Exercise 8.5, Problem 16

Mohit Singh

Abstract—**This document provides the solution to the problem no. 16 given in the exercise 8.5. The figures are constructed using python and LaTeXcodes.**

This documentation can be downloaded from

Parameter	Value	Decription
r	2	Radius of circle
O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of circle
θ	45°	Argument of C

TABLE 2.1: Input Table for construction

1 PROBLEM No. 16

AB is a diameter of the circle, ***CD*** is a chord equal to the radius of the circle. ***AC*** and ***BD*** when extended intersect at point ***E*** . Prove that $\angle AEB = 60°$.
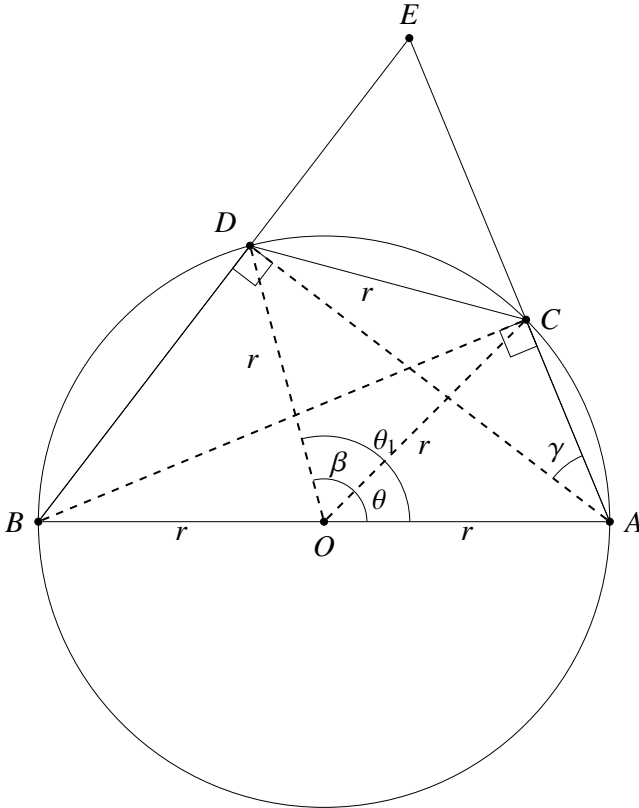
2 CONSTRUCTION

Fig. 2.0: Using Latex-Tikz

2.1. The following inputs were taken for constructing the figure:

2.2. The coordinates of **A** and **B** such that **AB** is a diameter as in Fig. 2.0 are:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \tag{2.2.1}$$

$$\mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{2.2.2}$$

2.3. Let **C** be a point on circle such that its coordinates are:

$$\mathbf{C} = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \tag{2.3.1}$$

where $r = 2$ units (radius of circle) and $\theta = 45°$

Now **D** is a point on the circle with $\|D\| = r = 2$ units (radius of the circle). Let,

$$\mathbf{D} = r \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} \tag{2.3.2}$$

$$= 2 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} \tag{2.3.3}$$

D should be such that CD = r = 2 units. The vector **CD** is given by

$$\mathbf{CD} = (\mathbf{D} - \mathbf{C}) \tag{2.3.4}$$

Applying the distance formula using matrix multiplication,

$$(\mathbf{D} - \mathbf{C})^T (\mathbf{D} - \mathbf{C}) = (CD)^2 \tag{2.3.5}$$

$$\tag{2.3.6}$$

On futher expanding this expression,

$$\left(\mathbf{D}^T - \mathbf{C}^T\right)(\mathbf{D} - \mathbf{C}) = (CD)^2 \quad (2.3.7)$$

$$\mathbf{D}^T\mathbf{D} - \mathbf{D}^T\mathbf{C} - \mathbf{C}^T\mathbf{D} + \mathbf{C}^T\mathbf{C} = (CD)^2 \quad (2.3.8)$$

$$\|D\|^2 - \mathbf{D}^T\mathbf{C} - \mathbf{C}^T\mathbf{D} + \|C\|^2 = (CD)^2 \quad (2.3.9)$$

Substituting the magnitudes, we get

$$\mathbf{D}^T.\mathbf{C} + \mathbf{C}^T\mathbf{D} = 4 \quad (2.3.10)$$

The two terms on L.H.S is the scalar product of vectors. Fron Fig. 2.0 , the angle between vectors $\mathbf{C}$ and $\mathbf{D}$ is β.

$$\left\|D^T\right\|\|C\|\cos\beta + \left\|C^T\right\|\|D\|\cos\beta = 4 \quad (2.3.11)$$

$$4\cos\beta + 4\cos\beta = 4 \quad (2.3.12)$$

$$\cos\beta = \frac{1}{2} \quad (2.3.13)$$

We get β as ± 60 °. Taking the positive value, $\beta = 60°$. From Fig. 2.0 ,

$$\theta_1 = \beta + \theta \quad (2.3.14)$$

$$= 60° + 45° \quad (2.3.15)$$

$$= 105° \quad (2.3.16)$$

Substituting θ_1 in (2.3.3), we get $\mathbf{D}$.

