

Exercise 8.5, Problem 16

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Abstract—This document provides the solution to the problem no. 16 given in the exercise 8.5. The figures are constructed using python and \LaTeX codes.

This documentation can be downloaded from

svn co <https://github.com/mohit-singh-9/Summer-2020/tree/master/geometry/circle.git>

Parameter	Value	Decription
r	2	Radius of circle
O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of circle
θ	45°	Argument of C

TABLE 2.1: Input Table for construction

1 PROBLEM No. 16

AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at point E . Prove that $\angle AEB = 60^\circ$.

2 CONSTRUCTION

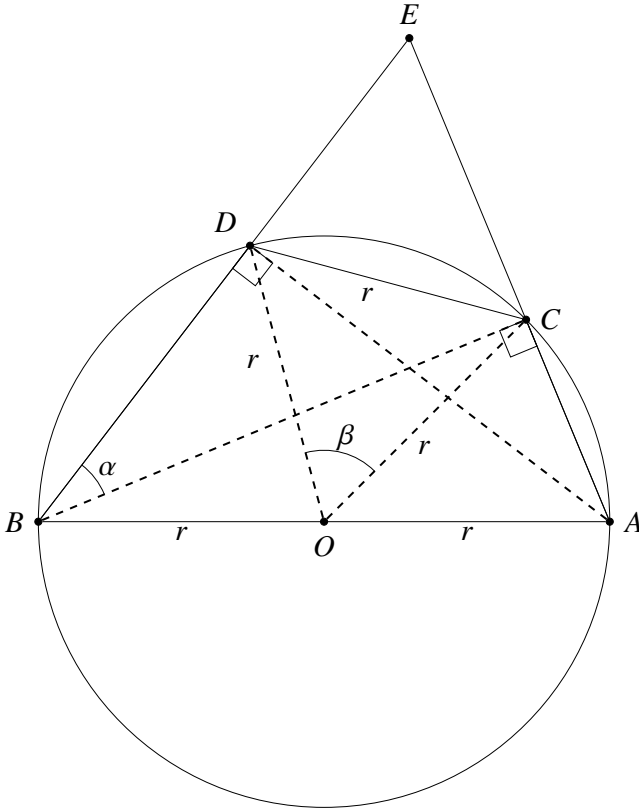


Fig. 2.0: Using Latex-Tikz

2.1. The following inputs were taken for constructing the figure:

2.2. The coordinates of A and B such that AB is a diameter as in Fig. 2.0 are:

$$A = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad (2.2.1)$$

$$B = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (2.2.2)$$

2.3. Let C be a point on circle such that its coordinates are:

$$C = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (2.3.1)$$

where $r = 2$ units (radius of circle) and $\theta = 45^\circ$

Now D is a point on the circle with $\|D\| = r = 2$ units (radius of the circle). Let,

$$D = r \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} \quad (2.3.2)$$

$$= 2 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} \quad (2.3.3)$$

D should be such that $\|CD\| = r = 2$ units. The vector CD is given by

$$CD = (D - C) \quad (2.3.4)$$

Applying the distance formula using matrix multiplication,

$$(D - C)^T (D - C) = \|CD\|^2 \quad (2.3.5)$$

$$(2.3.6)$$

On further expanding this expression,

$$(\mathbf{D}^T - \mathbf{C}^T)(\mathbf{D} - \mathbf{C}) = \|\mathbf{CD}\|^2 \quad (2.3.7)$$

$$\mathbf{D}^T \mathbf{D} - \mathbf{D}^T \mathbf{C} - \mathbf{C}^T \mathbf{D} + \mathbf{C}^T \mathbf{C} = \|\mathbf{CD}\|^2 \quad (2.3.8)$$

$$\|\mathbf{D}\|^2 - \mathbf{D}^T \mathbf{C} - \mathbf{C}^T \mathbf{D} + \|\mathbf{C}\|^2 = \|\mathbf{CD}\|^2 \quad (2.3.9)$$

Using (2.3.1), (2.3.3), $\|\mathbf{C}\|=2$ units and $\|\mathbf{D}\|=2$ units, we get

$$\sin \theta_1 + \cos \theta_1 = 0.707 \quad (2.3.10)$$

Using the identity

$$\sin^2 \theta_1 + \cos^2 \theta_1 = 1 \quad (2.3.11)$$

and (2.3.10) we get

$$\sin \theta_1 - \cos \theta_1 = \pm 1.225 \quad (2.3.12)$$

From (2.3.10), (2.3.12) we get θ_1 as 105.011° and 345.016° .

