MA1511 Cheat Sheet

Mixed Derivatives Theorem

$$f_{xy} = f_{yx} \tag{1}$$

Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt} \tag{2}$$

Gradient Vector

$$\nabla f = \langle f_x, f_y, f_z \rangle \tag{3}$$

The gradient vector is always perpendicular to the level surface $f = c$.

$$\vec{r} \cdot \nabla f = \nabla f \cdot \langle x_0, y_0, z_0 \rangle \tag{4}$$

Use this to find the equation of the tangent plane when $f(x, y, z) = c$ at the point (x_0, y_0, z_0).

Directional Derivatives

$$D_u f(x, y) = \nabla f \cdot \vec{u} \tag{5}$$

where $\vec{u}$ is the unit vector in the desired direction.

Derivatives Test

Critical points of f can be found by setting $f_x = 0$ and $f_y = 0$.

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 \tag{6}$$

Local Maximum	$D > 0$ and $f_{xx} < 0$
Local Minimum	$D > 0$ and $f_{xx} > 0$
Saddle	otherwise

Lagrange Multipliers

The minimum/maximum of a function $f(x, y)$ subject to the constraint $g(x, y) = c$ occurs at points that satisfy these three equations.

$$f_x = \lambda \cdot g_x \tag{7}$$

$$f_y = \lambda \cdot g_y \tag{8}$$

$$g(x, y) = c \tag{9}$$

where λ is some constant called the lagrange multiplier.

Iterated Integrals

If $f(x, y) = a(x) \cdot b(y)$ for functions a and b,

$$\int_b^a \int_d^c f(x, y)dxdy = (\int_b^a a(x)dx) \cdot (\int_d^c b(y)dy) \tag{10}$$

Type 1 Domain $(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)$
Type 2 Domain $(x, y) : c \leq y \leq d, g(y) \leq x \leq h(y)$

Polar Coordinates

$$x = r \cos\theta \tag{11}$$

$$y = r \sin\theta \tag{12}$$

$$\iint_D f(x, y)dA = \int_{r_s}^{r_e} \int_{\theta_s}^{\theta_e} f(r\cos\theta, r\sin\theta) r dr d\theta \tag{13}$$

Vector Differentiation

Let **C** be a constant vector and $\mathbf{u}(t)$ and $\mathbf{u}(t)$ be vector-valued functions.

$$\begin{aligned}
\frac{d}{dt}\vec{C} &= \vec{0} \\
\frac{d}{dt} f(t)\vec{u}(t) &= f'(t)\vec{u}(t) + f(t)\vec{u'}(t) \\
\frac{d}{dt}\vec{u}(t) \cdot \vec{v}(t) &= \vec{u'}(t) \cdot \vec{v}(t) + \vec{v'}(t) \cdot \vec{u}(t) \\
\frac{d}{dt}\vec{u}(t) \times \vec{v}(t) &= \vec{u'}(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v'}(t) \\
\frac{d}{dt}\vec{u}(f(t)) &= f'(t)\vec{u'}(f(t))
\end{aligned}$$

Vector Integration

$$r(t) = (f(t))\vec{i} + (g(t))\vec{j} + (h(t))\vec{k} \tag{14}$$

$$\int r(t)dt = (\int f(t)dt)\vec{i} + (\int g(t)dt)\vec{j} + (\int h(t)dt)\vec{k} \tag{15}$$

The same applies for definite integrals.

Arc Length

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt \tag{16}$$

Line Integrals

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2} dt \tag{17}$$

Parameterizing Line Segments

The line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) has the following parametric equation:

$$r(t) = (1 - t)\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, 0 \leq t \leq 1 \tag{18}$$

Parametric Surfaces

$$r(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)k \tag{19}$$

Surfaces explicitly defined by $z = g(x, y)$ can be parameterized as follows:

$$x = u, y = v, z = g(u, v) \tag{20}$$

To find the equation of the plane tangent to a parametric surface $r(u, v)$ at the point $u = u_0, v = v_0$,

$$\vec{n} = r_u \times r_v \tag{21}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \vec{n} = \vec{n} \cdot r(u_0, v_0) \tag{22}$$

Line Integrals of Vector Fields

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot r'(t)dt \tag{23}$$

If $\vec{F} = P(x, y)\vec{i} + Q(x, y)\vec{j}$ and $r(t) = x(t)\vec{i} + y(t)\vec{j}$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)dt \tag{24}$$

Conservative Fields

If there exists a scalar function f such that $\vec{F} = \nabla f$, the vector field $\vec{F}$ is said to be conservative. The scalar function f is the potential function.

$$\int_C \vec{F} \cdot d\vec{r} = f(r(b)) - f(r(a)) \tag{25}$$

For a closed curve, the above quantity is always 0 for a conservative field.
$\vec{F}$ is conservative if $P_y = Q_x$, $Q_z = R_y$ and $R_x = P_z$.

