Lecture XV: Games with Incomplete Information

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Readings for this class: Fudenberg and Tirole, pages 209-215; Dutta chapter 20.

1 Introduction

Informally, a game with incomplete information is a game where the game being played is not common knowledge. This idea is tremendously important in practice where its almost always a good idea to assume that something about the game is unknown to some players. What could be unknown?

1. **Payoffs:** In a price or quantity competition model you may know that your rival maximizes profits but now what his costs are (and hence his profits).

2. **Identity of other players:** R&D race between drug companies - who else will come up with the same drug?

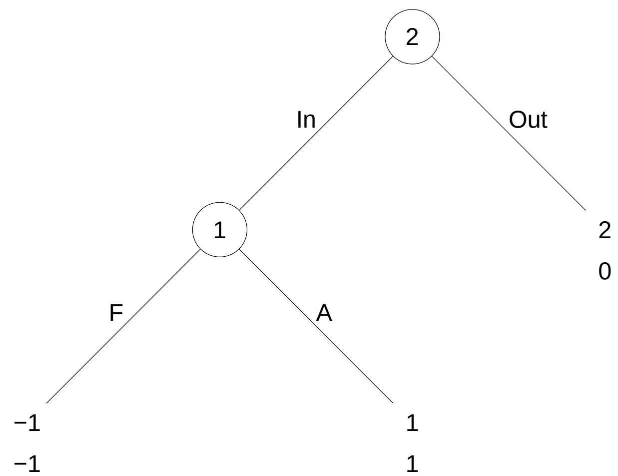
3. **What moves are possible:** What levels of quality can rivals in a quality competition choose?

4. **How does the outcome depend on action:** Workers work/shirk don't know probability of getting caught because product fails.

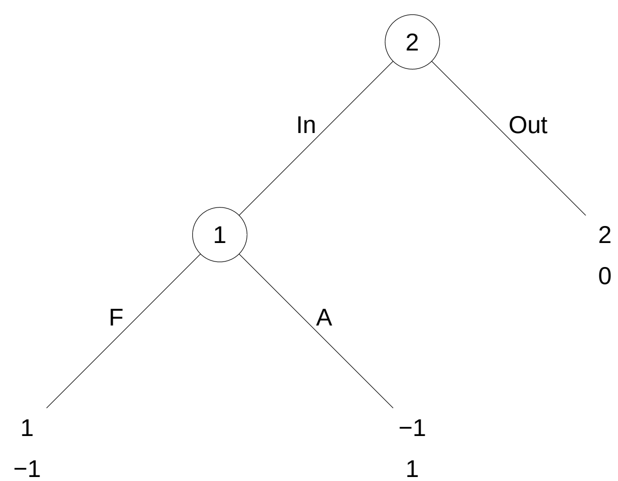
2 Examples

2.1 Example I: Crazy Incumbent

Think of a standard entry game where the incumbent is 'crazy' with probability $1 - p$ and rational with probability p. The normal incumbent faces the standard entry game from lecture 11:



If the incumbent is crazy he will always want to fight because is is facing a different subgame:



2.2 Example II: Auction

Two bidders are trying to purchase the same item at a sealed bid auction. The players simultaneously choose b_1 and b_2 and the good is sold to the highest bidder at his bid price (assume coin flip if $b_1 = b_2$). Suppose that the players' utilities are

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - b_i & \text{if } b_i > b_{-i} \\ \frac{1}{2}(v_i - b_i) & \text{if } b_i = b_{-i} \\ 0 & \text{if } b_i < b_{-i} \end{cases}$$

The crucial incomplete information is that while each player knows his own valuation, he does not know his rival's. Assume, each has a prior that his rival's valuation is uniform on $[0, 1]$ and that this is common knowledge.

2.3 Example III: Public Good

Two advisors of a graduate student each want the student to get a job at school X. Each can ensure this by calling someone on the phone and lying about how good the student is. Suppose the payoffs are as shown because each advisor gets utility 1 from the student being employed but has a cost of making the phone call.

	Call	Dont
Call	$1-c_1, 1-c_2$	$1-c_1, 1$
Dont	$1, 1-c_2$	$0,0$

Assume that the actions are chosen simultaneously and that players know only their own costs. They have prior that $c_{-i} \in U[\underline{c}, \overline{c}]$.

Alternatively, we could have player 1's cost known to all ($c_1 = \frac{1}{2}$) but $c_2 \in \{\underline{c}, \overline{c}\}$ known only to player 2.

