Lecture XIII: Analysis of Infinite Games

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Readings for this class: OR (sections 6.5, 8.1 and 8.2) Fudenberg and Tirole, chapters 4.2 and 4.3 are most useful; Dutta, chapter 15

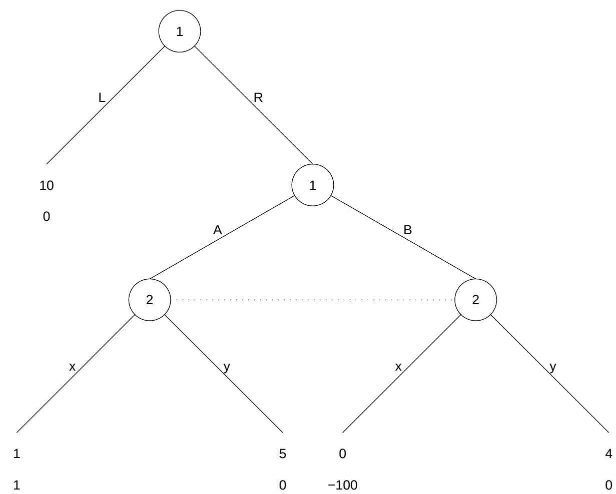
1 Introduction - Critique of SPE

The SPE concept eliminates non-credible threats but it's worth to step back for am minute and ask whether we think SPE is reasonable or in throwing out threats we have been overzealous.

Practically, for this course the answer will be that SPE restrictions are OK and we'll always use them in extensive form games. However, it's worth looking at situations where it has been criticized. Some of the worst anomalies disappear in infinite horizon games which we study next.

1.1 Rationality off the Equilibrium Path

Is it reasonable to play NE off the equilibrium path? After all, if a player does not follow the equilibrium he is probably as stupid as a broomstick. Why should we trust him to play NE in the subgame? Let's look at the following game to illustrate that concern:



Here (L, A, x) is the unique SPE. However, player 2 has to put a lot of trust into player 1's rationality in order to play x. He must believe that player 1 is smart enough to figure out that A is a dominant strategy in the subgame following R. However, player 2 might have serious doubts about player 1's marbles after the guy has just foregone 5 utils by not playing L.[1]

Remark 1 *The main reason for the breakdown of cooperation in the finitely repeated Prisoner's Dilemma is not so much SPE by itself by the fact that there is a final period in which agents would certainly defect. This raises the question whether an infinitely repeated PD game would allow us to cooperate. Essentially, we could cooperate as long as the other player does, and if there is defection, we defect from then on. This still looks like a SPE - in any subgame in which I have defected before, I might as well defect forever. If I haven't defected yet, I can jeopardize cooperation by defection, and therefore should not do it as long as I care about the future sufficiently.*

1.2 Multi-Stage Games

Lemma 1 *The unique SPE of the finitely repeated Prisoner's Dilemma game in which players get the sum of their payoffs from each stage game has every player defecting at each information set.*

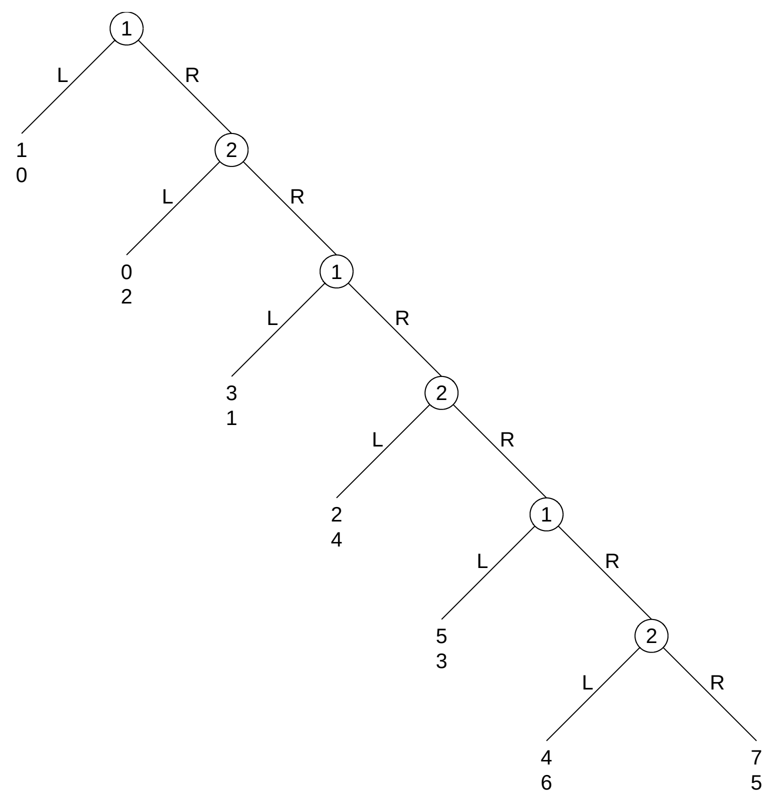
[1] After all any strategy in which L is played strictly dominates any strategy in which R is played in the normal form.

