Exercise 8.5, Problem 16

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***Abstract*—This document provides the solution to the problem no. 16 given in the exercise 8.5. The figures are constructed using python and LaTeXcodes.**

This documentation can be downloaded from

```
svn co https://github.com/mohit−singh−9/Summer
    −2020/tree/master/geometry/circle.git
```

1 PROBLEM No. 16

AB is a diameter of the circle, ***CD*** is a chord equal to the radius of the circle. ***AC*** and ***BD*** when extended intersect at point ***E*** . Prove that $\angle AEB = 60°$.

2 CONSTRUCTION

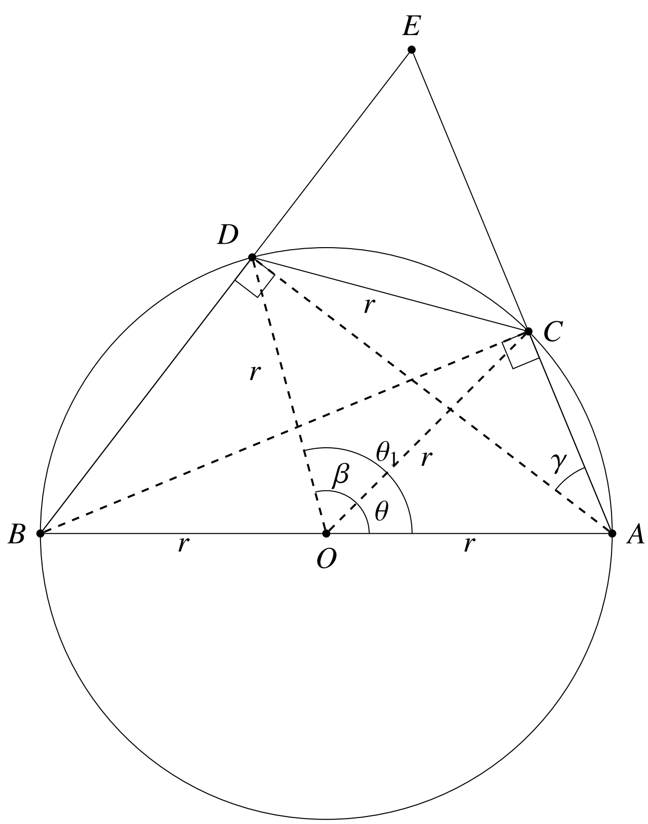

Fig. 2.0: Using Latex-Tikz

2.1. The following inputs were taken for constructing the figure:

Parameter	**Value**	**Decription**
r	2	Radius of circle
O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of circle
θ	45°	Argument of C

TABLE 2.1: Input Table for construction

2.2. The coordinates of **A** and **B** such that **AB** is a diameter as in Fig. 2.0 are:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \tag{2.2.1}$$

$$\mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{2.2.2}$$

2.3. Let **C** be a point on circle such that its coordinates are:

$$\mathbf{C} = r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.3.1}$$

where $r = 2$ units (radius of circle) and $\theta = 45°$

Now **D** is a point on the circle with $\|D\| = r = 2$ units (radius of the circle). Let,

$$\mathbf{D} = r\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} \tag{2.3.2}$$

$$= 2\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} \tag{2.3.3}$$

D should be such that $\|CD\| = r = 2$ units. The vector **CD** is given by

$$\mathbf{CD} = (\mathbf{D} - \mathbf{C}) \tag{2.3.4}$$

Applying the distance formula using matrix multiplication,

$$(\mathbf{D} - \mathbf{C})^T . (\mathbf{D} - \mathbf{C}) = \|CD\|^2 \tag{2.3.5}$$

$$\tag{2.3.6}$$

On futher expanding this expression,

$$\left(\mathbf{D}^T - \mathbf{C}^T\right).(\mathbf{D} - \mathbf{C}) = \|CD\|^2 \tag{2.3.7}$$

$$\mathbf{D}^T.\mathbf{D} - \mathbf{D}^T.\mathbf{C} - \mathbf{C}^T.\mathbf{D} + \mathbf{C}^T.\mathbf{C} = \|CD\|^2 \tag{2.3.8}$$

$$\|D\|^2 - \mathbf{D}^T.\mathbf{C} - \mathbf{C}^T.\mathbf{D} + \|C\|^2 = \|CD\|^2 \tag{2.3.9}$$

