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NAME - MOHIT KUMAR
SR NO - 04-01-03-10-51-21-1-19825
COMPUTATIONAL METHODS OF OPTIMIZATION
ASSIGNMENT - 4

(Q1) (a) As can be seen by running the program the initial working set $W = [4]$. Plotting of the said points has been done.

(Q1) (b). $d = \begin{bmatrix} -1.0976 \\ 0.8780 \end{bmatrix}$

$W = [4, 9]$.

$w^1 = \begin{bmatrix} 0.3750 \\ -0.5 \end{bmatrix}$

(Q1) (c). $d = \begin{bmatrix} -0.4726 \\ 0.3780 \end{bmatrix}$

$w^2 = \begin{bmatrix} 0.3333 \\ -0.4667 \end{bmatrix}$

(Q2)
(a) Let $u_t = v_t - v_{min}$ $\quad t \in \{0, 1, \ldots, T-1\}$
The LP in standard form is given by

$$\begin{aligned} \text{argmin} \quad & c^T x \\ \text{s.t} \quad & Ax = b. \\ & x \geq 0. \end{aligned}$$

$c = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ is a length T vector.

Orchies

$$x = \begin{pmatrix} u_0 \\ u_1 \\ | \\ | \\ \vdots \\ u_{T-1} \end{pmatrix} \text{ is a length } T \text{ vector.}$$

$$A = \begin{pmatrix} \sin\theta_0 & \sin\theta_1 & --- & \sin\theta_{T-1} \\ \cos\theta_0 & \cos\theta_1 & --- & \cos\theta_{T-1} \end{pmatrix}$$

$$b = \begin{pmatrix} y_f - y_0 - \sum_{i=0}^{T-1} v_{min}\sin\theta_i \\ x_f - x_0 - \sum_{j=0}^{T-1} v_{min}\cos\theta_j \end{pmatrix}$$

(Q2) (c). $v_{min} \leq \dfrac{x_f + y_f}{T\sqrt{2}}$

According to Farka's Lemma, exactly one of the following two systems has a solution.

(i) $Ax \leq 0 \quad c^T x > 0$ for some x.
$A \in R^{m \times n} \qquad x \in R^n$.

(ii) $A^T y = c \quad y \geq 0$ for some $y \in R^m$

Let $x = (y_f, x_f)^T$

$$Ax = \begin{pmatrix} \sin\theta_0 & \sin\theta_1 & --- & \sin\theta_{T-1} \\ \cos\theta_0 & \cos\theta_1 & -- & \cos\theta_{T-1} \end{pmatrix} \begin{pmatrix} y_f \\ x_f \end{pmatrix}$$

$$= \sum_{i=0}^{T-1} \left(x_f \cos\theta_i + y_f \sin\theta_i \right)$$

$$c^T x = \begin{pmatrix} x_f - \sum_{i=0}^{T-1} v_{min}\cos\theta_i \\ y_f - \sum_{i=0}^{T-1} v_{min}\sin\theta_i \end{pmatrix}^T \begin{pmatrix} x_f \\ y_f \end{pmatrix}$$

$$c^T x = x_f^2 - x_f \sum_{i=0}^{T-1} v_{min}\cos\theta_i + y_f^2 - y_f \sum_{i=0}^{T-1} v_{min}\sin\theta_i$$

$$c^T x = x_f^2 + y_f^2 - \left(x_f V_{min} \sum_{i=0}^{T-1} \cos\theta_i + y_f V_{min} \sum_{i=0}^{T-1} \sin\theta_i \right)$$

Also note that for $z_1, z_2 > 0$, a and b positive constants we have

$$a z_1 + b z_2 \geq (a+b) \min(z_1, z_2)$$

$$c^T x \leq x_f^2 + y_f^2 - V_{min}(x_f + y_f) \min\left\{ \sum_{i=0}^{T-1} \cos\theta_i, \sum_{i=0}^{T-1} \sin\theta_i \right\}$$

$$c^T x \leq x_f^2 + y_f^2 - \frac{(x_f + y_f)^2}{T\sqrt{2}}$$

$$c^T x \leq 0 .$$

Hence $A^T y = c \quad y \geq 0$ has a feasible solution for some $y \in R^m$

Therefore if $V_{min} \leq \frac{x_f + y_f}{T\sqrt{2}}$ the linear program in part (a) has a feasible solution.

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(Q3) (a) $\operatorname{argmin} \frac{1}{2} ||Aw - b||^2$

$$\text{s.t } ||w - w^\circ||^2 \leq r^2$$

$$\mathcal{L}(w, \lambda) = \frac{1}{2} ||Aw - b||^2 + \lambda(||w - w^\circ||^2 - r^2)$$

$$\begin{cases} \nabla_w \mathcal{L}(w, \lambda) = A^T(Aw - b) + 2\lambda(w - w^\circ) = 0 \\ \lambda(||w - w^\circ||^2 - r^2) = 0 \\ \lambda \geq 0. \\ ||w - w^\circ||^2 \leq r^2 \end{cases}$$

KKT conditions.

