

Linear Algebra

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Abstract—This document provides the solution to the problem no. 7 of each section under linear algebra. The figures are constructed using python.

This documentation can be downloaded from

svn co https://github.com/mohit-singh-9/Summer-2020/tree/master/geometry/linear_algebra.git

Input values	
A	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$
B	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
C	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

TABLE 1.0: Input Table for construction

1 TRIANGLE EXERCISE

1.1 Problem

Find the area of the triangle formed by joining the midpoints of the sides of the triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

1.2 Solution

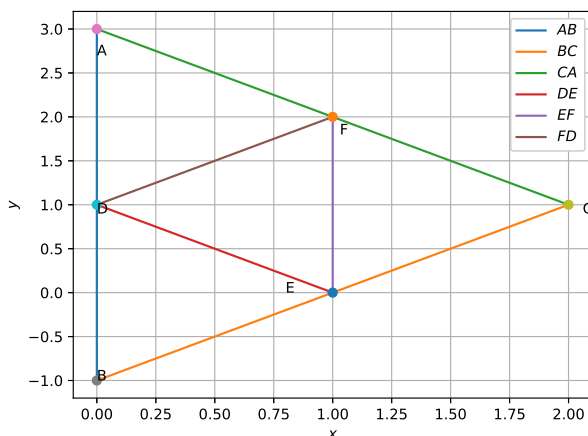


Fig. 1.0: Triangle DEF formed from midpoints of Triangle ABC

1.1. Let $\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Derived values	
D	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
E	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
F	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

TABLE 1.0: Derived values

1.2. The midpoints of each side is given by

$$\mathbf{D} = \frac{\mathbf{A} + \mathbf{B}}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.2.1)$$

$$\mathbf{E} = \frac{\mathbf{B} + \mathbf{C}}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.2.2)$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{C}}{2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.2.3)$$

$$(1.2.4)$$

1.3. Area of a $\triangle ABC$ is given by

$$= \frac{1}{2} \|(\mathbf{E} - \mathbf{D}) \times (\mathbf{F} - \mathbf{D})\| \quad (1.3.1)$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \quad (1.3.2)$$

On solving we get area of $\triangle DEF = 1$ sq.units

1.4. Download the python code for finding a triangle's area from

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codes\triangle\area_tri_area.py
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and the figure from

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figs\triangle\draw_triangle.py
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2 QUADRILATERAL EXERCISE

2.1 Problem

The two opposite vertices of a square are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of other two vertices.

2.2 Solution

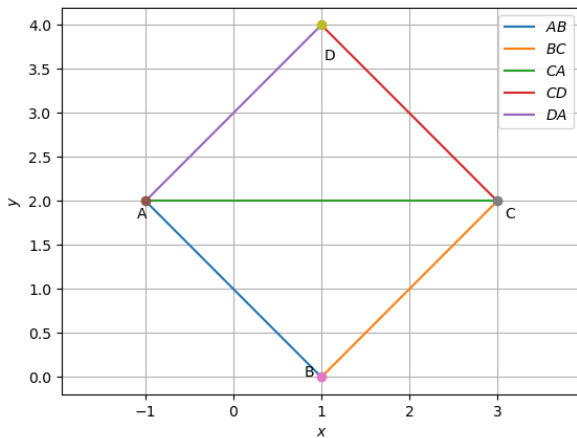


Fig. 2.0: Square ABCD

- 2.1. From inspection we see that the opposite vertices forms a diagonal which is parallel to x-axis. Then the diagonal formed by other two vertices is parallel to y-axis(i.e. their x coordinates are equal). Let $\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
- 2.2. Diagonals bisect each other at 90° . Let \mathbf{B} and \mathbf{D} be other two vertices.
- 2.3. Using the property that diagonals bisect each other at 90° , we can obtain other vertices by rotating diagonal AC by 90° about \mathbf{E} in clockwise or anticlockwise direction.
- 2.4. The rotation matrix for a rotation of angle 90° about origin in anticlockwise direction is given by

$$\begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (2.4.1)$$

The \mathbf{E} is given by

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \quad (2.4.2)$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.4.3)$$

- 2.5. To make the rotation we need to shift the \mathbf{E} to origin. So the change in other vectors are

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (2.5.1)$$

$$\mathbf{C} - \mathbf{E} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (2.5.2)$$

The required matrix now is $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$. Multiplying this with rotation matrix

