# Lecture X: Extensive Form Games

Markus M. Möbius

April 9, 2003

**Readings for this class: OR (section 6.1), FT (sections 3.2.1 and 3.3.1), Gibbons (Chapter 2), Dutta (Chapter 11).**

## 1 Introduction

While models presented so far are fairly general in some ways it should be noted that they have one main limitation as far as accuracy of modeling goes - in each game each player moves once and moves simultaneously.

This misses common features both of many classic games (bridge, chess) and of many economic models.[1] A few examples are:

- auctions (sealed bid versus oral)
- executive compensation (contract signed; then executive works)
- patent race (firms continually update resources based on where opponent are)
- price competition: firms repeatedly charge prices
- monetary authority and firms (continually observe and learn actions)

Topic today is how to represent and analyze such games.

[1] Gibbons is a good reference for economic applications of extensive form games.

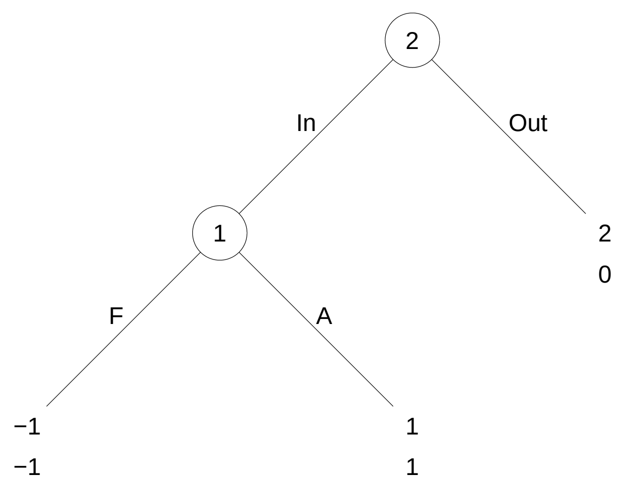
# 2 Extensive Form Games

The extensive form of a game is a complete description of

1. The set of players.
2. Who moves when and what their choices are.
3. The players' payoffs as a function of the choices that are made.
4. What players know when they move.

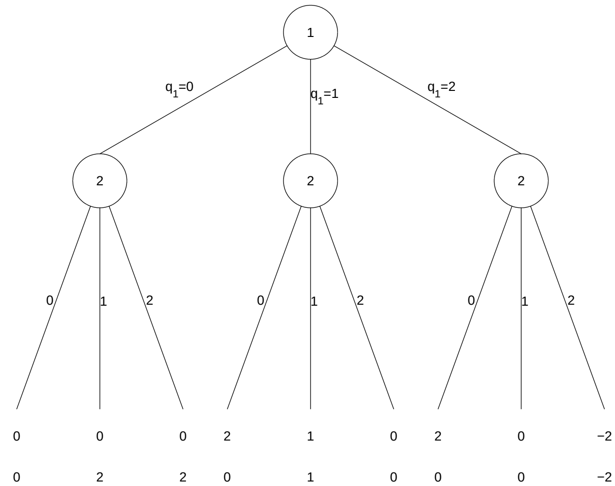
## 2.1 Example I: Model of Entry

Currently firm 1 is an incumbent monopolist. A second firm 2 has the opportunity to enter. After firm 2 makes the decision to enter, firm 1 will have the chance to choose a pricing strategy. It can choose either to *fight* the entrant or to *accommodate* it with higher prices.




## 2.2 Example II: Stackelberg Competition

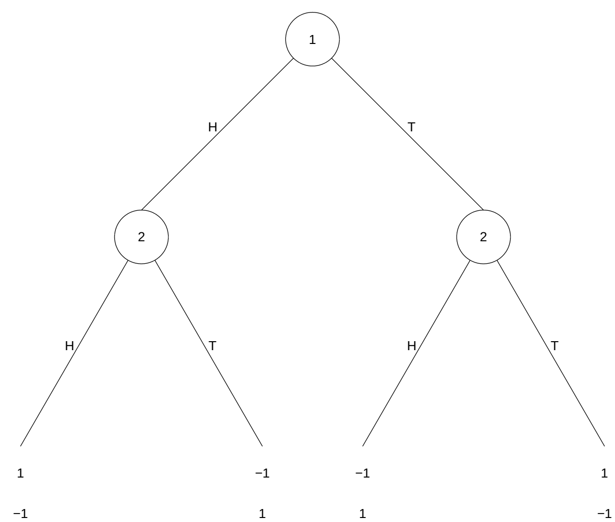
Suppose firm 1 develops a new technology before firm 2 and as a result has the opportunity to build a factory and commit to an output level $q_1$ before firm 2 starts. Firm 2 then observes firm 1 before picking its output level $q_2$. For concreteness suppose $q_i \in \{0, 1, 2\}$ and market demand is $p(Q) = 3 - Q$. The marginal cost of production is 0.



