Lecture I-II: Motivation and Decision Theory

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1 Motivating Experiment: Guess the average

Setup: Each of you (the students in this course) have to declare an integer between 0 and 100 to guess "2/3 of the average of all the responses". More precisely, each student who guesses the highest integer which is not higher than 2/3 of the average of all responses, will receive a prize of 10 Dollars.

This game can be solved by iterated dominance. However, being smart is not enough. Those who bid 0 *typically lose badly. You have to guess how 'dumn' all the other students in the class are. Even if everyone is supersmart, but has low confidence in the smartness of others, the winning value can be quite high. In our class we got a winning bid of 17 and 1 person walked away with 10 Dollars.*

2 What is game theory?

Definition 1 *Game theory is a formal way to analyze interaction among a group of rational agents who behave strategically.*

This definition contains a number of important concepts which are discussed in order:

Group: In any game there is more than one decision maker who is referred to as player. If there is a single player the game becomes a decision problem.

Interaction: What one individual player does directly affects at least one other player in the group. Otherwise the game is simple a series of independent decision problems.

Strategic: Individual players account for this interdependence.

Rational: While accounting for this interdependence each player chooses her best action. This condition can be weakened and we can assume that agents are boundedly rational. Behavioral economics analyzes decision problems in which agents behave boundedly rational. Evolutionary game theory is game theory with boundedly rational agents.

Example 1 *Assume that 10 people go into a restaurant. Every person pays for her own meal. This is a decision problem. Now assume that everyone agrees before the meal to split the bill evenly amongst all 10 participants. Now we have a game.*

Game theory has found numerous applications in all fields of economics:

1. *Trade:* Levels of imports, exports, prices depend not only on your own tariffs but also on tariffs of other countries.
2. *Labor:* Internal labor market promotions like tournaments: your chances depend not only on effort but also on efforts of others.
3. *IO:* Price depends not only on your output but also on the output of your competitor (market structure ...).
4. *PF:* My benefits from contributing to a public good depend on what everyone else contributes.
5. *Political Economy:* Who/what I vote for depends on what everyone else is voting for.

3 Decision Theory under Certainty

It makes sense to start by discussing trivial games - those we play against ourselves, e.g. decision problems. Agents face situations in which they have to make a choice. The actions of other agents do not influence my preference ordering over those choices - therefore there is no strategic interaction going on. Proper games will be discussed in the next lectures.

A decision problem $(A, \preceq)$ consists of a finite set of outcomes $A = \{a_1, a_2, .., a_n\}$ and a preference relation $\preceq$. The expression $a \preceq b$ should be interpreted as "b is at least as good as a". We expect the preference relation to fulfill two simple axioms:

Axiom 1 *Completeness. Any two outcomes can be ranked, e.g.* $a \preceq b$ *or* $b \preceq a$*.*

Axiom 2 *Transitivity implies that if* $a \geq b$ *and* $b \geq c$ *then* $a \geq c$*.*

Both axioms ensure that all choices can be ordered in a single chain without gaps (axiom 1) and without cycles (axiom 2).

Although the preference relation is the basic primitive of any decision problem (and generally observable) it is much easier to work with a consistent utility function $u : A \to \Re$ because we only have to remember n real numbers $\{u_1, u_2, .., u_n\}$.

Definition 2 *A utility function* $u : A \to \Re$ *is consist with the preference relationship of a decision problem* $(A, \preceq)$ *if for all* $a, b \in A$*:*

$$a \preceq b \quad \textit{if and only if} \quad u(a) \leq u(b)$$

Theorem 1 *Assume the set of outcomes is finite. Then there exists a utility function u which is consistent.*

Proof: The proof is very simple. Simple collect all equivalent outcomes in equivalence classes. There are finitely many of those equivalence classes since there are only finitely many outcomes. Then we can order these equivalence classes in a strictly increasing chain due to completeness and transitivity.

Note that the utility function is not unique. In fact, any monotonic transformation of a consistent utility function gives another utility function which is also consistent.

We can now define what a rational decision maker is.

Definition 3 *A rational decision maker who faces a decision problem* $(A, \preceq)$ *chooses an outcome* $a^* \in A$ *which maximizes his utility (or, equivalently, for each* $a \in A$ *we have* $a \preceq a^*$*).*

Remark 1 *When there are infinitely many choices we want to make sure that there is a continuous utility function. This requires one more axiom which makes sure that preferences are continuous. For that purpose, one has to define topology on the set of outcomes. We won't deal with that since we won't gain much insight from it.*

4 Decision Theory under Uncertainty

Lotteries are defined over the of outcomes A (which is again assumed to be finite to keep things simple).

