MATH-221: Probability and Statistics

Tutorial # 1 (Countable & uncountable sets, Properties of Probability Measure, Conditional Probability, Total Probability Theorem, Baye's Theorem)

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1. Consider the random experiment of tossing a coin indefinitely. Show that the corresponding sample space is uncountable.

Solution: the same space Ω consists infinite sequences of Hs and Ts, i.e.,

$$\Omega = \{\omega = (\omega_1, \omega_2, \cdots) : \omega_i \in \{H, T\} \text{ for each } i = 1, 2, \cdots\}$$

We could just view heads and tails in a coin toss as 1 and 0, respectively. Then the set Ω could be rewritten as $\Omega = \{(a_n)_{n\geq 1} : a_n \in \{0, 1\} \text{ for each } n = 1, 2, \cdots\}$.

Claim. *The set Ω is uncountable.*

It is clear that Ω is infinite. Now suppose contrary that Ω is countable. Then we can enumerate it's element in a sequence $s_1, s_2, s_3, \cdots$. Now we construct a sequence s as follows: If the nth term in s_n is 1, we let nth term of s be 0, and vice versa. Then the sequence s differs from each of s_n. But by construction $s \in \Omega$, which is a contradiction.

2. Let A, B, C be events such that $P(A) = 0.7, P(B) = 0.6, P(C) = 0.5, P(A \cap B) = 0.4, P(A \cap C) = 0.3, P(C \cap B) = 0.2$ and $P(A \cap B \cap C) = 0.1$. Find $P(A \cup B \cup C), P(A^c \cap C)$ and $P(A^c \cap B^c \cap C^c)$.

Solution:

$$\begin{aligned}
P(A \cup B \cup C) &= P(A \cup B) + P(C) - P((A \cup B) \cap C) \\
&= P(A) + P(B) - P(A \cap B) + 0.5 - P((A \cap C) \cup (B \cap C)) \\
&= 0.7 + 0.6 + 0.5 - 0.4 - [P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))] \\
&= 1.4 - [0.3 + 0.2 - P(A \cap B \cap C)] \\
&= 1.4 - 0.4 = 1 \\
P(A^c \cap C) &= P(C \setminus (A \cap C)) \\
&= P(C) - P(A \cap C) = 0.5 - 0.3 = 0.2 \\
P(A^c \cap B^c \cap C^c) &= P\left((A \cup B \cup C)^c =\right) = 1 - P(A \cup B \cup C) = 0
\end{aligned}$$

3. Prove or disprove: If $P(A \cap B) = 0$ then A and B are mutually exclusive events.

Solution: Statement is False:
Consider $\Omega = [0, 1]$ and "length" be the probability measure P. Then $P[0, 1/2] = 0.5, P[1/2, 1] = 0.5$. Then $A \cap B = \{1/2\}$ hence $P(A \cap B) = 0$.

4. Does there exists a probability measure (or function) P such that the events A, B, C satisfies $P(A) = 0.6, P(B) = 0.8, P(C) = 0.7, P(A \cap B) = 0.5, P(A \cap C) = 0.4, P(C \cap B) = 0.5$ and $P(A \cap B \cap C) = 0.1$?

Solution: No.

$$\begin{aligned}P((A \cap B) \cup C) &= P(A \cap B) + P(C) - P((A \cap B) \cap C)\\ &= P(A \cap B) + P(C) - P(A \cap B \cap C) = 1.1\end{aligned}$$

Which is a contradiction. Similarly one can show $P((A \cap C) \cup B) = 1.1$.

5. For any events A and B, show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence conclude

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

6. Let $\Omega = \mathbb{N}$. Define a set function P as follows: For $A \subset \Omega$,

$$P(A) = \begin{cases} 0 & \text{if} \quad A \text{ is finite} \\ 1 & \text{if} \quad A \text{ is infinite} \end{cases}.$$

Is P a probability measure (or function)?

Solution: P is not a probability measure. Let us assume that P is a probability measure. Take $A = \{2n : n \in \mathbb{N}\}$ and $B = \{2n - 1 : n \in \mathbb{N}\}$. Then $A \cap B = \phi$ hence $P(A \cup B) = P(\mathbb{N}) = P(A) + P(B) = 2$, which is absurd. Hence P is not a probability measure.

7. **(Continuity of Probability Measure)** Let $A_n, n \geq 1$ be a sequence of events. Then prove the following:

   (a) If $A_1 \subset A_2 \subset \cdots$ Then $P\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} P(A_k)$.

