MA1512 Cheat Sheet

Separable Equations

$$M(x)\,dx = N(y)\,dy \tag{1}$$

Homogeneous Functions

If $y' = f(x,y)$ and $f(tx,ty) = t^n f(x,y)$ for some n, then

$$\frac{dz}{f(1,z)-z} = \frac{dx}{x} \tag{2}$$

where $z = \frac{y}{x}$.

Linear Change of Variable

If $y' = f(ax+by+c)$, then set $u = ax+by+c$ and solve the resulting equation.

Exact Equations

If $M(x,y)\,dx + N(x,y)\,dy = 0$ and $M_y = N_x$ (by the Mixed Derivatives Theorem), then let $f_x = M(x,y)$ and $f_y = N(x,y)$ and solve for $f(x,y)$.

Or, to make the equation exact, let $g(x) = \frac{M_y - N_x}{N}$ and multiply by the integrating factor $R(x) = e^{\int g(x)\,dx}$.

Linear First Order ODE

$$y' + P(x)y = Q(x) \tag{3}$$

Multiply both sides of the equation by the integrating factor R:

$$R(x) = e^{\int P\,dx} \tag{4}$$

We now have

$$(Ry)' = RQ \tag{5}$$

Reduction of Order

If $f(x, y', y'') = 0$, set $y' = p$ and $y'' = p'$.
If $f(y, y', y'') = 0$, set $y' = p$ and $y'' = pp'$.

Bernoulli's Equation

If an equation has the form

$$y' + p(x)y = q(x)y^n \tag{6}$$

divide by y^n and let $z = y^{1-n}$. You'll get

$$z' + (1-n)p(x)z = (1-n)q(x) \tag{7}$$

Homogeneous ODE with Constant Coefficients

$$y'' + py' + qy = 0 \tag{8}$$

Characteristic equation: $r^2 + pr + q = 0$

Case 1: Real and distinct roots $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: Distinct complex roots If solution is $a \pm ib$, then $y = e^{ax}(c_1 cos(bx) + c_2 sin(bx))$

Case 3: Equal real roots $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Method of Undetermined Coefficients

The solution to the equation

$$y'' + py' + qy = R(x) \tag{9}$$

is of the form

$$y = y_g + y_p \tag{10}$$

where y_g is the general solution found by letting $R(x)$ to be 0 and y_p is the particular solution that is to be determined.

Case 1: $R(x) = P(x)e^{kx}$. Substitute $y_p = u(x)e^{kx}$.

Case 2: $R(x)$ is a trigonometric function with angular frequency b. Substitute $y_p = u(x)e^{a+ib}$ and take the real or imaginary component of the resultant solution based on whether $R(x)$ has sin or cos. Alternatively, let $y_p = u(x)(A\sin bx + B\cos bx)$

Case 3: $R(x)$ is a polynomial. Substitute $y_p = A_0 + A_1 x + \cdots A_n x^n$, $x(A_0 + A_1 x + \cdots A_n x^n)$, etc where the degree of the polynomial is the degree of $R(x)$.

Method of Variation of Parameters

To solve

$$y'' + p(x)y' + q(x)y = r(x) \tag{11}$$

Let $W(y_1, y_2) = y_1 y_2' - y_1' y_2$.

$$u = -\int \frac{y_2 r(x)}{W(y_1,y_2)}\,dx \tag{12}$$

$$v = \int \frac{y_1 r(x)}{W(y_1,y_2)}\,dx \tag{13}$$

The solution is $y_p = uy_1 + vy_2$.

Determining One Solution from Another

If y_1 is a known solution to a homogeneous second-order differential equation, then $y_2 = vy_1$ where

$$v = \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \tag{14}$$

Superposition

If y_1 is the solution to the equation

$$y'' + p(x)y' + q(x)y = g(x)$$

and y_2 is the solution to the equation

$$y'' + p(x)y' + q(x)y = h(x)$$

then for all constants C_1 and C_2, the function $y = C_1 y_1 + C_2 y_2$ is a solution to the equation

$$y'' + p(x)y' + q(x)y = g(x) + h(x)$$

General Laplace Transforms

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$1 \to \frac{1}{s} \qquad t^n \to \frac{n!}{s^{n+1}}$$

$$\sqrt{t} \to \frac{\pi}{2s^{3/2}} \qquad \cos at \to \frac{s}{s^2+a^2}$$

$$\sin at \to \frac{a}{s^2+a^2} \qquad \cosh at \to \frac{s}{s^2-a^2}$$

$$\sinh at \to \frac{a}{s^2-a^2} \qquad t\sin at \to \frac{2as}{s^2+a^2}$$

$$\sin(at+b) \to \frac{s\sin b + a\cos b}{s^2+a^2} \qquad \cos(at+b) \to \frac{s\cos b - a\sin b}{s^2+a^2}$$

$$f(ct) \to \frac{1}{c}L(f(s-c)) \qquad u(t-c) \to \frac{e^{-cs}}{s}$$

Transforms of Derivatives and Integrals

$$\begin{aligned} L(f') &\to sL(f) - f(0) \\ L(f'') &\to sL(f') - f'(0) \\ &\to s^2 L(f) - sf(0) - f'(0) \\ L\left(\int_0^t f(\tau)\,d\tau\right) &\to \frac{1}{s}L(f) \end{aligned}$$

s-Shifting

If $L(f) = F(s)$,

$$L(e^{ct} f(t)) = F(s-c)$$

t-Shifting

If $L(f) = F(s)$,

$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

Dirac Delta

$$\begin{aligned} \delta(t) &\to 1 \\ \delta(t-a) &\to e^{-as} \\ \int_0^\infty \delta(t)\,dt &= 1 \end{aligned}$$

