Lecture VI: Existence of Nash equilibrium

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Readings for this class: Osborne and Rubinstein, section 2.4 and Fudenberg and Tirole, chapter 1.3

1 Nash's Existence Theorem

When we introduced the notion of Nash equilibrium the idea was to come up with a solution concept which is stronger than IDSDS. Today we show that NE is not too strong in the sense that it guarantees the existence of at least one mixed Nash equilibrium in most games (for sure in all finite games). This is reassuring because it tells that there is at least one way to play most games.[1]

Let's start by stating the main theorem we will prove:

Theorem 1 ***(Nash Existence)****Every finite strategic-form game has a mixed-strategy Nash equilibrium.*

Many game theorists therefore regard the set of NE for this reason as the lower bound for the set of reasonably solution concept. A lot of research has gone into refining the notion of NE in order to retain the existence result but get more precise predictions in games with multiple equilibria (such as coordination games).

However, we have already discussed games which are solvable by IDSDS and hence have a unique Nash equilibrium as well (for example, the two thirds of the average game), but subjects in an experiment will not follow those equilibrium prescription. Therefore, if we want to *describe* and *predict* the behavior of real-world people rather than come up with an explanation

[1]Note, that a pure Nash equilibrium is a (degenerate) mixed equilibrium, too.

of how they *should* play a game, then the notion of NE and even even IDSDS can be too restricting.

Behavioral game theory has tried to weaken the joint assumptions of rationality and common knowledge in order to come up with better theories of how real people play real games. Anyone interested should take David Laibson's course next year.

Despite these reservation about Nash equilibrium it is still a very useful benchmark and a starting point for any game analysis.

In the following we will go through three proofs of the Existence Theorem using various levels of mathematical sophistication:

- existence in 2×2 games using elementary techniques
- existence in 2×2 games using a fixed point approach
- general existence theorem in finite games

You are only required to understand the simplest approach. The rest is for the intellectually curious.

2 Nash Existence in 2×2 Games

Let us consider the simple 2×2 game which we discussed in the previous lecture on mixed Nash equilibria:

	L	R
U	1,1	0,4
D	0,2	2,1

We next draw the best-response curves of both players. Recall that player 1's strategy can be represented by a single number α such that $\sigma_1 = \alpha U + (1 - \alpha) D$ while player 2's strategy is $\sigma_2 = \beta L + (1 - \beta) R$.

Let's find the best-response of player 2 to player 1 playing strategy α:

$$\begin{aligned} u_2(L, \alpha U + (1-\alpha)D) &= 2 - \alpha \\ u_2(R, \alpha U + (1-\alpha)D) &= 1 + 3\alpha \end{aligned} \tag{1}$$

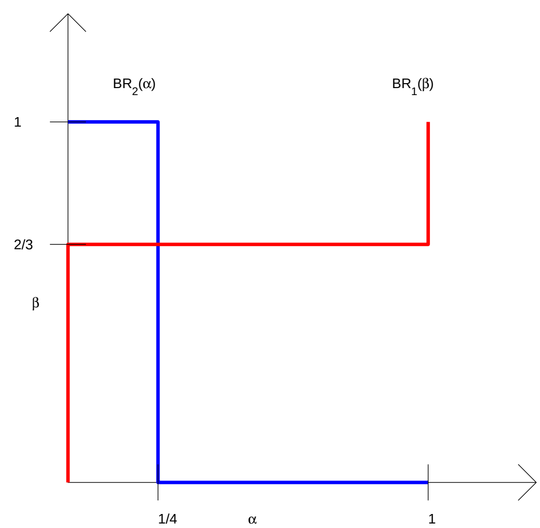
Therefore, player 2 will strictly prefer strategy L iff $2 - \alpha > 1 + 3\alpha$ which implies $\alpha < \frac{1}{4}$. The best-response correspondence of player 2 is therefore:

$$BR_2(\alpha) = \begin{cases} 1 & \text{if } \alpha < \frac{1}{4} \\ [0,1] & \text{if } \alpha = \frac{1}{4} \\ 0 & \text{if } \alpha > \frac{1}{4} \end{cases} \tag{2}$$

