Lecture V: Mixed Strategies

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Readings for this class: Osborne and Rubinstein, section 3.1 and 3.2

1 The Advantage of Mixed Strategies

Consider the following Rock-Paper-Scissors game: *Note that RPS is a zero-sum game.*

	R	P	S
R	0,0	−1,1	1,−1
P	1,−1	0,0	−1,1
S	−1,1	1,−1	0,0

This game has no pure-strategy Nash equilibrium. Whatever pure strategy player 1 chooses, player 2 can beat him. A natural solution for player 1 might be to randomize amongst his strategies.

Another example of a game without pure-strategy NE is matching pennies. As in RPS the opponent can exploit his knowledge of the other player's action.

	H	T
H	1,−1	−1,1
T	−1,1	1,−1

Fearing this what might the opponent do? One solution is to randomize and play a mixed strategy. Each player could flip a coin and play H with probability $\frac{1}{2}$ and T with probability $\frac{1}{2}$.

Note that each player cannot be taken advantage of.

Definition 1 *Let G be a game with strategy spaces S_1,S_2,..,S_I. A mixed strategy σ_i for player i is a probability distribution on S_i i.e. for S_i finite a mixed strategy is a function $\sigma_i : S_i \to \Re^+$ such that $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.*

Several notations are commonly used for describing mixed strategies.

1. Function (measure): $\sigma_1(H) = \frac{1}{2}$ and $\sigma_1(T) = \frac{1}{2}$
2. Vector: If the pure strategies are s_{i1},..s_{iN_i} write $(\sigma_i(s_{i1}), .., \sigma_i(s_{iN_i}))$ e.g. $\left(\frac{1}{2}, \frac{1}{2}\right)$.
3. $\frac{1}{2}H + \frac{1}{2}T$

Class Experiment 1 *Three groups of two people. Play RPS with each other 30 times. Calculate frequency with which each strategy is being played.*

- Players are indifferent between strategies if opponent mixes equally between all three strategies.
- In games such as matching pennies, poker bluffing, football run/pass etc you want to make the opponent guess and you worry about being found out.

2 Mixed Strategy Nash Equilibrium

Write Σ_i (also $\Delta(S_i)$) for the set of probability distributions on S_i.

Write Σ for $\Sigma_1 \times .. \times \Sigma_I$. A mixed strategy profile $\sigma \in \Sigma$ is an I-tuple $(\sigma_1, .., \sigma_I)$ with $\sigma_i \in \Sigma_i$.

We write $u_i(\sigma_i, \sigma_{-i})$ for player i's expected payoff when he uses mixed strategy σ_i and all other players play as in σ_{-i}.

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i, s_{-i} u_i(s_i, s_{-i}) \sigma_i(s_i) \sigma_{-i}(s_i)} \tag{1}$$

Remark 1 *For the definition of a mixed strategy payoff we have to assume that the utility function fulfills the VNM axioms. Mixed strategies induce lotteries over the outcomes (strategy profiles) and the expected utility of a lottery allows a consistent ranking only if the preference relation satisfies these axioms.*

Definition 2 *A mixed strategy NE of G is a mixed profile $\sigma^* \in \Sigma$ such that*

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(\sigma_i, \sigma_{-i}^*\right)$$

for all i and all $\sigma_i \in \Sigma_i$.

3 Testing for MSNE

The definition of MSNE makes it cumbersome to check that a mixed profile is a NE. The next result shows that it is sufficient to check against pure strategy alternatives.

Proposition 1 *σ^* is a Nash equilibrium if and only if*

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(s_i, \sigma_{-i}^*\right)$$

for all i and $s_i \in S_i$.

Example 1 *The strategy profile $\sigma_1^* = \sigma_2^* = \frac{1}{2}H + \frac{1}{2}T$ is a NE of Matching Pennies.*

Because of symmetry is it sufficient to check that player 1 would not deviate. If he plays his mixed strategy he gets expected payoff 0. Playing his two pure strategies gives him payoff 0 as well. Therefore, there is no incentive to deviate.

Note: Mixed strategies can help us to find MSNE when no pure strategy NE exist.

4 Finding Mixed Strategy Equilibria I

Definition 3 *In a finite game, the support of a mixed strategy σ_i, $supp(\sigma_i)$, is the set of pure strategies to which σ_i assigns positive probability*

$$supp(\sigma_i) = \{s_i \in S_i | \sigma_i(s_i) > 0\}$$

Proposition 2 *If σ^* is a mixed strategy Nash equilibrium and $s_i', s_i'' \in supp(\sigma_i^*)$ then*

$$u_i\left(s_i', \sigma_{-i}^*\right) = u_i\left(s_i'', \sigma_{-i}^*\right)$$

Proof: Suppose not. Assume WLOG that

$$u_i\left(s_i', \sigma_{-i}^*\right) > u_i\left(s_i'', \sigma_{-i}^*\right)$$

with $s_i', s_i'' \in supp(\sigma_i^*)$.