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \quad (2.5.3)$$

$$= \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \quad (2.5.4)$$

Now we obtained the coordinates as $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. To obtain the final coordinates we will add \mathbf{E} to shift to the actual position.

$$\mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.5.5)$$

$$\mathbf{D} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.5.6)$$

Thus

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.5.7)$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (2.5.8)$$

- 2.6. The python code for the figure can be downloaded from

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codes/quad/quad.py
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3 LINE EXERCISE

3.1 Matrix

3.1.1 Problem:

3.1. Given $A = \begin{pmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{pmatrix}$. Find $A+B$.

3.1.2 Solution:

3.1. Since the two matrices have equal number of rows and columns, they are summable. Every element of a matrix gets added to its corresponding element in other matrix.

3.2. So

$$A + B = \begin{pmatrix} \sqrt{3} + 2 & \sqrt{5} + 1 & 0 \\ 0 & 6 & \frac{1}{2} \end{pmatrix} \quad (3.2.1)$$

3.3. The python code for matrix addition can be downloaded from

codes/line/matrix/matrix_add.py

3.2 Complex Numbers

3.2.1 Problem:

1. Find the modulus and argument of the complex numbers:

a) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3.2.2 Solution:

1. A complex number $z = a + ib$ where $i = \sqrt{-1}$ is represented in vector notation as $\begin{pmatrix} a \\ b \end{pmatrix}$.

2. The multiplication of two complex numbers is not same as the multiplication of two vectors. It involves rotation of axes.

3. Suppose $z_1 = r_1 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}$ and $z_2 = r_2 \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}$ be two complex numbers, then $z_1 \cdot z_2 = r_1 r_2 \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix}$. Through vectors and matrices it can be realised through

$$z_1 \cdot z_2 = r_1 r_2 \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix} \quad (3.2.2.3.1)$$

where $\mathbf{R} = \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}$ is the rotation matrix.

4. Similarly division of two complex numbers is given by $z_1 \cdot z_2^{-1} = \frac{r_1}{r_2} \begin{pmatrix} \cos(\theta_1 - \theta_2) \\ \sin(\theta_1 - \theta_2) \end{pmatrix}$ and through matrices multiplication as

$$z_1 \cdot z_2^{-1} = \frac{r_1}{r_2} \begin{pmatrix} \cos \theta_1 & \sin \theta_1 \\ \sin \theta_1 & -\cos \theta_1 \end{pmatrix} \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix} \quad (3.2.2.4.1)$$

where $\mathbf{S} = \begin{pmatrix} \cos \theta_1 & \sin \theta_1 \\ \sin \theta_1 & -\cos \theta_1 \end{pmatrix}$ is the rotation matrix.

5. First converting the given vectors in polar form

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \frac{\sqrt{2} \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix}}{\sqrt{2} \begin{pmatrix} \cos(-45^\circ) \\ \sin(-45^\circ) \end{pmatrix}} \quad (3.2.2.5.1)$$

Since this is the division of two complex numbers

$$= \frac{\sqrt{2}}{\sqrt{2}} \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ \sin 45^\circ & -\cos 45^\circ \end{pmatrix} \begin{pmatrix} \cos(-45^\circ) \\ \sin(-45^\circ) \end{pmatrix} \quad (3.2.2.5.2)$$

$$= 1 \cdot \begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \end{pmatrix} \quad (3.2.2.5.3)$$

The magnitude is 1 and argument is 90° .

6. Here the numerator can be made a vector by taking y coordinate as 0. Also converting the vectors in polar form

$$\frac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad (3.2.2.6.1)$$

$$= \frac{1 \begin{pmatrix} \cos 0^\circ \\ \sin 0^\circ \end{pmatrix}}{\sqrt{2} \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix}} \quad (3.2.2.6.2)$$

Since its a division of two complex numbers, it can solved by

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 0^\circ & \sin 0^\circ \\ \sin 0^\circ & -\cos 0^\circ \end{pmatrix} \begin{pmatrix} \cos(45^\circ) \\ \sin(45^\circ) \end{pmatrix} \quad (3.2.2.6.3)$$

$$= 1 \cdot \begin{pmatrix} \cos(-45^\circ) \\ \sin(-45^\circ) \end{pmatrix} \quad (3.2.2.6.4)$$

The magnitude is $\frac{1}{\sqrt{2}}$ and argument is -45° .

