CHAPTER FOUR

Interpolation and Polynomial Approximation

Objectives

- Newton Polynomials

Newton Polynomials

- Recursive Pattern

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_2(x) = \underbrace{a_0 + a_1(x - x_0)}_{P_1(x)} + a_2(x - x_0)(x - x_1)$$

$$P_3(x) = \underbrace{a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)}_{P_2(x)} + a_3(x - x_0)(x - x_1)(x - x_2)$$

Newton Polynomials

$$P_N(x) = a_0 + a_1(x - x_0) + \dots\dots + a_N\,(x - x_0)\,\dots..(\,x - x_{N-1})$$

Newton Polynomials

$$P_N(x) = P_{N-1}(x) + a_N\,(x - x_0)\,\dots..(\,x - x_{N-1})$$

Newton Polynomials

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_1(x_0) = a_0 + a_1(x_0 - x_0) = \boxed{a_0 = f(x_0)}$$

$$P_1(x_1) = a_0 + a_1(x_1 - x_0) = \boxed{f(x_0) + a_1(x_1 - x_0) = f(x_1)}$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Newton Polynomials

$$P_2(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$$

$$a_2 = \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$a_2 = \frac{f(x_2) - f(x_0) - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

Divided Difference

$$a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{f\,[x_1,\ x_2] - f\,[x_0, x_1]}{x_2 - x_0}$$

Divided Difference

$$f[x_k] = f(x_k)$$

$$f[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}}$$

$$f[x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-1}, x_k] - f[x_{k-2}, x_{k-1}]}{x_k - x_{k-2}}$$

$$f[x_{k-3}, x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-2}, x_{k-1}, x_k] - f[x_{k-3}, x_{k-2}, x_{k-1}]}{x_k - x_{k-3}}$$

$$\boxed{a_k = f[x_0, x_1, \ldots., x_k]}$$

Divided Difference

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

Divided Difference Table

x_k	$f[x_k]$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_{k-1}, x_k]$	$f[x_{k-3}, \dots, x_k]$	$f[x_{k-4}, \dots, x_k]$
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, \dots, x_4]$

Divided Difference Table

x_k	$f[x_k]$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_{k-1}, x_k]$	$f[x_{k-3}, \dots, x_k]$	$f[x_{k-4}, \dots, x_k]$
x_0	a_0 $\boxed{f[x_0]}$				
x_1	$f[x_1]$	a_1 $\boxed{f[x_0, x_1]}$			
x_2	$f[x_2]$	$f[x_1, x_2]$	a_2 $\boxed{f[x_0, x_1, x_2]}$	a_3	
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$\boxed{f[x_0, x_1, x_2, x_3]}$	a_4
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$\boxed{f[x_0, x_1, \dots, x_4]}$

Newton Polynomials - Example

- Let $f(x) = x^3 - 4x$, construct divided-difference table based on the nodes: $x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 5,$ and $x_5 = 6$

x_k	$f[x_k]$	1st Divided difference	2nd Divided difference	3rd Divided difference	4th Divided difference	5th Divided difference
$x_0 = 1$	-3					
$x_1 = 2$	0	3				
$x_2 = 3$	15	15	6			
$x_3 = 4$	48	33	9	1		
$x_4 = 5$	105	57	12	1	0	
$x_5 = 6$	192	87	15	1	0	0

Newton Polynomials – Example (Cont'd)

- Find Newton Polynomial $P_3(x)$ based on $x_0 = 1,\ \ x_1 = 2,\ \ x_2 = 3,\ \ x_3 = 4,\ \ x_4 = 5,$ and $x_5 = 6$

From divided- difference table we have: $a_0 = -3,\quad a_1 = 3,\ \ a_2 = 6,\ \ a_3 = 1$

Then,

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= -3 + 3(x - 1) + 6(x - 1)(x - 2) + 1(x - 1)(x - 2)(x - 3)$$

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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