CHAPTER SIX

Numerical Differentiation

Objectives

- Central-difference formula of order $O(h^4)$
- Optimum step-size for central-difference formula of order $O(h^4)$

Central-difference formula of order $O(h^4)$

- Start with fourth degree Taylor expressions about x for $f(x+h)$ and $f(x-h)$
- $f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \frac{f^{(4)}(x)h^4}{4!} + O(h^5)$
- $f(x-h) = f(x) - f'(x)h + \frac{f''(x)h^2}{2!} - \frac{f'''(x)h^3}{3!} + \frac{f^{(4)}(x)h^4}{4!} - O(h^5)$
- $f(x+h) - f(x-h) = 2f'(x)h + 2\frac{f'''(x)h^3}{3!} + 2O(h^5) \;\ldots\ldots(1)$
- $f(x+2h) - f(x-2h) = 4f'(x)h + 16\frac{f'''(x)h^3}{3!} + 64O(h^5) \;\ldots\ldots(2)$

Central-difference formula of order $O(h^4)$ – Cont'd

- $f(x+h) - f(x-h) = 2f'(x)h + 2\frac{f'''(x)h^3}{3!} + 2O(h^5)$ ……(1)
- $f(x+2h) - f(x-2h) = 4f'(x)h + 16\frac{f'''(x)h^3}{3!} + 64O(h^5)$ ……(2)
- Now, multiply (1) by 8 and subtract (2) from it. We will got:
- $8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h) = 12f'(x)h + O(h^5)$
- $\boxed{f'(x) = \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}} + \boxed{O(h^4)}$ $\boxed{= \frac{h^4 f^{(5)}(c)}{30}}$

Central-difference formula of order $O(h^4)$

Central-difference formula of order $O(h^4)$ - Example

- Let $f(x) = \cos(x)$. Use central-difference formula of order $O(h^4)$ with step size $h = 0.1,\ 0.01,\ 0.001,$ and 0.0001 to approximate $f'(0.8)$
- Note that the exact value of $f'(0.8) = -\sin(0.8) = -0.717356090899$

- If $h = 0.01$ →
- $f'(0.8) = \frac{-f(0.82)+8f(0.81)-8f(0.79)+f(0.78)}{0.12}$

$$= -0.717356108$$

Table 6.2 Numerical Differentiation

Step size	Approximation by formula (10)	Error using formula (10)
0.1	−0.717353703	−0.0000023889
0.01	−0.717356108	0.000000017
0.001	−0.717356167	0.000000076
0.0001	−0.717360833	0.000004742

[1]

Optimum Step Size

- $f(x_0 + kh) = y_k + e_k$
- $f'(x) = \frac{-y_2+8y_1-8y_{-1}+y_{-2}}{12h} + E(f,h)$
- $E(f,h) = \frac{-e_2+8e_1-8e_{-1}+e_{-2}}{12h} + \frac{f^{(5)}(c)h^4}{30}$
- $|E(f,h)| \leq \boxed{\frac{3\epsilon}{2h} + \frac{Mh^4}{30}}$ $g(h)$
- For optimal value of h: $g'(h) = 0$ ➔ $\boxed{h = \left(\frac{45\epsilon}{4M}\right)^{\frac{1}{5}}}$

Optimum Step Size - Example

- Let $f(x) = \cos(x)$, and $\epsilon = 0.5 \times 10^{9}$. Find optimum step size for central-difference formula of order $O(h^4)$.

- $\left|f^{(5)}(x)\right| \leq |\sin(x)| \leq 1 \quad \rightarrow M = 1$

- $h = (\frac{45\epsilon}{4M})^{\frac{1}{5}} = (\frac{45\times0.5\times10^{-9}}{4})^{\frac{1}{5}} = 0.022388475$

- Note that from previous example, optimum step size $h = \ 0.01$

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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