Taking $\theta_1 = 105.011^\circ$, we get \mathbf{D} .

- 2.4. The point \mathbf{E} is obtained by the intersection of the extended lines of \mathbf{AC} and \mathbf{BD} .

Using the corollary that angle subtended by diameter at any point on the circle is 90° ; $\angle ACB = \angle BDA = 90^\circ$. This implies that

$$\mathbf{BC} \perp \mathbf{AC} \quad (2.4.1)$$

$$\mathbf{AD} \perp \mathbf{BD} \quad (2.4.2)$$

The equations of \mathbf{AC} and \mathbf{BD} are

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \quad (2.4.3)$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{x} - \mathbf{B}) = 0 \quad (2.4.4)$$

Since \mathbf{E} lies on both these lines it will satisfy both the equations. Therefore,

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{E} - \mathbf{A}) = 0 \quad (2.4.5)$$

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{E} - \mathbf{B}) = 0 \quad (2.4.6)$$

On solving the above two equations you can find the coordinates of \mathbf{E} .

- 2.5. The derived values are listed in Table. 2.5
 2.6. For solving the problem, join \mathbf{OC} , \mathbf{OD} , \mathbf{BC} and \mathbf{AD} .
 2.7. To get the python code for Fig 2.7, download it from

codes/circle.py

and the equivalent latex-tikz code for Fig. 2.0 from

Derived Values	
	Coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$
A	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
B	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
C	$\begin{pmatrix} 1.414 \\ 1.414 \end{pmatrix}$
D	$\begin{pmatrix} -0.518 \\ 1.932 \end{pmatrix}$
E	$\begin{pmatrix} 0.597 \\ 3.385 \end{pmatrix}$

TABLE 2.5: To construct the figure

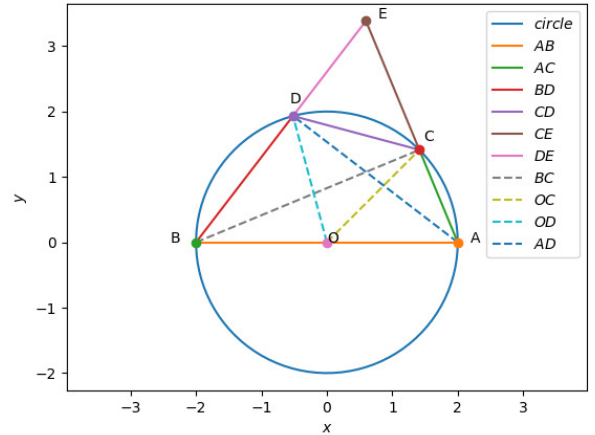


Fig. 2.7: Using Python

figs/circle.tex

The above latex code can be compiled as a standalone document as

figs/circle_fig.tex

3 SOLUTION

3.1. In $\triangle OCD$

$$OD = OC = r \quad (3.1.1)$$

$$CD = r \quad (3.1.2)$$

$$(3.1.3)$$

$\therefore \triangle OCD$ is an equilateral triangle ,

$$\angle COD = \beta = 60^\circ \quad (3.1.4)$$

3.2. In $\triangle CBD$

Using the Theorem : Angle subtended by chord at the centre of circle is twice the angle subtended by it at any other point on the circle, we get

$$\angle CBD = \frac{\angle COD}{2} \quad (3.2.1)$$

$$= \frac{60^\circ}{2} \quad (3.2.2)$$

$$= 30^\circ \quad (3.2.3)$$

$$\implies \alpha = 30^\circ \quad (3.2.4)$$

3.3. In $\triangle BCA$,

We know that, angle subtended by a diameter at any point on circle is 90° .

$$\angle BCA = 90^\circ \quad (3.3.1)$$

$$\implies \angle BEC = 90^\circ \quad (3.3.2)$$

3.4. Applying the sum of interior angles in $\triangle EBC$

$$\angle BCE + \alpha + \angle BEC = 180^\circ \quad (3.4.1)$$

Using (3.2.4), (3.3.2) and (3.4.1), we get

$$\angle BEC = 60^\circ \quad (3.4.2)$$

$$\therefore \angle AEB = 60^\circ \quad (3.4.3)$$

Hence proved.