The proof proceeds by analyzing the last stage game where we would see defection for sure. But then we would see defection in the second to last stage game etc. In the same way we can show that the finitely repeated Bertrand game results in pricing at marginal cost all the time.

These results should make us suspicious. Axelrod's experiments (see future lecture) showed that in the finitely repeated Prisoner's Dilemma people tend to cooperate until the last few periods when the 'endgame effect' kicks in. Similarly, there are indications that rival firms can learn to collude if they interact repeatedly and set prices above marginal cost.

This criticisms of SPE is reminiscent of our criticism of IDSDS. In both cases we use an iterative procedure to find equilibrium. We might have doubts where real-world subjects are able (and inclined) to do this calculation.

A famous example for the perils of backward induction is Rosenthal's centipede game:



The game can be extended to even more periods. The unique SPE of the game is to drop out immediately (play L) at each stage. However, in experiments people typically continue until almost the last period before they drop out.

2 Infinite Horizon Games

Of the criticism of SPE the one we will take most seriously is that long finite horizon models do not give reasonable answers. Recall that the problem was that the backward induction procedure tended to unravel 'reasonably' looking strategies from the end. It turns out that many of the anomalies go away once we model these games as infinite games because there is not endgame to be played.

The prototypical model is what Fudenberg and Tirole call an infinite horizon multistage game with observed actions.

- At times $t = 0, 1, 2, ..$ some subset of the set of players simultaneously chooses actions.
- All players observe the period t actions before choosing period $t + 1$ actions.
- Players payoffs maybe any function of the infinite sequence of actions (play does not end in terminal nodes necessarily any longer)

2.1 Infinite Games with Discounting

Often we assume that player i's payoffs are of the form:

$$u_i(s_i, s_{-i}) = u_{i0}(s_i, s_{-i}) + \delta u_{i1}(s_i, s_{-i}) + \delta^2 u_{i2}(s_i, s_{-i}) + .. \quad (1)$$

where $u_{it}(s_i, s_{-i})$ is a payoff received at t when the strategies are followed.

2.1.1 Interpretation of δ

1. Interest rate $\delta = \frac{1}{1+r}$. Having two dollars today or two dollars tomorrow makes a difference to you: your two dollars today are worth more, because you can take them to the bank and get $2(1+r)$ Dollars tomorrow where r is the interest rate. By discounting future payoffs with $\delta = \frac{1}{1+r}$ we correct for the fact that future payoffs are worth less to us than present payoffs.

2. Probabilistic end of game: suppose the game is really finite, but that the end of the game is not deterministic. Instead given that stage t is reached there is probability δ that the game continues and probability $1-\delta$ that the game ends after this stage. Note, that the expected number of periods is then $\frac{1}{1-\delta}$ and finite. However, we can't apply backward induction directly because we can never be sure that any round is the last one. The probabilistic interpretation is particularly attractive for interpreting bargaining games with many rounds.

