EC14

STATISTICAL AVERAGES

To determine the average behaviour of the outcomes of random experiment

MEAN VALUE / EXPECTED VALUE OF RV X

also called first moment of X

$$m_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$E[\ ]$ = Expectation operator

PROPERTIES:

* $E[cX] = cE[X]$, c = constant
* $E[X+c] = E[X] + c$
* $E[c] = c$

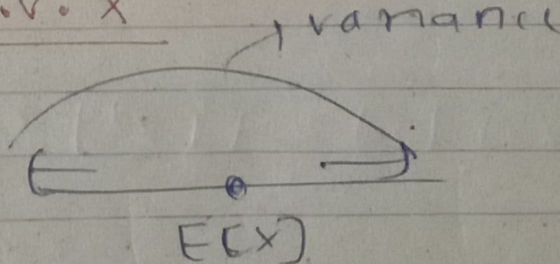
n^{th} moment of prob distribution of R.V. X

$$\boxed{E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx}$$

Mean square value of X :
(average power)

$$\boxed{E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx}$$

VARIANCE OF R.V. X



✓
spread of R.V. (X) around mean value E[X]

$$Var(X) = E[(X - m_X)^2]$$

✓ mean / E[X]

$$\boxed{var(x) = \int_{-\infty}^{\infty} (x - m_x)^2 f_x(x)\, dx}$$

$$Var[X] = E[(X - m_X)^2]$$

$$= E[x^2 + m_x^2 - 2x m_x]$$

$$= E[x^2] + m_x^2 - 2m_x E[x]$$

$$\left(E[C] = C, \; E[m_x^2] = m_x^2 \right)$$

$$= E[x^2] + m_x^2 - 2m_x^2$$

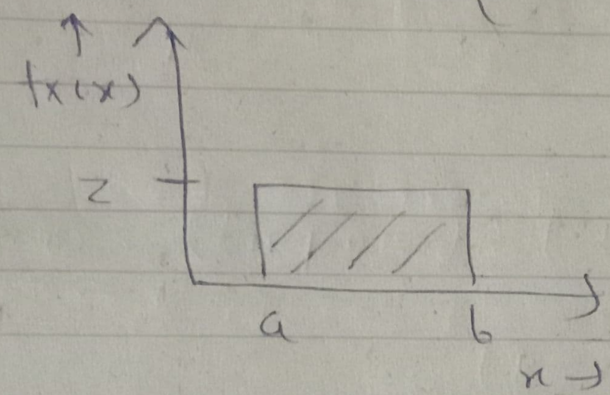
$$\boxed{Var[x] = E[x^2] - m_x^2}$$

Q1) Find mean & variance of R.V. X which is uniformly distributed between a & b, b > a ?

Soln:

$$X \sim U(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \end{cases}$$



$$z(b-a) = 1$$

$$z = \frac{1}{b-a}$$

$$m_X = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \times (b-a) \times \frac{1}{2}(b+a)$$

$$E[X] = \frac{b+a}{2}$$

$$Var[X] = E[X^2] - m_x^2$$

$$E[x^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$= \cancel{\frac{b^2+a^2+ab}{2}}$$

$$= \frac{b^2+a^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4b^2+4a^2+4ab-3a^2-3b^2-6ab}{12}$$

$a^3 - b^3 = (a-b)(a^2+b^2+ab)$
$= a^3 + \cancel{ab^2} - a^2 b - ba^2$
$- b^3 + \cancel{ab^2}$

$a^3 + \cancel{ab^2} + \cancel{a^2 b} - \cancel{ba^2} - b^3 - \cancel{ab^2}$

$$= \frac{a^2 + b^2 - 2ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$\boxed{var(X) = \frac{(a-b)^2}{12}} = \frac{(b-a)^2}{12}$$

$\to \sigma_x^2$

MEAN & VARIANCE OF SUM OF R.V.

