Lecture-8:A2

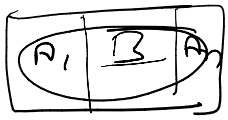
15 January 2021 11:49

Lecture-8

Total Probability thm. and Baye's form

$S = A_1 \cup A_2 \cup \text{———} A_n$ where $A_i A_j = \phi$
$i \neq j$.

then
$$\boxed{P(B) = \sum_{k=1}^{n} P(B|A_k)\,P(A_k)}$$



$$\boxed{P(A_k|B) = \frac{P(B|A_k)\,P(A_k)}{\sum_{k=1}^{n} P(B|A_k)\,P(A_k)}}$$ Baye's formula.

Ex: A person belongs to a population of 100,000 out of which 2000 persons are known to suffer for a particular cancer. A diagnostic test for this disease is known to be 95% accurate. The test results on a person X chosen randomly from this popn of 100,000 people is +ive. What is the prob. that X has this particular cancer?

Ans: Test is 95% accurate $\Longrightarrow$ 95% of all +ive tests are correct and 95% of all -ive tests are correct.

Let $\{T > 0\}$: test report is +ive

$\{T < 0\}$: test report is -ive.

$\{T<0\}$: test report is -ive.

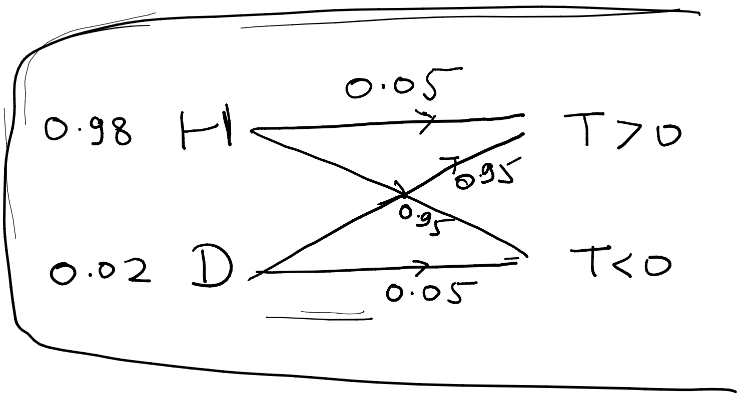
H: set of healty people
D: set of diseased people.

$$P(H) = 0.98, \quad P(D) : 0.02.$$

$$P(T>0|D) = 0.95, \quad P(T>0/H) = 0.05$$

$$P(T<0|H) = 0.95, \quad P(T<0|D) = 0.05$$

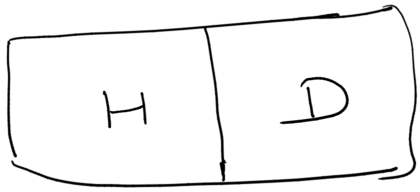
$$P(D|T>0) = ?$$



i.e. what is the prob. that the person suffers from cancer given that the test is +ive. We use Baye's thm.

$$P(D/T>0) = \frac{P(T>0/D)\,P(D)}{P(T>0)}$$

$$= \frac{P(T>0|D)\,P(D)}{P(T>0/D)\,P(D) + P(T>0/H)\,P(H)}$$



$HD = \phi$
$S = H \cup D$
H & D partitioned S here.

$$= \frac{0.95 \times 0.02}{0.95 \times 0.02 + 0.05 \times 0.98}$$

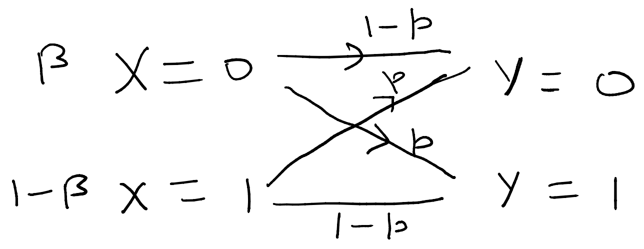
$$= \boxed{0.278}$$

$= \boxed{0.218}$

This implies that if the test is taken by someone from this pop^n without knowing whether that person has disease or not, then even a test is true, only there is 27.6% chance of having the disease. However, if a person knows that he or she has the disease, then test is 95% accurate.

<u>Remark</u> One application of Baye's theorem can be found in "Binary symmetric channel" which is a topic of Communication theory.

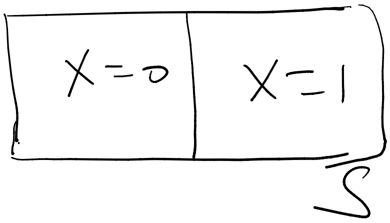
A BSC is a channel with binary input and binary output with <u>crossover</u> probability p. That is probability of error p, i.e. if X is the transmitted variable and Y is the received variable, then the channel is characterized by the conditional probability



$$\left.\begin{aligned} P(Y=0/X=0) &= 1-p \\ P(Y=0 \mid X=1) &= p \\ P(Y=1/X=1) &= 1-p \end{aligned}\right\}$$