Malthus Model

$$\begin{aligned} \frac{dN}{dt} &= BN - DN = kN \\ N(t) &= \hat{N}e^{kt} \end{aligned}$$

Logistic Model

$$\begin{aligned}\frac{dN}{dt} &= BN - DN \\ &= BN - (sN)N \\ &= BN - sN^2\end{aligned}$$

$$N_\infty = \frac{B}{s}$$

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}} \qquad (\hat{N} < N_\infty)$$

$$N(t) = \frac{N_\infty}{1 - \left(1 - \frac{N_\infty}{\hat{N}}\right) e^{-Bt}} \qquad (\hat{N} > N_\infty)$$

$$N(t) = N_\infty \qquad (\hat{N} = N_\infty)$$

Harvesting Model

$$\frac{dN}{dt} = (B - sN)N - E$$

The quadratic curve has no solution when $E > \frac{B^2}{4s}$. This means the derivative will always be negative and population would dwindle to zero.
The quadratic curve has one solution when $E = \frac{B^2}{4s}$. This means there is one unstable equilibrium at $\frac{B}{2s}$.
In the last case, there is a stable and unstable equilibrium at two roots of equation when the derivative is zero.

Wave Equations

$$c^2 y_{xx} = y_{tt} \qquad y(t,0) = 0$$
$$y(t,\pi) = 0 \qquad y(0,x) = f(x)$$
$$y_t(0,x) = 0 \qquad y(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

Heat Equations

$$u_t = c^2 u_{xx} \qquad u(0,t) = 0$$
$$u(L,t) = 0$$

$$u_n(x,t) = A_n e^{-c^2\pi^2 n^2 t/L^2} \sin\frac{\pi n x}{L} \tag{15}$$

This provided boundary is from 0 to L.

Trigonometric Integration

$$\begin{aligned}\frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\cos^{-1} x &= \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\tan^{-1} x &= \frac{1}{1+x^2}\end{aligned}$$

Integration Rules

$$\int uv = u\int vdx - \int \frac{du}{dx}\int vdxdx, LIATE \tag{16}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{u'v - uv'}{v^2} \tag{17}$$

Trigonometric Identities

Degrees	0	30	45	60	90
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-

$$1 + \tan^2 u = \sec^2 u, 1 + \cot^2 u = \csc^2 u$$

$$\begin{aligned}\sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \\ \tan(-x) &= -\tan x\end{aligned}$$

$$\sin(\frac{\pi}{2} - x) = \cos x \qquad \cos(\frac{\pi}{2} - x) = \sin x$$
$$\tan(\frac{\pi}{2} - x) = \cot x \qquad \cot(\frac{\pi}{2} - x) = \tan x$$
$$\sec(\frac{\pi}{2} - x) = \csc x \qquad \csc(\frac{\pi}{2} - x) = \sec x$$

$$\begin{aligned}\sin(x \pm y) &= \sin x\cos y \pm \cos x\sin y \\ \cos(x \pm y) &= \cos x\cos y \mp \sin x\sin y \\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}\end{aligned}$$

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \\ \tan(2x) &= \frac{2\tan x}{1 - \tan^2 x}\end{aligned}$$

$$\begin{aligned}\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\ \cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\ \tan\frac{x}{2} &= \frac{1-\cos x}{\sin x} \\ &= \frac{\sin x}{1+\cos x}\end{aligned}$$

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \\ \tan^2 x &= \frac{1-\cos 2x}{1+\cos 2x}\end{aligned}$$

$$\begin{aligned}\sin x\sin y &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \\ \cos x\cos y &= \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \\ \sin x\cos y &= \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] \\ \tan x\tan y &= \frac{\tan x + \tan y}{\cot x + \cot y} \\ \tan x\cot y &= \frac{\tan x + \cot y}{\cot x + \tan y}\end{aligned}$$

$$\begin{aligned}\sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \sin x - \sin y &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\ \cos x + \cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \cos x - \cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\ \tan x \pm \tan y &= \frac{\sin(x \pm y)}{\cos x\cos y}\end{aligned}$$

$$\begin{aligned}a\cos\theta \pm b\sin\theta &= R\cos(\theta \mp \alpha) \\ a\sin\theta \pm b\cos\theta &= R\sin(\theta \pm \alpha) \\ \alpha &= \arctan\frac{b}{a} \\ R &= \sqrt{a^2 + b^2}\end{aligned}$$

Hyperbolic Functions

$$\begin{aligned}\cosh t &= \frac{e^t + e^{-t}}{2} \\ \sinh t &= \frac{e^t - e^{-t}}{2} \\ \tanh t &= \frac{\sinh t}{\cosh t} \\ \cosh^2 t - \sinh^2 t &= 1 \\ (\sinh x)' &= \cosh x \\ (\cosh x)' &= \sinh x \\ (\tanh x)' &= \text{sech}^2 x \\ (\sinh^{-1} x)' &= \frac{1}{\sqrt{1+x^2}} \\ (\cosh^{-1} x)' &= \frac{1}{\sqrt{x^2-1}} \\ (\tanh^{-1} x)' &= \frac{1}{1-x^2}\end{aligned}$$

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