Lec-22

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Lecture-22

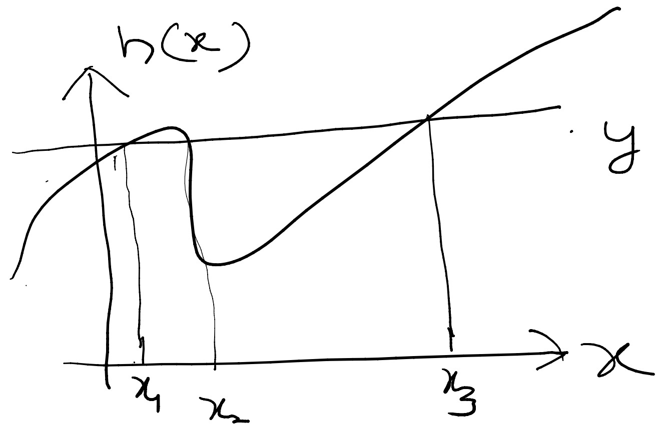
X is discrete, $h(x)$ Cont / discon $\quad Y = h(X)$ discrete.

X is continuous $h(x)$ discontinuous

$Y = h(X)$ discrete

X : Continuous, h : Continuous

$\{Y \le y\}$ corresponds to the event $\{X \le x_1$ and $x_2 \le x \le x_3\}$



Thus

$$P\{Y \le y\} = P\{x \text{ values yielding } Y \le y\}$$

$$= P\{x \mid Y \le y\}$$

$$\Rightarrow F_Y(y) = P\{Y \le y\} = P\{x \mid Y \le y\}$$

$$= \int_{\{x \mid Y \le y\}} f_X(x)\, dx$$

$$\boxed{F_Y(y) = \int_{\{x \mid Y \le y\}} f_X(x)\, dx}$$

If $F_Y(y)$ is differentiable, then

$$\boxed{f_Y(y) = \frac{d}{dy} \int_{\{x \mid Y \le y\}} f_X(x)\, dx}$$

Ex: Let $X \sim \exp(5)$

Find pdf if it exists for the random variable $Y = \min\{X, 10\}$.

X : Continuous

h : Continuous

$$y = h(x) = \min\{x, 10\}$$

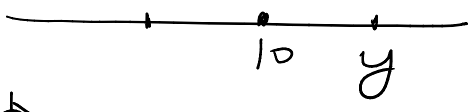
$R_X = [0, \infty)$

$R_Y = [0, 10]$

Note that for $y \in \mathbb{R}$

$$\{Y \leq y\} = \{X \leq y\} \cup \{10 \leq y\}$$

Note that if



$10 > y$, then $\{10 \leq y\} = \phi$

and if $y \geq 10$ then $\{10 \leq y\} = S$.

$$\Rightarrow \{Y \leq y\} = \begin{cases} X \leq y, & \text{if } y < 10 \\ S, & \text{if } y \geq 10 \end{cases}$$

$$\Rightarrow P\{Y \leq y\} = \begin{cases} P\{X \leq y\} & \text{if } y < 10 \\ P(S) & \text{if } y \geq 10 \end{cases}$$

i.e.

$$F_Y(y) = \begin{cases} F_X(y), & \text{if } y < 10 \\ 1 & \text{if } y \geq 10 \end{cases}$$

$$= \begin{cases} 0, & y < 0 \\ 1 - e^{-5y} & \text{if } 0 \leq y < 10 \\ 1 & \text{if } y \geq 10 \end{cases}$$

Here, $F_Y(y)$ is discontinuous at $y = 10$

hence $Y = \min\{X, 10\}$ does not possess p.d.f.

pd.f.

Ex:- $X \sim U[0,1]$.
Find the pdf. of r.v. $Y = X^2$ if exists.

X : Continuous, $y = h(x) = x^2$
Continuous

$\underline{Y = X^2}$

Since $X \sim U[0,1]$, so

$$F_X(x) = P(X \leq x)$$

$$= \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

For $y \in \mathbb{R}$,

$$F_Y(y) = P(Y \leq y) = P\{X^2 \leq y\}$$

$$= \begin{cases} 0 & \text{if } y < 0 \\ P(\underline{-\sqrt{y} < X} < \sqrt{y}) & \text{if } y \geq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 0 \\ F_X(\sqrt{y}) - \underbrace{F_X(-\sqrt{y})}_{=0} & \text{if } y \geq 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } y < 0 \\ F_X(\sqrt{y}), & \text{if } y \geq 0 \end{cases},$$

$$= \begin{cases} 0, & \text{if } y < 0 \\ \sqrt{y} & \text{if } 0 \leq y \leq 1 \\ 1 & \text{if } y > 1 \end{cases}$$

