CHAPTER TWO

The Solution of Nonlinear Equations $f(x) = 0$

Objectives

- What is Numerical Analysis.
- Bracketing methods for finding a root
- Bisection method.

Introduction

- ***Numerical analysis*** is the study of algorithms that use numerical approximation for the problems of mathematical analysis.
- The overall ***goal*** of the field of numerical analysis is the design and analysis of techniques to give approximate but accurate solutions to hard problems

The solution of nonlinear equations f(x)=0

- Numerical methods to find numerical approximation for the root of an equation.
- Iterative techniques are used.
- Iteration: process repeated until an answer is achieved.
- Need starting value and a rule or function for computing successive terms.

Bracketing methods for finding a root

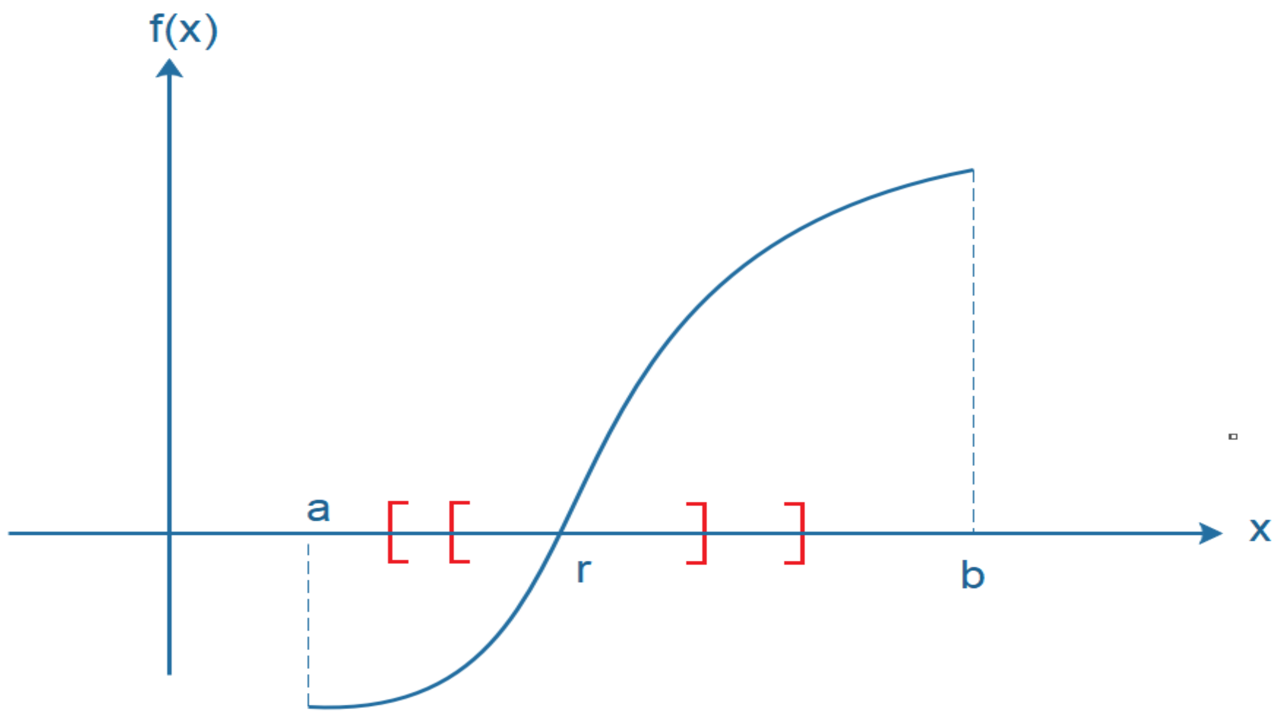
- Root of an equation or zero of a function:
  - Assume that $f(x)$ is a continuous function ➔ any number **r** for which $f(r) = 0$ is called a root of a function(equation) $f(x) = 0$.

Example

- The equation $2x^2 + 5x - 3 = 0$ has two roots.
- Using quadratic formula
  - $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Then, $x_1 = \dfrac{-5 + \sqrt{25 - 4 \times 2 \times -3}}{2 \times 2} = \dfrac{2}{4}$, $x_1 = \dfrac{1}{2}$
- And, $x_2 = \dfrac{-5 - \sqrt{25 - 4 \times 2 \times -3}}{2 \times 2} = \dfrac{-12}{4}$, $x_2 = -3$

➔ $2x^2 + 5x - 3 = (2x - 1)(x + 3)$

The Bisection Method of Bolzano



The Bisection Method of Bolzano

- Step 1: Start with initial interval $[a, b]$, where *f(a)* and *f(b)* have opposite signs.

  ➔ $f(a) \times f(b) < 0$

  - Note : $y = f(x)$is a continuous function.

    →The graph will be unbroken.

