CHAPTER SEVEN

Numerical Integration

Objectives

- Derivation of Simpson’s Rule

Derivation of Simpson's Rule

$$\boxed{\int_{x_0}^{x_2} P_2(x)dx = \frac{h}{3}(f_0 + 4f_1 + f_2)}$$

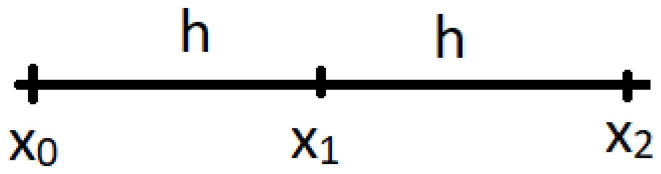
- Start with Lagrange Polynomial $P_2(x)$ based on x_0, x_1, and x_2.
- $P_2(x) = f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$
- $\int_{x_0}^{x_2} P_2(x)dx = f_0 \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} dx + f_1 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} dx$

$$+f_2 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} dx$$

Derivation of Simpson's Rule

- Change of variables:

$x = x_0 + ht \;\rightarrow\; dx = hdt$



For $x = x_0 \;\rightarrow t = 0$

For $x = x_1 \;\rightarrow t = 1$

For $x = x_2 \;\rightarrow t = 2$

And using: $x_k = x_0 + kh \quad \rightarrow \quad \boxed{x_k - x_j = (k - j)h} \;\rightarrow (x_0 + kh) - (x_0 + jh)$

$\rightarrow \quad \boxed{x - x_k = (t - k)h} \;\rightarrow (x_0 + ht) - (x_0 + kh)$

- Cont'd

$$\int_{x_0}^{x_2} f(x)dx = f_0 \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} dx + f_1 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} dx + f_2 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} dx$$

$$\int_{x_0}^{x_2} f(x)dx = f_0 \int_0^2 \frac{h(t-1)h(t-2)}{(-h)(-2h)} hdt + f_1 \int_0^2 \frac{h(t-0)h(t-2)}{(h)(-h)} hdt + f_2 \int_0^2 \frac{h(t-0)h(t-1)}{(2h)(h)} hdt$$

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{2} f_0 \int_0^2 (t^2-3t+2)dt - hf_1 \int_0^2 (t^2-2t)\, dt + \frac{h}{2} f_2 \int_0^2 (t^2-t)\, dt$$

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{2} f_0 \left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Big|_{t=0}^{t=2} - hf_1 \left(\frac{t^3}{3} - \frac{2t^2}{2}\right)\Big|_{t=0}^{t=2} + \frac{h}{2} f_2 \left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_{t=0}^{t=2}$$

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{2} f_0 \left(\frac{2}{3}\right) - hf_1 \left(\frac{-4}{3}\right) + \frac{h}{2} f_2 \left(\frac{2}{3}\right) = \boxed{\frac{h}{3}(f_0 + 4f_1 + f_2)}$$

Newton-Cotes Quadrature Formulas - Example

- Let $f(x) = 1 + e^{-x}\sin(4x)$, and the nodes: $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$
- Apply Newton-Cotes Quadrature Formulas. (Note: $h = 0.5$)

$$\int_0^{0.5} f(x)dx = \frac{0.5}{2}\big(f(0) + f(0.5)\big) = 0.63788$$

$$\int_0^{1.0} f(x)dx = \frac{0.5}{3}(f(0) + 4f(0.5) + f(1.0)) = 1.32128$$

$$\int_0^{1.5} f(x)dx = \frac{3(0.5)}{8}(f(0) + 3f(0.5) + 3f(1.0) + f(1.5)) = 1.64193$$

$$\int_0^{2} f(x)dx = \frac{2(0.5)}{45}(7f(0) + 32f(0.5) + 12f(1.0) + 32f(1.5)) + 7f(2) = 2.29444$$

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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