Lecture III: Normal Form Games, Rationality and Iterated Deletion of Dominated Strategies

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1 Definition of Normal Form Game

Game theory can be regarded as a multi-agent decision problem. It's useful to define first exactly what we mean by a *game*.

Every normal form (strategic form) game has the following ingredients.

1. There is a list of players $D = \{1, 2, .., I\}$. We mostly consider games with just two players. As an example consider two people who want to meet in New York.

2. Each player i can choose actions from a strategy set S_i. To continue our example, each of the players has the option to go the Empire State building or meet at the old oak tree in Central Park (where ever that is ...). So the strategy sets of both players are $S_1 = S_2 = \{E, C\}$.

3. The outcome of the game is defined by the 'strategy profile' which consists of all strategies chosen by the individual players. For example, in our game there are four possible outcomes - both players meet at the Empire state building (E, E), they miscoordinate, (E, C) and (C, E), or they meet in Central Park (C, C). Mathematically, the set of strategy profiles (or outcomes of the game) is defined as

$$S = S_1 \times S_2$$

   In our case, S has order 4. If player 1 can take 5 possible actions, and player 2 can take 10 possible actions, the set of profiles has order 50.

Figure 1: General 2 by 2 game

	E	C
E	1,1	0,0
C	0,0	1,1

4. Players have preferences over the outcomes of the play. You should realize that players cannot have preferences over the actions. In a game my payoff depends on your action. In our New York game players just want to be able to meet at the same spot. They don't care if they meet at the Empire State building or at Central Park. If they choose E and the other player does so, too, fine! If they choose E but the other player chooses C, then they are unhappy. So what matters to players are outcomes, not actions (of course their actions influence the outcome - but for each action there might be many possible outcomes - in our example there are two possible outcomes per action). Recall, that we can represent preferences over outcomes through a utility function. Mathematically, preferences over outcomes are defined as:

$$u_i : S \to R$$

In our example, $u_i = 1$ if both agents choose the same action, and 0 otherwise.

All this information can be conveniently expressed in a game matrix as shown in figure 1:

A more formal definition of a game is given below:

Definition 1 *A normal (strategic) form game G consists of*

- *A finite set of agents $D = \{1, 2, .., I\}$.*
- *Strategy sets $S_1, S_2, .., S_I$*
- *Payoff functions $u_i : S_1 \times S_2 \times ..S_I \to R$ $(i = 1, 2, .., n)$*

We'll write $S = S_1 \times S_2 \times .. \times S_I$ and we call $s \in S$ a strategy profile ($s = (s_1, s_2, .., s_I)$). We denote the strategy choices of all players except player i with s_{-i} for $(s_1, s_2, .., s_{i-1}, s_{i+1}, ..s_I)$.

2 Some Important Games

We already discussed coordination games. These are interesting games, because players have an incentive to work together rather than against each other. The first games analyzed by game theorists were just the opposite - zero sum games, where the sum of agents' utilities in each outcome sums up to zero (or a constant).

2.1 Zero-Sum Games

Zero-sum games are true games of conflict. Any gain on my side comes at the expense of my opponents. Think of dividing up a pie. The size of the pie doesn't change - it's all about redistribution of the pieces between the players (tax policy is a good example).

The simplest zero sum game is matching pennies. This is a two player game where player 1 get a Dollar from player 2 if both choose the same action, and otherwise loses a Dollar:

	H	T
H	1,−1	−1,1
T	−1,1	1,−1

2.2 Battle of the Sexes

This game is interesting because it is a coordination game with some elements of conflict. The idea is that a couple want to spend the evening together. The wife wants to go to the Opera, while the husband wants to go to a football game. Each get at least some utility from going together to at least one of the venues, but each wants to go their favorite one (the husband is player 1 - the column player).

