Lecture XI: Subgame Perfect Equilibrium

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Readings for this class: OR (section 6.2), FT (section 3.2.2), Gibbons (Chapter 2), Dutta (Chapter 13).

1 Introduction

Last time we discussed extensive form representation and showed that there are typically lots of Nash equilibria. Many of them look unreasonable because they are based on out of equilibrium threats. For example, in the entry game the incumbent can deter entry by threatening to flood the market. In equilibrium this threat is never carried out. However, it seems unreasonable because the incumbent would do better accommodating the entrant if entry in fact occurs. In other words, the entrant can call the incumbent's bluff by entering anyway.

Subgame perfection is a refinement of Nash equilibrium. It rules out non-credible threats.

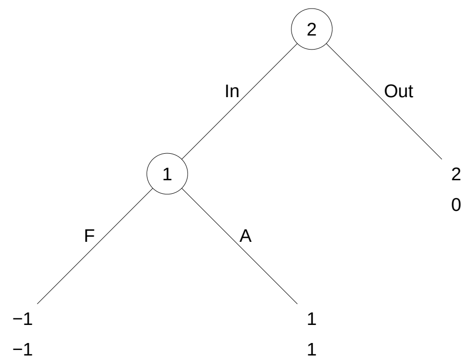
2 Subgames

Definition 1 *A subgame G' of an extensive form game G consists of*

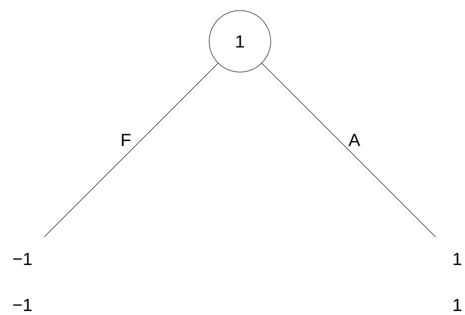
1. *A subset T' of the nodes of G consisting of a single node x and all of its successors which has the property that $t \in T'$, $t' \in h(t)$ then $t' \in T'$.*

2. *Information sets, feasible moves and payoffs at terminal nodes as in G.*

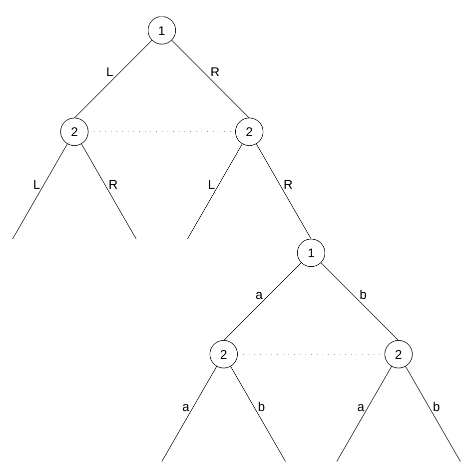
2.1 Example I: Entry Game



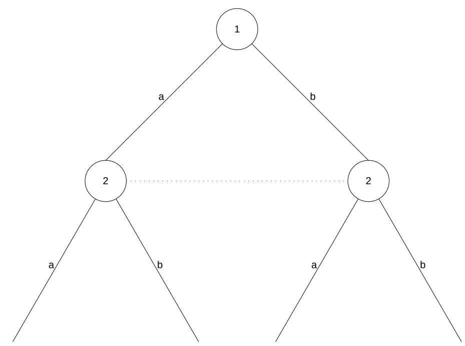
This game has two subgames. The entire game (which is *always* a subgame) and the subgame which is played after player 2 has entered the market:



2.2 Example II



This game has also two subgames. The entire game and the subgame (a simultaneous move game) played after round 2:



This subgame has no further subgames: otherwise the information set of player 2 would be separated which is not allowed under our definition.

3 Subgame Perfect Equilibrium

Definition 2 *A strategy profile s^* is a subgame perfect equilibrium of G if it is a Nash equilibrium of every subgame of G.*

Note, that a SPE is also a NE because the game itself is a (degenerate) subgame of the entire game.

