MATH-221: Probability and Statistics

Tutorial # 3 (Random Variables, PMF, PDF, CDF, Functions of Random variable)

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1. Define $F : \mathbb{R} \to [0,1]$ by $F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{x^2}{4} & \text{if} \quad 0 \leq x \leq 2 \\ 1 & \text{if} \quad x > 2 \end{cases}$. Show that F is a distribution function. Find pdf or pmf (if exists). Also compute $P(1 \leq X < 3)$, where X has distribution function F.

Solution: It is easy to verify (students please do verify!) that F satisfies three properties

(a) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$.

(b) F is non-decreasing.

(c) F is right-continuous.

Hence F is a distribution function of some random variable X. Since F is a continuous function on $\mathbb{R}$ and differentiable everywhere except at $x = 2$, also the derivative is continuous everywhere except at $x = 2$ hence F has pdf as well, which is given by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if} \quad 0 \leq x < 2 \\ 0 & \quad \text{otherwise} \end{cases}.$$

Now

$$\begin{aligned} P(1 \leq X < 3) &= P(X < 3) - P(X < 1) = F(3-) - F(1-) \\ &= F(3) - F(1) (\because F \text{ is continuous}) \\ &= 1 - 1/4 = 3/4 \end{aligned}$$

2. Let X be a random variable with distribution function F. Find the distribution function of the following random variables in terms of F: (a) $\max\{X, a\}$, where $a \in \mathbb{R}$ (b) $|X|^{\frac{1}{3}}$ (c) $|X|$ (d) e^X (e) $-\ln|X|$.

Solution: (a) For $x \in \mathbb{R}$,

$$\{\max\{X, a\} \leq x\} = \{X \leq x\} \cap \{a \leq x\}$$

Note that if $a > x$ then $\{a \leq x\} = \emptyset$ and if $a \leq x$ then $\{a \leq x\} = \Omega$. Hence

$$\{\max\{X, a\} \leq x\} = \begin{cases} \emptyset & \text{if} \quad x < a \\ \{X \leq x\} & \text{if} \quad x \geq a \end{cases}$$

Hence distribution function of $Y = \max\{X, a\}$ is denoted by F_Y

$$F_Y(x) = \begin{cases} P(\emptyset) = 0 & \text{if} \quad x < a \\ P\{X \leq x\} = F(x) & \text{if} \quad x \geq a \end{cases}$$

(b) For $x \in \mathbb{R}$,

$$\{|X|^{\frac{1}{3}} \leq x\} = \begin{cases} \emptyset & \text{if } \; x < 0 \\ \{|X| \leq x^3\} = \{-x^3 \leq X \leq x^3\} & \text{if } \; x \geq 0 \end{cases}$$

Hence distribution function of $Y = |X|^{\frac{1}{3}}$ is

$$F_Y(x) = \begin{cases} 0 & \text{if } \; x < 0 \\ F(x^3) - F((-x^3)-) & \text{if } \; x \geq 0 \end{cases}$$

(c) For $x \in \mathbb{R}$,

$$\{|X| \leq x\} = \begin{cases} \emptyset & \text{if } \; x < 0 \\ \{-x \leq X \leq x\} & \text{if } \; x \geq 0 \end{cases}$$

Hence distribution function of $|X|$ is

$$F_{|X|}(x) = \begin{cases} 0 & \text{if } \; x < 0 \\ F(x) - F((-x)-) & \text{if } \; x \geq 0 \end{cases}$$

(d) For $x \in \mathbb{R}$,

$$\{e^X \leq x\} = \begin{cases} \emptyset & \text{if } \; x \leq 0 \\ \{X \leq \ln x\} & \text{if } \; x > 0 \end{cases}$$

Hence distribution function of e^X is

$$F_{e^X}(x) = \begin{cases} 0 & \text{if } \; x \leq 0 \\ F(\ln x) & \text{if } \; x > 0 \end{cases}$$

(e) We need to assume that $X \neq 0$. For $x \in \mathbb{R}$,

$$\{-\ln|X| \leq x\} = \{\ln|X| \geq -x\} = \{|X| \geq e^{-x}\} = \{X \geq e^{-x}\} \cup \{X \leq -e^{-x}\}$$

Hence distribution function of $Y = -\ln|X|$ is

$$F_Y(x) = [1 - F(e^{-x}-)] + F(-e^{-x})$$

3. Let X be the uniform random variable on $[0, 1]$. Then Determine pdf of (a) $\sqrt{X}$ (b) $X^{\frac{1}{4}}$.

Solution: (a) Let $x \in \mathbb{R}$ be given.

