The LNM Institute of Information Technology
Jaipur, Rajasthan

MTH 222 Probability and Statistics

Tutorial-2

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1. Let X and Y be Poisson random variables. Assume that they are independent. Their respective parameters are λ_1 and 2. Then the conditional distribution of Y, given that $X + Y = n$ is:

$$P\{Y=k \mid X+Y=n\} = \frac{P\{Y=k,\ X+Y=n\}}{P\{X+Y=n\}}$$

$$= \frac{P\{Y=k,\ X=n-k\}}{P\{X+Y=n\}}$$

$$= \frac{P\{Y=k\}\,P\{X=n-k\}}{P\{X+Y=n\}} \quad [\because X \,\&\, Y \text{ are indep}]$$

Now $X+Y \sim \text{Poisson}(\lambda_1+2)$.

$$\therefore P\{Y=k \mid X+Y=n\} = \frac{e^{-2}\,2^k}{k!}\,\frac{e^{-\lambda_1}\,\lambda_1^{n-k}}{(n-k)!}\left[\frac{e^{-(\lambda_1+2)}\,(\lambda_1+2)^n}{n!}\right]^{-1}$$

$$= \frac{n!}{(n-k)!\,k!}\,\frac{2^k\,\lambda_1^{n-k}}{(\lambda_1+2)^n}$$

$$= \binom{n}{k}\left(\frac{2}{\lambda_1+2}\right)^k\left(\frac{\lambda_1}{\lambda_1+2}\right)^{n-k}$$

$\therefore$ Conditional distribution of Y given $X+Y=n$ is the Binomial distribution with parameters n and $\frac{2}{\lambda_1+2}$ Ans.

2. X is uniformly distributed over $(0,1)$ and Y is exponentially distributed with parameter $\lambda = 1$. Also assume that X and Y are independent. Then the distribution (i.e., cdf) of $Z = X + Y$ are:

PDF of X:

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

PDF of Y:

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0 & \text{o.w.} \end{cases}$$

$\because$ X & Y are independent, we have

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \begin{cases} 1.e^{-y}, & 0 < x < 1,\ y > 0 \\ 0 & \text{o.w.} \end{cases}$$

Let $Z := X+Y$

$$\therefore \text{CDF of } Z = F_Z(z) = P(Z \le z) = P(X+Y \le z)$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{z-x} e^{-y}\, dy \right) dx$$

$$= -\int_{x=0}^{1} \left[e^{-y} \right]_0^{z-x} dx$$

$$= \int_{x=0}^{1} (1 - e^{x-z})\, dx = \left[x - e^{-z} e^{x} \right]_0^1$$

$$= 1 - e^{-z}(e-1)$$ **Ans**

3. Two friends Rohit and Virat decide to meet at Wankhede Stadium for net practice. If each of them arrives independently at a time uniformly distributed between 7 a.m. to 8 a.m., then the probability that the first to arrive has to wait longer than 20 minutes is:

Let us denote by X and Y respectively, the time past 7 a.m. that Rohit and Virat arrive. Then X and Y are independent random variables, each of which is uniformly distributed over $(0, 60)$. [60 minutes]. 1hr

<u>Want</u>: $P\{X + 20 < Y\} + P\{Y + 20 < X\}$

$$= 2P\{X + 20 < Y\} \quad \text{(By symmetry)}$$

$$\therefore 2P\{X + 20 < Y\} = 2 \iint_{x+20<y} f(x,y)\,dx\,dy = 2 \iint_{x+20<y} f_X(x) f_Y(y)\,dx\,dy$$

$$= 2 \int_{20}^{60} \int_{0}^{y-20} \left(\frac{1}{60}\right)^2 dx\,dy = \frac{2}{(60)^2} \int_{20}^{60} (y-20)\,dy$$

$$= \frac{2}{60\times 60}\left[\frac{y^2}{2} - 20y\right]_{20}^{60}$$

$$= \frac{2}{60\times 60}\left[\frac{60\times 60}{2} - 20\times 60 - \frac{20\times 20}{2} + 20\times 20\right]$$

$$= 1 - \frac{4}{6} - \frac{1}{9} + \frac{2}{9}$$

$$= 1 - \frac{2}{3} + \frac{1}{9} = \frac{9-6+1}{9} = \frac{4}{9}$$

<u>Ans</u>

4. Let $f(x, y, z) = kxyz^2$ with $0 < x < 1$, $0 < y < 1$, $0 < z < 2$ be the joint density function of three random variables X, Y and Z. Then $P(Z > XY)$ is:

$$\int_{z=0}^{2}\int_{y=0}^{1}\int_{x=0}^{1} kxyz^2 \, dx\, dy\, dz = 1 \Rightarrow k\left[\frac{z^3}{3}\right]_0^2 \left[\frac{y^2}{2}\right]_0^1 \left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\Rightarrow k = \frac{3}{2}$$

$$\therefore P(Z > XY) = \int_0^1 \left(\int_0^1 \left(\int_{xy}^{2} kxyz^2 \, dz \right) dy \right) dx$$

$$= \int_0^1 \int_0^1 \frac{3}{2} xy \left[\frac{z^3}{3}\right]_{xy}^{2} dy\, dx$$

$$= \int_0^1 \int_0^1 \frac{3}{2} \frac{xy}{3} \left(8 - x^3y^3\right) dy\, dx$$

$$= \int_0^1 \int_0^1 \left(4xy - \frac{x^4y^4}{2}\right) dy\, dx$$

$$= \int_0^1 \left[2xy^2 - \frac{x^4y^5}{10}\right]_0^1 dx$$

$$= \left. x^2 - \frac{x^5}{50} \right]_0^1 = 1 - \frac{1}{50} = \frac{49}{50} \quad \underline{\text{Ans}}$$

5. The joint density of X and Y is given by:

$$f(x,y) = \begin{cases} e^{-x-y}, & \text{for } x > 0,\ y > 0 \\ 0, & \text{elsewhere.} \end{cases}.$$

Then the density function of $Z = X/Y$ is:

CDF of $Z := X/Y = F_{X/Y}(a) = P\left\{ \frac{X}{Y} \le a \right\}$

$$= \iint_{x/y \le a} e^{-x-y}\, dx\,dy = \int_0^{\infty}\int_0^{ay} e^{-(x+y)}\, dx\,dy$$

$$= \int_0^{\infty} \left(1 - e^{-ay}\right) e^{-y}\, dy = \left[-e^{-y} + \frac{e^{-(a+1)y}}{a+1} \right]_0^{\infty}$$

$$= 1 - \frac{1}{a+1}$$

Differentiation gives the pdf of X/Y as:

$$f_{X/Y}(a) = \frac{1}{(a+1)^2}, \quad a > 0$$ Ans