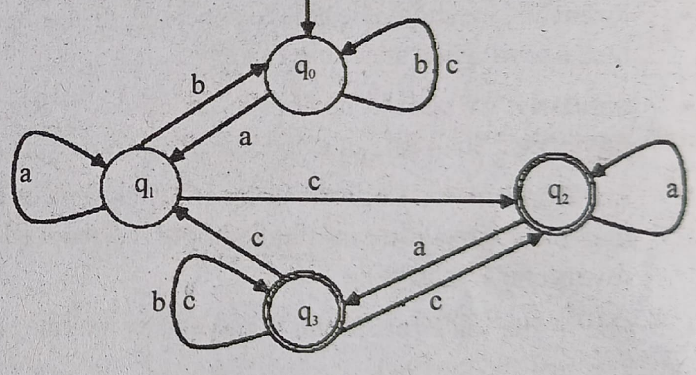


Fig.

□□□

POSETS, HASSE DIAGRAM AND LATTICES

3

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 ***Prove that α^2 is an even integer, then α is an even integer.*** ***[R.T.U. 2019]***

Ans. Let a α^2 is an even integer

$\alpha^2 = 2k$, for some integer k

$$\alpha = \frac{2k}{\alpha}$$

So there is an integer $J = \frac{k}{\alpha}$, such that $\alpha = 2J\ \alpha$ is an even.

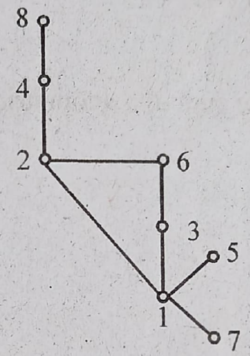
Q.2 ***Give an example of a partially ordered set which is not a lattice.*** ***[R.T.U. 2019]***

Ans. A partially ordered set $(X, \leq)$ is called a lattice if for every pair of elements $x, y \in X$ both the infimum and suprememum of the set $\{x, y\}$ exists. I am trying to get an intuition for how a partially ordered set can fail to be a lattice. In R, for example, once two elements are selected the completeness of the real numbers guarantees the existence of both the infimum and supremum. Now, if we restrict our attention to a nondegenerate interval (a, b) it is clear that no two points in (a, b) have either a suprememum or infimum in (a, b).

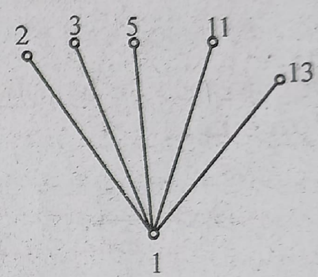
Q.3 ***Draw a Hasse Diagram for (A), (divisibility relation), where***
(i) $A = \{1,2,3,4,5,6,7,8\}$
(ii) $A = \{1,2,3,5,11,13\}$;
(iii) $A = \{2,3,4,5,6,30,60\}$;
(iv) $S = \{1,2,3,6,12,24,\}$;

Ans.

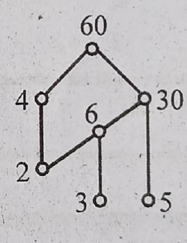
(i) $A = \{1,2,3,4,5,6,7,8\}$



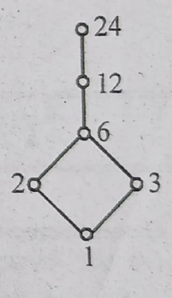
(ii) $A = \{1,2,3,5,11,13\}$



(iii) $A = \{2,3,4,5,6,30,60\}$



(iv) $S = \{1,2,3,6,12,24,\}$



Q.4 ***Solve the following:***

(a) In a city, the bus route numbers consists of a natural number less than 100, followed by one of the letters A,B,C,D,E and F. How many different bus routes are possible?

(b) There are 3 question in a question paper. If the questions have 4,3 and 2 solutions respectively, find the total number of solutions.

Ans.(a) The number can be any me of the natural numbers from 1 to 99.

There are 99 choices for the number.

The letter can be chooses in 6 ways

Number of possible bus routes are : $99 \times 6 = 594$

Ans.(b) Here question 1 has 4 solutions, question 2 has 3 solutions and question 3 has 2 solutions.

By the multiplication rule,

Total number of solutions: $4 \times 3 \times 2 = 24$

Q.5 ***Find the generating functions for the following sequences:***

(a) 1,1,1,1,1,1,0,0,0,0,...

(b) 1,3,3,1,0,0,0,0,...

Ans.(a) The generating function is:

$$G(x) = 1+1x+1x^2+1x^3+1x^5+0x^6+0x^7+\ldots$$
$$= 1+x+x^2+x^3+x^4+x^5$$

We can apply the formula for the sum of a geometric series to rewrite G(x) as

$$G(x) = \frac{1-x^6}{1-x}$$

Ans.(b) The generating function is

$$G(x) = 1+3x+3x^2+1$$

Using formula

$$G(x) = (1+x)^3$$

PART-B

Q.6 ***(a) Solve the recurrence relations-***

$$a_n - 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$$

(b) Prove by induction that sum of the cubes of three consecutive integers is divisible by 9. ***[R.T.U. 2019]***

Ans.(a) $a_n - 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

Particular solution of the above is if the form

$$a_n = P_1n^2 + P_2n + P_3$$

$$P_1n^2 + P_2n + P_3 - 5P_1(n-1)^2 - 5P_2(n-1) - 5P_3 + 6P_1(n-2)^2 + 6P_2(n-2) + 6P_3 = 3n^2 - 2n + 1$$

$$\Rightarrow P_1n^2 + P_2n + P_3 - 5P_1(n^2 - 2n + 1) - 5P_2n + 5P_2 - 5P_3 + 6P_1(n^2 - 4n + 4) + 6P_2n - 12P_2 + 6P_3 = 3n^2 - 2n + 1$$

$$\Rightarrow P_1n^2 + P_2n + P_3 - 5P_1n^2 - 10P_1n - 5P_1 - 5P_2n + 5P_2 - 5P_3 + 6P_1n^2 - 24P_1n + 24P_1 + 6P_2n - 12P_2 + 6P_3$$
$$= 3n^2 - 2n + 1$$

$$\Rightarrow n^2(P_1 - 5P_1 + 6P_1) + n(P_2 + 10P_1 - 5P_2 - 24P_1 + 6P_2) + (P_3 - 5P_1 + 5P_2 - 5P_3 + 24P_1 - 12P_1 + 6P_3)$$
$$= 3n^2 - 2n + 1$$

Equating coefficients of n^2, n and constant term, we get

$$P_1 - 5P_1 + 6P_1 = 3$$
$$P_1 + P_1 = 3$$
$$2P_1 = 3$$
$$P_1 = \frac{3}{2}$$

$$P_2 + 10P_1 - 5P_2 - 24P_1 + 6P_2 = -2$$
$$P_2 + 10 \times \frac{3}{2} - 5P_2 - 24 \times \frac{3}{2} + 6P_2 = -2$$
$$P_2 + 15 - 5P_2 - 36 + 6P_2 = -2$$
$$2P_2 = -2 - 15 + 36$$
$$P_2 = \frac{19}{2}$$

$$P_3 - 5P_1 + 5P_2 - 5P_3 + 24P_1 - 24P_1 + 6P_3 = 1$$
$$19P_1 + 2P_3 - 7P_2 = 1$$
$$19 \times \frac{3}{2} + 2P_3 - 7 \times \frac{19}{2} = 1$$
$$19 \times 3 + 4P_3 - 7 \times 19 = 2$$
$$4P_3 = 2 + 7 \times 19 - 19 \times 3$$
$$P_3 = \frac{78}{4}$$

so the particular solution is

$$a_nP = \frac{3}{2}n^2 + \frac{19}{2}n + \frac{78}{4}$$

Ans. (b) $P(n) = m^3 + (m+1)^3 + (m+2)^3$ is divide by 9.

$$P(n) = m^3 + (m+1)^3 + (m+2)^3 = a\lambda$$
$$P(1) = 1^3 + (1+1)^3 + (1+2)^3$$
$$= 1 + 8 + 27 = 36 = 9 \times 4$$

$aX* \rightarrow P(1)$ is true

Let P(m) be true

$$P(m) : m^3 + (m+1)^3 + (m+2)^3 = a\lambda$$

$$G\ P(m+1) : (m+1)^3 + (m+2)^3 + (m+3)^3 = aK$$

$(m+1)^3+(m+2)^3+(m+3)^3=(m+1)^3+(m+2)^3+m^3 +9m^3+27m+27$

$\Rightarrow\ m^3+(m+1)^3+(m+2)^3+9m^3+27m+27=a\lambda+am^2 +27m+27$

$\Rightarrow\ a(\lambda+m^2+3m+3)=9k$

$\Rightarrow \dfrac{P(1)}{P(m+1)}$ is true.

Q.7 *If the coefficient of $(2r+4)^{th}$ and $(r-2)^{th}$ terms in the expansion of $(1+\chi)^{18}$ are equal, then find the value of r.*

Ans. The general term of $(1+x)^n$ is $T_{r+1}\ C_r x^r$

Hence coefficient of $(2r+4)^{th}$ term will be

$$T_{2r+4}=T_{2r+3+1}={}^{18}C_{2r+3}$$

and coefficient or $(r-2)^{th}$ term will be

$$T_{r-2}=T_{r-3+1}={}^{18}C_{r-3}$$

$${}^{18}C_{2r+3}={}^{18}C_{r-3}$$

$$(2r+3)+(r-3)=18$$

$$(\because {}^nC_r={}^nC_k=r=k \text{ or } r+k=n)$$

$\therefore\ r=6$

Q.8 *In a lattice defined by the following Hasse Diagram, how many complements does the element 'e' have?*

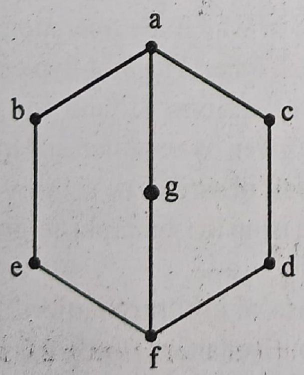


Fig.

Ans. The element e has 3 components – g,c and d.

$e\vee g=a$ and $e\wedge g=f$

$e\vee c=a$ and $e\wedge c=f$

$e\vee d=a$ and $e\wedge d=f$

Q.9 ***Which of the following are posets? Explain.***

(a) (Z, =)

(b) (Z, ≠)

(c) (Z, ≥)

(d) (Z, +)

Ans.(a) (Z, =) : This is a poset. The only ordered pairs we will have in this relation is (a,a) for all $a\in z$.

This would mean that the relation is reflexive, anti symmetric and transitive.

(b) (Z, ≠) : This is not a poset because it is not reflexive. We cannot have the order pair (a,a) for all $a\in z$. This relation is also not anti symmetric and not transitive.

(c) (Z, ≥) : This is a poset. For reflexive, we can have the ordered pair (a,a) for all $a\in z$. This is also antisymmetric because consider the ordered pair (a,b) and $a\neq b$, this would mean that $a>b$. If this is the case, then $b>a$ is not tree and you cannot have (b, a). This is also transitive because if $a>b$, $b>c$, and $a\neq b\neq c$. Then it follows that $a>c$ for all $a,b,c\in z$.

(d) (Z, +) : This is not a poset because it is not reflexive. Consider 2+2, since this is not true, we cannot have (2,2). This relation is also not antisymmetric and not transitive.

Q.10 ***Use iteration to solve the recurrence relation***

$$a_n=a_{n-1}+n$$

with $a_0=4$

Ans. $a_1=a_0+1$

Now $a_2=a_1+2$

But we know what a_1 is

By substitution, we get

$a_2=(a_0+1)+2$

Now, $a_3=a_2+3$

Using our known value of a_2.

$a_3=((a_0+1)+2)+3$

We notice a pattern. Each time, we take the previous term and add the current index. So,

$$a_n=((((a_0+1)+2)+3)+\ldots+n-1)+n$$

Regrouping terms, we notice that a_n is just a_0 plus the sum of the integers from 1 to n. So, since $a_0=4$

$$a_n=4+\frac{n(n+1)}{2}$$

Q.11 Solve the recurrence relation.

$$F_n = 5F_{n-1} - 6F_{n-2}$$

Where $F_0 = 1$ and $F_1 = 4$

Ans. The characteristic equation of the recurrence relation is

$$x^2 - 5x + 6 = 0$$

So, $(x-3)(x-2) = 0$

Hence, the roots are.

$$x_1 = 3$$

and $x_2 = 2$

The roots are real and distinct

Hence $F_n = ax_1^n + bx_2^n$

Here, $F_n = a3^n + b2^n$ (As $x_1 = 3$ and $x_2 = 2$)

Therefore,

$$1 = F_0 = a3^0 + b2^0 = a+b$$

$$4 = F_1 = a3^1 + b2^1 = 3a+2b$$

Solving these two equations, we got a=2 and b = –1

Hence, the final solution is

$$F_n = 2.3^n + (-1).2^n = 2.3^n - 2^n$$

PART-C

Q.12 (a) Define Bounded lattices, complement of an element of a lattices and distributive lattices.

(b) Let $(L, \leq)$ be a bounded distributive Lattice, if an element $a \in L$, has a complement then it is unique. *[R.T.U. 2019]*

Ans.(a) Bounded Lattice : A bounded lattice is an algebraic structure $\mathbb{L} = (L, \wedge, \vee, 0, 1)$, such that $(L, \wedge, \vee)$ is a lattice, and the constants $0, 1 \in L$ satisfy the following:

1. for all $x \in L$, $x \wedge 1 = x$ and $x \vee 1 = 1$,
2. for all $x \in L$, $x \wedge 0 = 0$ and $x \vee 0 = x$.

The element 1 is called the upper bound, or top of $\mathbb{L}$ and the element 0 is called the lower bound or bottom of $\mathbb{L}$.

There is a natural relationship between bounded lattices and bounded lattice-ordered sets. In particular, given a bounded lattice, $(L, \wedge, \vee, 0, 1)$, the lattice-ordered set $(L, \leq)$ that can be defined from the lattice $(L, \wedge, \vee)$ is a bounded lattice-ordered set with upper bound 1 and lower bound 0. Also, one may produce from a bounded lattice-ordered set $(L, \leq)$ a bounded lattice $(L, \wedge, \vee, 0, 1)$ in a pedestrian manner, in essentially the same way one obtains a lattice from a lattice-ordered set. Some authors do not distinguish these structures, but here is one fundamental difference between them: A bounded lattice-ordered set $(L, \leq)$ can have bounded subposets that are also lattice-ordered, but whose bounds are not the same as the bounds of $(L, \leq)$; however, any subalgebra of a bounded lattice $\mathbb{L} = (L, \wedge, \vee, 0, 1)$ is a bounded lattice with the same upper bound and the same lower bound as the bounded lattice $\mathbb{L}$.

Complemented Lattice : A complemented lattice is a bounded lattice (with least element 0 and greatest element 1), in which every element a has a complement, i.e. an element b such that

$$a \vee b = 1 \text{ and } a \wedge b = 0.$$

Complement of an Element : In general an element may have more than one complement. However, in a (bounded) distributive lattice every element will have at most one complement.A lattice in which every element has exactly one complement is called a uniquely complemented lattice

A lattice with the property that every interval (viewed as a sublattice) is complemented is called a relatively complemented lattice. In other words, a relatively complemented lattice is characterized by the property that for every element a in an interval [c, d] there is an element b such that

$$a \vee b = d \text{ and } a \wedge b = c.$$

Such an element b is called a complement of *a* relative to the interval.

A distributive lattice is complemented if and only if it is bounded and relatively complemented.The lattice of subspaces of a vector space provide an example of a complemented lattice that is not, in general, distributive.

Distributive Lattice : In mathematics, a distributive lattice is a lattice in which the operations of join and meet distribute over each other. The prototypical examples of such structures are collections of sets for which the lattice operations can be given by set union and intersection. Indeed, these lattices of sets describe the scenery completely: every distributive lattice is-up to isomorphism-given as such a lattice of sets.

As in the case of arbitrary lattices, one can choose to consider a distributive lattice L either as a structure of order theory or of universal algebra. In the present situation, the algebraic description appears to be more convenient:

A lattice $(L, \vee, \wedge)$ is distributive if the following additional identity holds for all x, y, and z in L:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Viewing lattices as partially ordered sets, this says that the meet operation preserves non-empty finite joins. It is a basic fact of lattice theory that the above condition is equivalent to its dual:

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, and z in L.