Let 'x' & 'y' be two independent R.V. with m_x & m_y

let $z = m_x + m_y$

find m_z in terms of m_x & m_y?

$$\boxed{m_z = m_x + m_y}$$

$$z = x + y$$

$$m_z = E[(x+y)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y)\, f_{xy}(x,y)\, dx\, dy$$

x & y are independent R.V.

$$\therefore \boxed{f_{xy}(x,y) = f_x(x) \cdot f_y(y)}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{xy}(x,y)\, dx\, dy$$

$$+ \int_{-\infty}^{\infty} \int_{\infty}^{\infty} y\, f_{xy}(x y)\, dx\, dy$$

$$\boxed{m_z = m_x + m_y}$$

Var[Z] :

$$E[Z^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)^2 \, f_{xy}(x,y)\, dx\, dy$$

$\hookrightarrow f_x(x) f_y(y)$

$$= \int_{-\infty}^{\infty} x^2 f_x(x) dx \int_{-\infty}^{\infty} f_y(y)\, dy$$

$$+ \int_{-\infty}^{\infty} y^2 f_x(y)\, dy \int_{-\infty}^{\infty} f_x(x)\, dx$$

$$+ 2 \int_{-\infty}^{\infty} x f_x(x)\, dx \int_{-\infty}^{\infty} y f_y(y)\, dy$$

$$= E[x^2] + E[y^2] + 2 m_x m_y$$

$$Var(Z) = E[Z^2] - m_z^2$$

= [illegible]

$$E[x^2] = \sigma_x^2 + m_x^2$$
$$E[y^2] = \sigma_y^2 + m_y^2$$
$$E[Z^2] = \sigma_x^2 + m_y^2 + \sigma_y^2 + m_y^2 + 2 m_x m_y$$

$$E[Z^2] = \sigma_x^2 + \sigma_y^2 + (m_x + m_y)^2$$

$$E[Z^2] = \sigma_x^2 + \sigma_y^2 + m_z^2$$

$$E[Z^2] - m_z^2 = \sigma_x^2 + \sigma_y^2$$

$$\Downarrow$$

$$\sigma_z^2$$

$$\boxed{\sigma_z^2 = \sigma_x^2 + \sigma_y^2}$$

lec 15

* PDF of sum of RV's

$X, Y = 2$ independent R.V.

$Z = X + Y$

* $m_Z = m_X + m_Y$

* $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$

$$\boxed{\begin{aligned} f_X(x) &= \text{pdf of } X \\ f_Y(y) &= \text{pdf of } Y \end{aligned}}$$

find PDF of Z ?, $Z = X + Y$

$$\boxed{f_Z(z) = f_X(x) * f_Y(y)}$$

convolution

(Normal distribution)

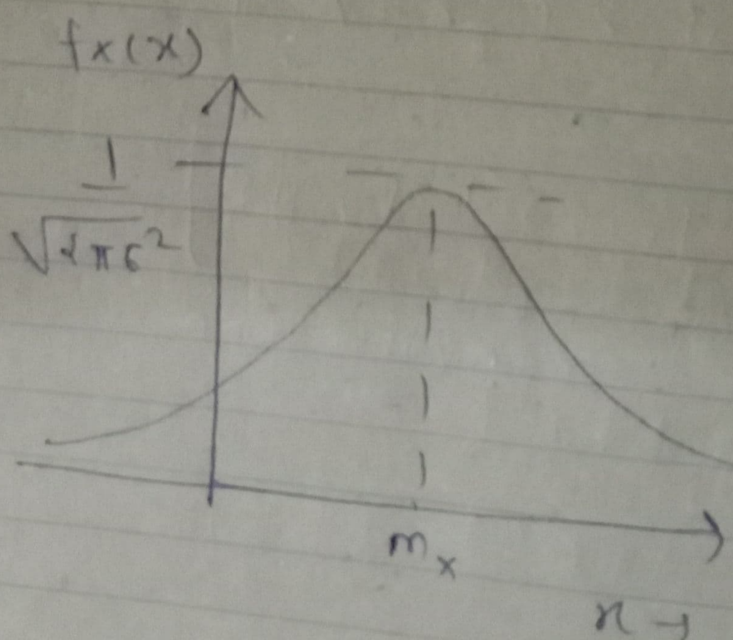
GAUSSIAN DISTRIBUTION

Gaussian PDF is of great importance in communication scenario becouse many naturally occurring experiments are characterized by R.V. with gaussian density.

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m_x)^2/2\sigma^2}}$$

σ^2 = variance

m_x = mean



$$\boxed{X \sim N(\mu, \sigma^2)}$$

$$\boxed{X \sim N(m_x, \sigma_x^2)}$$

Notation

TRANSFORMATION OF R.V.

