# Lecture XVII: Dynamic Games with Incomplete Information

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**Readings for this class: Osborne and Rubinstein, Fudenberg and Tirol, chapter 8.2**

## 1 Introduction

In the last lecture I introduced the idea of incomplete information. We analyzed some important simultaneous move games such as sealed bid auctions and public goods.

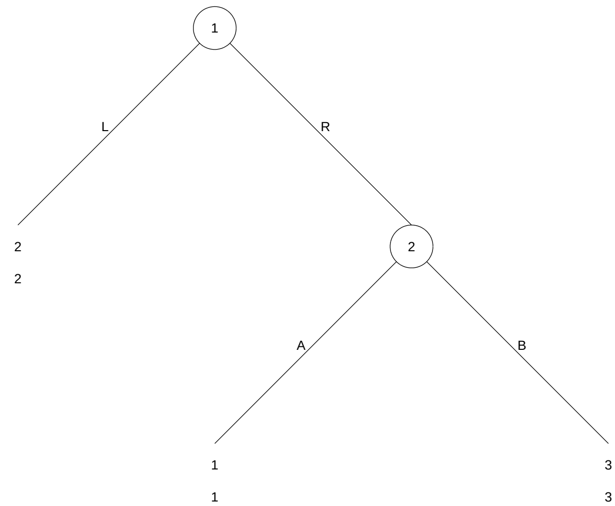
In practice, almost all of the interesting models with incomplete information are dynamic games also. Before we talk about these games we'll need a new solution concept called Perfect Bayesian Equilibrium.

Intuitively, PBE is to extensive form games with incomplete games what SPE is to extensive form games with complete information. The concept we did last time, BNE is a simply the familiar Nash equilibrium under the Harsanyi representation of incomplete information. In principle, we could use the Harsanyi representation and SPE in dynamic games of incomplete information. However, dynamic games with incomplete information typically don't have enough subgames to do SPE. Therefore, many 'non-credible' threats are possible again and we get too many unreasonable SPE's. PBE allows subgame reasoning at information sets which are not single nodes whereas SPE only applies at single node information sets of players (because only those can be part of a proper subgame).

The following example illustrates some problems with SPE.

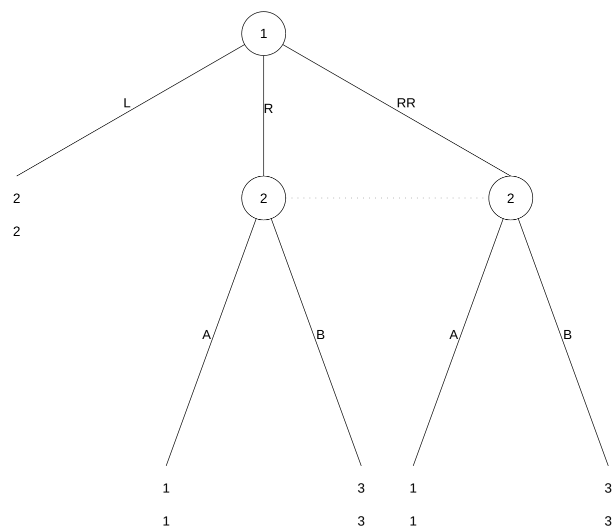
## 1.1 Example I - SPE

Our first example has no incomplete information at all.




Its unique SPE is (R,B).

The next game looks formally the same - however, SPE is the same as NE because the game has no proper subgames.




The old SPE survives - all $(pR + (1-p)\,RR, B)$ for all $p$ is SPE. But there are suddenly strange SPE such as $(L, qA + (1-q)\,B)$ for $q \geq \frac{1}{2}$. Player 2's strategy looks like an non-credible threat again - but out notion of SPE can't rule it out!

**Remember:** *SPE can fail to rule out actions which are not optimal given any 'beliefs' about uncertainty.*

**Remark 1** *This problem becomes severe with incomplete information: moves of Nature are not observed by one or both players. Hence the resulting extensive form game will have no or few subgames. This and the above example illustrate the need to replace the concept of a 'subgame' with the concept of a 'continuation game'.*

## 1.2 Example II: Spence's Job-Market Signalling

The most famous example of dynamic game with incomplete information is Spence's signalling game. There are two players - a firm and a worker. The worker has some private information about his ability and has the option of acquiring some education. Education is always costly, but less so for more able workers. **However, education does not improve the worker's productivity at all!** In Spence's model education merely serves as a signal to firms. His model allows equilibria where able workers will acquire education and less able workers won't. Hence firms will pay high wages only to those who acquired education - however, they do this because education has revealed the type of the player rather than improved his productivity.