In every lattice, defining $p \le q$ as usual to mean $p \wedge q = p$, the inequality $x \wedge (y \vee z) \ge (x \wedge y) \vee (x \wedge z)$ holds as well as its dual inequality $x \vee (y \wedge z) \le (x \vee y) \wedge (x \vee z)$. A lattice is distributive if one of the converse inequalities holds, too. More information on the relationship of this condition to other distributivity conditions of order theory can be found in the article on distributivity (order theory).

Ans.(b) Bounded Lattice : Let $(L, \le)$ is a lattice. An element I will be used for denote the upper bound (UB) of the set L $(a \in L, a \le L)$. Obviously I is unique in the lattice, If it exist 0 to denote lower bound (LB) of the set L $(a \in L, 0 \le a)$. In another words the greatest element in (L, z) if exist.

Complement of a lattice : Let $(L, \le)$ be a bounded lattice, whose greatest and least elements are 1 and 0 respectively. Then an element $a' \in L$ is called a element of an element $a \in L$ if

$a \vee a' = 1$ and $a \wedge a' = 0$

Also $0' = 1$ and $1' = 0$

Distributive Lattice : A lattice $(L, \wedge, \vee)$ is called a distributive lattice if for any $a, b, c \in L$

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Q.13 ***The matching problem***

N guests arrive at a party. Each person is wearing a hat. We collect all hats and then randomly redistribute the hats, giving each person one of the N hats randomly. What is the probability that at least one person receives his/her own hat?

Hint : Use the inclusion – exclusion principle.

Ans. Let A_i be the event that i^{th} person receives his/her own hat. Then we are interested in finding P(E), where $E = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_N$. To find P(E), we use the inclusion – exclusion principle. We have.

$$P(E) = P(\cup_{i=1}^{N} A_i)$$

$$= \sum_{i=1}^{N} P(A_i) - \sum_{i,j:i<j} P(A_i \cap A_j)$$

$$+ \sum_{i,j,K:i<j<K} P(A_i \cap A_j \cap A_K) -$$

$$\ldots + (-1)^{N-1} P(\cap_{i=1}^{N} Ai)$$

Note that there is complete symmetry here, that is, we can write

$P(A_1) = P(A_2) = P(A_3) = \ldots = P(A_N);$

$P(A_1 \cap A_2) = P(A_1 \cap A_3) = \ldots = P(A_2 \cap A_4) = \ldots$

$P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2 \cap A_4)$

$= \ldots = P(A_2 \cap A_4 \cap A_5) = \ldots;$

Thus we have

$$\sum_{i=1}^{N} P(A_i) = NP(A_1);$$

$$\sum_{i,j:i<j=1} P(A_i \cap A_j) = \binom{N}{2} P(A_1 \cap A_2)$$

$$\sum_{i,j,K:i<j<K} P(A_i \cap A_j \cap A_K) = \binom{N}{2} P(A_1 \cap A_2 \cap A_3)$$

Therefore, we have

$$P(E) = NP(A_1) - \binom{N}{2} P(A_1 \cap A_2) + \binom{N}{3} P(A_1 \cap A_2 \cap A_3)$$

$$- \ldots + \ldots (-1)^{N-1} P(A_1 \cap A_2 \cap A_3 \ldots \cap A_N) \qquad \ldots(1)$$

Now, we only need to find $P(A_1)$, $P(A_1 \cap A_2)$, $P(A_1 \cap A_2 \cap A_3)$, etc. to finish solving the problem. To find $P(A_1)$, we have

$$P(A_1) = \frac{|A_1|}{|S|}$$

Here, the sample spare S consists of all possible permutations of N objects (hats). Thus, we have

$|S| = N!$

On the other hand, A_1 consists of all possible permutations of N-1 objects (because the first object is fixed). Thus

$|A_1| = (N-1)!$

Therefore, we have

$$P(A_1) = \frac{|A_1|}{|S|} = \frac{(N-1)!}{N!} = \frac{1}{N}$$

Similarly, we have

$|A_1 \cap A_2| = (N-2)!$

Thus

$$P(A_1 \cap A_2) = \frac{|A_1 \cap A_2|}{|S|} = \frac{(N-2)!}{N!} = \frac{1}{P_{N-2}^{N}}$$

Similarly,

$$P(A_1 \cap A_2 \cap A_3) = \frac{|A_1 \cap A_2 \cap A_3|}{|S|}$$

$$= \frac{(N-3)!}{N!} = \frac{1}{P_{N-3}^{N}}$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{|A_1 \cap A_2 \cap A_3 \cap A_4|}{|S|}$$

$$= \frac{(N-4)!}{N!} = \frac{1}{P^N_{N-4}}$$

Thus using equation (1) we have

$$P(E) = N.\frac{1}{N} - \binom{N}{2}.\frac{1}{P^N_{N-2}} + \binom{N}{3}.\frac{1}{P^N_{N-2}} - .... + (-1)^{N-1}\frac{1}{N!} \quad ....(2)$$

By simplifying a little bit, we obtain

$$P(E) = 1 - \frac{1}{2!} + \frac{1}{3!} + ... + (-1)^{N-1}\frac{1}{N!}$$

We are done. It is interesting to note what happens when N becomes large. To see that, we should remember the Taylor series expansion of e^x. In particular,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Letting $x = -1$, we have

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...$$

Thus, we conclude that as N becomes large, P(E) approaches

$$1 - \frac{1}{e}$$

Q.14 *Solve the following recurrence relations:-*

(a) $f_n = f_{n-1} + f_{n-2}$

With $f_0 = 0$ *and* $f_1 = 1$

(b) $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$

With $a_0 = 8$, $a_1 = 6$ *and* $a_2 = 26$

(c) $a_n = a_{n-1} + 2a_{n-2}$

With $a_0 = 2$ *and* $a_1 + 7$

Ans.(a) Since it is linear homogeneous recurrence, first find its characteristic equation

$$r^2 - r - 1 = 0$$

$$r_1 = (1+\sqrt{5})/2$$

and $r_2 = (1-\sqrt{5})/2$

So, by theorem

$$f_n = \alpha_1((1-\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$$

is a solution

Now, we should find α_1 and α_2 using initial conditions

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1(1+\sqrt{5})/2 + \alpha_2(1-\sqrt{5})/2 = 1$$

So, $\alpha_1 = 1 + /\sqrt{5}$ and $\alpha_2 = -1 + /\sqrt{5}$

$$a_n = 1/\sqrt{5}\ ((1+\sqrt{5})/2^n - 1/\sqrt{5}((1-\sqrt{5})/2)^n$$

is a solution.

Ans.(b) Since it is a linear homogeneous recurrence, first find its characteristic equation

$$r^3 + r^2 - 4r - 4 = 0$$

$$(r+1)(r+2)(r-2) = 0$$

$$r_1 = -1,\ r_2 = -2 \text{ and } r_3 = 2$$

So, by theorem $a_n = \alpha_1 (-1)^n + \alpha_2 (-2)^n + \alpha_3 2^n$ is a solution.

Now, we should find α_1, α_2 and α_3 using initial conditions.

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$$

$$a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$$

$$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$$

So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$

$a_n = 2(-1)^n + (-2)^n + 5 . 2^n$ is a solution.

Ans.(c) Since it is linear homogeneous recurrence, first find its characteristic equation.

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0$$

$$r_1 = 2 \text{ and } r_2 = -1$$

So, by theorem $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ is a solution.

Now, we should find α_1 and α_2 using initial conditions.

$$a_1 = \alpha_1 + \alpha_2 = 2$$

$$a_1 = \alpha_1 2 + \alpha_2 (-1) = 7$$

So, $\alpha_1 = 3$ and $\alpha_2 = -1$

$a_n = 3.2^n - (-1)^n$ is a solution.

Q.15 *Solve for* a_n ***given that*** $a_0 = 0$, $a_1 = 6$ ***and*** $a_n = -3a_{n-1} + 10a_{n-2} + 3.2^n$. ***For*** $n \geq Z$.

Ans. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for

$a_0, a_1, a_2, ...$

$$G(x) = 6x + \sum_{n=2}^{\infty} a_n x^n$$

$$= 6x + \sum_{n=2}^{\infty} -3a_{n-1} + 10a_{n-2} + 3.2^n \Big) x^n$$

$$=6x-3+\sum_{n=2}^{\infty}a_{n-1}x^n+10\sum_{n=1}^{\infty}a_{n-2}x^n-3\sum_{n=2}^{\infty}2^nx^n$$

$$= 6x - 3x\,(a_1x + a_2x^2 + \ldots)$$

$$+ 10x^2\,(a_0 + a_1x + a_2x^2 + \ldots)$$

$$+3((2x)^2 + (2x)^3 + (2x)^4 + \ldots)$$

$$= 6x - 3x\,G(x) + 10x^2\,G(x)$$

$$+ 3(2x)^2\,(1 + (2x) + (2x)^2 \ldots)$$

From the above, we see that

$$G(x) = 6x + G(x)\,(-3x + 10x^2) + 12x^2.\frac{1}{1-2x}$$

Therefore,

$$G(x)(1+3x-10x^2) = 6x + \frac{12x^2}{1-2x}$$

$$G(x) = \frac{6x}{1+3x-10x^2} + \frac{12x^2}{(1-2x)(1+3x-10x^2)}$$

Note that $1 + 3x - 10x^2 = (1 - 2x)(1 + 5x)$

Therefore

$$G(x) = \frac{6x}{(1-2x)(1+5x)} + \frac{12x^2}{(1-2x)^2(1+5x)}$$

Using partial fraction, we obtain that

$$\frac{1}{(1-2x)(1+5x)} = \frac{2}{7}.\frac{1}{(1-2x)} + \frac{5}{7}.\frac{1}{(1+5x)}$$

From this, we also obtain that

$$\frac{1}{(1-2x)^2(1+5x)} = \frac{2}{7}.\frac{1}{(1-2x)^2} + \frac{10}{49}.\frac{1}{(1-2x)} + \frac{25}{49}.\frac{1}{(1+5x)}$$

Therefore,

$$G(x) = 6x\left(\frac{2}{7}.\frac{1}{(1-2x)} + \frac{5}{7}.\frac{5}{(1+5x)}\right) + 12x^2\left(\frac{2}{7}.\frac{1}{(1-2x)^2}\right.$$

$$\left.+\frac{10}{49}.\frac{1}{(1-2x)} + \frac{25}{49}.\frac{1}{(1+5x)}\right)$$

The coeff. of x^n in $6x\left(\frac{2}{7}\frac{1}{(1-2x)} + \frac{5}{7}\frac{5}{(1+5x)}\right)$ equals

$6x$ coeff. of x^{n-1} in $\frac{2}{7}\frac{1}{(1-2x)} + \frac{5}{7}\frac{5}{(1+5x)}$ which equals

$$6\left(\frac{2}{7}.2^{n-1} + \frac{5}{7}.(-5)^{n-1}\right)$$

The coeff. of x^n in

$$12x^2\left(\frac{2}{7}\frac{1}{(1-2x)^2} + \frac{10}{49}\frac{1}{(1-2x)} + \frac{25}{49}\frac{1}{(1+5x)}\right)$$

equal $12x$ coeff. of x^{n-2} in

$$\frac{2}{7}\frac{1}{(1-2x)^2} + \frac{10}{49}\frac{1}{(1-2x)} + \frac{25}{49}\frac{1}{(1+5x)}$$

which equals

$$12\left(\frac{2}{7}(n-1).2^{n-2} + \frac{10}{49}2^{n-2} + \frac{25}{49}(-5)^{n-2}\right)$$

Putting the above together, we obtain that.

$$a_n = 6\left(\frac{2}{7}.2^{n-1} + \frac{5}{7}.(-5)^{n-1}\right) + 12\left(\frac{2}{7}(n-1).2^{n-2}\right.$$

$$\left.+\frac{10}{49}2^{n-2} + \frac{25}{49}(-5)^{n-2}\right), n \ge 2$$

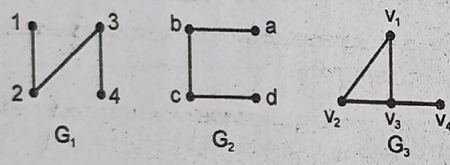
□□□

ALGEBRAIC STRUCTURES 4

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 *Prove that these graphs G_1, G_3 and G_2, G_3 are non-isomorphic*



[R.T.U. 2019]

Ans. No of edges in $G_1 = 3$

No of edges in $G_2 = 3$

No of edges in $G_3 = 4$

As the size of G_1, G_3 and G_2, G_4 are not same so these set of graphs are non-isomorphic.

Q.2 *Show that the multiplicative group $G = \{1, -1, i, -i\}$ is cyclic. Find its generators.* *[R.T.U. 2019]*

Ans. $G = \{1, -1, i, -i\}$

$\{i^4, i^2, i^1, (i)^3\}$

$i \in G$

At that same time

$G = \{1, -1, i, -i\}$

$= \{(-i)^4, (-i)^2, (-i)^3, (-i)^1\}$

$-i \in G$

i.e. $\{i$ & $-i\}$ generators of G.

Q.3 *Write short note on Group.* *[R.T.U 2012]*

Ans. Group : A group is an algebraic structure $\{G, \perp\}$, where G is a non-empty set and $\perp$ denotes a binary operation $\perp : G \times G \to G$, called the group operation. The notation $\perp(x, y)$ is normally shortened to the infix notation $x \perp y$.

A group must obey the following rules. Let a, b, c be arbitrary elements of G. Then :

- *CLOSURE*- $a \perp b \in G$. This axiom is often omitted because a binary operation is closed by definition.
- *ASSOCIATIVITY*- $(a \perp b) \perp c = a \perp (b \perp c)$.
- *IDENTITY*- There exists an identity (or neutral) element $e \in G$ such that $a \perp e = e \perp a = a$. The identity of G is unique.
- *INVERSE*- For each $a \in G$, there exists an inverse element $x \in G$ such that $a \perp x = x \perp a = e$. The inverse of a is unique.

An abelian group also obeys the additional rule :

- *COMMUTATIVITY*- $a \perp b = b \perp a$.

Q.4 *Write short note on Field.* *[R.T.U 2012]*

Ans. Field : Field theory considers sets, such as the real number line, on which all the usual arithmetic properties hold --- those governing addition, subtraction, multiplication and division.

Specifically, a field is a commutative ring in which every nonzero element is assumed to have a multiplicative inverse. Examples include the real number field R, the complex numbers C, the rational numbers.

$S(0) = 2, \; S(1) = 5$

Fields are important objects of study in algebra since they provide a useful generalization of many number systems, such as the rational numbers, real numbers, and complex numbers. In particular, the usual rules of associativity, commutativity and distributivity hold.

The concept of a field is of use, for example, in defining vectors and matrices, two structures in linear algebra whose components can be elements of an arbitrary field.

Q.5 ***Write short note on Ring.*** *[R.T.U 2012]*

Ans. Ring : A Ring is a structure that abstracts and generalizes the basic arithmetic operations, specifically the addition and the multiplication.

Briefly, a ring is an abelian group with a second binary operation that is associative and is distributive over the abelian group operation. The abelian group operation is called "addition" and the second binary operation is called "multiplication" in analogy with the integers. One familiar example of a ring is the set of integers. The integers are a commutative ring, since a times b is equal to b times a. The set of polynomials also forms a commutative ring. An example of a non-commutative ring is the ring of square matrices of the same size. Finally, a field (such as the real numbers) is a commutative ring in which one can do "division" by any nonzero element.

The most familiar example of a ring is the set of all integers, Z, consisting of the numbers

$\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

Mathematically a ring is a set R equipped with two binary operations

$+ : R \times R \to R$ and $: R \times R \to R$ (where $\times$ denotes the Cartesian product), *called addition* and *multiplication*, such that:

- (R, +) is an abelian group with identity element 0, meaning that for all a, b, c in R, the following axioms hold :
  - $(a + b) + c = a + (b + c)$ (+ is associative)
  - $0 + a = a$ (0 is the identity)
  - $a + b = b + a$ (+ is commutative)
  - For each a in R there exists $-a$ in R such that $a + (-a) = (-a) + a = 0$ ($-a$ is the inverse element of a)
- Multiplication $\cdot$ is associative:
  - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Multiplication distributes over addition:
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
  - $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Q.6 ***Write short note on Galois field.*** *[R.T.U 2012]*

Ans. Galois Field: A finite field is a field with a finite field order (i.e., number of elements), also called a Galois field. For example, GF(p) is called the prime field of order p, and is the field of residue classes modulo p, where the p elements are denoted $0, 1, \ldots, p - 1$. $a = b$ in GF(p) means the same as $a \equiv b$. (mod p)

The finite field GF(2) consists of elements 0 and 1 which satisfy the following addition and multiplication tables.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

$H = (3Z, +)$

$G = (Z, +)$

Q.7 ***Explain Subgroup.***

Ans. Let G be a group and $4 \subseteq G$ be non-empty. If H is also a group under the same operation as G, then H is a sub group of G. if $\{e\} \subseteq H \subseteq G$ then H is proper subgroup of G.

