Sets Homework

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5.1. What are these sets?:

5.1a: $\{\{2,4,6\} \cup \{6,4\}\} \cap \{4,6,8\}$

$$\begin{aligned}
&\{\{2,4,6\} \cup \{6,4\}\} \cap \{4,6,8\} \\
&\{2,4,6\} \cap \{4,6,8\} \\
&\{4,6\}
\end{aligned}$$

5.1b: $\mathcal{P}(\{7,8,9\}) - \mathcal{P}(\{7,9\})$

$$\begin{aligned}
&\mathcal{P}(\{7,8,9\}) - \mathcal{P}(\{7,9\}) \\
&\{\phi, \{7\}, \{8\}, \{9\}, \{7,8\}, \{8,9\}, \{7,9\}, \{7,8,9\} - \{\phi, \{7\}, \{9\}, \{7,9\}\} \\
&\{\{8\}, \{7,8\}, \{8,9\}, \{7,8,9\}\}
\end{aligned}$$

5.1c: $\mathcal{P}(\phi)$

$$\begin{aligned}
&\mathcal{P}(\phi) \\
&\{\phi\}
\end{aligned}$$

5.1d: $\{1,3,5\} \times \{0\}$

$$\begin{aligned}
&\{1,3,5\} \times \{0\} \\
&\{(1,0), (3,0), (5,0)\}
\end{aligned}$$

5.1e: $\{2,4,6\} \times \phi$

$$\begin{aligned}
&\{2,4,6\} \times \phi \\
&\phi
\end{aligned}$$

5.1f:

$$\begin{gathered}\mathcal{P}(\{0\}) \times \mathcal{P}(\{1\}) \\ \{\phi, \{0\}\} \times \{\phi, \{1\}\} \\ \{(\phi, \phi), (\phi, \{1\}), (\{0\}, \phi), (\{0\}, \{1\})\}\end{gathered}$$

5.1g:

$$\begin{aligned}\mathcal{P}(\mathcal{P}(\{2\})) \\ \mathcal{P}(\{\phi, \{2\}\}) \\ \{\phi, \{\phi\}, \{\{2\}\}, \{\phi, \{2\}\}\}\end{aligned}$$

5.3.: Show that if A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

Base case—ϕ:

$$\begin{aligned}\mathcal{P}(\phi) &= \{\phi\} \\ |\mathcal{P}(\phi)| &= 2^0\end{aligned}$$

Induction hypothesis—Suppose that for any $|A| \geq 0$, $|\mathcal{P}(A)| = 2^{|A|}$.

Induction step—$|A'| = |A| + 1$:

A' contains all the elements in A, except for one additional member. Therefore for every member of $\mathcal{P}(A)$, there are $2^{|A|}$ distinct combinations of members that include this new element. As a result, the cardinality of the powerset of A' is:

$$\begin{aligned}|\mathcal{P}(A')| &= 2^{|A|} + 2^{|A|} \\ |\mathcal{P}(A')| &= 2^1 * 2^{|A|} \\ |\mathcal{P}(A')| &= 2^{|A|+1} \\ |\mathcal{P}(A')| &= 2^{|A'|}\end{aligned}$$

5.5. Suppose:

5.5a: Compare $|\mathcal{P}(A \times B)|$ and $|\mathcal{P}(A)| * |\mathcal{P}(B)|$. Under what circumstances is one larger than the other, and what is their ratio?

Only when $|A|, |B| = 0$ or $|A|, |B| = 2$ will $|\mathcal{P}(A \times B)| = |\mathcal{P}(A)| * |\mathcal{P}(B)|$. Otherwise:

- If both have more than 2 members each, $|\mathcal{P}(A \times B)|$ will be higher.
- If one set has at most 1 member, $|\mathcal{P}(A)| * |\mathcal{P}(B)|$ will be higher.

5.5b: Is it inevitably true that $(A - B) \cap (B - A) = \phi$?

Yes. Since $A - B$ is A without any elements also in B, and $B - A$ is the inverse, no elements exist within both of these sets.

5.7. Decide whether each statement is true or false and why:

5.7a: $\phi = \{\phi\}$

False. ϕ has a cardinality of 0 while $\{\phi\}$ has a cardinality of 1.

5.7b: $\phi = \{0\}$

False. ϕ has a cardinality of 0 while $\{0\}$ has a carindality of 1.

5.7c: $|\phi| = 0$

True. ϕ is an empty set, meaning that it has a cardinality of 0.

5.7d: $|\mathcal{P}(\phi)| = 0$

False. $\mathcal{P}(\phi)$ results in $\{\phi\}$, a set of 1 member.

5.7e: $\phi \in \{\}$

True. ϕ and $\{\}$ have the same cardinality, so, being equal, ϕ can be a subset of $\{\}$.

5.7f: $\phi = \{x \in \mathbb{N} : x \leq 0$ and $x > 0\}$

False. This set contains all natural numbers, while ϕ contains no members.

5.9. Prove the following:

5.9a: $A \cap (A \cup B) = A$

$$\begin{aligned}
A \cap (A \cup B) &= A \\
(A \cap A) \cup (A \cap B) &= A \\
A \cup (A \cap B) &= A
\end{aligned}$$

$A \cap B$ is a subset of A, meaning that its union with A is equal to the set itself.

5.9b: $A - (B \cap C) = (A - B) \cup (A - C)$

In $A - (B \cap C)$, the elements in B and C are removed from A, resulting in a set that has all elements of A except for the ones in common with *both* B and C.

In $(A-B) \cup (A-C)$, a union is made between two sets: A, without any elements in common with B, and A, without any in common with C. In both sets, the intersection of B and C is excluded from A. As a result, the union is a set that has all element of A except for the ones in common with *both* B and C.

5.11.: Defining $\langle x, y \rangle$ to be $\{x, \{x, y\}\}$, prove that $\langle x, y \rangle = \langle u, v \rangle$ iff $x = u$ and $y = v$.

$$\begin{aligned} \langle x, y \rangle &= \langle u, v \rangle \\ \{x, \{x, y\}\} &= \langle u, v \rangle \\ \{x, \{x, y\}\} &= \{u, \{u, v\}\} \\ \{x, \{x, y\}\} &= \{x, \{x, y\}\} \end{aligned}$$