(b) constraint is inactive $\Rightarrow ||w - w^\circ||^2 < r^2$

$\Rightarrow \lambda = 0$ (complementary slackness)

$\nabla_w \mathcal{L}(w, \lambda) = 0$

$$\Rightarrow A^T(Aw - b) = 0.$$

$$\Rightarrow A^TAw = A^Tb.$$

$$\Rightarrow w = (A^TA)^{-1} A^Tb.$$

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(Q4) $\operatorname{argmin} \; x^T A_0 x + 2b_0^T x + c_0$

$\quad$ s.t $\; x^T A_i x + 2b_i^T x + c_i \leq 0 \quad i = 1, 2, \ldots, m$

$A_i \in R^{n\times n}$ psd $\quad b_i \in R^n \quad c_i \in R$

$$\mathcal{L}(x,\lambda) = x^T A_0 x + 2b_0^T x + c_0 + \sum_{i=1}^{m} \lambda_i \left(x^T A_i x + 2b_i^T x + c_i\right)$$

$$= x^T\left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right)x + 2\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)^T x$$

$$+ \left(c_0 + \sum_{i=1}^{m} \lambda_i c_i\right.$$

$$\nabla_x \mathcal{L}(x,\lambda) = 2\left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right)x + 2\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right) = 0$$

$$x^* = -\left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right)^{-1}\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)$$

Dual function $g(\lambda)$

$$g(\lambda) = \left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)^T\left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right)\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)$$

$$+ 2b_0^T\left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right)^{-1}\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)$$

$$+ c_0 + \sum_{i=1}^{m} \lambda_i c_i$$

Dual problem

$$\max_{\lambda \geq 0} g(\lambda)$$

(Q5) (a) $\underset{x \in R^n}{\operatorname{argmin}} \frac{1}{2} \|x-y\|^2$

$$Ax \leq b.$$

$$\mathcal{L}(x, \lambda) = \frac{1}{2}\|x-y\|^2 + \lambda^T(Ax-b)$$

$$\nabla_x \mathcal{L}(x,\lambda) = x - y + A^T\lambda = 0.$$

$$\Rightarrow x = y - A^T\lambda. \qquad \lambda \geq 0.$$

dual function $g(\lambda) = \min_x \frac{1}{2}\|x-y\|^2 + \lambda^T(Ax-b)$

$$= \min_x \frac{1}{2}\|A^T\lambda\|^2 + \lambda^T(A(y-A^T\lambda)-b)$$

$$= \min_x \frac{1}{2}\|A^T\lambda\|^2 + \lambda^T Ay - \lambda^T AA^T\lambda - \lambda^T b.$$

$$= \min_x \lambda^T(Ay-b) - \frac{1}{2}\|A^T\lambda\|^2$$

dual problem $\max_\lambda \lambda^T(Ay-b) - \frac{1}{2}\|A^T\lambda\|^2.$

$$\lambda \geq 0.$$

dual problem

$$\min_{\lambda \geq 0} \frac{1}{2}\lambda^T AA^T\lambda - (Ay-b)^T\lambda$$

(Q5) (b) gradient of the dual problem

$$\nabla f = AA^T\lambda - (Ay-b).$$

The general iteration step of the gradient projection algorithm is given by:

$$\lambda_{k+1} = P_c\left(\lambda_k - \frac{1}{L}AA^T\lambda_k + \frac{1}{L}A(y-b)\right)$$

P_c is the projection onto the positive orthant.

$L \rightarrow$ Lipschitz constant of the gradient of the objective function is given by the maximum eigenvalue of AA^T

(Q5)(c). The projection of the point $(3,-1,2)$ after 100 iterations was found to be $(3, 0.5, 0.5)$

(Q5)(d) In my opinion step size $\frac{1}{L}$ is better because we see faster convergence in this case as compared to the step size $\frac{2}{L}$.

(Q5)(e) Analytical solution

$$\min \frac{1}{2}(x^2+y^2+z^2)$$

$$\text{s.t} \quad x+3y=1$$
$$2y+z=-1$$

$$\mathcal{L}(x,y,z,\mu) = \frac{1}{2}(x^2+y^2+z^2) + \mu_1(x+3y-1) + \mu_2(2y+z+1)$$

$$\nabla_x \mathcal{L} = x+\mu_1 = 0. \quad \Rightarrow x = -\mu_1$$
$$\nabla_y \mathcal{L} = y+3\mu_1+2\mu_2 = 0 \quad \Rightarrow y = -3\mu_1-2\mu_2$$
$$\nabla_z \mathcal{L} = z+\mu_2 = 0. \quad \Rightarrow z = -\mu_2$$

$$x+3y=1.$$
$$-\mu_1-9\mu_1-6\mu_2=1.$$
$$-10\mu_1-6\mu_2=1.$$
$$2y+z=-1.$$
$$-6\mu_1-4\mu_2-\mu_2=-1.$$
$$-6\mu_1-5\mu_2=-1$$
$$6\mu_1+5\mu_2=1$$
$$10\mu_1+6\mu_2=-1$$
$$36\mu_1+30\mu_2=6$$
$$50\mu_1+30\mu_2=-5$$
$$- \quad - \quad +$$
$$-14\mu_1 = 11$$

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$$\mu_1 = \frac{-11}{14}$$

$$6\mu_2 + 5\mu_2 = 1.$$

$$-\frac{66}{14} + 5\mu_2 = 1.$$

$$\mu_2 = \frac{16}{14}$$

$$x = \frac{11}{14}, \quad y = \frac{33}{14} - \frac{32}{14} = \frac{1}{14}, \quad z = \frac{-16}{14}$$

Hence the projection of origin is found to be

$$\left(\frac{11}{14}, \frac{1}{14}, \frac{-16}{14}\right)$$