Exercise 8.5, Problem 16

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***Abstract*—This document provides the solution to the problem no. 16 given in the exercise 8.5. The figures are constructed using python and LaTeXcodes.**

This documentation can be downloaded from

```
svn co https://github.com/mohit-singh-9/Summer
    -2020/tree/master/geometry/circle.git
```

1 PROBLEM No. 16

AB is a diameter of the circle, ***CD*** is a chord equal to the radius of the circle. ***AC*** and ***BD*** when extended intersect at point ***E*** . Prove that $\angle AEB = 60°$.

2 CONSTRUCTION

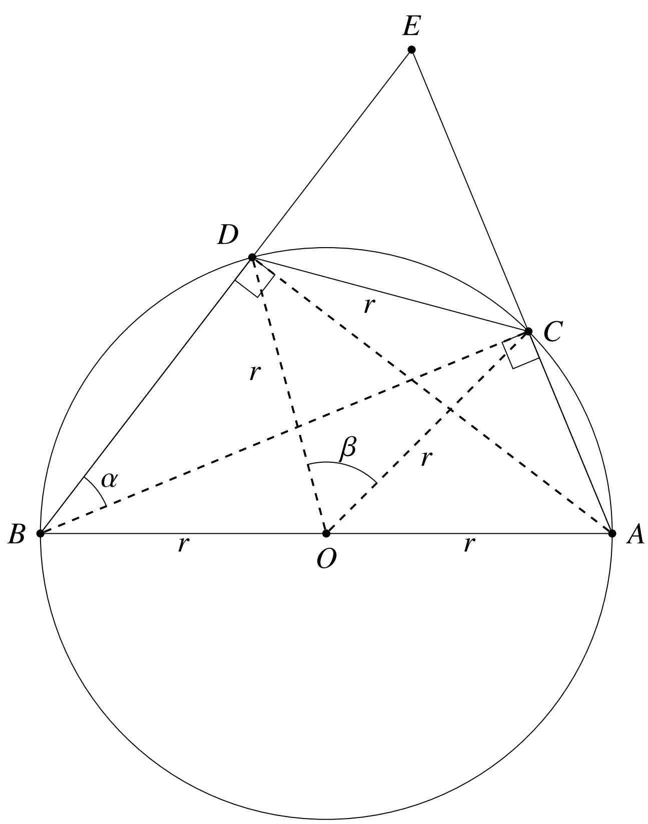

Fig. 2.0: Using Latex-Tikz

2.1. The following inputs were taken for constructing the figure:

Parameter	Value
Radius (r)	2
Centre (O)	(0,0)

TABLE 2.1: Input Table for construction

2.2. The coordinates of **A** and **B** such that **AB** is a diameter as in Fig. 2.0 are:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \tag{2.2.1}$$

$$\mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{2.2.2}$$

2.3. Let **C** be a point on circle such that its coordinates are:

$$\mathbf{C} = \begin{pmatrix} rcos\theta \\ rsin\theta \end{pmatrix} \tag{2.3.1}$$

where r = 2 units (radius of circle) and $\theta = 45°$ On Substituting,

$$\mathbf{C} = \begin{pmatrix} 1.414 \\ 1.414 \end{pmatrix} \tag{2.3.2}$$

Now **D** should be a point on the circle such that **CD** = 2 units (radius of the circle). Let,

$$\mathbf{D} = \begin{pmatrix} rcos\theta \\ rsin\theta \end{pmatrix} \tag{2.3.3}$$

$$= \begin{pmatrix} 2cos\theta \\ 2sin\theta \end{pmatrix} \tag{2.3.4}$$

D should be such that the magnitude of vector **CD** = 2 units. The vector **CD** is given by

$$\mathbf{CD} = (\mathbf{D} - \mathbf{C}) \tag{2.3.5}$$

Applying the distance formula using matrix multiplication,

$$(\mathbf{D} - \mathbf{C})^T (\mathbf{D} - \mathbf{C}) = \mid \mathbf{CD} \mid^2 \tag{2.3.6}$$