3.3 Points and Vectors

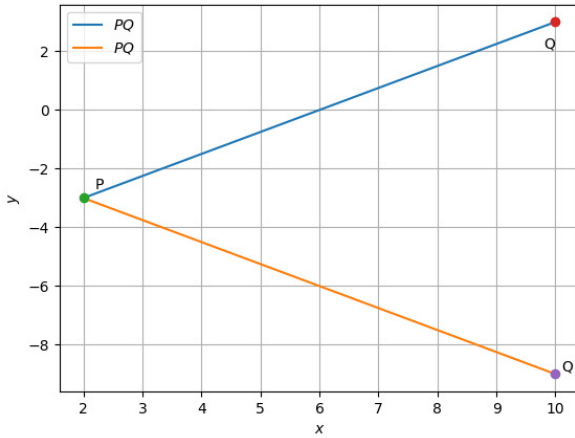
3.3.1 Problem:

- Find the values of y for which distance between points

$$P = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, Q = \begin{pmatrix} 10 \\ y \end{pmatrix} \quad (3.3.1.1.1)$$

is 10 units.

3.3.2 Solution:



- The distance between two points is given by equation

$$(\mathbf{P} - \mathbf{Q})^T (\mathbf{P} - \mathbf{Q}) = 10^2 \quad (3.3.2.1.1)$$

$$(\mathbf{P}^T - \mathbf{Q}^T)(\mathbf{P} - \mathbf{Q}) = 100 \quad (3.3.2.1.2)$$

$$\|\mathbf{P}\|^2 - \mathbf{P}^T \mathbf{Q} - \mathbf{Q}^T \mathbf{P} + \|\mathbf{Q}\|^2 = 100 \quad (3.3.2.1.3)$$

On substituting

$$y^2 + 6y - 27 = 0 \quad (3.3.2.1.4)$$

$$(y + 9)(y - 3) = 0 \quad (3.3.2.1.5)$$

The roots of the above quadratic equation are -9 and 3. Therefore the values of $y = -9, 3$

- Both the lines in the graph have length equal to 10 units. \mathbf{Q} can have two values $\begin{pmatrix} 10 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ -9 \end{pmatrix}$.
- The python code to find the roots of the quadratic equation can be downloaded from

[codes/line/point_vec/roots.py](#)

- The python code for the figure can be downloaded from

[codes/line/point_vec/point_vec.py](#)

3.4 Points on a Line

3.4.1 Problem:

- Find the coordinates of points which divide the line segment joining $A = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ into four equal parts.

3.4.2 Solution:

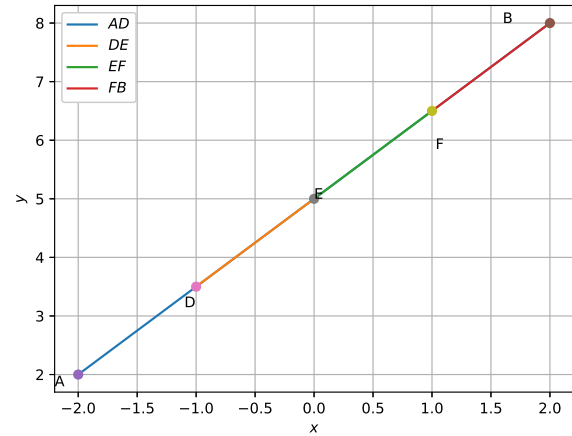


Fig. 3.4.2.0: Line segment AB

Input values	
A	$\begin{pmatrix} -2 \\ 2 \end{pmatrix}$
B	$\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

TABLE 3.4.2.0: Input Table for construction

Derived values	
D	$\begin{pmatrix} -1 \\ 7/2 \end{pmatrix}$
E	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$
F	$\begin{pmatrix} 1 \\ 13/2 \end{pmatrix}$

TABLE 3.4.2.0: Derived values

- Let **D, E, F** be the points that divide the line segment into four equal parts.

