Joint distribution.

It allows us to compute probabilities of events involving both (or more) variables & the relationship betⁿ the variables.

Simplest: When variables are independent.

If not, we use covariance & correlation as measures of the nature of dependence betⁿ them.

Defn: Let $(\Omega, \mathcal{F}, P)$ be a probability space. A map $X : \Omega \longrightarrow \mathbb{R}^n$,

$$X(\omega) = (X_1(\omega), \ldots, X_n(\omega))$$

is called a <u>n-dimensional random vector</u> on $(\Omega, \mathcal{F}, P)$ if each X_i is a random variable on $(\Omega, \mathcal{F}, P)$.

$\boxed{n=2} \longrightarrow$ Our interest

Example: Toss two fair dice.

$$\Omega = \{(1,1),(1,2),\ldots,(1,6),\ldots \\ \ldots,(6,1),(6,2),\ldots,(6,6)\}$$

$$= \{(i,j) : i,j = 1,\ldots,6\}.$$

All outcomes are equally likely.

(36 possible outcomes).

σ-algebra := power set

$$X(i,j) = i+j, \qquad Y(i,j) = |i-j|$$

$(X,Y) \to$ Random vector

$$P(X=5, Y=3) = ?$$

$i+j = 5$

$|i-j| = 3$

$(i,j) = ?$ $\quad \{(1,4),(4,1)\}$

$$\frac{1}{36} + \frac{1}{36} = \frac{1}{18}$$

Defn: Discrete random vector: We say that a random vector $X = (X_1, X_2)$ is a discrete random vector if both X_1 and X_2 are discrete random variables.

Recall: Range of a discrete random variable is either finite or countably infinite.

Hence, Range of a discrete r. vector is also finite or countably infinite.

(1) If range of X_1 & X_2 are both finite, then $X = (X_1, X_2)$ has finite range.

(2) - - - range of X_1 is finite, but range of X_2 is countably infinite, then range of X is countably infinite.

(3) - - - X_1 is count. inf, ... $X_2 \to$ fin,
- - - X is count. inf.

(4) - - - Both count. inf, - - - $X \to$ count. inf

Defn: <u>Joint pmf</u> (probability mass function)

Let $X = (X_1, X_2)$ be a discrete random vector.

$\Omega \to \mathbb{R}^2 \to \mathbb{R}$

<u>Define:</u> $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = P\{X_1 = x_1,\ X_2 = x_2\} \quad \forall\, x_1, x_2 \in \mathbb{R}.$$

Then f is called joint pmf of X.

<u>Importance:</u> The joint pmf completely determines the probability distribution of the discrete random vector (X_1, X_2).

<u>Notation:</u> $f_{X_1, X_2}(x_1, x_2)$.

<u>Previous example:</u>

$$X_1(i,j) = i + j$$

$$X_2(i,j) = |i - j|$$

$P(X_1 = 4) = ?$

$X_2 \backslash X_1$	2	3	4	5	6	7	8	9	10	11	12
0	1/36	0	1/36	0	1/36	0	1/36	0	1/36	0	1/36
1	0	1/18	0	1/18							
2											
$P(X_2=3) =$ 3											
4											
5											

$f(2,0) = ?$

$$f(2,0) = P(X_1 = 2, X_2 = 0) = \frac{1}{36}$$

$\{(1,1)\}$

$$f(3,0) = P(X_1 = 3, X_2 = 0) = 0$$

ϕ

HW: Complete the table

Properties of the joint pmf: (Very similar to the single variable case).

(i) $f(x_1, x_2) \geq 0 \quad \forall (x_1, x_2) \in \mathbb{R}^2$

(ii) The set $\{(x_1, x_2) : f(x_1, x_2) \neq 0\}$ is at most countably infinite (subset of $\mathbb{R}^2$).

(iii) $$\sum_{(x_1,x_2) \in R(X_1,X_2)} f(x_1, x_2) = 1$$

$R(X_1, X_2) :=$ Range of (X_1, X_2)

HW: Check the above properties for the previous example.

Thus the joint pmf determines the prob. of any event that can be specified in terms of the discrete random variables X_1 and X_2.

Thm.: Let $X = (X_1, X_2)$ be a discrete random vector with the joint pmf f. Then for any $A \subseteq \mathbb{R}^2$,

$$P\{(X_1, X_2) \in A\} = \sum_{(x_1, x_2) \in A} f(x_1, x_2)$$

Example: $X_1 \to$ r. var. s.t. $R(X_1) = \{-2, 1, 3\}$
$X_2 \to$ r. var. s.t. $R(X_2) = \{-1, 0, 4, 6\}$

Suppose the joint pmf of (X_1, X_2):

$X_1 \backslash X_2$	-1	0	4	6
-2	$1/9$	$1/27$	$1/27$	$1/9$
1	$2/9$	0	$1/9$	$1/9$
3	0	0	$1/9$	$\frac{4}{27}$

$P(X_1 X_2 = \text{odd}) = ?$

HW $P(X_1 X_2 = \text{even}) = ?$

Soln: $\{X_1 X_2 \text{ is odd}\} = \{X_1 = 1, X_2 = -1\} \cup \{X_1 = 3, X_2 = -1\}$

$P(X_1 X_2 \text{ is odd}) = f(1, -1) + f(3, -1) = \frac{2}{9}$

Marginal PMF:

Even if we are considering a r. vector (X_1, X_2), there may be probabilities of interest that involve only one of the random variables in the random vector.

<u>E.g.</u> $P(X_1 = 2) = ?$ (in the previous example)

<u>Notation:</u> $f_{X_1}(x_1)$

The marginal pmf can be easily calculated from the joint pmf of (X_1, X_2).

<u>Proposition:</u> If f is the joint pmf of X_1 and X_2, then

$$f_{X_1}(x_1) = \sum_{x_2 \in R(X_2)} f(x_1, x_2)$$

$$f_{X_2}(x_2) = \sum_{x_1 \in R(X_1)} f(x_1, x_2)$$

Proof: For each $x_1 \in \mathbb{R}$, the events

$$\{X_1 = \underset{\text{fixed}}{x_1},\ X_2 = x_2\},\quad x_2 \in R(X_2)$$

are disjoint and their union is the event $\{X_1 = x_1\}$.

$$\therefore f_{X_1}(x_1) = P(X_1 = x_1)$$

$$= P\left(\bigcup_{x_2 \in R(X_2)} \{X_1 = x_1,\ X_2 = x_2\}\right)$$

$$= \sum_{x_2 \in R(X_2)} P(X_1 = x_1,\ X_2 = x_2)$$

$$= \sum_{x_2 \in R(X_2)} f(x_1, x_2)$$

$f_{X_2}(x_2) = \text{- - - -}$ (HW)