$Y = f(X)$

* Pdf of Y given Pdf of X :

- $h(y) = f^{-1}(y)$

$$\boxed{f_Y(y) = f_X(h(y)) \times \left| \frac{dh}{dy} \right|}$$

Q1) $X \sim N(0, 1)$

mean ↙ ↳ variance

$$Y = e^X$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Find $f_Y(y)$?

Soln:

$$Y = e^X$$

$$h(y) = f^{-1}(y) = \ln y = X$$

$$\frac{dh}{dy} = \frac{1}{y}$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} \cdot \frac{1}{y}}$$

gives variation in R.V.

COVARIANCE B/W 2 R.V. (μ)

x & y

$$\mu = E\{(x - m_x)(y - m_y)\}$$

m_x = mean of 'x'

m_y = mean of 'y'

(1) when 'x' & 'y' are independent R.V.

$$= \int_{-\infty}^{\infty} (x - m_x) f_x(x)\, dx \int_{-\infty}^{\infty} (y - m_y) f_y(y)\, dy$$

$$\mu = (m_x - m_x)(m_y - m_y) = 0$$

(2) 'x' & 'y' are dependent R.V.

$X = Y$ OR $X = -Y$

$m_x = m_y = 0$

$$\mu = E[xy] = E[x^2] = E[y^2]$$

$$= \sigma_x^2 = \sigma_y^2 = \sigma_x \sigma_y$$

extent to which x & y
dependent

* ρ = correlation coefficient

$$\boxed{\rho = \frac{\mu}{\sigma_x \sigma_y}}$$

$$\boxed{-1 \leq \rho \leq 1}$$

If $\rho = 0$, x & y = independent

when x & y are independent in ~~nature~~ ($\mu = 0$), x & y are uncorrelated but vice-versa not true.

✓

cannot say that when x & y = uncorrelated, they may not be independent

Q1) Let Z = RV with pdf

$$f_Z(z) = \frac{1}{2}, \quad -1 \le z \le 1.$$

Let $X = Z$ & $Y = Z^2$
obviously X & Y not independent, since $X^2 = Y$. Show that X & Y are uncorrelated.

Soln:

$$E[Z] = \int_{-1}^{1} \frac{1}{2} \cdot z = \frac{1}{2}[z^2]_{-1}^{1} = 0$$

$$E[X] = E[Z] = 0$$

$$E[Y] = E[Z^2] = \int_{-1}^{1} \frac{1}{2} z^2 \, dz$$

$$= 1/3$$

$$\mu = E[(X - m_X)(Y - m_Y)]$$

$$= E[X(Y - 1/3)]$$

$$= E[XY - 1/3 X]$$

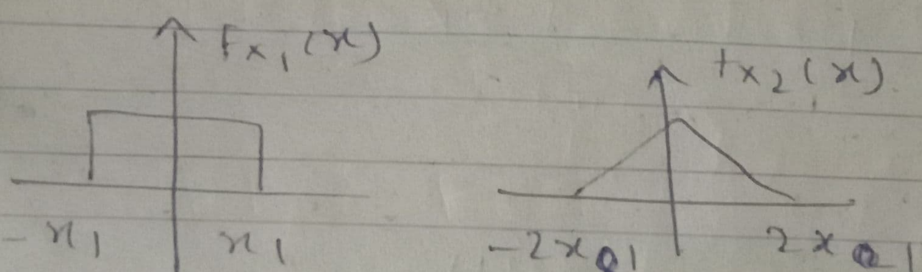
$$= E[Z^3 - Z/3]$$

Lec 16

CENTRAL LIMIT THEOREM

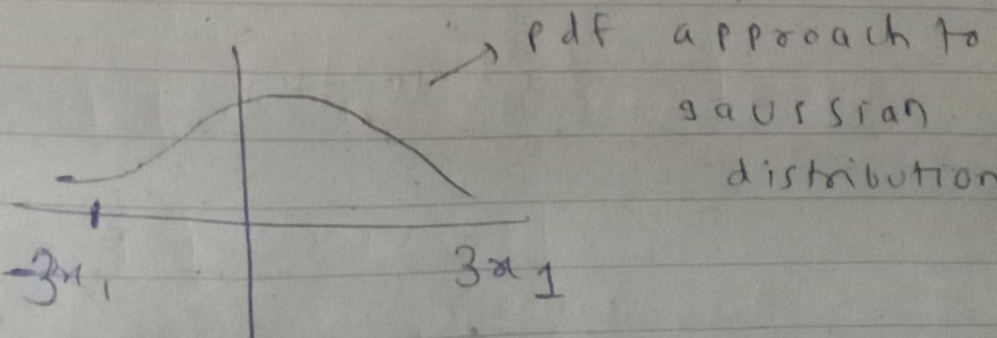
→ Based on concept of pdf of sum of R.v.

→ states that PDF of sum of 'n' independent RV tends to approach gaussian distribution as 'n' increases



$z = x_1 + x_2$

⇓



Random var = outcome of each RV is number

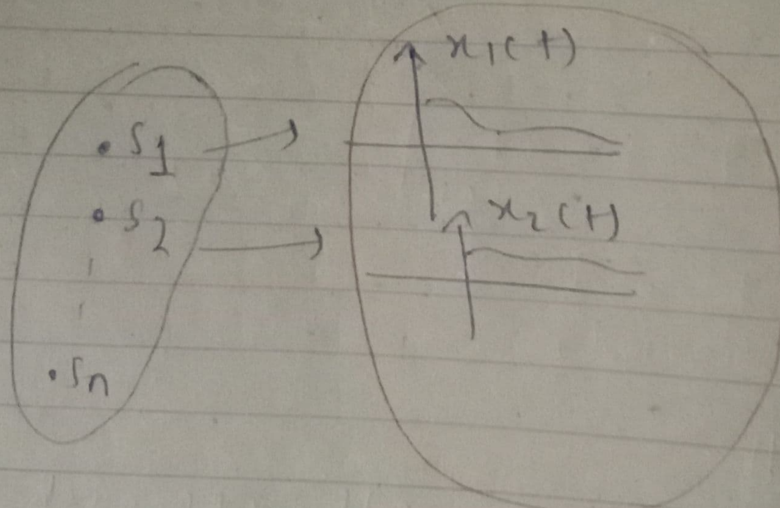
outcome of each process = waveform

RANDOM PROCESS

* Time domain representation of RV

* It is a RV which is function of time

* In describing Random signals, each sample point in our sample space = function of time



Ensemble

R.P = Random process

ENSEMBLE AVERAGE

$$= E[n^2(t_1)] = \overline{n^2(t_1)}$$

TIME AVERAGE $= \langle n^2(t) \rangle$

In general time average & ensemble average different

STATIONARITY = If the statistical characteristics does not change with time, Random process is stationary

ERGODIC R.P. = ensemble average & time average same

& stationary in nature

"An ergodic R.P. is stationary but vice-versa not true"

PSD = Power spectral density
ACF = Autocorrelation function

MEAN OF R.P.

$$\boxed{\bar{X}(t) = \int_{-\infty}^{\infty} x\, f_x(x, t)\, dx}$$

ACF & PSD of RP

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

$$\boxed{R_X(0) = E[X^2(t)]}$$

mean square value of the random process can be obtained from ACF at $\tau = 0$

* $R_X(\tau)$ is even function of τ

$$\boxed{R_X(\tau) = R_X(-\tau)}$$

* $R_X(\tau)$ has maximum magnitude at $\tau = 0$

$$\boxed{|R_x(\tau)| \leq R_x(0)}$$

Wiener-Khintchin theorem

PSD = FT (ACF)

FT = Fourier transform

$$\boxed{R_x(\tau) \xleftrightarrow{F.T.} PSD}$$

Classification of Random Process

(1) Stationary R.P.

* A random process whose statistical characteristics does not change with time

* ACF of stationary R.P. must depend on the time difference

$$\boxed{R_x(t_1, t_2) = R_x(t_2 - t_1)}$$

(1) Wide-sense (weakly) stationary R.P.

* process that is not stationary
* ~~only mean~~
in strict sense & may have mean & ACF that are independent of shift of time origin

* mean $(\bar{x}(t)) =$ constant

* $R_X(\tau) = R_X(t_1, t_2)$

↳ depends on time difference