PART-B

Q.8 ***Define and explain the following by suitable examples-***
(i) Cyclic group
(ii) Order of an element in a group
(iii) Field
(iv) Zero divisor of a ring *[R.T.U. 2019]*

Ans.(i) Cyclic Group : In group theory, a branch of abstract algebra, a cyclic group or monogenous group is a group that is generated by a single element. That is, it is a set of invertible elements with a single associative binary operation, and it contains an element g such that every other element of the group may be obtained by repeatedly applying the group operation to g or its inverse. Each element can be written as a power of g in multiplicative notation, or as a multiple of g in additive notation. This element g is called a generator of the group.

Example:

Integer and modular addition

The set of integers Z, with the operation of addition, forms a group. It is an infinite cyclic group, because all integers can be written by repeatedly adding or subtracting the single number 1. In this group, 1 and "1 are the only generators. Every infinite cyclic group is isomorphic to Z.

For every positive integer n, the set of integers modulo n, again with the operation of addition, forms a finite cyclic group, denoted Z/nZ. A modular integer i is a generator of this group if i is relatively prime to n, because these elements can generate all other elements of the group through integer addition. (The number of such generators is $\varphi(n)$, where φ is the Euler totient function.) Every finite cyclic group G is isomorphic to Z/nZ, where n = |G| is the order of the group.

The addition operations on integers and modular integers, used to define the cyclic groups, are the addition operations of commutative rings, also denoted Z and Z/nZ or Z/(n). If p is a prime, then Z/pZ is a finite field, and is usually denoted F_p or GF(p).

(ii) Order of a group : In group theory, a branch of mathematics, the order of a group is its cardinality, that is, the number of elements in its set. The order of an element a of a group, sometimes also period length or period of a, is the smallest positive integer m such that $a^m = e$, where e denotes the identity element of the group, and a^m denotes the product of m copies of a. If no such m exists, a is said to have infinite order.

The order of a group G is denoted by ord(G) or |G|, and the order of an element a is denoted by ord(a) or |a|. The order of an element a is equal to the order of its cyclic subgroup $\langle a \rangle = \{a^k$ for k an integer$\}$, the subgroup generated by a. Thus, $|a| = |\langle a \rangle|$.

Lagrange's theorem states that for any subgroup H of G, the order of the subgroup divides the order of the group: |H| is a divisor of |G|. In particular, the order |a| of any element is a divisor of |G|.

Example. The symmetric group S_3 has the following multiplication table.

•	e	s	t	u	v	w
e	e	s	t	u	v	w
s	s	e	v	w	t	u
t	t	u	e	s	w	v
u	u	t	w	v	e	s
v	v	w	s	e	u	t
w	w	v	u	t	s	e

This group has six elements, so $\text{ord}(S_3) = 6$. By definition, the order of the identity, e, is one, since $e^1 = e$. Each of s, t, and w squares to e, so these group elements have order two: |s| = |t| = |w| = 2. Finally, u and v have order 3, since $u^3 = vu = e$, and $v^3 = uv = e$.

(iii) Field : *Refer to Q.4.*

Example :

Rational Numbers : Rational numbers have been widely used a long time before the elaboration of the concept of field. They are numbers that can be written as fractions a/b, Where a and b are integers, and $b \neq 0$. The additive inverse of such a fraction is –a/b, and the multiplicative inverse (provided that $a \neq 0$) is b/a, which can be seen as follows :

$$\frac{b}{a}\cdot\frac{a}{b} = \frac{ba}{ab} = 1$$

The abstractly required field axioms reduce to standard properties of rational numbers. For example, the law of distributivity can be proven as follows :

$$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right)$$

$$= \frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{f}{f}+\frac{e}{f}\cdot\frac{d}{d}\right)$$

$$= \frac{a}{b}\cdot\left(\frac{cf}{df}+\frac{ed}{fd}\right) = \frac{a}{b}\cdot\frac{cf+ed}{df}$$

$$= \frac{a(cf+ed)}{bdf} = \frac{acf}{bdf}+\frac{aed}{bdf} = \frac{ac}{bd}+\frac{ae}{bf}$$

$$= \frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f}$$

(iv) Zero Divisors in Rings

Definition : Let (R,+, *) be a ring where $0 \in R$ is the identity of +. The element $a \in R\setminus\{0\}$ is said to be a Zero-Divisor of R if there exists a $b \in R\setminus\{0\}$ such that a * b = 0 or b * a = 0.

For example, consider the ring (M_{22}, +, *) of 2 × 2 matrices with real coefficients and with the operations of standard matrix additiion +, and standard matrix multiplication *. Recall that the identity of + is the 2 × 2 zero matrix

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Further consider the matrices $A, B \in M_{22}$ given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad ...(1)$$

When we multiply the matrices A and B together we have that

$$A * B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \ldots(2)$$

Notice that A is not the identity for + and B is not the identity for +. Therefore, the matrices A and B are zero divisors of M_{22}.

We should bee clear that a ring (R, + , *) need not have any zero divisors. For example, consider the ring (C, +, *) of complex numbers where + is standard addition and * is standard multiplication. We note that the identity of + is $0 = 0 + 0i \in C$. Since (C, + , *) is a commutative ring, then for x, y $\in C\backslash\{0\}$ and where $x = a + bi$ and $y = c + di$ for a, b, c, d $\in$ R we need to consider the following equation.

$$x * y = (a + bi)(c + di) = (ac - bd) + (ad + bc)i = 0 + 0i \quad \ldots(3)$$

Note that this equality holds if and only if :

$$ac = bd \text{ and } ad + bc = 0 \quad \ldots(4)$$

Without loss of generality, assume $x \neq 0$. Then either $a \neq 0$ or $b \neq 0$ or both. Assume $a \neq 0$. Then we can divide both equations by a to get :

$$c = \frac{bd}{a}(*) \text{ and } d + \frac{bc}{a} = 0(**) \quad \ldots(5)$$

Substituting the first equation into the second yields:

$$d + c\frac{b}{a} = 0(*) \quad \ldots(6)$$

$$d + \frac{bd}{a}.\frac{b}{a} = 0$$

$$d + \frac{b^2 d}{a^2} = 0(*)$$

$$d\left(1 + \frac{b^2}{a^2}\right) = 0$$

There are two possibilities in the equation above. Either $d = 0$ or $1 + \frac{b^2}{a^2} = 0$. Clearly $1 + \frac{b^2}{a^2} = 0$ since this would imply that $\left(\frac{b}{a}\right)^2 = -1$. Therefore $d = 0$.

Looking at (**) we see that then $\frac{bc}{a} = 0$ so either $b = 0$ or $c = 0$. If $c = 0$, we have that then $y = c + di = 0$. Meanwhile, if $c \neq 0$ then $b = 0$ and by (*) this implies that $c = 0$ so then $y = c + di = 0$ again. In either case, we see that if $x = a + bi \neq 0 + 0i$ then $y = c + di = 0 + 0i$. Therefore, there exists no zero divisors in the ring (c, + , *)

Q.9 *If {G, *} is a finite cyclic group generated by on element $a \in G$ and is of order n, then $a^n = e$ so that $G = \{a, a^2, \ldots, a^n(=e)\}$. Also n is the least positive integer for which $a^n = e$.* *[R.T.U. Dec. 2016]*

OR

*Prove that if (G, *) is a finite cyclic group generated by an element $a \in G$ and is of order n, then $a^n = e$. Also n is the least positive integer for which $a^n = e$.*

Ans. G = <a>, group is generated by a.

G has a order n :

$\Rightarrow G = \{a^0, a^1, a^2, \ldots a^{n-1}\}$

Here, $a^n = a^0$ (because G is cyclic)

Let $a \in G$ such that $a^n \neq e$. Because $|G| = n$, we cannot have order (a) > n, else $\{e, a, \ldots\ldots., a^n\}$ would all be distinct and this contradicts the order of G.

Let order (a) = m $\leq$ n. Then $|\langle a \rangle| = m$, the cyclic group generated by a.

By Lagrange theorem

$|\langle a \rangle| / |G| = m/n$

Therefore, n = dm . Then,

$a^n = (a^m)^d = e^d = e$

But, since G is cyclic and G = <a>

Therefore, order (a) = |G|

$G = \{ e, a, a^2, \ldots\ldots, a^{n-1}\}$

$a^n = e$. Hence, order (a) = n.

Since it is order, it must be minimum such that $a^n = e$

But $p < n$ such that $a^p = e$

Then order (a) = p which is not true as order (a) = n.

Hence, $a^n = e$ and n is the least positive number for which this is possible.

Q.10 *If S is the set of ordered pairs (a, b) of real numbers and if the binary operations $\oplus$ and $\odot$ are defined by the equations-*

$$(a, b) \oplus (c, d) = (a + c, b + d)$$

and $$(a, b) \odot (c, d) = (ac - bd, bc + ad)$$

prove that (S, $\oplus$, $\odot$) is a field. *[R.T.U. 2016]*

Ans. S is a set of ordered pairs

$(a, b) \oplus (c, d) = (a+c, b+d)$

$(a, b) \odot (c, d) = (ac - bd, bc + ad)$

1. Closure of S under addition and multiplication :

$(a, b), (c, d) \in S \Rightarrow a, b, c, d \in IR$

$a + c, b + d \in IR \Rightarrow (a + c, b + d) \in S$

$ac, bd, bc, ad \in IR \Rightarrow ac - bd, bc + a\propto \in IR$

$\Rightarrow (ac - bd, bc + ad) \in S$

Hence, it is closure w.r.t addition and multiplication

2. Associativity of addition and multiplication :

$(a, b) \oplus ((c,d) \oplus (e, f)) = (a, b) \oplus (c+ e, d+ f)$

$= (a + c + e, b+ d + f) = ((a + c, b + d)) \oplus (e, f)$

$= ((a, b) \oplus (c,d)) \oplus (e, f)$

$(a, b) \odot ((c,d) \odot (e,f)) = (a,b) \odot (ce - df, de + cf)$

$= (ace - adf - bde - bcf, bce - bdf, + ade + acf)$

$= (ace - bde - dcf - adf, bce + ade + acf - bdf)$

$= (ac - bd, bc + ad), \odot (e, f)$

$= ((a,b) \odot (c,d)) \odot (e,f)$

3. Commutativity of addition and multiplication :

$(a,b) \oplus (c,d) = (a+c, b+d) = (c + a, d + b)$

$= (c, d) \oplus (a,b)$

$(a,b) \odot (c,d) = (ac - bd, bc + ad) = (ca - db, da + cb)$

$= (c, d) \odot (a, b)$

4. Existence of additive and multiplicative identity :

$(a, b) \oplus (0, 0) = (a + 0, b + 0) = (a, b)$

$(a,b) \odot (1,0) = (a.1 - b\times 0, b.1 + a.0)$

$= (a,b)$

(0,0) and (1,0) are additive and multiplicative identity respectively.

5. Existence of additive and multiplicative inverse :

$(a, b) \oplus (-a, -b) = (a - a, b - b) = (0,0)$

$$(a,b)\odot\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$=\left(\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}, \frac{ab}{a^2+b^2}-\frac{ab}{a^2+b^2}\right)=(1,0)$$

Hence, $(-a,-b)$ and $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ are additive and multiplicative inverse.

6. Distributivity of multiplication over addition :

To prove $(a, b), (c,d), (e,f) \in S$

$(a, b) \odot ((e, d) \oplus (c, f)) = ((a, b) \odot (c, d)) \oplus ((a, b) \odot (e, f))$

$LHS = (a, b) \odot (c + e, d + f)$

$= (ac + ae - bd - bf, bc + be + ad + af)$

$RHS = ((a,b) \odot (c, d)) \oplus ((a, b) \odot (e, f))$

$= (ac - bd, bc + ad) \oplus (ae - bf, be + af)$

$= (ac - bd + ae - bf, bc + ad + be + af)$

$LHS = RHS$ **Hence Proved**

Q.11 ***The necessary and sufficient condition for a non-empty subset H of a group {G, *} to be a subgroup is a, $b \in H \Rightarrow a * b^{-1} \in H$.*** *[R.T.U. 2016]*

Ans. The condition is necessary. Suppose H is a subgroup of G and let $a \in H, b \in H$. Now each element of H must posses inverse because H itself is a group.

$b \in H \Rightarrow b^{-1} \in H$

Also, H is closed under composition * in G. Therefore,

$a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$

The condition is sufficient. If it is given $a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$, then we have to prove that H is a subgroup.

(i) Closure property : Let a, b ∈ H then $b \in H \Rightarrow b^{-1} \in H$.

Therefore, $a \in H, b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H$.

$\Rightarrow a * b \in H$.

H is closed with respect to composition * in G.

(ii) Associative property : Since elements of H are also the elements of G, the composition is associative in H.

(iii) Existence of identity : Since,

$a \in H, a^{-1} \in H \Rightarrow a * a^{-1} \in J = e \in H$.

(iv) Existence of inverse : Let $a \in H$ then

$e \in H, a \in H \Rightarrow e * a^{-1} \in H \Rightarrow a^{-1} \in H$.

Hence, H itself is a group for the composition * in group G.

Q.12 ***Show that $z_5 = \{0,1,2,3,4\}$ is an abelian group for the operation $+_5$ defined as.***

$$a +_5 b = \begin{cases} a+b & \text{if } a+b<5 \\ a+b-5 & \text{if } a+b \geq 5 \end{cases}$$

[R.T.U. 2015]

Ans. Composition table is

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\because$ All entries in the composition table are elements of z_5. So z_5 is closed with respect to addition modulo 5 *i.e.* $+_5$

The composition $+_5$ is associative. If a, b, c are there elements of z_5, then

$a +_5 (b +_5 c) = a +_5 (b + c)$ [$\because b +_5 c \equiv (b+c)$ mod5]

= least non-negative remainder when a+ (b+c) is divided by 5.

= least non-negative remainder when (a+b)+c is divided by 5.

$= (a+b) +_5 c = (a +_5 b) +_5 c$

Existence of identity : we have $0 \in z_5$. If a is any element of z_5 than from the composition table, we have $0 +_5 a = a +_5 0$

Thus 0 is the identity element .

Existence of inverse : Form the table we get that the inverses of 0, 1, 2, 3, 4 are 0, 4, 3, 2, 1 respectively.

The composition is commutative as the corresponding rows and columns in the table are identical.

So $(z_5, +_5)$ is an abelian group.

Q.13 ***Prove that a non-void subset H of a group G is a subgroup***

if $a \in H$, $b \in H \Rightarrow ab^{-1} \in H$ ***[R.T.U. 2015]***

Ans. Necessary condition : Suppose H is a subgroup of G. Let $a \in H$, $b \in H$. Now each element of H must possess inverse because H itself is a group.

$\therefore$ $b \in H \Rightarrow b^{-1} \in H$

Further H must closed w.r. to multiplication i.e. the composition in G. Therefore

$a \in H$, $b^{-1} \in H \Rightarrow ab^{-1} \in H$.

Sufficient condition : Now it is given that $a \in H$, $b \in H \Rightarrow ab^{-1} \in H$. We are prove that H is a subgroup of G.

Existence of identity : We have

$a \in H$, $a \in H \Rightarrow aa^{-1} \in H$

$\Rightarrow e \in H$

Thus the identity e is an element of H.

Existence of inverse: Let a be any element of H. Then by the given condition we have

$e \in H$, $a \in H \Rightarrow e\, a^{-1} \in H \Rightarrow a^{-1} \in H$

Thus each element of H possess inverse.

Closure property : Let a, b $\in H$

$\because$ $b \in H \Rightarrow b^{-1} \in H$

Therefore applying the given condition, we have

$a \in H$, $b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H \Rightarrow ab \in H$

Associativity : The elements of H are also the elements of G. The composition in G is associative. Therefore it must also be associative in H.

Hence H itself is a group for the composition in G. Therefore H is a subgroup of G.

PART-C

Q.14 ***(a) Let (m, *) be a semi group and a $\in$ m such that the equations a * u = x and v * a = x have solutions in M for all x $\in$ M. Show that (M, *) is a monoid.***

(b) Let $\Delta(G)$ be the maximum of the degrees of the vertices of a graph G then $K(G) \leq 1 + \Delta(G)$ where K(G) is the chromatic number of graph.

(c) Let G be the set of all non-zero real numbers and Let $a * b = \frac{ab}{2}$, then show that $(G_1 *)$ is an abelian group.

[R.T.U. 2019]

Ans.(a) Given a * u = x

and v * a = x; $\forall x \in A$ ...(i)

$a \in A$ if we take x = a equation (i) are satisfied.