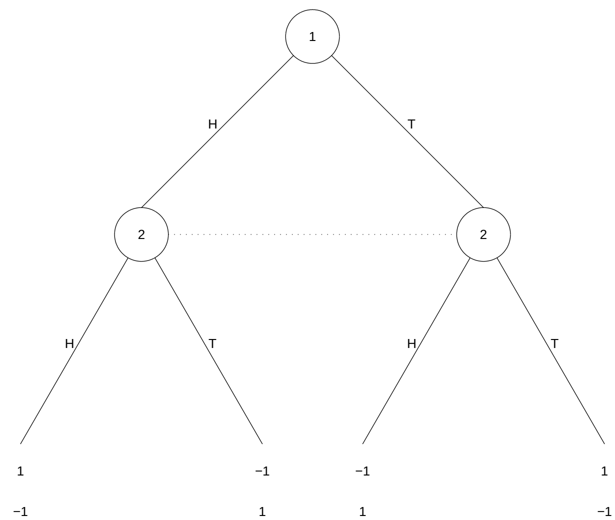

## 2.3 Example III: Matching Pennies

So far we assumed that players can observe all previous moves. In order to model the standard matching pennies game in extensive form we have to assume that the second player cannot observe the first player's move.

Sequential matching pennies is represented as follows:




If we want to indicate that player 2 cannot observe the move of player 1 we depict the game as follows:




The extensive form representation allows that players can 'forget' information. For example we can assume that in a game with 4 rounds player 2 can observe player 1's move in round 1, but in round 4 he has forgotten the move of player 1. In most cases, we assume *perfect recall* which rules out that players have such 'bad memory'.[2]

# 3 Definition of an Extensive Form Game

Formally a finite extensive form game consists of

1. A finite set of players.

2. A finite set $T$ of nodes which form a tree along with functions giving for each non-terminal node $t \notin Z$ ($Z$ is the set of terminal nodes)

   - the player $i(t)$ who moves
   - the set of possible actions $A(t)$
   - the successor node resulting from each possible action $N(t, a)$

[2]It becomes difficult to think of a solution concept of a game where players are forgetful. Forgetfulness and rational behavior don't go well together, and concepts like Nash equilibrium assume that players are rational.

3. Payoff functions $u_i : Z \to \Re$ giving the players payoffs as a function of the terminal node reached (the terminal nodes are the outcomes of the game).

4. An information partition: for each node $x$, $h(x)$ is the set of nodes which are possible given what player $i(x)$ knows. This partition must satisfy

$$x' \in h(x) \Rightarrow i(x') = i(x), A(x') = A(x), \text{and} \quad h(x') = h(x)$$

We sometimes write $i(h)$ and $A(h)$ since the action set is the same for each node in the same information set.

It is useful to go over the definition in detail in the matching pennies game where player 2 can't observe player 1's move. Let's number the non-terminal nodes $x_1$, $x_2$ and $x_3$ (top to bottom).

1. There are two players.

2. $S_1 = S_2 = \{H, T\}$ at each node.

3. The tree defines clearly the terminal nodes, and shows that $x_2$ and $x_3$ are successors to $x_1$.

4. $h(x_1) = \{x_1\}$ and $h(x_2) = h(x_3) = \{x_2, x_3\}$

# 4 Normal Form Analysis

In an extensive form game write $H_i$ for the set of information sets at which player $i$ moves.

$$H_i = \{S \subset T | S = h(t) \text{ for some } t \in T \text{ with } i(t) = i\}$$

Write $A_i$ for the set of actions available to player $i$ at any of his information sets.

**Definition 1** *A pure strategy for player $i$ in an extensive form game is a function $s_i : H_i \to A_i$ such that $s_i(h) \in A(h)$ for all $h \in H_i$.*

Note that a strategy is a **complete contingent plan** explaining what a player will do in *any* situation that arises. At first, a strategy looks overspecified: earlier action might make it impossible to reach certain sections of a tree. Why do we have to specify how players would play at nodes which can never be reached if a player follows his strategy early on? The reason is that play off the equilibrium path is crucial to determine if a set of strategies form a Nash equilibrium. Off-equilibrium threats are crucial. This will become clearer shortly.

**Definition 2** *A mixed behavior strategy for player $i$ is a function $\sigma_i : H_i \to \Delta(A_i)$ such that $supp(\sigma_i(h)) \subset A(h)$ for all $h \in A_i$.*

Note that we specify an independent randomization at each information set![3]

## 4.1 Example I: Entry Game

We can find the pure strategy sets $S_1 = \{\text{Fight}, \text{Accomodate}\}$ and $S_2 = \{\text{Out}, \text{In}\}$. We can represent the game in normal form as:

| | Out | In |
|---|---|---|
| F | 2,0 | −1,−1 |
| A | 2,0 | 1,1 |

[3] You might think that a more natural definition would simply define mixed strategies as a randomization over a set of pure strategies (just as in simultaneous move games). It can be shown that for games with perfect recall this definition is equivalent to the one given here, i.e. a mixed strategy is a mixed behavior strategy and vice versa. In games without perfect recall this is no longer true - it is instructive to convince yourself that in such games each mixed behavior strategy is a mixed strategy but not vice versa.

