MA1508E Cheat Sheet

Equivalent Statements

1. A is non-singular.
2. A is invertible.
3. $Ax = b$ has exactly one solution given by $x = A^{-1}b$ for every $n \times 1$ matrix b.
4. $Ax = 0$ has only the trivial solution.
5. The reduced row echelon form of A is I_n.
6. A can be expressed as a product of elementary matrices.
7. The determinant of A is not zero.
8. The columns of A span $\mathbb{R}^n$.
9. The rows of A span $\mathbb{R}^n$.
10. The columns of A are linearly independent.
11. The rows of A are linearly independent.
12. The columns of A form a basis for $\mathbb{R}^n$.
13. The rows of A form a basis for $\mathbb{R}^n$.
14. The column space of A is $\mathbb{R}^n$.
15. The row space of A is $\mathbb{R}^n$.
16. The null space of A is $\{0\}$.
17. A is of full rank; that is, $\text{rank}(A) = n$.
18. A has nullity 0.
19. $A^T A$ is invertible.
20. All the eigenvalues of A are non-zero.

Theorems from Lectures

Thm If augmented matrices of two linear systems are row-equivalent, then the two linear systems have the same solution set.
Prop Every system of linear equations has zero, one or infinitely many solutions.
Prop All homogeneous linear systems are consistent.
Thm An invertible matrix has exactly one inverse.
Thm Every elementary matrix is invertible and its invserse is another elementary matrix.
Thm Let A be a square matrix. If the matrix B satisfies either $AB = I$ or $BA = I$, then the other condition holds automatically.
Thm Let A and B be square matrices of the same order. If A is singular, then both AB and BA are singular.
Cor Let A and B be square matrices. If AB is invertible, then A and B must be invertible as well.
Cramer's If A is a non-singular matrix of order n, then the linear system $Ax = b$ has a unique solution for $x = (x_1, \cdots x_n)$ given by

$$x_i = \frac{detA_i}{detA}$$

where A_i is the matrix obtained by replacing the i-th column of A by b.
Cauchy-Schwarz Let u, v be vectors in $\mathbb{R}^n$. Then

$$|u \cdot v| \leq \|u\|\|v\|$$

where equality holds if and only if u and v are parallel.
Thm You need at least n vectors to span $\mathbb{R}^n$.
Thm Let S_1 and S_2 be subsets of $\mathbb{R}^n$. Then $\text{span}(S_1) \subseteq \text{span}(S_2)$ iff every vector in S_1 is a linear combination of the vectors of S_2.
Thm If u_k is a linear combination of $u_1, \cdots, u_{k-1}$, then $\text{span}\{u_1, \cdots, u_{k-1}\} = \text{span}\{u_1, \cdots, u_k\}$.
Thm In $\mathbb{R}^n$, a set with more than n vectors is always linearly dependent.
Thm If u_{k+1} is not a linear combination of $u_1, \cdots, u_k$ (linearly independent vectors), then $u_1, \cdots u_k, u_{k+1}$ are linearly independent.
Thm Every vector in a subspace is a unique combination of the basis vectors of the subspace.
Def The dimension of a vector space is the number of vectors in its basis.
Thm Set of vectors that spans the solution space of a homogeneous system is always linearly independent.
Thm Let V be a vector space of dimension k and S a subset of V. These statements are equivalent:

1. S is a basis for V.
2. S is linearly independent and has k vectors.
3. S spans V and has k vectors.

Thm A vector is orthogonal to a vector space iff it is orthogonal to all its basis vectors.
Thm Let V be a subspace of $\mathbb{R}^n$. Every vector u in $\mathbb{R}^n$ can be uniquely written as $u = n + p$ where n is orthogonal to V and p is in V. The vector p is also called the orthogonal projection.
Thm If $u_1, \cdots, u_k$ for an orthogonal basis for V, then the projection of any vector w onto V is given by

$$\frac{w \cdot u_1}{\|u_1\|^2} u_1 + \cdots + \frac{w \cdot u_k}{\|u_2\|^2} u_2$$

Results from Tutorials

Let A, B, C, D be block matrices.