For some $u = e_l$ and $v = e_m$

$a * b\, e_l = a$ and $e_m * a = a$

Again Let $y \in A$

$y * e_l = (v * a) * e_l = v * (a * e_l) = v * a = y$

$e_m * y = e_m * (a * u) = (e_l * a) * u$

$= a * u = y$

$y * e_l = y$ and $e_m * y = y$

e_l and e_m are right and left identity in A

$e_l = e_m = e$

identity element $e \in A$.

Ans.(b) Proof : Let the no of vertex in a graph is denoted by $|V|$. If $|V| = 1$ then $A(G) = 0$, $K(G) = 1$. So the results holds. Now let k be an integer. and $k \geq 1$. Assume that results hold for all graph with $|V| = k$ vertex.

Let G be a graph with $(k + 1)$ vertex

Let V be any vertex of G and $G_0 = G\{v\}^2$

is a subgraph of G obtained by deleting V from G.

Since G_0 has k vertex so we use induction

$k(G_0) \leq 1 + \Delta(G_0)$

$\Delta(G_0) \leq \Delta(G_0)$

$K(G_0) \leq 1 + \Delta(G)$

So G_0 can be colored with atmost $1 + \Delta(G)$ colors. Since there can be almost $\Delta(G)$ vertices adjacent to V, one of the available $1 + \Delta(G)$ colors remains for V. Thus G can be colored with atmost $1 + \Delta(G)$ colors.

Ans.(c) Let a, b $\in$ G Here $a \not> \beta = \frac{ab}{2}, \forall a, b \in G$

(i) **Closure :** a and b are non zero real numbers ab is also a non zero real number

$\frac{ab}{2}$ is also a non zero real number

$\frac{ab}{2} \in G \Rightarrow (a * b) \in G$

G is closed

(ii) **Associativity :** Let e a, b, c $\in$ G

Then

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{2.2} = \frac{abc}{4} \quad ...(1)$$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{a(bc)}{2.2} = \frac{abc}{4} \quad ...(2)$$

From (1) and (2)

$a * (b * c) = (a * b) * c$

G is associative

(iii) Identity : Let e be the identity element is G.

$a * e = a \ \forall a \in G$.

$\frac{ae}{2} = a \Rightarrow e = 2 \in G$

2 is identity element is G.

(iv) Inverse : Let a' b c the inverse of $a \in G$.

$a * a' = 2$

$\frac{aa'}{2} = 2 \Rightarrow a' = \frac{4}{a} \in G(a \neq 0)$

$a \in G$, So each elements of G has its inverse in G.

(v) Abelian : Let a, b $\in$ G then

$$a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$$

$a * b = b * a. \ \forall a, b \in G$

Therefore (G, *) is an abelian.

Q.15 *(a) Prove that every infinite cyclic group has two and only two generators.*

(b) Show that the set $t(g) = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} a \in z, b \in z\right\}$ *is an ideal of the ring* $R = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} a, b, c \in z\right\}$.

Matrix addition and matrix multiplication being the operations of the system. *[R.T.U. 2014]*

Ans.(a) Let $G = \{a\}$ be an infinite cyclic group generated by a. The elements of G will be integral power of a.

We claim that no two distinct integral power of a can be equal.

For, if possible, let $a^r = a^s, r > s$

$\Rightarrow \ a^r \cdot a^{-s} = a^{s-s}$

$\Rightarrow \ a^{r-s} = a^0$

$\Rightarrow \ a^{r-s} = 0$

Since r–s is positive integer

$a^{r-s} = 1 \Rightarrow a^0 = r - s$ finite

So 'a' can't be a generator of an infinite cyclic group G.

Hence $a^r \neq a^s$ unless $r = s$

Therefor we can write

$$G = \{...a^{-4}, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, ...\}$$

If a^r is any elment of G we can write $a^r = (a^{-1})^{-r}$

Thus a^{-1} is also a generator of G. To show that a and a^{-1} generator Now if $m \neq 1$ or -1 then a^m can't be generator of G. If a^m is to a generator of G, there must exist an integer k such that $(a^m)k = a$ i.e. $a^{mk} = a$

Now $m = 1$ or -1 Þ $mk \neq 1$

Therefore two distinct integral powers of 'a' are equal and this contracts that statement we have just proved. Hence a^m cannot be a generator of G if $m \neq 1$ or $- 1$. Thus G has exactly two generators.

Ans.(b) A subset of ring R is said to be ideal of R.

(let $S \subset R$) is ideal of R

If (I) $a \in S\ b \in S \Rightarrow a - b \in S$

(II) $rS \in S$, $Sr \in S\ \forall\ r \in R, S \in S$

Let $\alpha = \begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix}$

$\beta = \begin{bmatrix} a_2 & 0 \\ b_2 & 0 \end{bmatrix} \in t(g)$

$\alpha - \beta \begin{bmatrix} a_1 - a_2 & 0 \\ b_1 - b_2 & 0 \end{bmatrix} \in t(g)$

Since $a_1 - a_2, b_1 - b_2 \in Z$

Let $r = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in R$

$$\alpha r = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ ab & 0 \end{bmatrix} \in t(g)$$

$$r\alpha = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ ab + bc & 0 \end{bmatrix} \in t(g)$$

We have above so that

$\alpha, \beta \in t(g) \Rightarrow \alpha - \beta \in t(g)$

$r\alpha \in t(g)\ \alpha r \in t(g)$

$\forall r \in R, a \in S$

Q.16 ***(a) Explain permutation group with example.***

(b) Write out all of the permutations of the set {1, 2, 3, 4}. How many are there in all? Find a sensible way to organize you list.

Ans.(a) A permutation of n distinct object is just a listing of the objects in some order. For example, [c, b, a] is a permutation of the set {a, b, c} of three objects. Likewise, [triangle, melon, airplane] is a permutation of three objects as well. From our mathematical point of view, the objects we use don't actually matter; all we care about is the order they are arranged in. So usually we'll just talk about permutations of the numbers 1 through n. It can think of each number as just counting the objects involved : first object, second object, n^{th} object.

Permutations arise in the world in a many, many ways. For example, suppose you are asked to list preferences amongst a bunch of presidential candidates. The list makes up, from favorite to least favorite, is a permutation of the candidates. In fact, it can use the mathematics of permutations to learn interesting things about different kinds of voting systems.

Another example is a deck of playing cards. In a standard deck, each card appears exactly once. When you shuffle the deck, you are just creating a random permutation of the cards. One can use mathematics related to permutations to answer interesting questions about cards. Like : "How many times do I need to shuffle the deck before it is truly randomized?" The answer, by the way, seems to be 7 for a standard riffle shuffle. But proving that is well beyond the scope fo these notes.

Because permutations are so common, problems involving permutations tend to be very applicable! For example, suppose you have two hundred students in a class and they all hand in an exam. The stack of exams they give you is a permutation of the students; most likely, the list of student scores you keep is alphabetical. This suggests a problem : What is the fastest way to sort the exams? (In fact, sorting is a fundamental problem in computer science.)

How many permutations are there of a set of n objects? Suppose we try to build a permutation by successively choosing objects. Then there are n choices for the first object, n – 1 choices for the second, and so on, until there is only one choice for the last object. Then to get the total number of possible permutations, we multiply these numbers together, and get n(n – 1) (n – 2) . . . 1. This number, if you haven't seen it before, is called n-factorial written n!

Ans.(b) Suppose we have some initial ordering of our objects. The letters {a, b, c}, for example, can be organized alphabetically. Then every permutation we can think of as a mising-up of this initial order. In this sense, the permutation is a special kind of function from the set of objects back to itself. A permutation of these objects is then the list [σ(a), σ(b), σ(c)]; this list is called the one line notation for σ.

These permutations-as-functions can be composed : if we think of two permutations σ and τ as different ways to mix up the set, can be mix them up according to σ and then according to τ. Then the composition is specified by the list [$\tau(\sigma$ (a)), $\tau(\sigma$ (b)), $\tau(\sigma$ (c))].

For example, if $\sigma = [2, 3, 1, 4]$ and $\tau = [3, 4, 1, 2]$, then $\tau o \sigma = [4, 3, 1, 2]$. (In particular $\sigma(1) = 2$, and $\tau(2) = 4$; so the first entry of $\tau o \sigma$ is 4. The other three entries are computed similarly.) On the other hand, $\sigma o \tau = [3, 1, 4, 2]$. This is different from $\tau o \sigma$. So we see that the group of permutations has elements where fog $\neq$ gof; we say that S_n is non-commutative. (But remember that nothing in our group definition says that the group needs to be commutative, so this is ok.)

A very nice way to keep track of this mixing-up is the braid notation for a permutation. This simply writes the list of objects in two lines, and draws a line connecting an object on the top to the object it is sent to under the permutation.

Braid diagrams for some permutations. At this point, we can ask whether the permutations with the composition operation are in fact a group. In fact, they are! Let's check. Let σ, τ be permutations of the set $X = \{1, 2, 3, \ldots, n\}$. Then we can specify σ by the list $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$.

Composition of two permutations is again a permutation. Since each permutation contains every element of X exactly once, the composition $\tau o \sigma$ must also contain each element of X exactly once. Identity: The permutation $[1, 2, \ldots, n]$ acts as the identity. Inverses : Roughly speaking, if we can mix things up, we can just as easily sort them back out. The 'sorting permutation' of σ is exactly σ^{-1}. Associativity : Suppose we compose three permutations, σ, τ and ρ. In the braid notation, this just means placing the three braids on top of each other top-to-bottom, and then 'forgetting' the two sets of intermediate dots. (TODO : a picture!) Associativity is tantamount to forgetting the two sets of dots in two different orders; the resulting picture is the same either way, so composition of permutations is associative.

Q.17 ***Write short notes on following :***

(a) Monoid

(b) Cosets

(c) Homomorphism and isomorphism.

Ans.(a) Monoid : In abstract algebra, a branch of mathematics, a monoid is an algebraic structure with a single associative binary operation and an identity element.

Monoids are studied in semigroup theory, because they are semigroups with identity. Monoids occur in several branches of mathematics; for instance, they can be regarded as categories with a single object. Thus, they capture the idea of function composition within a set. In fact, all functions from a set into itself from naturally a monoid with respect to function composition. Monoids are also commonly used in computer science, both in its foundational aspects and in practical programming. The set of string built from a given set of character is a free monoid. The transition monoid and systactic monoid are used in describing finite-state machines,, whereas trace monoids and history monoids provide a foundation for process calculi and concurrent computing. Some of the more important results in the study of monoids are the Krohn-Rhodes theorem.

Ans.(b) Cosets

Left Coset : It is very important in group theory (and in abstract algebra more generally). Incidentally, this definition talks about left cosets. Right cosets are important too, but we will consider them another time.

Definition : Let G be a group and $H \subset G$ a subgroup. A left coset of H in G is a subset of G of the form gH for some $g \in G$.

Multiplying element and sets : Of course, the expression gH does not make immediate sense from the group axioms. What is means, by definition, is

$$gH = \{gh \mid h \in H\}$$

To put this another way, the golden rule is this : if you know that $f \in gH$, then we can conclude that there is some $h \in H$ so that $f = gh$.

Here is an example of how the golden rule works.

Applying the golden rule :

Consider $G = S_4$ and $H = \{id, (1, 2)\}$. If $g = (2, 3, 4)$, then $gH = \{(2, 3, 4)\,(2, 3, 4)\,(1, 2)\} = \{(2, 3, 4)\}, (1, 3, 4, 2)\}$. Now let $f = (3, 4, 2, 1)$ – this is an element of gH. Which $h \in H$ satisfies $f = gh$?

Or if $g = (1, 3)\,(2, 4)$, then $gH = \{(1, 3)\,(2, 4), (1, 4, 2, 3)\}$. If you let $f = (1, 4, 2, 3)$, which $h \in H$ satisfies $f = gh$ this time?

n the box below, compute the two cosets $g_1H \subset S_4$ and $g_2H \subset S_4$ for

$$H = \{id, (1, 2, 3, 4), (1, 3)\,(2, 4), (1, 4, 3, 2)\}$$

and $g_1 = (1, 3, 2)$, $g_2 = (1, 2, 3\ 4)$

One of two cosets is equal to H itself; the other is disjoint from H.

One coset can have many representatives : Compute another example – in this case the group operation is addition, so we write a + b rather than ab, and similarly cosets are written a + H rather than aH.

Cosets in Z

Let G = Z and $H = 5Z = \{5n \mid n \in Z\} = \{..., -5, 0, 5, 10, ...\}$. The coset 2 + H is the set $\{2 + 5n \mid n \in Z\} = \{..., -8, -3, 2, 7, 12, \ldots\}$, which is of course just the set of numbers congruent to 2 mod 5.

We say that 2 is a representative of the coset 2 + H.

Which of the numbers 17, 152, 21, –18, –2 lie in the set 2 + H?

Now calculate 7 + H =

We should see that 7 + H is exactly the same subset of Z as 2 + H. Therefore we can also say that 7 is a representative of 2 + H. The point is that 7 and 2 are the same mod 5. In fact, so are –8, 12, 152 or indeed any other element of the set 2 + H. We can refer to any of them as a representative of this coset.

Ans.(c) Homomorphism and Isomorphism : Let $(G, \cdot)$ and $(G', *)$ be groups. A homomorphism is a set map $\phi : G \to G'$ that preserves the group operation in the respective groups; that is,

$$\phi(a \cdot b) = \phi(a) * \phi(b)$$

for all $a, b \in G$.

Example :

1. Consider the groups $(GL_2(R)$, matrix multiplication), $(R\backslash\{0\}, \times)$. Define $\phi : GL_2(R) \to R\backslash\{0\}$ by

$\phi(A) = \det(A)$.

Then $\phi(AB) = \det(AB) = \det(A)\det(B) = \phi(A)\,\phi(B)$.

2. If V and W are vector spaces, then (V, +) and (W, +) are groups. Let $T : V \to W$ be any linear transformation. Then T is a group homomorphism. Indeed,

   $$T(v_1 + v_2) = T(v_1) + T(v_2)$$

   by linearity.

   $$\phi(a) = [a]$$

   Then $\phi(a + b) = [a + b] = [a] + [b] = \phi(a) + \phi(b)$.
3. Consider the groups (R, +) and $(R_{>0}, \times)$.
   Define $\phi : R \to R_{>0}$ by

   $$\phi(a) = 2^a.$$

   Then $\phi(a + b) = 2^{a+b} = 2^a \,.\, 2^b = \phi(a)\,.\,\phi(b)$.
4. For any group $(G, \cdot)$, we have identity homomorphism $I : G \to G$ given by

   $$I(g) = g$$

   for all $g \in G$.
5. Define $\phi : S_n \to Z/2Z$ by :
   $\phi(\sigma) = 1$ if σ is an odd permutation, $\phi(\sigma) = 0$ if σ is an even permutation. Exercise : ϕ is a homomorphism.

Definition : Let $\phi : G \to G'$ be a homomorphism. We say G is the domain of ϕ and G' is the codomain of ϕ.

We now discuss some properties of homomorphisms.

Theorem : Let $\phi : G \to G'$ be a homomorphism. For all $a \in G, n \in Z$, the following hold

1. $\phi(e_G) = e_{G'}$
2. $\phi(a)^{-1} = \phi(a^{-1})$
3. $\phi(a_1 \ldots a_n) = \phi(a_1) \ldots \phi(a_n)$
4. $\phi(a)^n = \phi(a^n)$

Proof :

1. $\phi(e_G) = \phi(e_G)\,\phi(e_G)$ so multiplying by $\phi(e_G)^{-1}$ (on any side) implies $\phi(e_G) = e_{G'}$.
2. $e_{G'} = \phi(a \cdot a^{-1}) = \phi(a)\,\phi(a^{-1})$, so $\phi(a)^{-1} = \phi(a^{-1})$.
3. Induction on multiplicative definition.
4. Part 3, where $a_i = a$ for all i.

Definition : A group homomorphism $\phi : G \to G'$ is an isomorphism if ϕ is a bijection. If there is an isomorphism between G and G' we say G and G' are isomorphic. This is denoted by $G \cong G'$.

Example : In the example with different groups, 4 and 5 are isomorphisms.