Define a new mixed strategy $\hat{\sigma}_i$ for player i by

$$\hat{\sigma}_i(s_i) = \begin{cases} \sigma_i^*(s_i') + \sigma_i^*(s_i'') & \text{if } s_i = s_i' \\ 0 & \text{if } s_i = s_i'' \\ \sigma_i^*(s_i) & \text{otherwise} \end{cases}$$

We can calculate the gain from playing the modified stratgey:

$$\begin{aligned} u_i\left(\hat{\sigma}_i, \sigma_{-i}^*\right) &- u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \\ &= \sum_{s_i \in S_i} u_i\left(s_i, \sigma_{-i}^*\right)\hat{\sigma}_i(s_i) - \sum_{s_i \in S_i} u_i\left(s_i, \sigma_{-i}^*\right)\sigma_i^*(s_i) \\ &= \sum_{s_i \in S_i} u_i\left(s_i, \sigma_{-i}^*\right)\left[\hat{\sigma}_i(s_i) - \sigma_i^*(s_i)\right] \\ &= u_i\left(s_i', \sigma_{-i}^*\right)\sigma_i^*(s_i'') - u_i\left(s_i'', \sigma_{-i}^*\right)\sigma_i^*(s_i'') \\ &> 0 \end{aligned}$$

Note that a mixed strategy NE is never strict.
The proposition suggests a process of finding MSNE.

1. Look at all possible supports for mixed equilibria.

2. Solve for probabilities and check if it works.

Example 2 *Find all the Nash equilibria of the game below.*

	L	R
U	1,1	0,4
D	0,2	2,1

It is easy to see that this game has no pure strategy Nash equilibria. For a mixed strategy Nash equilibrium to exist player 1 has to be indifferent between strategies U and D and player 2 has to be indifferent between L and R. Assume player 1 plays U with probability α and player 2 plays L with probability β.

$$\begin{aligned} u_1(U, \sigma_2^*) &= u_1(D, \sigma_2^*) \\ \beta &= 2(1-\beta) \\ u_2(L, \sigma_1^*) &= u_2(R, \sigma_1^*) \\ \alpha + 2(1-\alpha) &= 4\alpha + (1-\alpha) \end{aligned}$$

We can deduce that $\alpha = \frac{1}{4}$ and $\beta = \frac{2}{3}$. There is a unique mixed Nash equilibrium with $\sigma_1^* = \frac{1}{4}U + \frac{3}{4}D$ and $\sigma_2^* = \frac{2}{3}L + \frac{1}{3}R$

Remark 2 *Recall the Battle of the Sexes experiments from last class. It can be shown that the game has a mixed NE where each agent plays her favorite strategy with probability $\frac{2}{3}$. This was not quite the proportion of people playing it in class (but pretty close to the proportion of people choosing it in the previous year)! In many instances of this experiment one finds that men and women differed in their 'aggressiveness'. Does that imply that they were irrational? No. In a mixed NE players are indifferent between their strategies. As long as men and women are matched completely randomly (i.e. woman-woman and man-man pairs are also possible) it only matters how players mix in the aggregate! It does NOT matter if subgroups (i.e. 'men' and 'women') mix at different states, although it would matter if they* *would play only against players within their subgroup. Interestingly, that suggests that letting women 'segregate' into their own communities should make them more aggressive, and men less aggressive. The term 'aggressive' is a bit misleading because it does not result in bigger payoffs. However, you could come up with a story of lexicographic preferences - people care first of all about payoffs, but everything else equal they want to fit gender stereotypes - so playing 'football' is good for men's ego.*

5 Finding Mixed Strategy Equilibria II

Finding mixed NE in 2 by 2 games is relatively easy. It becomes harder if players have more than two strategies because we have to start worrying about supports. In many cases it is useful to exploit iterated deletion in order to narrow down possible supports.

Proposition 3 *Let σ^* be a NE of G and suppose that $\sigma^*(s_i) > 0$ then $s_i \in S_i^\infty$, i.e. strategy s_i is not removed by ISD.*

Proof: see problem set 2

Having introduced mixed strategies we can even define a tighter notion of IDSDS. Consider the next game. No player has a strategy which is strictly dominated by another pure strategy. However, C for player 2 is strictly dominated by $\frac{1}{2}L + \frac{1}{2}R$. Thus we would think that C won't be used.

