MATH-221: Probability and Statistics

Tutorial # 2 (Random Variables, PMF, PDF, CDF, Functions of Random variable)

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1. Define $F : \mathbb{R} \to [0, 1]$ by

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{x^2}{4} & \text{if} \quad 0 \leq x \leq 2 \\ 1 & \text{if} \quad x > 2 \end{cases}.$$

Show that F is a distribution function. Find pdf or pmf (if exists). Also compute $P(1 \leq X < 3)$, where X has distribution function F.

2. Let X be a random variable with distribution function F. Find the distribution function of the following random variables in terms of F. (i) $\max\{X, a\}$, where $a \in \mathbb{R}$ (ii) $|X|^{\frac{1}{3}}$ (iii) $|X|$ (iv) e^X (v) $-\ln|X|$.

3. Let X be the uniform random variable on $[0, 1]$. Then Determine pdf of (i) $\sqrt{X}$ (ii) $X^{\frac{1}{4}}$.

4. Let X be a random variable with PMF

$$f_X(x) = \begin{cases} \dfrac{x^2}{a} & \text{if} \quad x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \quad \text{otherwise} \end{cases}$$

Find a. What is the PMF of the random variable $Z = (X - a)^2$.

5. Let X be a binomial random variable with parameters (n, p). What value of p maximizes $P(X = k)$, $k = 0, 1, \ldots, n$?

6. Let X be a Poisson random variable with parameter λ. If $P(X = 1 | X \leq 1) = 0.8$, what is the value of λ?

7. Let X be a normal random variable with parameters μ and σ^2. Find (a) $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$, (b) $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$.

8. Let X have a geometric distribution with $p = 0.8$. Compute (a) $P(X > 3)$ (b) $P(4 \leq X \leq 7 \text{ or } X > 9)$ (c) $P(3 \leq X \leq 5 \text{ or } 7 \leq X \leq 10)$.

9. Let the random variable X denote the decay time of some radioactive particle and follows the exponential distribution function. Suppose λ is such that $P(X \geq 0.01) = \frac{1}{2}$. Find a number t such that $P(X \geq t) = 0.9$.

10. Let X have a normal distribution with parameters μ and $\sigma^2 = 0.25$. Find a constant c such that $P(|X - \mu| \leq c) = 0.9$. (Hint: Use Table for standard normal distribution function).

11. Consider transmission of a single bit, 0 or 1, goes through a series of n relays. Each relays flips the bit independently with probability p. Prove that

$$\text{Prob(input} = \text{output)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} p^{2k} (1 - p)^{n-2k}.$$

Where $\lfloor x \rfloor = \text{floor}(x) =$ largest integer not greater than x. (**Hint.** You can receive input as output if number of flips in the n relays are even.)