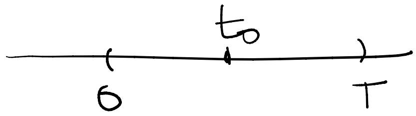
Lecture6 : A2

12 January 2021 09:26

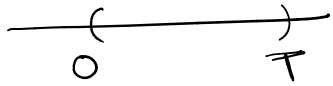
Lecture-6

Ex.:- A telephone call occurs "at random" in the interval $(0, T)$.

What does "at random" mean?



This means that the probability that it will occur in the interval $0 \le t \le t_0$ equals t_0/T.



i.e. $P(0 \le t \le t_0) = \frac{t_0}{T}$

Thus the outcomes of this experiment are all points in $(0, T)$. and

$$P(t_1 \le t \le t_2) = \frac{t_2 - t_1}{T}$$

In this case,

$$\alpha(t) = \begin{cases} \frac{1}{T}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \alpha(t)\,dt = \int_{-\infty}^{0} \alpha(t)\,dt + \int_{0}^{T} \alpha(t)\,dt + \int_{T}^{\infty} \alpha(t)\,dt$$

$$= \underbrace{\int_{-\infty}^{0} 0\,dt}_{=0} + \underbrace{\int_{0}^{T} \frac{1}{T}\,dt}_{=1} + \underbrace{\int_{T}^{\infty} 0\,dt}_{=0} = 1$$

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By "at random" basically we specifying prob. in (S, F, P) here.

Conditional Probability :- To describe it, Consider an experiment of rolling two dice and assume that each of 36 possible outcomes is equally likely to occur and hence has the prob. $\frac{1}{36}$.

Suppose that we observe that the first die is a 3. Then given this information, what is the prob. that sum of the two dice is 8?

$\Rightarrow$ Given that the initial die is a 3, there are at most 6 possible outcome of our experiment namely $(3,1), (3,2), (3,3)$ $(3,4), (3,5), (3,6)$. Since each of these outcomes originally had the same prob. of occuring, the outcomes should still have equal probabilities. That is, given that the first die is 3, the (Conditional) probability of each of the outcomes $(3,1), (3,2), (3,3), (3,4), (3,5)$ $(3,6)$ is $\frac{1}{6}$, whereas the (Conditional)

(5, 6) is $\frac{1}{6}$, whereas the (Conditional) probability of the other 30 points in ~~origi~~ sample space is 0.

$\Rightarrow$ The desired prob. will be $\frac{1}{6}$.

If we let A and M denote respectively the event that the sum of two dice is 8 and the event that the first is a 3.

A: Sum of two dice is 8.
M: first die is 3.

Then the prob. just obtained is called the conditional prob. that A occurs given that M has occured and is denoted by

$$P(A/M) \text{ or } P_M(A).$$

If the event M occurs, then in order for A to occur, it is necessary that the actual occurence be a point in both A and in M i.e., it must be in AM. Now as we know that M has occured, it follows that M becomes our new or reduced sample space. Hence the prob. that the event A/M occurs will equal the prob. of AM relative will ealual the prob. of AM relative to the prob. of M i.e. we have the following defn

$$\boxed{\text{If } P(M) > 0, \text{ then } P(A|M) = P_M(A) = \frac{P(AM)}{P(M)}}$$

$$P(A|M) = \frac{1/36}{6/36} = \frac{1}{6}.$$

* if $M \subset A$ then $P(A|M) = 1$.

Independent Events: In special case, where

$$P(A|M) = P(A)$$

we say that A is independent of M. That is A is independent of M if occurence of M does not change the prob. that A occurs,

Since $P(A|M) = \dfrac{P(AM)}{P(M)}$

$\Rightarrow$ we see that A is independent of M if

$$\boxed{P(AM) = P(A)P(M)}$$

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In the next example, we want to discuss that the complicated events can be expressed as the union of simpler mutually exclusive events.

<u>Ex 1</u>

m white	n black

Balls are drawn "at random" one at a time <u>without replacement</u>. Find the prob. of encountering a white ball <u>by the</u> k-th draw.

<u>Ans 1</u> Let $W_k = \{\text{a white ball is drawn by the } k\text{-th draw}\}$

The event W_k can occur in the following <u>mutually exclusive ways</u>:

a white ~~wa~~ ball is drawn on the first draw, or a black ball followed by a white ball or two black balls followed by a white ball and so on ——. Let

$X_i = \{ i$ black ball followed by a white ball are drawn$\}$, $i = 0, 1, 2, — n$

Then

$$W_k = X_0 \cup X_1 \cup - - - \cup X_{k-1}$$

$$\Rightarrow P(W_k) = \sum_{i=0}^{k-1} P(X_i)$$ since X_i are mutualy exclusive

Now $P(X_0) = \frac{m}{m+n}$

H_i : i-th drawn ball is white

B_i : i-th drawn ball is black

$$P(X_1) = P(B_1 \cap H_2)$$
$$= P(B_1) P(H_2 / B_1)$$
$$= \frac{n}{m+n} \cdot \frac{m}{m+n-1}$$

$$P(X_2) = P(B_1 \cap B_2 \cap H_3)$$

$=$

$$= P(B_1) P(B_2 / B_1) P(H_3 / B_1 \cap B_2)$$
$$= \frac{n}{m+n} \cdot \frac{n-1}{m+n-1} \cdot \frac{m}{m+n-2}$$

$$P(X_{k-1}) = \frac{n}{m+n} \cdot \frac{n-1}{m+n-1} - - - \frac{n-(k-2)}{m+n-(k-2)} \cdot \frac{m}{m+n-(k-1)}$$
$$= \frac{n(n-1) — (n-k+2)\, m}{(m+n)(m+n-1) — (m+n-k+1)}$$

$$= \frac{\qquad}{(m+n)(m+n-1) \cdots (m+n-k+1)}$$

$$\Rightarrow \boxed{P(W_k) = \frac{m}{m+n}\left[1 + \frac{n}{m+n-1} + \frac{n(n-1)}{(m+n-1)(m+n-2)} + \cdots + \frac{n(n-1)\cdots(n-k+2)}{(m+n-1)\cdots(m+n-k+1)}\right]} \quad (*)$$

By the $(n+1)$th draw, we must have a white ball, and hence

$$P(W_{n+1}) = 1$$

Thus from $(*)$, we have

$$1 = \frac{m}{m+n}\left[1 + \frac{n}{m+n-1} + \frac{n(n-1)}{(m+n-1)(m+n-2)} + \cdots + \frac{n(n-1)\cdots 2\cdot 1}{(m+n-1)\cdots(m+1)m}\right]$$

That gives an interesting identity

$$\boxed{1 + \frac{n}{m+n-1} + \frac{n(n-1)}{(m+n-1)(m+n-2)} + \cdots + \frac{n(n-1)\cdots 2\cdot 1}{(m+n-1)(m+n-2)\cdots(m+1)m} = \frac{m+n}{m}}$$