Remark : If G is a group, the set of bijection $\{\phi : G \to G \mid \phi$ is a bijection$\}$ is a group under composition. Indeed, one can check that if $\phi : G \to G$ and $\psi : G \to G$ are isomorphisms, then $\phi o \psi$ is an isomorphism. The fact that $\phi o \psi$ is a bijection has nothing to do with the group structure. To see that $\phi o \psi$ is a homomorphism, observe

$$\phi(\psi(ab)) = \phi(\psi(a)\ \psi(b)) = \phi(\psi(a))\ \phi(\psi(b)).$$

Furthermore If $\phi : G \to G$ is a bijective homomorphism and we define $\psi : G \to G$ by assigning $\psi(a)$ to the unique value $a' \in G$ such that $\phi(a') = a$, then by definition $\psi o \phi = \phi o \psi = I$, (where I is the identity map). We need to justify that ψ, the candidate inverse for ϕ, is indeed a homomorphism. In this light, if $a', b' \in G$ then there exist a, b such that $\psi(a') = a$ and $\psi(b') = b$, and this means $\phi(a) = a'$ and $\phi(b) = b'$ and hence $\phi(ab) = a'b'$, so $\psi(a'b') = ab = \psi(a')\psi(b')$. Hence $\phi = \psi^{-1}$ an isomorphism of G.

This group of isomorphism from a group G to itself is called the automosphism group of G, and is denoted Aut (G).

Given a homomorphism $\phi : G \to G'$ there are subgroup of each that can indicate to us whether ϕ is injective or surjective.

Definition : Let $\phi : G \to G'$ be a homomorphism. Define $\ker(\phi) = \{g \in G : \phi(g) = e_{G'}\}$. This is called the kernel of ϕ. Define $\mathrm{im}(\phi) = \{\phi(g) : g \in G\}$. This is called the image of ϕ. We usually use the notation $\phi(G)$ for im(G).

Example : If $\phi : GL_2(R) \to R\backslash\{0\}$ is given by $\phi(A) = \det(A)$ then $\ker(\phi(A)) = SL_2(R)$ and $\mathrm{im}(\phi) = R\backslash\{0\}$.

□□□

GRAPH THEORY 5

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 *Write short note on isomorphism of graphs.* *[R.T.U. 2019]*

OR

Define the isomorphic graph with example. *[R.T.U. 2014]*

Ans. Isomorphic Graphs : Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be Isomorphic to each other if there exists a bijection mapping f from V_1 to V_2.

i.e., $f : V_1 \to V_2$ such that for each of the vertices v_i, v_j of V_1,

$\{v_i, v_j\} \in E_1 \Rightarrow \{f(v_i), f(v_j)\} \in E_2$

The function f is called an Isomorphism from G_1 to G_2.

It is immediately apparent by the definition of isomorphism that two isomorphic graphs must have

(a) The same number of vertices

(b) The same number of edges

(c) An equal number of vertices with a given degree *i.e.*, same degree sequence.

However, these conditions are by no means sufficient. For instance, the two graphs (given 'below) satisfy all three conditions, yet they are not isomorphic.

(a) (b)

Fig. : Two graphs that are not isomorphic

The graphs in fig. (a) and (b) are not isomorphic can be shown as follows : If the graph (a) were to be isomorphic to the one in (b), vertex x must correspond to y, because there are no other vertices of degree three. Now in (b) there is only one pendant vertex, w, adjacent to y. While in (a) there are two pendant vertices, u and v, adjacent to x. Thus the adjacency relationship is not preserved. Hence (a) and (b) are not isomorphic.

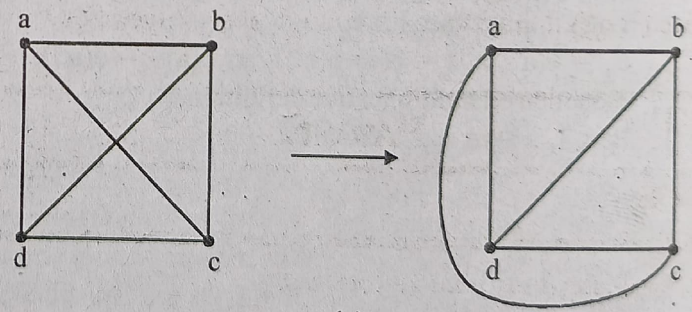
Q.2 *Write short note on planar graphs.* *[R.T.U. 2019]*

OR

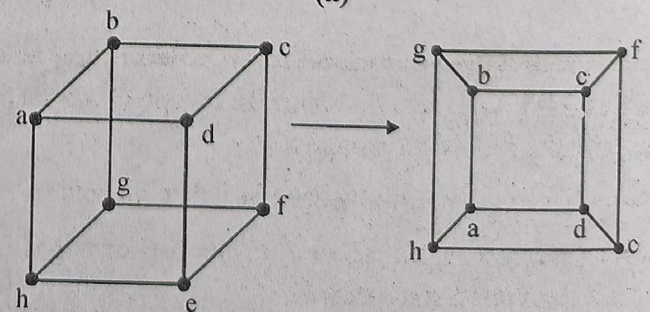
Define the planar graph with example. *[R.T.U. 2014]*

Ans. Planar Graphs : A graph is called planar if it can be drawn in a plane such that no two edges intersect except at their common end vertices, if any.

Note that, if a graph G has been drawn with crossing edges, it does not mean that G is non–planar. There may be other planar representation of G. For example, following are planar graphs.



(a)



(b)

Fig.

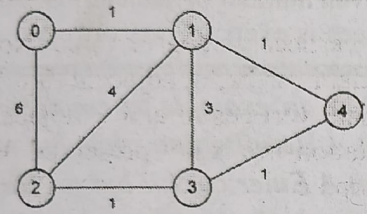
Drawing of a geometric representation of a graph on any surface such that no edges interest is called embedding.

Q.3 *Define the weighted graph with example.*

[R.T.U. 2015]

Ans. Weighted Graph : A graph having a weight (a number), associated with each edge. Some algorithms require all weights to be nonnegative, integral, positive, etc. These are also known as edge-weighted graph.

Example: The edge weights are 6,1,4,1,3,1,1 from left to right.



Q.4 *Write short note on Eulerian and Hamiltonian Graphs.* ***[R.T.U. 2012]***

Ans. Eulerian Graph : A closed walk in a graph G, which passes through every edge of G exactly once and which ends at the first vertex thus forming a closed circuit is called an *Euler line* in G, and a graph that consists of an Euler line is called an *Eulerian graph*.

$$v_1 \, e_1 \, v_2 \, e_2 \, v_3 \, e_3 \, v_4 \, e_4 \, v_2 \, e_5 \, v_5 \, e_6 \, v_1$$

Hamiltonian Graph : The Hamiltonian graph G is that which has a closed path passing through every vertex exactly once though it may not pass through every edge of G.

PART-B

Q.5 *Write short note on cut sets.*

[R.T.U. 2019]

Ans. Cut Set : A cut set of a connected graph G is a set S of edges with the following properties

- The removal of all edges in S disconnects G.
- The removal of some (but not all) of edges in S does not disconnects G.

As an example consider the following graph

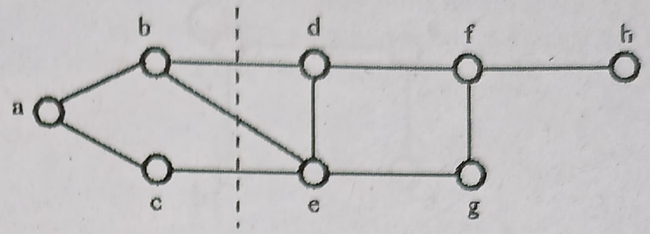


Fig.

We can disconnect G by removing the three edges bd, bc, and ce, but we cannot disconnect it by removing just two of thes edges. Note that a cut set is a set of edges in which no edges is redundant.

Q.6 *Write short note on vertex connectivity.*

[R.T.U. 2019]

Ans. Vertex Connectivity : The connectivity (or vertex connectivity) K(G) of a connected graph G (other than a complete graph) is the minimum number of vertices whose removal disconnects G. When $K(G) \geq k$, the graph is said to be k-connected (or k-vertex connected). When we remove a vertex, we must also remove the edges incident to it. As an example consider following graphs.

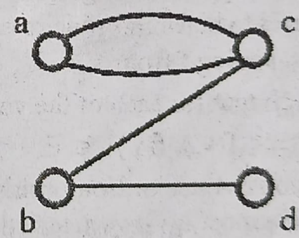


Fig.

The above graph G can be disconnected by removal of single vertex (either b or c). The G has connectivity 1.

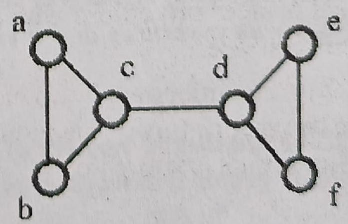


Fig.

The above graph G can be disconnected by removal of single vertex (either c or d). The vertex c or d is a cut-vertex. The G has connectivity 1.

The above G can be disconnected by removing just one vertex i.e., vertex c. The vertex c is the cut-vertex. The G has connectivity 1.

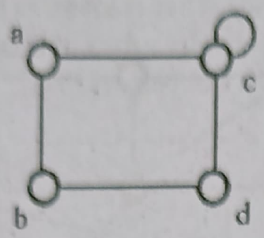


Fig.

The above G cannot be disconnected by removing a single vertex, but the removal of two non-adjacent vertices (such as b and c) disconnects it. The G has connectivity 2.

Q.7 ***Prove that the chromatic number of a graph will not exceed by more than one, the maximum degree of the vertices in a graph.*** *[R.T.U. 2017, 2011]*

Ans. Proof : Let $\Delta(G)$ be the maximum of the degrees of the vertices of a graph G.

Let the number of vertices in a graph is denoted by $|V|$. If $|V| = 1$, then $\Delta(G) = 0$ and $K(G) = 1$, so the result holds. Now let K be an integer and $K \geq 1$. Assume that the result holds for all graph with $|V| = K$ vertices.

Let G be a graph with $(K + 1)$ vertices. Let v be any vertex of G and let $G_0 = G/\{v\}$ is a subgraph of G obtained by deleting v from G.

Since G_0 has K vertices so we can use the induction hypothesis to conclude that

$$K(G_0) \leq 1 + \Delta(G_0)$$

Also, $\Delta(G_0) \leq \Delta(G)$

$\therefore \quad K(G_0) \leq 1 + \Delta(G)$

So G_0 can be colored with at most $1 + \Delta(G)$ colors. Since there can be atmost $\Delta(G)$ vertices adjacent to v, one of the available $1 + \Delta(G)$ colors remains for v. Thus G can be colored with atmost $1 + \Delta(G)$ colors. **Hence proved.**

Q.8 ***Show the total number of odd degree vertices of a (p, q), graph (graph with p vertices and q edges) is even.*** *[R.T.U. 2012]*

OR

Prove that the number of vertices of odd degrees in an undirected graph is always even. *[R.T.U. 2016]*

OR

Prove that the number of odd degree vertices in a graph G is always even.

[R.T.U. 2011, 2009; Raj. Univ. 2007, 2006, 2005]

Ans. Let G (V, E) be a graph. $V_e \subset V$ and $V_o \subset V$ be the set of vertices of even degree and odd degree respectively. The $V_e \cup V_o$. Also let *n* be the number of edges.

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v) = 2n$$

Since deg(v) is even for $v \in V_e$ $\sum_{v \in V_e} \deg(v) = m$ (say) is also even.

$$\Rightarrow \sum_{v \in V_o} \deg(v) = 2n - m = 2n - 2k$$

$$(m = 2k \text{ for some integer } k)$$

$$= 2(n-k) = 2l \; (l = n - k \text{ is an integer})$$

$$= \text{an even number}$$

Since all the terms in the sum $\sum_{v \in V_o} \deg(v)$ are odd, there must be an even number of such terms. Thus, number of odd degree vertices is even.

Q.9 ***(a) Give an example of connected graph that has***
(i) A Hamiltonian cycle but no Euler circuit
(ii) A Euler circuit but no Hamiltonian cycle
(b) What is the length of shortest path between the vertices a to z in the following weighted graph.

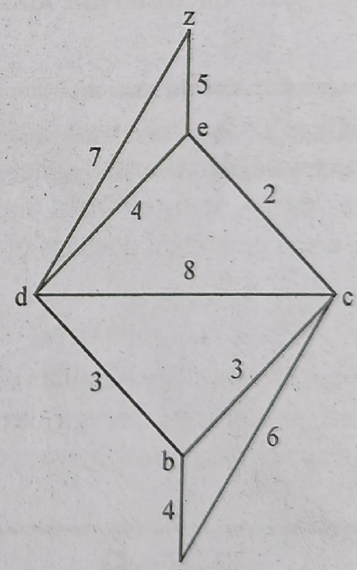


Fig.

[R.T.U. 2016]

Ans.(a)(i) A Hamiltonian cycle but no Euler circuit

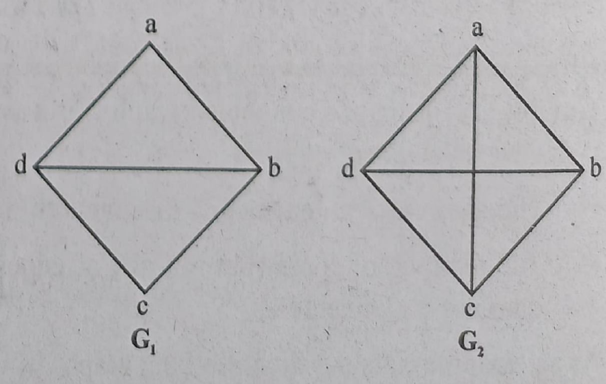


Fig. 1

(ii) Euler circuit but no Hamiltonian cycle

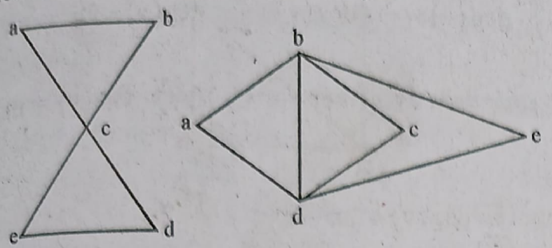


Fig. 2

Ans.(b) The weighted graph is as follows:

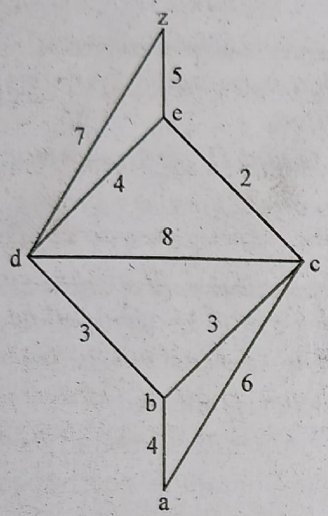


Fig.

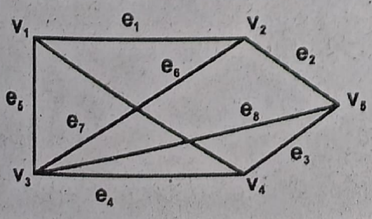
Initially, let $L(v_1) = 0$, $L(v) = \infty$, $v \neq v_1$

Constructing temporary and permanent labels as follows

v	v_1	v_2	v_3	v_4	v_5	v_6
$L_0(v)$	$\boxed{0}$	∞	∞	∞	∞	∞
$L_1(v)$	$\boxed{0}$	$\boxed{4}$	6	∞	∞	∞
$L_2(v)$	$\boxed{0}$	$\boxed{4}$	$\boxed{6}$	7	∞	∞
$L_3(v)$	$\boxed{0}$	$\boxed{4}$	$\boxed{6}$	$\boxed{7}$	8	∞
$L_4(v)$	$\boxed{0}$	$\boxed{4}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	14
$L_5(v)$	$\boxed{0}$	$\boxed{4}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	$\boxed{13}$

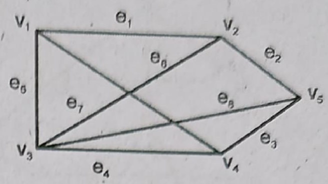
Thus, the length of the shortest path is 13 and the path is $v_1 \to v_3 \to v_5 \to v_6$

Q.10 ***Define spanning tree in a graph. Find five spanning trees for the graph shown in figure and write the sets of branches and chords corresponding to these spanning trees.***

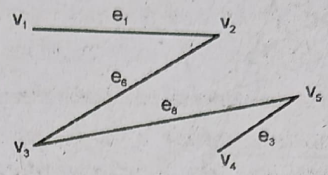


[R.T.U. 2015]

Ans. Spanning tree : A tree is spanning tree of graph G if it spans G, i.e., include every vertex of G and is a subgraph of G (every edge in the tree belongs to G).