Or, player 1 is a senior faculty member who knows from experience the cost of such phone calls ($c_1 = \frac{1}{2}, c_2 = \frac{2}{3}$). Player 2 is new assistant professor who has priors $c_1, c_2 \in U[0, 2]$.

3 Definitions

Definition 1 *A game with incomplete information $G = (\Phi, S, P, u)$ consists of*

1. *A set $\Phi = \Phi_1 \times ... \times \Phi_I$ where Φ_i is the (finite) set of possible types for player i.*[1]
2. *A set $S = S_1 \times ... \times S_I$ giving possible strategies for each player.*
3. *A joint probability distribution $p(\phi_1, .., \phi_I)$ over the types. For finite type space assume $p(\phi_i) > 0$ for all $\phi_i \in \Phi_i$.*
4. *A payoff function $u_i : S \times \Phi \to \Re$.*

It's useful to discuss the types in each of our examples.

- **Example I:** $\Phi_1 = \text{normal}$, $\Phi_2 \in \{\pi \text{ maximizer}, \text{crazy}\}$
- **Example II:**$\Phi_1 = \Phi_2 = (0, 1)$
- **Example III:**$\Phi_1 = \Phi_2 = [\underline{c}, \overline{c}]$

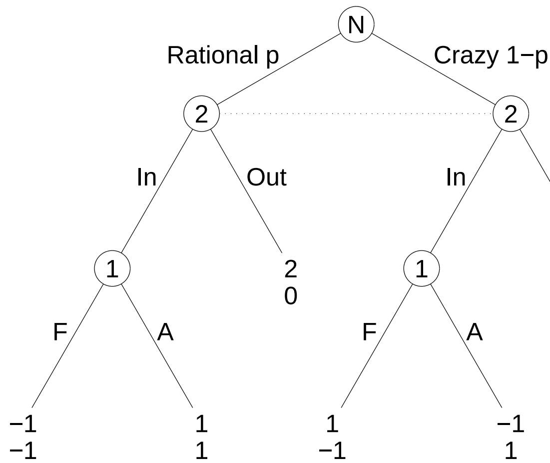
Note that payoffs can depend not only on your type but also on your rival's type as well.

Players know their own types but not the other players' types.

To analyze games of incomplete information we rely on the following observation (Harsanyi):

Observation: *Games of incomplete information can be thought of as games of complete but imperfect information where nature makes the first move and not everyone is informed about nature's move, i.e. nature chooses Φ but only reveals ϕ_i to player i.*

[1] Note, that the player i knows his type.



Think of nature simply as another player who rather than maximizing chooses a fixed mixed strategy.

This observation should make all of the following definitions seem completely obvious. They just say that to analyze these games we may look at NE of the game with Nature as another player.

Definition 2 *A Bayesian strategy for player i in G is a function $f_i : \Phi_i \to \Sigma_i$. Write S^{Φ_i} for the set of Bayesian strategies.*[2]

Definition 3 *A Bayesian strategy profile $(f_1^*, .., f_I^*)$ is a Bayesian Nash equilibrium if*

$$f_i^* \in \arg\max_{f_i \in S_i^{\Phi_i}} \sum_{\phi_i, \phi_{-i}} u_i\left(f_i(\phi_i), f_{-i}^*(\phi_{-i}); \phi_i, \phi_{-i}\right) p(\phi_i, \phi_{-i})$$

for all i or equivalently if for all i,ϕ_i,s_i

$$\sum_{\phi_{-i}} u_i\left(f_i^*(\phi_i), f_{-i}^*(\phi_{-i}); \phi_i, \phi_{-i}\right) p(\phi_i, \phi_{-i}) \geq \sum_{\phi_{-i}} u_i\left(s_i, f_{-i}^*(\phi_{-i}); \phi_i, \phi_{-i}\right) p(\phi_i, \phi_{-i})$$

This just says that your maximize expected payoff, and given that you know your type (that all have positive probability) this is equivalent to saying you maximize conditional on each possible type.

Remark 1 *A Bayesian Nash equilibrium is simply a Nash equilibrium of the game where Nature moves first, chooses $\phi \in \Phi$ from a distribution with probability $p(\phi)$ and reveals ϕ_i to player i.*

[2] Note, that a Bayesian strategy is simply an extensive form strategy where each type is treated as a distinct information set.