   (b) If $A_1 \supset A_2 \supset \cdots$ Then $P\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} P(A_k)$.

Solution: (a) Suppose $A_1 \subset A_2 \subset \cdots$ and $A := \bigcup_{k=1}^{\infty} A_k$. Set $B_1 = A_1$, and for each $n \geq 2$, let B_n denote those points which are in A_n but not in A_{n-1} , i.e., $B_n = A_n \setminus A_{n-1}$. By definition, the sets B_n are disjoint. Also $A_n = \bigcup_{k=1}^{n} B_k$ and $A = \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$. Hence

$$P(A_n) = \sum_{k=1}^{n} P(B_k)$$

Since the left side above cannot exceed 1 for all n, $P(B_k) \geq 0$ for all k, so sequence of partial sums is increasing and bounded above hence the series on the right side must converge. Hence we obtain

$$\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k) =: \sum_{k=1}^{\infty} P(B_k) = P(A). \tag{1}$$

(b) Now if $A_1 \supset A_2 \supset \cdots$ Then $A_1^c \subset A_2^c \subset \cdots$. Hence by part (a),

$$\begin{aligned} P\left(\bigcup_{k=1}^{\infty} A_k^c\right) &= \lim_{k\to\infty} P(A_k^c) \\ 1 - P\left[\left(\bigcup_{k=1}^{\infty} A_k^c\right)^c\right] &= \lim_{k\to\infty} [1 - P(A_k)] \\ 1 - P\left(\bigcap_{k=1}^{\infty} A_k\right) &= 1 - \lim_{k\to\infty} P(A_k) \\ P\left(\bigcap_{k=1}^{\infty} A_k\right) &= \lim_{k\to\infty} P(A_k) \end{aligned}$$

8. Let $A_1, A_2, \cdots$ be a sequence of events then show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

Solution: By finite sub-additivity of probability measure we have

$$P\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} P(A_k) \text{ for every } n \in \mathbb{N}.$$

Since $P(A_k) \geq 0$ for all $k \in \mathbb{N}$,

$$\sum_{k=1}^{n} P(A_k) \leq \sum_{k=1}^{\infty} P(A_k) \text{ for every } n \in \mathbb{N}.$$

Therefore we have

$$P\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{\infty} P(A_k) \text{ for every } n \in \mathbb{N}. \tag{2}$$

Now in order show that $\lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} A_k\right) = P\left(\bigcup_{k=1}^{\infty} A_k\right)$ we appeal to continuity property of probability measure (Problem 7). Set $B_n := \bigcup_{k=1}^{n} A_k$. Then $B_1 \subseteq B_2 \subseteq \cdots$ and are in $\mathcal{F}$. Also observe that

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Hence

$$\begin{aligned} P\left(\bigcup_{n=1}^{\infty} A_n\right) &= P\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \lim_{n\to\infty} P\left(B_n\right) \\ &= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} A_k\right) \end{aligned}$$

Taking limit as $n \to \infty$ in (2) we get the desired result.

Note that the infinite series $\sum_{k=1}^{\infty} P(A_k)$ may diverge to $+\infty$. Of course inequality remains true in this case also !!

9. Let Ω be a nonempty set and P be a function from set of subsets of Ω to $[0,1]$ such that

   (a) $P(\Omega) = 1$.

   (b) For A and B disjoint, $P(A \cup B) = P(A) + P(B)$.

   (c) If (A_n) is a decreasing sequence of events such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then

   $$\lim_{n\to\infty} P\left(A_n\right) = 0.$$

   Show that P is a probability measure.

Solution: Let $A_1, A_2, \cdots \in \mathcal{F}$ be pairwise disjoint. Then define the following sequence of set

$$B_1 = \bigcup_{n=1}^{\infty} A_n, B_2 = \bigcup_{n=2}^{\infty} A_n, \cdots B_k = \bigcup_{n=k}^{\infty} A_n, \cdots$$

Clearly $B_1 \supseteq B_2 \cdots \supseteq B_n \supseteq \cdots$ and each $B_n \in \mathcal{F}$. Also $\bigcap_{n=1}^{\infty} B_n = \phi$. Hence we have

$$0 = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right)$$

Writing for $n \geq 2$

$$\begin{aligned} P\left(\bigcup_{k=1}^{\infty} A_k\right) &= P\left(\bigcup_{k=1}^{n-1} A_k\right) + P\left(\bigcup_{k=n}^{\infty} A_k\right) \quad (P \text{ is given to be finitely additive}) \\ &= \sum_{k=1}^{n-1} P(A_k) + P\left(\bigcup_{k=n}^{\infty} A_k\right) \end{aligned}$$

Left hand side is a constant (does not depend on n) and $\lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) = 0$, so series $\sum_{k=1}^{\infty} P(A_k)$ converges. Hence taking limit as $n \to \infty$ above we get the countable additivity of P.

