1

Sets and Set Theory

OBJECTIVES

- Introduction
- Sets
- Different Types of Sets
- Set Operations
- Cardinality of A Finite Set
- Recursive Definition of A Set
- Disjoint Sets
- Comparable Sets
- Non–Comparable Set
- Venn Diagrams
- Algebra of Sets
- Partitions
- Addition Principle

1.1 Introduction

In almost all branches of mathematics every mathematical study requires a set of objects, to begin with. Therefore the basic concepts of sets and set theory are given in this chapter.

1.2 Sets

A pack of cards, a bunch of grapes, a block of pigeons are all examples of sets and, the objects are called members of the sets. So, a set is just a collection of objects. The objects constituting a set are called elements or members. An element of a set may be a card, a grape or a pigeon. A particular element has to satisfy a property to be the member of a particular set. If we say a set of vowels of English alphabet (*a*, *e*, *i*, o, *u*), then we can immediately say that the letters *p*, *q*, *r* do not belong to the set. Mathematics is full of examples of sets. A line, for instance, is a set of points; a plane is a set of lines.

Definition

A set is a collection of well–defined objects (called its elements) each of which has to satisfy a property by which it can be decided whether a given object belongs to the set or not.

Notation

We shall generally denote a set by capital letter A, B, C, D etc. while the objects of the set by lower case alphabets.

If A be any set of objects and a be any member or element of A, we write,

$a \in A$

and read it as "a belongs to A" or "a is an element of A" or "a is a member of A" or "a is contained in A" or "a is in A".

If the given object a is not in the set A, then we write $a \notin A$.

Remark 1. The words class, set, aggregate, collection, family are all used as synonymns.

Some of the set that occur very often in our study are :

(i) The set of all real numbers

(ii) The set of all rational numbers

(iii) The set of all natural numbers

(iv) The set of all integers, and

(v) The set of all complex numbers

These sets shall be denoted respectively by **R, Q, N, Z** (or **I**) and **C**.

1.3 Representation of a Set

There are two ways of representing a set :

1. In which all (or some) elements of the set are listed and enclosed in curly brackets,
2. In which elements are specified by some property P(x).

Symbolically, the set is denoted as

$$S = \{x : x \text{ satisfies P}\}$$

or

$$S = \{x : P(x)\}$$

which means that S is a set of objects x which satisfy the condition P(x).

The symbol : is read " such that"

If the set S be of industrial town of India, then we write

$$S = \{x : x \text{ is an industrial town and } x \text{ is in India}\}$$

EXAMPLES

(i) $1 \in N$ and $100 \in N$ since 1 and 100 are natural numbers; $\frac{1}{3}, -7 \notin N$ since $\frac{1}{3}$ and -7 are not natural numbers.

(ii) $S = \{1, 2, 3, 4\}$ is the set of natural numbers less than 5.

or $S = \{x : x \in N, x < 5\}$

(iii) $S_1 = \{....., -15, -10, -5, 0, 5, 10, 15, ...\}$ is the set of all integers having 5 as a factor

or $S_1 = \{x : x \in Z, x \text{ is divisible by } 5\}$

Remark 2. The symbol $\in$ indicates membership and may be translated as "belongs to", "in", "is in", "are in", " be in" according to context.

1.4 Different Types of Sets

1.4.1 Equal Sets

When two sets S and T consist of the same elements, they are called equal and we shall write S = T. To indicate that S and T are not equal, we shall write $S \neq T$.

Example 1. When S = {Ram, Shyam, Mohan} and T = {Mohan, Shyam, Ram}, then S = T.

Note that a variation in the order in which the elements of a set are tabulated is immaterial.

Example 2. When $S = \{1, 2, 3\}$ and $T = \{2, 1, 1, 2, 2, 3, 1, 3\}$, then S = T since each element of S is in T and each element of T is in S. Note that a set is not changed by repeating one or more of its elements.

Example 3. When $S = \{1, 2, 3\}$ and $T = \{0, 1, 2, 3\}$, then $S \neq T$ since 0 is an element of T but not of S.

1.4.2. Subsets

If S and T are two sets such that every element of T is an element of S, we say that T is a subset of S.

Symbolically, we write it as

$T \subseteq S$.

Thus $T \subseteq S$ iff $x \in T \Rightarrow x \in S$.

1.4.2.1 Properties of Subsets

We now mention few properties of subsets which can easily be proved by using the definition of subset.

1. Every set is a subset of itself, i.e., $S \subseteq S$ for any set S.

2. The null set is a subset of every set, i.e., $\phi \subset S$
3. If S is a subset of T and T is a subset of K, then S is a subset of K, i.e., $S \subseteq T$ and $T \subseteq K \Rightarrow S \subseteq K$
4. A finite set having n element has 2^n subsets.

Remark 3. If T is a subset of S then we also say that S is a **super set** of T.

Example 4. If S = {0, 1, 2, 3}, and T = {0, 1}, then T is a subset of S.

Example 5. The subsets of the set S = {0, 1, 2, 3} are

ϕ, {0}, {1}, {2}, {3}, {0, 1}, {0, 2}, {0, 3}, {1, 2}, {1, 3}, {2, 3}, {0, 1, 2}, {0, 2, 3}, {0, 1, 3}, {1, 2, 3}, {0, 1, 2, 3}.

1.4.2.2 Proper Subset

If T be a subset of S but $T \neq S$, then T is called a proper subset of S.

Symbolically, we write it as $T \subset S$.

Example 6. If S = {1, 2, 3} and T = {1, 2}, then T is a proper subset of S, because T is a subset of S and S contains the element 3 while T does not.

Example 7. If S = {2, 5, 9, 11}, then the sets

{2}, {2, 5}, {2, 5, 9}, {2, 5, 11}, {5}, {5, 9}, ... are proper subsets of S.

1.4.3 Null Set or Empty Set or Void Set

A set having no elements is called the null set or empty set. It is denoted by ϕ or an empty curly bracket = { }. There follow readily

1. ϕ is a subset of every set S
2. ϕ is a proper subset of every set $S \neq \phi$
3. $o(\phi) = 0$.

Example 8. $S = \{x : x \in R \text{ and } x^2 + 2 = 0\} = \phi$, is an empty set, as there is no real number satisfying the equation $x^2 + 2 = 0$.

Example 9. $S = \{x | x \in R \text{ and } x > 3 \text{ and also } x < 2\} = \phi$ is an empty set.

1.5 Finite Set

A set is said to be a fintie set if the number of its distinct elements is finite.

Example 10. Set $S = \{x : x \in N \text{ and } x \leq 10\}$ is a finite set.

1.6 Infinite Set

A set having infinite number of distinct elements is called the infinite set

Example 11. Set $S = \{x \mid x \in N \text{ and } x \geq 10\}$ is an infinite set.

1.7 Cardinality of A Finite Set

The number of elements in a finite set S is called its cardinal number and is denoted by $n(S)$ or $|S|$.

Example 12. Let S = {1, 2, 3, 4, 5, 6},

$\therefore$ $n(S) = |S| = 6$, as it has six elements. So the cardinal number is 6.

1.8 Set of Sets

A set itself may sometimes be an element of another set. Then the latter set is called the set of sets.

Example 13. Set of all lines in a plane since the line itself is nothing but a set of points.

1.9 Singleton Set or Singlet

A set having only one element is known as a singleton set or simply singlet.

Example 14. $\{\phi\}$ is a set whose only element is a null set, therefore $\{\phi\}$ is a singleton set or singlet.

Example 15. A set of positive integers between 5 and 7 will be a singleton set, consisting of one element 6 i.e., {6}.

1.10 Universal Set

If all the sets under consideration are subsets of a fixed set, then this fixed set is called universal set and is denoted by U.

Example 16. If U = {1, 2, 3, 4, 5, 6, 7, 8, ...}, S = {1, 3, 5, 7}, T = {2, 4, 6, 8, ...}, then U is the Universal set for A and B.

Example 17. The set of natural numbers is the universal set for the set of positive odd numbers and the set of positive even numbers.

1.11 Complement of a Set

If U be the universal set then complement of a set S (denoted by S' or S^c) is the set defined as

$U - S = S' = S^c = \{x : x \in U \text{ but } x \notin S\}$

1.11.1 Properties of Complementary Operation

Following are some of the important properties of the complementry operation :

(i) $U' = \phi$

(ii) $\phi' = U$

(iii) $A \cup A' = U$

(iv) $(A')' = A$

(v) $A \cap A' = \phi$

(vi) $(A \cup B)' = A' \cap B'$

(vii) $(A \cap B)' = A' \cup B'$ **[Raj. 2004]**

Properties (vi) and (vii) are called **De Morgan's Laws.** To show the method of proof, we prove (vii) only. The others can be similarly proved.

In order to prove (vii), we have to prove

(1) $(A \cap B)' \subseteq A' \cup B'$ and

(2) $A' \cup B' \subseteq (A \cap B)'$

Proof of (1) $x \in (A \cap B)' \Rightarrow x \notin A \cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

which proves (1)

Proof of (2) $x \in A' \cup B' \Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \notin A$ or $A \notin B$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \in (A \cap B)'$

which proves (2)

Thus, the proof of (vii) is complete.

Remark 4. We easily find that

$U^c = \phi$ and $\phi^c = U$

1.12 Set S–T (Difference of Two Sets)

For two sets S and T, set $S - T$ is defined as

$S - T = \{x : x \in S \text{ but } x \notin T\}$.

Similarly, $T - S = \{x : x \in T \text{ but } x \notin S\}$.

Example 18. If $S = \{1, 2, 3, 4, 5, 6\}$

and $T = \{5, 6, 7, 8, 9, 10\}$

then $S - T = \{1, 2, 3, 4\}$ and $T - S = \{7, 8, 9, 10\}$.

1.13 Symmetric Difference of Two Sets

If S and T are two sets, then the set $(S - T) \cup (T - S)$ is known as the symmetric difference of S and T and is denoted by $S \oplus T$, i.e.,

$S \oplus T = (S - T) \cup (T - S) = (S \cup T) - (S \cap T)$.

Thus, the symmetric difference of two sets S and T is the set of all elements of S and T which are not common to both S and T.

