Lecture-21

Thm:- If $X \sim N(\mu, \sigma^2)$, then $Y = \alpha X + \beta$ is also normally distributed with parameters $(\alpha\mu + \beta, \alpha^2 \sigma^2)$.

Proof: We prove it for $\alpha > 0$. Let F_Y denote the c.d.f. of r.v. Y. Then

$$F_Y(y) = P(Y \leq y) = P(\alpha X + \beta \leq y)$$

$$= P\left\{ X \leq \frac{y-\beta}{\alpha} \right\}$$

$$= F_X\left(\frac{y-\beta}{\alpha}\right) = \int_{-\infty}^{\frac{y-\beta}{\alpha}} f_X(x)\, dx$$

$$= \int_{-\infty}^{\frac{y-\beta}{\alpha}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

by changing the variable from x to v by using

$$v = \alpha x + \beta$$

$$dv = \alpha\, dx$$

So that we get

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\frac{v-\beta}{\alpha} - \mu}{\sigma}\right)^2} \frac{dv}{\alpha}$$

$F_Y(y) = \int_{-\infty} \frac{}{\sigma\sqrt{2\pi}} \cdots \alpha$

$$= \int_{-\infty}^{y} \frac{1}{\alpha\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{v-(\alpha\mu+\beta)}{\alpha\sigma}\right)^2} dv$$

$$\Rightarrow Y \sim N(\alpha\mu+\beta, \alpha^2\sigma^2) \qquad \left(\begin{array}{l} X \sim N(\mu, \sigma^2) \\ F_X(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \end{array} \right.$$

Similarly, we can consider the case, when $\alpha < 0$.

Functions of a r.v. :- Suppose that X is a r.v. and $h(x)$ is a function of the real variable x. The expression

$$Y = h(X)$$

is a new r.v.

In general;

X : discrete, continuous or mixed r.v.

$h : \mathbb{R} \to \mathbb{R}$ well behaved function, h can be linear, non-linear, continuous monotonic decreasing, or increasing etc.

$$\underset{f_X(x)}{X} \longrightarrow \boxed{y = h(x)} \longrightarrow Y$$

$f_Y(y) = ?$

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This problem can be viewed as a "black box" with input X and transfer characteristic $Y = h(X)$.

Ex 1:- Let X be a discrete r.v. with p.m.f.

$$p_X(x) = P(X=x) = \frac{1}{9}, \text{ for } x = -4, -3, \cdots 3, 4,$$

$Y = |X|$ then $p_Y(y) = ?$, $h(x) = |x|$

$R_X : -4, -3, -2, -1, 0, 1, 2, 3, 4$

$R_Y : 0, 1, 2, 3, 4$

$$\{Y=0\} = \{X=0\} \Rightarrow P\{Y=0\} = P(X=0)$$
$$= \frac{1}{9} = \sum_{x:|x|=0} P(X=x)$$

i.e. $p_Y(0) = \frac{1}{9}$

$$\{Y=1\} = \{X=1\} \cup \{X=-1\}$$
$$\Rightarrow P\{Y=1\} = P\{X=1\} + P\{X=-1\}$$
$$= \frac{1}{9} + \frac{1}{9} = \frac{2}{9} = \sum_{x:|x|=1} P(X=x)$$

i.e. $p_Y(1) = \frac{2}{9}$

Similarly $p_Y(2) = \frac{2}{9} = p_Y(3) = p_Y(4)$.

PMF of function of a random Variable

If X is discrete random variable then we can obtain pmf of $Y = h(X)$ as follows:

$$\boxed{P\{Y=y\} = \sum_{x:\,h(x)=y} P(X=x)}$$

Note that if X is discrete, then $Y = h(X)$ is necessarily discrete.

Ex:- Suppose $X \sim N(0,1)$ Continuous

and $h(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$ discontin

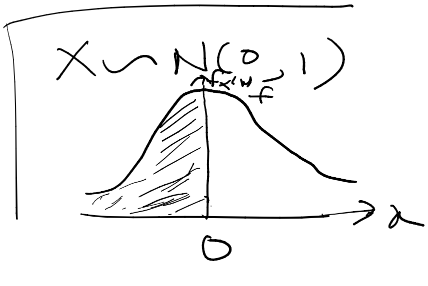
$Y = h(X)$. Then determine pmf of $h(X)$.

$R_Y : -1, 0, 1$ Y is discrete.

$p_Y(-1) = P(Y=-1) = P(X<0) = \frac{1}{2}$

$p_Y(0) = P(Y=0) = P(X=0) = 0$

$p_Y(1) = P(Y=1) = P(X>0) = \frac{1}{2}$



In above example, X is continuous, h is a real valued function, then $Y = h(X)$ ~~was~~ is not a continuous r.v. i.e, does not have pdf.

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There exists example, where, X is continuous, h is also continuous, but $Y = h(X)$ does not have p.d.f.

Maximum and Minimum of two functions :- Let $f, g : I \to \mathbb{R}$. Then

$$\text{Max}\{f(x), g(x)\} = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

and

$$\text{Min}\{f(x), g(x)\} = \frac{f(x)+g(x)-|f(x)-g(x)|}{2}$$

So if f and g are continuous function of on I then $\text{Max}\{f(x), g(x)\}$ and $\text{Min}\{f(x), g(x)\}$ are also continuous on I.

Ex :- Let $X \sim \exp(5)$, $\lambda = 5$

$$f(x) = x, \quad g(x) = 10$$

$$h(x) = \text{Min}\{f(x), g(x)\} = \min\{x, 10\}$$

$$Y = h(X) = \text{Min}\{X, 10\}$$

Y is continuous r.v.

First we determine the distribution function of a r.v. $Y = \text{Min}\{X, 10\}$.

$$R_X = [0, \infty)$$

$$R_Y = [0, 10]$$

$$\{Y \leq y\} = \{X \leq y\} \cup \{10 \leq y\}$$

No