Lec-23B

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Lecture - 23

X : Continuous

$y = h(x)$ continuous and having countable number of solutions say $x_1, x_2, \text{———}$ i.e.

$$y = h(x_i), \quad i = 1, 2, 3 \text{ ———}$$

then $Y = h(X)$

$$f_Y(y) = \sum_i \frac{f_X(x_i)}{|h'(x_i)|}$$

here x_i need to be written in terms of y.

Ex Let X be a random variable

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}$$

here X is called Cauchy random variable.

$$y = h(x) = \frac{1}{x}$$

$$Y = \frac{1}{X} \qquad f_Y(y) = ?$$

Ans $R_X : (-\infty, \infty)$

$y = h(x)$ i.e. $y = \frac{1}{x}$ have a

$y = h(x)$ i.e. $y = \frac{1}{x}$ have a unique sol^n say $x_1 = \frac{1}{y}$.

$Y = \frac{1}{X}$ is defined everywhere except $X = 0$. But $P(X = 0) = 0$ as X is absolutely continuous.

$R_Y : (-\infty, \infty) \setminus \{0\}$

$$\Rightarrow f_Y(y) = \frac{f_X(x_1)}{|h'(x_1)|} \quad \text{where } x_1 = \frac{1}{y}$$

$$= \frac{f_X\left(\frac{1}{y}\right)}{\left|h'\left(\frac{1}{y}\right)\right|} \qquad h'(x) = -\frac{1}{x^2}$$

$$= \frac{\frac{1}{\pi\left(1+\frac{1}{y^2}\right)}}{\left|-\frac{1}{\left(\frac{1}{y}\right)^2}\right|}$$

$$= \frac{\frac{y^2}{\pi(1+y^2)}}{y^2} = \frac{1}{\pi(1+y^2)}$$

if $y \neq 0$

pdf. is not affected if we change

p.d.f. is not affected if we change its value at a point.

$\Rightarrow f_Y(0) = \frac{1}{\pi}$

$\Rightarrow f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad \underline{y \in (-\infty, \infty)}$

Here, we can see that Y is also Cauchy's r.v.

Ex:- $X \sim \exp(\lambda)$, $y = h(x) = \sqrt{x}$

$Y = \sqrt{X} \quad f_Y(y) = ?$

$R_X : [0, \infty)$

$R_Y : [0, \infty)$

$y = h(x)$ has a unique sol^n s.t.

$x^2 = y$.

$h'(x) = \frac{1}{2\sqrt{x}}$,

$y = 0$, $h'(y^2) = \frac{1}{2y}$

| $y = \sqrt{x}$
| $x_1 \to$ any x_1 s.t.
| $y^2 = x_1$

for any $y \in R_Y$ we have a unique

sol^n $x = ?$ 2

For any $y \in R_Y$ we have a unique

sol^n $x \in R_X$ s.t. $y = x^2$.

$$f_Y(y) = \frac{f_X(x_1)}{|h'(x_1)|}$$

$$= \frac{f_X(y^2)}{|h'(y^2)|}$$

$$= \frac{\lambda e^{-\lambda y^2}}{\frac{1}{2y}} \qquad y > 0$$

$$= 2\lambda y e^{-\lambda y^2}, \quad y > 0.$$

Note that a pdf is not affected if we change its value at a point. Thus we define

$$f_Y(0) = 0.$$

and for $y < 0$, we have

$$f_Y(y) = 0$$

$$\Rightarrow f_Y(y) = \begin{cases} 0, & y \leq 0 \\ 2\lambda y e^{-\lambda y^2} & y > 0. \end{cases}$$

$R_Y : [0, \infty)$

$(0, \infty)$

Ex: Consider

$$X : U[-1, 2]$$

$y = h(x) = x^2, \quad Y = X^2 \quad f_Y(y) = ?$

If $y < 0$, $y = x^2$ has no real solⁿ so.

$$f_Y(y) = 0, \text{ if } y < 0.$$

When $y \geq 0$, then $y = x^2$ has two solⁿ say $x_1 = \sqrt{y}$, and

$x_2 = -\sqrt{y}$.

$y = h(x)$ has derivative $h'(x) = 2x$.

$\Rightarrow h'(x_1) = 2\sqrt{y}$ & $h'(x_2) = -2\sqrt{y}$.

