Chapter 2 Homework

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2.1: Is -1 an odd integer, as we have defined the term? Why or why not?

An odd integer is represented by $2k+1$, where $k \in \mathbb{Z}$. If we assume -1 is an odd integer:

$$\begin{aligned}
-1 &= 2k+1 \\
-2 &= 2k \\
-1 &= k
\end{aligned}$$

Then $-1 \in \mathbb{Z}$ proves that -1 is an odd integer.

2.3: Prove that the product of two odd numbers is an odd number.

Assuming for contradiction that given odd integers a and b, ab is even:

$$\begin{aligned}
j, k, l &\in \mathbb{Z} \\
a &= 2k+1 \\
b &= 2j+1 \\
ab &= 2l \\
(2k+1)(2j+1) &= 2l \\
4jk+2j+2k+1 &= 2l \\
2jk+j+k+0.5 &= l
\end{aligned}$$

The 0.5 in the formula means that $l \notin \mathbb{Z}$, a contradiction. Thus, the product of two odd numbers must always be an odd number.

2.5: Prove that $\sqrt[3]{2}$ is irrational.

Assume for contradiction: $\sqrt[3]{2}$ is rational.

All rational numbers can be represented by the fully reduced fraction $\frac{a}{b}$, where at most one of the integers a or b can be even.

$$\sqrt[3]{2} = \frac{a}{b}$$
$$\sqrt[3]{2}b = a$$
$$2b^3 = a^3$$

At this point, it can be deduced that $2b^3$, being a multiple of 2, is always even. As a result, since raising an integer to any whole and positive power results in an integer of the same parity, then $2b^3 = a^3$ also proves that a is even and can be represented by $2k$, where $k \in \mathbb{Z}$.

$$2k = a$$
$$2b^3 = (2k)^3$$
$$2b^3 = 8k^3$$
$$b^3 = 4k^3$$

Similarly with $2b^3$, it can be assumed that $4k^3$, a multiple of 4, is also always even. Considering the same rule about integers and their parity when raised to a whole and positive power, the problem presents itself when $b^3 = 4k^3$ proves b to be even. a and b both being even contradicts the assumption that $\frac{a}{b}$, representing $\sqrt[3]{2}$, is irreducible. Due to this contradiction, $\sqrt[3]{2}$ must be an irrational number.

2.7: Show that there is a seven-sided die; that is, a polyhedron with seven sides that is equally likely to fall on any one of its faces.

The distribution of probabilities on a die is most influenced by how much surface area each face has. Given a rod of equilateral pentagonal ends, there is a ratio between the pentagon's side length and the rod's length where:

- If the side length is too long, the 2 faces of the pentagon will be larger
- If the rod is too long, the 5 sides of the rod will be larger

This indicates that there is a point where the surface area (and probability) of the die will be evenly distributed between the 2 faces of the pentagon and the 5 faces of the rod.

Solved with help from: `https://rpg.stackexchange.com/a/123787`

2.9. Prove or provide a counterexample:

2.9a: If c and d are perfect squares, then cd is a perfect square.

True. If c and d are perfect squares, then, respectively, the factors that make them up are $\sqrt{c}$ and $\sqrt{d}$. To ensure a lack of redundancy, though:

$$\begin{aligned} a &= \sqrt{c}, a \in \mathbb{Z} \\ b &= \sqrt{d}, b \in \mathbb{Z} \\ cd &= a^2 * b^2 \\ cd &= (aa)(bb) \\ cd &= (ab)(ab) \\ cd &= (ab)^2 \\ \sqrt{cd} &= ab \end{aligned}$$

As $\sqrt{cd}$ is ab, whose factors are integers, cd must be a perfect square.

2.9b: If cd is a perfect square and $c \neq d$, then c and d are perfect squares.

