Lecture-7:A2

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Lecture-7

Remark: If we have a prob. model (S, F, P) and $M \in F$ with $P(M) > 0$ then we can define a different prob. model $(M, F \cap M, P_M)$ for conditional probability i.e.

$$(S, F, P) \longrightarrow (M, F \cap M, P_M)$$

Also note that conditional probability satisfy the following axioms for a specific M.

(1) $P_M(E) \geq 0$

(2) $P_M(S) = \dfrac{P(S \cap M)}{P(M)} = \dfrac{P(M)}{P(M)} = 1$

(3) If $E \cap F = \phi$, then

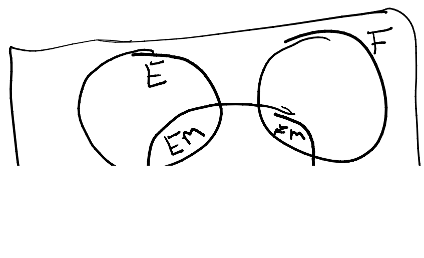
$P_M(E \cup F) = P_M(E) + P_M(F)$

Means conditional prob. f^n satisfy all 3 axioms of the ~~def~~ axiomatic def^n of the prob. that's why $(M, F \cap M, P_M)$ is also a prob. model or prob. space.

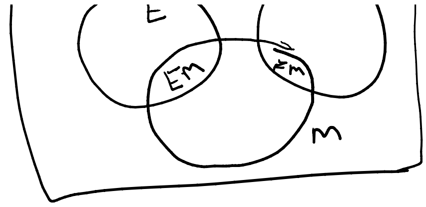
(3) axiom also can be proved easily.

$$P_M(E \cup F) = \frac{P((E \cup F)M)}{P(M)}$$

$$= \frac{P(EM \cup FM)}{P(M)}$$



$$= \frac{P(EM \cup FM)}{P(M)}$$



$$= \frac{P(EM) + P(FM)}{P(M)}$$

$(EM) \cap (FM) = \phi$

$$= \frac{P(EM)}{P(M)} + \frac{P(FM)}{P(M)}$$

$$= P(E/M) + P(F/M)$$

$$= P_M(E) + P_M(F)$$

Ex:- Consider some population records of a country.

Let t : age of a person when he/she dies.

The prob. that $t \leq t_0$ is given by

$$P(t \leq t_0) = \int_0^{t_0} p(t)\, db$$

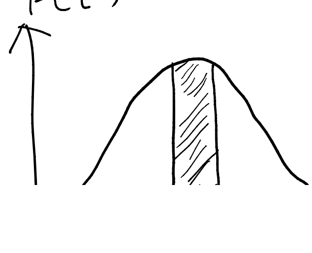
where $p(t)$ is a function determined from mortality records. We shall assume that

$$p(t) = \begin{cases} \frac{3}{10^9} t^2 (100-t)^2, & 0 \leq t \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

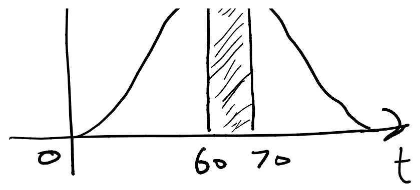
Now pick up a man at random and find the prob. that person will dies between the ages of 60 and 70.

$P(60 \leq t \leq 70)$

$$= \int^{70} p(t)\, dt = \boxed{0.154}$$



$$= \int_{60} p(t)\,dt = \boxed{0.154}$$



This can be considered equals the number of people who die between 60 and 70 divided by the total pop^n.

Now, let me assume

$$E = \{60 \le t \le 70\}, \quad M = \{t \geq 60\}$$

$$EM = E$$

The prob. that a person chosen randomly will die between the ages of 60 and 70 provided he/she was alive at 60

$$\therefore \quad P(60 \le t \le 70 / t \geq 60) = P(E/M)$$

$$= \frac{P(EM)}{P(M)} = \frac{P(E)}{P(M)}$$

$$= \frac{P(60 \le t \le 70)}{P(t \geq 60)} = \frac{\int_{60}^{70} p(t)\,dt}{1 - P(t < 60)}$$

$$= \frac{\int_{60}^{70} p(t)\,dt}{1 - \int_{0}^{60} p(t)\,dt} = \boxed{0.486}$$

This equals the number of people who die between the ages of 60 and 70 divided by number of people that are alive at age 60.

Ex 1:- A box contains white and black balls. When two balls are drawn without replacement, suppose the prob. that both are white is $\frac{1}{3}$. Then

(a) Find the smallest number of balls in the box.