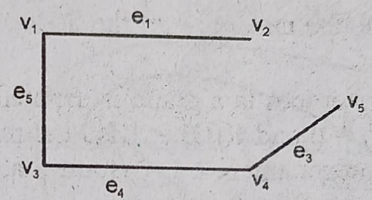


Spanning Tree 1 :



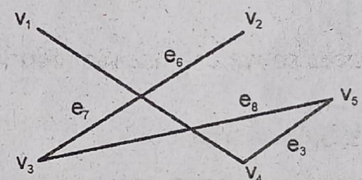
$E = \{e_1, e_3, e_6, e_8\}$

Spanning Tree 2 :



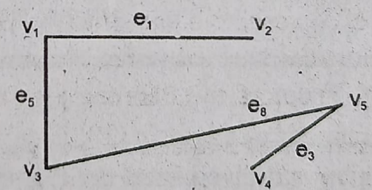
$E = \{e_1, e_3, e_4, e_5\}$

Spanning Tree 3 :



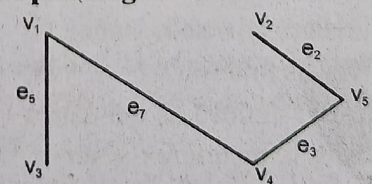
$E = \{e_3, e_6, e_7, e_8\}$

Spanning Tree 4 :



$E = \{e_1, e_3, e_5, e_8\}$

Spanning Tree 5 :



$E = \{e_2, e_3, e_5, e_7\}$

All the 5 are minimum spanning trees (trees with minimum edges) and have no branching sets.

Q.11 ***State the Kuratowski's theorem.*** *[R.T.U. 2013, 2012]*

Ans. Kuratowskis Theorem : A simple graph is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or K_5.

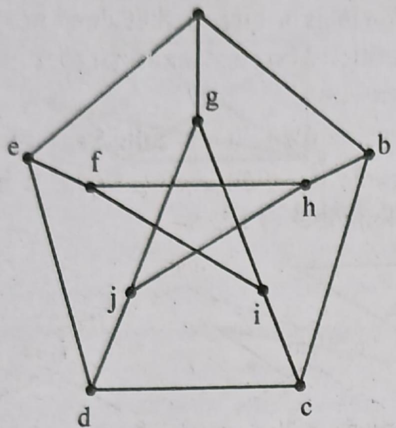


Fig. : Peterson Graph G

For example, the Peterson graph is not planar as it has a subgraph homeomorphic to $K_{3,3}$

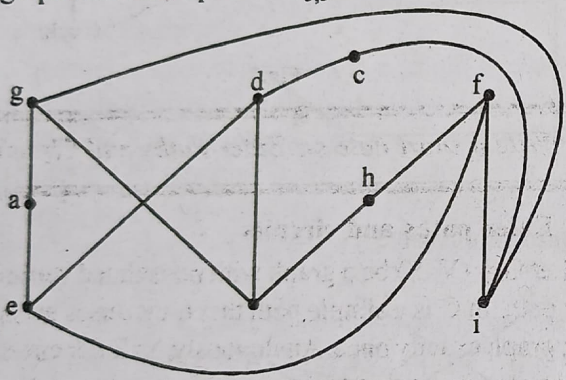


Fig. : Subgraph of G Homeomorphic to $K_{3,3}$

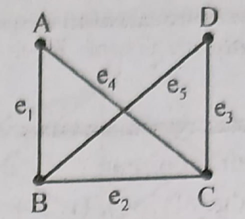
Q.12 ***Show the sum of the degrees of all the vertices in a graph is equal to twice the number of edges in the graph.*** ***[R.T.U. 2011]***

Ans. The given statement, "sum of the degree of all the vertices in G is twice the number of edges in G", is called **Handshaking Lemma** which is derived from the fact that at a gathering the total no. of handshakes is equal to twice the number of hands involved. Replacing 'handshakes' with 'edges' and 'hands' with 'vertices' gives the lemma.

Now proof can be given as follows :

Since degree of vertex is defined as the number of edges (or arcs) connected with it, therefore the sum of the degree counts the total no. of times an edge is connected with a vertex. As every edge is connected with exactly two vertices, so each edge is counted twice at each of its ends. This implies that the sum of the degree equals twice the number of edges.

For example : In the given graph G



deg (A) = 2 deg (B) = 3
deg (C) = 3 deg (D) = 2

The sum of degree equals 10, which as expected is twice the number of edges.

Q.13 ***(a) Show that isomorphism of simple graphs is an equivalence relation.***
(b) Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs $\bar{G}$ and $\bar{H}$ are also isomorphic. ***[Raj.Univ. 2005]***

Ans.(a) Reflexive : Let G be any simple graph then G is isomorphic to itself by the identity function.

$\therefore$ Isomorphism is reflexive.

Symmetric : Let G is isomorphic to another simple graph H. Then there exists a one-to-one correspondence f from G to H that preserves adjacency and non-adjacency. It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency as well as non adjacency.

Hence isomorphism is symmetric.

Transitive : Let G is isomorphic to H and H is isomorphic to K, then there exist bijections f and g from G to H and from H to K that preserve adjacency and non adjacency. It follows that gof is a bijection from G to K that preserves adjacency and non adjacency.

Hence isomorphism is transitive.

Thus, isomorphism is an equivalence relation.

Ans.(b) Let f be an isomorphism from G onto H. Then f is a one-to-one correspondence from G onto H and f also preserves adjacency as well as non adjacency of vertices.

Let ϕ be a mapping from $\bar{G}$ to $\bar{H}$. Further the adjacent vertices in $\bar{G}$ are not adjacent in G also the non adjacent vertices in $\bar{G}$ are adjacent in G, similar statement holds for the graph H.

As G and H are simple so $\bar{G}$ and $\bar{H}$ must be simple graph. Thus ϕ is a one-to-one correspondence from $\bar{G}$ onto $\bar{H}$ and ϕ also preserves the adjacency as well as non-adjacency of vertices.

Hence ϕ is an isomorphism from $\bar{G}$ onto $\bar{H}$. Therefore, $\bar{G}$ and $\bar{H}$ are isomorphic to each other.

Q.14 *Write short note on Graph Homeomorphism, paths and circuits.*

Ans. Graph Homeomorphism : If a graph G has a vertex v of degree 2 and edges (v, v_1), (v, v_2) with $v_1 \neq v_2$, we say that the edges (v, v_1) and (v, v_2) are in series. Deleting such vertex v and replacing (v, v_1) and (v, v_2) with (v_1, v_2) is called a series reduction. For instance, in the third graph of Fig. the edges (v_6, v_2) and (v_6, v_4) are in series. By removing vertex v_6 we get the first graph in the left.

The opposite of a series reduction is an elementary subdivision. It consists of replacing an edge (u, v) with two edges (u, w) and (w, v), where w is a new vertex.

Two graphs are said to be homeomorphic if they are isomorphic or can be reduced to isomorphic graphs by a sequence of series reductions. Equivalently; two graphs are homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.

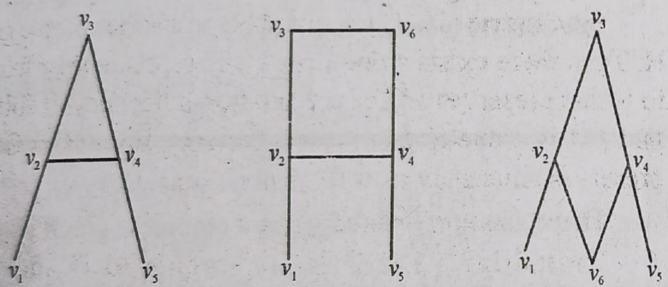


Fig. : Three Homeomorphic Graphs

Note that if a graph G is planar, then all graphs homeomorphic to G are also planar.

Paths and Circuits : A path from v_0 to v_n of length n is a sequence of $(n + 1)$ vertices (v_k) and n edges (e_k) of the form $v_0, e_1, v_1, e_2, v_2, \ldots e_n, v_n$, where each edge e_k connects v_{k-1} with v_k (and points from v_{k-1} to v_k if the edge is directed). The path may be specified by giving only the sequence of edegs $e_1 \ldots e_n$. If there are no multiple edges we can specify the path by giving only the vertices: $v_0, v_1, \ldots, v_n$. The path is a circuit (or cycle) if it begins and ends at the same vertex, *i.e.*, $v_0 = v_n$ and has length greater than zero. A path or circuit is simple if it does not contain the same edge twice.

A graph that does not contain any circuit is called acyclic. The number of edges appearing in the sequence is called the length of the path. A path with no repeated edges is called a trail. A (path or trail) is trivial if it has only one vertex and no edges, otherwise non-trivial. A non–trivial closed trail from a vertex u to itself is called a circuit or a circuit is a closed path of non–zero length from a vertex u to u with no repeated edges. A cycle is a closed path of non zero length from a vertex u to u with no repeated edges and no repeated vertices except initial and terminal vertices. Thus,

- A cycle is a circuit that does not contain any repetition of vertices except the initial and terminal vertices.
- A cycle of length n is called a n–cycle.
- A cycle is called even (or odd) if it contains an even (or odd) number of edges.

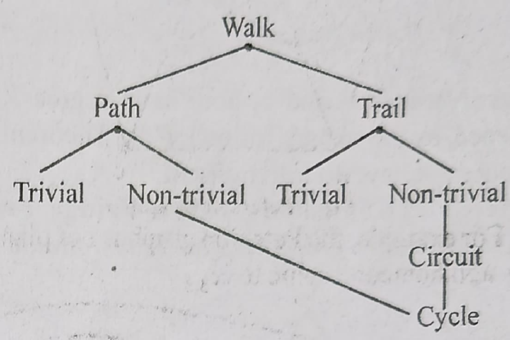


Fig.

Q.15 *Write a short note on Euler Paths and Circuits.*

Ans. Euler paths and circuits

Let G = (V, E) be a graph with no isolated vertices. An Euler path, in G is a simple path that transverses every edge of the graph exactly once. Analogously, an Euler circuit in G is a simple circuit that transverses every edge of the graph exactly once.

Theorem 1 : *If a graph G has an Euler circuit then every vertex of the graph has even degree.*

Proof : Let G be a graph with an Euler circuit. Start at some vertex on the circuit and follow the circuit from vertex to vertex, erasing each edge as you go along it. When you go through a vertex you erase one edge going in and one edge going out, or else you erase a loop. Either way, the erasure reduces the degree of the vertex by 2. Eventually every edge gets erased and all the vertices have degree 0. So all vertices must have had even degree to begin with.

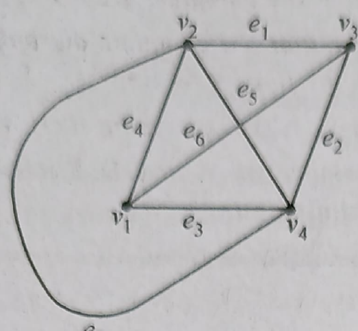
It follows from the above theorem that if a graph has a vertex with odd degree then the graph can not have an Euler circuit.

The following provide a converse to the above theorem.

Let G be a connected multigraph. Then C contains an Euler circuit if and only if its (G) vertices have even degree. Also, G contains an Euler path from vertex a to vertex b ($\neq a$) if and only if a and b have odd degree, and all its other vertices have even degree.

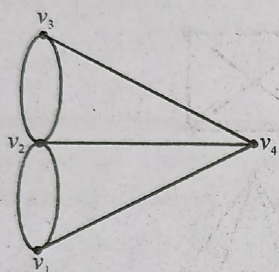
Theorem 2 : (Euler Theorem) : *If all the vertices of a connected graph have even degree, then the graph has an Euler circuit.*

Example 1 : *Show that the following graph has no Euler circuit :*



Solution: Vertices v_1 and v_3 both have degree 3, which is odd. Hence, by the remark following the Theorem 3.11, this graph does not have an Euler circuit.

Example 2 : (Königsberg Seven Bridge Problem) : *Does the following graph has an Euler circuit ?*



Solution: Here deg $(v_1) = 3$, deg $(v_2) = 5$, deg $(v_3) = 3$, deg $(v_4) = 3$.

Since more than two vertices are of odd degree, the given graph does not have an Euler path or circuit.

Procedure to Determine whether a Graph G has an Euler Circuit :

Step I : First list the degrees of all the vertices in the graph.

Step II :

(a) If degree of any vertex is zero, the graph is not connected therefore it cannot have an Euler path or circuit.

(b) If all the degrees are even then G has both Euler path and Euler circuit.

(c) If exactly two vertices are of odd degree then the graph has Euler path only, no Euler circuit.

(d) If degree of more than two vertices is odd, then graph does not have any Euler path or circuit.

FLEURY'S ALGORITHM

This algorithm is used to construct an Euler circuit.

Let G = (V, E) be a graph and C be the Euler circuit.

Step I.: Choose a vertex u as the starting vertex.

Step II : Select an edge $e_1 = \{u, v_1\}$. Let $V_C : u, v_1$ and $E_C: e_1$. Remove e_1 from E and let $G_1 = (V, E-\{G\})$ be the resulting subgraph of G.

Step III : Suppose that $V_C = v, u, v_1, v_2, ...v_n$ and E_C: $e_1, e_2,.....,e_m$ have been constructed so far and that all of these edges and any resulting isolated vertices have been removed from V and E to form G_m.

Since v_m is of even degree and e_m ends at v_m there is an edge e_{m+1} in G_m that incident on v_m. If there are more than one such edges, select one that is not a bridge for G_m, denote the vertex of e_{m+1} other than v_m by v_{m+1} and extend V_C to V_C : $u, v_1, v_2,.....v_m, v_{m+1}$ and E_C to E_C: $e_1, e_2, ... e_m, e_{m+1}$. Then delete e_{m+1} and any isolated vertices from G_m to G_{m+1}.

Step IV : Repeat step III untill no edges remain in E.

PART-C

Q.16 *In a complete graph with n - vertices there are $\dfrac{(n-1)}{2}$ edge disjoint Hamiltonian circuits, if n is an odd number ≥ 3.*

[R.T.U. 2019]

Ans. There are $\dfrac{n(n-1)}{2}$ edges in a complete graph with n-vertex and a Hamilton circuit in G consist of n edges, So number of edges disjoint Hamilton circuit is G cannot exceed $\dfrac{n-1}{2}$. That these are $\dfrac{n-1}{2}$ edges disjoint Hamilton circuits.

When n is odd the subgraph is diagram given below is a Hamilton.

Circuits vertex fixed on a circle rotate polygon pattern clockwise $\dfrac{360}{n-1}, \dfrac{2\times 360}{n-1}, ... \dfrac{n-3}{2} \times \dfrac{360}{n-1}$ edges.

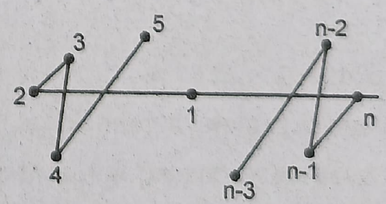
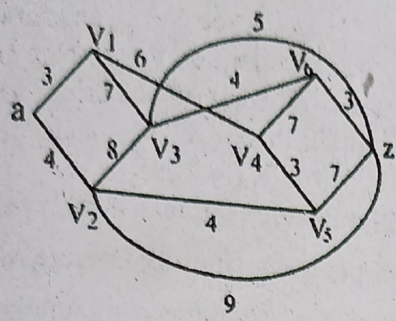


Fig.

Hamilton circuits has no edges in common with any of the previous one. Therefore $\dfrac{n-3}{2}$ new Hamilton circuits all edges disjoint form fig. and also disjoint among themselves.

Q.17 (a) *Find the shortest path from a to z in the following graph-*



(b) *Suppose that a connected planar graph has 30 vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph.* ***[R.T.U. 2019]***

Ans.(a) Find the shortest path from a to z

(i) $av_1, v_1v_4, v_4v_6, v_6z$

$3+6+7+3=19$

(ii) $av_1, v_1v_3, v_3v_6, v_6z$

$3+7+4+3=17$

(iii) $av_1, v_1v_4, v_4v_5, v_5z$

$3+6+3+7=19$

(iv) $av_2, v_2v_3, v_3v_6, v_6z$

$4+8+4+3=19$

(v) $av_2, v_2v_3, v_3v_6, v_6v_4, v_4v_5, v_5z$

$4+8+4+7+3+7=33$

(vi) av_2, v_2z

$4+9=13$

(vii) av_1, v_1v_3, v_3z

$3+8+5=16$

(viii) av_2, v_2v_3, v_3z

$4+8+5=17$

shortest path (av_2, v_2z)

Ans.(b) $n=3$ and deg $(v)=3\times 30=90$

sum of degrees of vertex $= 2 \times$ No. of edges

$90=2e$

$e=45$

Region according Euler's formula

$n-e+r+2=30$

$r=17$ regions

Q.18 (a) *Sketch the complete graphs $k_n, 1 \le n \le 6$.*

(b) *Show that the complete digraph with n-nodes has the maximum number of edges i.e. n(n–1) edges, assuming there are no loops.*

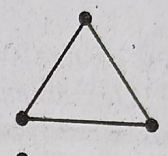
(c) *Draw graph which is Eulerian as well as Hamiltonian.* ***[R.T.U. 2017]***

Ans.(a) The graphs of k_1, k_2, k_3, k_4, k_5 and k_6 are as follows :

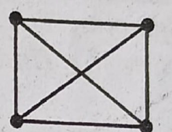
(i) k_1

(ii) k_2

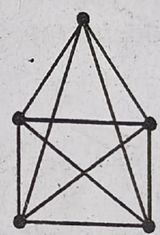
(iii) k_3



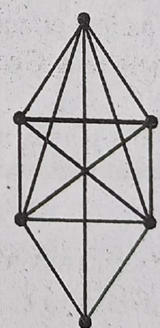
(iv) k_4



(v) k_5



(vi) k_6



Ans.(b) There are two types of complete digraph, i.e. complete asymmetric digraph and complete symmetric digraph. Out of these two a complete symmetric digraph has more edges than that of complete asymmetric.