We can similarly find the best-response correspondence of player 1:

$$BR_1(\beta) = \begin{cases} 0 & \text{if } \beta < \frac{2}{3} \\ [0,1] & \text{if } \beta = \frac{2}{3} \\ 1 & \text{if } \beta > \frac{2}{3} \end{cases} \tag{3}$$

We draw both best-response correspondences in a single graph (the graph is in color - so looking at it on the computer screen might help you):



We immediately see, that both correspondences intersect in the single point $\alpha = \frac{1}{4}$ and $\beta = \frac{2}{3}$ which is therefore the unique (mixed) Nash equilibrium of the game.

What's useful about this approach is that it generalizes to a proof that any two by two game has at least one Nash equilibriu, i.e. its two best response correspondences have to intersect in at least one point.

An informal argument runs as follows:

1. The best response correspondence for player 2 maps each α into at least one β. The graph of the correspondence connects the left and right side of the square $[0, 1] \times [0, 1]$. This connection is continuous - the only discontinuity could happen when player 2's best response switches from L to R or vice versa at some α^*. But at this switching point player 2 has to be exactly indifferent between both strategies - hence the graph has the value $BR_2(\alpha^*) = [0, 1]$ at this point and there cannot be a discontinuity. Note, that this is precisely why we need mixed strategies - with pure strategies the BR graph would generally be discontinuous at some point.

2. By an analogous argument the BR graph of player 1 connects the upper and lower side of the square $[0, 1] \times [0, 1]$.

3. Two lines which connect the left/right side and the upper/lower side of the square respectively have to intersect in at least one point. Hence each 2 by 2 game has a mixed Nash equilibrium.

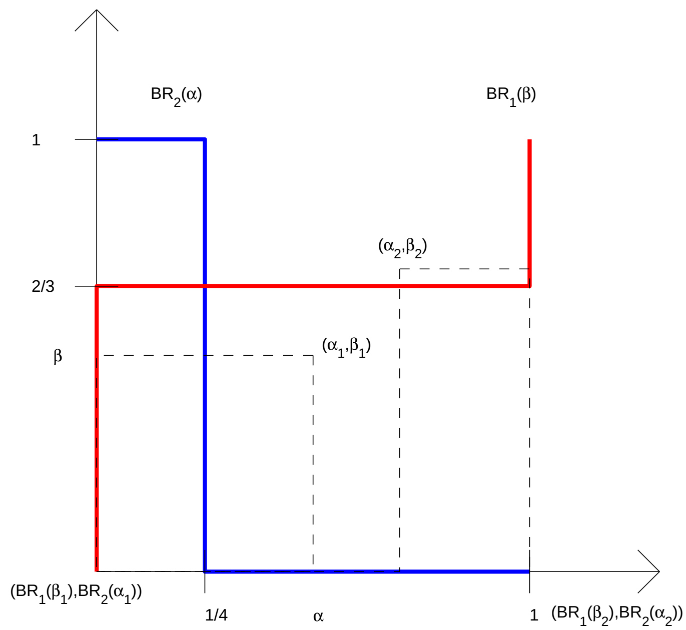
3 Nash Existence in 2×2 Games using Fixed Point Argument

There is a different way to prove existence of NE on 2×2 games. The advantage of this new approach is that it generalizes easily to general finite games.