Symbolically, $S \oplus T = \{x : x \in S \text{ and } x \notin T \text{ or } x \in T \text{ and } x \notin S\}$

Example 19. If S = {1, 2, 3, 4} and T = {4, 5, 6, 7}, then

$$S \oplus T = \{1, 2, 3, 5, 6, 7\}.$$

Remark 5. In example 17, obviously, $S \oplus T = T \oplus S$.

1.14 Power Set

The set of all subsets of a given set S is called the power set of S and is denoted by P(S)

i.e., $P(S) = \{T : T \subseteq S\}$

ϕ and S are both members of P(S).

Example 20. Let $S = \{a, b\}$ then $P(S) = \{\{a, b\}, \{a\}, \{b\}, \phi\}$.

Theorem 1. If S be a finite set of order n, then P(S) is a finite set of order 2^n.

Proof : Consider the subsets of S which contain exactly r of the n elements of S. There are n_{C_r} such subsets (because out of n objects, r objects can be selected in n_{C_r} ways, if the order is immaterial). Furthermore, r can take the values 0, 1, 2, 3, 4, ..., n.

Therefore, the total number of subsets of S is

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots.. + {}^nC_n \qquad \ldots..(1)$$

By Binomial Theorem,

$$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 + \ldots.. + {}^nC_n\, x^0 y^n$$

On taking x = y = 1, we get

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots.. + {}^nC_n$$

Thus, the total number of subsets of S is

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots.. + {}^nC_n = 2^n$$

Since the elements of P(S) are subsets of S, therefore, the order of P(S) is 2^n.

1.15 Recursive Definition of A Set

In some cases it is not possible to define an object in explicit manner. Such an object can be defined in terms of itself. This process is called **recursion**. We can define sequences, functions and sets with the help of recursion.

Recursive definitions can be used to define sets. A recursive definition of a set S consists of three clauses:

1. **Basis Clause** explicitly lists atleast one element in S.
2. **Recursive Clause** provides rules used to generate new elements of the set S from the known elements.
3. **Terminal Clause** ensures that the first two clauses are the only means to determine the elements of S.

   Sets described in this manner are well defined.

Example 21. Let X be defined recursively by

$3 \in X$;

$a + b \in X$ if $a \in X$ and $b \in X$.

Show that X is the set of positive integers divisible by 3.

Solution. Let S be the set of all positive integers divisible by 3. Now, to prove that S = X, it is sufficient to show that $S \subseteq X$ and $X \subseteq S$.

First, we will show that $S \subseteq X$ by mathematical induction.

Let P(n): $3n$ belongs to X, $n \in N$.

Basis Step: P(1) is the statement that $3 \times 1 = 3$ belongs to X, which is true.

Inductive Step: Let P(k) be true, i.e., $3k$ belongs to X, $k \in N$. Now we have to show that P(k+1) is also true. Since $3k \in X$ and $3 \in X$ so by second part of the recursive definition of X, it follows that $3k + 3 = 3(k+1) \in X$, which is P(k+1). Thus by mathematical induction P(n) is true i.e., all positive integers divisible by 3 belong to X.

Hence $S \subseteq X$ .....(1)

Next, we will show that $X \subseteq S$.

It is given that $3 \in X$. Since $3 = 3 \times 1$, each element specified to be in X in this step is divisible by 3. To complete the proof, we have to show that all integers in X generated using the second part of the recursive definition are in S. For this we have to show that whenever a and b are in X, also assumed to be in S, then $a+b \in S$. Now if a and b are in S, then a and b must be divisible by 3 and then $(a+b)$ is also divisible by 3.

Hence $X \subseteq S$ .....(2)

(1) and (2) $\Rightarrow$ S = X

1.16 Disjoint Sets

Two sets S and T are said to be disjoint when they have no element in common, i.e., when no element of S is in T and no element of T is in S.

Example 22. Let S be the set of positive numbers and T be the set of negative numbers. Then S and T are disjoint, because no number is both positive and negative.

Example 23. Let S be the set $\{a, b, c\}$ and T be the set $\{e, f, g\}$. Then S and T are disjoint.

1.17 Comparable Sets

Two sets S and T are said to be comparable if $S \subset T$ or $T \subset S$, i.e., if one of the sets is a subset of the other.

Example 24. Let S = {1, 2, 3} and T = {1, 2, 3, 4}. Then S is comparable to T, because $S \subset T$.

1.18 Non–Comparable Set

Two sets S and T are said to be non–comparable if $S \not\subset T$ and $T \not\subset S$, i.e., none of the set is subset of each other.

Example 25. If S = $\{a, b\}$ and T = $\{b, d, e\}$, then S and T are not comparable, because $a \in S$ and $a \notin T$ and $d \in T$ and $d \notin S$.

1.19 Important Sets of Numbers

Some important special sets of numbers are denoted by **fixed letters.** These sets are the following :

1. N, the set of natural numbers

N = {1, 2, 3, 4, 5, ...} is called the set of natural numbers. Its elements are called natural numbers and are used for counting. The number of elements in the set N is infinite. The set N is identical with the set Z^+ of positive integers.

2. W, the set of whole numbers

The set W = {0, 1, 2, 3, 4, 5, ...} is called the set of whole numbers. It includes zero and all the elements of N.

3. Z or I, the set of integers

The set Z (or I) = {, ..., –3, –2, –1, 0, 1, 2, 3, ...,} is called the set of integers. It includes zero and all positive and negative integers.

4. Q, the set of rational numbers

The set Q is the set of elements of the form $\frac{p}{q}$ called rational numbers, where p and q are integers but q is not zero.

Thus, $$Q = \left\{\frac{p}{q}\middle| p,q \in Z \text{ but } q \neq 0\right\}$$

This set is countably infinite. The method of their ordering is not explained here as it is beyond the scope of this book.

5. R, the set of real numbers and number line

The line extending from $-\infty$ to $+\infty$, having origin at its middle point is called the number line. After selecting a scale, every point on this line corresponds to a number. Points on the line which corresponds to rational numbers are called rational points. There are also points on this line which corresponds to numbers like $\sqrt{2}, \sqrt{3}$, etc.. As these numbers can not be expressed as terminating decimals, they are called **irrational numbers. Both rational and irrational numbers taken together are called real number.** The number line thus corresponds to the set of real numbers R. The set R is uncountably infinite.

6. C, the set of complex numbers

The set C of complex numbers is defined as

$$C = \{x + i\,y : x, y \in R\} \text{ with } i = \sqrt{-1}$$

and i is called an imaginary quantity. x and y are called the real and the imaginary parts respectively, of the complex number $z = x + iy$.

Example 26. Are the sets ϕ, {0} and {ϕ} different ?

Solution :

(i) ϕ is the empty set and so ϕ is a set consisting of no element.

(ii) {0} is a set containing only one element 0.

(iii) {ϕ} is a family of sets contain only the element ϕ.

Example 27. Which of the sets are equal ?

(i) $\{x : x$ is a letter in the word "follow"$\}$.

(ii) The letters f, l, o, w

(iii) The letters which appear in the word "Wolf".

(iv) $\{x :$ is a letter in the word flow$\}$.

Solution : Firstly we shall write tabular forms of every set. We know that order and repetition of elements do not alter the nature of a set.

The set $(i) = \{f, o, l, l, o, w\}$,

$= \{f, o, l, w\}$,

$= \{f, l, o, w\}$,

The set(ii) $= \{f, l, o, w\}$,

The set (iii)$= \{w, o, l, f\}$,

$= \{f, l, o, w\}$,

The set (iv)$= \{f, l, o, w\}$.

From what has been done, it follows that all the given sets are equal.

Example 28. Prove that $\phi \subset A$ for every set A.

Proof. Since ϕ does not contain any element and therefore the condition,

$$\text{any } x \in \phi \Rightarrow x \in A$$

is fulfilled and hence the result follows.

Example 29. Prove $A \subset \phi \Rightarrow A = \phi$.

Proof. Let $A \subset \phi$.

To prove that $A = \phi$

Since ϕ is a subset of every set and hence in particular ϕ is a subset of A, i.e., $\phi \subset A$.

Now, $\phi \subset A$ and $A \subset \phi \Rightarrow A = \phi$.

Example 30. Is a set A comparable with itself.

Solution : Since $A \subset A$ is true for every set A. By definition, A is comparable with itself.

Example 31. Find the power set of $\{\{a, b\}, c\}$

Solution : Let $A = \{\{a, b\}, c\}$.

To determine P(A).

Since A contains two elements $\{a, b\}$, c.

Hence P(A) will have $2^2 = 4$ elements. The elements of P(A) are

ϕ, A, $\{a, b\}$, $\{c\}$.

Example 32. Prove that

$A \subset B, B \subset C \Rightarrow A \subset C$

Solution : Let $A \subset B$, $B \subset C$

$\therefore$ any $x \in A \Rightarrow x \in B$ for $A \subset B$

$\Rightarrow x \in C$ for $B \subset C$

Finally, any $x \in A \Rightarrow x \in C$. Thus $A \subset C$.

1.20 Venn Diagrams

Swiss mathematician Euler, first of all gave an idea to represent a set by the points in a closed curve (usually a circle but not necessarily circle). Later on British mathematician Venn brought this idea to practice and to represent sets diagrammatically which found very useful to illustrate various concepts and properties of sets.

Pictorial representation of sets by area within circles (or discs), are called their Venn diagrams. Conventionally, the universal set (the set, of which all the sets under consideration are subsets) is generally represented by the interior of a rectangular region.

Example 33. If U = $\{x : x \in N, x \leq 10\}$, A = {1, 3, 5, 7}, B = {3, 5}; C = {6, 7} and D = {8, 9, 10}. Then these sets can be represented by Venn diagrams as shown in the adjoining Figure 1.

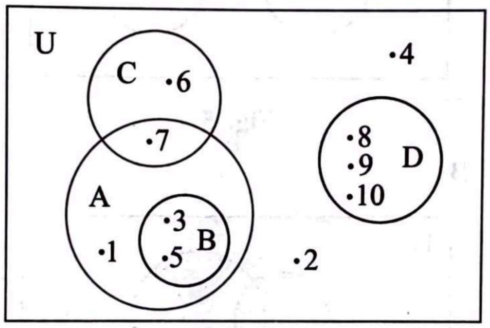


Fig. 1

Example 34. Represent by Venn diagram, $A \subset B$ and $A \neq B$.