    →The graph will cross the x-axis at a root $x = r$

- Step 2: Choose a midpoint to halve the interval

$$c = \frac{a+b}{2}$$

The Bisection Method of Bolzano

- Step 3: analyze the possibilities:

1. *f(a)* and *f(c)* have opposite signs:
   → root $\underline{\boldsymbol{r}}$ lies in $[a, c]$ → set $b = c$ and return to step 2
2. f(c) and f(b) have opposite signs:
   → root $\underline{\boldsymbol{r}}$ lies in $[c, b]$ → set $a\ = c$ and return to step 2
3. $f(c) = 0$

   → root $\boldsymbol{r} = c$

Example

- The function $f(x) = x \times sin(x) - 1$, is continuous at [0,2].
- Then: $a_0 = 0, \; b_0 = 2$
- $f(a_0) = f(0) = -1, \; f(b_0) = f(2) = 0.818595$ (opposite signs) Step 1
- $c_0 = \frac{a_0+b_0}{2} = \frac{0+2}{2} = 1$ Step 2
- $f(c_0) = f(1) = -0.158529$ *(Note: x is in radians)*

Example - continued

- $f(c_0)\,f(b_0) < 0$ → then, root ***r*** lies in the interval [c_o,b_o] Step 3
- Then, $[a_1, b_1] = [c_0, b_0] = [1,2]$
- Now, ***start new iteration***:
- $f(a_1) = f(1) = -0.158529,\;\; f(b_1) = f(2) = 0.818595$ *(opposite signs)*
- $c_1 = \frac{a_1+b_1}{2} = \frac{1+2}{2} = 1.5$
- $f(c_1) = f(1.5) = 0.496242$
- $f(a_1)\,f(c_1) < 0$ → then, root ***r*** lies in the interval [a_1,c_1]
- Then, $[a_2, b_2] = [a_1, c_1] = [1,1.5]$

Example - continued

- The following table show the calculations for 8 iterations.

Table 2.1 Bisection Method Solution of $x \sin(x) - 1 = 0$

k	Left end point, a_k	Midpoint, c_k	Right end point, b_k	Function value, $f(c_k)$
0	0	1.	2.	−0.158529
1	1.0	1.5	2.0	0.496242
2	1.00	1.25	1.50	0.186231
3	1.000	1.125	1.250	0.015051
4	1.0000	1.0625	1.1250	−0.071827
5	1.06250	1.09375	1.12500	−0.028362
6	1.093750	1.109375	1.125000	−0.006643
7	1.1093750	1.1171875	1.1250000	0.004208
8	1.10937500	1.11328125	1.11718750	−0.001216
⋮	⋮	⋮	⋮	⋮

[1]

- After many iterations the root will converge to $\boldsymbol{r} \approx 1.114157141$

Error bound $|r - c_n|$ - Bisection Method

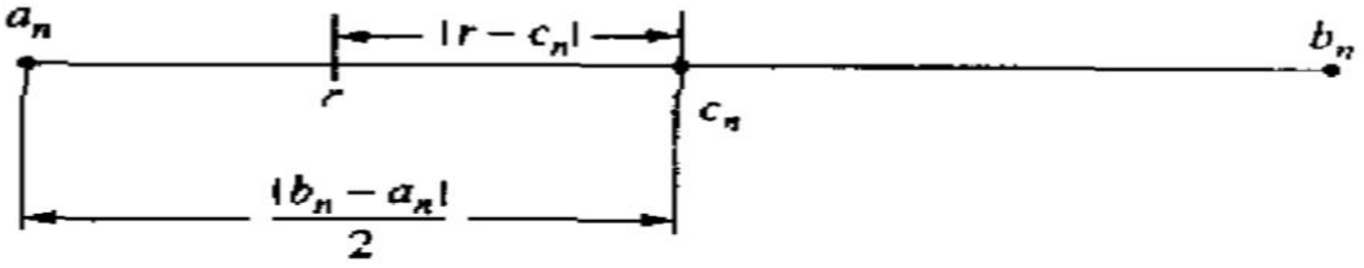


Figure 2.7 The root r and midpoint c_n of $[a_n, b_n]$ for the bisection method.

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- $|r - c_n| \le \frac{b_n - a_n}{2}$
- Then, $error\ bound = \delta = |r - c_n| \le \frac{b_0 - a_0}{2^{n+1}}$
- δ delta → tolerance value

Error bound $|r - c_n|$ - Bisection Method

- The number $\boldsymbol{n}$ of repeated bisections needed to guarantee that the $\boldsymbol{n^{th}}$ midpoint $\boldsymbol{c_n}$ is an approximation to a root that has an error less than the pre assigned value δ is:

$$n = \ int\left(\frac{\ln(b_0 - a_0) - \ln(\delta)}{\ln(2)}\right)$$

Termination Criterion

- When to terminate the iterations.
- The iterations can be terminated when the following condition occurs during any iteration.

$$(b - a) < \delta$$

- δ sufficiently small.

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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