# Lec - 27

**Recall:**
- → Random vector
- → Joint distribution (discrete)
- → Joint pmf
- → Marginal pmf

**Example:** ① X, Y → r. variables

$R(X) = \{1, 2\}$

$R(Y) = \{1, 2, 3, 4\}$

Joint pdf of $(X, Y)$:

| X \ Y | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1/4 | 1/8 | 1/16 | 1/16 |
| 2 | 1/16 | 1/16 | 1/4 | 1/8 |

$P(Y=1)$

$f_X(x) = ?$

$f_Y(y) = ?$

**Soln:** $f_X(1) = P(X=1) = f(1,1) + f(1,2) + f(1,3) + f(1,4)$

$= \frac{1}{2}$

$f_X(2) = P(X=2) = \cdots = \frac{1}{2}$

$f_Y(1) = P(Y=1) = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$

$f_Y(2) = P(Y=2) = \frac{3}{16} =$

$f_X(1) = \frac{1}{2}$ $\quad$ $f_Y(1) = 5/16 = f_Y(3)$

$f_X(2) = \frac{1}{2}$ $\quad$ $f_Y(2) = \frac{3}{16} = f_Y(4)$

$\Rightarrow$ X is uniformly distributed, whereas Y is not.

This means that marginal pmf's ~~$f$~~ $f_X$ and $f_Y$ do not completely determine the joint distribution / joint pmf of X and Y.

(i.e., the converse may not be true!)

Indeed, there are many different joint pmfs that have the same marginal pmfs.

Example ②: Define a joint pmf by:

$f(0,0) = \frac{1}{12}$, $f(1,0) = \frac{5}{12}$, $f(0,1) = f(1,1) = \frac{3}{12}$

The marginal pmf of Y is $f_Y(0) = f_Y(1) = \cancel{\frac{3}{1}}\ \frac{1}{2}$

The marginal pmf of X is: $f_X(0) = \frac{1}{3}$

$f_X(1) = \frac{2}{3}$

Define another joint pmf by: $f(0,0) = f(0,1) = \frac{1}{6}$

$f(1,0) = f(1,1) = \frac{1}{3}$.

$f_Y(0) = f_Y(1) = \frac{1}{2}$.

$f_X(0) = \frac{1}{3}$

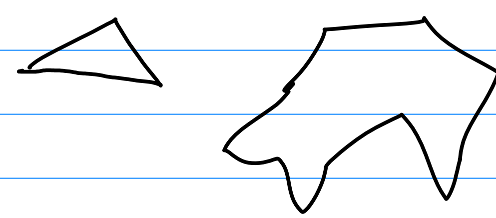
$f_X(1) = \frac{2}{3}$.

$\therefore$ Joint pmf unequal, but marginal pmfs equal.

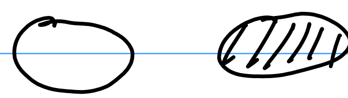
## Random Vectors with density.

**Borel set:** Examples of Borel sets in $\mathbb{R}^2$

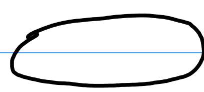
- polygons



- disks



- ellipses



- Finite or countable unions of such shapes.
- Open sets, closed sets, their finite or countable unions or intersections etc.

Recall: (1) A collection $\mathcal{A}$ of subsets of $X$ (universal set) is called an algebra of sets or a

Boolean algebra if (i) $A \cup B \in \mathcal{Q}$ whenever $A$ & $B$ are in $\mathcal{Q}$.

(ii) $A^c \in \mathcal{Q}$ whenever $A \in \mathcal{Q}$.

(2) An algebra $\mathcal{Q}$ of sets is called a $\sigma$-algebra or a Borel field if every union of countable collection of sets in $\mathcal{Q}$ is again in $\mathcal{Q}$. That is, if $\langle A_i \rangle$ is a sequence of sets, then $\bigcup_{i=1}^{\infty} A_i$ must again be in $\mathcal{Q}$.

$\therefore \bigcap_{i=1}^{\infty} A_i \in \mathcal{Q}$. (By De Morgan's law & prop (2) of defn (1))

(3) A Borel set is an elt. of a Borel $\sigma$-algebra.

Roughly speaking, Borel sets are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections.

Formally, the class $\mathcal{B}$ of Borel sets in $\mathbb{R}^2$ is the smallest collection of sets that includes the open and closed sets such that if

$E, E_1, E_2, \ldots$ are in $\mathcal{B}$, then so are $\bigcup_{i=1}^{\infty} E_i$, $\bigcap_{i=1}^{\infty} E_i$ and $E^c = \mathbb{R}^2 \setminus E$.