2.4. The point $\mathbf{E}$ is obtained by the intersection of the extended lines of $\mathbf{AC}$ and $\mathbf{BD}$.

Using the corrollary that angle subtended by diameter at any point on the circle is 90°; $\angle ACB = \angle BDA = 90°$. This implies that

$$\mathbf{BC} \perp \mathbf{AC} \quad (2.4.1)$$

$$\mathbf{AD} \perp \mathbf{BD} \quad (2.4.2)$$

The equations of AC and BD are

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \quad (2.4.3)$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{x} - \mathbf{B}) = 0 \quad (2.4.4)$$

Since E lies on both these lines it will satisfy both the quations. Therefore,

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{E} - \mathbf{A}) = 0 \quad (2.4.5)$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{E} - \mathbf{B}) = 0 \quad (2.4.6)$$

On soving the above two equations you can find the coordinates of $\mathbf{E}$.

2.5. The derived values are listed in Table. 2.5
2.6. For solving the problem, join $\mathbf{OC}$, $\mathbf{OD}$, $\mathbf{BC}$ and $\mathbf{AD}$.

Derived Values		
	Coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$	
A	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	
B	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	
C	$\begin{pmatrix} 1.414 \\ 1.414 \end{pmatrix}$	
D	$\begin{pmatrix} -0.518 \\ 1.932 \end{pmatrix}$	
E	$\begin{pmatrix} 0.597 \\ 3.385 \end{pmatrix}$	

TABLE 2.5: To construct the figure

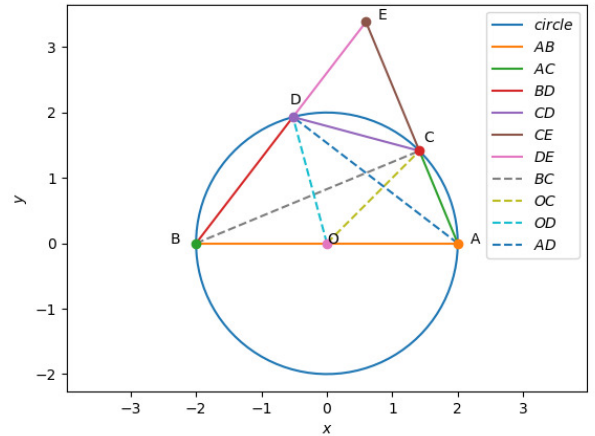2.7. To get the python code for Fig 2.7, download it from

codes/circle.py

Fig. 2.7: Using Python

and the equivalent latex-tikz code for Fig. 2.0 from

figs/circle.tex

The above latex code can be compiled as a standalone document as

figs/circle_fig.tex

3 SOLUTION

3.1. Now we know the following values:

$$A = \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{3.1.1}$$

$$B = \begin{pmatrix} -r \\ 0 \end{pmatrix} \tag{3.1.2}$$

$$C = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \tag{3.1.3}$$

$$D = r \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} \tag{3.1.4}$$

where $\theta = 45°$, $\theta_1 = 105°$ and $r = 2$ units.

3.2. Let the $\angle AEB = \alpha$. Consider the below equation,

$$(\mathbf{D} - \mathbf{B})^T (\mathbf{C} - \mathbf{A}) = BD.AC.\cos \alpha \tag{3.2.1}$$

3.3. Now finding BD

$$(BD)^2 = (\mathbf{D} - \mathbf{B})^T (\mathbf{D} - \mathbf{B}) \tag{3.3.1}$$

$$= r (\cos \theta_1 + 1 \quad \sin \theta_1) r \begin{pmatrix} \cos \theta_1 + 1 \\ \sin \theta_1 \end{pmatrix} \tag{3.3.2}$$

$$= r^2 \left(1 + \cos^2 \theta_1 + 2 \cos \theta_1 + \sin^2 \theta_1 \right) \tag{3.3.3}$$

Substituting the values of θ_1 and r, we get

$$BD = 2.435 \tag{3.3.4}$$

3.4. Finding AC

$$(AC)^2 = (\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) \tag{3.4.1}$$

$$= r (\cos \theta - 1 \quad \sin \theta) r \begin{pmatrix} \cos \theta - 1 \\ \sin \theta \end{pmatrix} \tag{3.4.2}$$

$$= r^2 \left(1 + \cos^2 \theta + 2 \cos \theta + \sin^2 \theta \right) \tag{3.4.3}$$

Substituting the values of θ and r, we get

$$AC = 1.531 \tag{3.4.4}$$

3.5. Expanding the L.H.S. of (3.2.1)

$$= r (\cos \theta_1 + 1 \quad \sin \theta_1) r \begin{pmatrix} \cos \theta - 1 \\ \sin \theta \end{pmatrix} \tag{3.5.1}$$

$$= r^2 ((\cos \theta_1 + 1)(\cos \theta - 1) + \sin \theta \sin \theta_1) \tag{3.5.2}$$

$$= r^2 (\cos (\theta_1 - \theta) - \cos \theta_1 + \cos \theta - 1) \tag{3.5.3}$$

$$= r^2 (\cos \beta - \cos \theta_1 + \cos \theta - 1) \tag{3.5.4}$$

Substituting value of β, θ_1, θ and r, we get

$$(\mathbf{D} - \mathbf{B})^T (\mathbf{C} - \mathbf{A}) = 1.864 \tag{3.5.5}$$

3.6. Substituting values from (3.4.4),(3.3.4) and (3.5.5) in (3.2.1), we get

$$\cos \alpha = 0.500 \tag{3.6.1}$$

$$\alpha = 60° \tag{3.6.2}$$

$$\angle AEB = 60° \tag{3.6.3}$$

Hence proved.