CHAPTER THREE

The Solution of Linear Systems **AX=B**

Objectives

- Iterative Methods for linear systems: Jacobi and Gauss-Seidel

Iterative Methods for linear systems: Jacobi Iteration

Having the following linear system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

After reordering the equations, we will have:

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} = g(x_2, x_3)$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} = g(x_1, x_3)$$

$$x_3 = \frac{b_3 - a_{31}x_1 + a_{32}x_2}{a_{33}} = g(x_1, x_2)$$

Jacobi Iteration

- On general:

$$x_{1(k+1)} = \frac{b_1 - a_{12}x_{2(k)} - a_{13}x_{3(k)}}{a_{11}} = g(x_{2(k)}, x_{3(k)})$$

$$x_{2(k+1)} = \frac{b_2 - a_{21}x_{1(k)} - a_{23}x_{3(k)}}{a_{22}} = g(x_{1(k)}, x_{3(k)})$$

$$x_{3(k+1)} = \frac{b_3 - a_{31}x_1(k) + a_{32}x_{2(k)}}{a_{33}} = g(x_{1(k)}, x_{2(k)})$$

Jacobi Iteration - Example

- Having the following linear system, and starting with: $\boldsymbol{(x_0 , y_0 , z_0)} = (\mathbf{1} , \mathbf{2} , \mathbf{2})$, use Jacobi Iteration to find the solution $\boldsymbol{P} = (\mathbf{2} , \mathbf{4} , \mathbf{3})$

$$4x - y + z = 7 \qquad x_{k+1} = \frac{7+y_k-z_k}{4}$$

$$4x - 8y + z = -21 \qquad y_{k+1} = \frac{-21-4x_k-z_k}{-8} = \frac{21+4x+z}{8}$$

$$-2x + y + 5z = 15 \qquad z_{k+1} = \frac{15+2x_k-y_k}{5}$$

Jacobi Iteration – Example (Cont'd)

$x_1 = \frac{7+2-2}{4} = 1.75$

$y_1 = \frac{21+4+2}{8} = 3.375$

$z_1 = \frac{15+2-2}{5} = 3.00$

- $p_1 = \ (1.75\ , 3.375\ , 3.00\ )$

$x_2 = 1.84375$

$y_2 = 3.875$

$z_2 = 3.025$

- $p_2 = \ (1.84375,\ \ 3.875\ , 3.025)$

.....

- $p_{19} = \ (2.000,\ \ 4.000\ , 3.000)$

Gauss-Seidel Iteration

- Same as Jacobi except:

$$x_{1(k+1)} = \frac{b_1 - a_{12}x_{2(k)} - a_{13}x_{3(k)}}{a_{11}} = g(x_{2(k)}, x_{3(k)})$$

$$x_{2(k+1)} = \frac{b_2 - a_{21}x_{1(k+1)} - a_{23}x_{3(k)}}{a_{22}} = g(x_{1(k+1)}, x_{3(k)})$$

$$x_{3(k+1)} = \frac{b_3 - a_{31}x_1(k+1) + a_{32}x_{2(k+1)}}{a_{33}} = g(x_{1(k+1)}, x_{2(k+1)})$$

Gauss-Seidel Iteration - Example

- Having the following linear system, and starting with: $(\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0}) = (\mathbf{1}, \mathbf{2}, \mathbf{2})$, use Jacobi Iteration to find the solution $\boldsymbol{P} = (\mathbf{2}, \mathbf{4}, \mathbf{3})$

$$4x - y + z = 7 \qquad x_{k+1} = \frac{7 + y_k - z_k}{4}$$

$$4x - 8y + z = -21 \qquad y_{k+1} = \frac{21 + 4x_{k+1} + z_k}{8}$$

$$-2x + y + 5z = 15 \qquad z_{k+1} = \frac{15 + 2x_{k+1} - y_{k+1}}{5}$$

Gauss-Seidel Iteration – Example (Cont'd)

$$x_1 = \frac{7+2-2}{4} = 1.75$$

$$y_1 = \frac{21+4(1.75)+2}{8} = 3.75$$

$$z_1 = \frac{15+2(1.75)-3.75}{5} = 2.95$$

- $p_1 = (1.75 , 3.75 , 2.95)$

$$x_2 = 1.95$$

$$y_2 = 3.96875$$

$$z_2 = 2.98625$$

- $p_2 = (1.95 , 3.96875 , 2.98625)$

.....

- $p_{10} = (2.000, \; 4.000 , 3.000)$

Convergence

- Sufficient condition for convergence (Jacobi and gauss-seidel)
- For $N \times N$ matrix
  - $|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{N} |a_{kj}|$ for $k = 1,2, \ldots \ldots , n$
- If A is 3×3 matrix: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
- $|a_{11}| > |a_{12}| + |a_{13}|$ $|a_{22}| > |a_{21}| + |a_{23}|$ $|a_{33}| > |a_{31}| + |a_{32}|$

Convergence

- Sufficient:

1) Convergence is guaranteed if the condition is satisfied.
2) Method may work even if condition is not satisfied.

- See example 3.27 on textbook (Jacobi doesn’t work)

References

- [1] Mathews J. H. and Fink K. D. (1999). Numerical Methods using MATLAB, NJ: Prentice Hall

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