The LNM Institute of Information Technology
Jaipur, Rajsthan

P&S

Quiz-II February 19, 2020

Time: 40 Minutes Maximum Marks: 10

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1. Let X be a Poisson random variable with $P(X=1)+2P(X=0)=12P(X=2)$. Then $P(X=0)$ is

   (a) $e^{-\frac{4}{3}}$.

   (b) * $e^{-\frac{2}{3}}$.

   (c) $e^{\frac{1}{2}}$.

   (d) $e^{-\frac{1}{2}}$.

2. Let X be a Binomial random variable with parameters $(5,p)$. The value of p for which $P(|X-E(X)|\leq 3)=1$ are given by

   (a) $\dfrac{1}{5}\leq p\leq\dfrac{2}{5}$.

   (b) $\dfrac{3}{5}\leq p\leq\dfrac{4}{5}$.

   (c) $\dfrac{4}{5}\leq p\leq 1$.

   (d) * $\dfrac{2}{5}\leq p\leq\dfrac{3}{5}$.

Solution:

3. In a factory, instruments are tested one at a time until a good instrument is found. Let X denote the number of instruments that need to be tested in order to find a good one. Given that $P(X>1)=\frac{1}{2}$, then $E(X)=$

   (a) * 2.

   (b) $\dfrac{1}{2}$.

   (c) $\left(\dfrac{1}{2}\right)^2$.

(d) 0.

Solution:

4. For what value of c, the following function is a probability density function:

$$f(x) = \begin{cases} \frac{15}{64} + \frac{x}{64}, & -2 \le x \le 0, \\ \frac{3}{8} + cx, & 0 \le x \le 3, \\ 0, & \text{elsewhere} \end{cases}$$

(a) $\dfrac{1}{4}$.

(b) $-\dfrac{1}{4}$.

(c) * $-\dfrac{1}{8}$.

(d) $\dfrac{1}{2}$.

5. Let X be the time (in hours) required to repair a car, which is exponentially distributed with an average of 4 hours. Then $P(X > 10|X > 8) =$

(a) $1 - e^{-\frac{1}{4}}$.

(b) $e^{-\frac{1}{4}}$.

(c) $1 - e^{-\frac{1}{4}}$.

(d) * $e^{-\frac{1}{2}}$.

6. The random variable X has the distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \dfrac{2x^2+1}{10}, & 0 \le x < 1, \\ \dfrac{4}{5}, & 1 \le x < 2, \\ \dfrac{(x-2)^2+16}{20}, & 2 \le x < 3, \\ 1, & x \ge 3 \end{cases}$$

Then the respective values of $P(1 \le X \le 2.5)$ and $P(0.5 < X < 3)$ are

(a) * $\dfrac{41}{80}$ and $\dfrac{7}{10}$

(b) $\frac{41}{80}$ and $\frac{17}{20}$

(c) $\frac{1}{80}$ and $\frac{7}{10}$

Solution:

$$\begin{aligned}
P(1 \le X \le 2.5) &= P(X \le 2.5) - P(X < 1) = F(2.5) - F(1-) = \frac{1/4 + 16}{20} - \frac{3}{10} \\
&= \frac{65}{80} - \frac{3}{10} = \frac{41}{80}
\end{aligned}$$

$$P(0.5 < X < 3) = P(X < 3) - P(X \le 0.5) = F(3-) - F(0.5) = \frac{17}{20} - \frac{3}{20} = \frac{14}{20} = \frac{7}{10}$$

7. Let $X \sim$ Binomail $\left(2, \frac{1}{2}\right)$. Then $E\left[\dfrac{2}{1+X}\right] =$

(a) * $\frac{7}{6}$

(b) 1

(c) $\frac{6}{7}$

(d) $\frac{2}{3}$

Solution:

$$\begin{aligned}
E\left[\frac{2}{1+X}\right] &= \sum_{k=0}^{2} \frac{2}{1+k} P(X = k) = 2P(X = 0) + P(X = 1) + \frac{2}{3} P(X = 2) \\
&= 2 \cdot \frac{1}{4} + \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{4} = \frac{7}{6}
\end{aligned}$$

8. Suppose that F is a cumulative distribution function, where

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-x}, & 0 \le x < 1, \\ c, & 1 \le x < 2, \\ 1 - e^{-x}, & x \ge 2. \end{cases}$$

Then possible values of c are?

(a) * $1 - e^{-1}$

(b) $1 - e^{-0.5}$

(c) * $1 - e^{-1.5}$

(d) $1 - e^{2.5}$

Solution: For CDF F $F(1) \geq F(1-) \implies c \geq 1 - e^{-1}$ and $F(2) \geq F(2-) \implies c \leq 1 - e^{-2}$. Since e^{-x} is an strictly deceasing function on $\mathbb{R}$. $c = 1 - e^{-x}$ for any $x \in [1, 2]$

9. Let X be a normal random variable with mean 2 and variance 4, and $g(a) = P(a \leq X \leq a+2)$. The value of a that maximizes the $g(a)$ is

(a) * 1

(b) 2

(c) 0

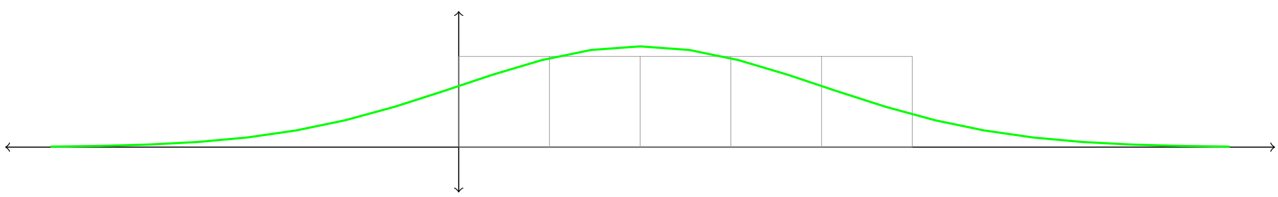
(d) 4

Solution: The CDF of random variable X is $\Phi(x) := \int_{-\infty}^{x} \frac{1}{2\sqrt{2\pi}} e^{-\frac{(t-2)^2}{8}} dt$, which differentiable on $\mathbb{R}$.

$$
\begin{aligned}
&g(a) = P(X \leq a+2) - P(X < a) = \Phi(a+2) - \Phi(a-) = \Phi(a+2) - \Phi(a) \\
&\implies g'(a) = \Phi'(a+2) - \Phi'(a) = 0 \implies \frac{1}{2\sqrt{2\pi}} e^{-\frac{a^2}{8}} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(a-2)^2}{8}} \\
&\implies a^2 = (a-2)^2 \implies a = 1
\end{aligned}
$$

So there is only one critical point $a = 1$ of g on $\mathbb{R}$.

Below is the graph of density of X, i.e. of $\Phi'(a)$ (idea about the shape is enough)



Since g' changes from $+$ to $-$ at $a = 1$, by first derivative test, g has local maximum at $a = 1$. Also we conclude that g is strictly increasing on $(-\infty, 1)$ and strictly decreasing on $(1, \infty)$. Also

$$
\begin{aligned}
\lim_{a \to -\infty} g(a) &= \lim_{a \to -\infty} \Phi(a+2) - \lim_{a \to -\infty} \Phi(a) = 0 - 0 = 0 \\
\lim_{a \to \infty} g(a) &= \lim_{a \to \infty} \Phi(a+2) - \lim_{a \to \infty} \Phi(a) = 1 - 1 = 0
\end{aligned}
$$

Therefore the point of local maximum is the point of global maximum.