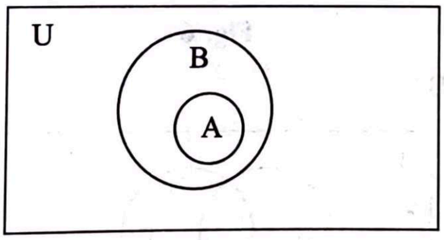


Fig. 2

Example 35. Represent the following by Venn diagram

(i) $A \subseteq B$, when A = {1, 2, 3} and B = {1, 2, 3, 4}.

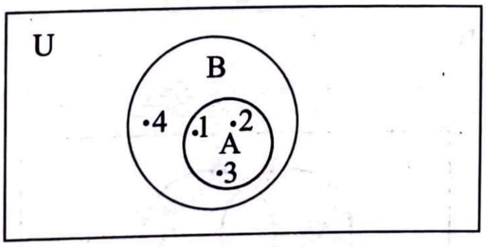


Fig. 3

(ii) $A \cap B \neq \phi$, where A = $\{b, c, d\}$ and B = $\{d, e\}$.

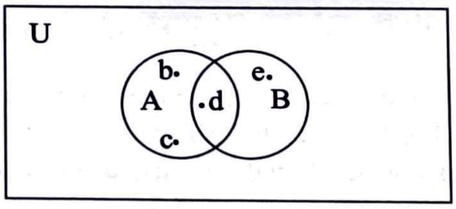


Fig. 4

(iii) $A \cap B = \phi$, where $A = \{a, b, c\}$ and $B = \{d, e\}$.

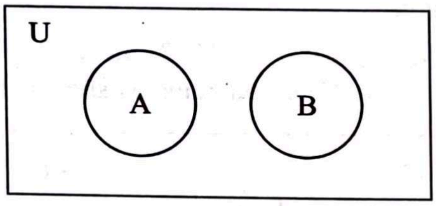


Fig. 5

(iv) $A \cup B$ for the sets A and B

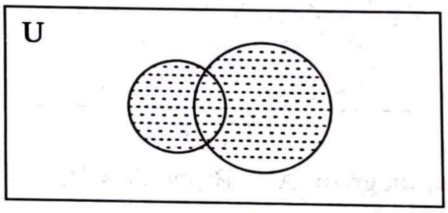


Fig. 6

(v) $A \cap B$ for the sets A and B

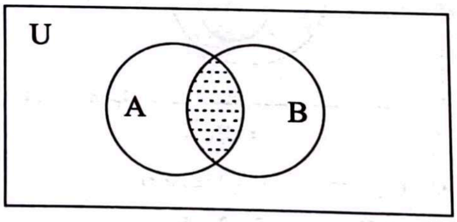


Fig. 7

(vi) $A - B$ for the sets A and B

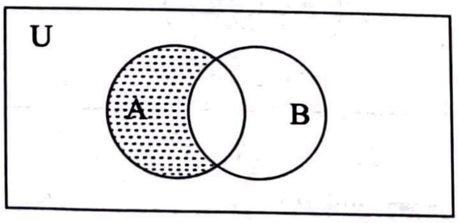


Fig. 8

(vii) Symmetric difference $= A \oplus B$ or $A \Delta B = (A - B) \cup (B - A)$ for the sets A and B

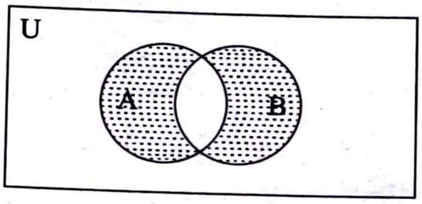


Fig. 9

(viii) U – A = A' or A^c for the set A

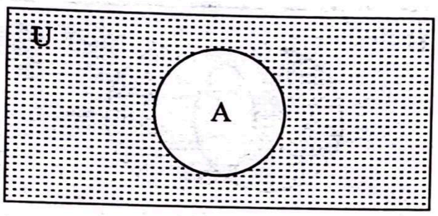


Fig. 10

(ix) A ∪ B ∪ C for the sets A, B and C

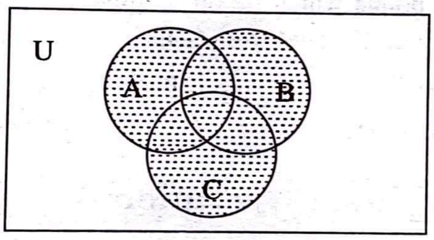


Fig. 11

1.21 Algebra of Sets

While studying real numbers, we associate with any two given real numbers, a and b say, other real numbers like $a + b$, $a - b$, $\frac{a}{b}$, ab. Similarly, we could like to associate some sets with any two given sets. In this section we will study different ways in which we can associate one set with any two given sets. In what follows all sets that we talk about are subsets of some specified universal set, U say.

1.21.1 Union Operation

The union of two sets A and B, denoted by A ∪ B, is defined as the set of those elements which either belong to A or to B.

Symbolically,

$A \cup B = \{x : x \in A \text{ or } x \in B\}$,

A ∪ B is usually read as 'A union B'.

Remark 6. The following words have the same meaning : Sum, Logical Sum, Join, Union.

The union of a finite number of sets, $A_1, A_2, \ldots\ldots, A_n$ is denoted by

$A_1 \cup A_2 \cup \ldots\ldots \cup A_n$

or by $\bigcup_{r=1}^{n} A_r$.

Similarly, $A_1 \cup A_2 \cup \ldots\ldots \cup A_n = \bigcup_{r=1}^{n} A_r$

Also $\bigcup_{r=1}^{n} A_r = \{x : x \in A_r \text{ for some value of } r, 1 \leq r \leq n\}$.

In the diagram the shaded portion represents $A \cup B$.

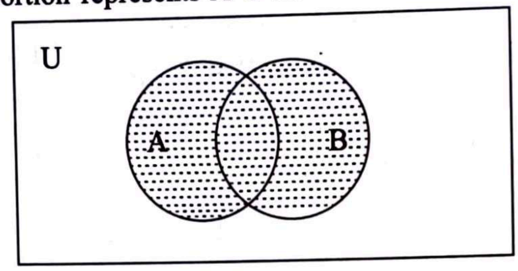


Fig. 12

Example 36. If P is the set of points which lie on the line AB and Q that of the line CD, then $P \cup Q$ is the set of these points which lie either on AB or CD.

Example 37. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$

then $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

1.21.2 Properties of Union Operation

Some of the fundamental properties that the union operation defined in § 1.22.1 obeys are :

(i) $A \cup A = A$

(ii) $A \cup \phi = A$

(iii) $A \cup U = U$

(iv) $A \cup B = B \cup A$

(v) $(A \cup B) \cup C = A \cup (B \cup C)$

The above properties are true for any subsets A, B and C of the universal set U. The proofs are direct consequence of the definition of union and equality of sets. To indicate the method of proof, we prove below property (v).

Observe that proving property (v) (in fact any one of the five properties) requires us to prove the equality of two sets, which in the case of property (v) are

$(A \cup B) \cup C$ and $A \cup (B \cup C)$

And to prove the equality of $(A \cup B) \cup C$ and $A \cup (B \cup C)$, we have simply to prove two inclusion relations, viz.

(1) $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

and (2) $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

Proof of (1)

$x \in (A \cup B) \cup C \Rightarrow [x \in (A \cup B)]$ or $[x \in C]$

$\Rightarrow$ $[x \in A$ or $x \in B]$ or $[x \in C]$

$\Rightarrow$ $[x \in A]$ or $[x \in B]$ or $[x \in C]$

$\Rightarrow$ $[x \in A]$ or $[x \in B \cup C]$

$\Rightarrow$ $x \in A \cup (B \cup C)$.

Thus, we have proved that if x is any arbitrary element in $(A \cup B) \cup C$, then x is also in $A \cup (B \cup C)$. Hence, we have

$(A \cup B) \cup C \subseteq A \cup (B \cup C)$

Proof of (2)

$x \in A \cup (B \cup C) \Rightarrow [x \in A]$ or $[x \in B$ or $x \in C]$

$\Rightarrow [x \in A$ or $x \in B]$ or $[x \in C]$

$\Rightarrow [x \in A \cup B]$ or $[x \in C]$

$\Rightarrow x \in (A \cup B) \cup C$

This argument proves that

$A \cup (B \cup C) \subseteq (A \cup B) \cup C$

1.21.3 Intersection Operation

The intersection of two sets A and B, denoted by $A \cap B$, is defined as the set containing those elements which belong to A and B both.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \cap B$ reads "A intersection B".

The intersection of a finite number of the set $A_1, A_2, ..., A_n$ is denoted by

$A_1 \cap A_2 \cap A_3 \cap ...... \cap A_n$ or by $\bigcap_{r=1}^{n} A_r$.

By definition, $\bigcap_{r=1}^{n} A_r = \{x : x \in A_r, 1 \le r \le n\}$.

Example 38. If P is the set of points lying on a line AB and Q that of the line CD, and the lines AB and CD intersect at the points (1, 2), then $P \cap Q = \{(1, 2)\}$. In the diagram

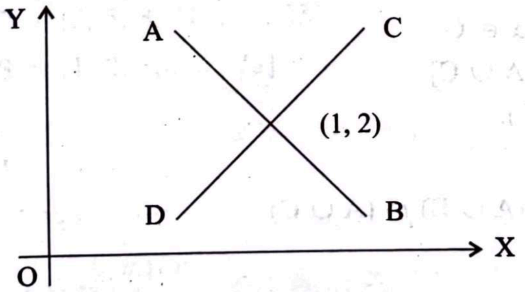


Fig. 13

Example 39. If $A = \{x : x \text{ is multiple of } 3\} = \{3, 6, 9, 12, 15, 18, ...\}$
and $B = \{x : x \text{ is multiple of } 4\} = \{4, 8, 12, 16, 20, 24, ...\}$, $A \subseteq N$, $B \subseteq N$
then $A \cap B = \{x : x \text{ is a common multiple of 3 and 4}\}$
$= \{12, 24, 36, 48, 60, ...\}$.

1.21.4 Properties of Intersection Operation

The intersection operation, as may easily be proved, satisfies the following properties.

1. $A \cap A = A$
2. $A \cap \phi = \phi$
3. $A \cap U = A$
4. $A \cap B = B \cap A$
5. $(A \cap B) \cap C = A \cap (B \cap C)$

The proofs of these properties are simple.