False. The square roots of all perfect squares are integers, so given:

$$\begin{aligned} cd &= 36 \\ \sqrt{cd} &= \sqrt{36} = 6 \Rightarrow \sqrt{cd} \in \mathbb{Z} \\ c &= 3 \\ d &= 12 \\ 3 * 12 &= 36 = cd \\ \sqrt{c} &= \sqrt{3} \approx 1.73 \Rightarrow \sqrt{c} \notin \mathbb{Z} \\ \sqrt{d} &= \sqrt{12} \approx 3.46 \Rightarrow \sqrt{d} \notin \mathbb{Z} \end{aligned}$$

As $\sqrt{c}, \sqrt{d} \notin \mathbb{Z}$, c and d are not always perfect squares.

2.9c: If c and d are perfect squares such that $c > d$, and $x^2 = c$ and $y^2 = d$, then $x > y$.

False. All positive numbers have one positive and one negative square root, so given:

$$
\begin{aligned}
c &= 25 \\
d &= 16 \\
x^2 &= 25 \\
y^2 &= 16 \\
x &= \sqrt{25} = \pm 5 \\
y &= \sqrt{16} = \pm 4
\end{aligned}
$$

Then even if $c > d$, the value of x can be -5 while y can be 4, meaning that $x < y$.

2.11: Critique the following "proof":

$$
\begin{aligned}
x &> y \\
x^2 &> y^2 \\
x^2 - y^2 &> 0 \\
(x+y)(x-y) &> 0 \\
x + y &> 0 \\
x &> -y
\end{aligned}
$$

2.11.1!: Line 2

$$
\begin{aligned}
x > y &\not\Rightarrow x^2 > y^2 \\
x &= 2 \\
y &= -4 \\
x &= 2 > -4 = y \\
x^2 &= 4 < 16 = y^2
\end{aligned}
$$

2.11.2!: Line 3

$$
\begin{aligned}
x > y &\not\Rightarrow x^2 - y^2 > 0 \\
x &= 3 \\
y &= -3 \\
x &= 3 > -3 = y \\
x^2 - y^2 &= 9 - 9 = 0
\end{aligned}
$$

2.11.3!: Line 4

$$
\begin{aligned}
x > y &\not\Rightarrow (x+y)(x-y) > 0 \\
x &= 5 \\
y &= -5 \\
x = 5 &> -5 = y \\
(x+y)(x-y) &= (5-5)(5+5) = 0
\end{aligned}
$$

2.11.4!: Line 5

$$
\begin{aligned}
x > y &\not\Rightarrow x + y > 0 \\
x &= 7 \\
y &= -8 \\
x = 7 &> -8 = y \\
x + y &= 7 - 8 < 0
\end{aligned}
$$

2.11.5!: Line 6

$$
\begin{aligned}
x > y &\not\Rightarrow x > -y \\
x &= 1 \\
y &= -2 \\
x = 1 &> -2 = y \\
x = 1 &< 2 = -y
\end{aligned}
$$

2.13: Write the following statements in terms of quantifiers and implications:

2.13a: Every positive real number has two distinct square roots.

$$\forall\, n \in \mathbb{R}, n > 0 \Rightarrow \exists\, \sqrt{n}, -\sqrt{n} : \sqrt{n} \neq -\sqrt{n}$$

2.13b: Every positive even number can be expressed as the sum of two prime numbers.

$$\forall\, k \in \mathbb{N}, k \neq 0 \Rightarrow \exists\, p_1, p_2 \in \mathbb{P} : 2k = p_1 + p_2$$

2.15: Using concepts developed in Chapter 1, explain that one "of the remaining 5 people, there must be at least 3 whom X knows, or at least 3 whom X does not know."

In Chapter 1, the Extended Pigeonhole Principle is introduced as a formula that can be used to find the "lowest maximum" amount of members with the same mapping in a pigeonhole-ish situation.

In order to prove that the lowest maximum is 3, assume a set of people P and of relationships R:

$$
\begin{aligned}
P &= \{1, \ldots, 6\} \\
R &= \{1, 2\} \\
\left\lceil \frac{|P|}{|R|} \right\rceil &= k + 1 \\
\left\lceil \frac{6}{2} \right\rceil &= k + 1 \\
\lceil 3 \rceil &= k + 1 \\
3 &= k + 1
\end{aligned}
$$

Then, in the most evenly distributed case, the lowest maximum must be 3 people that X knows or does not know.