Consider any strategy profile $(\alpha U + (1 - \alpha)D, \beta L + (1 - \beta)R)$ represented by the point (α, β) inside the square $[0, 1] \times [0, 1]$. Now imagine the following: player 1 assumes that player 2 follows strategy β and player 2 assumes that player 1 follows strategy α. What should they do? They should play their BR to their beliefs - i.e. player 1 should play $BR_1(\beta)$ and player 2 should play $BR_2(\alpha)$. So we can imagine that the strategy profile (α, β) is mapped onto $(BR_1(\beta), BR_2(\alpha))$. This would describe the actual play of both players if their beliefs would be summarizes by (α, β). We can therefore define a giant correspondence $BR : [0,1] \times [0,1] \rightarrow [0,1] \times [0,1]$ in the following way:

$$BR(\alpha, \beta) = BR_1(\beta) \times BR_2(\alpha) \tag{4}$$

The following figure illustrates the properties of the combined best-response map BR:



The neat fact about BR is that the Nash equilibria σ^* are precisely the fixed points of BR, i.e. $\sigma^* \in BR(\sigma^*)$. In other words, if players have beliefs σ^* then σ^* should also be a best response by them. The next lemma follows directly from the definition of mixed Nash equilibrium:

Lemma 1 *A mixed strategy profile σ^* is a Nash equilibrium if and only if it is a fixed point of the BR correspondence, i.e. $\sigma^* \in BR(\sigma^*)$.*

We therefore look precisely for the fixed points of the correspondence BR which maps the square $[0,1] \times [0,1]$ onto itself. There is well developed mathematical theory for these types of maps which we utilize to prove Nash existence (i.e. that BR has at least one fixed point).

3.1 Kakutani's Fixed Point Theorem

The key result we need is Kakutani's fixed point theorem. You might have used Brower's fixed point theorem in some mathematics class. This is not sufficient for proving the existence of nash equilibria because it only applies to functions but not to correspondences.

Theorem 2 ***Kakutani*** *A correspondence $r : X \rightarrow X$ has a fixed point $x \in X$ such that $x \in r(x)$ if*

1. *X is a compact, convex and non-empty subset of $\Re^n$.*

2. *$r(x)$ is non-empty for all x.*

3. *$r(x)$ is convex for all x.*

4. *r has a closed graph.*

There are a few concepts in this definition which have to be defined:

Convex Set: A set $A \subseteq \Re^n$ is convex if for any two points $x, y \in A$ the straight line connecting these two points lies inside the set as well. Formally, $\lambda x + (1 - \lambda) y \in A$ for all $\lambda \in [0, 1]$.

Closed Set: A set $A \subseteq \Re^n$ is closed if for any converging sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \rightarrow x^*$ as $n \rightarrow \infty$ we have $x^* \in A$. Closed intervals such as $[0, 1]$ are closed sets but open or half-open intervals are not. For example $(0, 1]$ cannot be closed because the sequence $\frac{1}{n}$ converges to 0 which is not in the set.

Compact Set: A set $A \subseteq \Re^n$ is compact if it is both closed and bounded. For example, the set $[0, 1]$ is compact but the set $[0, \infty)$ is only closed but unbounded, and hence not compact.

Graph: The graph of a correspondence $r : X \rightarrow Y$ is the set $\{(x, y) \,| y \in r(x)\}$. If r is a real function the graph is simply the plot of the function.

Closed Graph: A correspondence has a closed graph if the graph of the correspondence is a closed set. Formally, this implies that for a sequence of point on the graph $\{(x_n, y_n)\}_{n=1}^{\infty}$ such that $x_n \rightarrow x^*$ and $y_n \rightarrow y^*$ as $n \rightarrow \infty$ we have $y^* \in r(x^*)$.[2]

It is useful to understand exactly why we need each of the conditions in Kakutani's fixed point theorem to be fulfilled. We discuss the conditions by looking correspondences on the real line, i.e. $r : \Re \rightarrow \Re$. In this case, a fixed point simply lies on the intersection between the graph of the correspondence and the diagonal $y = x$. Hence Kakutani's fixed point theorem tells us that

[2] If the correspondence is a function then the closed graph requirement is equivalent to assuming that the function is continuous. It's easy to see that a continuous function has a closed graph. For the reverse, you'll need Baire's category theorem.

a correspondence $r : [0,1] \to [0,1]$ which fulfills the conditions above always intersects with the diagonal.