2. If a point **X** divides a line segment(here AB) in the ratio of m:n then its coordinates are given by

$$\mathbf{X} = \frac{n\mathbf{B} + m\mathbf{A}}{m + n} \quad (3.4.2.2.1)$$

3. From figure, points **D, E, F** divides AB in the ratio of 1:3, 2:2, 3:1 respectively. Thus there coordinates are given by

$$\mathbf{D} = \frac{1\mathbf{B} + 3\mathbf{A}}{4} = \begin{pmatrix} -1 \\ 7/2 \end{pmatrix} \quad (3.4.2.3.1)$$

$$\mathbf{E} = \frac{2\mathbf{B} + 2\mathbf{A}}{4} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad (3.4.2.3.2)$$

$$\mathbf{F} = \frac{3\mathbf{B} + 1\mathbf{A}}{4} = \begin{pmatrix} 1 \\ 13/2 \end{pmatrix} \quad (3.4.2.3.3)$$

4. Download the python code for figure from

codes/line/point_line/line_division.py

3.5 Lines and Plane

3.5.1 Problem:

1. Check which of the following are solutions of the equation

$$(1 \quad -2)\mathbf{x} = 4 \quad (3.5.1.1)$$

- | | |
|---|--|
| a) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ | d) $\begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix}$ |
| b) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ | e) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ |
| c) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ | |

3.5.2 Solution:

1. Substitute given vectors from options in the above line equation and check which will satisfy it.
2. Answer = $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
3. Also from Fig.3.5.0, the line passes through the point $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.
4. The python code for the figure

codes/line/lines_planes/lines_planes.py

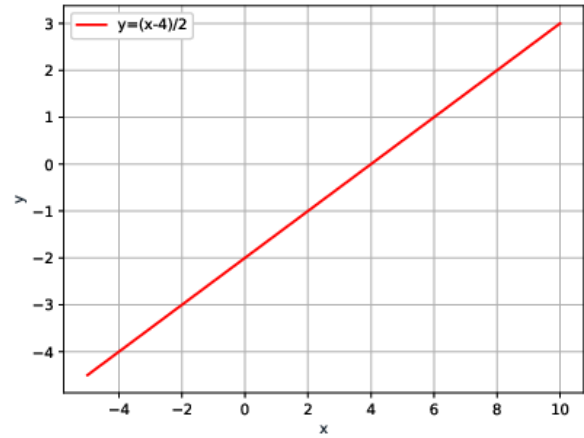


Fig. 3.5.0: Line equation: $y = (x-4)/2$

3.6 Motion in a Plane

3.6.1 Problem:

1. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?

3.6.2 Solution:

1. Let +x axis be east and +y be north direction. Also let \mathbf{v}_b and \mathbf{v}_w represent the velocity of boat and wind respectively along.
2. Then

$$\mathbf{v}_w = \begin{pmatrix} 72 \cos 45^\circ \\ 72 \sin 45^\circ \end{pmatrix} \quad (3.6.2.1)$$

$$\mathbf{v}_b = \begin{pmatrix} 0 \\ 51 \end{pmatrix} \quad (3.6.2.2)$$

3. The direction of the flag on the boat will be the relative velocity of wind w.r.t boat. So let \mathbf{v}_{wb} represent the direction of flag. Then

$$\mathbf{v}_{wb} = \mathbf{v}_w - \mathbf{v}_b \quad (3.6.3.1)$$

$$= \begin{pmatrix} 36\sqrt{2} \\ 36\sqrt{2} - 51 \end{pmatrix} = \begin{pmatrix} 50.91 \\ -0.09 \end{pmatrix} \quad (3.6.3.2)$$

4. Let the angle made by \mathbf{v}_{wb} w.r.t x-axis(east) be α . Then

$$\alpha = \tan^{-1}\left(\frac{-0.09}{50.91}\right) \quad (3.6.4.1)$$

$$= -0.1^\circ \quad (3.6.4.2)$$

5. The direction of flag on the boat is 0.1° w.r.t east.

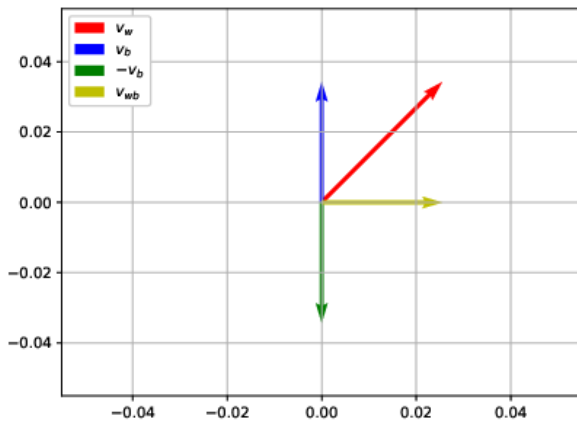


Fig. 3.6.5: Vectors representing different velocities

6. The python code for the figure can be downloaded from

`codes/line/motion/motion.py`

3.7 Determinants

3.7.1 Problem:

1. Find values of x , if

$$\text{a) } \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} \quad \text{b) } \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$