1. $A \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} AB_1 & AB_2 \end{bmatrix}$
2. $\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} A = \begin{bmatrix} D_1 A \\ D_2 A \end{bmatrix}$
3. $\begin{vmatrix} A & 0 \\ 0 & I \end{vmatrix} = |A|$
4. $\begin{vmatrix} I & 0 \\ C & D \end{vmatrix} = |D|$
5. $\begin{vmatrix} A & 0 \\ C & D \end{vmatrix} = \begin{vmatrix} A & C \\ 0 & D \end{vmatrix} = |A||D|$

QR Factorization

1. Run Gram-Schmidt to obtain orthogonal basis for the column space of A.
2. Make orthogonal basis orthonormal.
3. Express the columns of A as linear combination of the orthonormal vectors.
4. Now, $A = QR$ where the columns of Q are the orthonormal vectors and R is an upper-triangular matrix where the columns are the coordinate vectors calculated in the previous step.
5. Now, the least square solution to an inconsistent system $Ax = b$ is simply $x' = R^{-1}Q^T b$.

Matrix Multiplication

Note that $(AB)^2 \neq A^2B^2$ in general.
Let x_i be column vectors.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = ax_1 + bx_2 + cx_3$$

In words, the result is a linear combination of the column vectors.
Let A be a matrix and x_i be column vectors.

$$A \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix}$$

Let y_i be row vectors.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} A = \begin{bmatrix} y_1 A \\ y_2 A \\ y_3 A \end{bmatrix}$$

In words, the rows of A are weighted by the vector y_i and this weighted row forms the i-th row in the result matrix.
Also, if A is a $m \times n$ matrix and B is a $n \times p$ matrix,

$$AB_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$

Matrix Inverses and Transposes

A is said to be invertible if there exists a matrix B such that $AB = BA = I$.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(\alpha A)^T = \alpha A^T$
4. $(AB)^T = B^T A^T$
5. $(cA)^{-1} = \frac{1}{c} A^{-1}$
6. $(A^T)^{-1} = (A^{-1})^T$
7. $(A^{-1})^{-1} = A$
8. $(AB)^{-1} = B^{-1} A^{-1}$
9. If $E_k \cdots E_2 E_1 A = I$, then $A^{-1} = E_k \cdots E_2 E_1$ and $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.
10. Symmetric: $A = A^T$
11. Anti-symmetric: $A = -A^T$
12. $A^2 = 0 \rightarrow (I - A)/(I + A)$ is invertible.

Determinants

1. $det(A) = det(A^T)$
2. The determinant of a square matrix with two identical rows or columns is zero.
3. Row swapping multiplies determinant by -1.
4. Row scaling scales determinant by the scaling factor.
5. Row linear combination (ERO) doesn't change determinant.
6. $det(AB) = det(A)det(B)$
7. $det(cA) = c^n det(A)$
8. $det(A^{-1}) = \frac{1}{det(A)}$

Euclidean Vectors

1. $\|u\| = \sqrt{u_1^2 + \cdots + u_n^2}$
2. $d(u,v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$
3. $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$
4. $u \cdot v = \|u\|\|v\| \cos\theta$

Solving Ax = b

1. Create an augmented matrix $\left(A \mid b\right)$ and reduce it to REF or RREF.
2. If the augmented column has a pivot entry, the system is inconsistent and therefore not solvable.
3. If all columns but the last have pivot entries, there exists a unique solution. Back-substitute to determine the solution.
4. Else, let variables corresponding to non-pivot columns be free variables. Back-substitute to find the solution of other variables in terms of the free variables.

Determining Number of Solutions

1. Reduce the augmented matrix to REF or RREF.
2. Systematically set each pivot (including augmented column if necessary) to zero and non-zero values.
3. Check if any row can be a multiple of another for particular pivot values.

LU Factorization

1. Let $E_1, \cdots, E_k$ be elementary matrices such that $E_k \cdots E_1 A = U$.
2. Then, $L = E_1^{-1} \cdots E_k^{-1}$ such that $A = LU$.
3. Note that $Ax = (LU)x = L(Ux) = b$. Therefore, let $Ux = y$ and solve $Ly = b$ using forward substitution.
4. Now, solve $Ux = y$ using back substitution.

