Lec - 35

Recall: Conditional pmf & pdf.

Example (1): Let X and Y be two random variables having the joint pdf

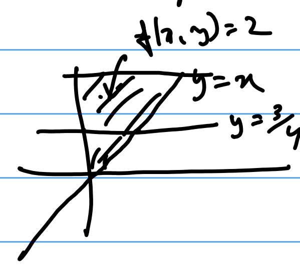
$$f(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Then find the conditional probability $P\left(X \le \frac{2}{3} \,\middle|\, Y = \frac{3}{4}\right)$.

Soln: We are supposed to use the following defn:

$$P(X \in B \mid Y = y) = \int_B f_{X|Y}(x \mid y)\, dx$$

i.e., we need to compute the conditional density $f_{X|Y}\left(x \mid \frac{3}{4}\right)$ and for this we need to compute $f_Y\left(\frac{3}{4}\right)$.



$$f_Y\left(\tfrac{3}{4}\right) = \int_{-\infty}^{\infty} f\left(x, \tfrac{3}{4}\right) dx = \int_0^{3/4} 2\, dx$$

$$= 2 \times \frac{3}{4} = \frac{3}{2}.$$

Since $f_Y\left(\frac{3}{4}\right) > 0$, therefore

$$f_{X|Y}\left(x \mid \tfrac{3}{4}\right) = \begin{cases} \dfrac{f\left(x, \frac{3}{4}\right)}{f_Y\left(\frac{3}{4}\right)} = \dfrac{2}{\frac{3}{2}} = \dfrac{4}{3} & \text{if } 0 < x < \frac{3}{4} \\ 0 & \text{o.w.} \end{cases}$$

Hence $P\left(X \le \frac{2}{3} \middle| Y = \frac{3}{4}\right) = \int_{-\infty}^{2/3} f_{X|Y}\left(x \middle| \frac{3}{4}\right) dx$

$$= \int_0^{2/3} \frac{4}{3}\, dx = \frac{8}{9}$$

Ans.

Example ②: Let X and Y be independent continuous random variables with pdf f_X and f_Y respectively. Let $Z = X + Y$. Determine the conditional density of Z given X.

$\left[\begin{array}{l} x \in \mathbb{R} \\ X = x \end{array}\right]$

Soln: Basically we first determine the conditional distribution fn of Z given X, i.e,

$$P(Z \le z \mid X = x)$$

Then we have the relation

$$P(Z \le z \mid X = x) = \int_{-\infty}^{x} f_{Z|X}(t \mid x)\, dt$$

Now $P(Z \le z \mid X = x) = P(X + Y \le z \mid X = x)$

$$= P(x + Y \le z \mid X = x)$$

$$= P(x + Y \le z) \quad [\because X, Y \text{ are independent}]$$

$$= P(Y \le z - x)$$

$$= \int_{-\infty}^{z-x} f_Y(y)\,dy = \int_{-\infty}^{x} f_Y(t-x)\,dt \qquad y = t-x$$

Hence $f_{Z|X}(z|x) = f_Y(z-x)$.

Remark: In the last example, if we try to compute conditional density of $X+Y$ given X by definition, then we are required to compute the joint density of $X+Y$ and X.

Law of Total Probability

Proposition: Let Y be a discrete random var. on the prob. sp. $(\Omega, \mathcal{F}, P)$. Then for any event $B \in \mathcal{F}$,

$$P(B) = \sum_{y \in R(Y)} P(B \mid Y = y)\, f_Y(y)$$

where f_Y is the pmf of Y.

Proof: If Y is a discrete ran. var. with range $R(Y) \subseteq \mathbb{R}$, then the collection of events $\{\{Y = y\}\}_{y \in R(Y)}$ form a partition of the sample space Ω (i.e, a disjoint union).

$$P(B) = P\left(B \cap \{Y=y\}_{y \in R(Y)}\right)$$

$$= \sum_{y \in R(Y)} P(B \mid Y = y)\, P(Y = y)$$

$$= \sum_{y \in R(Y)} P(B \mid Y = y)\, f_Y(y)$$

We state the law of total probability for continuous ran. var. as well (which is completely analogous to the discrete case).

