Normal Forms Homework

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10.1. Pair each statement with its negation.

		a	b	c	d	e
p	q	$p \oplus q$	$\neg p \land q$	$p \Rightarrow (q \Rightarrow p)$	$p \Rightarrow q$	$p \land \neg q$
F	F	F	F	T	T	F
F	T	T	T	T	T	F
T	F	T	F	T	F	T
T	T	F	F	T	T	F

		f	g	h	i	j
p	q	$q \land (p \land \neg p)$	$p \lor \neg q$	$p \Leftrightarrow q$	$p \land (q \lor \neg q)$	$(p \Rightarrow q) \Rightarrow p$
F	F	F	T	T	F	F
F	T	F	F	F	F	F
T	F	F	T	F	T	T
T	T	F	T	T	T	T

Statement	Negation
a	h
b	g
c	f
d	e
e	d
f	c
g	b
h	a
i	¬j
j	¬i

10.3. Decide whether each formula is a tautology, satisfiable, or unsatisfiable:

10.3a: $(p \vee q) \vee (q \Rightarrow p)$

This is a tautology.

p	q	$p \vee q$	$q \Rightarrow p$	$(p \vee q) \vee (q \Rightarrow p)$
F	F	F	T	T
F	T	T	F	T
T	F	T	T	T
T	T	T	T	T

10.3b: $(p \Rightarrow q) \Rightarrow p$

This is satisfiable.

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \Rightarrow p$
F	F	T	F
F	T	T	F
T	F	F	T
T	T	T	T

10.3c: $p \Rightarrow (q \Rightarrow p)$

This is a tautology.

p	q	$q \Rightarrow p$	$p \Rightarrow (q \Rightarrow p)$
F	F	T	T
F	T	F	T
T	F	T	T
T	T	T	T

10.3d: $(\neg p \wedge q) \wedge (q \Rightarrow p)$

This is unsatisfiable.

p	q	$\neg p \wedge q$	$q \Rightarrow p$	$(\neg p \wedge q) \wedge (q \Rightarrow p)$
F	F	F	T	F
F	T	T	F	F
T	F	F	T	F
T	T	F	T	F

10.3e: $(p \Rightarrow q) \Rightarrow (\neg p \Rightarrow \neg q)$

This is satisfiable.

p	q	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$	$(p \Rightarrow q) \Rightarrow (\neg p \Rightarrow \neg q)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	T
T	T	T	T	T

10.3f: $(\neg p \Rightarrow \neg q) \Leftrightarrow (q \Rightarrow p)$

This is a tautology.

p	q	$\neg p \Rightarrow \neg q$	$q \Rightarrow p$	$(\neg p \Rightarrow \neg q) \Leftrightarrow (q \Rightarrow p)$
F	F	T	T	T
F	T	F	F	T
T	F	T	T	T
T	T	T	T	T

10.5.:

10.5a: Show that for any formulas α, β, γ, $(\alpha \wedge \beta) \vee \alpha \vee \gamma \equiv \alpha \vee \gamma$.

α	β	γ	$\alpha \wedge \beta$	$\alpha \vee \gamma$	$(\alpha \wedge \beta) \vee \alpha \vee \gamma$
F	F	F	F	F	F
F	F	T	F	T	T
F	T	F	F	F	F
F	T	T	F	T	T
T	F	F	F	T	T
T	F	T	F	T	T
T	T	F	T	T	T
T	T	T	T	T	T

10.5b: Give the corresponding rule for simplifying $(\alpha \vee \beta) \wedge \alpha \wedge \gamma$.

$$(\alpha \vee \beta) \wedge \alpha \wedge \gamma \equiv \alpha \wedge \gamma$$

α	β	γ	$\alpha \vee \beta$	$\alpha \wedge \gamma$	$(\alpha \vee \beta) \wedge \alpha \wedge \gamma$
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	T	F	F
F	T	T	T	F	F
T	F	F	T	F	F
T	F	T	T	T	T
T	T	F	T	F	F
T	T	T	T	T	T

10.5c: Find the simplest possible disjunctive and conjunctive normal forms of the formula $(p \wedge q) \Rightarrow (p \oplus q)$.

p	q	$p \wedge q$	$p \oplus q$	$(p \wedge q) \Rightarrow (p \oplus q)$
F	F	F	F	T
F	T	F	T	T
T	F	F	T	T
T	T	T	F	F

Disjunctive normal form: $\neg p \vee \neg q$

Conjunctive normal form: $\neg p \vee \neg q \wedge \neg p \vee \neg q$

10.7:

10.7a: Write out a conjunctive normal form of the formula in the case $n = 3$.

$$(p_1 \wedge q_1) \vee (p_2 \wedge q_2) \vee (p_3 \wedge q_3)$$

Using the distributive law, this formula will be turned into one of 2^3 clauses.