(b) How small can the total number of balls be if black balls are even in number.

a	b
white balls	black balls

W_k : "a white ball is drawn at the k-th draw"

given that $P(W_1 \cap W_2) = \frac{1}{3}$.

$$P(W_1 \cap W_2) = P(W_2 \cap W_1)$$
$$= P(W_2 | W_1)\, P(W_1)$$
$$= \frac{a-1}{a+b-1} \cdot \frac{a}{a+b} = \frac{1}{3} \quad \text{--- (1)}$$

because $\frac{a}{a+b} > \frac{a-1}{a+b-1}$, $b > 0$

So from eqn (1), we have the following inequality

$$\left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \left(\frac{a}{a+b}\right)^2$$

$$\left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \left(\frac{a}{a+b}\right)^2$$

$$\Rightarrow \underbrace{\frac{a-1}{a+b-1} < \frac{1}{\sqrt{3}}}, \underbrace{\frac{1}{\sqrt{3}} < \frac{a}{a+b}},$$

$$\Rightarrow \boxed{\frac{(\sqrt{3}+1)b}{2} < a < 1 + \frac{(\sqrt{3}+1)b}{2}}$$

$$\frac{a-1}{a+b-1} < \frac{1}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}(a-1) < a+b-1$$

$$\Rightarrow (\sqrt{3}-1)a < b+\sqrt{3}-1$$

$$\Rightarrow a < \frac{b}{\sqrt{3}-1} + 1$$

$$= \frac{b(\sqrt{3}+1)}{2} + 1$$

For $b = 1$, this gives

$1.36 < a < 2.36$

$\Rightarrow a = 2$

$a = 2, \quad b = 1$

we get $P(W_2 \cap W_1) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

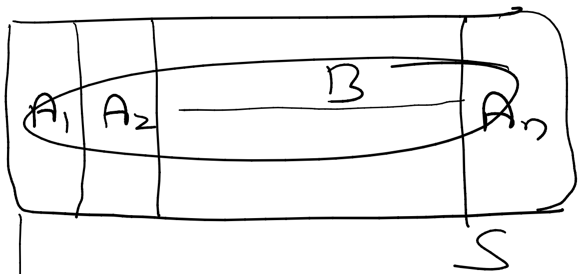
$\Rightarrow$ Thus the smallest number of balls required in the box is 3.

b	a	$P(W_2 W_1)$
× 2	3	$\frac{3}{5} \times \frac{2}{4} = \frac{3}{10} \neq \frac{1}{3}$
✓ 4	6	$\frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$

$\Rightarrow$ For even value of b, we can use above table. 10 is the smallest number of balls ($a = 6$, $b = 4$) that gives number of balls ($a = 6$, $b = 4$) that gives the desired probability.

Total Probability thm: Consider $S = A_1 \cup A_2 \cup \text{——} \cup A_n$, where $A_i A_j \neq \phi$, $i \neq j$. This is called the partition of a sample space.

Let B be an arbitrary event i.e. $B \in F$, then



$$\boxed{P(B) = \sum_{k=1}^{n} P(B|A_k) P(A_k)}$$

Proof: $B = BS = B(A_1 \cup A_2 \cup \text{——} \cup A_n)$

$$= BA_1 \cup BA_2 \cup \text{——} \cup BA_n$$

$$\Rightarrow P(B) = \sum_{k=1}^{n} P(BA_k) \qquad (BA_i)(BA_j) = \phi$$

$$= \sum_{k=1}^{n} P(B|A_k) P(A_k)$$

This result is known as Total probability thm.

Baye's Theorem: $S = A_1 \cup A_2 \cup \text{——} A_n$, where $A_i A_j = \phi$ $i \neq j$

where $A_i A_j = \phi$, $i \neq j$

Let $B \in F$, then

$$\boxed{P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{k=1}^{n} P(B|A_k)P(A_k)}}$$

Baye's formula

Proof: By Total Prob. thm, we have

$$P(B) = \sum_{k=1}^{n} P(B|A_k)P(A_k)$$

$$P(A_k|B) = \frac{P(BA_k)}{P(B)} = \frac{P(B|A_k)P(A_k)}{P(B)}$$

$$= \frac{P(B|A_k)P(A_k)}{\sum_{k=1}^{n} P(B|A_k)P(A_k)}$$

$\Rightarrow$ i.e. to evaluate n-pieces of information, we need & 2n pieces of information.