Clearly this is an extreme assumption - in reality education has presumably dual roles: there is some signalling and some productivity enhancement. But it is an intriguing insight that education might be nothing more than a costly signal which allows more able workers to differentiate themselves from less able ones.

Let's look at the formal set-up of the game:

- **Stage 0:** Nature chooses the ability $\theta$ of a worker. Suppose $\Theta = \{2, 3\}$ and that $Prob\,(\theta = 2) = p$ and $Prob\,(\theta = 3) = 1 - p$.

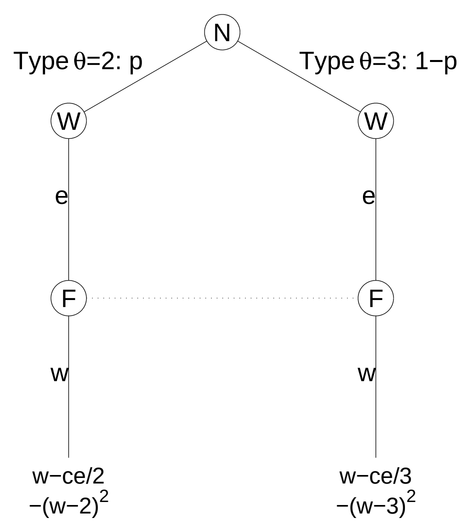
- **Stage I:** Player 1 (worker) observes his type and chooses eduction level $e \in \{0, 1\}$. Education has cost $\frac{ce}{\theta}$. Note, that higher ability workers have lower cost and that getting no education is costless.

- **Stage II:** Player 2 (the competitive labor market) chooses the wage rate $w\,(e)$ of workers after observing the education level.

Suppose that $u_1(e, w, ; \theta) = w - \frac{ce}{\theta}$ and that $u_2(e, w; \theta) = -(w - \theta)^2$. Note, that education does not enter the firm's utility function. Also note, that the BR of the firm is to set wages equal to the expected ability of the worker under this utility function. This is exactly what the competitive labor market would do if $\theta$ is equal to the productivity of a worker (the dollar amount of output he produces). If a firm pays above the expected productivity it will run a loss, and if it pays below some other firm would come in and offer more to the worker. So the market should offer exactly the expected productivity. The particular (rather strange-looking) utility function we have chosen implements the market outcome with a single firm - it's a simple shortcut.

Spence's game is a *signalling game*. Each signalling game has the same three-part structure: nature chooses types, the sender (worker) observes his type and takes and action, the receiver (firm) sees that action but not the worker's type. Hence the firm tries to deduce the worker's type using his action. His action therefore serves as a signal. Spence's game is extreme because the signal (education) has no value to the firm except for its signalling function. This is not the case for all signalling models: think of a car manufacturer who can be of low or high quality and wants to send a signal to the consumer that he is a high-quality producer. He can offer a short or long warranty for the car. The extended warranty will not only signal his type but also benefit the consumer.

The (Harsanyi) extensive form representation of Spence's game (and any other signalling game) is given below.



### 1.3 Why does SPE concept together with Harsanyi representation not work?

We could find the set of SPE in the Harsanyi representation of the game. The problem is that the game has no proper subgame in the second round when the firm makes its decision. Therefore, the firm can make unreasonable threats such as the following: both workers buy education and the firms pays the educated worker $w = 3 - p$ (his expected productivity), and the uneducated worker gets $w = -235.11$ (or something else). Clearly, every worker would get education, and the firm plays a BR to a worker getting education (check for yourself using the Harsanyi representation).

However, the threat of paying a negative wage is unreasonable. Once the firm sees a worker who has no education it should realize that the worker has a least ability level 2 and should therefore at least get a wage of $w = 2$.

## 2 Perfect Bayesian Equilibrium

Let G be a multistage game with incomplete information and observed actions in the Harsanyi representation. Write $\Theta_i$ for the set of possible types for player $i$ and $H_i$ to be the set of possible information sets of player $i$. For each information set $h_i \in H_i$ denote the set of nodes in the information set with $X_i(h_i)$ and $X \bigcup_{H_i} X_i(h_i)$.

A strategy in $G$ is a function $s_i : H_i \rightarrow \Delta(A_i)$. Beliefs are a function $\mu_i : H_i \rightarrow \Delta(X_i)$ such that the support of belief $\mu_i(h_i)$ is within $X_i(h_i)$.

