CHAPTER TWO

The Solution of Nonlinear Equations $f(x) = 0$

Objectives

- Speed of Convergence
- Accelerated Newton’s Iteration for Multiple Roots

Speed of Convergence

- Order of convergence: measure of how rapidly a sequence converges.
- Assume:
  - Sequence $\{p_n\}, (where\ n = 0, 1, \dots . \infty)$ converges to root p
  - Error $e_n = p - p_n$ for $n \geq 0$
  - Two positive constants $A \neq 0, R > 0$ exists

Speed of Convergence

- Then,

$$\lim_{n\to\infty} \frac{|p-p_{n+1}|}{|p-p_n|^R} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^R} = A$$

- The sequence is said to <u>converge to p with order of convergence R</u>

$$|e_{n+1}| \cong \boldsymbol{A}|e_n|^{\boldsymbol{R}}$$

Speed of Convergence

- $R = 1$ → Linear Convergence
- $R = 2$ → Quadratic Convergence
- With larger $\boldsymbol{R}$, the sequence will converge rapidly to $\boldsymbol{p}$

- Example:
  - If $R = 2$, and $|e_n| \approx 10^{-2}$
  - ➔ $|e_{n+1}| \approx A \times 10^{-4}$

Convergence Rate for Newton's Iteration

- Suppose that Newton-Raphson iteration produces a sequence $\{p_n\}$, $(n = 0, 1, \dots\dots, \infty)$ that converges to root p of $f(x)$

- If p is a simple root → convergence is Quadratic ($R = 2$)

$$|e_{n+1}| \approx \frac{|f''(x)|}{2|f'(x)|} |e_n|^2$$

- If p is a multiple root of order M → convergence is Linear ($R = 1$)

$$|e_{n+1}| \approx \frac{M-1}{M} |e_n|$$

Convergence Rate for Newton's Iteration

- Example:
  - If M=2. What is the speed of convergence?
  - Answer: Double root, then convergence is linear.

- *Note*: we say that $f(x) = 0$ has a root of order M at $x = p$, if and only if $f(p) = 0$, $f'(p) = 0$, $f''(p) = 0$, ……, $f^{(M)}(p) \neq 0$

Example – Simple Root

- Let $f(x) = x^3 - 3x + 2 = (x+2)(x-1)^2$.
- Start with $p_0 = -2.4$ to find the root $p = -2$ (Simple Root)

- $p_k = p_{k-1} - \frac{f(p_{k-1})}{f\prime(p_{k-1})} = p_{k-1} - \frac{p_{k-1}^3 - 3p_{k-1} + 2}{3p_{k-1}^2 - 3}$

- $p_k = \frac{2p_{k-1}^3 - 2}{3p_{k-1}^2 - 3}$

Example – Simple Root (cont'd)

- Now, check for quadratic convergence.

Table 2.5 Newton's Method Converges Quadratically at a Simple Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2}$
0	−2.400000000	0.323809524	0.400000000	0.476190475
1	−2.076190476	0.072594465	0.076190476	0.619469086
2	−2.003596011	0.003587422	0.003596011	0.664202613
3	−2.000008589	0.000008589	0.000008589	
4	−2.000000000	0.000000000	0.000000000	

[1]

- $|p - p_3| = 0.000008589$
- $|p - p_2|^2 = 0.003596011^2 = 0.000012931$

Example – Simple Root (cont'd)

- $|p - p_3| = 0.000008589$
- $|p - p_2|^2 = \ 0.003596011^2 = 0.000012931$

- $A = \frac{1}{2}\ \frac{|f''(-2)|}{|f'(-2)|} = \frac{1}{2}\ \frac{|-12|}{|9|} = \frac{2}{3}$

- $A = \frac{|p-p_3|}{|p-p_2|^2} = 0.664202613\ \approx\ \frac{2}{3}$

Example – Double Root

- Let $f(x) = \; x^3 \; - 3x + 2 = (x + 2)(x - 1)^{\;2}$.
- Start with $p_0 \; = 1.2$ to find the root $p \; = 1$ (Double Root)

- $p_k = \; p_{k-1} \; - \; \frac{f(p_{k-1})}{f\prime(p_{k-1})} \;\; = p_{k-1} \; - \; \frac{p_{k-1}^3 - 3p_{k-1} + 2}{3p_{k-1}^2 - 3}$

- $p_k = \; \frac{2p_{k-1}^3 - 2}{3p_{k-1}^2 - 3}$

Example – Double Root (cont'd)

Table 2.6 Newton's Method Converges Linearly at a Double Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert}$
0	1.200000000	−0.096969697	−0.200000000	0.515151515
1	1.103030303	−0.050673883	−0.103030303	0.508165253
2	1.052356420	−0.025955609	−0.052356420	0.496751115
3	1.026400811	−0.013143081	−0.026400811	0.509753688
4	1.013257730	−0.006614311	−0.013257730	0.501097775
5	1.006643419	−0.003318055	−0.006643419	0.500550093
⋮	⋮	⋮	⋮	⋮

[1]

Example – Double Root (cont'd)

- $|p - p_5| = 0.006643419$
- $|p - p_4| = 0.013257730$

- $A = \frac{|p-p_5|}{|p-p_4|} = 0.501097775 \;\approx \frac{1}{2}$

- $A = \frac{M-1}{M} = \;\frac{1}{2}$

Accelerated Newton Iteration for Multiple Roots

- We hope there are a root-finding techniques which converge faster than linearly when p is a root of order M.
- Modify Newton's method to converge faster
  - Become quadratic at multiple root
- For root p of order $M > 1$, Modified Newton-Raphson formula is:

$$p_n = p_{n-1} - \frac{Mf(p_{n-1})}{f\prime(p_{n-1})}$$

Now, the sequence will converge quadratically to p

Example – Double Root

- Let $f(x) = x^3 - 3x + 2 = (x+2)(x-1)^2$.
- Start with $p_0 = 1.2$ to find the root $p = 1$ (Double Root)
- Here, $M = 2$
- $p_k = p_{k-1} - \frac{2f(p_{k-1})}{f'(p_{k-1})} = \frac{p_{k-1}^3 - 3p_{k-1} - 4}{3p_{k-1}^2 - 3}$

Table 2.8 Acceleration of Convergence at a Double Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2}$
0	1.200000000	−0.193939394	−0.200000000	0.151515150
1	1.006060606	−0.006054519	−0.006060606	0.165718578
2	1.000006087	−0.000006087	−0.000006087	
3	1.000000000	0.000000000	0.000000000	

[1]

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

جامعة العلوم والتكنولوجيا الأردنية

١٤٠٦ هـ ١٩٨٦م