$$P(\sqrt{X} \leq x) = \begin{cases} P(\emptyset) & \text{if } \; x < 0 \\ P\{X \leq x^2\} = P(0 \leq X \leq x^2) & \text{if } \; x \geq 0 \end{cases}$$

Now if $x \leq 1$ then $x^2 \leq 1$ hence

$$P(0 \leq X \leq x^2) = \int_0^{x^2} dt = x^2.$$

and if $x > 1$ then $x^2 > 1$ hence

$$P(0 \leq X \leq x^2) = \int_0^1 dt = 1.$$

Hence CDF of $Y = \sqrt{X}$ is

$$F_Y(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad x > 1 \end{cases}$$

Since F is a continuous function on $\mathbb{R}$ and differentiable everywhere except at $x = 1$, also the derivative is continuous everywhere except at $x = 1$ hence F has pdf, which is given by

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x < 1 \\ 0 & \quad\text{otherwise} \end{cases} .$$

(b) For $x \in \mathbb{R}$,

$$\{X^{\frac{1}{4}} \leq x\} = \begin{cases} \emptyset & \text{if} \quad x < 0 \\ \{X \leq x^4\} = \{0 \leq X \leq x^4\} & \text{if} \quad x \geq 0 \end{cases}$$

Hence distribution function of $Y = X^{\frac{1}{4}}$ is

$$F_Y(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ P(0 \leq X \leq x^4) & \text{if} \quad x \geq 0 \end{cases}$$

Now if $x \leq 1$ then $x^4 \leq 1$ hence

$$P(0 \leq X \leq x^4) = \int_0^{x^4} dt = x^4.$$

and if $x > 1$ then $x^4 > 1$ hence

$$P(0 \leq X \leq x^4) = \int_0^1 dt = 1.$$

Hence

$$F_Y(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^4 & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad x > 1 \end{cases}$$

Since F is a continuous function on $\mathbb{R}$ and differentiable everywhere except at $x = 1$, also the derivative is continuous everywhere except at $x = 1$ hence F has pdf as well, which is given by

$$f(x) = \begin{cases} 4x^3 & \text{if} \quad 0 \leq x < 1 \\ 0 & \quad \text{otherwise} \end{cases}.$$

4. Let X be a random variable with PMF

$$f_X(x) = \begin{cases} \dfrac{x^2}{a} & \text{if} \quad x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \quad \text{otherwise} \end{cases}$$

Find a. What is the PMF of the random variable $Z = (X - a)^2$.

Solution: Since f_X is a pmf so

$$1 = \sum_{x=-3}^{3} f_X(x) = \sum_{x=-3}^{3} \frac{x^2}{a} = \frac{2(1+4+9)}{a} = \frac{28}{a} \implies a = 28$$

Range of $X - 28$ is $\{-31, -30, -29, -28, -27, -26, -25\}$. Hence range of Z would be $\{n^2 | n = 25, \cdots, 31\}$. Now

$$P(Z = n^2) = P(X - 28 = n) + P(X - 28 = -n)$$

5. Let X be a binomial random variable with parameters (n, p). What value of p maximizes $P(X = k)$, $k = 0, 1, \ldots, n$?

6. Let X be a Poisson random variable with parameter λ. If $P(X = 1 | X \leq 1) = 0.8$, what is the value of λ?

7. Let X be a normal random variable with parameters μ and σ^2. Find (a) $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$, (b) $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$.

8. Let X have a geometric distribution with $p = 0.8$. Compute: (a) $P(X > 3)$; (b) $P(4 \leq X \leq 7 \text{ or } X > 9)$; (c) $P(3 \leq X \leq 5 \text{ or } 7 \leq X \leq 10)$.

Solution: We have $p = 0.8,\ q = 0.2$ and the PMF as $f_X(x) = \begin{cases} q^{i-1}p \text{ if } x = i \text{ and } i = 1, 2, \cdots \\ 0, \quad\quad \text{otherwise} \end{cases}$.