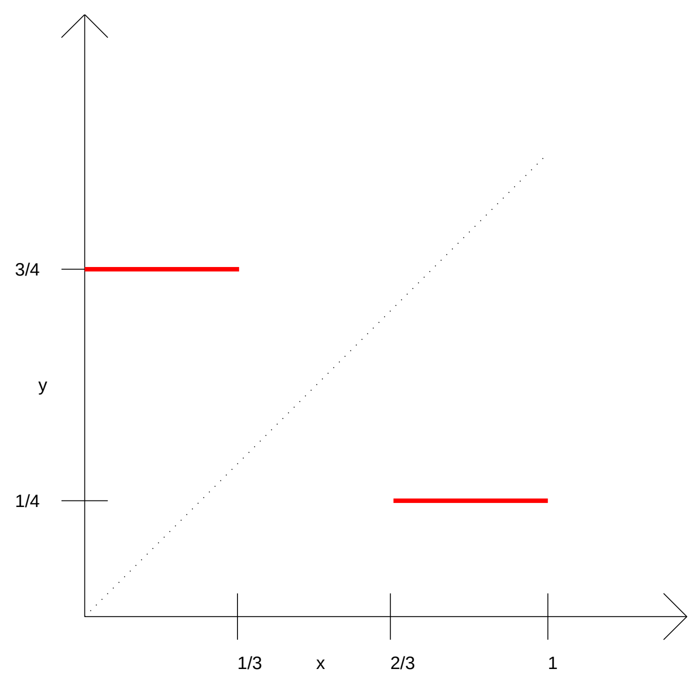
3.1.1 Kakutani Condition I: X is compact, convex and non-empty.

Assume X is not compact because it is not closed - for example $X = (0,1)$. Now consider the correspondence $r(x) = x^2$ which maps X into X. However, it has no fixed point. Now consider X non-compact because it is unbounded such as $X = [0, \infty)$ and consider the correspondence $r(x) = 1 + x$ which maps X into X but has again no fixed point.

If X is empty there is clearly no fixed point. For convexity of X look at the example $X = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ which is not convex because the set has a hole. Now consider the following correspondence (see figure below):

$$r(x) = \begin{cases} \frac{3}{4} & \text{if } x \in [0, \frac{1}{3}] \\ \frac{1}{4} & \text{if } x \in [\frac{2}{3}, 1] \end{cases} \tag{5}$$

This correspondence maps X into X but has no fixed point again.



From now on we focus on correspondences $r : [0,1] \to [0,1]$ - note that $[0,1]$ is closed and bounded and hence compact, and is also convex.

3.1.2 Kakutani Condition II: $r(x)$ is non-empty.

If $r(x)$ could be empty we could define a correspondence $r : [0,1] \longrightarrow [0,1]$ such as the following:

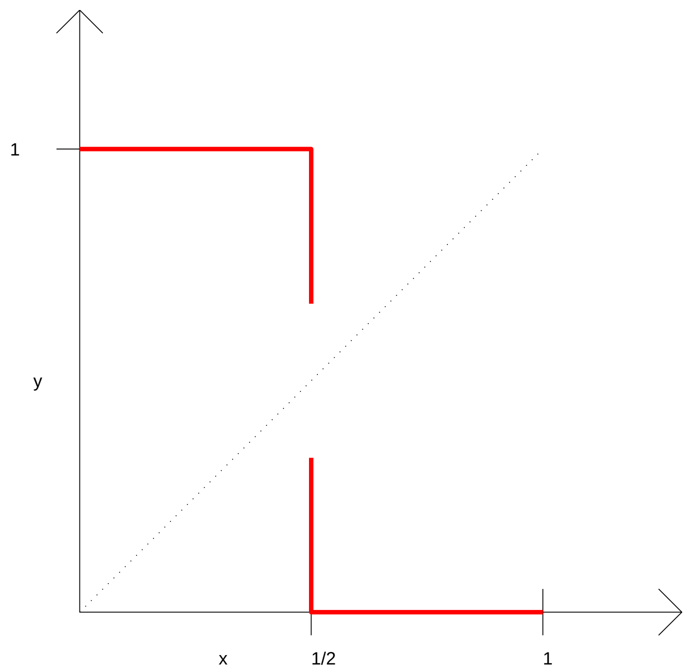
$$r(x) = \begin{cases} \frac{3}{4} & \text{if } x \in [0, \frac{1}{3}] \\ \emptyset & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{4} & \text{if } x \in [\frac{2}{3}, 1] \end{cases} \tag{6}$$