1.21.5 Distributive Properties

The two operations of union and intersection, introduced already, are related to each other as follows:

1. Distributive property of union over intersection

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. Distributive property of intersection over union

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

We prove below (1) only. The proof of (2) can be carried through in a similar manner. To prove (1), we once again emphasise that we are required to prove

(i) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

and (ii) $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Proof of (i)

$x \in A \cup (B \cap C) \Rightarrow [x \in A]$ or $[x \in B \cap C]$.

Let $x \in A$. Then, we have

$x \in A \Rightarrow x \in A \cup B$

and $x \in A \Rightarrow x \in A \cup C$ and

therefore,

$x \in A \Rightarrow (x \in A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

Similarly,

$x \in B \cap C \Rightarrow x \in B$ and $x \in C$

$\Rightarrow [x \in A \cup B]$ and $[x \in A \cup C]$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$.

Thus,

$x \in A \cup (B \cap C) \Rightarrow x \in (A \cup B) \cap (A \cup C)$

which proves (i).

Proof of (ii).

$x \in (A \cup B) \cap (A \cup C) \Rightarrow [x \in A \cup B]$ and $[x \in A \cup C]$

$\Rightarrow [x \in A$ or $x \in B]$ and $[x \in A$ or $x \in C]$.

Now there are four different possibilities, etc.

(a) $x \in A$ and $x \in A \Rightarrow x \in A \cup (B \cap C)$

(b) $x \in A$ and $x \in C \Rightarrow x \in A \cup (B \cap C)$

(c) $x \in B$ and $x \in A \Rightarrow x \in A \cup (B \cap C)$

(d) $x \in B$ and $x \in C \Rightarrow x \in A \cup (B \cap C)$

Thus, we have

$x \in (A \cup B) \cap (A \cup C) \Rightarrow x \in A \cup (B \cap C)$

which proves (2).

This completes the proof of the distributive property of union over intersection.

1.22 Partitions

A set {A, B, C, ...} of non–empty subsets of a set S is called the **partition** of S

if (i) $A \cup B \cup C \cup \ldots = S$,

(ii) the intersection of every pair of distinct subsets is the empty set,

where the subsets A, B, C, ... are called its members (elements) or blocks.

There is no restriction pertaining to the number of elements in every partition.

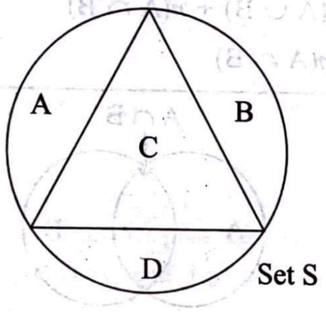


Fig. 14

Example 40. Consider the subsets

$$A = \{3, 6, 9, 12, \ldots\ldots, 24\}$$

$$B = \{1, 4, 7, 10, \ldots\ldots, 25]$$

and $$C = \{2, 5, 8, 11, \ldots\ldots, 23\}$$

of $$S = \{1, 2, 3, \ldots\ldots, 25\}$$

Obviously, $A \cup B \cup C = S$

and $A \cap B = A \cap C = B \cap C = \phi$,

so that {A, B, C} is a partition of S.

1.23 Addition Principle

Theorem 2. For two finite sets A and B which are disjoint, prove that

$$n(A \cup B) = n(A) + n(B)$$ **[Raj. 2002, RTU 2010]**

Proof. Suppose that A have m_1 elements than $n(A) = m_1$

Suppose that B have m_2 elements then $n(B) = m_2$

Since A and B are disjoint (having no element in common) therefore $A \cup B$ will have all the elements of A and all the elements of B. So number of elements in $A \cup B$ is $m_1 + m_2$.

Thus, we have
$$n(A \cup B) = m_1 + m_2$$
$$= n(A) + n(B).$$
Proved.

Theorem 3. Prove, for finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Proof. We know that

$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B$ .....(1)

and $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint

therefore, $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$ .....(2)

Further $A = (A - B) \cup (A \cap B)$

and $(A - B) \cap (A \cap B) = \phi$

So $n(A) = n(A - B) + n(A \cap B)$ .....(3)

Likewise $n(B) = n(A \cap B) + n(B - A)$ .....(4)

Hence, adding (3) and (4), we have

$$n(A) + n(B) = \{n(A - B) + n(A \cap B) + n(B - A)\} + n(A \cap B)$$
$$= n(A \cup B) + n(A \cap B)$$ [using (2)]

Thus, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

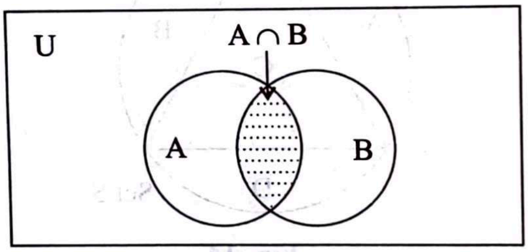


Fig. 15

Obviously in Venn diagram we see that sets A and B overlap, as shown in the Figure 15, then elements in $A \cap B$ belongs to both the sets, and the sum $|A| + |B|$ counts these elements twice.

To correct regarding this double counting, we substract $|A \cap B|$ so that we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

that is the cardinality of $A \cup B$ is derived by including the cardinality of A and B both and excluding the cardinality of $A \cap B$. Because of this, the addition principle is also called as **Principle of Inclusion and Exclusion.**

Theorem 4. If A, B and C be finite sets then

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$ **[Raj. 2000]**

Proof. Suppose that $A \cup B = X$

then $$n(X) = n(A) + n(B) - n(A \cap B)$$ .....(1)

Also $$n(A \cup B \cup C) = n(X \cup C)$$
$$= n(X) + n(C) - n(X \cap C)$$

$= n(X) + n(C) - n[(A \cup B) \cap C]$

$= n(X) + n(C) - n[(A \cap C) \cup (B \cap C)]$

$= n(X) + n(C) - n(A \cap C) + n(B \cap C) - n(A \cap C \cap B)$ .....(2)

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$

$-n(C \cap A) + n(A \cap B \cap C)$ **Hence the theorem.**

ILLUSTRATIVE EXAMPLES

Example 41. Let A, B, C be three sets, then $(A - B) \cap (A - C) = A - (B \cup C)$.

Solution : We have to prove that

(i) $(A - B) \cap (A - C) \subseteq A - (B \cup C)$

(ii) $A - (B \cup C) \subseteq (A - B) \cap (A - C)$

Let $x \in (A - B) \cap (A - C)$ then

$x \in (A - B) \cap (A - C) \Leftrightarrow x \in (A - B)$ and $x \in (A - C)$

$\Leftrightarrow x \in A, x \notin B$ and $x \in A, x \notin C$

$\Leftrightarrow x \in A, x \notin (B \cup C)$

$\Leftrightarrow x \in A - (B \cup C)$

which implies that

$(A - B) \cap (A - C) = A - (B \cup C)$.

Example 42. In a class containing 50 students, 15 play Tennis, 20 play Cricket and 20 play Hockey, 3 play Tennis and Cricket, 6 play Cricket and Hockey, and 5 play Tennis and Hockey. 7 play no game at all. How many play Cricket, Tennis and Hockey.

Solution : Let the sets of Tennis, Cricket and Hockey players be denoted by T, C and H respectively then $n(U) = 50$, $n(T) = 15$, $n(C) = 20$, $n(H) = 20$, $n(T \cap C) = 3$, $n(C \cap H) = 6$, $n(T \cap H) = 5$, $n(T \cup C \cup H)' = 7$.

We want to find $n(T \cap C \cap H)$.

Now $n(T \cup C \cup H) = n(T) + n(C) + n(H) - n(T \cap C) - n(C \cap H) - n(T \cap H) + n(T \cap C \cap H)$

$\therefore\ n(T \cup C \cup H) = 15 + 20 + 20 - 3 - 6 - 5 + n(T \cap C \cap H)$

or $n(T \cup C \cup H) = 41 + n(T \cap C \cap H)$

But $n(T \cup C \cup H)' = n(U) - n(T \cup C \cup H)$

$\therefore\ 7 = 50 - [41 + n(T \cap C \cap H)]$

or $n(T \cap C \cap H) = 2$.

Remark 7. 'n' before the set has been used to indicate the cardinal number of a set.

Example 43. In a class of 50 students, 30 are studying Hindi and 25 English language and 10 are studying both languages. How many students are studying either language ?

Solution : Let U be the class of 50 students, A is a set of those students studying Hindi and B be the set of those studying English. We have to find $n(A \cup B)$

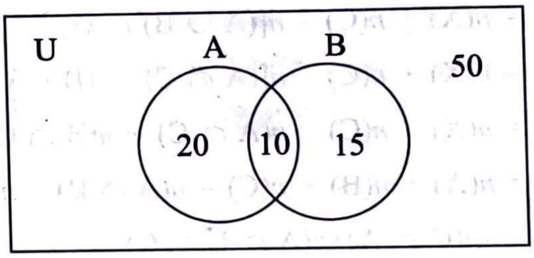


Fig. 16

Given that $n(A) = 30$, $n(B) = 15$, $n(A \cap B) = 10$

we have,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\therefore \quad n(A \cup B) = 30 + 25 - 10 = 45$$

Therefore, the number of students who studying either of languages are 45.

