CHAPTER FOUR

Interpolation and Polynomial Approximation

Objectives

- Lagrange Approximation

Lagrange Approximation

- Estimating a missing function value by taking a weighted average of known values for neighboring points.

  - $m = \frac{y_1 - y_0}{x_1 - x_0}$
  - $y = m(x - x_0) + y_0$
  - $Y = P(x) = y_0 + (y_1 - y_0)\,\frac{x - x_0}{x_1 - x_0}$

  - $P(x_0) = y_0 + (y_1 - y_0)\,(0) = y_0$
  - $P(x_1) = y_0 + (y_1 - y_0)\,(1) = y_1$

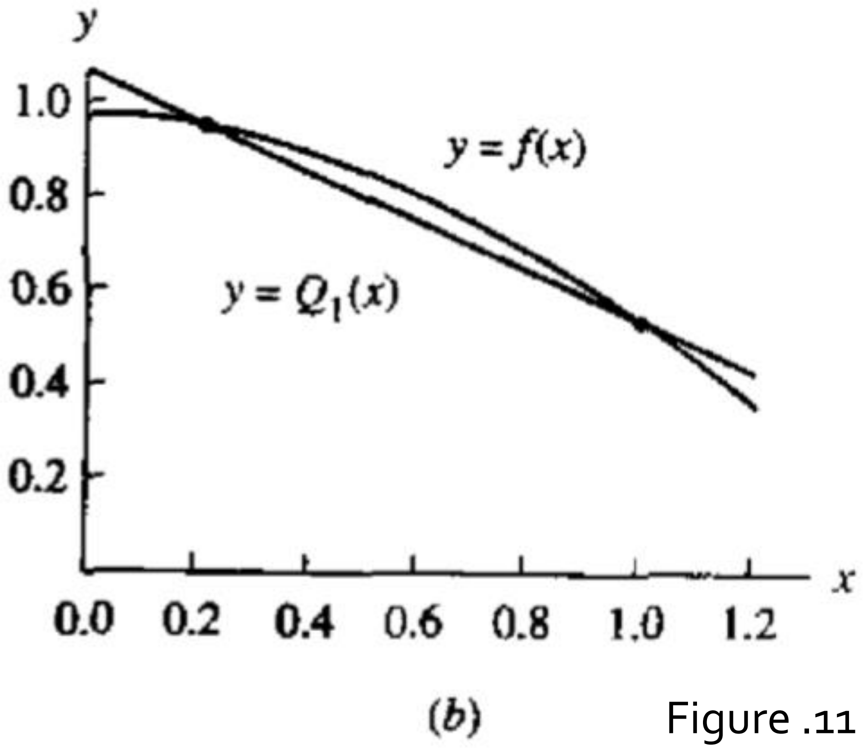

Figure .11 [1]

Lagrange Approximation

- $y = P_1(x) = y_0 \underbrace{\frac{x-x_1}{x_0-x_1}}_{L_{1,0}(x)} + y_1 \underbrace{\frac{x-x_0}{x_1-x_0}}_{L_{1,1}(x)}$

- $P_1(x) = y_0 L_{1,0}(x) + y_1 L_{1,1}(x)$

  $L_{1,0}(x_0) = 1$ $\quad L_{1,1}(x_0) = 0$ $\quad \rightarrow P_1(x_0) = y_0$

  $L_{1,0}(x_1) = 0$ $\quad L_{1,1}(x_1) = 1$ $\quad \rightarrow P_1(x_1) = y_1$

- $P_1(x) = \sum_{k=0}^{1} y_k\, L_{1,k}(x)$

Lagrange Approximation

- If $P_1(x)$ used to approximate $f(x)$ over interval $[x_0, x_1]$, we call the process **interpolation.**

- If $x < x_0$ (or $x_1 < x$) then using $P_1(x)$ called **extrapolation.**

Lagrange Approximation

- Generalization : Polynomial of degree (N)

- $P_N(x) = \sum_{k=0}^{N} \; y_k \, L_{N,k}\,(x)$

- $L_{N,k}\,(x) = \frac{(x-x_0)\ldots\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)\ldots\ldots\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots\ldots(x_k-x_n)}$

- $L_{N,k}\,(x) = \frac{\prod_{\substack{\mathrm{j=0} \\ j\neq k}}^{\mathrm{N}} \; (x-x_j)}{\prod_{\substack{\mathrm{j=0} \\ j\neq k}}^{\mathrm{N}} \; (x_k-x_j)}$

Lagrange Approximation - Example

- Let $y = f(x) = \cos(x)$ over $[0.0\ , 1.2]$. And use $x_0 = 0$, $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$

to construct cubic interpolation poly $P_3(x)$.