	F	O
F	2,1	0,0
O	0,0	1,2

2.3 Chicken or Hawk versus Dove

This game is an anti-coordination game. The story is that two teenagers drive home on a narrow road with their bikes, and in opposite directions. None of them wants to go out of the way - whoever 'chickens' out loses his pride, while the tough guy wins. But if both stay tough, then they break their bones. If both go out of the way, none of the them is too happy or unhappy.

	t	c
t	−1,−1	10,0
c	0,10	5,5

2.4 Prisoner's Dilemma

This game might be the most famous of all. It's the mother of all cooperation games. The story is that two prisoners are interrogated. If both cooperate with the prosecution they get of with 1 year in prison. If both give each other away (defect) they get 3 years in prison each. If one cooperates and the other guy defects, then the cooperating guy is thrown into prison for 10 years, and the defecting guy walks free.

	C	D
C	3,3	−1,4
D	4,−1	0,0

Note, that the best outcome in terms of welfare is if both cooperate. The outcome (D, D) is worst in welfare terms, and is also Pareto dominated by (C, C) because both players can do better. So clearly, (D, D) seems to be a terrible outcome overall.

Some examples of Prisoner's dilemmas are the following:

- Arms races. Two countries engage in an expensive arms race (corresponds to outcome D,D). They both would like to spend their money on (say) healthcare, but if one spends the money on healthcare and the other country engages in arms build-up, the weak country will get invaded.

- Missile defence. The missile defence initiative proposed by the administration is interpreted by some observers as a Prisoner's dilemma. Country 1 (the US) can either not build a missile defence system (strategy C) or build one (strategy D). Country 2 (Russia) can either not build any more missiles (strategy C) or build lots more (strategy D). If the US does not build a missile system, and Russia does not build more missiles then both countries are fairly well off. If Russia builds more missiles and the US has no defence then the US feels very unsafe. If the US builds a missile shield, and Russia does not missiles then the US is happy but Russia feels unsafe. If the US builds missile defence and Russia builds more missiles then they are equally unsafe as in the (C,C) case, but they are much less well off because they both have to increase their defence budget.

- Driving a big SUV is a Prisoner's Dilemma. I want my car to be as safe as possible and buy an SUV. However, my neighbors who has a Volkswagen Beetle suddenly is much worse off. If she also buys an SUV

2.5 Cournot Competition

This game has an infinite strategy space. Two firms choose output levels q_i and have cost function $c_i(q_i)$. The products are undifferentiated and market demand determines a price $p(q_1 + q_2)$. Note, that this specification assumes that the products of both firms are perfect substitutes, i.e. they are homogenous products.

$$
\begin{aligned}
D &= \{1, 2\} \\
S_1 &= S_2 = R^+ \\
u_1(q_1, q_2) &= q_1 p(q_1 + q_2) - c_1(q_1) \\
u_2(q_1, q_2) &= q_2 p(q_1 + q_2) - c_2(q_2)
\end{aligned}
$$

2.6 Bertrand Competition

Bertrand competition is in some ways the opposite of Cournot competition. Firms compete in a homogenous product market but they set prices. Consumers buy from the lowest cost firm.

Remark 1 *It is interesting to compare Bertrand and Cournot competition with perfect competition analyzed in standard micro theory. Under perfect competition firms are price takers i.e. they cannot influence the market. In this case there is not strategic interaction between firms - each firm solves a simple profit maximization problem (decision problem). This is of course not quite true since the auctioneer does determine prices such that demand and supply equalize.*

3 Experiment: Prisoner's Dilemma

Students are asked which strategy they would play in the Prisoner's dilemma. The class was roughly divided in half - we calculated the expected payoff from both strategies if people in the class would be randomly matched against each other. We found that strategy D was better - this is unsurprising as we will see later since strategy C is strictly dominated by strategy D.

4 Experiment: Iterated Deletion Game

Students play the game given below and are asked to choose a strategy for player 1.

	A	B	C	D
A	5,2	2,6	1,4	0,4
B	0,0	3,2	2,1	1,1
C	7,0	2,2	1,5	5,1
D	9,5	1,3	0,2	4,8

In class no student chooses strategy A which is weakly dominated by C. 2 students chose B, 9 students chose C because it looked 'safe' and 16 students chose D because of the high payoffs in that row. It turns out that only (B,B) survives iterated deletion (see below).

