CHAPTER TWO

The Solution of Nonlinear Equations $f(x) = 0$

Objectives

- Newton-Raphson Method

Newton-Raphson and Secant Methods Slope Methods for Finding Roots

- If $f(x)$, $f'(x)$, and $f''(x)$ are continuous near root $\boldsymbol{p}$, then an algorithm can be developed to produce sequences $\{p_k\}$ that converge **faster** to $\boldsymbol{p}$ than either the Bisection or False position methods.

- Assume that the initial approximation $\boldsymbol{p_0}$ is near to the root $\boldsymbol{p}$.

Newton-Raphson Method

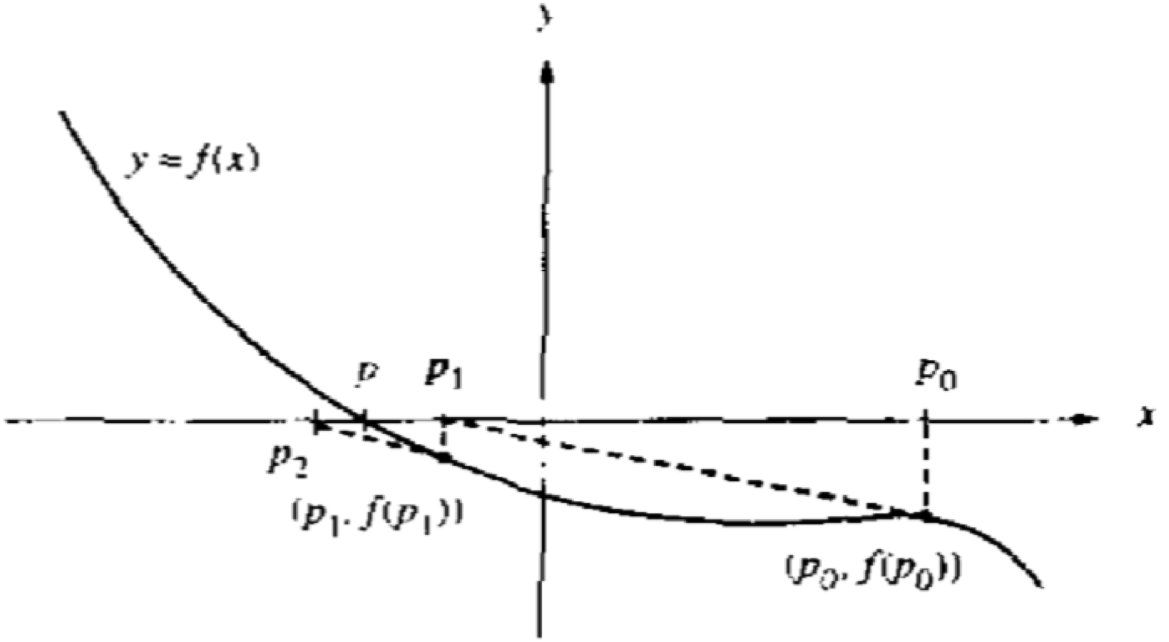

Figure 2.13 The geometric construction of p_1 and p_2 for the Newton-Raphson method. [1]

- Here p_1 will be closer to p than p_0
- Dotted-line on the figure is the tangent line (L) for the curve at p_0

Newton-Raphson Method (cont'd)

- Let us find the slope of the tangent line (L) by relating p_1 and p_0.

$$m = \frac{0 - f(p_0)}{p_1 - p_0}$$

➔ Slope of $(p_1, 0)$ and $(p_0, f(p_0))$

- Also,

$$m = f'(p_0)$$

➔ Slope at point $(p_0, f(p_0))$

- Then, $$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

Newton-Raphson Method (cont'd)

- On general ➔

$$p_k = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})} \quad \text{, for } k = 1, 2, 3, \dots\dots.$$

- Now, let $p_{k-1} = x$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

➔ Newton-Raphson iteration function

Example

- Let $f(t) = 4800\left[1 - e^{-t/_{10}}\right] - 320t$, and $p_0 = 8$. Find the root of the function.

- $f'(t) = 480e^{-t/_{10}} - 320$
- $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 8 - \frac{f(8)}{f'(8)} = 8 - \frac{83.220972}{-104.3220972} = 8.797731010$

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- $p_4 = 8.74217466$, $f(p_4) \cong 0$, $f(p_3) = 1 \times 10^{-6}$

Example (cont'd)

Table 2.4 Finding the Time When the Height $f(t)$ Is Zero

k	Time, p_k	$p_{k+1} - p_k$	Height, $f(p_k)$
0	8.00000000	0.79773101	83.22097200
1	8.79773101	−0.05530160	−6.68369700
2	8.74242941	−0.00025475	−0.03050700
3	8.74217467	−0.00000001	−0.00000100
4	8.74217466	0.00000000	0.00000000

[1]

$$|p_4 - p_3| = 10^{-8} \rightarrow \delta\ (delta)$$

Corollary 2.2: Newton's Iteration for Finding Square Roots

- Assume
  - A>o is a real number
  - $p_0 > 0$ is initial approximation to $\sqrt{A}$
- Then, $p_k = \frac{p_{k-1} + {}^{A}\!/_{p_{k-1}}}{2}$, for $k = 1, 2, 3, \dots .$
- The sequence $\{p_k\}$, $k = 0,1, \dots , \infty$ converges to $\sqrt{A}$

  → $\lim_{k \to \infty} p_k = \sqrt{A}$

Outline of the Proof (Corollary 2.2)

- Start with function $f(x) = x^2 - A$
  - Roots of equation $x^2 - A$ = 0 are $\pm\sqrt{A}$

- Then, using Newton-Raphson Iteration Function:

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - A}{2x}$$

$$\rightarrow \quad g(x) = \frac{x + A/x}{2}$$

Example

- Find $\sqrt{5}$, where $p_0 = 2$.

- $p_1 = \frac{2+5/2}{2} = 2.25$
- $p_2 = \frac{2.25+5/2.25}{2} = 2.236111111$
- $p_3 = 2.236067978$
- $p_4 = 2.236067978$
- $\rightarrow p_k \approx 2.236067978 \quad , for\ k > 4$
- Convergence accurate to nine significant digits has been achieved.

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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