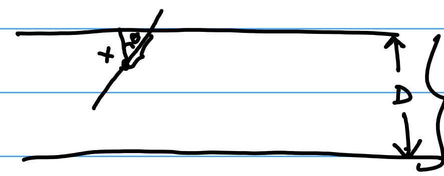
Lec-48

Tutorial - 3 (Contd...)

① Buffon's Needle Problem:



Let us determine the position of the needle by specifying the distance X from the mid point of the needle to the nearest parallel line.

$\measuredangle$ betn the projected length (X) & the needle $= \theta$.

The needle will intersect a line if the hypotenuse of the right angled triangle $< \frac{L}{2}$.

i.e, if $\frac{X}{\cos\theta} < \frac{L}{2}$ or $X < \frac{L}{2}\cos\theta$.

So X varies betn 0 and $\frac{D}{2}$.

& θ varies betn 0 and $\pi/2$.

We may assume that they are independent. Also they are uniformly distributed ran. vars over there respective ranges.

$$f_X(x) = \begin{cases} \frac{2}{D} & x \in \left(0, \frac{D}{2}\right) \\ 0 & \text{o.w.} \end{cases}$$

$$f_\Theta(y) = \begin{cases} \frac{2}{\pi}, & y \in (0, \frac{\pi}{2}) \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore P\left\{X < \frac{L}{2}\cos\Theta\right\} = \iint\limits_{x < \frac{L}{2}\cos y} f_X(x) f_\Theta(y)\,dx\,dy$$

$$= \int_{y=0}^{\pi/2} \int_{x=0}^{L/2 \cos y} \frac{2}{D} \times \frac{2}{\pi}\, dx\, dy$$

$$= \frac{4}{\pi D} \int_{y=0}^{\pi/2} \int_{x=0}^{\frac{L}{2}\cos y} dx\, dy$$

$$= \frac{2L}{\pi D} \quad \underline{\underline{\text{Ans}}}.$$

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(2) X, Y, Z ind & $U(0,1)$.

$$f_{X,Y,Z}(x,y,z) = \begin{cases} 1, & 0 < x, y, z < 1 \\ 0 & \text{o.w.} \end{cases}$$

$P(X > YZ)$

$$= \iiint\limits_{x > yz} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=yz}^{1} dx\,dy\,dz = \int_0^1\int_0^1 (1-yz)\,dy\,dz$$

$$= \int_0^1 \left(1 - \frac{z}{2}\right) dz = \frac{3}{4} \text{ Ans.}$$

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(3) Done (See Quiz-3 Soln).

(4) Let X denote the prob. that given trial is a success, then X is uniform $(0,1)$ ran. var.

Also given $X = x$, the $(n+m)$ trials are independent with common prob. of success x.

So N which is the no. of successes is a binomial ran. var. with parameters $(n+m, x)$.

Hence the conditional density of X given that $N = n$ is:

$$f_{X|N}(x|n) = \frac{P\{N=n \mid X=x\}\, f_X(x)}{P\{N=n\}}$$

$$= \frac{\binom{n+m}{n} x^n (1-x)^m}{P\{N=n\}} \qquad 0<x<1$$

$$= c\, x^n (1-x)^m.$$

where c does not depend on x.

(5) Done (Lec - 47)

(6) $E(X) = 1$, $Var(X) = 4$, $E(Y) = 2$, $Var(Y) = 1$

$\because \rho(X,Y) \le 1$

$\Rightarrow \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \le 1$

$\Rightarrow E(XY) - E(X)E(Y) \le \sqrt{4 \times 1}$

$\Rightarrow E(XY) \le 2 + 2 = 4.$

4 is the max^m possible value of $E(XY)$.

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(7), (8) Done (Lec - 47)

(9) To show $Cov(I_A, I_B) = Cov(I_{A^c}, I_{B^c})$

$Cov(I_A, I_B) = E(I_A I_B) - E(I_A)\,E(I_B)$

$= P(A \cap B) - P(A)\,P(B)$ —— (1)

$Cov(I_{A^c}, I_{B^c}) = E[I_{A^c} I_{B^c}] - E[I_{A^c}]\,E[I_{B^c}]$

$= P(A^c \cap B^c) - P(A^c)\,P(B^c)$

$= P(A \cup B)^c - (1 - P(A))(1 - P(B))$

$= (1 - P(A \cup B)) - (1 - P(A))(1 - P(B))$

$= \cancel{1} - P(A \cup B) - \cancel{1} + P(B) + P(A) - P(A)P(B)$

$= P(A \cap B) - P(A)P(B)$ $\left[\because P(A \cup B) = P(A) + P(B) - P(A \cap B)\right]$

$= \mathrm{Cov}(I_A, I_B)$ [By (1)]

$\mathrm{Var}(I_A) = E(I_A^2) - (E(I_A))^2$

$= P(A) - (P(A))^2 = P(A)(1 - P(A))$
$= P(A)\,P(A^c)$.