- $P_3(x) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} +$

$$y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Lagrange Approximation – Example (cont'd)

- $y_0 = \cos(0.0) \; = \; 1.0$
- $y_1 = \cos(0.4) \; = \; 0.921061$
- $y_2 = \cos(0.8) \; = \; 0.696707$
- $y_3 = \cos(1.2) \; = \; 0.362358$

- $P_3(x)$= $-2.604167(x-0.4)(x-0.8)(x-1.2) \; + 7.195789\,(x-0.0)(x-0.8)(x-1.2)$

$$-5.443021\,(x-0.0)(x-0.4)(x-1.2) + 0.943641(x-0.0)(x-0.4)(x-0.8)$$

Lagrange Approximation – Example (cont'd)

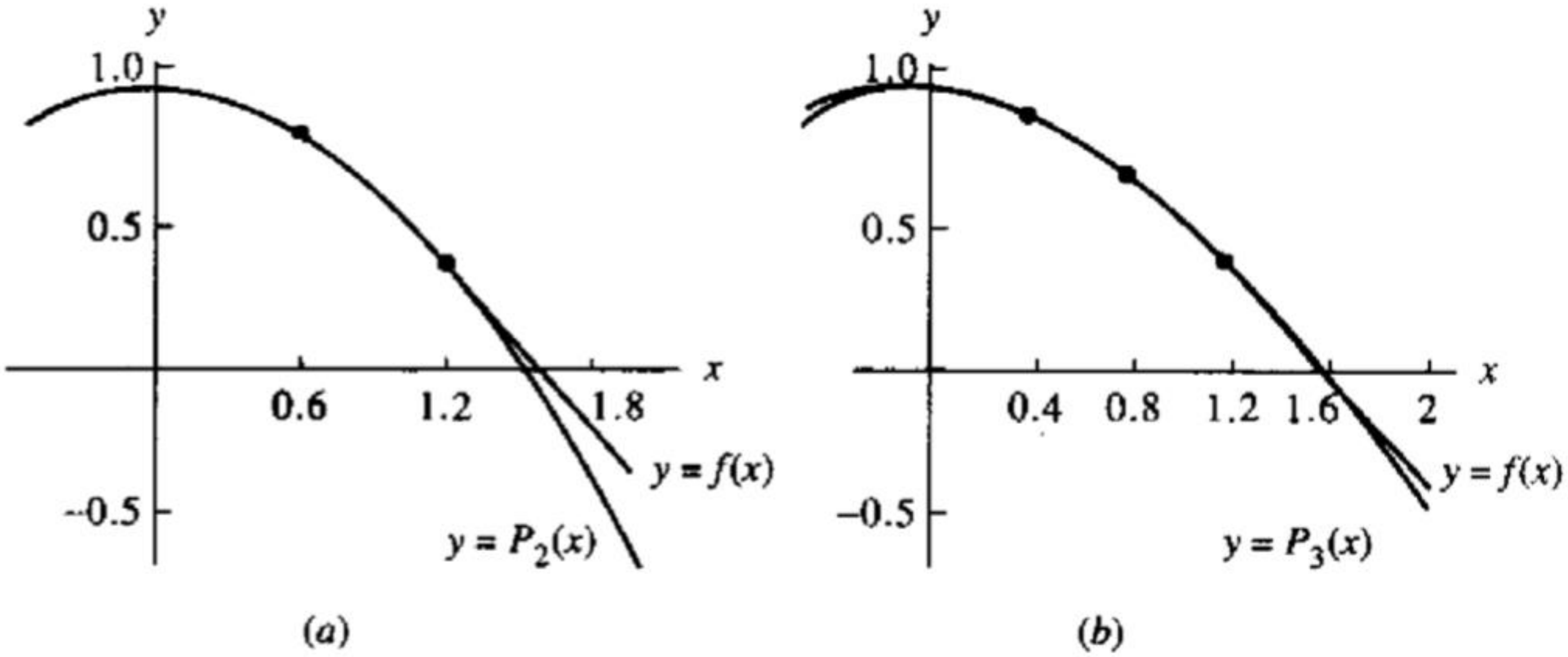

Figure 4.12 (a) The quadratic approximation polynomial $y = P_2(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 0.6$, and $x_2 = 1.2$. (b) The cubic approximation polynomial $y = P_3(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, and $x_3 = 1.2$. [1]

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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