Lec-47

Revision:

Computing Expectations by Conditioning:

Let $E[X|Y]$ be a fntn of the ran. var. Y whose value at $Y=y$ is $E[X|Y=y]$.

Note: $E[X|Y]$ is itself a random variable.

Property (1): $\boxed{E[X] = E[E[X|Y]]}$

Proof: Y discrete: $E(X) = \sum_y E[X|Y=y]\,P\{Y=y\}$
(To show)

$$RHS = \sum_y E[X|Y=y]\,P\{Y=y\}$$

$$= \sum_y \left(\sum_x x\,P\{X=x|Y=y\}\right) P(Y=y)$$

$$= \sum_y \sum_x x\,\frac{P\{X=x, Y=y\}}{\cancel{P\{Y=y\}}}\,\cancel{P\{Y=y\}}$$

$$= \sum_y \sum_x x\,P\{X=x, Y=y\}$$

$$= \sum_x x \sum_y P\{X=x, Y=y\} = \sum_x x\,P(X=x)$$

$$= E(X).$$

Y continuous: (Exercise!) $E(X) = \int_{-\infty}^{\infty} E[X \mid Y=y] f_Y(y)\, dy$.

<u>Example (1)</u>: Suppose that the no. of people entering a department store on a given day is a ran. var. with mean 50. Suppose further that the amounts of money spent by these customers are independent ran. var-s having a common mean of \$8. Assume also that the money spent by a customer is also independent of the total no. of customers to enter the store. What is the expected amount of money spent in the store on a given day?

<u>Soln</u>: Let N denote the no. of customers that enter the store & X_i the amount of money spent by the i^{th} such customer.

$\therefore$ The total amount of money spent $= \sum_{i=1}^{N} X_i$.

So $E\left[\sum_{i=1}^{N} X_i\right] = E\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right]$

But $E\left[\sum_{1}^{N} X_i \mid N=n\right] = E\left[\sum_{1}^{n} X_i \mid N=n\right]$

$= E\left[\sum_{1}^{n} X_i\right]$ [$\because$ X_i's and N are indep.]

$= n\,E(X)$ where $E(X) = E(X_i) = 8$

$$\Rightarrow E\left[\sum_{1}^{N} X_i \mid N\right] = N\,E(X).$$

Thus $E\left[\sum_{i=1}^{N} X_i\right] = E\left[N\,E(X)\right] = E(N)\,E(X)$

$\therefore$ The expected amount of money spent in the store is $50 \times 8 = 400\$$.

Property (2): Computing variance with conditional variance.

We have:

$$\mathrm{Var}(X \mid Y) = E\left[\left(X - E[X \mid Y]\right)^2 \mid Y\right]$$

That is, $\mathrm{Var}(X \mid Y)$ is equal to the (conditional) expected square of the difference betn X and its (conditional) mean when the value of Y is given.

Or in other words, $\mathrm{Var}(X \mid Y)$ is exactly analogous to the usual defn of variance but now all expectations are conditional on the fact that Y is known.

We use this to obtain $\mathrm{Var}(X)$.

$$\because \mathrm{Var}(X) = E(X^2) - (E(X))^2$$

$$\therefore \mathrm{Var}(X \mid Y) = E(X^2 \mid Y) - (E(X \mid Y))^2$$

Taking expectation on both sides:

$$E\left[Var(X|Y)\right] = E\left[E[X^2|Y]\right] - E\left[(E(X|Y))^2\right]$$

$$= E(X^2) - E\left(\left(E[X|Y]\right)^2\right) \quad \text{— } ①$$

(By property ①
$E(X^2) = E(E(X^2|Y))$)

Also as $E\left[E(X|Y)\right] = E(X)$, we have:

$$Var\left(E[X|Y]\right) = E\left[\left(E[X|Y]\right)^2\right] - \left(E[X]\right)^2 \quad \text{— } ②$$

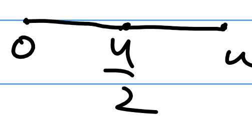
$\therefore$ ① + ② gives:

$$\boxed{Var(X) = E\left[Var(X|Y)\right] + Var\left(E[X|Y]\right)}$$

Tutorial - 3

Q.⑤ $U \sim \text{Uniform}(0,1)$ & $V \sim \text{Uniform}(0,U)$

ⓐ $E[V|U=u] = \dfrac{u}{2}$.



ⓑ $Var[V|U=u] = \dfrac{u^2}{12}$.