Example 44. A computer company must hire 25 programmers to handle systems programming jobs and 40 for the application programming. Of the hired persons, 10 will have to do the jobs of both types. Find how many programmers must be hired ? **[RTU 2010]**

Solution : Accordingly to diagram and the given conditions,

$$x_1 + x_2 = 25$$

$$x_2 + x_3 = 40$$

$$x_2 = 10$$

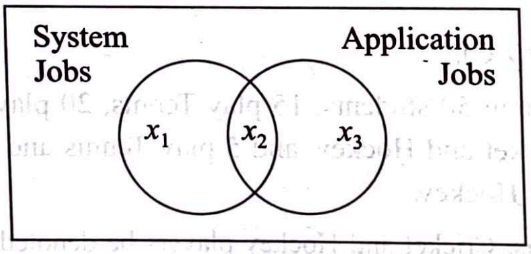


Fig. 17

So, $x_1 = 25 - 10 = 15$

$x_3 = 40 - 10 = 30$

Hence $x_1 + x_2 + x_3 = 15 + 10 + 30 = 55$

Example 45. In a survey of 60 people it was found that 25 read News week, 26 read Time and 26 read the magazine Fortune. Also 9 read both News week and Fortune, 11 read News week and Time and 8 read both Time and Fortune. If 8 read none of the three magazines, determine the number of people who read exactly one magazine. **[CE(RTU)-2007, Raj. 2001]**

Solution : Using the given data and the number of elements in various parts of the sets, we have

$V = 8$ .....(1)

$(x_1 + x_3 + x_6) + (x_2 + x_4 + x_5) + k = 60 - 8 = 52$ .....(2)

$x_1 + (x_2 + x_4) + k = 25$ .....(3)

$x_4 + (x_5 + x_6) + k = 26$ .....(4)

$x_2 + (x_3 + x_5) + k = 26$ .....(5)

$x_2 + k = 9$ .....(6)

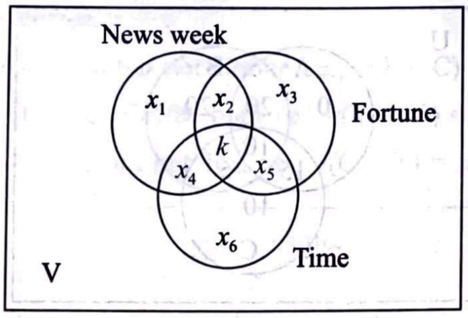


Fig. 18

$x_4 + k = 11$ .....(7)

$x_5 + k = 8$ .....(8)

$n(U) = 60$

Adding (6), (7) and (8)

$(x_2 + x_4 + x_5) + 3k = 28$ .....(9)

Adding (3), (4) and (5),

$(x_1 + x_3 + x_6) + 2(x_2 + x_4 + x_5) + 3k = 77$ .....(10)

Let $x_1 + x_3 + x_6 = p$ and $x_2 + x_4 + x_5 = q$

then (2) gives, $p + q + k = 52$ .....(11)

(9) $\Rightarrow q + 3k = 28$ .....(12)

and (10) $\Rightarrow p + 2q + 3k = 77$ .....(13)

Now (11) + (12) – (13) gives

$k = 52 + 28 - 77 = 3$ .....(14)

Hence, from (12)

$q = 28 - 3 \times 3 = 19$ .....(15)

From (11), $p = 52 - q - k$

$= 52 - 19 - 3 = 30$ .....(16)

$\therefore$ Those who read news week only $= x_1$

Those who read Fortune only $= x_3$

Those who read Time only $= x_6$

Those who read only one magazine $= x_1 + x_3 + x_6 = p = 30$ [by (16)]

Example 46. In a town 45% read magazine A, 55% read magazine B, 40% read magazine C, 30% read magazines A and B, 15% read magazines B and C, 25% read C and A, 10% read all the three magazines. Find what percentage do not read any magazine? What percentage reads exactly two of the magazines?

Solution : Let A, B and C denote the set of all those who read magazines A, B and C respectively. Then $n(A) = 45$, $n(B) = 55$, $n(C) = 40$, $n(A \cap B) = 30$, $n(B \cap C) = 15$, $n(A \cap C) = 25$, $n(A \cap B \cap C) = 10$

$n(U) = 100$ (say)

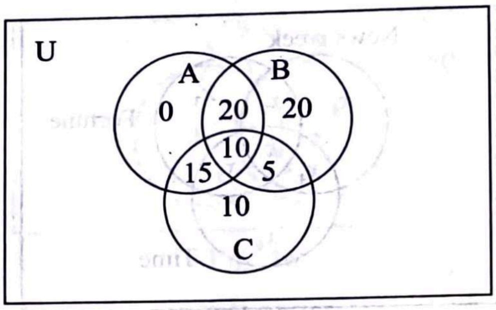


Fig. 19

The diagram given here make the understanding clear the following. The number of persons who read only A and B but not C = 30 – 10 = 20.

The number of persons who read B and C but not A = 15 – 10 = 5

The number of persons who read only C and A but not B = 25 – 10 = 15

The number of persons who read only A = 45 – (20 + 10 + 15) = 0

The number of persons who read only B = 55 – (20 + 10 + 5) = 20

The number of persons who read only C = 40 – (15 + 10 + 5) = 10

Thus the number of persons who read at least one magazine

$$= 0 + 20 + 10 + 20 + 15 + 5 + 10 = 80.$$

Hence those who do not read any magazine are 20 in number i.e., 20% do not read any magazine. The number of persons who read exactly two magazines = 20 + 5 + 15 = 40 i.e., 40% read two of the magazines.

Example 47. Let 100 of the 120 students of Mathematics at a college take at least one of the languages Hindi, English and German. Also, let 65 study Hindi, 45 study English and 42 German. If 20 study Hindi and English, 25 study English and German and 15 study Hindi and German. Find the number of students who study all the three languages. **[RTU 2011, 2009]**

Solution. Let A, B, C denote the set of students who study Hindi, English and German language, respectively. Then given that,

$|A \cup B \cup C| = 100, |A| = 65, |B| = 45, |C| = 42,$

$|A \cap B| = 20, |B \cap C| = 25, |A \cap C| = 15.$

We have to find $|A \cap B \cap C|$.

By Principle of Inclusion – Exclusion, we have

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$\Rightarrow 100 = 65 + 45 + 42 - 20 - 15 - 25 + |A \cap B \cap C|$

$\Rightarrow |A \cap B \cap C| = 160 - 152 = 8$

Hence 8 students study all the three languages.

Example 48. In a group of 52 persons 16 drink tea but not coffee and 33 drink tea. Find

(i) how many drink tea and coffee both.

(ii) how many drink coffee but not tea. **[Raj. 2005]**

Solution : Let U be the set of 52 persons. Let T denotes the set of persons who drink tea and C denotes the set of persons who drink coffee.

Let number of person who drink Tea but not Coffee i.e., $n(T - C) = x_1$

number of person who drink Tea and Coffee both i.e., $n(T \cap C) = x_2$

number of person who drink Coffee but not Tea, i.e., $n(C - T) = x_3$

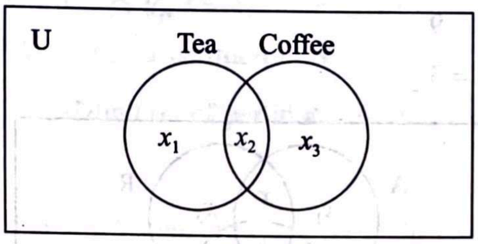


Fig. 20

then, we have

$x_1 + x_2 + x_3 = 52$

$x_1 = 16$

$x_1 + x_2 = 33$

$\therefore \; x_2 = 33 - 16 = 17$ and $x_3 = 52 - 33 = 19$

$\therefore$ (i) Those who drink tea and coffee both $= x_2 = 17$

(ii) Those who drink coffee but not tea $= x_3 = 19$

Example 49. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air conditioning (A), ratio (R) and power window (W), were already installed in a car. The survey found

15 had air conditioning (A),

12 had radio (R),

11 had power windows (W),

5 had both air conditioning (A), and power windows (W)

4 had R and W

9 had A and R and

3 had all the three, A, R and W.

Find the number of cars that had :

(i) only power windows (W),

(ii) only air conditioning (A),

(iii) only radio (R),

(iv) R and W,

(v) A and R but not W

(vi) Only one option

(vii) at least one option

(viii) none of the three

Solution. Using the data given in the problem, we have the following equations

$(x_1 + x_3 + x_6) + (x_2 + x_4 + x_5) + k + V = 25$ .....(1)

$x_1 + (x_2 + x_4) + k = 15$ .....(2)

$$x_3 + (x_2 + x_5) + k = 12 \quad .....(3)$$
$$x_6 + (x_4 + x_5) + k = 11 \quad .....(4)$$
$$x_4 + k = 5 \quad .....(5)$$
$$x_5 + k = 4 \quad .....(6)$$
$$x_2 + k = 9 \quad .....(7)$$
$$k = 3 \quad .....(8)$$

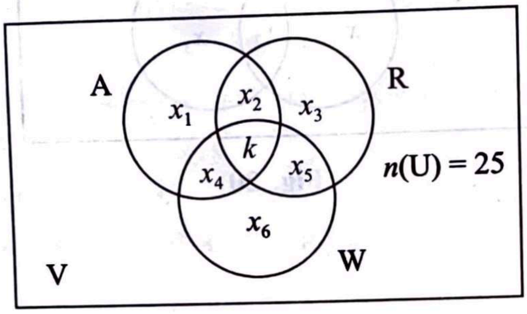


Fig. 21

Let $x_1 + x_3 + x_6 = p$, and $x_2 + x_4 + x_5 = q$. Using (8) in (5), (6) and (7), we get

$$x_4 = 5 - 3 = 2 \quad .....(9)$$
$$x_2 = 9 - 3 = 6 \quad .....(10)$$
$$x_5 = 4 - 3 = 1 \quad .....(11)$$

$(9) + (10) + (11) \Rightarrow q = x_2 + x_4 + x_5 = 9$ .....(12)

$(2) + (3) + (4) \Rightarrow$

$$p + 2q + 3k = 15 + 12 + 11 = 38$$

$\Rightarrow$ $$p = 38 - 2 \times 9 - 3 \times 3 = 11 \quad .....(13)$$

Thus, from (1),

$$p + q + k + V = 25$$

or $$11 + 9 + 3 + V = 25$$

$\therefore$ $$V = 2$$

From (2), $x_1 + 2 + 6 + 3 = 15$ .....(14)

$\therefore$ $x_1 = 4$

From (3), $x_3 + 6 + 1 + 3 = 12$, ...(15)

$\therefore$ $x_3 = 2$ .....(16)

From (4), $x_6 + 1 + 2 + 3 = 11$

$\therefore$ $x_6 = 5$ .....(17)

Answer : (i) $n(\text{only power window W}) = x_6 = 5$

(ii) $n(\text{only air conditioning A}) = x_1 = 4$

(iii) $n(\text{only radio R}) = x_3 = 2$

(iv) $n(R \cap W) = k + x_5 = 3 + 1 = 4$

(v) $n(A \cap R - W) = q + k - k = q = 6$

(vi) n(only one option) $= x_1 + x_3 + x_6 = p = 11$

(vii) n(at least one option) $= n(U) - V = 25 - 2 = 23$

(viii) n(none of the three) $= V = 2$

Example 50. Let $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_m\}$ be partition of a set X. Show that collection of sets, $P = \{A_i \cap B_j, i = 1, ..., n; j = 1, 2, ..., m\} - \phi$ (called the **cross partition** of X) is also a partition of X. **[Raj. 2002]**

Solution : Let $x \in X$, then by the definition of partition $x \in A_i$ for some i and $x \in B_j$ for some j, therefore $x \in A_i \cap B_j$. Thus $\bigcup_{i,j} A_i \cap B_j = X$.