$\mathrm{Var}(I_{A^c}) = E[I_{A^c}^2] - E[I_{A^c}]^2$

$= P(A^c) - (P(A^c))^2$

$= 1 - P(A) - (1 - P(A))^2$

$= 1 - P(A) - 1 + 2P(A) - (P(A))^2$

$= P(A)(1 - P(A)) = P(A)P(A^c)$.

S^{ly}, $\mathrm{Var}(I_B) = \mathrm{Var}(I_{B^c})$.

$\therefore \rho(I_A, I_B) = \rho(I_{A^c}, I_{B^c})$. [Proved].

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10 (a) $E(X) = 10$

Let $g(x) = \dfrac{1}{x+1}$.

$g'(x) = -\dfrac{1}{(x+1)^2}$ $\qquad g''(x) = \dfrac{2}{(x+1)^3} > 0$

for $x > 0$

$\therefore g$ is convex on $(0, \infty)$

$\therefore$ By Jensen's inequality:

$$g(E(X)) \leq E(g(X))$$

$$\therefore E\left[\frac{1}{X+1}\right] \geq \frac{1}{E(X)+1} = \frac{1}{11}.$$

10.b $\quad h(x) = e^x \qquad g(x) = \frac{1}{1+x}.$

h is convex and g is non-decreasing

$$\therefore (h \circ g)(\lambda x + (1-\lambda)y)$$

$$= h(g(\lambda x + (1-\lambda)y))$$

$$\leq h(\lambda g(x) + (1-\lambda)g(y)) \quad [\because g \text{ is convex on } [0,\infty)]$$

$$\leq \lambda(h \circ g)(x) + (1-\lambda)(h\circ g)(y) \quad [\because h \text{ is convex } \cancel{\text{on } [0,\infty)}]$$

$$\therefore E\left[e^{\frac{1}{1+x}}\right] \geq e^{\frac{1}{1+E(x)}} = e^{\frac{1}{11}}.$$

10.c $\quad g(x) = \ln\sqrt{X} = \frac{1}{2}\ln X$

$$g''(x) = -\frac{1}{2x^2} < 0$$

$\therefore g$ is concave on $(0,\infty)$. [Jensen's for concave]

$$\therefore E[\ln\sqrt{X}] = E\left[\frac{1}{2}\ln X\right] \leq \frac{1}{2}\ln E(X)$$

$$= \frac{1}{2}\ln 10$$

(11) (a) $X_1, \ldots, X_{20} \to$ Poisson(1) ind. ran. vars.

$\therefore X = X_1 + X_2 + \ldots + X_{20} \sim$ Poisson (20).

$\therefore$ By Markov inequality: $P(X \geq \varepsilon) \leq \frac{E(x^n)}{\varepsilon^n}$.

$$P(X > 15) \leq \frac{20}{15} = \frac{4}{3}.$$

(b) Using CLT:

Each X_i has mean 1, & variance 1.

$\therefore \mathrm{Var}(X) = 20$.

$\therefore$ Std. dev of $X = \sqrt{20}$.

$\because$ Each X_i is an integer, $\therefore X$ is an integer too.

$$\therefore P(X > 15) = P(X > 15.5)$$

$$= P\left(\frac{X-20}{\sqrt{20}} > \frac{15.5-20}{\sqrt{20}}\right)$$

$$= P(Z \geq -1.01)$$

$$= P(Z \leq 1.01)$$

$$= N(1.01) \approx 0.8438 \text{ (from table)}$$

(12) Let X_i denote the outcome of the i^{th} die.

To find the prob. that at least 80 rolls are necessary for the total sum of all rolls exceeds 300.

"Difficult".

So we will calculate the "opposite" probability.

That is, we will find the probability that 79 rolls are sufficient to exceed 300.

So in our context, we need to find

$$1 - P(X_1 + X_2 + \cdots + X_{79} > 300).$$

$$E(X_i) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{7}{2}$$

$$E(X_i^2) = 1^2 \times \frac{1}{6} + \cdots + 6^2 \times \frac{1}{6} = \frac{91}{6}$$

$$\mathrm{Var}(X_i) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

Now we use CLT:

$$P(X_1 + \cdots + X_{79} > 300) = P(X_1 + \cdots + X_{79} \geq 300.5)$$

$$= P\left(\frac{X_1 + \cdots + X_{79} - 79 \times \frac{7}{2}}{\sqrt{\frac{35}{12}} \times \sqrt{79}} \geq \frac{300.5 - 79 \times \frac{7}{2}}{\sqrt{\frac{35}{12}} \times \sqrt{79}}\right)$$

$$\approx P(Z \geq 1.58)$$

$\therefore$ Desired Probability :

$1 - P(Z \geq 1.58) = P(Z \leq 1.58) \approx 0.9429$

(from table).