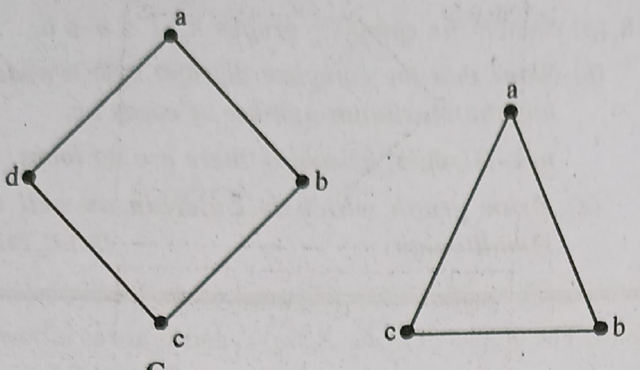
Also a complete symmetric digraph with n vertices has

$2\times {}^nC_2$ edges i.e.

$$2\times\frac{n(n-1)}{2}=n(n-1) \text{ edges}$$

Hence Maximum number of edges in a complete digraph is n(n – 1).

Ans.(c) Both Euler circuit and Hamiltonian cycle :



G_1 G_2

Fig.

In G_1 Euler circuit : a, b, c, d, a
Hamiltonian cycle : a, b, c, d, a

In G_2 Euler circuit and Hamiltonian cycle are a, b, c, a.

Q.19 *(a) Explain the Minimal Spanning Tree. Also write the Kruskal Algorithm for find Minimal Spanning tree.*

(b) Given the Graph in following figure. Apply Prim's algorithm to obtain the minimal spanning tree.

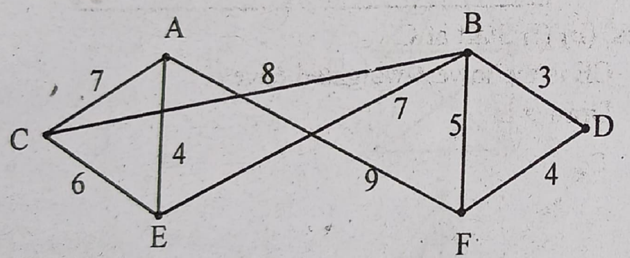


Fig. *[R.T.U. 2012]*

Ans. (a) Minimal Spanning Tree

If a connected weighted tree G, then its minimal spanning tree is a spanning tree of G such that the sum of the weights of its edges is minimum. For instance for the following graph of figure. The spanning tree, shown by thicker lines is the one of minimum weight.

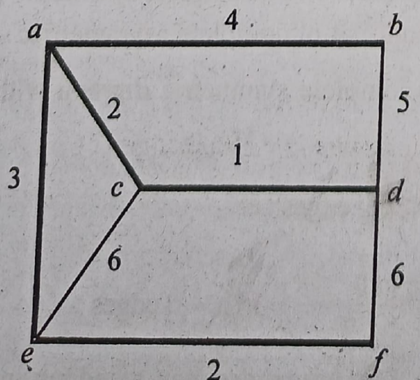


Fig. : Minimum Spanning Tree

Kruskal's algorithm

An algorithm in graph theory that finds a minimal spanning tree for a connected weighted graph. This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. If the graph is not connected, then it finds a minimal spanning forest (a minimal spanning tree for each connected component).

Kruskal's algorithm is an example of a greedy algorithm.

This algorithm was written by Joseph Kruskal in 1956.

An algorithm for computing a minimal spanning tree. It maintains a set of partial minimal spanning trees, and repeatedly adds the shortest edge in the graph whose vertices are in different partial minimal spanning trees.

Algorithm of finding minimal spanning tree by Kruskal's algorithm

Step 1 : Create a forest F (a set of trees), where each vertex in the graph is a separate tree

Step 2 : Create a set S containing all the edges in the graph

Step 3 : While S is nonempty

(a) remove an edge with minimum weight from S

(b) if that edge connects two different trees, then add it to the forest, combining two trees into a single tree

(c) otherwise discard that edge

At the termination of the algorithm, the forest has only one component and forms a minimal spanning tree of the graph.

For example determine the minimal spanning tree in the following graph by applying Kruskal's algorithm.

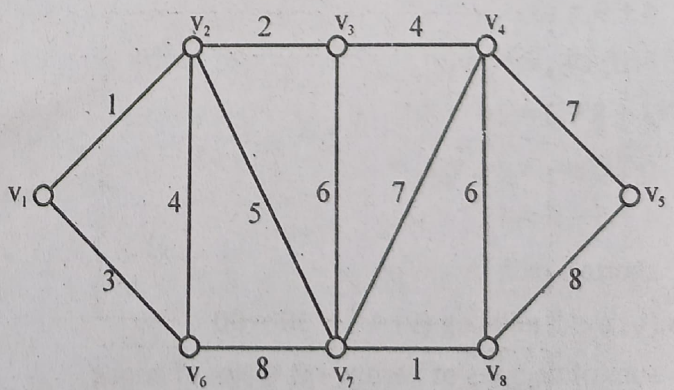
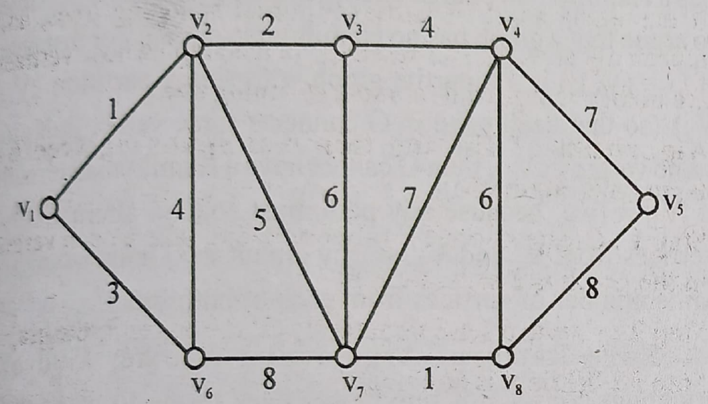


Fig.

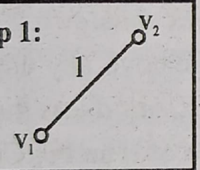
Using the above graph, here are the steps to the minimal spanning tree, using Kruskal's algorithm:

1. v_1 to v_2 –cost is 1 –add to tree
2. v_7 to v_8 –cost is 1 –add to tree
3. v_2 to v_3 – cost is 2 – add to tree

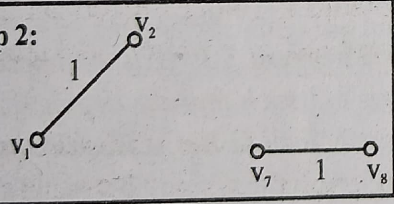
4. v_1 to v_6 – cost is 3 – add to tree
5. v_2 to v_6 – cost is 4 – reject because it forms a circuit
6. v_3 to v_4 – cost is 4 – add to tree
7. v_2 to v_7 – cost is 5 – add to tree
8. v_3 to v_7 – cost is 6 – reject because it forms a circuit
9. v_4 to v_8 – cost is 6 – reject because it forms a circuit
10. v_4 to v_7 – cost is 7 – reject because it forms a circuit
11. v_4 to v_5 – cost is 7 – add to tree
12. We stop here, because n – 1 edge has been added. We are left with the minimal spanning tree, with a total weight of 23.



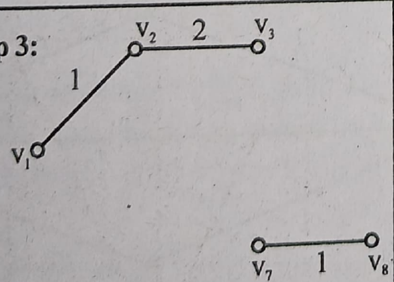
Step 1:



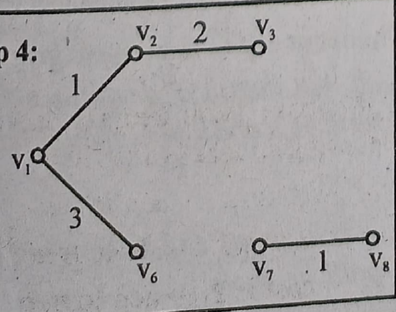
Step 2:



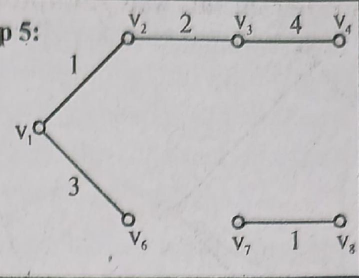
Step 3:



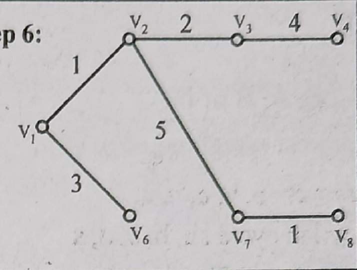
Step 4:



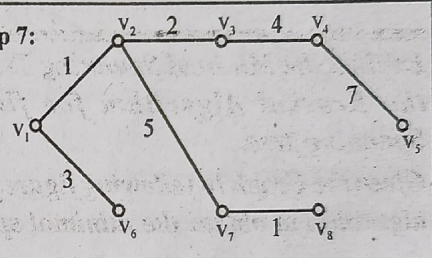
Step 5:



Step 6:



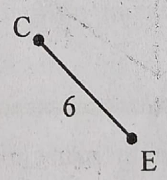
Step 7:



Ans. (b) (i) Start at C

CE is the lowest-weighted edge (6).

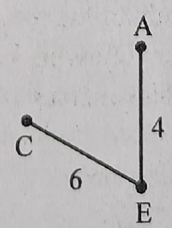
Draw it in.



(ii) From C or E

EA is the lowest-weighted edge (4).

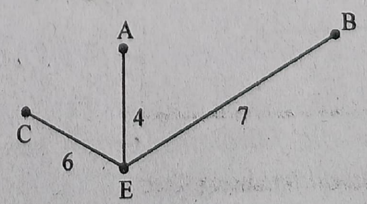
Draw it in.



(iii) From C, E or A

EB is the lowest-weighted edge (7).

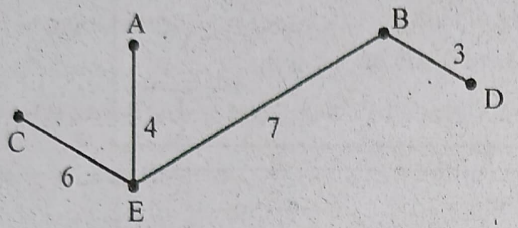
Draw it in.



(iv) From C, E, A or B

BD is the lowest-weighted edge (3).

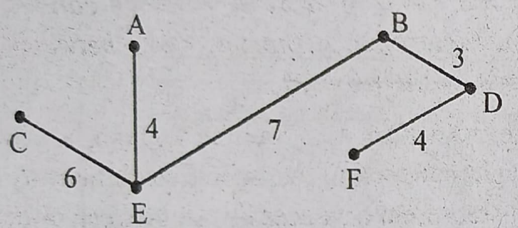
Draw it in.



(v) From C, E, A, B or D

DF is the lowest-weighted edge (4).

Draw it in.



All vertices have now been joined.

The minimum spanning tree is determined.

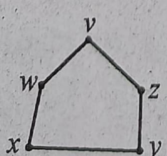
Minimum spanning tree length $= 6 + 4 + 7 + 3 + 4$

$= 24$ units.

Q.20 ***Write a detailed note on Hamiltonian path and circuits with example.***

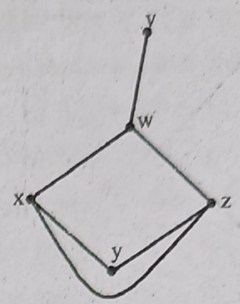
Ans. Hamiltonian Path and Circuits : A Hamiltonian circuit in a graph G is a circuit, that contains each vertex of G once (except for the starting and ending vertex, which occurs twice). A Hamiltonian path in G is a path (not a circuit) that contains each vertex of G once. Note that by deleting an edge in a Hamiltonian circuit we get a Hamiltonian path, so if a graph has a Hamiltonian circuit, then it also has a Hamiltonian path. The converse is not true, *i.e.*, *a* graph may have a Hamiltonian path but not a Hamiltonian circuit.

Example 1 : *Find a Hamiltonian circuit in the graph :*



Solution: *vwxyzv*

Example 2 : *Show that the following graph has a Hamiltonian path but no Hamiltonian circuit:*



Solution: *vwxyz* is a Hamiltonian path. There is no Hamiltonian circuit since no cycle goes through *v.*

In general it is not easy to determine if a given graph has a Hamiltonian path or circuit, although often it is possible to argue that a graph has no Hamiltonian circuit. For instance if $G = (V, E)$ is a bipartite graph with vertex partition $\{V_1, V_2\}$ (so that each edge in G connects some vertex in V_1 to some vertex in V_2), then G cannot have a Hamiltonian circuit if $|V_1| \neq |V_2|$, because any path must contain alternatively vertices from V_1 and V_2, so any circuit in G must have the same number of vertices from each of both sets.

Edge Removal Argument : Another kind of argument consists of removing edges trying to make the degree of every vertex equal two. For instance in the graph of Fig. we cannot remove any edge because that would make the degree of *b*, *e* or *d* less than 2, so it is impossible to reduce the degree of *a* and *c*. Consequently that graph has no Hamiltonian circuit.

Dirac's Theorem : *If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least n/2, then G has a Hamiltonian circuit.*

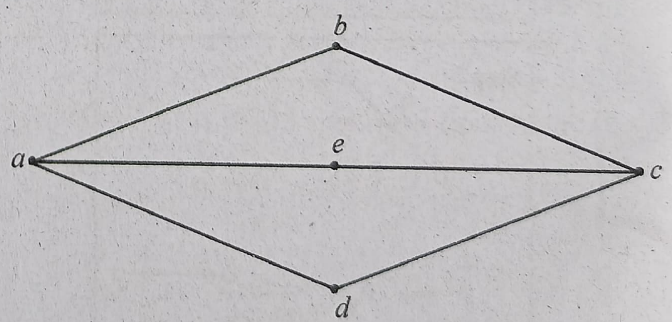
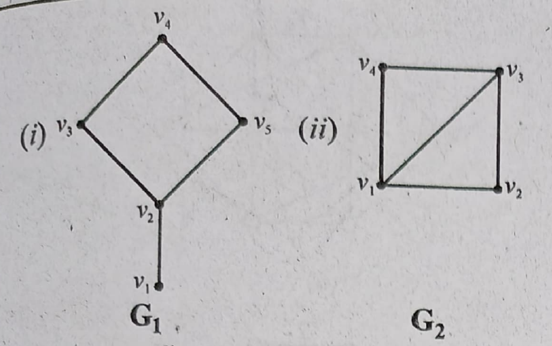
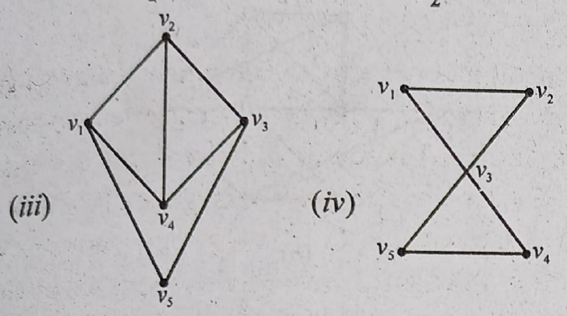


Fig. : Graph without Hamiltonian Circuit

Ore's Theorem : *If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G then G has a Hamiltonian circuit.*

Example 3 : *Which of the following graphs has a Hamiltonian path or cycle :*





Solution: (i) Graph G_1 has a Hamiltonian path v_1, v_2, v_3, v_4, v_5 but no Hamiltonian cycle.