3.7.2 Solution:

1. If $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$, then applying the same to above question and solve the equation

$$-18 = 2x^2 - 24 \quad (3.7.1.1)$$

$$x = \pm \sqrt{3} \quad (3.7.1.2)$$

2. Following the same steps as above we get,

$$-2 = 5x - 6x \quad (3.7.2.1)$$

$$x = 2 \quad (3.7.2.2)$$

3.8 Linear Inequalities

3.8.1 Problem:

- 3.1. Solve $3x - 6 \geq 0$ graphically in a two dimensional plane.

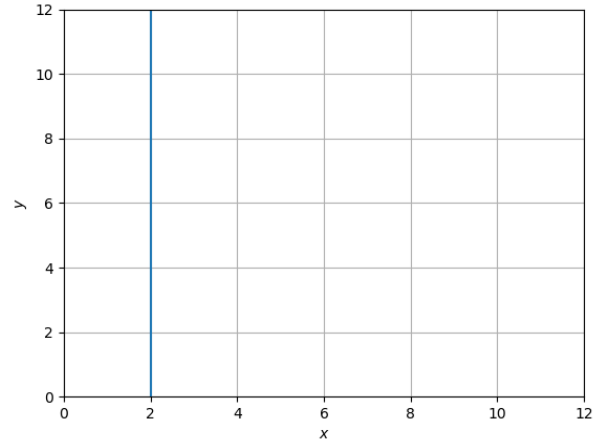


Fig. 3.0: Area satisfying $x \geq 2$

3.8.2 Solution:

- 3.1. If \mathbf{x} is a vector then the given inequality can be represented as

$$(3 \ 0)\mathbf{x} - 6 \geq 0 \quad (3.1.1)$$

On solving we get $x \geq 2$. No such constraint is on y . Graphically the solution is the whole region which lies to the right of line $(1 \ 0)\mathbf{x} = 2$ in a 2D plane.

- 3.2. The python code can be downloaded from

`codes/line/lin_ineq/lin_ineq1.py`

3.9 Miscellaneous

3.9.1 Problem:

1. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ when
- PQ is parallel to the y -axis.
 - PQ is parallel to the x -axis.

3.9.2 Solution:

1. If PQ is parallel to y axis then x coordinates doesn't change. Therefore $x_1 = x_2 = x$. Hence, $\mathbf{P} = \begin{pmatrix} x \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x \\ y_2 \end{pmatrix}$. Distance between \mathbf{P}

and \mathbf{Q} is given by

$$\sqrt{(\mathbf{P} - \mathbf{Q})^T (\mathbf{P} - \mathbf{Q})} \quad (3.9.1.1)$$

$$= \sqrt{\begin{pmatrix} 0 \\ y_1 - y_2 \end{pmatrix}^T \begin{pmatrix} 0 \\ y_1 - y_2 \end{pmatrix}} \quad (3.9.1.2)$$

$$= y_1 - y_2 \quad (3.9.1.3)$$

Distance between the points is $y_1 - y_2$

2. If PQ is parallel to x axis then y coordiantes doesn't change. Therefore $y_1 = y_2 = y$. Hence, $\mathbf{P} = \begin{pmatrix} x_1 \\ y \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 \\ y \end{pmatrix}$. Distance between \mathbf{P} and \mathbf{Q} is given by

$$\sqrt{(\mathbf{P} - \mathbf{Q})^T (\mathbf{P} - \mathbf{Q})} \quad (3.9.2.1)$$

$$= \sqrt{\begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix}^T \begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix}} \quad (3.9.2.2)$$

$$= x_1 - x_2 \quad (3.9.2.3)$$

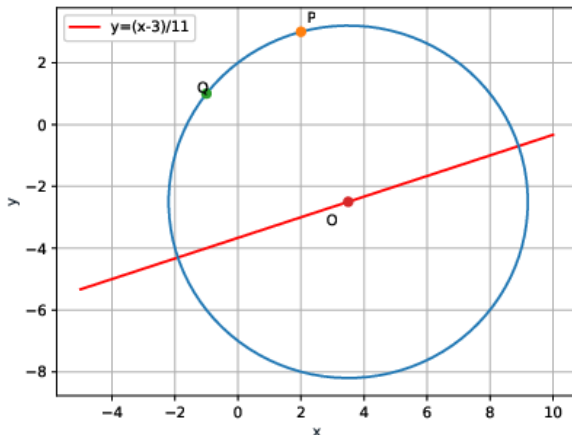
Distance between the points is $x_1 - x_2$