$\Rightarrow \quad h'(x_1) = 2\sqrt{y} \quad \& \quad h'(x_2) = -2\sqrt{y}.$

so

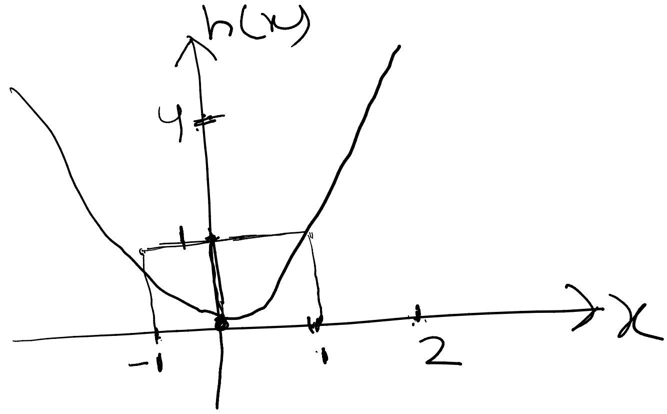
$$f_Y(y) = \sum_{i=1}^{2} \frac{f_X(x_i)}{|h'(x_i)|} \quad \text{for } y \geq 0.$$

Given $X \sim U[-1, 2]$

$$f_X(x) = \begin{cases} \frac{1}{3}, & -1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$R_X : [-1, 2].$

$R_Y : [0, 4]$



so

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right], \quad y \geq 0$$

$$= \begin{cases} \frac{1}{2\sqrt{y}} \left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right], & 0 \leq y \leq 1 \\ \frac{1}{2\sqrt{y}} \left[f_X(\sqrt{y}) + 0 \right] & 1 \leq y \leq 4 \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{y}}\left[\frac{1}{3} + \frac{1}{3}\right] & 0 \le y \le 1 \\ \frac{1}{2\sqrt{y}}\left[\frac{1}{3} + 0\right] & 1 \le y \le 4 \end{cases}$$

$$= \begin{cases} \frac{1}{3\sqrt{y}}, & 0 \le y \le 1 \\ \frac{1}{6\sqrt{y}}, & 1 \le y \le 4 \end{cases}$$

Thus we have

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}}, & 0 \le y \le 1 \\ \frac{1}{6\sqrt{y}}, & 1 \le y \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

$$\int_{\mathbb{R}} f_Y(y)\, dy = 1$$

$$\frac{1}{3}\int_0^1 \frac{1}{\sqrt{y}}\, dy + \frac{1}{6}\int_1^4 \frac{1}{\sqrt{y}}\, dy$$

$$= \frac{1}{3}\left[2\sqrt{y}\right]_0^1 + \frac{1}{6}\left[2\sqrt{y}\right]_1^4$$

$$= \frac{2}{3} + \frac{2}{3} - \frac{1}{3} = \frac{4-1}{3} = \underline{1}$$

Ex: $X : U[-1, 1]$

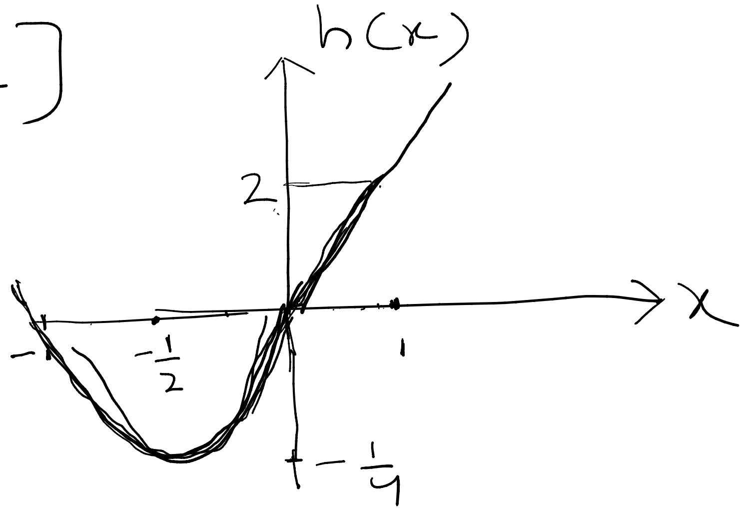
$h(x) = x^2 + x$, $\quad Y = X^2 + X$

$f_X(x) = \begin{cases} \frac{1}{2}, & -1 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$ $\quad f_Y(y) = ?$

$h(x) = x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$

$Y = \left(X + \frac{1}{2}\right)^2 - \frac{1}{4}$

$R_Y : \left[-\frac{1}{4}, 2\right]$



$y = h(x)$ has two solutions namely

$$x_1 = \frac{-1 - \sqrt{1 + 4y}}{2}$$

$$x_2 = \frac{-1 + \sqrt{1 + 4y}}{2}$$