Substituting the magnitudes, we get

$$\mathbf{D}^T.\mathbf{C} + \mathbf{C}^T.\mathbf{D} = 4 \tag{2.3.10}$$

The two terms on L.H.S is the scalar product of vectors. Fron Fig. 2.0 , the angle between vectors **C** and **D** is β.

$$\left\|D^T\right\| \|C\| \cos\beta + \left\|C^T\right\| \|D\| \cos\beta = 4 \tag{2.3.11}$$

$$4\cos\beta + 4\cos\beta = 4 \tag{2.3.12}$$

$$\cos\beta = \frac{1}{2} \tag{2.3.13}$$

We get β as ± 60 °. Taking the positive value, $\beta = 60°$. From Fig. 2.0 ,

$$\theta_1 = \beta + \theta \tag{2.3.14}$$

$$= 60° + 45° \tag{2.3.15}$$

$$= 115° \tag{2.3.16}$$

Substituting θ_1 in (2.3.3), we get **D**.

2.4. The point **E** is obtained by the intersection of the extended lines of **AC** and **BD**.
Using the corrollary that angle subtended by diameter at any point on the circle is 90°; $\angle ACB = \angle BDA = 90°$. This implies that

$$\mathbf{BC} \perp \mathbf{AC} \tag{2.4.1}$$

$$\mathbf{AD} \perp \mathbf{BD} \tag{2.4.2}$$

The equations of AC and BD are

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \tag{2.4.3}$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{x} - \mathbf{B}) = 0 \tag{2.4.4}$$

Since E lies on both these lines it will satisfy both the quations. Therefore,

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{E} - \mathbf{A}) = 0 \tag{2.4.5}$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{E} - \mathbf{B}) = 0 \tag{2.4.6}$$

On soving the above two equations you can find the coordinates of **E**.

2.5. The derived values are listed in Table. 2.5

Derived Values	
	Coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$
A	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
B	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
C	$\begin{pmatrix} 1.414 \\ 1.414 \end{pmatrix}$
D	$\begin{pmatrix} -0.518 \\ 1.932 \end{pmatrix}$
E	$\begin{pmatrix} 0.597 \\ 3.385 \end{pmatrix}$

TABLE 2.5: To construct the figure

2.6. For solving the problem, join **OC**, **OD**, **BC** and **AD**.

2.7. To get the python code for Fig 2.7, download it from

```
codes/circle.py
```

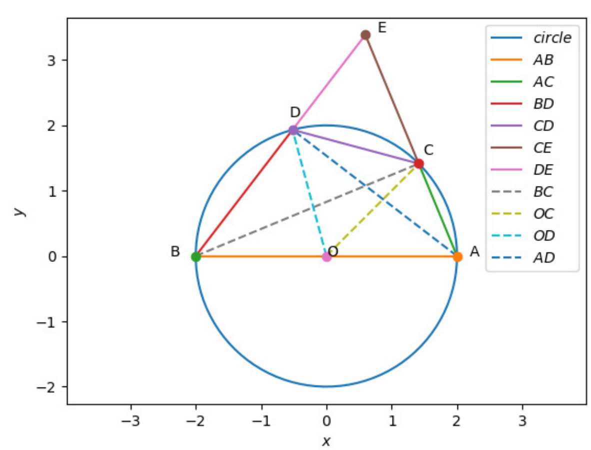

Fig. 2.7: Using Python

and the equivalent latex-tikz code for Fig. 2.0 from

```
figs/circle.tex
```

The above latex code can be compiled as a standalone document as figs/circle_fig.tex