4 CIRCLE EXERCISE

4.1 Problem

Find the equation of the circle passing through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and whose centre is on the line $(1 \ -3)\mathbf{x} = 11$.

4.2 Solution



Input values	
\mathbf{P}	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
\mathbf{Q}	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
\mathbf{O}	$\begin{pmatrix} 7/2 \\ -5/2 \end{pmatrix}$
Line eqn.	$(1 \ -3)\mathbf{x} = 11$

TABLE 4.0: Input Table for construction

Derived value	
r	5.7

TABLE 4.0: Derived values while construction

- 4.1. Let \mathbf{O} be the centre of the circle and r be the radius of the circle. Since centre lies on the line, it satisfies the line equation

$$(1 \ -3)\mathbf{O} = 11 \quad (4.1.1)$$

- 4.2. Also the circle passes through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Let these points be \mathbf{P} and \mathbf{Q} respectively. So the distance between centre and these points will be equal to the radius.

$$\|\mathbf{P} - \mathbf{O}\| = \|\mathbf{Q} - \mathbf{O}\| = r \quad (4.2.1)$$

On solving we get the equation

$$(6 \ 4)\mathbf{O} = 11 \quad (4.2.2)$$

- 4.3. The equations from (4.1.1) and (4.2.2), can be solved to get \mathbf{O} .

$$\begin{pmatrix} 1 & -3 \\ 6 & 4 \end{pmatrix} \mathbf{O} = \begin{pmatrix} 11 \\ 11 \end{pmatrix} \quad (4.3.1)$$

$$\mathbf{O} = \begin{pmatrix} 1 & -3 \\ 6 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 11 \\ 11 \end{pmatrix} \quad (4.3.2)$$

$$\mathbf{O} = \frac{1}{22} \begin{pmatrix} 77 \\ -55 \end{pmatrix} \quad (4.3.3)$$

$$\text{Hence } \mathbf{O} = \begin{pmatrix} 7/2 \\ -5/2 \end{pmatrix}$$

- 4.4. Substituting \mathbf{O} we get $r = 5.7$

- 4.5. Equation of circle is

$$\|\mathbf{x} - \mathbf{O}\| = 5.7 \quad (4.5.1)$$

- 4.6. The python code for the figure

codes/circle/circle.py

5 CONICS EXERCISE

5.1 Problem

Find the roots of the following quadratic equations:

- 1) $2x^2 - 7x + 3 = 0$
- 2) $2x^2 + x - 4 = 0$
- 3) $4x^2 + 4\sqrt{3}x + 3 = 0$.
- 4) $2x^2 + x + 4 = 0$.

5.2 Solution

5.1. A conic section has the following equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (5.1.1)$$

The equation is expressed in vector form is as follows

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + (D \ E) \mathbf{x} + F = 0 \quad (5.1.2)$$

a) $2x^2 - 7x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + (-7 \ 0) \mathbf{x} + 3 = 0 \quad (5.1.3)$$

If $\begin{pmatrix} k \\ 0 \end{pmatrix}$ satisfies (5.1.3) then k is the root of the equation (5.1.3).

From graph, the roots are the points where the quadratic equation cuts the x-axis. A quadratic equation can have a maximum of two distinct roots.

$$2k^2 - 7k + 3 = 0 \quad (5.1.4)$$

$$(k - 3)(2k - 1) = 0 \quad (5.1.5)$$

From the graph in 5.1, the roots are 3 and $\frac{1}{2}$.
The python code can be downloaded from

codes/conics/parabola1.py

b) $2x^2 + x - 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + (1 \ 0) \mathbf{x} - 4 = 0 \quad (5.1.6)$$