Now we have to prove that if $A_p \cap B_q$ and $A_p' \cap B_q'$ are two distinct members of P then they must be disjoint. Let these are not disjoint then there exists at least one element y such that

$y \in A_p \cap B_q$ and $y \in A_p' \cap B_q'$

$\Rightarrow$ y $\in A_p$ and y $\in A_p' \Rightarrow A_p = A_p'$ since $A_p \cap A_p' = \phi$ for $p \neq p'$ and also $y \in B_q$ and $y \in B_q' \Rightarrow B_q = B_q'$, since $B_q \cap B_q' = \phi$, for $q \neq q'$.

Accordingly, $A_p \cap B_q = A_p' \cap B_q'$

Hence the blocks are mutually disjoint or identical.

Hence P is a partition.

Example 51. Consider the following collection of subsets {A, B, C} of a set

S = {1, 2, 3, 4, 5, 6, 7, 8, 9}

(a) [{1, 5}, {2, 6, 3}, {4, 8, 9, 3}],

(b) [{1}, {2, 4, 8}, {5, 7, 9}], and

(c) [{1, 5}, {2, 4, 6, 8}, {3, 7, 9}]

Determine which one is a partition of the set.

Solution :

(a) is not a partition as $B \cap C = \{3\} \neq \phi$

(b) is not a partition as $A \cup B \cup C = \{1, 2, 4, 5, 7, 8, 9\} \neq S$

(c) is a partition as $A \cap B = \phi = B \cap C = C \cap A$ and $A \cup B \cup C = S$.

Example 52. Determine which one is a partition of the set N of positive integers

(i) $[A_1 = \{n : n > 5\}, A_2 = \{n : n < 5\}]$

(ii) $[A_3 = \{n : n > 5\}, A_4 = \{0\}, A_5 = \{1, 2, 3, 4, 5\}]$

(iii) $[A_6 = \{n : n^2 > 11\}, A_7 = \{n : n^2 < 11\}]$

Solution : (i) $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = N - \{5\}$ so it is not a partition.

(ii) is not a partition as $A_3 \cup A_4 \cup A_5$

$= \{0, 1, 2, 3, 4, 5, ...\} \neq N \qquad (\because\ 0 \notin N)$

(iii) is a partition as $A_6 \cup A_7 = \{4, 5, 6, ...\} \cup (1, 2, 3) = N$ and $A_6 \cap A_7 = \phi$.

Example 53. Determine whether or not each of the following is a partition of the set R of real numbers.

(a) $[\{x : x \geq 4\}, \{x : x \leq 5\}]$

(b) $[\{x : x > 0\}, \{0\}, \{x : x < 0\}]$

(c) $[\{x : x^2 > 12\}, \{x : x^2 < 12\}]$

Solution : (a) No, since the two blocks are not disjoint as 4, 5 belong to both blocks.

(b) Yes, since the three blocks are mutually disjoint and their union is R.

(c) No, since $\sqrt{12}$ in R does not belongs to either blocks.

Example 54. For any sets A, B and C prove the following mathematically : **[Raj. 2002]**

(a) $(A - B) \cup (B - A) = A \cup B - A \cap B$ **[Raj. 2005, 1999, MREC 2000]**

(b) $(A - B) - C = A - (B \cup C)$ **[Raj. 2005, 2003, MREC 2000]**

(c) $(A - B) - C = (A - C) - B$ **[Raj. 2005, 2003, MREC 2000]**

(d) $(A - B) - C = (A - C) - (B - C)$ **[Raj. 2000]**

(e) $A \subseteq A \cup B$ and $A \cap B \subseteq A$

Solution :

(a) Let $x \in (A - B) \cup (B - A)$

$\Rightarrow x \in (A - B)$ or $x \in (B - A)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin B$ or $x \notin A)$

$\Rightarrow x \in A \cup B$ and $x \notin A \cap B$

$\Rightarrow x \in (A \cup B) - (A \cap B)$ .....(1)

Hence $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$

Again, Let $x \in (A \cup B) - (A \cap B)$

$\Rightarrow x \in A \cup B$ and $x \notin A \cap B$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow x \in (A - B)$ or $x \in (B - A)$

$\Rightarrow x \in (A - B) \cup (B - A)$ .....(2)

Hence $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

(1) and (2) $\Rightarrow$

$(A - B) \cup (B - A) = A \cup B - A \cap B$

(b) Let $x \in (A - B) - C$

$\Rightarrow x \in (A - B)$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow x \in A - (B \cup C)$

Hence $(A - B) - C \subseteq A - (B \cup C)$ .....(1)

Again, Let $x \in A - (B \cup C)$

$\Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow x \in (A - B)$ and $x \notin C$

$\Rightarrow x \in (A - B) - C$

Hence $A - (B \cup C) \subseteq (A - B) - C$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow$

$(A - B) - C = A - (B \cup C)$

(c) Let $x \in (A - B) - C$

$\Rightarrow x \in (A - B)$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \notin C)$ and $x \notin B$

$\Rightarrow x \in (A - C)$ and $x \notin B$

$\Rightarrow x \in (A - C) - B$

Hence $(A - B) - C \subseteq (A - C) - B$ .....(1)

Again, Let $x \in (A - C) - B$

$\Rightarrow x \in (A - C)$ and $x \notin B$

$\Rightarrow (x \in A$ and $x \notin C)$ and $x \notin B$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow x \in (A - B)$ and $x \notin C$

$\Rightarrow x \in (A - B) - C$

Hence $(A - C) - B \subseteq (A - B) - C$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow$

$(A - B) - C = (A - C) - B.$

(d) Let $x \in (A - B) - C$

$\Rightarrow x \in (A - B)$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \notin C)$ and $x \notin B$

$\Rightarrow x \in (A - C)$ and $x \notin (B - C)$

$[\because (B - C) \subseteq B]$

$\Rightarrow x \in (A - C) - (B - C)$

Hence $(A - B) - C \subseteq (A - C) - (B - C)$ .....(1)

Again, Let $x \in (A - C) - (B - C)$

$\Rightarrow x \in (A - C)$ and $x \notin (B - C)$

$\Rightarrow (x \in A$ and $x \notin C)$ and $x \notin B$

$[\therefore x \notin C, x \notin (B - C) \Rightarrow x \notin B]$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \notin C$

$\Rightarrow x \in (A - B)$ and $x \notin C$

$\Rightarrow x \in (A - B) - C$

$\Rightarrow (A - C) - (B - C) \subseteq (A - B) - C$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow$

$(A - B) - C = (A - C) - (B - C)$

(e) Let $x \in A$

This implies that x will always lie in the set which results from the union of any other set with A.

$\Rightarrow x \in A \cup B$

Hence $A \subseteq A \cup B$.

Again, let $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

This means that if x lies in $(A \cap B)$ then x will always lie in the set A

$\Rightarrow x \in A$

Thus $A \cap B \subseteq A$.

Example 55. Given that $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$. Show that $A \subseteq B$. **[Raj. 2004]**

Solution : Let $x \in A$

$\Rightarrow x \in A$ and $(x \in C$ or $x \notin C)$

$\Rightarrow (x \in A$ and $x \in C)$ or $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A \cap C$ or $x \in A \cap \bar{C}$ $(\because x \notin C \Rightarrow x \in \bar{C})$

$\Rightarrow x \in B \cap C$ or $x \in B \cap \bar{C}$ (given $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$)

$\Rightarrow (x \in B$ and $x \in C)$ or $(x \in B$ and $x \in \bar{C})$

$\Rightarrow x \in B$ and $(x \in C$ or $x \in \bar{C})$

$\Rightarrow x \in B$

$\therefore x \in A \Rightarrow x \in B$

Thus $A \subseteq B$.

Example 56. Prove the following (Do not use examples)

(i) $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

(ii) $A \oplus B = (A \cup B) - (A \cap B)$ **[Raj. 2004]**

Solution :

(i) $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

Let $x \in \overline{A \cup (B \cap C)} \Rightarrow x \notin A \cup (B \cap C)$

$\Rightarrow x \notin A$ and $x \notin (B \cap C)$

$\Rightarrow x \notin A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow x \in \bar{A}$ and $(x \in \bar{B}$ or $x \in \bar{C})$

$\Rightarrow x \in \bar{A}$ and $x \in (\bar{C} \cup \bar{B})$

$\Rightarrow x \in (\bar{C} \cup \bar{B}) \cap \bar{A}$

$\therefore \overline{A \cup (B \cap C)} \subseteq (\bar{C} \cup \bar{B}) \cap \bar{A}$ .....(1)

Again, Let $x \in (\bar{C} \cup \bar{B}) \cap \bar{A}$

$\Rightarrow x \in (\bar{C} \cup \bar{B})$ and $x \in \bar{A}$

$\Rightarrow (x \in \bar{C}$ or $x \in \bar{B})$ and $x \in \bar{A}$

$\Rightarrow (x \notin C$ or $x \notin B)$ and $x \notin A$

$\Rightarrow x \notin A$ and $x \notin (B \cap C)$

$\Rightarrow x \notin A \cup (B \cap C)$

$\Rightarrow x \in \overline{A \cup (B \cap C)}$

$\therefore (\bar{C} \cup \bar{B}) \cap \bar{A} \subseteq \overline{A \cup (B \cap C)}$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow \overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$.