CDF of $Y = X^2$ is continuous on $\mathbb{R}$ and differentiable everywhere except the point $y = 0, 1$. Also the derivative of CDF $F_Y(y)$ is continuous everywhere except at $y = 0, 1$. Hence

$$f_Y(y) = \begin{cases} 0, & y \leq 0 \\ \frac{1}{2\sqrt{y}} & 0 \leq y < 1 \\ 0, & y \geq 1 \end{cases}$$

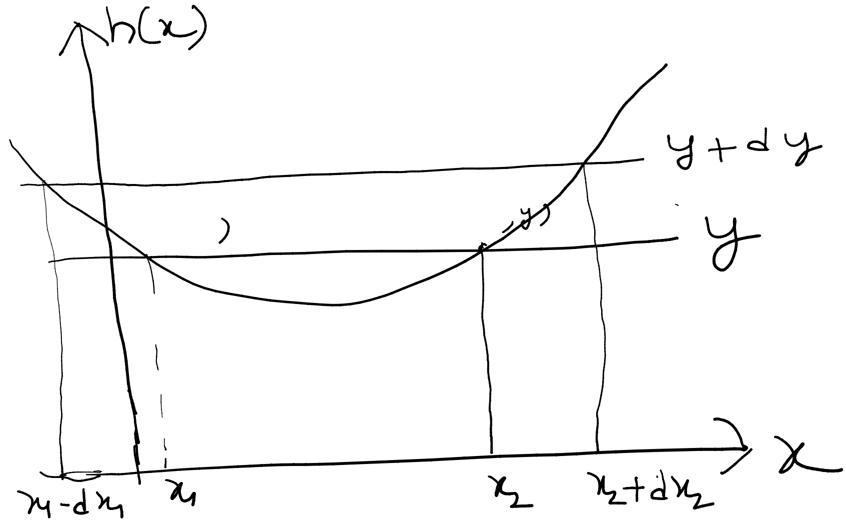
Thm: Suppose X is a random variable with pdf $f_X(x)$ and $h : I \to \mathbb{R}$

be a function s.t. $Y = h(X)$ and Y has a p.d.f. Here

$$\boxed{f_Y(y) = \sum_i \frac{f_X(x_i)}{|h'(x_i)|}}$$

where $x_1, x_2, \text{———}$ are the real roots of $y = h(x)$.

<u>Proof</u> To avoid the generalities, we assume that the eqn $y = h(x)$ has 2 roots say x_1 and x_2 i.e. $h(x_1) = h(x_2) = y$.



We know that

$$f_Y(y)\,|dy| = P\left\{ y \le Y \le y + |dy| \right\}$$

$$\int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f_X(x) \simeq \epsilon f_X(a)$$

$$= P\{ a-\tfrac{\epsilon}{2} < X < a+\tfrac{\epsilon}{2} \}$$

$$+ |dy|\}\Big| \quad = P\{a - \tfrac{\epsilon}{2} < X < a + \tfrac{\epsilon}{2}\}$$

$$= P\{y \le h(X) \le y + |dy|\}$$

$$= P\{x_1 - |dx_1| \le X \le x_1\}$$

$$+ P\{x_2 \le X \le x_2 + |dx_2|\}$$

$$= f_X(x_1)\,|dx_1| + f_X(x_2)\,|dx_2|$$

$$\Longrightarrow f_Y(y) = f_X(x_1)\frac{|dx_1|}{|dy|} + f_X(x_2)\frac{|dx_2|}{|dy|}$$

$$= \frac{f_X(x_1)}{\left|\frac{dy}{dx}\right|_{x=x_1}} + \frac{f_X(x_2)}{\left|\frac{dy}{dx}\right|_{x=x_2}}$$

$$= \frac{f_X(x_1)}{|h'(x_1)|} + \frac{f_X(x_2)}{|h'(x_2)|}$$

In general

$h(x_i) = y, \quad i = 1, 2, \ldots$

then

$$\boxed{f_Y(y) = \sum \frac{f_X(x_i)}{\;}}$$

$$\boxed{f_Y(y) = \sum_i \frac{f_X(x_i)}{|h'(x_i)|}}$$

Ex/ Let X be a cont. r.v. with p.d.f. $f_X(x)$.

$Y = aX + b$, where $a \neq 0$, $b \in \mathbb{R}$.

$f_Y(y) = ?$

$y = h(x) = ax + b$

then $x_1 = \frac{y-b}{a}$ is the only one real root of $y = h(x)$.

$$f_Y(y) = \frac{f_X(x_1)}{|h'(x_1)|}$$

$h'(x) = a$

$h'(x_1) = a$

$$= \frac{f_X\left(\frac{y-b}{a}\right)}{|a|}$$

$$= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

if $X \sim N(\mu, \sigma^2)$, $Y = aX + b$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$-\infty < x < \infty$

$$= \frac{1}{|a|} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{\frac{y-b}{a}-\mu}{\sigma}\right\}^2}$$

$$= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{y-(a\mu+b)}{a\sigma}\right\}^2}$$

$$Y = aX + b \sim N(a\mu+b,\ a^2\sigma^2).$$