From the 5.1, the roots are 1.186 and 1.686.
The python code can be downloaded from

codes/conics/parabola2.py

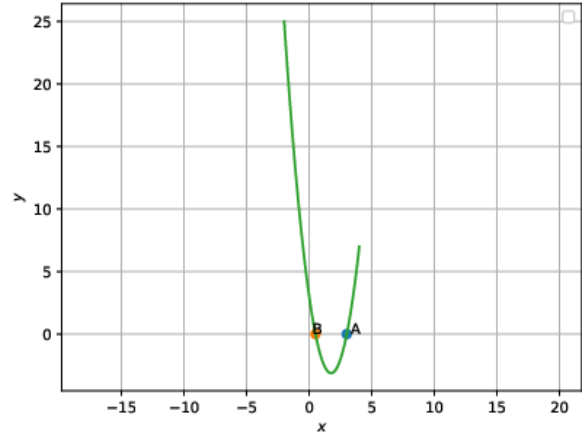


Fig. 5.1: Roots of $2x^2 - 7x + 3 = 0$

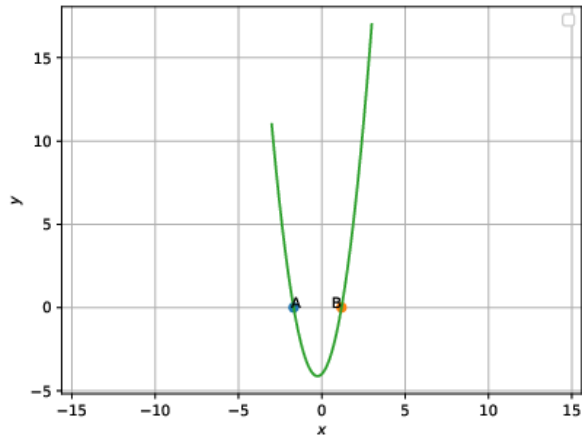


Fig. 5.1: Roots of $2x^2 + x - 4 = 0$

c) $4x^2 + 4\sqrt{3}x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + (4\sqrt{3} \ 0) \mathbf{x} + 3 = 0 \quad (5.1.7)$$

From the graph in 5.1, the roots are real and equal. The root is $-\frac{\sqrt{3}}{2}$. The python code can be downloaded from

codes/conics/parabola3.py

d) $2x^2 + x + 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + (1 \ 0) \mathbf{x} + 4 = 0 \quad (5.1.8)$$

From the graph 5.1, the quadratic equation doesn't intersect x-axis. Thus it doesn't have real roots. It has complex and conjugate

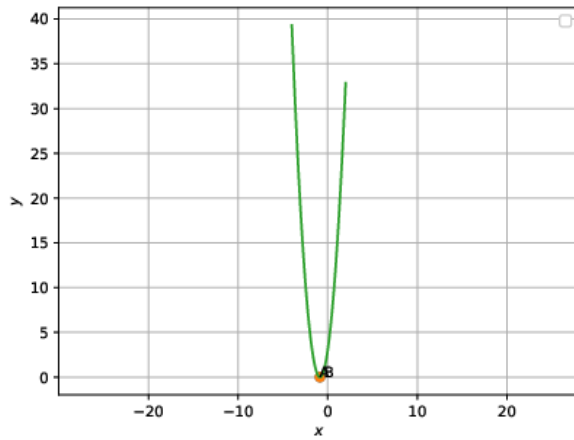


Fig. 5.1: Roots of $4x^2 + 4\sqrt{3}x + 3 = 0$

roots. The python code can be downloaded

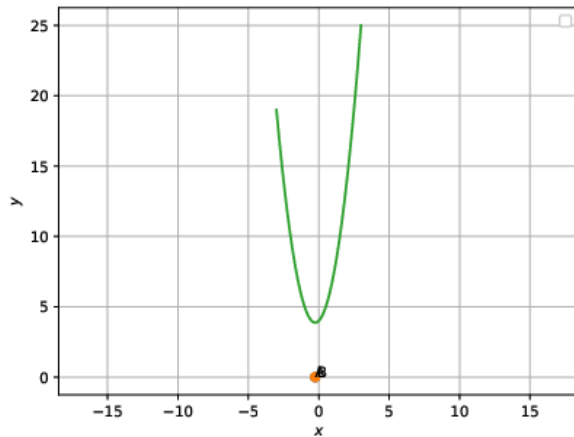


Fig. 5.1: Roots of $2x^2 + x + 4 = 0$

from

`codes/conics/parabola4.py`