Lec - 41

Recall: Complex random variables, Characteristic function, A lot of examples, Importance of char. fnt^{n}s.

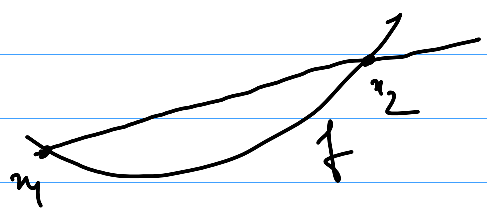
Inequalities

Defn: Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ be a function. We say that

① f is convex on I or concave upward if for any $x_1, x_2 \in I$ and $t \in (0,1)$ we have

$$f\left((1-t)x_1 + tx_2\right) \leq (1-t) f(x_1) + t f(x_2)$$

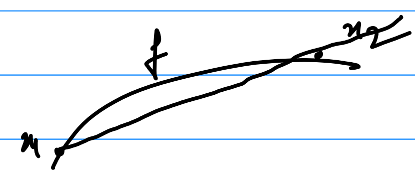
$f'' \geq 0$ on I



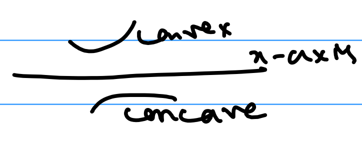
② f is concave on I or concave downward on I if for any $x_1, x_2 \in I$ and any $t \in (0,1)$ we have

$$f\left((1-t)x_1 + tx_2\right) \geq (1-t) f(x_1) + t f(x_2)$$

$f'' \leq 0$ on I



It follows from the defn that f is convex iff $-f$ is concave (reflection about the x-axis)



Theo: (Jensen's Inequality) Let $f : I \to \mathbb{R}$ be a convex function where $I \subset \mathbb{R}$ is an interval and X be a random variable s.t. X and $f(X)$ has finite mean. Then

$$f(E(X)) \leq E[f(X)].$$

If f is a concave function, then $-f$ is convex.

So by Jensen's inequality

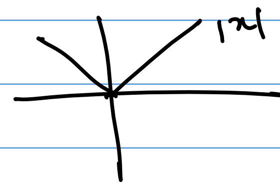
$$(-f)(E(X)) \leq E[(-f)(x)]$$

$$= -E[f(x)] \quad \text{(By linearity of expectation)}$$

$$\Rightarrow f(E(X)) \geq E[f(X)]$$

Example (1): Note that $f(x) = |x|$ is a convex fn.

$x^2, x^4, |x|$ Convex fn^{n}s.



Hence by Jensen's inequality:

$$E[X] \leq |E(X)| \leq E[|X|]$$

Defn: Let r be a positive real no. and X be a random variable. Then $E[X^r]$ is called the r^{th} moment of X about origin or central moment of X of order r.

$E[|X|^r]$ is called the rth absolute moment of X about origin or absolute central moment of X of order r.

We know from defn of expectation that, $E[X^r]$ exists and is a finite number if $E[|X|^r] < \infty$.

Therefore by example ①:

$$E[x^r] \leq |E[X^r]| \leq E[|X|^r].$$

Example ②: If the moment of order $q > 0$ exists for a random variable X, then show that moments of order p exist, where $0 < p < q$.

Soln: Let $f : (0, \infty) \to \mathbb{R}$ be defined as

$$f(x) = x^r, \text{ where } r > 1 \text{ is a real no.}$$

Then $f'(x) = r x^{r-1}$.

$f''(x) = r(r-1) x^{r-2}$.

$\because r > 1$, $f''(x) > 0$ on $(0, \infty)$.

$\Rightarrow f$ is a convex function on $(0, \infty)$.

Hence by Jensen's inequality

$$f(E(x)) = [E(x)]^r \leq [E|x|]^r \leq E[|X|^r]$$

$$\Rightarrow \left[E|X|\right] \le \left(E\left[|X|^r\right]\right)^{1/r} \quad \text{—— (1)}$$

Let $0 < p < q$. Then we take $r = \frac{q}{p} > 1$ in (1)

and we get

$$\left[E|X|\right] \le \left(E\left[|X|^{q/p}\right]\right)^{p/q} \quad \text{—— (2)}$$

Now replacing $|X|$ by $|X|^p$ in (2), we get

$$\left[E|X|^p\right] \le \left(E\left[|X|^q\right]\right)^{p/q}$$

If $E|X|^q < \infty$, then $\left(E\left[|X|^q\right]\right)^{p/q} < \infty$

and therefore $E\left(|X|^p\right) < \infty$.

<u>Example (3)</u>: Let X be a random variable with $E(X) = 10$. Show that

$$E\left[\ln\sqrt{X}\right] \le \frac{1}{2}\ln 10$$

<u>Soln:</u> Consider $f(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$, for $x \in (0,\infty)$.

Then $f'(x) = \frac{1}{2x}$ and $f''(x) = -\frac{1}{2x^2} < 0$ on $(0,\infty)$

Hence f is a concave function.

Therefore by Jensen's inequality:

$$f(E(X)) \geq E[f(X)]$$

$f(x) = \frac{1}{2}\ln x$

$$\left(\frac{1}{2}\ln(E(X))\right) \geq E\left[\frac{1}{2}\ln x\right]$$

$$\Rightarrow \frac{1}{2}\ln 10 \geq E\left[\ln\sqrt{X}\right]$$

Proved.

Markov Inequality & Chebyshev Inequality

→ They are primarily useful in situations where exact values or bounds for the mean and variance of a random variable X are easily computable, but the distribution of X is either unavailable or hard to calculate.

Theorem: (Markov Inequality) Let X be a non-negative random variable with finite n-th moment. Then we have for each $\varepsilon > 0$,

$$P\{X \geq \varepsilon\} \leq \frac{E[X^n]}{\varepsilon^n}.$$

Loosely speaking, Markov inequality asserts that if a non-negative random variable has a small n-th central moment, then the probability that it takes a large value must also be small.

As a corollary of Markov Inequality, we derive the Chebyshev Inequality.

Corollary: (Chebyshev's Inequality) Let X be a random variable with finite mean μ and finite variance σ^2. Then for every $\varepsilon > 0$,

$$P\{|X-\mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$$

Proof: Replace X by $|X-\mu|$ in Markov Inequality. Also note that

$$|X-\mu|^2 = [X-\mu]^2 .$$

Remark: ① Chebyshev's Inequality asserts that if a random variable has small variance, then the probability that it takes a value far from its mean is also small.

② Note that the Chebyshev's Inequality does not require the random variable to be non-negative.

Example (4): (Illustrating Chebyshev).

If we take $\varepsilon = 2\sigma$, then

$$P\{|X-\mu| \geq 2\sigma\} \leq \frac{\sigma^2}{4\sigma^2} = 0.25.$$

So there is at least a 75% chance that a random variable will be within 2σ of its mean, no matter what the distribution of X is.