<u>Theo:</u> Let X be a ran. var. on the prob. sp. $(\Omega, \mathcal{F}, P)$ with pdf f_X. Then for any event $B \in \mathcal{F}$,

$$P(B) = \int_{-\infty}^{\infty} P(B \mid X = x)\, f_X(x)\, dx$$

<u>Example ③</u>: Let X and Y be two independent uniform $(0,1)$ random variables. Find $P(X^3 + Y > 1)$.

Soln: **Method (I)**: One approach would be to form the joint pdf $f(x,y) = f_X(x) f_Y(y)$ and then compute the following integral:

$$P(X^3 + Y > 1) = \iint_A f(x,y)\,dx\,dy$$

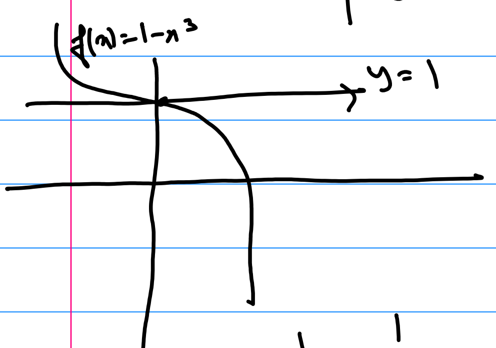
where $A = \{(x,y) \in \mathbb{R}^2 \mid x^3 + y > 1\}$. Since X & Y are independent, the joint pdf of X and Y is:

$$f(x,y) = f_X(x) f_Y(y) \quad \text{where}$$

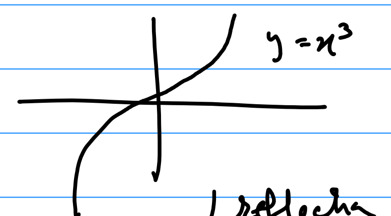
$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \quad \& \quad f_Y(y) = \begin{cases} 1 & \text{if } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Hence, $f(x,y) = \begin{cases} 1 & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$

We need to plot the graph of the fn $f(x) = 1 - x^3$.

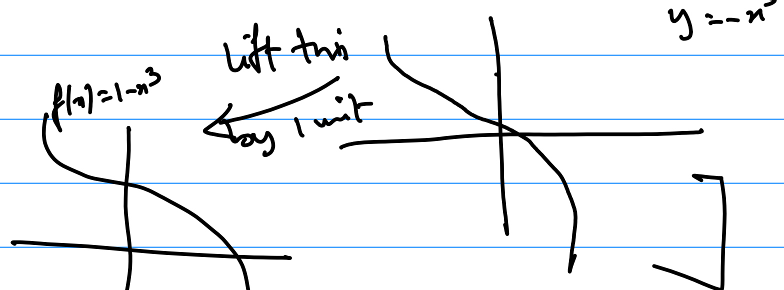


[Rough:



reflection about y-axis $y = -x^3$

$$\iint_A f(x,y)\,dx\,dy = \int_{x=0}^{1}\left(\int_{1-x^3}^{1} 1\,dy\right)$$



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$$= \int_0^1 \left(1 - (1-x^3)\right) dx = \int_0^1 x^3 \, dx = \frac{1}{4}$$

Method (II) : We now illustrate how the conditioning is useful: One can condition either w.r.t. $Y = y$ or $X = x$.

(a) We condition w.r.t. Y. Hence by total probability law:

$$P(X^3 + Y > 1) = \int_{-\infty}^{\infty} P(X^3 + Y > 1 \mid Y = y) \, f_Y(y) \, dy$$

$$= \int_0^1 P(x^3 + y > 1 \mid Y = y) \, dy$$

$$= \int_0^1 P(x^3 > 1 - y \mid Y = y) \, dy$$

$$= \int_0^1 P(X^3 > 1 - y) \, dy = \int_0^1 P\left(X > (1-y)^{1/3}\right) dy$$

$$= \int_0^1 \left(\int_{\sqrt[3]{1-y}}^1 1 . \, dx \right) dy$$

$\left[\because 0 < y < 1, \; X^3 > 1 - y \iff X > (1-y)^{1/3} \right]$

$$= \int_0^1 \left(1 - \sqrt[3]{1-y}\right) dy = \int_1^0 \left(1 - \sqrt[3]{u}\right)(-du) \quad u = 1-y$$