4 Solved examples

4.1 Public good I

Suppose player 1 is known to have cost $c_1 < \frac{1}{2}$. Player 2 has cost $\underline{c}$ with probability p, $\overline{c}$ with probability $1-p$. Assume that $0 < \underline{c} < 1 < \overline{c}$ and $p < \frac{1}{2}$.

	Call	Dont
Call	$1-c_1, 1-c_2$	$1-c_1, 1$
Dont	$1, 1-c_2$	0,0

Then the unique BNE is:

- $f_1^* = \text{Call}$
- $f_2^*(c) = \text{Don't Call for all } c$.

Proof: In a BNE each type of player must play a BR so for the type $\overline{c}$ of player 2 calling is strictly dominated:

$$u_2(s_1, \text{call}; \overline{c}) < u_2(s_1, \text{Dont}; \overline{c})$$

for all s_1.

So $f_2^*(\overline{c}) = \text{Dont}$.

For player 1:

$$\begin{aligned} u_1(\text{Call}, f_2^*; c_1) &= 1 - c_1 \\ u_1(\text{Dont}, f_2^*; c_1) &= p u_1(\text{Dont}, f_2^*(\underline{c}); c_1) + (1-p) u_1(\text{Dont}, \text{Dont}; c_1) \\ &\leq p + (1-p) 0 = p \end{aligned}$$

Because $1 - c_1 > p$ we know $f_1^*(c_1) = \text{Call}$.

For the type $\underline{c}$ of player 2 we have:

$$\begin{aligned} u_2\left(f_1^*, \text{Call}; \underline{c}\right) &= 1 - \underline{c} \\ u_2\left(f_1^*, \text{Dont}; \underline{c}\right) &= 1 \end{aligned}$$

because $f_1^* = Call$. So $f_2^*(\underline{c}) = \text{Dont}$.

This indicates this is the only possible BNE and our calculations have verified that it does actually work.

Process I've used is like iterated dominance.

4.2 Public Goods II

In the public good game suppose that c_1 and c_2 are drawn independently from a uniform distribution on $[0, 2]$. Then the (essentially) unique BNE is

$$f_i^*(c_i) = \begin{cases} \text{Call} & \text{if } c_i \leq \frac{2}{3} \\ \text{Don't} & \text{if } c_i > \frac{2}{3} \end{cases}$$

Proof: **Existence** is easy - just check that each is using a BR given that his rival calls with probability $\frac{1}{3}$ and doesn't call with probability $\frac{2}{3}$.

I'll show **uniqueness** to illustrate how to find the equilibrium.

Observation: *If $f_i^*(c_i) = Call$ then $f_i^*(c_i') = Call$ for all $c_i' < c_i$.*

To see this write z_{-i} for $Prob\left\{f_{-i}^*(c_{-i}) = \text{Call}\right\}$. If $f_i^*(c_i) = \text{call}$ then

$$\begin{aligned} E_{c_{-i}} u_i\left(\text{Call}, f_{-i}^*(c_{-i}); c_i\right) &\geq E_{c_{-i}} u_i\left(\text{Dont}, f_{-i}^*(c_{-i}); c_i\right) \\ 1 - c_i &\geq z_{-i} \end{aligned}$$

This clearly implies that for $c_i' < c_i$

$$E_{c_{-i}} u_i\left(\text{Call}, f_{-i}^*(c_{-i}); c_i'\right) = 1 - c_i' > E_{c_{-i}} u_i\left(\text{Dont}, f_{-i}^*(c_{-i}); c_i'\right) = z_{-i}$$

The intuition is that calling is more attractive if the cost is lower.

In light of observation a BNE must be of the form:[3]

$$f_1^*(c_1) = \begin{cases} \text{Call} & \text{if } c_1 \le c_1^* \\ \text{Don't} & \text{if } c_1 > c_1^* \end{cases}$$
$$f_2^*(c_2) = \begin{cases} \text{Call} & \text{if } c_2 \le c_2^* \\ \text{Don't} & \text{if } c_2 > c_2^* \end{cases}$$

For these strategies to be a BNE we need:

$$\begin{aligned} 1 - c_i &\ge z_{-i} \quad \text{for all } c_i < c_i^* \\ 1 - c_i &\le z_{-i} \quad \text{for all } c_i > c_i^* \end{aligned}$$

Hence $1 - c_i^* = z_{-i}$.

Because c_{-i} is uniform on $[0, 2]$ we get $z_{-i} = Prob\left\{c_{-i} < c_{-i}^*\right\} = \frac{c_{-i}^*}{2}$.