Look at the entry game again. We can show that $s_1 = A$ and $s_2 = \text{Entry}$ is the unique SPE. Accomodation is the unique best response in the subgame after entry has occurred. Knowing that, firm 2's best response is to enter.

3.1 Example: Stackelberg

We next continue the Stackelberg example from the last lecture. We claim that the unique SPE is $q_2^* = \frac{1}{2}$ and $q_1^*(q_2) = \frac{1-q_2}{2}$.

The proof is as follows. A SPE must be a NE in the subgame after firm 1 has chosen q_1. This is a one player game so NE is equivalent to firm 1 maximizing its payoff, i.e. $q_1^*(q_1) \in \arg\max q_1 [1 - (q_1 + q_2)]$. This implies that $q_1^*(q_2) = \frac{1-q_2}{2}$. Equivalently, firm 1 plays on its BR curve.

A SPE must also be a NE in the whole game, so q_2^* is a BR to q_1^*:

$$u_2(q_1, q_2^*) = q_2 \left(1 - \left(q_2 + q_1^*(q_2)\right)\right) = q_1 \frac{1 - q_1}{2}$$

The FOC for maximizing u_2 is $q_2^* = \frac{1}{2}$.

Note that firm 2 (the first mover) produces more than in Cournot.

There are many games which fit the Stackelberg paradigm such as monetary policy setting by the central bank, performance pay for managers etc. We will discuss general results for this class of games in the next lecture.

4 Backward Induction

The previous example illustrates the most common technique for finding and verifying that you have found the SPE of a game. Start at the end of the game and work your way from the start.

We will focus for the moment on extensive form games where each information set is a single node (i.e. players can perfectly observe all previous moves).

Definition 3 *An extensive form is said to have perfect information if each information set contains a single node.*

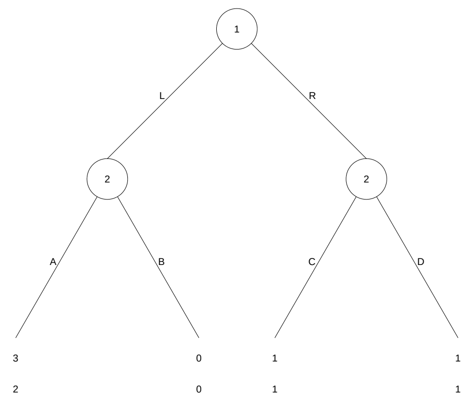
Proposition 1 *Any finite game of perfect information has a pure strategy SPE. For generic payoffs in a finite extensive form game with perfect information the SPE is unique.*

What does generic mean? With generic payoffs players are never indifferent between two strategies. If payoffs are randomly selected at the terminal nodes then indifference between two actions is a zero probability event. More mathematically, we can say that the results holds for almost all games.

Proof: I did it in class, and I do a more general proof in the next section for games with imperfect information. Intuitively, you solve the last rounds of the game, then replace these subgames with the (unique) outcome of the NE and repeat the procedure.

What happens if players are indifferent between two strategies at some point? Then there is more than one SPE, and you have to complete the backward induction for each possible outcome of the subgame.

Consider the following game:



After player 1 played R player 2 is indifferent between C and D. It is easy to see that there are infinitely many SPE such that $s_1^* = L$ and $s_2^* = (A, \alpha C + (1-\alpha) D)$ for $0 \leq \alpha \leq 1$.

Note however, that each of these SPE yields the same equilibrium outcome in which the left terminal node is reached. Hence equilibrium play is identical but off equilibrium pay differs. There are several SPE in this perfect information game because it is not generic.

5 Existence of SPE

The next theorem shows that the logic of backward induction can be extended to games with imperfect information.

