CHAPTER SIX

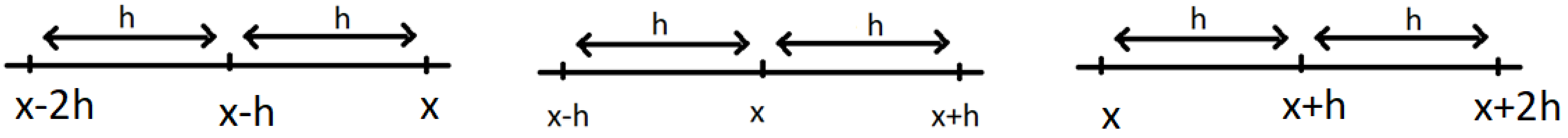
Numerical Differentiation

Objectives

- Forward- and backward-difference formulas

Forward- and Backward-Difference Formulas

- Forward- and backward-difference formulas are derived by differentiation of Lagrange interpolation polynomial.



Backward-Difference Central-Difference Forward-Difference

Forward- and Backward-Difference Formulas

- **Question**: Derive forward-difference formula for $f''(x)$.

$$f''(x_0) = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2}$$

- **Solution**: Start with Lagrange interpolation polynomial for $f(t)$ based on the points $x_0, x_1, x_2,\ and\ x_3$
- Then, differentiate the $f(x_0)$ twice to get $f''(x_0)$

- **Solution**:

$$f(t) = f_0 \frac{(t-x_1)(t-x_2)(t-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f_1 \frac{(t-x_0)(t-x_2)(t-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$
$$+f_2 \frac{(t-x_0)(t-x_1)(t-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f_3 \frac{(t-x_0)(t-x_1)(t-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f''(t) = f_0 \frac{2[(t-x_1)+(t-x_2)+(t-x_3)]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f_1 \frac{2[(t-x_0)+(t-x_2)+(t-x_3)]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$
$$+f_2 \frac{2[(t-x_0)+(t-x_1)+(t-x_3)]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f_3 \frac{2[(t-x_0)+(t-x_1)+(t-x_2)]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

- Now→ Substitute: $t = x_0$
- And note that→ $x_i - x_j = (i-j)h$ (remember: $x_k = x_0 + kh$)
  - i.e. $x_0 - x_1 = -h$ and $x_2 - x_0 = +2h$

- **Solution (cont'd)**:

$$f''(x_0) \approx f_0 \frac{2((x_0 - x_1) + (x_0 - x_2) + (x_0 - x_3))}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + f_1 \frac{2((x_0 - x_0) + (x_0 - x_2) + (x_0 - x_3))}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} + f_2 \frac{2((x_0 - x_0) + (x_0 - x_1) + (x_0 - x_3))}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + f_3 \frac{2((x_0 - x_0) + (x_0 - x_1) + (x_0 - x_2))}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

$$= f_0 \frac{2((-h) + (-2h) + (-3h))}{(-h)(-2h)(-3h)} + f_1 \frac{2((0) + (-2h) + (-3h))}{(h)(-h)(-2h)} + f_2 \frac{2((0) + (-h) + (-3h))}{(2h)(h)(-h)} + f_3 \frac{2((0) + (-h) + (-2h))}{(3h)(2h)(h)}$$ [1]

$$f''(x_0) = f_0 \frac{-12h}{-6h^3} + f_1 \frac{-10h}{2h^3} + f_2 \frac{-8h}{-2h^3} + f_3 \frac{-6h}{6h^3} \rightarrow \boxed{f''(x_0) = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2}}$$

Forward- and Backward-Difference Formulas

- For Backward–difference formulas:
  - Differentiate Lagrange interpolation polynomial based on the points: $x_{-3}, x_{-2}, x_{-1},\ and\ x_0$

- Check Example 6.7 on textbook [1] page 335

- Check Table 6.7 on textbook [1]: forward and Backward–difference formulas for $f'(x_0), f''(x_0),\ f^{(3)}(x_0),\ and\ f^{(4)}(x_0)$

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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