CHAPTER THREE

The Solution of Linear Systems **AX=B**

Objectives

- The solution on linear systems **AX=B**
- Upper-triangular linear systems.

The solution on linear systems **AX=B**

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$

$$. . . . .$$

$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

$$\begin{bmatrix} a_{11} & a_{12} & .. & a_{1N} \\ a_{21} & a_{22} & .. & a_{2N} \\ .. & .. & .. & .. \\ a_{N1} & a_{N2} & .. & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_N \end{bmatrix}$$

$\boldsymbol{AX = B}$
where $\boldsymbol{A}$ is $N \times N$ Matrix

Upper –Triangular linear systems

- Having the linear system $\boldsymbol{AX} = \boldsymbol{B}$, where $\boldsymbol{A}$ an upper-triangular matrix.
- Example: For $N = 3$, the following is an upper-triangular matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned}$$

Upper –Triangular linear systems

- Back substitution:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{33}x_3 = b_3$$

$$x_3 = \frac{b_3}{a_{33}}$$
$$x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}$$
$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

- General Step: $x_k = \dfrac{b_k - \sum_{j=k+1}^{N} a_{kj}x_j}{a_{kk}}$, for $k = N-1, N-2, \ldots\ldots, 1$

Upper –Triangular linear systems

- Condition:

- $\boldsymbol{A}$: Nonsingular matrix ($\det(A) \neq 0$)

  - $\mathrm{d}et\ (A) = \prod_{i=1}^{N} a_{ii}$ for any Upper/Lower-triangular Matrix

  - ***<u>OR</u>*** $a_{kk} \neq 0$ for all $k = 1, 2, \ldots, N$

Upper –Triangular linear systems - Example

- Use back-substitution to solve the following linear system

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ -2x_2 + 7x_3 - 4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

- Solution:

$$x_4 = \frac{6}{3} = 2$$

$$x_3 = \frac{4 - 5 * 2}{6} = -1$$

$$x_2 = \frac{-7 - 7(-1) + 4(2)}{-2} = -4$$

$$x_1 = \frac{20 - 4 - 2(-1) - 3(2)}{4} = 3$$

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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