5 Iterated Deletion of Dominated Strategies

How do agents play games? We can learn a lot by exploiting the assumption that players are rational and that each player knows that other players are rational. Sometimes this reasoning allows us to 'solve' a game.

5.1 Rational Behavior

Assume that agent i has belief μ_i about the play of her opponents. A belief is a probability distribution over the strategy set S_{-i}. Since we are only considering pure strategies right now this probability distribution is a point distribution (i.e. puts all its weight on a single s_{-i}.

Definition 2 *Player i is rational with beliefs μ_i if*

$$s_i \in \arg\max_{s_i'} E_{\mu_i(s_{-i})} u_i\left(s_i', s_{-i}\right),$$

or alternatively

$$s_i \mathit{maximizes} \sum_{s_{-i}} u_i\left(s_i', s_{-i}\right)\mu_i\left(s_{-i}\right).$$

Note, that player i faces a simple decision problem as soon as she has formed her belief μ_i.

Definition 3 *Strategy s_i is stricttly dominated for player i if there is some $s_i' \in S_i$ such that*

$$u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right)$$

for all $s_{-i} \in S_{-i}$.

Note that the inequality is strict for all s_{-i}. A strategy is weakly dominated if the inequality is weak for all s_{-i} and strict for at least one s_{-i}.

Proposition 1 *If player i is rational he will not play a strictly dominated strategy.*

Proof: If strategy s_i is strictly dominated by strategy s_i' we can deduce that for any belief of player i we have $E_{\mu_i(s_{-i})} u_i\left(s_i', s_{-i}\right) > E_{\mu_i(s_{-i})} u_i\left(s_i, s_{-i}\right)$.

5.2 Iterated Dominance

The hardest task in solving a game is to determine players' beliefs. A lot of games can be simplified by rationality and the *knowledge that my opponent is rational*. To see that look at the Prisoner's Dilemma.

Cooperating is a dominated strategy. A rational player would therefore never cooperate. This solves the game since every player will defect. Notice that I don't have to know anything about the other player. This prediction is interesting because it is the worst outcome in terms of joint surplus and it would be Pareto improving if both players would cooperate. This result highlights the value of commitment in the Prisoner's dilemma - commitment consists of credibly playing strategy C. For example, in the missile defence example the ABM treaty (prohibits missile defence) and the START II agreement (prohibits building of new missiles) effectively restrict both country's strategy sets to strategy C.

Now look at the next game.

	L	M	R
U	2,2	1,1	4,0
D	1,2	4,1	3,5

1. If the column player is rational he shouldn't play M

2. Row player should realize this if he know that the other player is rational. Thus he won't play D.

3. Column player should realize that R knows that C is rational. If he knows that R is rational he knows that R won't play D. Hence he won't play R. This leaves (U,L) as only outcome for rational players.

It's worth while to discuss the level of knowledge required by players. R has to know that C is rational. C has to know that R knows that C is rational. This latter knowledge is a 'higher order' form of knowledge. It's not enough to know that my opponent is rational - I also have to be sure that my opponent knows that I am rational. There are even higher order types of knowledge. I might know that my opponent is rational and that he knows that I am. But maybe he doesn't know that I know that he knows.

The higher the order of knowledge the more often the process of elimination can be repeated. For example, the game in section 4 of our second experiment can be solved by the iterated deletion of dominated strategies.

An important concept in game theory is common knowledge (see next lecture). We will assume throughout the course that rationality is common knowledge between both players. Therefore, the iteration process can be repeated arbitrarily often. However, the experiment showed that this assumption might be too strong.

5.3 Formal Definition Of Iterated Dominance

- **Step I:** Define $S_i^0 = S_i$

- **Step II:** Define

$$S_i^1 = \left\{ s_i \in S_i^0 \,\middle|\, \nexists s_i' \in S_i^0 u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right) \forall s_{-i} \in S_{-i}^0 \right\}$$

- **Step k+1:** Define

$$S_i^{k+1} = \left\{ s_i \in S_i^k \,\middle|\, \nexists s_i' \in S_i^k u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right) \forall s_{-i} \in S_{-i}^k \right\}$$

  S_i^{k+1} is the set still not strictly dominated when you know your opponent uses some strategy in S_{-i}^k.

