Lecture IV: Nash Equilibrium

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Readings for this class: Osborne and Rubinstein, Chapter 2.1-2.3; FT has a good section on the connection to IDSDS.

Iterated dominance is an attractive solution concept because it only assumes that all players are rational and that it is common knowledge that every player is rational (although this might be too strong an assumption as our experiments showed). It is essentially a constructive concept - the idea is to restrict my assumptions about the strategy choices of other players by eliminating strategies one by one.

For a large class of games iterated deletion of strictly dominated strategies significantly reduces the strategy set. However, only a small class of games are solvable in this way (such as Counot competition with linear demand curve).

Today we introduce the most important concept for solving games: Nash equilibrium. We will later show that all finite games have at least one Nash equilibrium, and that the set of Nash equilibria is a subset of the strategy profiles which survive iterated deletion. In that sense, Nash equilibrium makes stronger predictions than iterated deletion would but it is not excessively strong in the sense that it does not rule out any equilibrium play for some games.

Definition 1 *A strategy profile s^* is a pure strategy Nash equilibrium of G if and only if*

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$$

for all players i and all $s_i \in S_i$.

Definition 2 *A pure strategy NE is strict if*

$$u_i\left(s_i^*, s_{-i}^*\right) > u_i\left(s_i, s_{-i}^*\right)$$

A Nash equilibrium captures the idea of equilibrium. Both players know what strategy the other player is going to choose, and no player has an incentive to deviate from equilibrium play because her strategy is a best response to her belief about the other player's strategy.

1 Games with Unique NE

1.1 Prisoner's Dilemma

	C	D
C	3,3	−1,4
D	4,−1	0,0

This game has the unique Nash equilibrium (D,D). It is easy to check that each player can profitably deviate from every other strategy profile. For, example (C,C) cannot be a NE because player 1 would gain from playing D instead (as would player 2).

1.2 Example II

	L	M	R
U	2,2	1,1	4,0
D	1,2	4,1	3,5

In this game the unique Nash equilibrium is (U,L). It is easy to see that (U,L) is a NE because both players would lose from deviating to any other strategy.

To show hat there are no other Nash equilibria we could check each other strategy profile, or note that $S_1^\infty = \{U\}$ and $S_2^\infty = \{L\}$ and use:

Proposition 1 *If s^* is a pure strategy Nash equilibrium of G then $s^* \in S^\infty$.*

Proof: Suppose not. Then there exists T such that $s^* \in S_1^T \times ... \times S_I^T$ but $s^* \notin S_1^{T+1} \times ... \times S_I^{T+1}$. The definition of ISD implies that there exists $s_i' \in S_i^T \subseteq S_i$ such that $u_i(s_i', s_{-i}) > u_i(s_i^*, s_{-i})$ for all $s_{-i} \in S_{-i}^T$. Therefore there exists a $s_i' \in S_i$ such that $u_i\left(s_i', s_{-i}^*\right) > u_i\left(s_i^*, s_{-i}^*\right)$ which contradicts that s^* was a NE.

1.3 Cournot Competition

Using our new result it is easy to see that the unique Nash equilibrium of the Cournot game with linear demand and constant marginal cost is the intersection of the two BR functions since this was the only profile which survived IDSDS.

A more direct proof notes that any Nash equilibrium has to lie on the best response function of both players by the definition of NE:

Lemma 1 *(q_1^*, q_2^*) is a NE if and only if $q_i^* \in BR_i(q_{-i})$ for all i.*

We have derived the best response functions of both firms in previous lectures (see figure 1).

$$BR_i(q_j) = \begin{cases} \frac{\alpha - c}{2\beta} - \frac{q_j}{2} & \text{if } q_j \leq \frac{\alpha - c}{\beta} \\ 0 & \text{otherwise} \end{cases}$$