2.2 Example I: Repeated Games

Let G be a simultaneous move game with finite action spaces A_1,..,A_I. The infinitely repeated game G^∞ is the game where in periods $t = 0, 1, 2, ..$ the players simultaneously choose actions $(a_1^t, .., a_I^t)$ after observing all previous actions. We define payoffs in this game by

$$u_i\left(s_i, s_{-i}\right) = \sum_{t=0}^{\infty} \delta^t \tilde{u}_i\left(a_1^t, .., a_I^t\right) \tag{2}$$

where $(a_1^t, .., a_I^t)$ is the action profile taken in period t when players follow strategies $s_1, .., s_I$, and $\tilde{u}_i$ are the utilities of players in each stage game. For example, in the infinitely repeated Prisoner's Dilemma game the $\tilde{u}_i$ are simply the payoffs in the 'boxes' of the normal form representation.

2.3 Example II: Bargaining

Suppose there is a one Dollar to be divided up between two players. The following alternate offer procedure is used:

I. In periods 0,2,4,... player 1 offers the division $(x_1, 1-x_1)$. Player 2 then accepts and the game ends, or he rejects and play continues.

II. In period 1,3,5,... player 2 offers the division $(1-x_2, x_2)$. Player 1 then accepts or rejects.

Assume that if the division $(y, 1-y)$ is agreed to in period t then the payoffs are $\delta^t y$ and $\delta^t\left(1-y\right)$.[2]

[2] Note that this is not a repeated game. First of all, the stage games are not identical (alternate players make offers). Second, there is no per period payoff. Instead, players only get payoffs when one of them has agreed to an offer. Waiting to divide the pie is costly.

Remark 2 *There are finite versions of this game in which play end after period T. One has to make some assumption what happens if there is no agreement at time T - typically, one assumes that the pie simply disappears. If $T = 1$ then we get the simple ultimatum game.*

3 Continuity at Infinity

None of the tools we've discussed so far are easy to apply for infinite games. First, backward induction isn't feasible because there is no end to work backward from. Second, using the definition of SPE alone isn't very easy. There are infinitely many subgames and uncountably many strategies that might do better.

We will discuss a theorem which makes the analysis quite tractable in most infinite horizon games. To do so, we must first discuss what continuity at infinity means.

Definition 1 *An infinite extensive form game G is continuous at ∞ if*

$$\lim_{T\to\infty} \sup_{i,\sigma,\sigma' \text{ s. th. } \sigma(h) = \sigma'(h) \text{ for all } h \text{ in periods } t \leq T} \left|u_i(\sigma) - u_i(\sigma')\right| = 0$$

In words: compare the payoffs of two strategies which are identical for all information sets up to time T and might differ thereafter. As T becomes large the maximal difference between any two such strategies becomes arbitrarily small. Essentially, this means that distant future events have a very small payoff effect.

3.1 Example I: Repeated Games

If σ and σ' agree in the first T periods then:

$$\begin{aligned}
\left|u_i(\sigma) - u_i(\sigma')\right| &= \left|\sum_{t=T}^{\infty} \delta^t \left(\tilde{u}_i\left(a^t\right) - \tilde{u}_i\left(a'_t\right)\right)\right| \\
&\leq \sum_{t=T}^{\infty} \delta^t \max_{i,a,a'} \left|\tilde{u}_i(a) - \tilde{u}_i(a')\right|
\end{aligned}$$

For finite stage games we know that $M = \max_{i,a,a'} |\tilde{u}_i(a) - \tilde{u}_i(a')|$ is finite. This implies that

$$\lim_{T\to\infty} |u_i(\sigma) - u_i(\sigma')| \leq \lim_{T\to\infty} \sum_{t=T}^{\infty} \delta^t M = \lim_{T\to\infty} \frac{\delta^T}{1-\delta} M = 0.$$

3.2 Example II: Bargaining

It's easy to check that the bargaining game is also continuous.

3.3 Example III: Non-discounted war of attrition

This is an example for an infinite game which is NOT continuous at infinity. Players 1 and 2 choose $a_i^t \in \{\text{Out}, \text{Fight}\}$ at time $t = 0, 1, 2, ...$ The game ends whenever one player quits with the other being the 'winner'. Assume the payoffs are

$$u_i(s_i, s_{-i}) = \begin{cases} 1 - ct & \text{if player } i \text{ 'wins' in period } t \\ -ct & \text{if player } i \text{ quits in period } t \end{cases}$$

Note, that players in this game can win a price of 1 by staying in the game longer than the other player. However, staying in the game is costly for both players. Each player wants the game to finish as quickly as possible, but also wants the other player to drop out first.