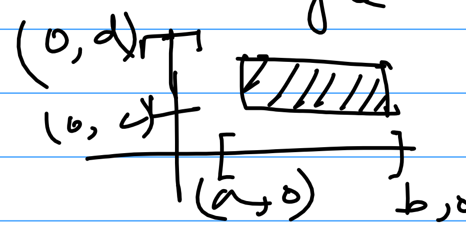
---

**Defn:** A random vector $(X, Y)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called <u>absolutely continuous</u> if there is a non-negative fnt$^n$ $f(x,y)$ defined on $\mathbb{R}^2$, called the <u>joint pdf</u> of $(X, Y)$ (sometimes just joint density of $(X, Y)$) s.t.

$$P\big((X,Y) \in S\big) = \iint_S f(x,y)\,dx\,dy,$$

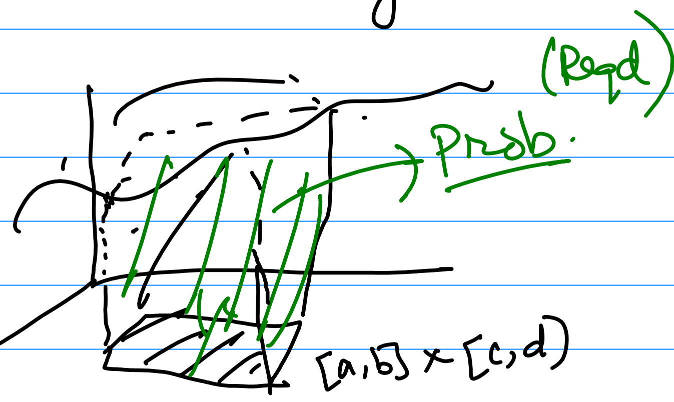
for every Borel subset $S$ of $\mathbb{R}^2$.

In particular, the probability that the value of $(X,Y)$ falls within a rectangle $[a,b] \times [c,d]$ is :-




$$P(a \le X \le b,\ c \le Y \le d) = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dx\,dy$$

This can be interpreted as the volume of the region lying below the surface $z = f(x,y)$ & above the rectangle $[a,b] \times [c,d]$.



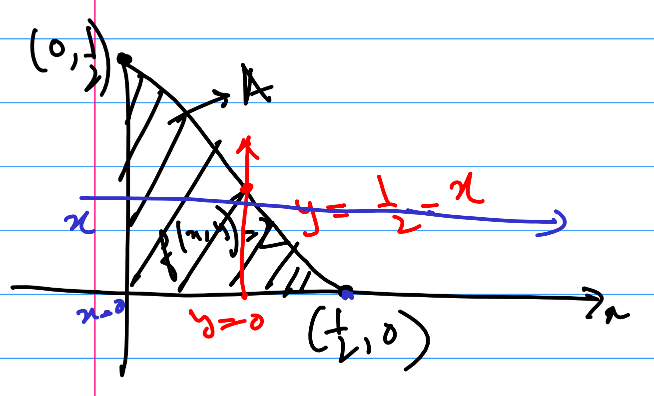

---

Example: The joint pdf of $(X, Y)$:

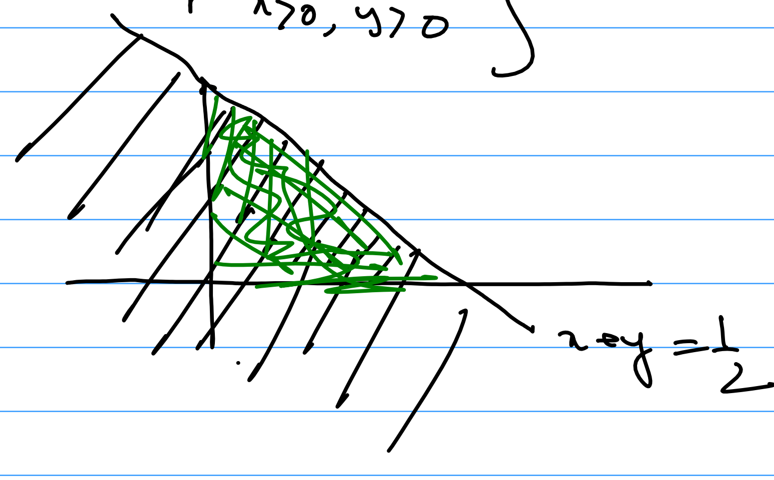
$$f(x,y) = \begin{cases} 2 & \text{if } x > 0,\ y > 0,\ 0 < x+y < 1 \\ 0 & \text{o-w.} \end{cases}$$

$$P\left(X + Y < \tfrac{1}{2}\right) = ?$$

Soln: Define: $A := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; 0 < x+y < \tfrac{1}{2},\ x > 0,\ y > 0 \right\}$

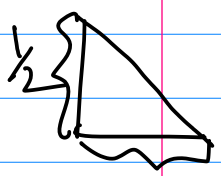



$f(x,y) = 0$



$$P\left(X+Y<\tfrac{1}{2}\right) = \iint_A f(x,y)\,dx\,dy$$

$$= \int_{x=0}^{1/2} \int_{y=0}^{\frac{1}{2}-x} (2\,dy)\,dx$$




$$= \int_{x=0}^{1/2} \left[2y\right]_0^{\frac{1}{2}-x} dx = 2\int_0^{1/2} \left(\tfrac{1}{2}-x\right) dx$$

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ = Area

Volume $= 2 \times \frac{1}{8}$

$= \frac{1}{4}$

$$= \dots = \frac{1}{4}$$ **Ans**

## Properties of joint density:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = ?$$

$$\parallel$$

$$P(-\infty < X < \infty,\ -\infty < Y < \infty)$$

$$= 1 = P(\Omega \cap \Omega)$$

Hence the joint pdf integrate to 1 on the entire plane.