$$= \left[u - \frac{3}{4} u^{4/3} \right]_0^1 = \frac{1}{4}$$

(b) We condition w.r.t. X. Hence by total probability law:

$$P(X^3+Y>1) = \int_{-\infty}^{\infty} P(X^3+Y>1 \mid X=x)\, f_X(x)\,dx$$

$$= \int_0^1 P(X^3+Y>1 \mid X=x)\,dx$$

$$= \int_0^1 P(x^3+Y>1 \mid X=x)\,dx$$

$$= \int_0^1 P(Y>1-x^3 \mid X=x)\,dx = \int_0^1 P(Y>1-x^3)\,dx = \int_0^1 \left(\int_{1-x^3}^1 dy\right)dx$$

$$= \int_0^1 x^3\,dx = \frac{x^4}{4}\Big]_0^1 = \frac{1}{4}$$

Conditional Expectation

Defn: Let X and Y be discrete random variables with conditional pmf $f_{X|Y}$ of X given Y. Then conditional expectation of X given $Y=y$ is defined as:

$$E[X|Y=y] = \sum_{x \in R(X)} x f_{X|Y}(x|y)$$

provided $$\sum_{x \in R(X)} |x| f_{X|Y}(x|y) < \infty$$

Example (4): Let X, Y be independent random variables with geometric distribution of parameter $0<p<1$. Calculate $E[Y \mid X+Y=n]$ where $n \geq 2$.

Soln: first we find the conditional pmf of Y given $X+Y=n$ where $n \geq 2$.

Since range of X and Y is $\mathbb{N}$, hence the range of the ran. var. $Z := X+Y$ is $\{2, 3, \ldots\}$. Let $n \geq 2$ is given. So if $X+Y=n$, then Y can only assume values in $\{1, 2, \ldots, n-1\}$. Therefore

$$P(Y=y \mid Z=n) = 0 \quad \text{for } y = n, n+1, n+2, \ldots$$

For $y \in \{1, 2, \ldots, n-1\}$,

$$P(Y=y \mid Z=n) = \frac{P(Y=y,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(Y=y,\ X+y=n)}{P\left(\bigcup_{k=1}^{n-1}\{X=k, Y=n-k\}\right)}$$

$$= \frac{P(Y=y,\ X=n-y)}{\sum_{k=1}^{n-1} P(X=k, Y=n-k)}$$

$$= \frac{P(Y=y)\,P(X=n-y)}{\sum_{k=1}^{n-1} P(X=k)\,P(Y=n-k)}$$ [$\because$ X & Y are independent]

$$= \frac{p(1-p)^{y-1}\ p(1-p)^{n-y-1}}{\sum_{k=1}^{n-1} p(1-p)^{k-1}\ p(1-p)^{n-k-1}} = \frac{p^2(1-p)^{n-2}}{\sum_{k=1}^{n-1} p^2(1-p)^{n-2}} = \frac{1}{n-1}$$

This shows that

$$f_{Y|X+Y}(y|n) = \begin{cases} \frac{1}{n-1} & \text{if } y = 1, \ldots, n-1 \\ 0 & \text{o.w.} \end{cases}$$

Hence Y is geometrically distributed in the original universe, but in the new universe (i.e., after conditioning) determined by the event $X+Y=n$, Y is uniformly (discrete) distributed over the set $\{1, 2, \ldots, n-1\}$.

Hence, $E[Y \mid Z=n] = \sum_y y \, f_{Y|Z}(y|n)$

$$= \sum_{y=1}^{n-1} y \frac{1}{n-1} = \frac{1}{n-1} \times \frac{(n-1)(n-1+1)}{2}$$

$$= \frac{n}{2}$$

Ans