This game is not continuous at ∞. Let

$$\begin{aligned} \sigma^T &= \text{Both fight for } T \text{ periods and then 1 quits} \\ \sigma'^T &= \text{Both fight for } T \text{ periods and then 2 quits.} \end{aligned}$$

Then we have

$$\left| u_i\left(\sigma^T\right) - u_i\left(\sigma'^T\right) \right| = 1.$$

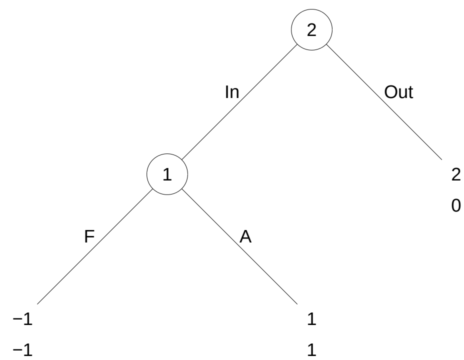
This expression does not go to zero as $T \to \infty$.

4 The Single-Period Deviation Principle

The next theorem makes the analysis of infinite games which are continuous at ∞ possible.

Theorem 1 *Let G be an infinite horizon multistage game with observed actions which is continuous at ∞. A strategy profile $(\sigma_1, .., \sigma_I)$ is a SPE if and only if there is no player i and strategy $\hat{\sigma}_i$ that agrees with σ_i except at a single information set h_i^t and which gives a higher payoff to player i conditional on h_i^t being reached.*

We write $u_i(\sigma_i, \sigma_{-i}|x)$ for the payoff conditional on x being reached. For example, in the entry game below we have $u_2(\text{Accomodate}, \text{Out}|\text{node 1 is reached}) = 1$.



Recall, that we can condition on nodes which are not on the equilibrium path because the strategy of each player defines play at each node.

4.1 Proof of SPDP for Finite Games

I start by proving the result for finite-horizon games with observed actions.

Step I: By the definition of SPE there cannot be a profitable deviation for any player at some information set in games with observed actions.[3]

Step II: The reverse is a bit harder to show. We want to show that $u_i(\hat{\sigma}_i, \sigma_{-i}|x_t) \leq u_i(\sigma_i, \sigma_{-i}|x_t)$ for all initial nodes x_t of a subgame (subgame at some round t).

We prove this by induction on T which is the number of periods in which σ_i and $\hat{\sigma}_i$ differ.

[3] We have to very careful at this point. We have defined SPE as NE in every subgame. Subgames can only originate at nodes and not information sets. However, in games with observed actions all players play simultaneous move games in each round t. Therefore any deviation by a player at an information set at round t which is not a singleton is on the equilibrium path of some subgame at round t.

T=1: In this case the result is clear. Suppose σ_i and $\hat{\sigma}_i$ differ only in the information set in period t'. If $t > t'$ it is clear that $u_i(\hat{\sigma}_i, \sigma_{-i}|x_t) = u_i(\sigma_i, \sigma_{-i}|x_t)$ because the two strategies are identical at all the relevant information sets. If $t \le t'$ then:

$$\begin{aligned} u_i(\hat{\sigma}_i, \sigma_{-i}|x_t) &= \sum_{h_{it'}} u_i(\hat{\sigma}_i, \sigma_{-i}|h_{it'})\, Prob\{h_{it'}|\hat{\sigma}_i, \sigma_{-i}, x_t\} \\ &\le \sum_{h_{it'}} \underbrace{u_i(\sigma_i, \sigma_{-i}|h_{it'})}_{\text{follows from one stage deviation criterion}} \underbrace{Prob\{h_{it'}|\sigma_i, \sigma_{-i}, x_t\}}_{\text{follows from } \sigma_i \text{ and } \hat{\sigma}_i \text{ having same play between } t \text{ and } t'} \\ &= u_i(\sigma_i, \sigma_{-i}|x_t) \end{aligned}$$

T $\rightarrow$ T+1: Assuming that the result holds for T let $\hat{\sigma}_i$ be any strategy differing from σ_i in $T+1$ periods. Let t' be the *last* period at which they differ and define $\tilde{\sigma}_i$ by:

$$\tilde{\sigma}_i(h_{it}) = \begin{cases} \hat{\sigma}_i(h_{it}) & \text{if } t < t' \\ \sigma_i(h_{it}) & \text{if } t \ge t' \end{cases}$$

In other words, $\tilde{\sigma}_i$ differs from σ_i only at T periods. Therefore we have for any x_t

$$u_i(\tilde{\sigma}_i, \sigma_{-i}|x_t) \le u_i(\sigma_i, \sigma_{-i}|x_t)$$

by the inductive hypothesis since we assumed that the claim holds for T.