10. Three switches connected in parallel operate independently. Each switches remains closed with probability p. Then (a) Find the probability of receiving an input signal at the output. (b) Find the probability that switch S_i is open given that an input signal is received at the output.

11. Suppose an individual applying to a college determines that he has an 80% chance of being accepted, and he knows that dormitory housing will only be provided for 60% of all of the accepted students. What is the probability that a student will be accepted and will receive dormitory housing?

12. An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities assumed to be known: $P(A \text{ fails}) = 0.20, P(A \text{ and} B \text{ both fail}) = 0.15, P(B \text{ fails alone}) = 0.15$. Evaluate the following conditional probabilities (a) $P(A \text{ fails } |B \text{ has failed })$ (b) $P(A \text{ fails alone } |A \text{ or } B \text{ fail})$.

13. In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question, given that he or she answered it correctly?

14. The completion of a construction job may be delayed because of a strike. The probabilities are 0.60 that there will be a strike, 0.85 that the construction job will be completed on time if there is no strike, and 0.35 that the construction will be completed on time if there is a strike. What is the probability that the construction job will be completed on time?

Ans 10:- (a) Let A_i = "Switch S_i is closed"

Then $P(A_i) = p$, $i = 1, 2, 3$

Since switches operate independently, we have

$P(A_i A_j) = P(A_i)P(A_j)$, $i \neq j$, $i, j = 1, 2, 3$.

$P(A_1 A_2 A_3) = P(A_1)P(A_2)P(A_3)$.

Note that here, $A_i A_j = A_i \cap A_j$ & $A_1 A_2 A_3 = A_1 \cap A_2 \cap A_3$.

Let R = event that input signal is received at output.

For the event R to occur, either switch 1 or switch 2 or switch 3 must remain closed, i.e.,

$$R = A_1 \cup A_2 \cup A_3 \Rightarrow \overline{R} = \overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$$

$$\Rightarrow P(R) = 1 - P(\overline{R})$$

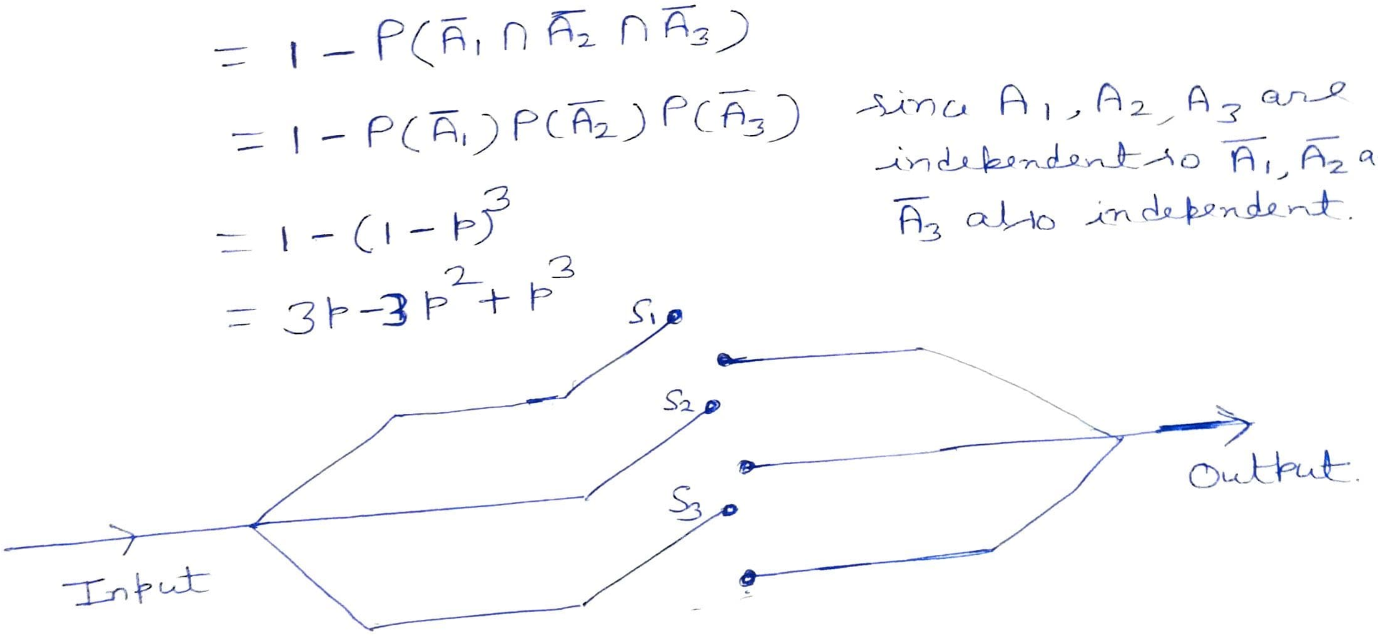
$$= 1 - P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3)$$