**Definition 1** *A PBE is a strategy profile $s^*$* ***together with*** *a belief system $\mu$ such that*

1. *At every information set strategies are optimal given beliefs and opponents' strategies (**sequential rationality**).*

$$\sigma_i^*(h) \quad \textit{maximizes} \quad E_{\mu_i(x|h_i)} u_i\left(\sigma_i, \sigma_{-i}^* | h, \theta_i, \theta_{-i}\right)$$

2. *Beliefs are always updated according to Bayes rule* ***when applicable****.*

The first requirement replaces subgame perfection. The second requirement makes sure that beliefs are derived in a rational manner - assuming that you know the other players' strategies you try to derive as many beliefs as possible. Branches of the game tree which are reached with zero probability cannot be derived using Bayes rule: here you can choose arbitrary beliefs. However, the precise specification will typically matter for deciding whether an equilibrium is PBE or not.

**Remark 2** *In the case of complete information and observed actions PBE reduces to SPE because beliefs are trivial: each information set is a singleton and the belief you attach to being there (given that you are in the corresponding information set) is simply 1.*

## 2.1 What's Bayes Rule?

There is a close connection between agent's actions and their beliefs. Think of job signalling game. We have to specify the beliefs of the firm in the second stage when it does not know for sure the current node, but only the information set.

Let's go through various strategies of the worker:

- *The high ability worker gets education and the low ability worker does not:* $e(\theta = 2) = 0$ and $e(\theta = 3) = 1$. In this case my beliefs at the information set $e = 1$ should be $Prob(\text{High}|e = 1) = 1$ and similarly, $Prob(\text{High}|e = 0) = 0$.

- *Both workers get education.* In this case, we should have:

$$Prob(\text{High}|e = 1) = 1 - p \tag{1}$$

  The beliefs after observing $e = 0$ cannot be determined by Bayes rule because it's a probability zero event - we should never see it if players follow their actions. This means that we can choose beliefs freely at this information set.

- *The high ability worker gets education and the low ability worker gets education with probability $q$.* This case is less trivial. What's the probability of seeing worker get education - it's $1 - p + pq$. What's the probability of a worker being high ability and getting education? It's $1 - p$. Hence the probability that the worker is high ability after we have observed him getting education is $\frac{1-p}{1-p+pq}$. This is the non-trivial part of Bayes rule.

Formally, we can derive the beliefs at some information set $h_i$ of player $i$ as follows. There is a probability $p(\theta_j)$ that the other player is of type $\theta_j$. These probabilities are determined by nature. Player $j$ (i.e. the worker) has taken some action $a_j$ such that the information set $h_i$ was reached. Each type of player $j$ takes action $a_j$ with some probability $\sigma_j^*(a_j|\theta_j)$ according to his equilibrium strategy. Applying Bayes rule we can then derive the belief of player $i$ that player $j$ has type $\theta_j$ at information set $h_i$:

$$\mu_i(\theta_j|a_j) = \frac{p(\theta_j)\,\sigma_j^*(a_j|\theta_j)}{\sum_{\tilde{\theta}_j\in\Theta_j} p\left(\tilde{\theta}_j\right)\sigma_j^*\left(a_j|\tilde{\theta}_j\right)} \tag{2}$$

1. In the job signalling game with separating beliefs Bayes rule gives us exactly what we expect - we belief that a worker who gets education is high type.

2. In the pooling case Bayes rule gives us $Prob\,(\text{High}|e=1) = \frac{1-p}{p\times 1+(1-p)\times 1} = 1-p$. **Note, that Bayes rule does NOT apply for finding the beliefs after observing $e=0$ because the denominator is zero.**

3. In the semi-pooling case we get $Prob\,(\text{High}|e=1) = \frac{(1-p)\times 1}{p\times q+(1-p)\times 1}$. Similarly, $Prob\,(\text{High}|e=0) = \frac{(1-p)\times 0}{p\times(1-q)+(1-p)\times 0} = 0$.

# 3 Signalling Games and PBE

It turns out that signalling games are a very important class of dynamic games with incomplete information in applications. Because the PBE concept is much easier to state for the signalling game environment we define it once again in this section for signalling games. You should convince yourself that the more general definition from the previous section reduces to the definition below in the case of signalling games.