Green's Theorem

$$\oint_C Pdx + Qdy = \iint_D Q_x - P_y dA \tag{26}$$

Curl and Divergence

$$curl(\vec{F}) = \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix} \tag{27}$$

$$div(\vec{F}) = P_x + Q_y + R_z \tag{28}$$

A vector field is incompressible if $div(\vec{F}) = 0$.

Standard Limits

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1, a \neq 0 \tag{29}$$

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \tag{30}$$

$$\lim_{n\to\infty} r^n = 0, -1 < r < 1 \tag{31}$$

$$\lim_{n\to\infty} (1 + \frac{a}{n})^n = e^a \tag{32}$$

Limit Laws

$$\lim_{n\to\infty} ca_n = cA \tag{33}$$

$$\lim_{n\to\infty} c + a_n = c + A \tag{34}$$

$$\lim_{n\to\infty} a_n b_n = AB \tag{35}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}, B \neq 0 \land b_n \neq 0 \forall n \tag{36}$$

$$\lim_{n\to\infty} f(a_n) = f(A) \tag{37}$$

Infinite Series

For $-1 < r < 1$, $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$.
$\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=m}^{\infty} a_k$ is convergent.

Divergence Test
If a_n does not converge to 0, then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

p-series test
The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Power Series

Ratio Test

$$L = \lim_{k\to\infty} \mid \frac{a_{k+1}}{a_k} \mid \tag{38}$$

Root Test

$$L = \lim_{k\to\infty} \mid a_k^{\frac{1}{k}} \mid \tag{39}$$

Let $a_k = c_k(x - a)^k$. Then the series $\sum_{k=0}^{\infty} a_k$ converges when $L < 1$ and diverges when $L > 1$. When $L = 1$, sit down and cry. To find interval of convergence, find the two extreme values of x, plug it into the equation separately and test for convergence.

Taylor Series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{40}$$

Maclaurin Series

(when $-1 < x < 1$)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(when $-1 < x < 1$)

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

(when $-1 < x < 1$)

$$\begin{aligned}(1+x)^p &= \sum_{k=0}^{\infty} \binom{p}{k} x^k \\ &= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \\ \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\end{aligned}$$

Trigonometric Integration

$$\begin{aligned}\frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\cos^{-1} x &= \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\tan^{-1} x &= \frac{1}{1+x^2}\end{aligned}$$

Integration Rules

$$\int uv = u\int vdx - \int \frac{du}{dx}\int vdxdx, LIATE \tag{41}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{u'v - uv'}{v^2} \tag{42}$$

Trigonometric Identities

Degrees	0	30	45	60	90
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-

$$1 + \tan^2 u = \sec^2 u, 1 + \cot^2 u = \csc^2 u$$

$$\begin{aligned}\sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \\ \tan(-x) &= -\tan x\end{aligned}$$

$$\begin{aligned}\sin(\frac{\pi}{2} - x) &= \cos x & \cos(\frac{\pi}{2} - x) &= \sin x \\ \tan(\frac{\pi}{2} - x) &= \cot x & \cot(\frac{\pi}{2} - x) &= \tan x \\ \sec(\frac{\pi}{2} - x) &= \csc x & \csc(\frac{\pi}{2} - x) &= \sec x\end{aligned}$$

$$\begin{aligned}\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}\end{aligned}$$

$$\begin{aligned}\sin(2x) &= 2\sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \\ \tan(2x) &= \frac{2\tan x}{1 - \tan^2 x}\end{aligned}$$

$$\begin{aligned}\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\ \cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\ \tan\frac{x}{2} &= \frac{1-\cos x}{\sin x} \\ &= \frac{\sin x}{1+\cos x}\end{aligned}$$

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \\ \tan^2 x &= \frac{1-\cos 2x}{1+\cos 2x}\end{aligned}$$

$$\begin{aligned}\sin x \sin y &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \\ \cos x \cos y &= \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \\ \sin x \cos y &= \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] \\ \tan x \tan y &= \frac{\tan x + \tan y}{\cot x + \cot y} \\ \tan x \cot y &= \frac{\tan x + \cot y}{\cot x + \tan y}\end{aligned}$$

$$\begin{aligned}\sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \sin x - \sin y &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\ \cos x + \cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \cos x - \cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\ \tan x \pm \tan y &= \frac{\sin(x \pm y)}{\cos x \cos y}\end{aligned}$$

$$\begin{aligned}a\cos\theta \pm b\sin\theta &= R\cos(\theta \mp \alpha) \\ a\sin\theta \pm b\cos\theta &= R\sin(\theta \pm \alpha) \\ \alpha &= \arctan\frac{b}{a} \\ R &= \sqrt{a^2 + b^2}\end{aligned}$$

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