CHAPTER FOUR

Interpolation and Polynomial Approximation

Objectives

- Interpolation and polynomial approximation
- Taylor polynomial approximation

Interpolation and Polynomial Approximation

- Polynomial of degree N

  P_N(x)= $a_0 + a_1\, x + a_2\, x^2$+ …. +$a_N\, x^N$

  where $a_0\,, a_1\,, \dots \dots .\,.\,, a_N$ are coefficients

- Note:
  - P_1(x) → Linear polynomial
  - P_2(x) → Quadratic polynomial
  - P_3(x) → Cubic polynomial

- OR: $P_N(x)$= b_0+$b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \dots\dots$

  + $b_N(x - x_0)(x - x_1) \dots .\,.\,(x - x_{N-1})$

Interpolation and Polynomial Approximation

- Polynomial to approximate :
  - Continuous function :
    - Simpler than original function.
    - Easier to handle analytically
    - Faster to evaluate numerically .
  - Function available only at discrete points
    - Finding value of function between available points (Interpolation)

Taylor Polynomial Approximation

- Assume we have the function as follows:

  $f \in C^{N+1}[a, b]$, $x_0 \in [a, b]$, $x \in [a, b]$

- Then, we have $f(x) = P_N(x) + E_N(x)$

- And, $\boxed{f(x) \approx P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}$ → Taylor polynomial .

- $\boxed{E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{N+1}}$,where c between x , x_0

Taylor Polynomial Approximation

Table 4.1 Taylor Series Expansions for Some Common Functions

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for all x
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	for all x
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	for all x
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 \le x \le 1$
$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$-1 \le x \le 1$
$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$	for $\lvert x \rvert < 1$

[1]

Taylor Polynomial Approximation - Example

- Show why 15 terms are all that we need to obtain 13-digit approximation

  $e \approx 2.718281828459.$

- Assume $f(x)= e^x$, $x_0 = 0$
- $P_{15}(x)= \sum_{k=0}^{15} \frac{f^{(k)}(0)}{k!} (x-0)^k$
- $f'(x) = f''(x)=........=f^{(16)}(x) = e^x$
- $\frac{f^{(k)}(0)}{k!} = \frac{e^0}{k!} = \frac{1}{k!}$

- $P_{15} = \sum_{k=0}^{15} \frac{1}{k!} x^k$

  $=1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} +\ldots+ \frac{x^{15}}{15!}$

- $E_{15}(x) = \frac{f^{(16)}(c)}{16!} x^{16}$
- $x_0 = 0$, $x = 1,$ $0 < c < 1$

  $e^c < e^1 \approx 2.71828 < 3$

- $|E_{15}(1)| = \left|\frac{f^{(16)}(c)}{16!}\right| .(1)^{16}$

  $= \frac{e^c}{16!} < \frac{3}{16!} =1.43384310^{-13}$

Taylor Polynomial Approximation – Error Bounds

- $|x - x_0| < R$ → $2R$ interval , where x_0 is in the center
- $-R < x - x_0 < R$
- $|error|$= $|E_n(x)| \leq \frac{M\, R^{N+1}}{(N+1)!}$
- $M \leq max \left\{ \left| f^{(n+1)}(z) \right| \right\}$, where $x_0 - R \leq z \leq x_0 + R$
- Note: R→0 makes $Error$→0

  N→∞ makes $Error$→0
- The accuracy decreases as we move far from x_0 (the center)
- The accuracy increase when N is large

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

جامعة العلوم والتكنولوجيا الأردنية

١٤٠٦ هـ ١٩٨٦م