Lecture XVI: Auctions

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Readings for this class: P. Klemperer - Auction Theory: A Guide to the Literature (especially parts of the appendix - the main text provides an excellent introduction to auction theory but is optional)

1 Introduction

Auctions are extremely common. Natural resources such as wireless spectrum, oil and minerals etc. are auctioned off.

There are several important types of auctions:

- ascending price auction
- second-price auction
- sealed bid (first price) auction
- all-pay auction (good model for legislative lobbying, war of attrition)

There are many possible scenarios for agents' types. The two most important ones are:

- Private Value environment: each agent's valuation of the good is drawn i.i.d. from some distribution F on $[\underline{v}, \overline{v}]$.
- Common value auction: agents' valuations are correlated (oil leases). Easy formulation:

$$v_i = V(t_1, ..., t_n) \text{ where } t_i \text{ are agents' signals.} \tag{1}$$

2 Some Simple Solved Examples

We frequently work with values (signals) drawn from the uniform distribution. In that case it is useful to know that the expected value of the *ith* order statistic is

$$\underline{v} + \frac{n+1-k}{n+1}\left(\overline{v} - \underline{v}\right) \tag{2}$$

2.1 Sealed Bid Auction (Private Value)

In the sealed bid auction where the players' valuations are independently uniformly distributed on $[0,1]$ the unique BNE is:

$$\begin{aligned} f_1^*(v_1) &= \frac{v_1}{2} \\ f_2^*(v_2) &= \frac{v_2}{2} \end{aligned}$$

Proof: To verify that this is a BNE is easy. We just show that each type of each player is using a BR:

$$E_{v_2}\left(u_1, f_2^*; v_1, v_2\right) = (v_1 - b_1)\, Prob\left(f_2^*(v_2) < b_1\right) + \frac{1}{2}(v_1 - b_1)\, Prob\left(f_2^*(v_2) = b_1\right)$$

We assume $b_1 \in \left[0, \frac{1}{2}\right]$. No larger bid makes sense given f_2^*. Hence:

$$E_{v_2}\left(u_1, f_2^*; v_1, v_2\right) = (v_1 - b_1)\, 2b_1$$

This is a quadratic equation which we maximize by the FOC:

$$0 = 2v_1 - 4b_1 \tag{3}$$

Hence $b_1 = \frac{v_1}{2}$.

To show **uniqueness** (or find the equilibrium if you don't know it) is harder. There are several methods:

- **Method 1:** Guess that $f_i^*(v_i) = a_i + c_i v_i$ then check to see which a_i and c_i work.

- **Method 2:** Guess that the f_i^* are increasing and differentiable and that $f_i^*(v_i)$ is always given by the FOC and that $f_1^* = f_2^*$:

$$
\begin{aligned}
f_1^*(v_1) &= \arg\max_{b_1} (v_1 - b_1)\, Prob\,(f_2^*(b_2) < b_1) \\
&= \arg\max_{b_1} (v_1 - b_1)\, Prob\left(b_2 < f_2^{*-1}(b_1)\right) \\
&= \arg\max_{b_1} (v_1 - b_1)\, f_2^{*-1}(b_1)
\end{aligned}
$$

The FOC for this is:

$$
\begin{aligned}
\left.(v_1 - b_1)\frac{d}{db_1} f_2^{*-1}(b_1) - f_2^{*-1}(b_1)\right|_{b_1 = f_1^*(v_1)} &= 0 \\
(v_1 - f_1^*(v_1))\frac{1}{f_2^{*\prime}\left(f_2^{*-1}(f_1^*(v_1))\right)} - f_2^{*-1}(f_1^*(v_1)) &= 0
\end{aligned}
$$

Using the symmetry assumption we get:

$$
(v_1 - f_1^*(v_1))\frac{1}{f_1'(v_1)} - v_1 = 0
$$

This gives us:

$$
v_1 = f_1^*(v_1) + f_1^{*\prime}(v_1)\, v_1
$$

This is a differential equation. Let's integrate on both sides:

$$
\begin{aligned}
\frac{1}{2}v_1^2 + K &= f_1^*(v_1)\, v_1 \\
f_1^*(v_1) = \frac{1}{2}v_1 + \frac{k}{v_1}
\end{aligned}
$$

The only true solution is $k = 0$. We need $k \leq 0$ to have increasing bids, and with 0 as a minimum bid $k < 0$ is impossible.

2.2 Second-Price Auction

In the second price auction where values v_i are drawn from some joint distribution F (not necessarily independent) bidding your own value $b_i = v_i$ is a weakly dominant strategy (and strict if the joint distribution is continuous and non-zero everywhere).

To see this, assume that $\tilde{b} = \max_{j \neq i} b_j$ is the maximal bid of all other agents. There are two cases:

$\tilde{b} < v_i$: In that case, choosing $b_i < v_i$ can only hurt me: I might not win the item even though it is profitable for me to buy it. Choosing $b_i > v_i$ has no effect. So $b_i = v_i$ is a weak best response.