	p_1	q_1
p_2	p_1, p_2	q_1, p_2
q_2	p_1, q_2	q_1, q_2

	p_1, p_2	q_1, p_2	p_1, q_2	q_1, q_2
p_3	p_1, p_2, p_3	q_1, p_2, p_3	p_1, q_2, p_3	q_1, q_2, p_3
q_3	p_1, p_2, q_3	q_1, p_2, q_3	p_1, q_2, q_3	q_1, q_2, q_3

$$(p_1 \vee p_2 \vee p_3) \wedge (q_1 \vee p_2 \vee p_3) \wedge (p_1 \vee q_2 \vee p_3) \wedge (q_1 \vee q_2 \vee p_3) \wedge (p_1 \vee p_2 \vee q_3) \wedge$$
$$(q_1 \vee p_2 \vee q_3) \wedge (p_1 \vee q_2 \vee q_3) \wedge (q_1 \vee q_2 \vee q_3)$$

10.7b: How long is the CNF? For general n, how long is the CNF as a function of n?

The CNF has a length of 47.

n	$\ldots \vee p_n \wedge q_n$ length	CNF length
1	3	3
2	7	15
3	11	47

The length of the CNF equations as a function of n can be described as the function $f(n) = 2^n * (2n - 1) + (2^n - 1)$, where

- 2^n represents the number of clauses

- $2n - 1$ represents the number of characters in each clause
- $2^n - 1$ represents the number of operators between each clause

This can then be simplified into:

$$f(x) = 2^n * (2n - 1) + (2^n - 1)$$
$$f(x) = 2^n * (2n) - 1$$
$$f(x) = 2^{n+1} * n - 1$$

10.7c: Show that putting a formula into disjunctive normal form may increase its length exponentially.

Given a similar formula of

$$(p_1 \vee q_1) \wedge \ldots \wedge (p_n \vee q_n),$$

then distributing it into DNF will yield an equation of the same length as the CNF of 10.7a and 10.7b. For example, using

$$(p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge (p_3 \vee q_3),$$

	p_1	q_1
p_2	p_1, p_2	q_1, p_2
q_2	p_1, q_2	q_1, q_2

	p_1, p_2	q_1, p_2	p_1, q_2	q_1, q_2
p_3	p_1, p_2, p_3	q_1, p_2, p_3	p_1, q_2, p_3	q_1, q_2, p_3
q_3	p_1, p_2, q_3	q_1, p_2, q_3	p_1, q_2, q_3	q_1, q_2, q_3

$$(p_1 \wedge p_2 \wedge p_3) \vee (q_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge q_2 \wedge p_3) \vee (q_1 \wedge q_2 \wedge p_3) \vee (p_1 \wedge p_2 \wedge q_3) \vee$$
$$(q_1 \wedge p_2 \wedge q_3) \vee (p_1 \wedge q_2 \wedge q_3) \vee (q_1 \wedge q_2 \wedge q_3)$$

has the same length as the CNF of the same n.

10.7d: Why is the algorithm mentioned in 10.7d exponentially costly?

Using the formula from 10.7c where $n = 2$, the equation and truth table are:

$$(p_1 \vee q_1) \wedge (p_2 \vee q_2)$$

p_1	q_1	p_2	q_2	$p_1 \lor q_1$	$p_2 \lor q_2$	$(p_1 \lor q_1) \land (p_2 \lor q_2)$
F	F	F	F	F	F	F
F	F	F	T	F	T	F
F	F	T	F	F	T	F
F	F	T	T	F	T	F
F	T	F	F	T	F	F
F	T	F	T	T	T	T
F	T	T	F	T	T	T
F	T	T	T	T	T	T
T	F	F	F	T	F	F
T	F	F	T	T	T	T
T	F	T	F	T	T	T
T	F	T	T	T	T	T
T	T	F	F	T	F	F
T	T	F	T	T	T	T
T	T	T	F	T	T	T
T	T	T	T	T	T	T

Distributing this formula results in the DNF form and a truth table of:

$$(p_1 \land p_2) \lor (q_1 \land p_2) \lor (p_1 \land q_2) \lor (q_1 \land q_2)$$

p_1	q_1	p_2	q_2	$p_1 \land p_2$	$q_1 \land p_2$	$p_1 \land q_2$	$q_1 \land q_2$	DNF form
F	F	F	F	F	F	F	F	F
F	F	F	T	F	F	F	F	F
F	F	T	F	F	F	F	F	F
F	F	T	T	F	F	F	F	F
F	T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	T	T
F	T	T	F	F	T	F	F	T
F	T	T	T	F	T	F	T	T
T	F	F	F	F	F	F	F	F
T	F	F	T	F	F	T	F	T
T	F	T	F	T	F	F	F	T
T	F	T	T	T	F	T	F	T
T	T	F	F	F	F	F	F	F
T	T	F	T	F	F	T	T	T
T	T	T	F	T	T	F	F	T
T	T	T	T	T	T	T	T	T

The issue with using this algorithm of checking each clause for contradictions is that when the formula is converted into DNF, the equation now has 2^n clauses to check, meaning that as the algorithm takes in larger formulas, it will take much more time than needed to determine if they are satisfiable.

10.9