Functions Homework

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October 26, 2023

6.1. Let f and the inverse relation f^{-1} be a function. Is f^{-1} a bijection?:

As long as f is a bijection, f^{-1} is a bijection. If:

- f is injective, but not surjective, its domain is smaller than its codomain and vice versa for f^{-1}, meaning that f^{-1} will lack a mapping for at least 1 of its domain values, invalidating it as a function.
- f is surjective, but not injective, its domain is larger than its codomain and vice versa for f^{-1}, meaning that at least 1 of the domain values in f^{-1} has more than 1 mapping, invalidating it as a function.
- f is neither surjective nor injective, the above cases apply.
- f is surjective and injective, it is bijective. Both f and f^{-1} have a domain and codomain of the same size. Since every domain value has a mapping and every codomain value has a single association in both functions, f^{-1} is bijective.

6.3.:

6.3a: Show that if two finite sets A and B are the same size, and r is an injective function from A to B, then r is a bijection.

Since r is injective, for any $a \in A$, if two $r(a)$ are the same, then they are associated with the same a.

Let I be the image of A under r. I is then a subset of B, the codomain. If

$I = B$, then all codomain values have an association, and r is surjective.

$$\begin{aligned}
I &= \{r(a_1), r(a_2), \ldots, r(a_{|A|})\} \\
|I| &= |B| \\
B - I &= \phi \\
I &= B
\end{aligned}$$

Thus, r is both injective and surjective.

6.3b: Give a counterexample showing that the conclusion of (a) does not necessarily hold if A and B are two bijectively related infinite sets.

If the cardinalities of A and B are different types of infinity, this conclusion can be countered. For example, if $|A| = \infty_{\mathbb{N}}$ and $|B| = \infty_{\mathbb{R}}$, the injection will not be able to associate every codomain value with a domain value. Thus, r would not be a surjection.

6.5. Suppose $f : A \to B$, $g : C \to D$, and $A \subseteq D$. Explain when $(f \circ g)^{-1}$ exists as a function from a subset of B to C:

$$\begin{aligned}
f &: A \to B \\
g &: C \to D \\
f \circ g &: C \to D, A \to B \\
f \circ g &= f[g(c)]
\end{aligned}$$

$f \circ g$ is constrained by whether both D and A contain the element mapped from C.

$$\begin{aligned}
f^{-1} &: B \to A \\
g^{-1} &: D \to C \\
(f \circ g)^{-1} &: B \to A, D \to C \\
(f \circ g)^{-1} &= g^{-1}[f^{-1}(b)]
\end{aligned}$$

In contrast, $(f \circ g)^{-1}$ does not have any constraints, since $A \subseteq D$ means that any element mapped from B to A is in D.

6.7. Given $f(n) = 2n$, $g(n) = 2n + 1$, and the function h from Theorem 6.4, what are f^{-1}, g^{-1}, and h^{-1}?:

f^{-1} is a bijection from the even integers to $\mathbb{Z}$:

$$\begin{aligned}
f(n) &= 2n \\
n &= 2f^{-1}(n) \\
\frac{n}{2} &= f^{-1}(n) \\
f^{-1}(n) &= \frac{n}{2}
\end{aligned}$$

g^{-1} is a bijection from the odd integers to $\mathbb{Z}$:

$$\begin{aligned}
g(n) &= 2n + 1 \\
n &= 2g^{-1}(n) + 1 \\
n - 1 &= 2g^{-1}(n) \\
\frac{n-1}{2} &= g^{-1}(n) \\
g^{-1}(n) &= \frac{n-1}{2}
\end{aligned}$$

h^{-1} is a bijection from $C \to B$, where A, B, and C are sets:

$$\begin{aligned}
f &: A \to B \\
f^{-1} &: B \to A \\
g &: A \to C \\
g^{-1} &: C \to A \\
h &: B \to A \to C \\
h &= g \circ f^{-1} \\
h^{-1} &: C \to A \to B \\
h^{-1} &= f \circ g^{-1}
\end{aligned}$$