Definition 4 *A simple lottery is defined as the set* $\{(a_1, p_1), (a_2, p_2), .. (a_n, p_n)\}$ *such that* $\sum_{i=1}^n p_i = 1$ *and* $0 \leq p_i \leq 1$*. In a simple lottery the outcome* a_i *occurs with probability* p_i*.*

When there are up to three outcomes we can conveniently describe the set of lotteries in a graphical way (see triangle).

Under certainty the preference relationship can still be written down explicitly for finite A (simply write down all of the $\frac{n(n+1)}{2}$ rankings). Under uncertainty there are suddenly infinitely many lotteries. This poses two problems. First of all, it's impractical to write a large number of lottery comparisons down. A second (and deeper) point is the observation that the preference relationship is in principle unobservable because of the infinite number of necessary comparisons.

John von Neumann and Oscar Morgenstern showed that under some additional restrictions on preferences over lotteries there exists a utility function over outcomes such that the expected utility of a lottery provides a consistent ranking of all lotteries.

Definition 5 *Assume a utility function* u *over the outcomes* A*. The expected utility of the lottery* $L = \{(a_1, p_1), (a_2, p_2), .. (a_n, p_n)\}$ *is defined as*

$$u(L) = \sum_{i=1}^{n} u(a_i) p_i$$

Before we introduce the additional axioms we discuss the notion of compound (two stage) lotteries.

Definition 6 *The compound lottery* $\tilde{L}$ *is expressed as* $\tilde{L} = \{(L_1, q_1), (L_2, 1 - q_1)\}$*. With probability* q_1 *the simple lottery* L_1 *is chosen and with probability* $1 - q_1$ *the simple lottery* L_2 *is chosen.*

Note, that we implicitly distinguish between simple and compound lotteries. Therefore, we allow that a simple lottery L might have the same outcome distribution as the compound lottery $\tilde{L}$ but $L \prec \tilde{L}$.

The first axiom assumes that only outcomes matter - the process which generates those outcomes is irrelevant.

Axiom 3 *Each compound lottery is equivalent to a simple lottery with the same distribution over final outcomes.*

In some books the equivalence of simple and compound lotteries is assumed in the definition of a lottery. However, it is useful to keep those types of lotteries separate because we know that the framing of a decision problem influences how people make choices (i.e. both the process and the final outcome distribution matter).

The next axiom is fairly uncontroversial.

Axiom 4 *Monotonicity. Assume that the lottery L_1 is preferred to lottery L_2. Then the compound lottery $\{(L_1, \alpha), (L_2, 1-\alpha)\}$ is preferred to $\{(L_1, \beta), (L_2, 1-\beta)\}$ if $\alpha > \beta$.*

Axiom 5 *Archimedian. For any outcomes $a < b < c$ there is some lottery $L = \{(a, \alpha), (c, 1-\alpha)\}$ such that the agent is indifferent between L and b.*

The substitution axiom (also know as independence of irrelevant alternatives) is the most critical axiom.

Axiom 6 *Substitution. If lottery L_1 is preferred to lottery L_2 then any mixture of these lotteries with any other lottery L_3 preserves this ordering:*

$$\{(L_1, \alpha), (L_3, 1-\alpha)\} \geq \{(L_2, \alpha), (L_3, 1-\alpha)\}$$

This axiom is also known as independence of irrelevant alternatives.

Under these axioms we obtain the celebrated result due to John von Neumann and Oskar Morgenstern.