3 SOLUTION

3.1. First we need to prove that angle subtended by a diameter at a point on circumference is 90°. This can be proved if the scalar product of **BD** and **AD** is 0.

$$
\begin{aligned}
(\mathbf{BD}) \,.\, (\mathbf{AD}) &= (\mathbf{D} - \mathbf{B}) \,.\, (\mathbf{D} - \mathbf{A}) && (3.1.1)\\
&= \|D\|^2 - \mathbf{D}.\mathbf{A} - \mathbf{B}.\mathbf{D} + \mathbf{B}.\mathbf{A} && (3.1.2)
\end{aligned}
$$

Since **A** and **B** are equal in magnitude but opposite in direction , **A** = - **B**. Also $\|A\|= \|B\| = \|D\| = r$. On substituting,

$$(\mathbf{BD}) \,.\, (\mathbf{AD}) = 0 \tag{3.1.3}$$

Hence **BD** ⊥ **AD**;

$$\angle BDA = \angle ADE = 90^\circ \tag{3.1.4}$$

3.2. To find γ , consider the equation

$$
\begin{aligned}
\|AC\|^2 &= (\mathbf{AC}) \,.\, (\mathbf{AC}) && (3.2.1)\\
&= (\mathbf{C} - \mathbf{A}) \,.\, (\mathbf{C} - \mathbf{A}) && (3.2.2)\\
&= \|C\|^2 + \|A\|^2 + 2.\, \|A\| \,.\, \|C\| \,.\, \cos\theta && (3.2.3)
\end{aligned}
$$

On sustituting the values, we get

$$\|AC\| = 1.53 \tag{3.2.4}$$

Similarly

$$
\begin{aligned}
\|AD\|^2 &= (\mathbf{AD}) \,.\, (\mathbf{AD}) && (3.2.5)\\
&= (\mathbf{D} - \mathbf{A}) \,.\, (\mathbf{D} - \mathbf{A}) && (3.2.6)\\
&= \|D\|^2 + \|A\|^2 + 2.\, \|A\| \,.\, \|D\| \,.\, \cos\theta_1 && (3.2.7)
\end{aligned}
$$

On sustituting the values, we get

$$\|AD\| = 3.174 \tag{3.2.8}$$

Consider the scalar product of **AC** and **AD**,

$$(\mathbf{AC}) \,.\, (\mathbf{AD}) = \|AC\| \, \|AD\| \cos\gamma \tag{3.2.9}$$

For L.H.S of (3.2.9),

$$
\begin{aligned}
(\mathbf{AC}) \,.\, (\mathbf{AD}) &= (\mathbf{C} - \mathbf{A}) \,.\, (\mathbf{D} - \mathbf{A}) && (3.2.10)\\
&= \mathbf{C}.\mathbf{D} - \mathbf{C}.\mathbf{A} - \mathbf{A}.\mathbf{D} + \|A\|^2 && (3.2.11)\\
&= 4\cos\beta - 4\cos\theta - 4\cos\theta_1 + 4 && (3.2.12)
\end{aligned}
$$

On substituting the angles , we get

$$(\mathbf{AC}) \,.\, (\mathbf{AD}) = 4.207 \tag{3.2.13}$$

Now substituting values from (3.2.13), (3.2.8), (3.2.4) in (3.2.9), we get γ.

$$
\begin{aligned}
\cos\gamma &= 0.866 && (3.2.14)\\
\gamma &= 30^\circ && (3.2.15)
\end{aligned}
$$

3.3. Applying the sum of interior angles in $\triangle ADE$

$$\angle ADE + \gamma + \angle AED = 180^\circ \tag{3.3.1}$$

Using (3.1.4), (3.2.15), we get

$$\angle AED = 60^\circ \tag{3.3.2}$$

$$\therefore \angle AEB = 60^\circ \tag{3.3.3}$$

Hence proved.