## 3.1 Definitions and Examples

Every signalling game has a sender, a receiver and two periods. The sender has private information about his type and can take an action in the first action. The receiver observes the action (signal) but not the type of the sender, and takes his action in return.

- **Stage 0:** Nature chooses the type $\theta_1 \in \Theta_1$of player 1 from probability distribution $p$.
- **Stage 1:** Player 1 observes $\theta_1$ and chooses $a_1 \in A_1$.
- **Stage 2:** Player 2 observes $a_1$ and chooses $a_2 \in A_2$.

The players utilities are:

$$\begin{aligned} u_1 &= u_1 (a_1, a_2; \theta_1) \quad (3) \\ u_2 &= u_2 (a_1, a_2; \theta_1) \quad (4) \end{aligned}$$

#### 3.1.1 Example 1: Spence's Job Signalling Game

- worker is sender; firm is receiver
- $\theta$ is the ability of the worker (private information to him)
- $A_1 = \{\text{educ}, \text{no educ}\}$
- $A_2 = \{\text{wage rate}\}$

#### 3.1.2 Example 2: Initial Public Offering

- player 1 - owner of private firm
- player 2 - potential investor
- $\Theta$ - future profitability
- $A_1$ - fraction of company retained
- $A_2$ - price paid by investor for stock

#### 3.1.3 Example 3: Monetary Policy

- player 1 = FED
- player 2 - firms
- $\Theta$ - Fed's preference for inflation/ unemployment
- $A_1$ - first period inflation
- $A_2$ - expectation of second period inflation

### 3.1.4 Example 4: Pretrial Negotiation

- player 1 - defendant
- player 2 - plaintiff
- $\Theta$ - extent of defendant's negligence
- $A_1$ - settlement offer
- $A_2$ - accept/reject

## 3.2 PBE in Signalling Games

A PBE in the signalling game is a strategy profile $(s_1^*(\theta_1), s_2^*(a_1))$ together with beliefs $\mu_2(\theta_1|a_1)$ for player 2 such that

1. Players strategies are optimal given their beliefs and the opponents' strategies, i.e.

$$s_1^*(\theta_1) \quad \text{maximizes} \quad u_1(a_1, s_2^*(a_1); \theta_1) \text{ for all } \theta_1 \in \Theta_1 \tag{5}$$

$$s_2^*(a_1) \quad \text{maximizes} \quad \sum_{\theta_1 \in \Theta_1} u_2(a_1, a_2; \theta_1)\, \mu_2(\theta_1|a_1) \quad \text{for all } a_1 \in A_1 \tag{6}$$

2. Player 2's beliefs are compatible with Bayes' rule. If any type of player 1 plays $a_1$ with positive probability then

$$\mu_2(\theta_1|a_1) = \frac{p(\theta_1)\, Prob(s_1^*(\theta_1) = a_1)}{\sum_{\theta_1' \in \Theta_1} p(\theta_1')\, Prob(s_1^*(\theta_1') = a_1)} \quad \text{for all } \theta_1 \in \Theta_1$$

## 3.3 Types of PBE in Signalling Games

To help solve for PBE's it helps to think of all PBE's as taking one of the following three forms"

1. **Separating** - different types take different actions and player 2 learns type from observing the action
2. **Pooling** - all types of player 1 take same action; no info revealed
3. **Semi-Separating** - one or more types mixes; partial learning (often only type of equilibrium

**Remark 3** *In the second stage of the education game the "market" must have an expectation that player 1 is type $\theta = 2$ and attach probability $\mu(2|a_1)$ to the player being type 2. The wage in the second period **must be** between 2 and 3. This rules out the unreasonable threat of the NE I gave you in the education game (with negative wages).*[1]

**Remark 4** *In the education game suppose the equilibrium strategies are $s_1^*(\theta = 2) = 0$ and $s_1^*(\theta = 3) = 1$, i.e. only high types get education. Then for any prior $(p, 1-p)$ at the start of the game beliefs must be:*

$$\begin{aligned}
\mu_2(\theta = 2|e = 0) &= 1 \\
\mu_2(\theta = 3|e = 0) &= 0 \\
\mu_2(\theta = 2|e = 1) &= 0 \\
\mu_2(\theta = 3|e = 1) &= 1
\end{aligned}$$