  Note restrictions S_{-i}^0, S_{-i}^1, ..

  Players know that opponents are rational, know that opponents know that they are rational ..., e.g. rationality is common knowledge.

- **Step ∞:** Let $S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k$.

  Note, that the process must stop after finitely many steps if the strategy set is finite because the sets can only get smaller after each iteration.

Definition 4 *G is solvable by pure strategy iterated strict dominance if S^∞ contains a single strategy profile.*

Most games are not dominance solvable (coordination game, zero sum game).

We have not specified the order in which strategies are eliminated. You will show in the problem set that the speed and order of elimination does not matter.

The same is not true for the elimination of *weakly dominated strategies* as the next example shows.

	L	R
T	1,1	0,0
M	1,1	2,1
B	0,0	2,1

We can first eliminate T and then L in which case we know that (2,1) is played for sure. However, if we eliminate B first and then R we know that (1,1) is being played for sure. So weak elimination does not deliver consistent results and is therefore generally a less attractive option than the deletion of strictly dominated strategies.

6 Example: Cournot Competition

Remark 2 *For the mathematically inclined: With finite strategy sets the set S^∞ is clearly non-empty because after each stage there must be some dominant strategy left. For infinite strategy sets this is not as obvious. However, one can show that for compact strategy sets each nested subset S_i^k is closed and non-empty. Therefore the intersection of all nested subsets cannot be empty.*

Cournot competition with two firms can be solved by iterated deletion in some cases. Specifically, we look at a linear demand function $p = \alpha - \beta(q_i + q_j)$ and constant marginal cost c such that the total cost of producing q_i units is cq_i. It will be usefull to calculate the 'best-response' function $BR(q_j)$ of each firm i to the quantity choice q_j of the other firm. By taking the first-order condition of the profit function you can easily show that the best-response function for both firms (there is symmetry!) is

$$BR_i(q_j) = \begin{cases} \frac{\alpha-c}{2\beta} - \frac{q_j}{2} & \text{if } q_j \leq \frac{\alpha-c}{\beta} \\ 0 & \text{otherwise} \end{cases}$$

The best-response function is decreasing in my belief of the other firm's action. Note, that for $q_j > \frac{\alpha-c}{\beta}$ firm i makes negative profits even if it chooses the profit maximizing output. It therefore is better off to stay out of the market and choose $q_i = 0$.

Initially, firms can set any quantity, i.e. $S_1^0 = S_2^0 = \Re^+$. However, the best-responses of each firm to any belief has to lie in the interval $\left[\underline{q}, \overline{q}\right]$ with $\underline{q} = 0$ and $\overline{q} = \frac{\alpha-c}{2\beta}$. All other strategies make negative profits, are therefore dominated by some strategy inside this interval, and eliminated.

In the second stage only the strategies $S_1^2 = S_2^2 = \left[BR_1(\overline{q}), BR_1(\underline{q})\right]$ survive, and in the third stage $S_1^3 = S_2^3 = \left[BR_2\left(BR_1(\underline{q})\right), BR_2\left(BR_1(\overline{q})\right)\right]$ (note, that the BR function is decreasing!).

Therefore in the $2k+1$th stage only strategies in $S_1^{2k+1} = S_2^{2k+1} = \left[BR_2\left(..BR_1\left(\underline{q}\right)\right), BR_2\left(..BR_1\left(\overline{q}\right)\right)\right]$ survive.

It's easy to show graphically that this interval shrinks in each iteration and that the two limits converge to the intersection $q_1^* = q_2^*$ of both best response functions where $q_2^* = BR_2\left(q_1^*\right)$. Therefore, the Cournot game is solvable through the iterated deletion of strictly dominated strategies.

Remark 3 *It can be shown that the same game with three firms is NOT dominance solvable. You have to show that on the problem set!*