Lec -39

Recall: Covariance & Correlation.

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

Properties:

(1) $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

(2) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

(3) $\mathrm{Cov}(X, aY+b) = a\,\mathrm{Cov}(X,Y)$

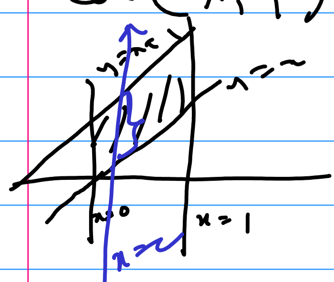
(4) $\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z)$

Recall from Lec-37:

Example (1) Joint pdf of (X,Y) is:

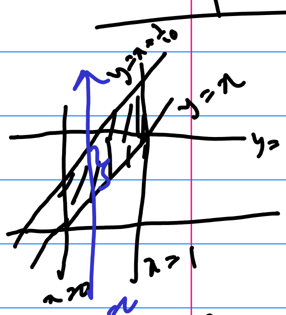
$$f(x,y) = \begin{cases} 1, & 0<x<1, \quad x<y<x+1 \\ 0 & \text{o.w.} \end{cases}$$

$\mathrm{Cov}(X,Y) = \frac{1}{12}$, Correlation $= \rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$



$= \frac{1}{\sqrt{2}}$

Example (2): Joint pdf of (X,Y) is:



$$f(x,y) = \begin{cases} 10, & 0<x<1, \quad x<y<x+\frac{1}{10} \\ 0 & \text{o.w.} \end{cases}$$

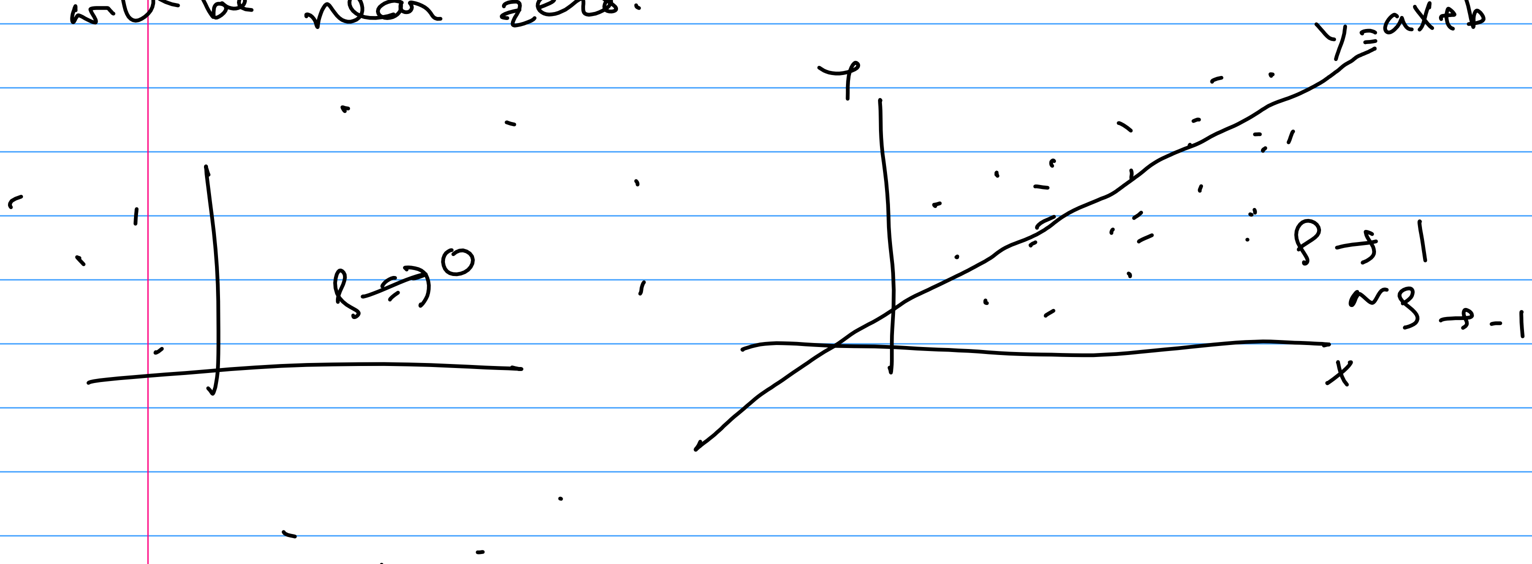
$\mathrm{Cov}(X,Y) = \frac{1}{12}$, $\rho(X,Y) = \sqrt{\frac{100}{101}}$

Theo: The correlation coefficient between two random variables X and Y satisfies the following properties:

① $|\rho(X,Y)| \leq 1$

② $|\rho(x,y)| = 1$ if and only if $\exists$ real nos. a, b with $a \neq 0$ s.t. $Y = aX + b$. If $\rho(X,Y) = 1$, then $a > 0$ and if $\rho(X,Y) = -1$, then $a < 0$.

