Lec - 34

Conditional Distributions

[When ran. var's are not independent]

We first look at discrete r v s.

Conditional PMF

Defn: Let X and Y be two discrete r.v. associated with the same random experiment. Then the conditional pmf $f_{X|Y}$ of X given $Y=y$ is defined as

$$f_{X|Y}(x|y) = \begin{cases} P(X=x \mid Y=y) & \text{if } P(Y=y) > 0 \\ 0 & \text{if } P(Y=y) = 0 \end{cases}$$

A conditional pmf can be thought of as an ordinary pmf over a new universe determined by the conditioning event.

For this, note that for fixed y, $f_{X|Y}(x|y) \geq 0 \;\; \forall x \in R(X)$.

Also if $P(Y=y) > 0$ then

$$\sum_{x \in R(X)} f_{X|Y}(x|y) = \sum_{x \in R(X)} P(X=x \mid Y=y) = P\Big(\bigcup_{x \in R(X)} \{X=x\} \Big| Y=y\Big)$$

$$= P(\Omega \mid Y=y) = 1$$

If X, Y have joint pmf f, then using the defn of conditional probability, we obtain

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{if } f_Y(y) = 0 \end{cases}$$

Conditional Distribution Function

Recall that the distr. fn F_X of any ran. var. X (discrete or cont.) is defined as:

$$F_X(x) = P\{X \leq x\} \quad \forall x \in \mathbb{R}$$

We define the conditional distr. fn of X given $Y=y$ as:

$$F_{X|Y}(x|y) = P(X \leq x \mid Y = y).$$

So conditional distribution fn is an ordinary (or unconditional) distr. fn in the new universe determined by the conditioning event.

Recall that if X is a discrete r.v. with pmf f_X then $F_X(x) = \sum_{t \in R(X): t \leq x} f_X(t)$

Similarly, if X is a discrete r.v. with conditional pmf $f_{X|Y}$, then $F_{X|Y}(x|y) = \sum_{t \in R(X): t \leq x} f_{X|Y}(t|y)$

Recall that if X is a discrete ran. var. with pmf f_X and $A \subset \mathbb{R}$, then

$$P(X \in A) = \sum_{x \in A \cap R(X)} f_X(x)$$

Similarly, if X is a discrete r.v. with conditional pmf $f_{X|Y}$ and $A \subset \mathbb{R}$, then we have

$$P(X \in A \mid Y = y) = \sum_{x \in R(X) \cap A} f_{X|Y}(x|y)$$

Example (1): Let the joint pmf of X and Y is given as follows:

X \ Y	-1	0	1
-1	0	$\frac{1}{4}$	0
0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	0	$\frac{1}{4}$	0

Q) Then compute the conditional pmf of X given $Y=0$. Also compute the conditional distr. fn^n of the same.

Soln: Note that $P(Y=0) = \frac{1}{4} + 0 + \frac{1}{4} = \frac{1}{2} > 0$

Hence

$$f_{X|Y}(x|0) = \begin{cases} \dfrac{f(x,0)}{f_Y(0)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} & \text{if } x = -1, 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Now the conditional distr. fn^n

$$F_{X|Y}(x|0) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Remark: ① We have said that the conditional pmf is a pmf in the new universe determined by the conditioning event.

In the previous example, the prob. distr. of X is

$$P(X=-1) = P(X=1) = \frac{1}{4}, \quad P(X=0) = \frac{1}{2}.$$

Whereas in the new universe determined by the event $\{Y=0\}$, the prob. distr. of X is revised as:

$$P(X=-1 \mid Y=0) = P(X=1 \mid Y=0) = \frac{1}{2}.$$

S'lly, the distr. fn of X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{4} & \text{if } -1 \le x < 0 \\ \frac{3}{4} & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

F_X got revised as $F_{X|Y}(x|0)$ in the new universe determined by the event $\{Y=0\}$. Also note that $F_{X|Y}(x|0)$ satisfies all the properties of a distr. fn.

① $\lim_{x \to -\infty} F_{X|Y}(x|0) = 0$, $\lim_{x \to +\infty} F_{X|Y}(x|0) = 1$

② $F_{X|Y}(x|0)$ is non-decreasing on $\mathbb{R}$.

③ $F_{X|Y}(x|0)$ is right continuous on $\mathbb{R}$.

Remark 2 The conditional pmf can also be used to calculate the marginal pmfs. In particular, we have by using the defns

$$f_X(x) = \sum_{y \in R(Y)} f(x,y) = \sum_y f_{X|Y}(x|y)\, f_Y(y).$$

Example 2: Suppose $f_Y(y) = \begin{cases} \frac{5}{6} & \text{if } y = 10^2 \\ \frac{1}{6} & \text{if } y = 10^4 \end{cases}$

$$f_{X|Y}(x|10^2) = \begin{cases} \frac{1}{2} & \text{if } x = \frac{1}{10^2} \\ \frac{1}{3} & \text{if } x = \frac{1}{10} \\ \frac{1}{6} & \text{if } x = 1 \end{cases}$$

(Q) Then find the pmf of X.

$$f_{X|Y}(x|10^4) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{3} & \text{if } x = 10 \\ \frac{1}{6} & \text{if } x = 10^2 \end{cases}$$

Soln: First of all by looking at conditional pmf $f_{X|Y}$ we see that X takes 5 values:

$\frac{1}{10^2}, \frac{1}{10}, 1, 10, 10^2$. Now,

$f_X\left(\frac{1}{10^2}\right) = \frac{1}{2} \times \frac{5}{6} = \frac{5}{12}$, $f_X\left(\frac{1}{10}\right) = \frac{1}{3} \times \frac{5}{6} = \frac{5}{18}$

$f_X(1) = \frac{1}{6} \times \frac{5}{6} + \frac{1}{2} \times \frac{1}{6} = \frac{8}{36} = \frac{2}{9}$

$f_X(10) = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

$f_X(10^2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Conditional Density.

Defn: Let X and Y be two ran. var. with pdf f.

The conditional density of X given $Y=y$ is defined as:

$$f_{X|Y}(x|y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{if } f_Y(y) = 0 \end{cases}$$

As the case of conditional pmf, conditional pdf can be thought of as an ordinary (or unconditional) pdf over a new universe. For fixed y, $f_{X|Y}(x|y) \geq 0 \ \forall x \in \mathbb{R}$. Also if $f_Y(y) > 0$, then

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

Recall that if X is a continuous ran. var. with pdf f_X and B is any Borel subset of $\mathbb{R}$, then

$$P(X \in B) = \int_B f_X(x)\,dx$$

The above motivated the following defn:

<u>Defn</u>: Let X, Y be jointly continuous ran. vars, with $f_{X|Y}$ denoting the conditional density of X given Y. Then for any Borel subset B of $\mathbb{R}$, we have

$$P(X \in B | Y = y) = \int_B f_{X|Y}(x|y)\,dx \qquad \text{--- (1)}$$

<u>Remark</u>: Conditional probability $P(X \in B | Y = y)$ were left undefined if the $P\{Y = y\} = 0$. But the above formula provides a natural way of defining such conditional probabilities in the present context.

In addition, it allows us to view the conditional pdf $f_{X|Y}$ (as a fn of x) as a description of the probability law of X, given that the event $\{Y = y\}$ has occurred.

In view of eqn (1), the conditional cdf of X given $Y = y$ is

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(t|y)\,dt$$

We have

$$\frac{d}{dx} F_{X|Y}(x|y) = f_{X|Y}(x|y),$$ where

equality holds at points (x,y) at which joint pdf f is continuous and $f_Y(y) > 0$ and f_Y is continuous at y.