However, we also know that $\tilde{\sigma}$ and $\hat{\sigma}$ only differ in a single deviation at round t'. Therefore, we can use exactly the same argument as in the previous step to show that

$$u_i(\hat{\sigma}_i, \sigma_{-i}|x_t) \le u_i(\tilde{\sigma}_i, \sigma_{-i}|x_t)$$

for any x_t.[4]

Combining both inequalities we get the desires result:

$$u_i(\hat{\sigma}_i, \sigma_{-i}|x_t) \le u_i(\sigma_i, \sigma_{-i}|x_t)$$

This proves the result for finite games with observed actions.

[4] It is important that we have defined $\tilde{\sigma}_i$ in differing only in the last period deviation. Therefore, after time t' strategy $\tilde{\sigma}_i$ follows σ_i. This allows us to use the SPDP.

4.2 Proof for infinite horizon games

Note, that the proof for finite-horizon games also establishes that we for a profile σ which satisfies SPDP player i cannot improve on σ_i in some subgame x_t by considering a new strategy $\hat{\sigma}_i$ with finitely many deviations from σ_i. However, it is still possible that deviations at infinitely many periods might be an improvement for player i.

Assume this would be the case for some $\hat{\sigma}_i$. Let's denote the gain from using this strategy with ϵ:

$$\epsilon = u_i\left(\hat{\sigma}_i, \sigma_{-i}|x_t\right) - u_i\left(\sigma_i, \sigma_{-i}|x_t\right)$$

Because the game is continuous at ∞ this implies that if we choose T large enough we can define some strategy $\tilde{\sigma}_i$ which agrees with $\hat{\sigma}_i$ up to period T and then follows strategy σ_i such that:

$$\left|u_i\left(\hat{\sigma}_i, \sigma_{-i}|x_t\right) - u_i\left(\tilde{\sigma}_i, \sigma_{-i}x_t\right)\right| < \frac{\epsilon}{2}$$

This implies that the new strategy $\tilde{\sigma}_i$ gives player i strictly more utility than σ_i. However, it can only differ from σ_i for at most T periods. But this is a contradiction as we have shown above. QED

Remark 3 *Games which are at continuous at ∞ are in some sense 'the next best thing' to finite games.*

5 Analysis of Rubinstein's Bargaining Game

To illustrate the use of the single period deviation principle and to show the power of SPE in one interesting model we now return to the Rubinstein bargaining game introduced before.

First, note that the game has many Nash equilibria. For example, player 2 can implement any division by adopting a strategy in which he only accepts and proposes a share x_2 and rejects anything else.

Proposition 1 *The bargaining game has a unique SPE. In each period of the SPE the player i who proposes picks $x_i = \frac{1}{1+\delta}$ and the other player accepts any division giving him at least $\frac{\delta}{1+\delta}$ and rejects any offer giving him less.*

Several observations are in order:

1. We get a roughly even split for δ close to 1 (little discounting). The proposer can increase his share the more impatient the other player is.

2. Agreement is immediate and bargaining is therefore efficient. Players can perfectly predict play in the next period and will therefore choose a division which makes the other player just indifferent between accepting and making her own proposal. There is no reason to delay agreement because it just shrinks the pie. Immediate agreement is in fact not observed in most experiments - last year in a two stage bargaining game in class we observed that only 2 out of 14 bargains ended in agreement after the first period. There are extensions of the Rubinstein model which do not give immediate agreement.[5]

3. The division becomes less equal for finite bargaining games. Essentially, the last proposer at period T can take everything for himself. Therefore, he will tend to get the greatest share of the pie in period 1 as well - otherwise he would continue to reject and take everything in the last period. In our two-period experiments we have in deed observed greater payoffs to the last proposer (ratio of 2 to 1). However, half of all bargains resulted in disagreement after the second period and so zero payoffs for everyone. Apparently, people care about fairness as well as payoffs which makes one wonder whether monetary payoffs are the right way to describe the utility of players in this game.