Theorem 2 *Under the above axioms an expected utility function exists.*

Proof: First of all we find the best and worst outcome b and w (possible because there are only finitely many outcomes). Because of the Archimedian axiom we can find a number α_a for each outcome a such that $L = \{(b, \alpha), (w, 1-\alpha)\}$. We can define a utility function over each outcome a such that $u(a) = \alpha$. Using the monotonicity axiom it can be shown that this number is unique. For each lottery we can now calculate its expected utility. It remains to be shown that this expected utility function is consistent with the original preferences. So take two lotteries L_1 and L_2 such that $L_1 \preceq L_2$. We can write $L_1 = \{(a_1, p_1), (a_2, p_2), .. (a_n, p_n)\}$

and $L_2 = \{(a_1, q_1), (a_2, q_2), .. (a_n, q_n)\}$. Now replace each outcome a_i by the above lotteries. The compound lottery can be rewritten as $L_1 = \{(b, \sum_{i=1}^n p_i u(a_i)), (w, 1 - \sum_{i=1}^n p_i u(a_i))\}$. Similarly, we get $L_2 = \{(b, \sum_{i=1}^n q_i u(a_i)), (w, 1 - \sum_{i=1}^n q_i u(a_i))\}$. By the monotonicity axiom we can deduce that $\sum_{i=1}^n p_i u(a_i) \leq \sum_{i=1}^n q_i u(a_i)$, e.g. $EU(L_1) \leq EU(L_2)$. QED

From now on all payoffs in our course will be assumed to represent vNM utility values. The expected payoff will be the expected utility.

4.1 Puzzles

EUT forms the basis of modern micro economics. Despite its success there are important behavioral inconsistencies related to it. Some of those we are going to discuss briefly before we turn our attention to proper games.

4.2 Allais Paradox

Consider the following choice situation (A) among two lotteries:

- Lottery A1 promises a sure win of 3000,
- Lottery A2 is a 80 percent chance to win 4000 (and zero in 20 percent of the cases).

Typically, A1 is strictly preferred to A2. Now, consider two further choice pairs (B) and (C):

- Lottery B1 promises a 90 percent chance of winning 3000,
- Lottery B2 is a 72 percent chance to win 4000.

This choice is included to see if there is a certainty effect.

- Lottery C1 promises a 25 percent chance of winning 3000,
- Lottery C2 is a 20 percent chance to win 4000.

Most people in our class now preferred C2 over C1.

It can be checked that the lotteries B_i and C_i are derived from A_i just by mixing the original lotteries with an irrelevant alternative - in the case of (B) there is a 10 percent chance of getting nothing and a 90 percent chance of getting (A), and in case of (C), there is a 75 percent chance of getting nothing.

The Allais paradox is the most prominent example for behavioral inconsistencies related to the von Neumann Morgenstern axiomatic model of choice under uncertainty. The Allais paradox shows that the significant majority of real decision makers orders uncertain prospects in a way that is inconsistent with the postulate that choices are independent of irrelevant alternatives.

There is an alternative explanation for the failure of EUT in this case. Assume, that agents face the compound lottery instead of the simple lotteries (B) and (C). Now the relationship to (A) is much more transparent - in fact, one could tell a story such as: "with 75 percent probability you are not invited to choose between these two outcomes, and with 25 percent probability you can choose either A1 or A2". It's likely that choices would be much more consistent now.

The standard explanation for the failure of EUT is peoples' inability to keep small probability differences apart. 80 percent and 100 percent 'looks' quite different and people focus on the probabilities. 20 percent and 25 percent 'looks' the same - so people focus on the values instead. Prospect theory (Kahnemann and Tversky) can deal with the Allais paradox by weighting probabilities accordingly.

4.3 Framing effects

Framing effects are preference reversals induced by changes in reference points.

Consider the following choice situation (A):

Pair 1: 600 people are struck with a disease that could kill. Vaccine 1 will save 400 lives for sure while the second one will either save no one (1/3) or will save everyone (with probability 2/3).

Pair 2: 600 people are struck with a disease that could kill. Vaccine 1 will kill 200 people for sure while the second one implies a 2/3 chance that no one will die and a 1/3 chance that everyone will die.

Note that both situations are identical because save is equal to not kill. However, people tend to be risk averse in saving lives and risk loving if it is

Table 1: McNeil, Pauker and Tversky (1988)

	Survival		Mortality		Both	
	Radiation	Surgery	Radiation	Surgery	Radiation	Surgery
immediate	100	90	0	10		
1 year	77	68	23	32		
5 year	22	34	78	66		
US	16	84	50	50	44	56
Israeli	20	80	45	56	34	66

phrased in terms of losses (kills).

Preference reversals have real effects and do not just appear in cute examples. McNeil, Pauker and Tversky (1988) asked American doctors and Israeli medical students about how they would choose between two cancer treatments (surgery and radiation) - they presented one group with survival statistics, a second group with mortality statistics and a third group with both. Table 1 sums up their choices.