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⇒ Continued fraction method

→ Continued fractions

Given a real number x, we construct its continued fraction expansion as follows:-

i) Let $a_0 = [x]$; where $[x]$ denote the greatest integer not ~~greater~~ greater than x and set $x_0 = x - a_0$.

ii) let $a_1 = \left[\frac{1}{x_0}\right]$ and set $x_1 = \frac{1}{x_0} - a_1$

iii) for $i > 1$, let $a_i = \left[\frac{1}{x_{i-1}}\right]$, and set $x_i = \frac{1}{x_{i-1}} - a_i$

iv) when $\frac{1}{x_{i-1}}$ is an integer then we find $x_i = 0$ and process stops.

→ It's not hard to see that process terminates if and only if x is rational {b/c in that case the x_i are rational numbers with ↓ denominators}, or ~~we can~~

⇒ <u>Notations</u>

→ Because of construction of $a_0, a_1, \ldots a_i$ for each i,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_i + x_i}}}$$

→ <u>Compact Notation</u> →

$$x = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \frac{1}{a_i + x_i}$$

→ Suppose x is irrational real no, If we carry out above expression to i^{th} term and then delete x_i, we obtain a rational no b_i/c_i called the i^{th} convergent of continued fraction for x:

$$\frac{b_i}{c_i} = a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}\cdots\frac{1}{a_{i-1} +}\frac{1}{a_i}$$

Note [I]

→ a) $\frac{b_0}{c_0} = \frac{a_0}{1}$; $\frac{b_1}{c_1} = \frac{a_0 a_1 + 1}{a_1}$; $\frac{b_i}{c_i} = \frac{a_i b_{i-1} + b_{i-2}}{a_i c_{i-1} + c_{i-2}}$

(for $i \geq 2$)

b) $b_i c_{i-1} - b_{i-1} c_i = (-1)^{i+1}$ for $i \geq 1$

Proof-(a) for this first we define sequence $\{b_i\}$ and $\{c_i\}$ and then ~~t~~ say that b_i/c_i is ~~t~~ i^{th} convergent.

- Now suppose that the claim is true for i^{th} convergent, and we prove the claim for $(i+1)^{th}$ convergent.

- We obtain $(i+1)^{th}$ convergent by replacing a_i by (a_{i+1}/a_{i+1}) in formula that expresses the nume. and denom. of i^{th} convergent in terms of ~~i+~~ $(i-1)^{th}$ & ~~a~~ $(i-2)^{th}$.

- i.e., $\Rightarrow$

$$\frac{\left(a_i + \frac{1}{a_{i+1}}\right) b_{i-1} + b_{i-2}}{\left(a_i + \frac{a}{a_{i+1}}\right) c_{i-1} + c_{i-2}}$$

$$\frac{(a_{i+1})\overbrace{(a_i b_{i-1} + b_{i-2})}^{b_i} + b_{i-1}}{(a_{i+1})\underbrace{(a_i c_{i-1} + c_{i-2})}_{c_i} + c_{i-1}} = \frac{a_{i+1} b_i + b_{i-1}}{a_{i+1} c_i + c_{i-1}}$$

by ~~induc~~ asumption. hence proved

(B)- Let the inequality hold for i, we will show for $(i+1)$

$$b_{i+1} c_i - b_i c_{i+1} = (a_{i+1} b_i + b_{i-1}) c_i - b_i (a_{i+1} c_i + c_{i-1})$$

$$= b_{i-1} c_i - b_i c_{i-1}$$

$$= -(-1)^{i-1} = (-1)^i$$

Proved

Note 1 → If we divide the eqⁿ in ~~last part of~~ by $c_i c_{i-1}$, we find that,

$$\frac{b_i}{c_i} - \frac{b_{i-1}}{c_{i-1}} = \frac{(-1)^{i-1}}{c_i c_{i-1}}.$$

Since c_i form a strictly increasing sequence of +ve integers, this inequality shows the sequence of convergents behaves like an alternating series, i.e. oscillates back and forth with shrinking Amplitude; Thus, the sequence of convergents converges to a limit.

Note 2 - limit of convergents is ~~the~~ the no. x which was expanded in the first place. To ~~see~~ see that, x can be obtained by forming $(i+1)^{th}$ convergent with a_{i+1} replaced by $1/x_i$. Thus using part (a) of above proposition ($i \to i+1$ and $a_{i+1} \longrightarrow 1/x_i$) we have,

$$x = \frac{\frac{b_i}{x_i} + b_{i-1}}{\frac{c_i}{x_i} + c_{i-1}} = \frac{b_i + x_i b_{i-1}}{c_i + x_i c_{i-1}}$$

which is strictly b/w $\left(\frac{b_i}{c_i}\right)$ and $\left(b_{i-1}/c_{i-1}\right)$

To see this,

Consider 2 vectors :- ~~v~~ $u = (b_i, c_i)$ & $v = (b_{i-1}, c_{i-1})$, both in same quadrant, and we can observe that the slope of vector $u + x_i v$ is intermediate b/w u and v.

Thus, sequence $\frac{b_i}{c_i}$ oscillates around x and converges to x.

Ex → To Expand $\sqrt{3}$ as continued fraction.

→ So Using above procedure,

$$\sqrt{3} = 1 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\cdots\frac{1}{2+}\cdots \quad — (1)$$

We might conjecture that a_i's alternat b/w 1 and 2,

To prove this, let x = RHS of (1),

i.e., $x = 1 + \cfrac{1}{\underbrace{1 + \left(1/(1+x)\right)}}$,

Replacing

↳ definition of Continued fraction

Simplifying expression on RHS and multiply both sides by $(2+x)$

↳ $2x + x^2 = 3 + 2x$

$\boxed{x = \sqrt{3}}$