$$P(Y=1/X=1) = 1-p$$
$$P(Y=0/X=0) = p$$
$$P(X=0) = \beta$$
$$P(X=1) = 1-\beta$$

$$P(X=1/Y=1) = ?$$



$$= \frac{P(Y=1/X=1)\,P(X=1)}{P(Y=1/X=1)\,P(X=1) + P(Y=1/X=0)\,P(X=0)}$$

$$= \boxed{\frac{(1-p)(1-\beta)}{(1-p)(1-\beta) + p\beta}}$$

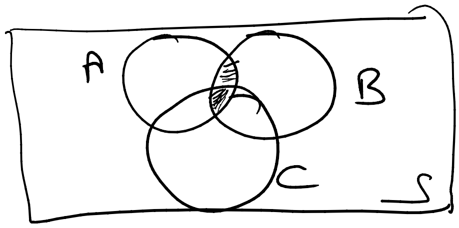
$$P(X=0/Y=1) = ? \qquad P(X=1/Y=0) = ?$$
$$P(X=0/Y=0) = ?$$

Few Properties

$$\circledast \quad P(A|B) = P(A|BC)\,P(C|B) + P(A|B\bar{C})\,P(\bar{C}|B)$$

Consider RHS

$$\frac{P(ABC)}{\cancel{P(BC)}}\,\frac{\cancel{P(BC)}}{P(B)} + \frac{P(AB\bar{C})}{\cancel{P(B\bar{C})}} \times \frac{\cancel{P(B\bar{C})}}{P(B)}$$



$$\times \frac{P(B\bar{C})}{P(B)}$$

$$= \frac{P(ABC)}{P(B)} + \frac{P(AB\bar{C})}{P(B)}$$

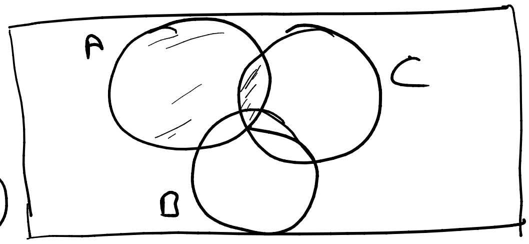
$$= \frac{P(AB)}{P(B)}$$

since $(ABC) \cap (AB\bar{C}) = \phi$
& $(ABC) \cup (AB\bar{C}) = AB$

$$= P(A|B) = \text{LHS}$$

(*) $P(A|C) > P(B|C)$ and $P(A|\bar{C}) > P(B|\bar{C})$

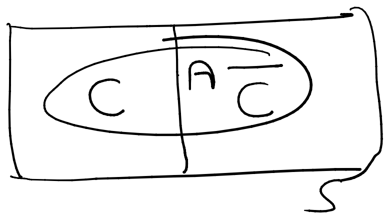
$$\implies P(A) > P(B)$$



$$P(A) = P(AC) + P(A\bar{C})$$

$$= P(A|C)P(C) + P(A|\bar{C})P(\bar{C}) \quad \text{— (1)}$$

Above can be obtainy through Total prob thm. also



Similarly,

$$P(B) = P(B|C)P(C) + P(B|\bar{C})P(\bar{C}) \quad \text{— (2)}$$

(1) — (2) $\implies$

$$P(A) - P(B) = \left[P(A|C) - P(B|C)\right] P(C)$$

$$+ \left[P(A|\bar{C}) - P(B|\bar{C})\right] P(\bar{C})$$

$$> 0$$

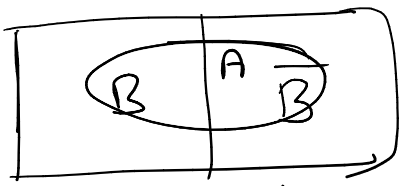
$$\Longrightarrow P(A) > P(B) \text{ —}$$

$$\circledast \quad \underline{P(A|B) > P(A)} \Longrightarrow \underline{P(A|\bar{B}) < P(A)}$$

By Total prob thm, we have

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$

$$> \underline{P(A)P(B)} + P(A|\bar{B})P(\bar{B})$$



$$\Longrightarrow P(A)[1 - P(B)] > P(A|\bar{B})\cancel{P(\bar{B})}$$

$$\underset{\cancel{P(\bar{B})}}{\shortparallel}$$

$B\bar{B} = \phi$

$B \cup \bar{B} = S$

$\Rightarrow B$ & $\bar{B}$ partitioned S.

$$\Longrightarrow P(A) > P(A|\bar{B}) = \text{RHS}$$

$$\circledast \quad P(A|B) > P(A|\bar{B}) \Longleftrightarrow P(AB) > P(A)P(B)$$

<u>LHS.</u>

$$P(A|B) > P(A|\bar{B}) \Longleftrightarrow \frac{P(AB)}{P(B)} > \frac{P(A\bar{B})}{P(\bar{B})}$$

$$\Longleftrightarrow \frac{P(AB)}{P(B)} > \frac{P(A) - P(AB)}{1 - P(B)}$$

$$\Longleftrightarrow P(AB)[1 - P(B)] > P(B)[P(A) - P(AB)]$$

$$\Longleftrightarrow P(AB) - \cancel{P(AB)P(B)} > P(A)P(B) - \cancel{P(B)P(AB)}$$

$$\Longleftrightarrow P(AB) > P(A)P(B)$$