Lecture-4:A2

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Lecture-4

Axiomatic Defn of the Probability: We assume that for each event E in the sample space S, $\exists$ a non-negative number P(E) is so chosen as to satisfy the following 3 axioms nothing else:

Axiom 1 : $P(E) \geq 0 \quad \forall E \subseteq S$

Axiom 2 : $P(S) = 1$

Axiom 3 : $E \subseteq S$, $F \subseteq S$ and $E \cap F = EF = \phi$

i.e. event E and F are mutually exclusive, then

$$P(E \cup F) = P(E) + P(F)$$

This approach to define the probability, is relatively recent. (A.N. Kolomogrov 1933)

Ex: Experiment tossing of a fair coin

$S = \{H, T\}$ 2 points finite

$\mathcal{P}(S) = \{\{H\}, \{T\}, \phi, S\}$ 2^2 events

$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$, $P(\phi) = 0$, $P(S) = 1$

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$$P : \mathcal{P}(S) \longrightarrow [0, 1] \subseteq \mathbb{R}^+$$

Thus $P : \mathcal{P}(S) \longrightarrow [0, 1]$ is a function satisfying above 3 axioms.

(*) If S is finite and consist n elements the $\mathcal{P}(S)$ will have 2^n elements.

In the development of the theory, all conclusions are based directly or indirectly on these 3 axioms and only on these 3 axioms.

Properties of Probability function P

(1) $\boxed{P(\phi) = 0}$

$A \cap \phi = A\phi = \phi \Rightarrow A$ & ϕ are mutually exclusive

& $A \cup \phi = A$

$\Rightarrow$ By axiom 3, $P(A) = P(A) + P(\phi)$

$$\Rightarrow P(\phi) = P(A) - P(A) = 0$$

$$\Rightarrow \boxed{P(\phi) = 0}$$

i.e. probability of impossible event is 0.

(2) $\boxed{P(A) \leq 1 \quad \forall A \subseteq S}$

$\bar{A}$ = Complement of A

$A \cup \bar{A} = S$ and $A\bar{A} = \phi$, hence by 3rd axiom

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$$P(S) = P(A \cup \bar{A})$$

$$\Rightarrow 1 = P(A) + P(\bar{A}) \Rightarrow P(A) = 1 - P(\bar{A}) \leq 1$$

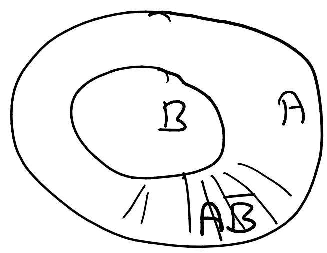
as $P(\bar{A}) \geq 0$ (by axiom 1)

$$\Rightarrow P(A) \leq 1 \quad \forall A \subseteq S$$

(3) $\boxed{B \subseteq A \Rightarrow P(B) \leq P(A)}$

$B = \{2\}$
$A = \{2, 4, 6\}$
$B \subseteq A$

$$A = B \cup (A\bar{B})$$



We can see $B \cap (A\bar{B}) = \phi$

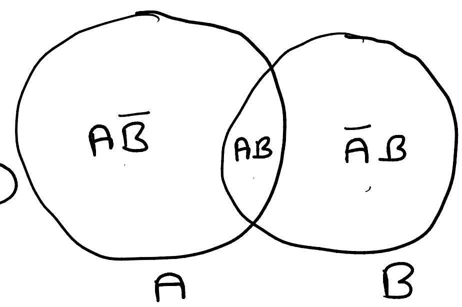
by third axiom, we have

$$P(A) = P(B) + P(A\bar{B})$$

$\Rightarrow P(A) \geq P(B)$ as $P(A\bar{B}) \geq 0$ by axiom 1.

(4) $\boxed{P(A \cup B) = P(A) + P(B) - P(AB)}$

$$A \cup B = (A\bar{B}) \cup (AB) \cup (\bar{A}B)$$



$\Rightarrow P(A \cup B) = P(A\bar{B} \cup AB) + P(\bar{A}B)$ ①

because $(A\bar{B} \cup AB) \cap (\bar{A}B) = \phi$

$$A = (A\bar{B}) \cup AB$$

$\Rightarrow P(A) = P(A\bar{B}) + P(AB)$ as $(A\bar{B}) \cap (AB) = \phi$

$$\Rightarrow P(A\bar{B}) = P(A) - P(AB)$$

Similary $P(\bar{A}B) = P(B) - P(AB)$

Substituting the value of $P(A\bar{B})$ and $P(\bar{A}B)$ in ①, we get

$$P(A \cup B) = P(A\bar{B}) + P(AB) + P(\bar{A}B)$$

$$= P(A) - \cancel{P(AB)} + \cancel{P(AB)} + P(B) - P(AB)$$

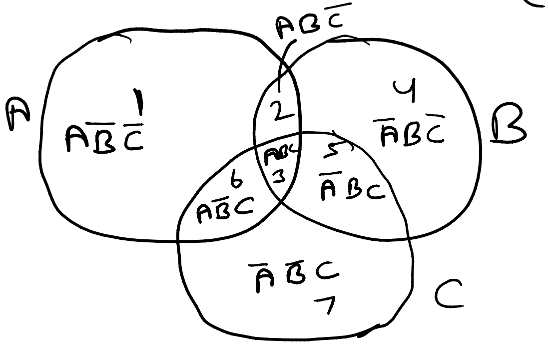
$$= P(A) + P(B) - P(AB)$$

$= P(A) + P(B) - P(AB)$

To find the $P(A \cup B)$, we need $2^2 - 1 = 3$ pieces of information.

(4) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$

$P(A \cup B \cup C)$
$= P(1) + P(2) + \text{——} + P(7)$



Other way is

$P(A \cup B \cup C) = P[(A \cup B) \cup C]$

$= P(A \cup B) + P(C) - P((A \cup B) C)$

$= P(A) + P(B) - P(AB) + P(C) - P(AC \cup BC)$

$= P(A) + P(B) - P(AB) + P(C) - [P(AC) + P(BC) - P(AC \cap BC)]$

$= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$

$2^3 - 1 = 7$ pieces of information

In general

$$\boxed{P(A_1 \cup A_2 \cup \text{——} \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) + \text{——} + (-1)^n P(A_1 A_2 \text{—} A_n)}$$

$2^n - 1$ pieces of information.