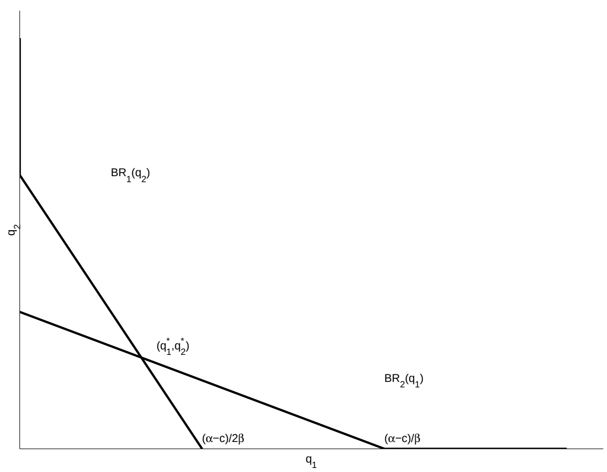
The NE is the solution to $q_1 = BR_1(q_2)$ and $q_2 = BR_2(q_1)$. This system has exactly one solution. This can be shown algebraically or simply by looking at the intersections of the BR graphs in figure 1. Because of symmetry we know that $q_1 = q_2 = q^*$. Hence we obtain:

$$q^* = \frac{\alpha - c}{\beta} - \frac{q^*}{2}$$

This gives us the solution $q^* = \frac{2(\alpha - c)}{3\beta}$.

If both firms are not symmetric you have to solve a system of two equations with two unknowns (q_1 and q_2).

Figure 1: BR functions of two firm Cournot game



1.4 Bertrand Competition

Recall the Bertrand price setting game between two firms that sell a homogenous product to consumers.

Two firms can simultaneously set any positive price p_i $(i = 1, 2)$ and produce output at constant marginal cost c. They face a downward sloping demand curve $q = D(p)$ and consumers always buy from the lowest price firm (this would not be true if the goods weren't homogenous!). Therefore, each firm faces demand

$$D_i(p_1, p_2) = \begin{cases} D(p_i) & \text{if } p_i < p_j \\ D(p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

We also assume that $D(c) > 0$, i.e. firms can sell a positive quantity if they price at marginal cost (otherwise a market would not be viable - firms couldn't sell any output, or would have to accumulate losses to do so).

Lemma 2 *The Bertrand game has the unique NE $(p_1^*, p_2^*) = (c, c)$.*

Proof: First we must show that (c,c) is a NE. It is easy to see that each firm makes zero profits. Deviating to a price below c would cause losses to the deviating firm. If any firm sets a higher price it does not sell any output and also makes zero profits. Therefore, there is no incentive to deviate.

To show uniqueness we must show that any other strategy profile (p_1, p_2) is not a NE. It's easiest to distinguish lots of cases.

Case I: $p_1 < c$ **or** $p_2 < c$ In this case one (or both players) makes negative losses. This player should set a price above his rival's price and cut his losses by not selling any output.

Case II: $c \leq p_1 < p_2$ **or** $c \leq p_2 < p_1$ Assume first that $c < p_1 < p_2$ or $c < p_2 < p_1$. In this case the firm with the higher price makes zero profits. It could profitably deviate by setting a price equal to the rival's price and thus capture at least half of his market, and make strictly positive profits. Now consider the case $c = p_1 < p_2$ or $c = p_2 < p_1$. Now the lower price firm can charge a price slightly above marginal cost (but still below the price of the rival) and make strictly positive profits.

Case III: $c < p_1 = p_2$ Firm 1 could profitably deviate by setting a price $p_1 = p_2 - \epsilon > c$. The firm's profits before and after the deviation are:

$$\pi_B = \frac{D(p_2)}{2}(p_2 - c)$$
$$\pi_A = D(p_2 - \epsilon)(p_2 - \epsilon - c)$$

Note, that the demand function is decreasing. We can therefore deduce:

$$\Delta\pi = \pi_A - \pi_B \geq = \frac{D(p_2)}{2}(p_2 - c) - \epsilon D(p_2)$$

This expression (the gain from deviating) is strictly positive for sufficiently small ϵ. Therefore, (p_1, p_2) cannot be a NE.

Remark 1 *In problem 11 of problem set 1 you had to solve for the unique Nash equilibrium when one firm (say 2) has higher marginal cost $c_2 > c_1$. Intuitively the price in the unique NE should be just below c_2 - this would keep firm 2 out of the market and firm 1 has no incentive to cut prices any further. However, if firms can set any real positive price there is no pure NE. Assume $c_2 = 10$. If firm 1 sets prices at 9.99 it can do better by setting them at 9.999 etc. Therefore, we have to assume that the pricing is discrete, i.e.*

can only be done in multiples of pennies say. In this way, the unique NE has firm 1 setting a price $p_1 = c_2$ *minus one penny.*

Food for Thought: How would you modify the Bertrand game to make it solvable through IDSDS? *Hint: You have to (a) discretize the strategy space, and (b) assume that* $D(p) = 0$ *for some sufficiently high price.*