	L	C	R
U	1,1	0,2	0,4
M	0,2	5,0	1,6
D	0,2	1,1	2,1

After we delete C we note that M is dominated by $\frac{2}{3}D+\frac{1}{3}U$. Using the above proposition we can conclude that the only Nash equilibria are the NE of the 2 by 2 game analyzed in the previous section. Since that game had a unique mixed strategy equilibrium we can conclude that the only NE of the 3 by 3 game is $\sigma_1^*=\frac{1}{4}U+\frac{3}{4}D$ and $\sigma_2^*=\frac{2}{3}L+\frac{1}{3}R$.

It is useful to adjust the formal definition of IDSDS and allow for mixed strategy domination:

Definition 4 *The set of strategy profiles surviving iterated strict dominance is* $S^\infty = S_1^\infty \times S_2^\infty \times .. \times S_I^\infty$ *where*

$$\begin{aligned} S_i^\infty &= \bigcap_{k=1}^{\infty} S_i^k \\ S_i^0 &= S_i \\ S_i^{k+1} &= \left\{ s_i \in S_i^k \middle| \not\exists \sigma_i \in \Delta\left(S_i^k\right) | u_i\left(\sigma_i, s_{-i}\right) > u_i\left(s_i, s_{-i}\right) \forall s_{-i} \in S_{-i}^k \right\} \end{aligned}$$

Remark 3 *Recall the above 3 by 3 game. If we would look for possible mixed NE with supports (U,M) and (L,C) respectively, we would find a potential NE* $\frac{2}{3}U+\frac{1}{3}M, \frac{5}{6}L+\frac{1}{6}C$. *However, this is NOT a NE because player 2 would play R instead.*

Remark 4 *In the RPS game we cannot reduce the set of strategies through IDSDS. Therefore we have to check all possible supports and check if it works.*

6 Finding Mixed Strategy Equilibria III

Definition 5 *A correspondence* $F: A \to B$ *is a mapping which associates to every element of* $a \in A$ *a subset* $F(a) \subset B$.

The mixed strategy best response correspondence for player i $BR_i : \Sigma_{-i} \to \Sigma_i$ is defined by

$$BR_i\left(\sigma_{-i}\right) = \arg\max_{\sigma_i \in \Sigma_i} u_i\left(\sigma_i, \sigma_{-i}\right)$$

Proposition 4 σ^* *is a Nash equilibrium if and only if* $\sigma_i^* \in BR_i\left(\sigma_{-i}^*\right)$ *for all* i.

In the 2 by 2 game we have:

$$BR_1(\beta L + (1-\beta) R) = \begin{cases} U & \text{if } \beta > \frac{2}{3} \\ \{\alpha U + (1-\alpha) D | \alpha \in [0,1]\} & \text{if } \beta = \frac{2}{3} \\ D & \text{if } \beta < \frac{2}{3} \end{cases}$$

$$BR_2(\alpha U + (1-\alpha) D) = \begin{cases} L & \text{if } \alpha < \frac{1}{4} \\ \{\beta L + (1-\beta) R | \beta \in [0,1]\} & \text{if } \alpha = \frac{1}{4} \\ R & \text{if } \alpha > \frac{1}{4} \end{cases}$$

We can graph both correspondences to find the set of Nash equilibria.

7 Interpretation of Mixed NE

1. Sometimes players explicitly flip coins. That fits games like Poker, soccer etc., where players have to randomize credibly.

2. Large populations of players with each player playing a fixed strategy and random matching. That's very related to the social norm explanation of pure Nash equilibrium.

3. Payoff uncertainty (Harsanyi, purification). Roughly their argument goes as follows in the matching penny game. There are two types of players - those who get slightly higher payoff from playing heads, and those who get higher payoff from getting tails (their preferences are almost the same - think of one guy getting 1 dollar and the other guy getting $1 + \epsilon = 1.01$ dollars from playing head). Also, we assume that there is an equal probability that my opponent is of type 1 or type 2. In this circumstances no player loses from just playing her favorite strategy (i.e. a pure strategy) because it will do best on average. To show that this is a NE we have to introduce the notion of incomplete information which we don't do just yet. Harsanyi-Selten then let the payoff uncertainty ϵ go to 0 such that the game in the limit approaches the standard matching pennies. Players' 'average' strategies converge to the mixed equilibrium.