*If player 1's strategy is $s_1^*(\theta = 2) = \frac{1}{2} \times 0 + \frac{1}{2} \times 1$ and $s_1^*(\theta = 3) = 1$:*

$$\begin{aligned}
\mu_2(\theta = 2|e = 0) &= 1 \\
\mu_2(\theta = 3|e = 0) &= 0 \\
\mu_2(\theta = 2|e = 1) &= \frac{\frac{p}{2}}{\frac{p}{2} + 1 - p} = \frac{p}{2-p} \\
\mu_2(\theta = 3|e = 1) &= \frac{2-2p}{2-p}
\end{aligned}$$

*Also note, that Bayes rule does NOT apply after an actions which should not occur in equilibrium. Suppose $s_1^*(\theta = 2) = s_1^*(\theta = 3) = 1$ then it's OK to assume*

$$\begin{aligned}
\mu_2(\theta = 2|e = 0) &= \frac{57}{64} \\
\mu_2(\theta = 3|e = 0) &= \frac{7}{64} \\
\mu_2(\theta = 2|e = 1) &= p \\
\mu_2(\theta = 3|e = 1) &= 1 - p
\end{aligned}$$

*The first pair is arbitrary.*

[1] It also rules out unreasonable SPE in the example SPE I which I have initially. Under any beliefs player 2 should strictly prefer B.

**Remark 5** *Examples SPE II and SPE III from the introduction now make sense - if players update according to Bayes rule we get the 'reasonable' beliefs of players of being with equal probability in one of the two nodes.*

# 4 Solving the Job Signalling Game

Finally, after 11 tough pages we can solve our signalling game. The solution depends mainly on the cost parameter $c$.

## 4.1 Intermediate Costs $2 \leq c \leq 3$

A separating equilibrium of the model is when only the able worker buys education and the firm pays wage 2 to the worker without education and wage 3 to the worker with education. The firm beliefs that the worker is able iff he gets educated.

- The beliefs are consistent with the equilibrium strategy profile.
- Now look at optimality. Player 2 sets the wage to the expected wage so he is maximizing.
- Player 1 of type $\theta = 2$ gets $3 - \frac{c}{2} \leq 2$ for $a_1 = 1$ and 2 for $a_1 = 0$. Hence he should not buy education.
- Player 1 of type $\theta = 3$ gets $3 - \frac{c}{3} \geq 2$ for $a_1 = 1$ and 2 for $a_1 = 0$. Hence he should get educated.

1. Note that for too small or too big costs there is no separating PBE.
2. There is no separating PBE where the $\theta = 2$ type gets an education and the $\theta = 3$ type does not.

## 4.2 Small Costs $c \leq 1$

A pooling equilibrium of the model is that both workers buy education and that the firm pays wage $w = 3 - p$ if it observes education, and wage 2 otherwise. The firm believes that the worker is able with probability $1 - p$ if it observes education, and that the worker is of low ability if it observes no education.

- The beliefs are consistent with Bayes rule for $e = 1$. If $e = 0$ has been observed Bayes rule does not apply because $e = 0$ should never occur - hence any belief is fine. The belief that the worker is low type if he does not get education makes sure that the worker gets punished for not getting educated.

- The firm pays expected wage - hence it's optimal response. The low ability guy won't deviate as long $2.5 - \frac{c}{2} \geq 2$ and the high ability type won't deviate as long as $2.5 - \frac{c}{3} \geq 2$. For $c \leq 1$ both conditions are true.

1. While this pooling equilibrium only works for small $c$ there is always another pooling equilibrium where no worker gets education and the firms thinks that any worker who gets education is of the low type.

### 4.3 $1 < c < 2$

Assume that $p = \frac{1}{2}$ for this section. In the parameter range $1 < c < 2$ there is a semiseparating PBE of the model. The high ability worker buys education and the low ability worker buys education with positive probability $q$. The wage is $w = 2$ if the firm observes no education and set to $w = 2 + \frac{1}{1+q}$ if it observes education. The beliefs that the worker is high type is zero if he gets no education and $\frac{1}{1+q}$ if he does.

- Beliefs are consistent (check!).

- Firm plays BR.

- Player 1 of low type won't deviate as long as $2 + \frac{1}{1+q} - \frac{c}{2} \leq 2$.

- Player 1 of high type won't deviate as long as $2 + \frac{1}{1+q} - \frac{c}{3} \geq 2$.

Set $1 + q = \frac{2}{c}$. It's easy to check that the first condition is binding and the second condition is strictly true. So we are done if we choose $q = \frac{2}{c} - 1$. Note, that as $c \to 2$ we get back the separating equilibrium and as $c \to 1$ we get the pooling one.