5.1 Useful Shortcuts

The hard part is to show that the Rubinstein game has a unique SPE. If we know that it is much easier to calculate the actual strategies.

5.1.1 Backward Induction in infinite game

You can solve the game by assuming that random payoff after rejection of offer at period T. The game then becomes a finite game of perfect information which can be solved through backward induction. It turns out that as $T \to \infty$

[5]For example, when there is uncertainty about a players discount factor the proposer might start with a low offer in order to weed out player 2 types with low discount factor. Players with high δ will reject low offers and therefore agreement is not immediate. To analyze these extension we have to first develop the notion of incomplete information games.

the backward solution converges to the Rubinstein solution. This technique also provides an alternative proof for uniqueness.

5.1.2 Using the Recursive Structure of Game

The game has a recursive structure. Each player faces essentially the same game, just with interchanged roles. Therefore, in the unique SPE player 1 should proposes some $(x, 1-x)$ and player 2 should propose $(1-x, x)$. Player 1's proposal to 2 should make 2 indifferent between accepting immediately or waiting to make her own offer (otherwise player 1 would bid higher or lower). This implies:

$$\begin{aligned} \underbrace{1-x}_{\text{2's payoff at period 1}} &= \underbrace{\delta x}_{\text{2's discounted payoff from period 2}} \\ x &= \frac{1}{1+\delta} \end{aligned}$$

5.2 Proof of Rubinstein's Solution

5.2.1 Existence

We show that there is no profitable single history deviation.

Proposer: If he conforms at period t the continuation payoff is $\delta^t \frac{1}{1+\delta}$. If he deviates and asks for more he gets $\delta^{t+1} \frac{\delta}{1+\delta}$. If he deviates and asks for less he gets less. Either way he loses.

Recipient: Look at payoffs conditional on y being proposes in period t.

- If he rejects he gets $\delta^{t+1} \frac{1}{1+\delta}$.
- If he accepts he gets $\delta^t y$.
- If $y \geq \frac{\delta}{1+\delta}$ the strategy says accept and this is better than rejecting.
- If $y < \frac{\delta}{1+\delta}$ the strategy says reject and accepting is not a profitable deviation.

5.2.2 Uniqueness

This proof illustrates a number of techniques which enable us to prove properties about equilibrium without actually constructing it.

Let $\overline{v}$ and $\underline{v}$ be the highest and lowest payoffs received by a proposer in any SPE. We first observe:

$$1 - \overline{v} \geq \delta \underline{v} \tag{3}$$

If this equation would not be true then no proposer could propose $\overline{v}$ because the recipient could always get more in any subgame. We also find:

$$\underline{v} \geq 1 - \delta \overline{v} \tag{4}$$

If not then no proposer would propose $\underline{v}$ - she would rather wait for the other player to make her proposal because she would get a higher payoff this way.

We can use both inequalities to derive bounds on $\underline{v}$ and $\overline{v}$:

$$\begin{aligned} \underline{v} &\geq 1 - \delta \overline{v} \geq 1 - \delta \left(1 - \delta \underline{v}\right) & (5) \\ \left(1 - \delta^2\right) \underline{v} &\geq 1 - \delta & (6) \\ \underline{v} &\geq \frac{1}{1+\delta} & (7) \end{aligned}$$

Similarly, we find:

$$\begin{aligned} 1 - \overline{v} &\geq \delta \underline{v} \geq \delta \left(1 - \delta \overline{v}\right) & (8) \\ \overline{v} &\leq \frac{1}{1+\delta} & (9) \end{aligned}$$

Hence $\overline{v} = \underline{v} = \frac{1}{1+\delta}$. Clearly, no other strategies can generate this payoff in every subgame.[6]

[6] While being a nice result it does not necessarily hold anymore when we change the game. For example, if both players make simultaneous proposals then any division is a SPE. Also, it no longer holds when there are several players.