(ii) $A \oplus B = (A \cup B) - (A \cap B)$

We know that $A \oplus B = (A - B) \cup (B - A)$ .....(1)

Let $x \in A \oplus B \Rightarrow x \in (A - B) \cup (B - A)$

$\Rightarrow x \in A - B$ or $x \in B - A$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow (x \in A$ or $x \in B)$ or $(x \notin A$ or $x \notin B)$

$\Rightarrow x \in A \cup B$ and $x \notin A \cap B$

$\Rightarrow x \in (A \cup B) - (A \cap B)$

$\therefore A \oplus B \subseteq (A \cup B) - (A \cap B)$ .....(2)

Again, let $x \in (A \cup B) - (A \cap B)$

$\Rightarrow x \in A \cup B$ and $x \notin A \cap B$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow x \in (A - B)$ or $x \in (B - A)$

$\Rightarrow x \in (A - B) \cup (B - A) \Rightarrow x \in A \oplus B$ [using (1)]

$\therefore (A \cup B) - (A \cap B) \subseteq A \oplus B$ .....(3)

$\therefore$ (2) and (3) $\Rightarrow A \oplus B = (A \cup B) - (A \cap B)$.

Example 57. Prove or disprove the following entities on sets :

(i) $A - (A - B) = A \cap B$ **[Raj. 2006]**

(ii) $(A - B) - (C - D) = (A - C) - (B - D)$

(iii) $|A \cap B| \leq |A \cup B|$

(iv) $A \cap B \subseteq B \subseteq A \cup B$ **[Raj. 2003, 2006]**

Solution:

(i) $A - (A - B) = A \cap B$

Let $x \in A - (A - B) \Rightarrow x \in A$ and $x \notin (A - B)$

$\Rightarrow x \in A$ and $x \in B$ $[\because x \in A, x \notin (A - B) \Rightarrow x \in B]$

$\Rightarrow x \in A \cap B$

$\therefore A - (A - B) \subseteq A \cap B$ .....(1)

Again, let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A$ and $x \notin (A - B)$

$\Rightarrow x \in A - (A - B)$

$\therefore A \cap B \subseteq A - (A - B)$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow A - (A - B) = A \cap B$.

(ii) $(A - B) - (C - D) = (A - C) - (B - D)$

Let $x \in (A - B) - (C - D)$

$\Rightarrow x \in (A - B)$ and $x \notin (C - D)$

$\Rightarrow (x \in A, x \notin B)$ and $[(x \in C, x \in D)$ or $(x \notin C, x \in D)$ or $(x \notin C, x \notin D)]$

$\Rightarrow (x \in A \text{ and } x \in C \text{ or } x \notin C) \text{ and } (x \notin B \text{ and } x \in D \text{ or } x \notin D)$

$\Rightarrow (x \in A - C \text{ or } x \notin A - C) \text{ and } (x \notin B - D)$ $[\because B - D \subseteq B]$

$\Rightarrow (x \in A - C \text{ and } x \notin B - D) \text{ or } (x \notin A - C \text{ and } x \notin B - D)$

$\Rightarrow x \in (A - C) - (B - D) \text{ or } x \notin (A - C) - (B - D)$

It shows that it is not necessary that every element of the set $(A - B) - (C - D)$ must belongs to the set $(A - C) - (B - D)$.

$\therefore \ (A - B) - (C - D) \not\subseteq (A - C) - (B - D)$

Thus the given entity is not true.

(iii) $|A \cap B| \leq |A \cup B|$

We know that if $A \subseteq B$ then $|A| \leq |B|$ .....(1)

Now, let $x \in A \cap B \Rightarrow x \in A$ and $x \in B \Rightarrow x \in A \cup B$

$\Rightarrow A \cap B \subseteq A \cup B$

$\Rightarrow |A \cap B| \leq |A \cup B|$ [using (1)]

(iv) $A \cap B \subseteq B \subseteq A \cup B$

Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$

$\therefore \ A \cap B \subseteq B$ .....(1)

Again, let $x \in B \Rightarrow x \in A \cup B$

$\therefore \ B \subseteq A \cup B$ .....(2)

Thus (1) and (2) $\Rightarrow A \cap B \subseteq B \subseteq A \cup B$

Example 58. Let A, B, C be sets. Under what conditions each of the following statements are true ?

(i) $(A - B) \cup (A - C) = A$

(ii) $(A - B) \cup (A - C) = \phi$

(iii) $(A - B) \oplus (A - C) = \phi$

Justify your answer. **[Raj. 2005]**

Solution :

(i) $(A - B) \cup (A - C) = A$

It is true if

(a) B or C is null set, or

(b) A and B are disjoint sets, or

(c) A and C are disjoint sets.

Since, if (a) is true then $B = \phi$ or $C = \phi \Rightarrow A - B = A$ or $A - C = A$ and then $(A - B) \cup (A - C) = A$.

If (b) is true then $A - B = A \Rightarrow (A - B) \cup (A - C) = A \cup (A - C) = A$

If (c) is true then $A - C = A \Rightarrow (A - B) \cup (A - C) = (A - B) \cup A = A$.

(ii) $(A - B) \cup (A - C) = \phi$

It is true if

(a) $A = B = C$ or (b) A is a null set i.e. $A = \phi$

As, if (a) is true i.e., $A = B = C$ then $A - B = \phi$ and $A - C = \phi$ so it follows that $(A - B) \cup (A - C) = \phi \cup \phi = \phi$.

If (b) is true i.e., $A = \phi$ then $A - B = \phi - B = \phi$ and $A - C = \phi - C = \phi$ so it follows that $(A - B) \cup (A - C) = \phi \cup \phi = \phi$.

(iii) $(A - B) \oplus (A - C) = \phi$

It is true if

(a) $A \subseteq B$ and $A \subseteq C$ or (b) $A \not\subseteq B$ and $A \not\subseteq C$ but $B = C$.

As, if (a) is true i.e., $A \subseteq B$ and $A \subseteq C \Rightarrow A - B = \phi$ and $A - C = \phi$ so it follows that $(A - B) \oplus (A - C) = \phi \oplus \phi = (\phi \cup \phi) - (\phi \cap \phi) = \phi - \phi = \phi$.

If (b) is true i.e., $A \not\subseteq B$,. $A \not\subseteq C$ but $B = C$ then it follows that

$$(A - B) \oplus (A - C) = (A - B) \oplus (A - B)$$
$$= [(A - B) \cup (A - B)] - [(A - B) \cap (A - B)]$$
$$= (A - B) - (A - B) = \phi$$

Example 59. Show that the necessary and sufficient condition for a set Y to be a subset of X is that $X \cup Y = X$.

Solution. Condition is necessary

Let Y be a subset of X i.e., $Y \subseteq X$.

Let $x \in X \cup Y \Rightarrow x \in X$ or $x \in Y$

$\Rightarrow x \in X$

$(\because Y \subseteq X)$

$\therefore X \cup Y \subseteq X$ .....(1)

Again, let $x \in X \Rightarrow x \in X \cup Y$

$\therefore X \subseteq X \cup Y$ .....(2)

$\therefore$ (1) and (2) $\Rightarrow X \cup Y = X$

So the condition is necessary.

Condition is sufficient

Let $X \cup Y = X$

Now, let $x \in Y \Rightarrow x \in X \cup Y$

$\Rightarrow x \in X$ $(\because X \cup Y = X)$

$\therefore Y \subseteq X$

Thus condition is sufficient.

Example 60. Let $A = \{a, b, c, d, e\}$ and $B = \{c, e, f, h, k, m\}$ then prove if A and B are finite sets then $|A \cup B| = |A| + |B| - |A \cap B|$. **[RTU 2011, 2009]**

Solution. Given $A = \{a, b, c, d, e\}$ and $B = \{c, e, f, h, k, m\}$

Then, $A \cup B = \{a, b, c, d, e, f, h, k, m\}$ and $A \cap B = \{c, e\}$

$\therefore$ $|A| = 5$, $|B| = 6$, $|A \cup B| = 9$, $|A \cap B| = 2$

L.H.S. $= |A \cup B| = 9$

R.H.S. $= |A| + |B| - |A \cap B| = 5 + 6 - 2 = 9$

$\therefore$ L.H.S. = R.H.S.

Hence $|A \cup B| = |A| + |B| - |A \cap B|$.

Example 61. A survey is taken on method of commuter travel. Each respondent is asked to check BUS, TRAIN or AUTOMOBILE as a major method of travelling to work. More than one answer is permitted. The results reported were as follows :

(i) 30 people checked BUS;

(ii) 35 people checked TRAIN;

(iii) 100 people checked AUTOMOBILE;

(iv) 15 people checked BUS and TRAIN;

(v) 15 people checked BUS and AUTOMOBILE;

(vi) 20 people checked TRAIN and AUTOMOBILE;

(vii) 5 people checked all three methods.

How many respondents completed their surveys ? **[Raj. 2001]**

Solution. Let B denotes the set of respondents who check BUS.

Let T denotes the set of respondents who check TRAIN.

Let A denotes the set of respondents who check AUTOMOBILE.

Given :

$|B| = 30, |T| = 35, |A| = 100$

$|B \cap T| = 15, |B \cap A| = 15, |T \cap A| = 20$

$|B \cap T \cap A| = 5.$

To find – $|B \cup T \cup A|$

We know that :

$$|B \cup T \cup A| = |B| + |T| + |A| - |B \cap T| - |B \cap A| - |T \cap A| + |B \cap T \cap A|$$

$$= 30 + 35 + 100 - 15 - 15 - 20 + 5 = 170 - 50 = 120$$

Hence 120 respondents completed their surveys.

EXERCISE 1

Q.1 Given $n(U) = 692$, $n(A) = 300$, $n(B) = 230$, $n(C) = 370$, $n(A \cap B) = 150$, $n(A \cap C) = 180$, $n(B \cap C) = 90$, $n(A \cap B' \cap C') = 10$, where $n(S)$ is the number of distinct elements in the set S, Find :

(a) $n(A \cap B \cap C)$

(b) $n(A' \cap B \cap C')$

(c) $n(A' \cap B' \cap C')$

(d) $n((A \cap B) \cup (A \cap C) \cup (B \cap C))$ **[Raj. 2007]**

Ans. (a) 40, (b) 30, (c) 172, (d) 340

Q.2 For any sets A and B, prove

(i) $(A - B) \cap B = \phi$

(ii) $A - B = A \cap B' = B' - A'$

(iii) $(A \cap B) \cup (A \cap B') = A$

(iv) $A \cup (A \cap B) = A$

(v) $A \cap (A \cup B)' = \phi$

Q.3 In a group athletic teams in a school, 21 are in the basket ball team, 26 in hockey, 28 in football team. If 14 play hockey and basket ball, 12 play foot ball and basket ball; 15 play hockey and football and 8 play all the three games. Find :

(a) how many players are there in all;

(b) how many play football only ?