As before, this correspondence has no fixed point because of the hole in the middle.

3.1.3 Kakutani Condition III: $r(x)$ is convex.

If $r(x)$ is not convex, then the graph does not have to have a fixed point as the following example of a correspondence $r : [0,1] \longrightarrow [0,1]$ shows:

$$r(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2} \\ \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2} \end{cases} \tag{7}$$

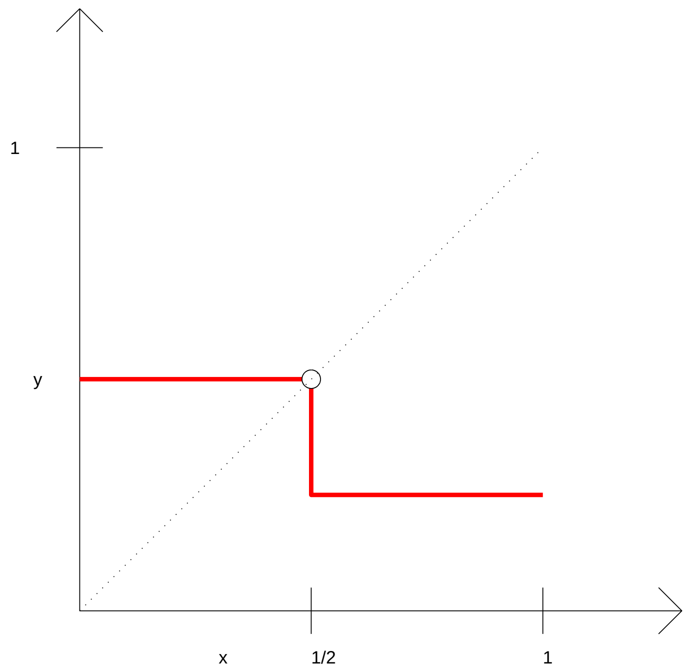


The graph is non-convex because $r(\frac{1}{2})$ is not convex. It also does not have a fixed point.

3.1.4 Kakutani Condition IV: $r(x)$ has a closed graph.

This condition ensures that the graph cannot have holes. Consider the following correspondence $r : [0,1] \to [0,1]$ which fulfills all conditions of Kakutani except (4):

$$r(x) = \begin{cases} \frac{1}{2} & \text{if } x < \frac{1}{2} \\ \left[\frac{1}{4}, \frac{1}{2}\right) & \text{if } x = \frac{1}{2} \\ \frac{1}{4} & \text{if } x > \frac{1}{2} \end{cases} \tag{8}$$



Note, that $r(\frac{1}{2})$ is the convex set $\left[\frac{1}{4}, \frac{1}{2}\right)$ but that this set is not closed. Hence the graph is not closed. For example, consider the sequence $x_n = \frac{1}{2}$ and $y_n = \frac{1}{2} - \frac{1}{n+2}$ for $n \geq 1$. Clearly, we have $y_n \in r(x_n)$. However, $x_n \to x^* = \frac{1}{2}$ and $y_n \to y^* = \frac{1}{2}$ but $y^* \notin r(x^*)$. Hence the graph is not closed.

3.2 Applying Kakutani

We now apply Kakutani to prove that 2×2 games have a Nash equilibrium, i.e. the giant best-response correspondence BR has a fixed point. We denote the strategies of player 1 with U and D and the strategies of player 2 with L and R.