## 4.2 Example II: Stackelberg Competition

Firm 1 chooses $q_1$ and firm 2 chooses a quantity $q_2(q_1)$. With three possible output levels, firm 1 has three strategies, while firm 2 has $3^3 = 9$ different strategies because it can choose three strategies at its three information sets.

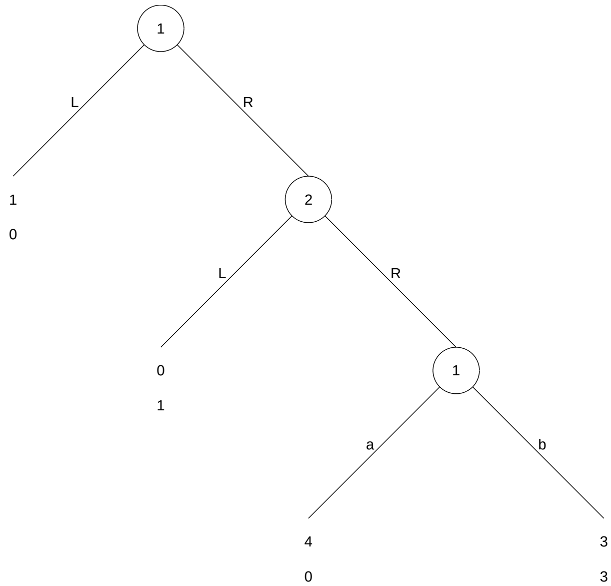
## 4.3 Example III: Sequential Matching Pennies

We have $S_1 = \{H, T\}$. Firm 2 has four strategies as it can choose two actions at two information sets. Strategy HH implies that firm 2 chooses H at both nodes, while $HT$ implies that it chooses H in the left node (after having observed H) and T in the right node (after having observed T).

| | HH | HT | TH | TT |
|---|---|---|---|---|
| H | 1,−1 | 1,−1 | −1,1 | −1,1 |
| T | −1,1 | 1,−1 | −1,1 | 1,−1 |

## 4.4 Example IV

Look at the following extensive form game:




One might be tempted to say that player 1 has three strategies because there are only three terminal nodes which can be reached. However, there are 4 because La and Lb are two distinct strategies. After player 1 plays L it is irrelevant for the final outcome what he would play in the bottom node. However, this off equilibrium pay is important for player 2's decision process which in turn makes 1 decide whether to play L or R.

| | L | R |
|---|---|---|
| La | 1,0 | 1,0 |
| Lb | 1,0 | 1,0 |
| Ra | 0,1 | 4,0 |
| Rb | 0,1 | 3,3 |

# 5 Nash Equilibrium in Extensive Form Games

We can apply NE in extensive form games simply by looking at the normal form representation. It turns out that this is not an appealing solution concept because it allows for too many profiles to be equilibria.

Look at the entry game. There are two pure strategy equilibria: (A,In) and (F,Out) as well as mixed equilibria $(\alpha F + (1-\alpha) A, \text{Out})$ for $\alpha \geq \frac{1}{2}$.

Why is (F,Out) a Nash equilibrium? Firm 2 stays out because he thinks that player 2 will fight entry. In other words, the threat to fight entry is sufficient to keep firm 2 out. Note, that in equilibrium this threat is never played since firm 2 stays out in the first place.

The problem with this equilibrium is that firm 2 could call firm 1's bluff and enter. Once firm 2 has entered it is in the interest of firm 1 to accommodate. Therefore, firm 1's threat is *not credible*. This suggests that only (A,In) is a reasonable equilibrium for the game since it does not rely on non-credible threats. The concept of subgame perfection which we will introduce in the next lecture rules out non-credible threats.

## 5.1 Example II: Stackelberg

We next look at a Stackelberg game where each firm can choose $q_i \in [0,1]$ and $p = 1 - q_1 - q_2$ and $c = 0$.

**Claim: For any $q_1' \in [0,1]$ the game has a NE in which firm 1 produces $q_1'$.**

Consider the following strategies:

$$\begin{aligned} s_1 &= q_1' \\ s_2 &= \begin{cases} \frac{1-q_1'}{2} & \text{if } q_1 = q_1' \\ 1 - q_1' & \text{if } q_1 \neq q_1' \end{cases} \end{aligned}$$

In words: firm 2 floods the market such that the price drops to zero if firm 1 does not choose $q_1'$. It is easy to see that these strategies form a NE. Firm 1 can only do worse by deviating since profits are zero if firm 2 floods the market. Firm 2 plays a BR to $q_1'$ and therefore won't deviate either.

Note, that in this game things are even worse. Unlike the Cournot game where we got a unique equilibrium we now have a continuum of equilibria. Second, we have even more disturbing non-credible threats. For instance, in the equilibrium where $q_1' = 1$ firm 2's threat is "if you don't flood the market and destroy the market I will". Not only won't the threat be carried out - it's also hard to see why it would be made in the first place.