Ans. (a) 42 (b) 9

Q.4 Participation in sports is compulsory in a school. In a class of 80 student, 60 play football and 40 play basket ball. Find :

(a) how many play both the games. (b) play foot ball only.

Ans. (a) 20 (b) 40

Q.5 A survey shows 74% of Indians like apples and 68% like oranges. What percentage like both apples and oranges ?

Ans. 42%

Q.6 In an examination candidates had the option to offer English or Hindi or both English and Hindi. Out of 1000 candidates who appeared in the examination 650 appeared in English and 200 appeared in both Hindi and English. Find :

(a) The number of candidates who offered Hindi,

(b) Number of candidates who offered English only,

(c) Number of candidates who offered Hindi only.

Ans. (a) 550 (b) 450 (c) 350

Q.7 Write the dual of each of the set equations

(a) $A = (B' \cap A) \cap (C' \cup A) \cap (A \cup B \cup C)$

(b) $(A \cap U) \cap (\phi \cup A') = \phi$

Ans. (a) $A = (B' \cup A) \cup (C' \cap A) \cup (A \cap B \cap C)$

(b) $(A \cup \phi) \cup (U \cap A') = U$

Q.8 Let A = [1, 2, 3, ..., 8, 9]. Find the cross partition P of the following partitions of A :

$P_1 = \{\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}\}$ and $P_2 = \{\{1, 2, 3, 4\}, \{5, 7\}, \{6, 8, 9\}\}$

Ans. $P = [\{\{1, 3\}, \{5, 7\}, \{9\}, \{2, 4\}, \{6, 8\}\}]$.

Q.9 Let $A = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not each of the following is a partition of A.

(a) $P_1 = [\{1, 2, 3\}, \{2, 3, 4, 5, 6\}]$ (b) $P_2 = [\{1, 2\}, \{3, 5, 6\}]$

(c) $P_3 = [\{1, 3, 5\}, \{2, 4\}, \{6\}]$ (d) $P_4 = [\{1, 3, 5\}, \{2, 4, 6, 7\}]$

Ans. (a) No (b) No (c) Yes (d) No

Q.10 Prove the following

(a) $A \cup \phi = A$ (b) $A \cup A = A$ (c) $A \cap A = A$ (d) $A \cap \phi = \phi$

Q.11 Given that $P \cup Q = P \cup R$, is it necessary that $Q = R$? Justify your answer.

Ans. No

Q.12 Given that $P \cap Q = P \cap R$, is it necessary that $Q = R$? Justify your answer.

Ans. No

Q.13 Given That $P \oplus Q = P \oplus R$, is it necessary that $Q = R$? Justify your answer.

Ans. Yes

Q.14 What can you say about P and Q if

(a) $P \cap Q = P$ (b) $P \cup Q = P$

(c) $P \oplus Q = P$ (d) $P \cap Q = P \cup Q$

Ans. (a) $P \subseteq Q$ (b) $Q = P$ or $Q \subset P$ (c) $Q = \phi$ (d) $P = Q$

Q.15 Let A and B be sets and Let A^c and B^c denote the complements of the sets A and B, the set $(a - b) \cup (b - a) \cup (a \cap b)$ is equal to. **[GATE 1996]**

(a) $A \cup B$ (b) $A^c \cup B^c$

(c) $A \cap B$ (d) $A^c \cap B^c$

Q.16 The number of elements in the power set P(S) of the set $S=\{(\phi),1,(2,3)\}$ **[GATE 1995]**

(a) 2 (b) 4

(c) 8 (d) None of the above

Q.17 Let S be an infinite set and $S_1, \ldots, S_n$ be sets such that $S_1 \cup S_2 \cup \ldots \cup S_n = S$. Then, **[GATE 1993]**

(a) at least one of the set S_i is a finite set (b) not more than one of the sets S_i can be finite

(c) at least one of the set S_i is an infinite set (d) not more than one of the sets S_i can be infinite

(e) None of the above

Q.18 Let E,F and G be finite sets $X = (E \cap F) - (F \cap G)$ and Y $(E-(E \cap G))-(E-F)$. Which one of the following is true? **[GATE 2006]**

(a) $X-Y = Q$ and $Y-X \neq Q$ (b) $X \supset Y$

(c) $X = Y$ (d) $X \subset Y$

Q.19 Let A, B and C are non empty sets and let

$X = (A-B) - C$ and $Y = (A-C)-(B-C)$

Which one is true? **[GATE 2005]**

(a) $X = Y$ (b) $Y \subset X$

(c) $X \subset Y$ (d) None of these

Q.20 In a room containing 28 people, there are 18 people who speak English, 15 people who speak Hindi and 22 people who speak Kannada. 9 persons speak both English and Hindi, 11 persons speak both Hindi and Kannada where as 13 persons speak both Kannada and English. How many people speak all three languages? **[GATE 1998]**

(a) 9 (b) 8

(c) 7 (d) 6

Q.21 25 persons are in a room. 15 of them play hockey, 17 of them play football and 10 of them play both hockey and football. Then the number of persons playing neither hockey nor football is: **[GATE 2010]**

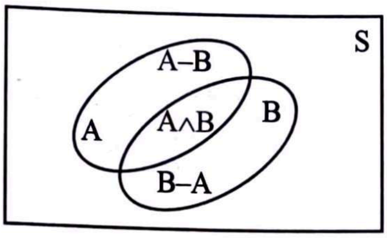
(a) 2 (b) 17

(c) 13 (d) 3

ANSWER KEY

15.	c	16.	b	17.	c	18.	c	19.	a	20.	d	21.	d

Sol.15 It is $A \cup B$
(from the figure)

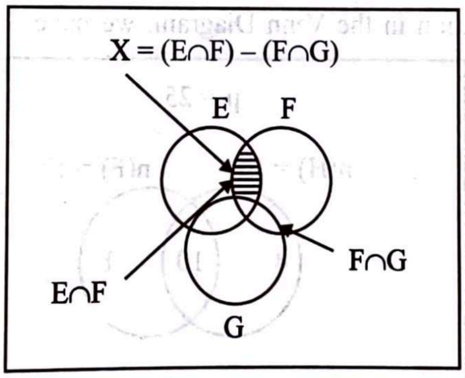


(Note: $(A - B) \cup (B - A) = A \Delta B$ is called symmetric difference of A and B.
We have $A \cup B = (A \Delta B) \cup (A \wedge B)$.

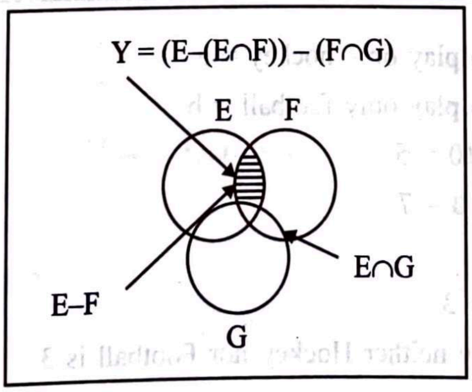
Sol.16 S has 3 elements.
$\Rightarrow$ P(S) has $2^3 = 8$ elements.
Note that $P(S) = \{\phi, \{\phi\}, \{1\}, \{(2, 3)\}, \{\{\phi\}, 1\}, \{\{\phi\}, (2, 3)\}, \{1, (2, 3)\}, \{\{\phi\}, 1, (2, 3)\}\}$.

Sol.17 Note that if one or more of the sets S_i be infinite, the
$\cup S_i = S$ will be infinite.
$\cup S_i = S$ will be finite iff all S_i be finite.
Hence it follows that least one of the sets S_i be infinite.

Sol.18



$Y = (E - (E\cap G)) - (E - F)$



So $X = Y$

$X = (E\cap F) - (F\cap G)$

Sol.19 $X = (A-B) - C$

$= (A\cap B') - C$

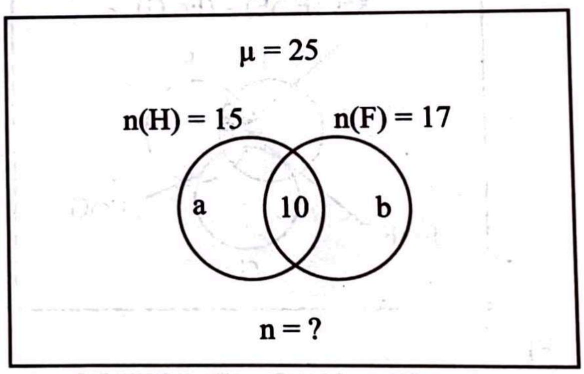
$= (A \cap B') \cap C'$
$= AB'C'$
$Y = (A–C) – (B–C)$
$= (A \cap C') – (B \cap C')$
$= (AC') – (BC')'$
$= (AC') \cap (B' + C)$
$= (AC') \cdot (B' + C)$
$= AC'B' + AC'C$
$= AC'\ B' = AB'C'$ (Since $C'C = 0$)
$X = Y$ (Commutative property)

Sol.20 By principle of inclusion and exclusion,

$n(E \cup H \cup K) – \Sigma b(E) – \Sigma n(E \cap H) + \Sigma n(E \cap H \cap K)$
$= n(E) + n(H) + n(K) – (n(E \cap H) + n(H \cap K) + n(K \cap E) + n(E \cap H \cap K)$
$\Rightarrow \quad 28 = 18 + 15 + 22 – (9 + 11 + 13) + n(E \cap H \cap K)$
$\Rightarrow \quad 28 = 55 – 33 + n(E \cap H \cap K)$
$\Rightarrow \quad n(E \cap H \cap K) = 6$

Sol.21 (d)

Representing the given information in the Venn Diagram, we have



Let, the number of people who play only hockey = a
The number of people who play only football = b
Now, $a = n(H) – 10 = 15 – 10 = 5$
$b = n(F) – 10 = 17 – 10 = 7$
Clearly, $a + b + 10 + n = 25$
$\Rightarrow n = 25 – 7 – 5 – 10 \Rightarrow n = 3$
The number of people who play neither Hockey nor Football is 3