We have to check (a) that BR is a map from some compact and convex set X into itself, and (b) conditions (1) to (4) of Kakutani.

- First note, that $BR : [0,1] \times [0,1] \to [0,1] \times [0,1]$. The square $X = [0,1] \times [0,1]$ is convex and compact because it is bounded and closed.

- Now check condition (2) of Kakutani - $BR(\sigma)$ is non-empty. This is true if $BR_1(\sigma_2)$ and $BR_2(\sigma_1)$ are non-empty. Let's prove it for BR_1 - the proof for BR_2 is analogous. Player 1 will get the following payoff $u_{1,\beta}(\alpha)$ from playing strategy α if the other player plays β:

$$\begin{aligned} u_{1,\beta}(\alpha) &= \alpha\beta u_1(U,L) + \alpha(1-\beta)u_1(U,R) + \\ &+ (1-\alpha)\beta u_1(D,L) + (1-\alpha)(1-\beta)u_1(D,R) \end{aligned} \tag{9}$$

  The function $u_{1,\beta}$ is continuous in α. We also know that $\alpha \in [0,1]$ which is a closed interval. Therefore, we know that the continuous function $u_{1,\beta}$ reaches its maximum over that interval (standard min-max result from real analysis - continuous functions reach their minimum and maximum over closed intervals). Hence there is at least one best response α^* which maximizes player 1's payoff.

- Condition (3) requires that if player 1 has tow best responses $\alpha_1^* U + (1-\alpha_1^*)D$ and $\alpha_2^* U + (1-\alpha_2^*)D$ to player 2 playing $\beta L + (1-\beta)R$ then the strategy where player 1 chooses U with probability $\lambda\alpha_1^* + (1-\lambda)\alpha_2^*$ for some $0 < \lambda < 1$ is also a best response (i.e. $BR_1(\beta)$ is convex). But since both the α_1 and the α_2 strategy are best responses of player 1 to the same β strategy of player 2 they also have to provide the same payoffs to player 1. But this implies that if player 1 plays strategy α_1 with probability λ and α_2 with probability $1-\lambda$ she will get exactly the same payoff as well. Hence the strategy where she plays U with probability $\lambda\alpha_1^* + (1-\lambda)\alpha_2^*$ is also a best response and her best response set $BR_1(\beta)$ is convex.

- The final condition (4) requires that BR has a closed graph. To show this consider a sequence $\sigma^n = (\alpha^n, \beta^n)$ of (mixed) strategy profiles and $\tilde{\sigma}^n = (\tilde{\alpha}^n, \tilde{\beta}^n) \in BR(\sigma^n)$. Both sequences are assumed to converge to $\sigma^* = (\alpha^*, \beta^*)$ and $\tilde{\sigma}^* = (\tilde{\alpha}^*, \tilde{\beta}^*)$, respectively. We now want to show that $\tilde{\sigma} \in BR(\sigma)$ to prove that BR has a closed graph.

  We know that for player 1, for example, we have

$$u_1(\tilde{\alpha}^n, \beta^n) \geq u_1(\alpha', \beta^n)$$

for any $\alpha' \in [0,1]$. Note, that the utility function is continuous in both arguments because it is linear in α and β. Therefore, we can take the limit on both sides while preserving the inequality sign:

$$u_1(\tilde{\alpha}^*, \beta^*) \geq u_2(\alpha', \beta)$$

for all $\alpha' \in [0,1]$. This shows that $\tilde{\alpha^*} \in BR_1(\beta)$ and therefore $\tilde{\sigma}^* \in BR(\sigma^*)$. Hence the graph of the BR correspondence is closed.

Therefore, all four Kakutani conditions apply and the giant best-response correspondence BR has a fixed point, and each 2×2 game has a Nash equilibrium.