$$= 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)$$

since A_1, A_2, A_3 are independent so $\overline{A}_1, \overline{A}_2$ and $\overline{A}_3$ also independent.

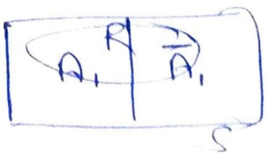
$$= 1 - (1-p)^3$$

$$= 3p - 3p^2 + p^3$$



(b) By Baye's theorem,

$$P(\overline{A}_1 | R) = \frac{P(R|\overline{A}_1)P(\overline{A}_1)}{P(R|\overline{A}_1)P(\overline{A}_1) + P(R|A_1)P(A_1)}$$



$$= \frac{P(A_2 \cup A_3)P(\overline{A}_1)}{P(A_2 \cup A_3)P(\overline{A}_1) + 1 \cdot P(A_1)} = \frac{(2p-p^2)(1-p)}{(2p-p^2)(1-p)+p}$$

$$= \frac{2-3p+p^2}{3-3p+p^2}.$$

Because of the symmetry of the switches we have $P(\overline{A}_1|R) = P(\overline{A}_2|R) = P(\overline{A}_3|R)$

Also one can notice that in computation of $P(\bar{A}_1|R)$, we have

$$P(R|\bar{A}_1)P(\bar{A}_1) + P(R|A_1)P(A_1) = P(R).$$

Ans 11:- Let

A = event that the application is accepted for admission.

B = event that student get dormitory.

Given $P(A) = 0.8, \Rightarrow P(\bar{A}) = 0.2$

$P(B|A) = 0.6.$

$$\Rightarrow P(AB) = P(B|A)P(A) = 0.6 \times 0.8 = \boxed{0.48}$$

Ans 12:- (a) Given

$P(A) = 0.2,\ P(AB) = 0.15,\ P(\bar{A}B) = 0.15$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(AB)}{P(\bar{A}B)+P(AB)}$$

$$= \frac{0.15}{0.15+0.15} = \boxed{\frac{1}{2}}$$

(b) $$P(A\bar{B}|A\cup B) = \frac{P((A\bar{B})\cap(A\cup B))}{P(A\cup B)}$$

$$= \frac{P(A\bar{B})}{P(A)+P(B)-P(A\cap B)} = \frac{P(A)-P(AB)}{P(A)+P(B)-P(AB)}$$

$$= \frac{0.2-0.15}{0.2+0.30-0.15} = \frac{0.05}{0.35}$$

$$= \boxed{\frac{1}{7}}$$

Ans 13 :- Let

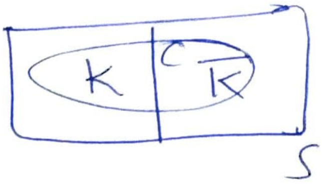
C : Student answers the questions correctly

K : Student knows the answers.

By Baye's thm,

$$P(K|C) = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|\bar{K})P(\bar{K})}$$

$$= \frac{1 \times p}{1 \times p + \left(\frac{1}{m}\right)(1-p)}$$



$$= \boxed{\frac{mp}{1+(m-1)p}}$$

Ans 14 :- A : Construction Job will be completed in time.

B : There will be a strike

So given

$$P(B) = 0.60, \quad P(A|\bar{B}) = 0.85, \quad P(A|B) = 0.35$$

So, by Total Probability thm, we have

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$

$$= 0.35 \times 0.60 + 0.85 \times 0.4$$

$$= \boxed{0.55}$$