# Countable and Uncountable Sets Homework

Mohibullah Meer

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## 7.1. Prove that there are infinitely many different sizes of infinite sets:

Assume for contradiction that all infinite sets are the same size. Thus, given $\mathbb{N}$ and $\mathbb{R}$, there should be a bijection from one to the other. In this example, $\mathbb{N} \to \mathbb{R}$ is used.

Without exactly using the diagonalization argument, the only acceptable association between the sets is at $0 \to 0$. After this, an assocation between $1 \to 1$ fails to account for the reals between 0 and 1. This issue continues to exist, regardless of how small the value becomes:

- $1 \to 0.1$ — fails to account for reals between 0 and 0.1
- $1 \to 0.01$ — fails to account for reals between 0 and 0.01
- $1 \to 0.001$ — fails to account for reals between 0 and 0.001
- ...

As a result of this, there is no bijection that accounts for every real number, and associates it with a single natural number. The assumption stated that $\mathbb{N}$ and $\mathbb{R}$ are the same size, however—they need to share some bijection to be the same size. As a result of this contradiction, there exist different sizes of infinite sets.

## 7.3. What is wrong with Johnny's argument?:

Using the diagonalization process on the list $T_0, T_1, \ldots$ will still yield a set unaccounted for in the new list. As a result, no matter how many times the indices are shifted to accomodate another set, there will be one unaccounted for in the function.

## 7.5. Which of the following are possible?:

### 7.5a: The set difference of two uncountable sets is countable.

This is possible.

$$A = \{x \in \mathbb{R} : x^2 \neq 2x\}$$
$$\mathbb{R} - A = \{0, 2\}$$

### 7.5b: The set difference of two countably infinite sets is countably infinite.

This is possible.

$$A = \mathbb{Z} - \mathbb{N}$$
$$A = \{x \in \mathbb{Z} : x < 0\}$$

A bijection $f : \mathbb{N} \to A$ can be represented as $f(x) = -(x + 1)$, meaning that $A$ is countably infinite.

### 7.5c: The power set of a countable set is countable.

This is possible when the set's cardinality is less than $\aleph_0$, beyond which the power set is uncountable.

### 7.5d: The union of a collection of finite sets is countably infinite.

This is possible if the *number* of finite sets is countably infinite. Suppose a countably infinite union of finite sets, the elements of which all exist in $\mathbb{N}$. If all these sets are distinct, the union would result in $\mathbb{N}$, a countably infinite set.

### 7.5e: The union of a collection of finite sets is uncountable.

This also follows the same nature as 7.5d, meaning that—given a union of uncountable cardinality of distinct finite sets, containing elements in $\mathbb{R}$—an uncountable set would be the result (in this case being $\mathbb{R}$).

### 7.5f: The intersection of two uncountable sets is empty.

This is possible.

$$N = \{x \in \mathbb{R} : x < 0\}$$
$$P = \{x \in \mathbb{R} : x > 0\}$$
$$N \cap P = \phi$$

## 7.7.:

### 7.7a: Show that there are as many ordered pairs of reals between 0 and 1 as there are reals in that interval.

Theorem 7.2's truth indicates that for any infinite set $A$, $|A * A| = |A|$. Since $\mathbb{N}$ cannot map to either $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$ or $\{x \in \mathbb{R} : 0 \leq x \leq 1\}^2$, they are the same size.

### 7.7b: Extend the result of part (a) to give a bijection between pairs of nonnegative real numbers and nonnegative real numbers.

## 7.9. State whether each set is finite, countably infinite, or uncountable:

### 7.9a: The set of all books.

Each character can be represented as a natural number, meaning that a book can be represented as a sequence of natural numbers. As a result, Since there are an infinite number of combinations that can be mapped to $\mathbb{N}$, this set is countably infinite.

### 7.9b: The set of all books of less than 500,000 symbols.

There is a finite number of combinations of symbols given a limit of 500,000. Thus, this set is finite.

### 7.9c: The set of all finite sets of books.

Similarly to 7.9a, each finite set can be represented by a set of bits (in which 0 or 1 indicates whether or not a book is in the set), enumerated by $\mathbb{N}$. Therefore, this set is countably infinite.

### 7.9d: The set of all irrational numbers greater than 0 and less than 1.

Given a set $A = \{x \in \mathbb{I} : 0 < x < 1\}$, if one tries to make a bijection from $\mathbb{N}$ to $A$ to prove its countability as an infinite set, they would encounter a similar problem to the one in 7.1:

- $0 \to \sqrt{0.1}$ — fails to account for irrationals between 0 and $\sqrt{0.1}$
- $0 \to \sqrt{0.001}$ — fails to account for irrationals between 0 and $\sqrt{0.001}$
- $0 \to \sqrt{0.00001}$ — fails to account for irrationals between 0 and $\sqrt{0.00001}$

- ...

Since there exists no bijection from $\mathbb{N}$ to $A$, it is not countable and thus the set of all irrational numbers greater than 0 and less than 1 is uncountable.

### 7.9e: The set of all sets of numbers that are divisible by 17.

The set of all multiples of 17 is countably infinite, by the function from $\mathbb{Z} \to \{x \in \mathbb{Z} : 17|x\}$, $f(x) = 17x$. All non-empty sets have more subsets than members, meaning that the set of all sets of numbers divisible by 17 is uncountable.

### 7.9f: The set of all sets of even prime numbers.

The only even prime number is 2, so the set of all sets of even prime numbers is countable.

### 7.9g: The set of all sets of powers of 2.

All powers of 2 can be defined by the function from $\mathbb{N} \to \{x : \sqrt{x} \in \mathbb{Z}\}$, $f(x) = x^2$, meaning that the set of powers of 2 is countably infinite. Since non-empty sets have more subsets than members, the set of all sets of powers of 2 is uncountable.

### 7.9h: The set of all functions from $\mathbb{Q}$ to $\{0, 1\}$.

## 7.11. Prove Theorem 7.3:

The union of sets includes only the elements in its unified sets. The individual sets all:

- do not contain a number of elements that makes them finite
- do not contain a number of elements that makes them uncountable
- contain a number of elements that makes them countably infinite

As a result, the union of countably many countably infinite sets does not contain a number of element that makes them finite or uncountable—it must be countably infinite.