We have:

$$\begin{aligned} 1 - c_1^* &= \frac{c_2^*}{2} \\ 1 - c_2^* &= \frac{c_1^*}{2} \end{aligned}$$

It is easy to see that the unique solution to this system of equations is $c_1^* = c_2^* = \frac{2}{3}$.

Remark 2 *The result here is common in public goods situations. We get inefficient underinvestment because of the free rider problem. Each wants the other to call.*

4.3 Auctions

In the sealed bid auction where the players' valuations are independently uniformly distributed on $[0, 1]$ the unique BNE is:

$$\begin{aligned} f_1^*(v_1) &= \frac{v_1}{2} \\ f_2^*(v_2) &= \frac{v_2}{2} \end{aligned}$$

[3] The cutoff values themselves are indeterminate - agents might or might not call. In this sense the equilibrium won't be unique. However, the cutoff values are probability zero events and hence the strategy at these points won't matter.

Proof: To verify that this is a BNE is easy. We just show that each type of each player is using a BR:

$$E_{v_2}\left(u_1, f_2^*; v_1, v_2\right) = \left(v_1 - b_1\right) Prob\left(f_2^*\left(v_2\right) < b_1\right) + \frac{1}{2}\left(v_1 - b_1\right) Prob\left(f_2^*\left(v_2\right) = b_1\right)$$

We assume $b_1 \in \left[0, \frac{1}{2}\right]$. No larger bid makes sense given f_2^*. Hence:

$$E_{v_2}\left(u_1, f_2^*; v_1, v_2\right) = \left(v_1 - b_1\right) 2b_1$$

This is a quadratic equation which we maximize by the FOC:

$$0 = 2v_1 - 4b_1 \tag{1}$$

Hence $b_1 = \frac{v_1}{2}$.

To show **uniqueness** (or find the equilibrium if you don't know it) is harder. There are several methods:

- **Method 1:** Guess that $f_i^*\left(v_i\right) = a_i + c_i v_i$ then check to see which a_i and c_i work.
- **Method 2:** Guess that the f_i^* are increasing and differentiable and that $f_i^*\left(v_i\right)$ is always given by the FOC and that $f_1^* = f_2^*$:

$$\begin{aligned} f_1^*\left(v_1\right) &= \arg\max_{b_1}\left(v_1 - b_1\right) Prob\left(f_2^*\left(b_2\right) < b_1\right) \\ &= \arg\max_{b_1}\left(v_1 - b_1\right) Prob\left(b_2 < f_2^{*-1}\left(b_1\right)\right) \\ &= \arg\max_{b_1}\left(v_1 - b_1\right) f_2^{*-1}\left(b_1\right) \end{aligned}$$

  The FOC for this is:

$$\begin{aligned} \left(v_1 - b_1\right)\frac{d}{db_1} f_2^{*-1}\left(b_1\right) - f_2^{*-1}\left(b_1\right)\Big|_{b_1 = f_1^*\left(v_1\right)} &= 0 \\ \left(v_1 - f_1^*\left(v_1\right)\right)\frac{1}{f_2^{*\prime}\left(f_2^{*-1}\left(f_1^*\left(v_1\right)\right)\right)} - f_2^{*-1}\left(f_1^*\left(v_1\right)\right) &= 0 \end{aligned}$$

  Using the symmetry assumption we get:

$$\left(v_1 - f_1^*\left(v_1\right)\right)\frac{1}{f_1'\left(v_1\right)} - v_1 = 0$$

This gives us:

$$v_1 = f_1^* (v_1) + f_1^{*'} (v_1) v_1$$

This is a differential equation. Let's integrate on both sides:

$$\begin{aligned} \frac{1}{2}v_1^2 + K &= f_1^* (v_1) v_1 \\ f_1^* (v_1) = \frac{1}{2}v_1 + \frac{k}{v_1} \end{aligned}$$

The only true solution is $k = 0$. We need $k \leq 0$ to have increasing bids, and with 0 as a minimum bid $k < 0$ is impossible.

4.4 Summary of Solution Methods

We have solved BNE in our games in several ways:

- **Discrete type and action space:** Iterated conditional dominance, or use equilibrium conditions (finite number of inequalities)
- **Discrete actions, continuous type space:** identify some simple cutoff rule
- **Continuous action and type space:** 'Guess' equilibrium *or* use FOC and differentiability of strategy to get some differential equation