$\tilde{b} > v_i$: In that case, choosing $b_i > v_i$ can only hurt me: I might win the item even though I lose by buying it. Choosing $b_i < v_i$ has no effect. So $b_i = v_i$ is a weak best response.

The intuition for this result is that my payment is determined by the other players' strategies.

2.3 Comparing the Second-Price and First Price Auction

Let's compare the revenue from both auctions. In the first price auction with 2 players the expected revenue is:

$$E(R) = \frac{1}{2}E(v^{1:2}) = \frac{1}{2}\frac{2}{3} = \frac{1}{3} \tag{4}$$

In the corresponding second price auction the expected revenue is:

$$E(R) = E(v^{2:2}) = \frac{1}{3} \tag{5}$$

This is no coincidence. The revenue equivalence theorem tells us that *any* auction mechanism (i.e. game) will give us the same revenue in a private value environment:

Theorem 1 *Assume each of n risk-neutral potential buyers has a privately-known value independently drawn from a common distribution F that is strictly increasing and atomless on $[\underline{v}, \overline{v}]$. No buyer wants more than one of the k available identical indivisible objects. Any auction mechanism in which (i) the objects always go to the k buyers with the highest values, and (ii) any bidder with value $\underline{v}$ expects zero surplus, yields the same expected revenue. and results in a buyer with value v making the same expected payment.*

The RET extends to our common-value environment - we substitute the v's for t's.

2.4 Proof of RET

Let $S_i(v_i)$ be the expected surplus that bidder i will expect in equilibrium. Let $P_i(v)$ be the probability of receiving the good.

Then the following holds in any NE:

$$S_i(v) \geq S_i(\tilde{v}) + (v - \tilde{v}) P_i(\tilde{v}) \tag{6}$$

That is, if player i decides to imitate type $\tilde{v}$ she will get the same surplus as that player - except that she will get extra value $v - \tilde{v}$ if she wins. In a NE she will prefer not to deviate.

In particular, this has to hold for small deviations $v + dv$:

$$S_i(v) \geq S_i(v + dv) + (-dv) P_i(v + dv) \tag{7}$$

Similarly:

$$S_i(v + dv) \geq S_i(v) + (dv) P_i(v) \tag{8}$$

This gives us:

$$P_i(v + dv) \geq \frac{S_i(v + dv) - S_i(v)}{dv} \geq P_i(v) \tag{9}$$

Then we take the limit and obtain:

$$\frac{dS_i}{dv} = P_i(v) \tag{10}$$

Integrating this expression gives us:

$$S_i(v) = S_i(\underline{v}) + \int_{x=\underline{v}}^{\overline{v}} P_i(x)\, dx \tag{11}$$

Note, that $S_i(v) = vP_i(v) - E(payments)$.

2.5 Applying RET

In an ascending auction the expected payment of a bidder of type v is $P_i(v)$ times the expectation of the highest of the remaining $n-1$ values conditional on all these values being below v:

$$\frac{\int^{v} x (n-1) f(x) F(x)^{n-2}}{\int^{v} (n-1) f(x) F(x)^{n-2}} \tag{12}$$

So this yields:

$$v - \frac{\int^{v} F(x)^{n-1}}{F(v)^{n-1}} \tag{13}$$

But this implies that in a first-price sealed bid auction we have bids:

$$b(v) = v - \frac{\int^{v} F(x)^{n-1}}{F(v)^{n-1}} \tag{14}$$

In an all-pay auction bids must satisfy:

$$b(v) = F(v)^{n-1} v - \int^{v} F(x)^{n-1} \tag{15}$$

3 Common Value Auctions

Let's look at the particularly simple auction where agents have values:

$$v_i = \alpha t_i + \beta \sum_{j \neq i} t_j \tag{16}$$

The private value case is $\beta = 0$ and the pure common value case is $\alpha = \beta$.

The symmetric equilibrium in an ascending auction looks as follows:

- The first drop out occurs with the lowest signal guy - she will drop out if $p = t^{n:n}$, which would be the value if all consumers had this signal.
- The next quit occurs at price $p = \beta t^{n:n} + (\alpha + (n-2)\beta) t^{n-1:n}$
- The final quit at the actual sale price is:

$$\hat{p} = \beta \sum_{j=3}^{n} t^{j:n} + (\alpha + \beta) t^{2:n} \tag{17}$$

Note: **Winner's Curse!** Each player realizes that in expectation the value of the good is higher than the actual price. However, the good is only useful if it is actually won. Conditional on winning at a higher price the bidder prefers to stick to her strategy and drop out now.

We can solve similarly for the second price auction. Bidder i is willing to bid up to his conditional valuation on winning the object and being tied with one other player (since she has to be indifferent between winning and not winning at that price). So she bids:

$$\beta\,(n-2)\,\frac{t_i}{2} + (\alpha+\beta)\,t_i = \left(\alpha + \frac{n}{2}\beta\right) t_i \tag{18}$$

The first-price auction is solved with RET. Her bid is:

$$b_i\,(t_i) = \left(\alpha + \frac{n}{2}\beta\right) \hat{t} = \left(\alpha + \frac{n}{2}\beta\right) \frac{n-1}{n} t_i \tag{19}$$