ⓒ $E[V] = E\left[E[V|U]\right] = E\left[\dfrac{U}{2}\right] = \dfrac{1}{2}E[U]$

$= \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{4}$

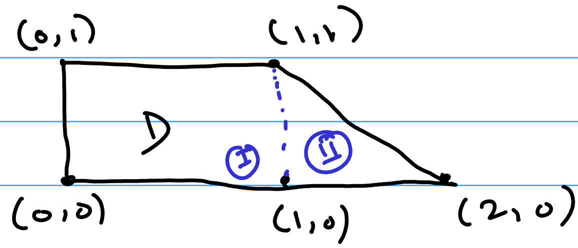
(d) Note that $E[U^2] = Var(U) + (E[U])^2$

$= \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

$Var(V) = E[Var(V|U)] + Var(E[V|U])$

$= E\left[\frac{U^2}{12}\right] + Var\left(\frac{U}{2}\right) = \frac{1}{36} + \frac{1}{4} \times \frac{1}{12} = \frac{7}{144}$

Q. 7



Area of D := Area of (I) + Area of (II)

$= 1 + \frac{1}{2} \times 1 \times 1$

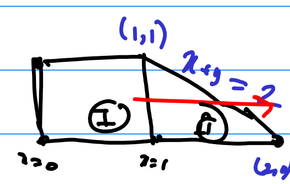
$= 1 + \frac{1}{2} = \frac{3}{2}$.

$\therefore$ PDF of (X, Y)

$f_{X,Y}(x,y) = \frac{1}{\text{Area of } D}$ for points inside D & 0 outside.

$$\therefore f_{X,Y}(x,y) = \begin{cases} \frac{2}{3} & (x,y) \in D \\ 0 & \text{o.w.} \end{cases}$$

$Cov(X,Y) = E(XY) - E(X)E(Y)$.



$$\therefore E(XY) = \underbrace{\int_{y=0}^{1}\int_{x=0}^{1} f(x,y)\,xy\,dx\,dy}_{\text{I}} + \underbrace{\int_{y=0}^{1}\int_{x=1}^{2-y} f(x,y)\,xy\,dx\,dy}_{\text{II}}$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} \frac{2}{3} xy \, dx\, dy + \frac{2}{3}\int_{y=0}^{1}\left(\int_{x=y}^{2-y} xy \, dx\right) dy$$

$$= \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} + \frac{2}{3}\int_{y=0}^{1}\left[\frac{x^2 y}{2}\right]_{1}^{2-y} dy$$

$$= \frac{11}{36}$$

$E(X) = \ldots$

$E(Y) = \ldots$

$\mathrm{Cov}(X,Y) = \ldots$

Q. 8

$X \sim \mathrm{Uni}[-a, a]$

$$\therefore f(x) = \begin{cases} \frac{1}{2a}, & x \in [-a, a] \\ 0 & \text{o.w.} \end{cases}$$

To show $\mathrm{Cov}(X,Y) = 0$ where $Y = X^2$.

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$
$$= E(X^3) - E(X)E(X^2).$$

$$E(X^3) = \int_{-a}^{a} x^3 \frac{1}{2a}\, dx = 0$$

$\because \frac{1}{2a}$ is const. in x & therefore symmetric about $x=0$, then every odd moment of X will be zero. That is,

$$E[X] = E[X^3] = \cdots = E[X^{2n+1}] = 0$$

$n = 0, 1, 2, 3, \ldots$

$\therefore \mathrm{Cov}(X,Y) = 0 - 0 \times E(X^2) = 0$,

$\therefore \rho = 0$, i.e., X & Y are uncorrelated.

Q. 9

$$\mathrm{Cov}(I_A, I_B) = E[I_A I_B] - E[I_A]E[I_B]$$

$$= P(A \cap B) - P(A)P(B)$$

$I_A = 1$ A occurs
$\quad\;\; 0$ A^c occurs

$E(I_A) = 1\,P(A) + 0\,P(A^c)$
$= P(A)$.

$E(I_A I_B) = 1\,P(A \cap B) + 0 \ldots$

$\therefore 1 - P(A \cup B)^c = 1 - P(A^c \cap B^c)$

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$\Rightarrow \mathrm{Cov}(I_A, I_B) = P(A) + P(B) - (1 - P(A^c \cap B^c)) - P(A)P(B)$

Now $P(A) = 1 - P(A^c)$
$P(B) = 1 - P(B^c)$

$\therefore \mathrm{Cov}(I_A, I_B) = P(A) + P(B) - 1 + P(A^c \cap B^c) - (1 - P(A^c)) \times (1 - P(B^c))$

.... Complete it.