Applying Cramer's Rule

1. Check if determinant is non-zero.
2. Apply Cramer's rule.

Checking for linear combination

1. To check if a vector x is a linear combination of u, v and w, reduce $\left(u \quad v \quad w \mid x\right)$.
2. If the system is consistent, x is indeed a linear combination.

Checking for linear independence

To check if a set of vectors is linearly independent,

1. Solve $c_1 u_1 + \cdots + c_2 u_k = 0$ where c_i are constants.
2. That is, reduce $\left(u_1 \quad \cdots \quad u_k \mid 0\right)$ and ensure that there are no non-pivot columns.

Coordinate Vectors

To find $(v)_S$ where S is a basis for the vector space,

1. Solve $\begin{bmatrix} u_1 & \cdots & u_k \end{bmatrix} x = v$ for x where u_i are the basis vectors.

Finding basis for row and column spaces

1. Reduce the matrix A to R and the non-zero rows form the basis for the row space.
2. The columns in the original matrix A corresponding to the pivot columns in R form the basis for the column space.
3. ERO preserves row space but no column space.

Gram-Schmidt Process

Let $u_1, \cdots, u_k$ be a basis for V.

1. Let $v_1 = u_1$.
2. Let $v_2 = u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1$.
3. Let $v_3 = u_3 - \frac{u_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{u_3 \cdot v_2}{\|v_2\|^2} v_2$.

Least Squares Solution

1. If $Ax = b$ is inconsistent, then x is a least squares solution if it is a solution to $A^T Ax = A^T b$.
2. This is the same as $Ax = p$ where p is the projection of b onto V.
3. This is the same as Ax is a projection of b onto V.
4. The error is $\|b - Ax\|$.

Diagonalizing

1. Derive the characteristic polynomial from $\det(\lambda I - A)$. Set it to zero and determine all eigenvalues.
2. For each eigenvalue, solve for $(\lambda I - A)x = 0$. That is, find a basis for the eigenspace E_λ. Ensure that there are n independent eigenvectors. When $\lambda =$, this is the same as finding the nullspace.
3. Let P be the matrix where the columns are the derived eigenvectors. Then, $P^{-1}AP = D$ or $A = PDP^{-1}$. D is a diagonal matrix whose entries are eigenvalues in the order the eigenvectors appear in P.

Differential Equations

To solve $Y' = AY$,

1. Find all eigenvalues and eigenvectors of A.
2. The general solution is of the form
$$Y = k_1 e^{\lambda_1 t} x_1 + k_2 e^{\lambda_2 t} x_2 + \cdots$$
3. If λ is a complex number, then $Y_1 = Re(e^{\lambda t} x)$ and $Y_2 = Im(e^{\lambda t} x)$

Properties of Complex Vectors

1. $\overline{ku} = \overline{k}\overline{u}$
2. $\overline{u \pm v} = \overline{u} \pm \overline{v}$
3. $u \cdot v = u_1 \overline{v_1} + u_2 \overline{v_2} + \cdots$
4. $\|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \cdots}$
5. $u \cdot v = \overline{v \cdot u}$
6. $u \cdot kv = \overline{k}(u \cdot v)$

Complex eigenvalues and eigenvectors always appear in pairs.

Properties of $A^T A$

1. $null(A) = null(A^T A)$
2. $nullity(A) = nullity(A^T A)$
3. $rank(A) = rank(A^T A)$

Discrete Systems

$$x(t+1) = Ax(t)$$
$$\text{where } x_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$$

Diagonalize A. Then,

$$x(t) = PD^t P^{-1} x_0$$

Alternatively,

$$x(t) = c_1 \lambda_1^t v_1 + c_2 \lambda_2^t v_2 + \cdots$$

Continuous Systems

$$X' = AX$$

Find eigenvalues and eigenvectors of A. If eigenvalues are not complex,

$$X = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots$$

Generalized Eigenvectors

If there is just a single eigenvalue whose eigenspace has a dimension of 1 for a 2 by 2 matrix,

1. Solve $(A - \lambda I)u = v$ for u where v is the eigenvector. Just choose a non-zero u.
2. Now, $y = c_1 e^{\lambda t} v + c_2 e^{\lambda t}(tv + u)$

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