From (2.3.2), (2.3.4) and (2.3.6) we get

$$(2cos\theta - 1.414)^2 + (2sin\theta - 1.414)^2 = 4 \tag{2.3.7}$$

$$sin\theta + cos\theta = 0.707 \tag{2.3.8}$$

Using the identity

$$sin^2\theta + cos^2\theta = 1 \tag{2.3.9}$$

and (2.3.8) we get

$$sin\theta - cos\theta = \pm 1.225 \tag{2.3.10}$$

From (2.3.8), (2.3.10) we get θ as 105.011°and 345.016°.
Taking θ = 105.011°, we get

$$\mathbf{D} = \begin{pmatrix} -0.518 \\ 1.932 \end{pmatrix} \tag{2.3.11}$$

2.4. The point **E** is obtained by the intersection of the extended lines of **AC** and **BD**.
Using the corrollary that angle subtended by diameter at any point on the circle is 90°; $\angle ACB = \angle BDA = 90°$. This implies that

$$\mathbf{BC} \perp \mathbf{AC} \tag{2.4.1}$$

$$\mathbf{AD} \perp \mathbf{BD} \tag{2.4.2}$$

The equations of AC and BD are

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \tag{2.4.3}$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{x} - \mathbf{B}) = 0 \tag{2.4.4}$$

Since E lies on both these lines it will satisfy both the quations. Therefore,

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{E} - \mathbf{A}) = 0 \tag{2.4.5}$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{E} - \mathbf{B}) = 0 \tag{2.4.6}$$

On soving the above two equations you can find the coordinates of E.

$$\mathbf{E} = \begin{pmatrix} 0.597 \\ 3.385 \end{pmatrix} \tag{2.4.7}$$

2.5. The derived values are listed in Table. 2.5
2.6. For solving the problem, join **OC**, **OD**, **BC** and **AD**.
2.7. To get the python code for Fig 2.7, download it from

```
codes/circle.py
```

Derived Values	
	Coordinates (x,y)
A	(2,0)
B	(-2, 0)
C	(1.414, 1.414)
D	(-0.518, 1.932)
E	(0.597, 3.385)

TABLE 2.5: To construct △ ABC

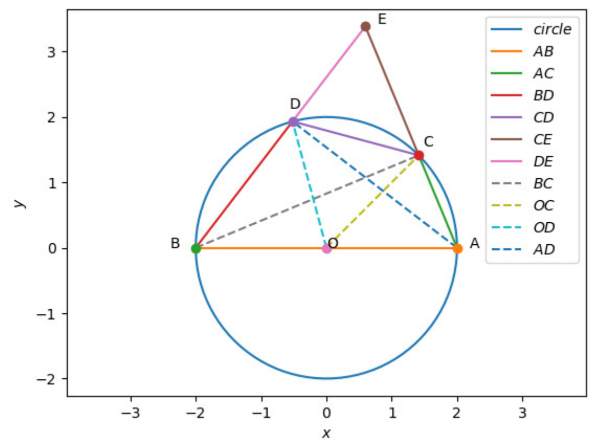

Fig. 2.7: Constructed using Python

and the equivalent latex-tikz code for Fig. 2.0 from

```
figs/circle.tex
```

The above latex code can be compiled as a standalone document as

```
figs/circle_fig.tex
```

3 SOLUTION

3.1. In $\triangle OCD$

$$OD = OC = r \tag{3.1.1}$$

$$CD = r \tag{3.1.2}$$

$$\tag{3.1.3}$$

$\therefore \triangle OCD$ is an equilateral triangle ,

$$\angle COD = \beta = 60° \tag{3.1.4}$$

3.2. In $\triangle CBD$
Using the Theorem : Angle subtended by chord

at the centre of circle is twice the angle subtended by it at any other point on the circle, we get

$$\angle CBD = \frac{\angle COD}{2} \tag{3.2.1}$$

$$= \frac{60^\circ}{2} \tag{3.2.2}$$

$$= 30^\circ \tag{3.2.3}$$

$$\implies \alpha = 30^\circ \tag{3.2.4}$$

3.3. In $\triangle BCA$,
We know that, angle subtended by a diameter at any point on circle is 90°.

$$\angle BCA = 90^\circ \tag{3.3.1}$$

$$\implies \angle BEC = 90^\circ \tag{3.3.2}$$

3.4. Applying the sum of interior angles in $\triangle EBC$

$$\angle BCE + \alpha + \angle BEC = 180^\circ \tag{3.4.1}$$

Using (3.2.4), (3.3.2) and (3.4.1), we get

$$\angle BEC = 60^\circ \tag{3.4.2}$$

$$\therefore \angle AEB = 60^\circ \tag{3.4.3}$$

Hence proved.