(ii) Graph G_2 has a Hamiltonian cycle v_1, v_2, v_3, v_4, v_1.

(iii) Graph G_3 has a Hamiltonian cycle $v_1, v_5, v_3, v_4, v_2, v_1$.

(iv) Graph G_4 has no Hamiltonian path.

Q.21 *Explain shortest path problem with example.*

Ans. Shortest Path : Shortest path between two vertices in a graph is the path of minimum length. Thus, if :

(a) The graph is without weights, the length of path denotes the number of edges in the path and shortest path between two vertices is the path with least number of edges.

(b) The graph is weighted graph, the shortest path between two vertices is the path of minimum length (weight).

Shortest Path Problem : With each edge e of G let there associate a real number $w(e)$, called its weight. Then G, together with these weights on its edges, is called a weighted graph.

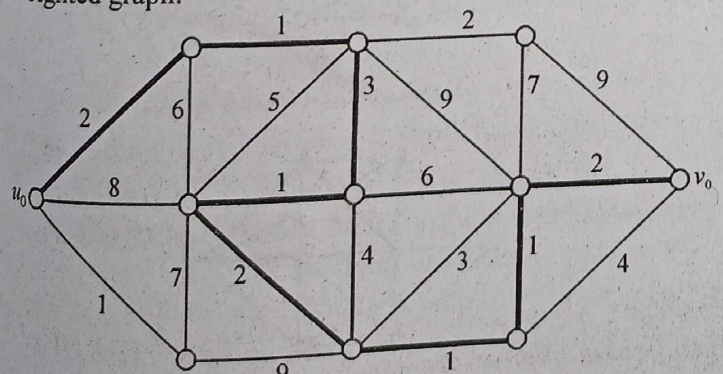


Fig. 1 : A (u_0, v_0) Path of Minimum Weight

Weighted graphs occur frequently in applications of graph theory. In the friendship graph, for example, weight might indicate intensity of friendship; in the communication graph, they could represent the construction of maintenance costs of the various communication links.

If H is a subgraph of a weighted graph, the weight $w(H)$ of H is the sum of the weights $\sum_{e \in E(H)} w(e)$ on its edges. Many optimization problems amount to finding, in a weighted graph, a subgraph of a certain type with minimum (or maximum) weight.

Problem : *Given a railway network connecting various towns, determine a shortest route between two specified towns in the network.*

Here one must find, in a weighted graph, a path of minimum weight connecting two specified vertices u_0 and v_0; the weights represent distances by rail between directly-linked towns, and are therefore non-negative. The path indicated in the graphs of figure 1 is a $(u_0–v_0)$-path of minimum weight.

We now present an algorithm for solving the shortest path problem. For clarity of exposition, we shall refer to the weight of a path in a weighted graph as its length; similarly the minimum weight of a (u, v)-path will be called the distance between u and v and denoted by $d(u, v)$. These definitions coincide with the usual notions of length and distance, when all the weights are equal to one.

It clearly suffices to deal with the shortest path problem for simple graphs; so we shall assume here that G is simple. We shall also assume that all the weights are positive. This, again, is not a serious restriction because, if the weight of an edge is zero, then its ends can be identified. We about the convention that $w(uv) = \infty$ if $uv \notin E$.

The algorithm to be described was discovered by Dijkstra (1959) and, independently, by Whiting and Hiller (1960). It finds not only a shortest (u_0, v_0)-path, but shortest paths from u_0 to all other of G. The basic idea is as follows :

Suppose that S is a proper subset of V such that $u_0 \in S$, and let $\overline{S}$ denote V\S. If P = $u_0 \ldots \overline{u}\,\overline{v}$ is a shortest path from u_0 to $\overline{S}$ then clearly $\overline{u} \in S$ and the $(u_0, \overline{u})$-section of path of P must be a shortest $(u_0, \overline{u})$-path. Therefore

$$d(u_0, \overline{v}) = d(u_0, \overline{u}) + w(\overline{u}\,\overline{v})$$

and the distance from u_0 to $\overline{S}$ is given by the formula

$$d(u_0, \overline{S}) = \min_{\substack{u \in S \\ v \in \overline{S}}} \{d(u_0, u) + w(uv)\} \qquad \ldots\text{(i)}$$

This formula is the basis of Dijkstra's algorithm. Starting with the set $S_0 = \{u_0\}$, an increasing sequence $S_0, S_1, \ldots, S_{v-1}$ of subsets of V is constructed, in such a way that, at the end of stage i, shortest paths from u_0 to all vertices in S_1 are known.

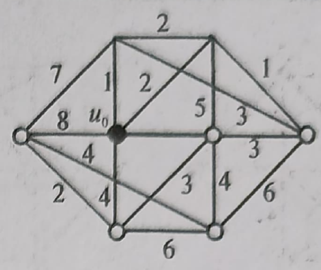
The first step is to determine a vertex nearest to u_0. This is achieved by computing $d(u_0, \overline{S}_0)$ and selecting a vertex $u_1 \in \overline{S}_0$ such that $d(u_0, u_1) = d(u_0, \overline{S}_0)$; by eq.(i)

$$d(u_0, \overline{S}_0) = \min_{\substack{u \in S_0 \\ v \in \overline{S}_0}} \{d(u_0, u) + w(uv)\} = \min_{v \in \overline{S}_0} \{w(u_0, v)\}$$

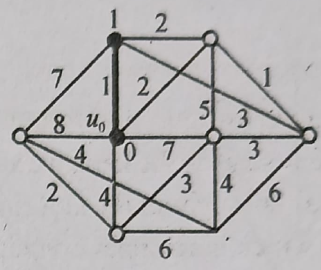
and so $d(u_0, \overline{S}_0)$ is easily computed. We now set $S_1 = \{u_0, u_1\}$ and let P_1 denotes the path u_0, u_1; this is clearly a shortest (u_0, u_1)-path. In general, if the set $S_k = (u_0, u_1, \ldots, u_k)$ and corresponding shortest paths $P_1, P_2, \ldots, P_k$ have already been determined, we compute $d(u_0, \overline{S}_k)$ using eq.(i) and select a vertex $u_{k+1} \in \overline{S}_k$ such that $d(u_0, u_{k+1}) = d(u_0, \overline{S}_k)$. By eq.(i), $d(u_0, u_{k+1}) = d(u_0, u_j) + w(u_j u_{k+1})$ for some $j \leq k$; we get a shortest (u_0, u_{k+1})-path by adjoining the edge $u_j u_{k+1}$ to the path P_j.

We illustrate this procedure by considering the weighted graph depicted in figure 2. Shortest paths from u_0 to the remaining vertices are determined in seven stages. At each stage, the vertices to which shortest paths have been found are indicated by solid dots, and each is labeled by its distance from u_0; initially u_0 is labeled 0. The actual shortest paths are indicated by solid lines. Notice that, at each stage, these shortest paths together form a connected graph without cycles; such a graph is called a tree, and we can think of the algorithm as a 'tree-growing' procedure. The final tree, in figure 2, has the property that, for each vertex v, the path connecting u_0 and v is a shortest (u_0, v)-path.

Dijkstra's algorithm is refinement of the procedure. This refinement is motivated by the consideration that, if the minimum in eq.(i) were to be computed from scratch at each stage, many comparisons would be repeated unnecessarily.



(a)



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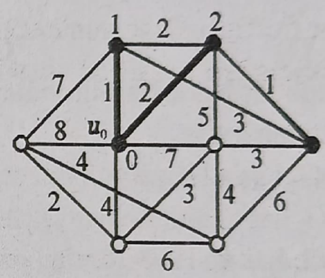
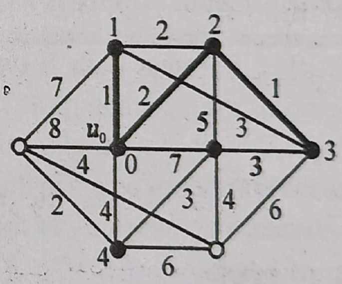


(c)



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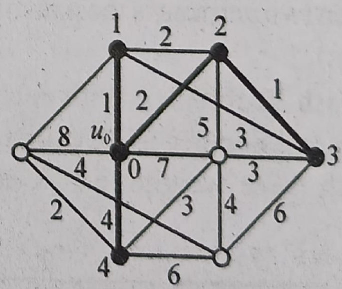
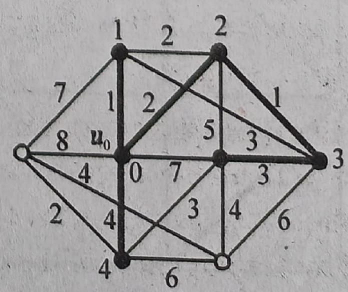


(e)



(f)

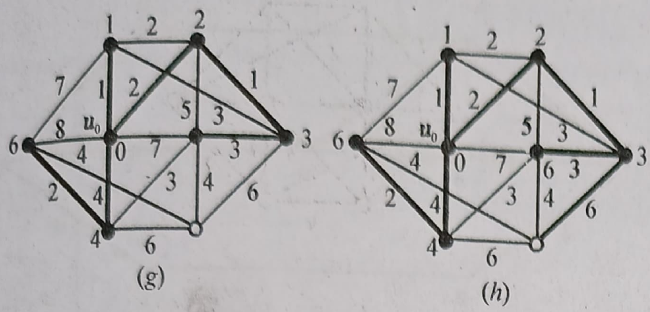


Fig. 2 : Shortest Path Algorithm

To avoid such repetitions, and to retain computational information from one stage to the next, we adopt the following labeling procedure. Throughout the algorithm, each vertex v carries a label $l(v)$ which is an upper bound; $d(u_0, v)$. Initially $l(u_0) = 0$ and $l(v) = \infty$ for $v \neq u_0$ (In actual computation ∞ is replaced by any sufficiently large number.) As the algorithm proceeds, these labels are modified so that, at the end of stage i,

$$l(u) = d(u_0, u) \text{ for } u \in S_i$$

$$\text{and } l(v) = \min_{u \in S_{i-1}} \{d(u_0, u) + w(uv)\} \text{ for } v \in \overline{S}_i$$

Dijkstra's Algorithm

1. Set $l(u_0) = 0, l(v) = \infty$ for $v \neq u_0, S_0 = \{u_0\}$ and $i = 0$
2. For each $v \in \overline{S}_i$, replace $l(v)$ by min $\{l(v), l(u_i) + w(u_i v)\}$. Compute $\min_{v \in \overline{S}_i}\{l(v)\}$ and let u_{i+1} denotes a vertex this minimum is attained.

   Set $S_{i+1} = S_i \cup \{u_{i+1}\}$.
3. If $i = v - 1$, stop. If $i < v - 1$, replace i by $i + 1$ and go to step 2.

When the algorithm terminates, the distance from u_0 to v is given by the final value of the label $l(v)$. (If our interest is in determining the distance to one specific vertex v_0, we stop as soon as some u_j equals v_0.) A flow diagram summarizing this algorithm is shown in figure 3.

As described above, Dijkstra's algorithm determines only the distances from u_0 to all the other vertices, and not the actual shortest paths. These shortest paths can, however, be easily determined by keeping track of the predecessors of vertices in the tree.

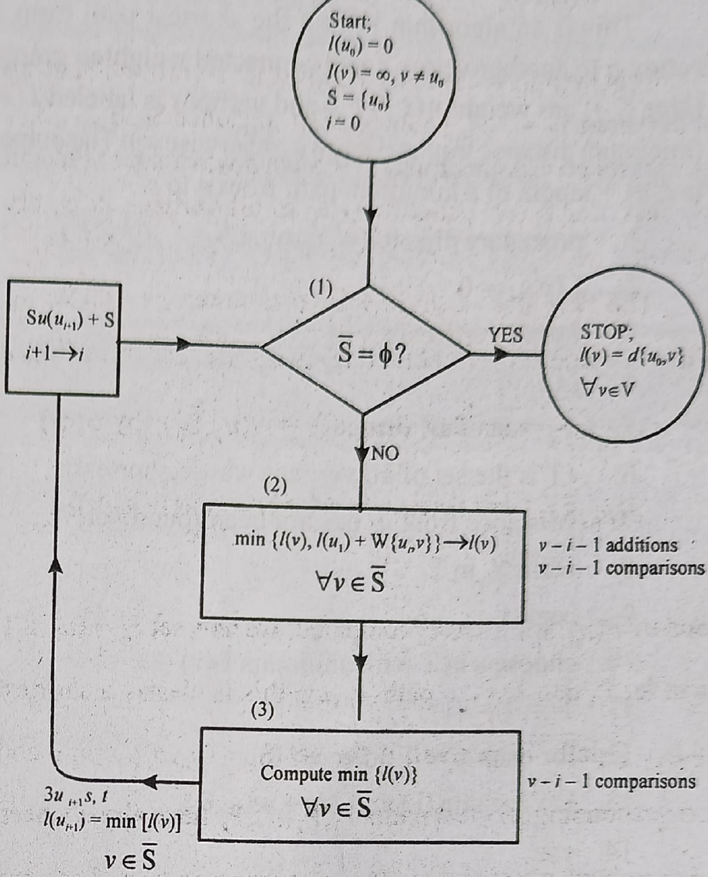


Fig. 3 : Dijkstra's Algorithm

Dijkstra's algorithm is an example of what Edmonds (1965) calls a good algorithm. A graph-theoretic algorithm is good. If the number of computational steps required for its implementation on any graph G is bounded above by a polynomial in v and $\in$ (such as $3v^2\in$). An algorithm whose implementation may require an exponential number of steps (such as 2) might be very inefficient for some large graphs.

To see that Dijkstra's algorithm is good, note that the computations involved in boxes 2 and 3 of the flow diagram, totalled over all iterations, require $v(v-1)/2$ additions and $v(v-1)$ comparisons. One of the questions that is not elaborated upon in the flow diagram is the matter of deciding whether a vertex belongs to 9 or not (box 1). Dreyfus (1969) reports a technique for doing this that requires a total of $(v-1)^2$ comparisons. Hence, if we regard either a comparison or an addition as a basic computational unit, the total number of computations required for this algorithm is approximately $5v^2/2$, and thus of order v^2. (A function $f(v, e)$ is of order $g(v, \in)$ if there exists positive constant c such that $f(v, \in)/g(v, \in) \leq c$ for all v and e.)

Although the shortest path problem can be solved by a good algorithm, there are many problems in graph theory for which no good algorithm is known.

Dijkstra's Shortest-Path Algorithm

This is an algorithm to find the shortest path from a vertex a to another vertex z in a connected weighted graph. Edge (i, j) has weight $w(i, j) > 0$ and vertex x is labeled $L(x)$ (minimum distance from a if known, otherwise ∞). The output is $L(z)$ = length of a minimum path from a to z.

```
procedure dijkstra(w, u0, v0, L)
L(u0) := 0
for all vertices x ≠ u0
L(x) := ∞
T := set of all vertices
{T is the set of all vertices whose shortest}
{distance from u0 has not been found yet}
while v0 in T
begin
choose v in T with minimum L(v)
T := T − {v}
for each x in T adjacent to v
L(x) := min{L(x), L(v) + w(v,x)}
end
return L(v0)
end dijkstra
```

For instance consider the graph in Fig.4.

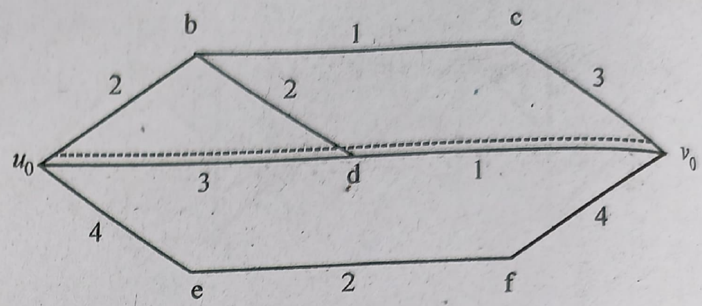


Fig. 4 : Shortest Path from u_0 to v_0

The algorithm would label the vertices in the following way in each iteration (the boxed vertices are the ones removed from T).

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Table

Iteration	u_0	b	c	d	e	f	v_0
0	0	∞	∞	∞	∞	∞	∞
1	$\boxed{0}$	2	∞	3	4	∞	∞
2	$\boxed{0}$	$\boxed{2}$	3	3	4	∞	∞
3	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	3	4	∞	6
4	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$	4	∞	4
5	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$	$\boxed{4}$	6	4
6	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$	$\boxed{4}$	6	$\boxed{4}$

At this point the algorithm returns the value 4.