Then (a) $P(X > 3) = 1 - P(X \leq 3) = 1 - \sum_{i=1}^{3} P(X = i) = 1 - (p + qp + q^2p) =$ $1 - 0.8(1 + 0.2 + 0.04) = 1 - 0.992 = 0.008$

(b) Since the events are disjoint, $P(4 \leq X \leq 7 \text{ or } X > 9) = P(4 \leq X \leq 7) + P(X > 9) = \sum_{i=4}^{7} P(X = i) + \sum_{i=10}^{\infty} P(X = i) = 1 - \sum_{i=1}^{3} P(X = i) - \sum_{i=8}^{9} P(X = i) = 1 - (p + qp + q^2p) - (q^7p + q^8p) = 1 - 0.8(1 + 0.2 + 0.04 + 0.0000128 + 0.00000256) = 1 - 0.992012288 \approx 0.0079877.$

(c) Since the events are disjoint, $P(3 \leq X \leq 5 \text{ or } 7 \leq X \leq 10) = P(3 \leq X \leq 5) + P(7 \leq X \leq 10) = \sum_{i=3}^{5} P(X = i) + \sum_{i=7}^{10} P(X = i) = \sum_{i=3(i\neq 6)}^{10} P(X = i) = (q^2p + q^3p + q^4p + q^6p + q^7p + q^8p + q^9p) = 0.8(0.04 + 0.008 + 0.0016 + 0.000064 + 0.0000128 + 0.00000256 + 0.000000512) = 0.0397438976.$

9. Let the random variable X denote the decay time of some radioactive particle and follows the exponential distribution function. Suppose λ is such that $P(X \geq 0.01) = \frac{1}{2}$. Find a number t such that $P(X \geq t) = 0.9$.

Solution: We have the PDF of a exponential distribution with parameter λ as $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$.

We calculate the CDF of it as $F_X(x) = P(X \leq x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$.

So, $\frac{1}{2} = P(X \geq 0.01) = e^{-\lambda 0.01}$. Or, $\lambda = 100 \log 2$. Then, $0.9 = P(X \geq t) = e^{-\lambda t} = e^{-100 \log 2t}$. Or, $-100 \log 2t = \log 0.9$. Or, $t = -\frac{\log 0.9}{100 \log 2} \approx 0.00152$

10. Let X have a normal distribution with parameters μ and $\sigma^2 = 0.25$. Find a constant c such that $P(|X - \mu| \leq c) = 0.9$. (Hint: Use Table for standard normal distribution function).

Solution: Let $Y = \frac{X-\mu}{\sigma}$. Then Y is a standard normal variable. Then $0.9 = P(|X - \mu| \leq c) = P(\frac{|X-\mu|}{\sigma} \leq \frac{c}{\sigma})$

$= P(|Y| \leq \frac{c}{\sigma}) = P(-\frac{c}{\sigma} \leq Y \leq \frac{c}{\sigma})$

$= F_Y(\frac{c}{\sigma}) - F_Y(-\frac{c}{\sigma}) = F_Y(\frac{c}{\sigma}) - (1 - F_Y(\frac{c}{\sigma}))$ (since here $F_Y(y)$ is symmetric)

$= 2F_Y(\frac{c}{\sigma}) - 1$. Or, $F_Y(\frac{c}{\sigma}) = 0.95$.

From standard normal table, we get $\frac{c}{\sigma} = 1.65$. Or, $c = 0.5 \times 1.65 = 0.825$.

et X : r.v. which counts the # flips in the n relays

R_X : $0, 1, 2, \text{—} \, n.$

$$\Longrightarrow \quad X \sim B(n, p)$$

$$P(X = j) = {}^{n}C_{j} \, p^{j} (1-p)^{n-j}$$

Input = Output $\Longleftrightarrow$ X is even

$$\therefore P(\text{Input} = \text{Output}) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P(X = 2k)$$

here $\lfloor x \rfloor$ = floor x = largest integer not greater than x.

$$\Rightarrow P(\qquad) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} {}^{n}C_{2k} \, p^{2k} (1-p)^{n-2k}$$

$$= (1-p)^{n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} {}^{n}C_{2k} \left(\frac{p}{1-p}\right)^{2k} = \frac{1 + (1-2p)^{n}}{2}$$

$$\left.\begin{aligned} (1+x)^{n} &= \sum_{k=0}^{n} {}^{n}C_{k} \, x^{k} \\ (1-x)^{n} &= \sum_{k=0}^{n} {}^{n}C_{k} (-x)^{k} \end{aligned}\right] \Longrightarrow \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} {}^{n}C_{2k} \, x^{2k} = \frac{1}{2}\left[(1+x)^{n} + (1-x)^{n}\right]$$