4 Nash Existence Proof for General Finite Case

Using the fixed point method it is now relatively easy to extend the proof for the 2×2 case to general finite games.The biggest difference is that we cannot represent a mixed strategy any longer with a single number such as α. If player 1 has three pure strategies A_1,A_2 and A_3, for example, then his set of mixed strategies is represented by two probabilities - for example, (α_1, α_2) which are the probabilities that A_1 and A_2 are chosen. The set of admissible α_1 and α_2 is described by:

$$\Sigma_1 = \{(\alpha_1, \alpha_2) | 0 \leq \alpha_1, \alpha_2 \leq 1 \quad \text{and} \quad \alpha_1 + \alpha_2 \leq 1\} \tag{10}$$

The definition of the set of mixed strategies can be straightforwardly extended to games where player 1 has a strategy set consisting of n pure strategies A_1,..,A_n. Then we need $n-1$ probabilities α_1,..,α_{n-1} such that:

$$\Sigma_1 = \{(\alpha_1, .., \alpha_{n-1}) | 0 \leq \alpha_1, .., \alpha_{n-1} \leq 1 \quad \text{and} \quad \alpha_1 + .. + \alpha_{n-1} \leq 1\} \tag{11}$$

So instead of representing strategies on the unit interval $[0,1]$ we have to represent as elements of the simplex Σ_1.

Lemma 2 *The set Σ_1 is compact and convex.*

Proof: It is clearly convex - if you mix between two mixed strategies you get another mixed strategy. The set is also compact because it is bounded (all $|\alpha_i| \leq 1$) and closed. To see closedness take a sequence $(\alpha_1^i, .., \alpha_{n-1}^i)$ of elements of Σ_1 which converges to $(\alpha_1^*, ..)$. Then we have $\alpha_i^* \geq 0$ and $\sum_{i=1}^{n-1} \alpha_i^* \leq 1$ because the limit preserves weak inequalities. QED

We can now check that all conditions of Kakutani are fulfilled in the general finite case. Checking them is almost 1-1 identical to checking Kakutani's condition for 2×2 games.

Condition 1: The individual mixed strategy sets Σ_i are clearly non-empty because every player has at least one strategy. Since Σ_i is compact $\Sigma = \Sigma_1 \times ... \times \Sigma_I$ is also compact. Hence the BR correspondence $BR : \Sigma \to \Sigma$ acts on a compact and convex non-empty set.

Condition 2: For each player i we can calculate his utiltiy $u_{i,\sigma_{-i}}(\sigma_i)$ for $\sigma_i \in \Sigma_i$. Since Σ_i is compact and $u_{i,\sigma_{-i}}$ is continuous the set of payoffs is also compact and hence has a maximum. Therefore, $BR_i(\Sigma_i)$ is non-empty.

Condition 3: Assume that σ_i^1 and σ_i^2 are both BR of player i to σ_{-i}. Both strategies have to give player i equal payoffs then and any linear combination of these two strategies has to be a BR for player i, too.

Condition 4: Assume that σ^n is a sequence of strategy profiles and $\tilde{\sigma}^n \in BR(\sigma^n)$. Both sequences converge to σ^* and $\tilde{\sigma}^*$, respectively. We know that for each player i we have

$$u_i\left(\tilde{\sigma}_i^n, \sigma_{-i}^n\right) \geq u_i\left(\sigma_i', \sigma_{-i}^n\right)$$

for all $\sigma_i' \in \Sigma_i$. Note, that the utility function is continuous in both arguments because it is linear.[3] Therefore, we can take the limit on both sides while preserving the inequality sign:

$$u_i\left(\tilde{\sigma}_i^*, \sigma_{-i}^*\right) \geq u_i\left(\sigma_i', \sigma_{-i}^*\right)$$

for all $\sigma_i' \in \Sigma_i$. This shows that $\tilde{\sigma}_i^* \in BR_i\left(\sigma_{-i}^*\right)$ and therefore $\tilde{\sigma}^* \in BR(\sigma^*)$. Hence the graph of the BR correspondence is closed.

So Kakutani's theorem applies and the giant best-response map BR has a fixed point.

[3] It is crucial here that the set of pure strategies is finite.