CHAPTER THREE

The Solution of Linear Systems **AX=B**

Objectives

- Gaussian Elimination and Pivoting

Gaussian Elimination and Pivoting

$$AX = B \xrightarrow{transformation} UX = Y$$, where U: Upper-triangular matrix

- Certain transformations do not change solution:
  - Interchange: the order of two equations (rows) can be changed.
  - Scaling : multiply equation by nonzero constant
  - Replacement :equation can be replaced by sum of itself and a nonzero multiple of any other equation

Note

- If pivot =0, switch the first row below the pivot with the pivot row (trivial pivoting).
- To reduce error propagation, move entry of greatest magnitude to main diagonal.
- See example 3.17 and 3.18 from textbook (partial pivoting)

Gaussian Elimination and Pivoting - Example

- Solve the following linear system

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$

$$2x_1 + 0x_2 + 4x_3 + 3x_4 = 28$$

$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$

$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6$$

- First, form augmented matrix $[A|B]$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

Now, we will zero everything under the diagonal of the matrix (red dashed line)

Example (cont'd)

$$\begin{array}{r} \text{Pivot} \rightarrow \\ m_{21} = 2/1 \\ m_{31} = 4/1 \\ m_{41} = -3/1 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

$$row_2 = row_2 - m_{21} \times row_1$$

$$\begin{array}{r} 2\ \ 0\ \ 4\ \ 3\ \ |28 \\ -2(1\ \ 2\ \ 1\ \ 4\,|13) \end{array}$$

$$row_2 = (0\ \ -4\ \ 2\ \ -5\ \ |2)$$

$$row_3 = row_3 - m_{31} \times row_1$$

$$\begin{array}{r} 4\ \ 2\ \ 2\ \ 1\ \ |20 \\ -4(1\ \ 2\ \ 1\ \ 4\,|13) \end{array}$$

$$row_3 = (0\ \ -6\ \ -2\ \ -15\ \ |-32)$$

$$row_4 = row_4 - m_{41} \times row_1$$

$$\begin{array}{r} -3\ \ 1\ \ 3\ \ 2\ \ |6 \\ +3(1\ \ 2\ \ 1\ \ 4\,|13) \end{array}$$

$$row_4 = (0\ \ 7\ \ 6\ \ 14\ \ |45)$$

Example (cont'd)

$$
\begin{array}{r}
\\
\text{Pivot} \rightarrow \\
m_{32} = -6/-4 \\
m_{42} = 7/-4
\end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 4 & 13 \\
0 & \color{red}{-4} & 2 & -5 & 2 \\
0 & -6 & -2 & -15 & -32 \\
0 & 7 & 6 & 14 & 45
\end{array}\right]
$$

$$row_3 = row_3 - m_{32} \times row_2$$

$$
\begin{array}{r}
0 \quad -6 \quad -2 \quad -15 \quad |-3 \\
-1.5(0 \quad -4 \quad 2 \quad -5 \quad |2)
\end{array}
$$

$$row_3 = (0 \;\; 0 \;\; -5 \;\; -7.5 \;\; |-35)$$

$$row_4 = row_4 - m_{42} \times row_2$$

$$
\begin{array}{r}
0 \quad 7 \quad 6 \quad 14 \quad |45 \\
+1.75(0 \quad -4 \quad 2 \quad -5 \quad |2)
\end{array}
$$

$$row_4 = (0 \;\; 0 \;\; 9.5 \;\; 5.25 \;\; |48.5)$$

Example (cont'd)

$$\begin{array}{r} \\ \\ \text{Pivot} \rightarrow \\ m_{43} = 9.5/-5 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & \color{red}{-5} & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array}\right]$$

$$row_4 = row_4 - m_{43} \times row_3$$

$$\begin{array}{rccccl} & 0 & 0 & 9.5 & 5.25 & |48.5 \\ +1.9\,(& 0 & 0 & -5 & -7.5 & |-35) \end{array}$$

$$row_4 = (0 \quad 0 \quad 0 \quad -9 \; | -18)$$

Example (cont’d)

Final Matrix: $UX = Y$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right]$$

Using Back-Substitution:

$$x_4 = \frac{-18}{-9} = 2$$

$$x_3 = \frac{-35 - 7.5(2)}{-5} = 4$$

$$x_2 = \frac{2 - (2)4 - (-5)(2)}{-4} = -1$$

$$x_1 = \frac{13 - 2(-1) - 1(4) - 4(2)}{1} = 3$$

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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