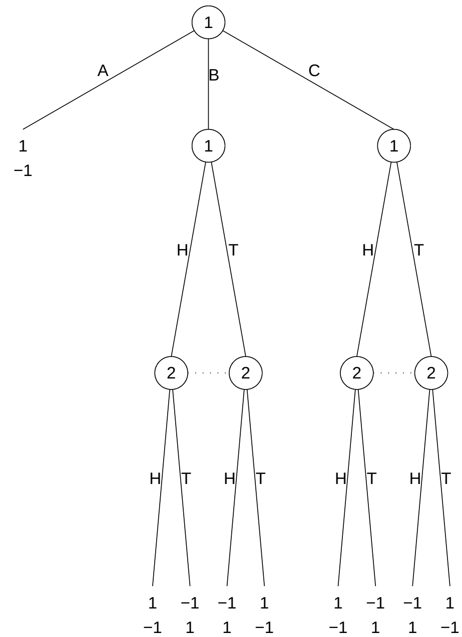
Theorem 1 *Every finite extensive form game has a SPE.*

This theorem is the equivalent of the Nash existence theorem for extensive form games. It establishes that SPE is not too strong in the sense that a SPE exists for each extensive form game. We have seen that NE is too weak in extensive form games because there are too many equilibria.

The proof of the theorem is a generalization of backward induction. In backward induction we solve the game from the back by solving node after node. Now we solve it backwards subgame for subgame.

Formally, define the set Γ of subgames of the game G. Γ is never empty because G itself is a member. We can define a partial order on the set Γ such that for two subgames G_1 and G_2 we have $G_1 \geq G_2$ if G_2 is a subgame of G_1.

Look at the following example.



This game has three subgames: the whole game G_1 and two matching pennies subgames G_2 and G_3. We have $G_1 \geq G_2$ and $G_1 \geq G_3$ but G_2 and G_3 are not comparable.

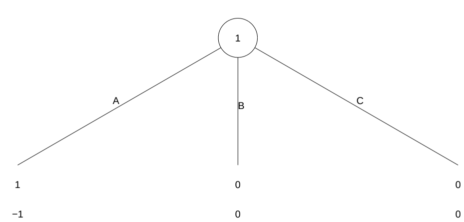
Step I *Identify the terminal subgames.* Terminal subgames are those which do not dominate another subgame (G' is terminal if there is no G'' such that $G' > G''$.

Step II *Solve the terminal subgames* These subgames have no further subgames. They have a Nash equilibrium by the Nash existence result (they are finite!).

In our example the matching pennies subgames have the unique NE $\frac{1}{2}H + \frac{1}{2}T$ for each player.

Step III *Calculate the Nash payoffs of the terminal subgames and replace these subgames with the Nash payoffs.*

In our example the matching pennies payoffs are 0 for each player. We get:



Step IV *Goto step I.* Repeat this procedure until all subgames have been exhausted. In this way we construct 'from the back' a SPE. In many cases the procedure does not produce a unique SPE if a subgame has multiple NE.

In our example we are lucky because matching pennies just has one NE. In the reduced game player 1 plays A which is his unique BR. The unique SPE is therefore $s_1^* = \left(A, \frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T\right)$ and $s_2^* = \left(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T\right)$.

6 Application of SPE to Behavioral Economics

In the first two lectures of the course we analyzed decision problems and later contrasted them to proper games where agents have to think strategically. Our decision problems were essentially static - people one action out of a number of alternatives.

In reality many decision problems involve taking decision over time: retirement decision, savings, when to finish a paper for class etc. are standard examples. Intertemporal decision making is different from static decision making because agents might want to revise past decisions in the future (they never have a chance to do so under static decision making). If agents revise decisions we say that they are *time-inconsistent*. In most economics classes you will never hear about such behavior - the decision making process of agents is assumed to be time-consistent. This reduces intertemporal decision making essentially to static decision making.

6.1 Time-Consistent Preferences

How do we model intertemporal decision making? Economists assume that the future counts less than the present - agents discount. Typically we assume that the utility in different time periods can be added up. So getting x_1 now and x_2 in the next period has total utility:

$$U = u(x_1) + \delta u(x_2) \tag{1}$$

For concreteness assume that agents want to spend 100 Dollars over two time periods. Their discount factor for next period's utility is 0.5. Their utility function is the square root function $u(x) = \sqrt{x}$. You can check that the agent would allocate 80 Dollar to today's consumption and the rest to tomorrow's consumption.

