Lec - 28

Recall: Joint pdf., Borel subsets of $\mathbb{R}^2$.

Properties of joint density

If f is the joint pdf of random vector (X,Y)

then $P(-\infty < X < \infty, -\infty < Y < \infty)$

$$= P(\Omega \cap \Omega) = 1$$

But by defn, we have

$$P(-\infty < X < \infty, -\infty < Y < \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$= 1$$

Hence joint pdf integrate to 1 on the entire plane.

Thm: (Characterization of joint pdf)

Let $f : \mathbb{R}^2 \xrightarrow{\text{fntn}} \mathbb{R}$ be such that

(a) $f(x,y) \geq 0 \;\forall\, (x,y) \in \mathbb{R}^2$

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Then $\exists$ a probability space $(\Omega, \mathcal{F}, P)$ and a random vector (X,Y) defined on it such that f is the joint pdf of (X,Y).

Example: Let $f(x,y) = c\, e^{-\frac{x^2 - xy + 4y^2}{2}}$, $x, y \in \mathbb{R}$. Find the value of c s.t. f is a joint pdf.

Soln: If f is a joint pdf then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,s)\, dt\, ds = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c\, e^{-\left(\frac{x^2 - xy + 4y^2}{2}\right)} dx\, dy$$

$$= c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\cdot x\cdot\frac{y}{2} + \frac{y^2}{4} + 4y^2 - \frac{y^2}{4}}{2}}\, dx\,dy$$

$$= c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{(x-\frac{y}{2})^2 + \frac{15y^2}{4}}{2}\right)}\, dx\,dy$$

$$= c\int_{-\infty}^{\infty} e^{-\frac{15y^2}{8}}\left(\int_{-\infty}^{\infty} e^{-\frac{(x-\frac{y}{2})^2}{2}}\, dx\right) dy$$

$u = x - \frac{y}{2}$

$du = dx$

$$= c\int_{-\infty}^{\infty} e^{-\frac{15}{8}y^2}\left(\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\, du\right) dy$$

$$= c\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-\frac{15}{8}y^2}\underbrace{\left(\int_{-\infty}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\, du\right)}_{=1} dy$$

$$= c\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-\frac{v^2}{2}}\frac{2}{\sqrt{15}}\, dv$$

$v^2 = \frac{15y^2}{4}$

$dv = \frac{\sqrt{15}}{2}\, dy$

$$= \cdots = \frac{4}{\sqrt{15}}\pi \times c = 1 \implies c = \frac{\sqrt{15}}{4\pi} \text{ Ans.}$$

Proposition: If f is the joint pdf of X and Y, then

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad , \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Proof: $P\{X \le x\} = P\{X \le x, Y < \infty\}$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f(x,y)\,dy \right] dx$$

$$= \int_{-\infty}^{x} g(x)\,dx \qquad g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

Hence $f_X(x) = g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

$f_Y(y) = ?$ (HW)

Example: The joint pdf of (X,Y) is given as:

$$f(x,y) = \begin{cases} 6(1-x), & 0 < y < x, \quad 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

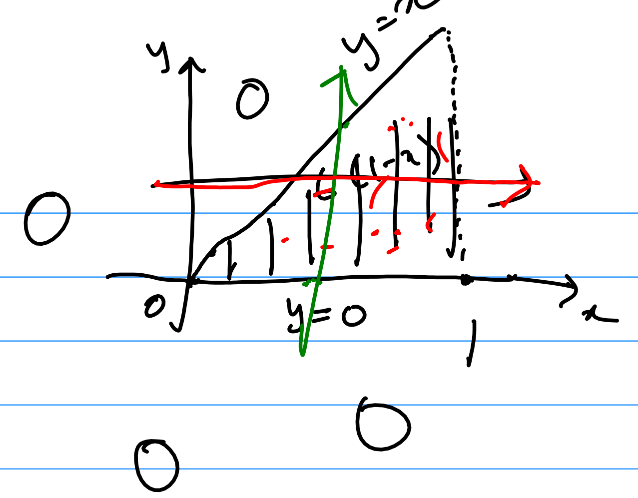
Then $f_X(x) = ?$ & $f_Y(y) = ?$

Sol^n: Density of Y. We have $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$
$\forall y \in \mathbb{R}$

$$f_Y(y) = \int_y^1 6(1-x)\,dx$$

Red - X

Green - Y



$$= 6\left(x - \frac{x^2}{2}\right)\Big]_y^1$$

$$= 3(y-1)^2$$

Hence, $f_Y(y) = \begin{cases} 3(y-1)^2, & 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$

<u>Density of X</u>: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_{y=0}^{x} 6(1-x)\,dy = 6(y - xy)\Big]_0^x = 6(x - x^2)$$

$\therefore$ $f_X(x) = \begin{cases} 6(x - x^2), & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

<u>Joint Distribution Function (Joint CDF)</u>

(It can be defined for any kind of random vector: discrete or continuous or mixed).

Let (X, Y) be a random vector on $(\Omega, \mathcal{F}, P)$.
Then the fn $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

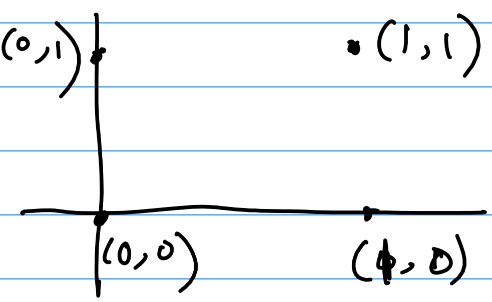
$$F(x, y) = P\{X \le x, Y \le y\}$$ is called the joint distribution function of (X, Y).

<u>Example:</u> Suppose the joint pmf of (X, Y) is given as:

$$f(0,0) = f(0,1) = \frac{1}{6}, \quad f(1,0) = f(1,1) = \frac{1}{3}$$

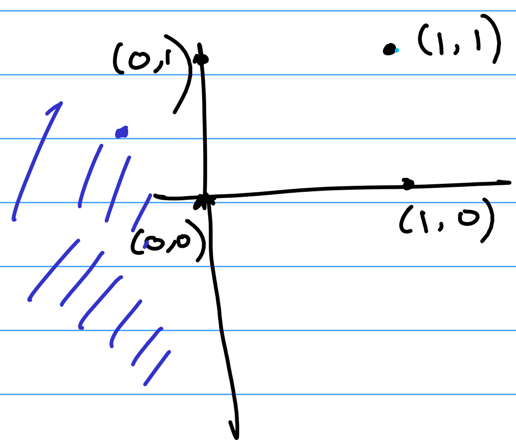
Then joint CDF of X & Y = ?

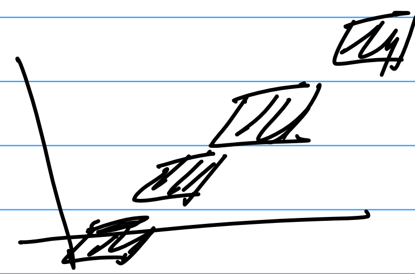
<u>Soln:</u> We have



$$F(x, y) = \sum_{(i,j) : i \le x, j \le y} P(X = i, Y = j)$$

$F(x,y) = \cancel{0 \; x<0 \text{ or } y<0}$





$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ \frac{1}{6} & 0 \le x < 1,\ 0 \le y < 1 \\ \frac{1}{3} & 0 \le x < 1,\ y \ge 1 \\ \frac{1}{2} & x \ge 1,\ 0 \le y < 1 \\ 1 & x \ge 1,\ y \ge 1 \end{cases}$$