Remark: Intuitively, if there is a line $y = ax + b$, with $a \neq 0$ s.t. values of (X,Y) have high probability of being near to this line, then the correlation between X and Y will be near 1 or -1. But if no such line exist, the correlation will be near zero.



Example ③: A standard normal ran. var. X satisfies: $E(X) = 0$, $E(X^2) = 1$, $E(X^3) = 0$,

$E(X^4) = 3$. Let $Y = a + bX + cX^2$, find the correlation coefficient $\rho(X, Y)$.

<u>Soln:</u> $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$= E(aX + bX^2 + cX^3) - 0$$

$$= aE(X) + bE(X^2) + cE(X^3)$$

$$= 0 + b \times 1 + 0 = b$$

$$Var(X) = E(X^2) - [E(X)]^2 = 1$$

$$Var(Y) = E(Y^2) - (E(Y))^2$$

$$= E(a^2 + b^2X^2 + c^2X^4 + 2abX + 2acX^2 + 2bcX^3) - (E(a + bX + cX^2))^2$$

$$= \cancel{a^2} + b^2 + 3c^2 + \cancel{2ac} - \cancel{a^2} - c^2 - \cancel{2ac}$$

$$= b^2 + 2c^2$$

$$\therefore \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{b}{\sqrt{b^2 + 2c^2}} \text{ Ans}$$

<u>When correlation fails:</u>

Covariance & correlation measures only a particular kind of linear relationship.

But it may happen that X and Y have a strong relationship but their covariance & correlation are small or even zero, because the relationship is not linear.

In the above example (3), we saw that

$$|\rho(x,y)| = \left|\frac{b}{\sqrt{b^2+2c^2}}\right| \le \frac{|b|}{\sqrt{2}|c|}$$

$\therefore$ If b is small and c is large, then correlation is small. $\left[\because |\rho(x,y)| \le \frac{|b|}{\sqrt{2}|c|} \quad \begin{matrix} b \to \text{small} \\ c \to \text{large} \end{matrix}\right.$

If $b=0$, then $Cov(X,Y)=0$ and $\rho(X,Y)=0$

but $Y = a + \underset{0}{\underset{\shortparallel}{b}}X + cX^2 = a + cX^2$

X & Y are not linear.

Complex-valued Random Variables

A complex valued random variable

$Z : \Omega \longrightarrow \mathbb{C}$ can be written in the form $Z = X + iY$, where X & Y are real-valued random

(Ω, F, P)

$\Omega \to \mathbb{R}$. Till now

What if $F: \Omega \to \mathbb{C}$.

variables. Its expectation $E(Z)$ is defined as: $E(Z) = E(X+iY) = E(X) + i\,E(Y)$, provided $E(X)$ and $E(Y)$ are well-defined and finite. The formula $E(a_1 Z_1 + a_2 Z_2)$

$$= a_1 E(Z_1) + a_2 E(Z_2)$$

is valid whenever a_1 & a_2 are complex constants and Z_1 and Z_2 are complex-valued random variables having finite expectation.

Characteristic Function

→ Serves as an important tool for analyzing random phenomenon.

Defn: The characteristic function of a random variable X is defined as:

$$\Phi_X(t) = E\left[e^{itX}\right], \quad t \in \mathbb{R}.$$

So basically $\Phi_X : \mathbb{R} \to \mathbb{C}$.

The advantage of the characteristic fntn is that it is defined for all real-valued random variables. Because for any real-valued random variable X and for any real no. t, the ran. var. $\cos tX$, $\sin tX$ are bounded by 1.

Therefore, both have finite expectation bounded by 1, hence $\Phi_X(t)$ is defined for all t and for all X.

Characteristic Function of a Discrete random variable:

If X is a discrete random variable, then

$$\Phi_X(t) = E[e^{itX}] = E[\cos tX + i\sin tX]$$

$$= E[\cos tX] + i\,E[\sin tX]$$

$$= \sum_{x\in R(X)} \cos(tx)\,P(X=x) + i\sum_{x\in R(X)} \sin(tx)\,P(X=x)$$

$$= \sum_{x\in R(X)} [\cos(tx) + i\sin(tx)]\,P(X=x)$$

$$= \sum_{x\in R(X)} e^{itx}\,P(X=x)$$

Characteristic Function of a random variable with density:

If the ran. var. X has density f_X, then

$$\Phi_X(t) = E[e^{itX}] = E[\cos(tX)] + i\,E[\sin(tX)]$$

$$= \int_{-\infty}^{\infty} \cos(tx) f_X(x)dx + i \int_{-\infty}^{\infty} \sin(tx) f_X(x)dx$$

$$= \int_{-\infty}^{\infty} [\cos(tx) + i\sin(tx)] f_X(x)dx = \int_{-\infty}^{\infty} e^{itx} f_X(x)dx$$

Example ④: Let $X \sim \text{Bernoulli}(p)$. Find its characteristic function.

Soln: $\Phi_X(t) = E[e^{itX}] = e^{itx}P(\underset{x}{X=1}) + e^{itx}P(\underset{x}{X=0})$

$$= e^{it}P(X=1) + e^{0}P(X=0)$$

$$= e^{it}p + (1-p)$$

Ans.