The agent is necessarily time-consistent in a two-period model because she cannot reallocate any resources in period 2. However this is no longer true in a 3-period model.

Let's first look at *exponential discounting* where consumption in period t is discounted by δ^t:

$$U_1 = u(x_1) + \delta u(x_2) + \delta^2 u(x_3) \tag{2}$$

Let's assume the same parameters as above. It's easy to check that the agent would want to allocate $\frac{2100}{16} \approx 76$ Dollars to the first period, 19 Dollars to the second and 5 to the third.

Now the agent could potentially change her allocation in period 2. Would she do so? The answer is no. Her decision problem in period 2 can be written as maximizing U_2 given that $x_2 + x_3 = 100 - 76$:

$$U_2 = u(x_2) + \delta u(x_3) \tag{3}$$

Note, that:

$$U_1 = u(x_1) + \delta U_2 \tag{4}$$

Therefore, if an agent would just a different consumption plan in period 2 she would have done so in period 1 as well.

We say that there is no conflict between different selves in games with exponential discounting. Agents are time-consistent. Time-consistent preferences are assumed in most of micro and macro economics.

6.2 Time-Inconsistent Preferences

Let's now look at a difference discounting rule for future consumption which we refer to as *hyperbolic discounting*. Agents discount at rate δ between all future time periods. However, they use an additional discount factor $\beta < 1$ to discount future versus present consumption. The idea here is, that consumers discount more strongly between period 1 and 2 than between period 2 and 3:

$$U = u(x_1) + \beta\delta u(x_2) + \beta\delta^2 u(x_3) \tag{5}$$

For simplicity we assume $\delta = 1$ from now on.

Let's assume $\beta = \frac{1}{2}$. In this case a period 1 agent would allocate 50 Dollars to today and 25 Dollars to both tomorrow and the day after tomorrow.

What would the period 2 agent do with her remaining 50 Dollars? Her decision problem would look as follows:

$$U = u(x_2) + \beta u(x_3) \tag{6}$$

So she would allocate 40 Dollars to period 2 and only 10 Dollars to the third period.

Therefore there is conflict between agent 1's and agent 2's preferences! Agent 1 would like agent 2 to save more, but agent 2 can't help herself and splurges!

6.3 Naive and Sophisticated Agents

There are two ways to deal with the self-control problem of agents. First, agents might not be aware of their future self's self-control problem - we say that they are *naive*. In this case you solve a different decision problem in each period and the consumption plans of agents get continuously revised.

If agents **are** aware of their self-control problem we call them *sophisticated*. Sophisticated agents play a game with their future self, are aware that they do so, and use SPE to solve for a consumption plan.

Let's return to our previous problem. A period 2 agent would always spend four times as much on this period than on period 3 (sophisticated or naive). Period 1 agent realizes this behavior of agent 2 and therefore takes the constraint $x_2 = 4x_3$ into account when allocating her own consumption. She maximizes:

$$U_1 = \sqrt{x_1} + \frac{1}{2}\left[\sqrt{\frac{4}{5}(1-x_1)} + \sqrt{\frac{1}{5}(1-x_1)}\right] \tag{7}$$

She would now spend 68 Dollars in the first period and predict that her future self spends 24 Dollars in the second period such that there are 6 left in the last period.

What has happened? Effectively, self 1 has taken away resources from self 2 by spending more in period 1. Self 1 predicts that self 2 would splurge - so self 1 might as well splurge immediately.

6.4 The Value of Commitment

If agents are time-inconsistent they can benefit from commitment devices which effectively constrain future selves. For example, a sophisticated hyperbolic agent could invest 50 Dollars in a 401k from which he can't withdraw more than 25 Dollars in the second period. While a time-consistent agent would never enter such a bargain (unexpected things might happen - so why should he constrain his future choices), a time-inconsistent agent might benefit.

7 Doing it Now or Later (Matt Rabin, AER, 1999)

This is a nice little paper which analyzes procrastination.

7.1 Salient Costs

Example 1 *Assume you go to the cinema on Saturdays. The schedule consists of a mediocre movie this week, a good movie next week, a great movie in two weeks and (best of all) a Johnny Depp movie in three weeks. Also assume that you must complete a report during the next four weeks so that you have to skip one of the movies. The benefit of writing the report is the same in each period (call it* $(\overline{v}, \overline{v}, \overline{v}, \overline{v})$*. The cost of not seeing a movie is* $(c_1, c_2, c_3, c_4) = (3, 5, 8, 13)$*. When do you write the report?*

Let's assume that there are three types of agents. Time consistent agents (TC) have $\delta = 1$ and $\beta = 1$. Naive agents have $\beta = \frac{1}{2}$ and sophisticated agents are aware of their self-control problem.

TC agents will write the report immediately and skip the mediocre movie. Generally, TC agents will maximize $\overline{v} - c$.

Naive agents will write the report in the last period. They believe that they will write the report in the second period. In the second period, they assume to write it in the third period (cost 4 versus 5 now). In the third period they again procrastinate.

Sophisticated agents use backward induction. They know that period 3 agent would procrastinate. Period 2 agent would predict period 3's procrastination and write the report. Period 1 agent knows that period 2 agent will write the report and can therefore safely procrastinate.

This example captures the idea that sophistication can somehow help to overcome procrastination because agents are aware of their future selfs tendencies to procrastinate.

7.2 Salient Rewards

Example 2 *Assume you can go to the cinema on Saturdays. The schedule consists of a mediocre movie this week, a good movie next week, a great movie in two weeks and (best of all) a Johnny Depp movie in three weeks. You have no money but a coupon to spend on exactly one movie. The benefit of seeing the movies are* $(v_1, v_2, v_3, v_4) = (3, 5, 8, 13)$. *Which movie do you see?*

The TC agent would wait and see the Depp movie which gives the highest benefit.

The naive agent would see the third movie. He would not the mediocre one because he would expect to see either the Depp movie later. He would also not see the week 2 movie for the same reason. But in week 3 he caves in to his impatience and spends the coupon.

The sophisticated agent would see the mediocre movie! She would expect that period 3 self caves in to her desires. Period 2 self would then go to the movies expecting period 3 self to cave in. But then period 1 self should go immediately because $3 > 2.5$.

The result is the opposite of the result we got for the procrastination example. Sophistication hurts agents! The intuition is that naive agents can pretend that they see the Depp movie and therefore are willing to wait. Sophisticated agents know about their weakness and therefore don't have the same time horizon. This makes them cave in even earlier.

A sophisticated agent would not be able to withstand a jar of cookies because she knows that she would cave in too early anyway, so she might as well cave in right away.

7.3 Choice and Procrastination: The Perfect as the Enemy of the Good

In a related paper (Choice and Procrastination, QJE 2002, forthcoming) Rabin points out that greater choice can lead to more procrastination. In the previous example, there was just procrastination on a single action. Now assume, that agents have the choice between two actions.

Example 3 *Assume you are a naive hyperbolic discounter* ($\beta = \frac{1}{2}$). *You can invest 1,000 Dollar in your 401k plan which gives you a yearly return of 5 percent. Once the money is invested it is out of reach for the next 30 years. Just when you want to sign the forms your friend tells you that he has invested* *his 1,000 Dollars at a rate of 6 percent. You make ca quick calculation and decide that you should do more research before signing on because research only causes you a disutility of 30 Dollars and the compounded interest gain over 30 years far exceed this amount. What will you do?*

You won't do anything and still have your 1000 Dollars after 30 years. Waiting a year has a cost of 50 Dollars (lost interest) which is discounted by $\beta = \frac{1}{2}$ and thus is below the salient cost of doing research. So you wait. and wait. and wait.

This is a nice example of why the perfect can be the enemy of the good: more choice can lead to procrastination. For a naive decision maker choices are determined by long-term benefits (just as for TC decision maker). However, procrastination is caused by small period to period costs of waiting which exceed salient costs of doing research.