Lec - 29

Recall: (Yesterday's example again)

Example: Joint pmf of (X, Y) : $f(0,0) = \frac{1}{6} = f(0,1)$

$f(1,0) = \frac{1}{3} = f(1,1)$

Joint CDF of $(X, Y) = ?$

Soln:

$$F(x,y) = \sum_{(i,j)\,:\, i \le x,\, j \le y} P(X=i, Y=j)$$

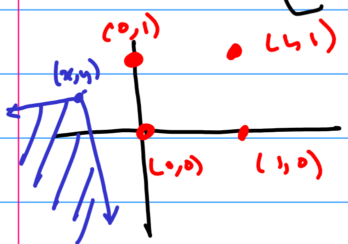
Hence,

$$F(x,y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{1}{6}, & 0 \le x < 1,\ 0 \le y < 1 \\ \frac{2}{6}, & 0 \le x < 1,\ y \geq 1 \\ \frac{1}{2}, & x \geq 1,\ 0 \le y < 1 \\ 1, & x \geq 1,\ y \geq 1 \end{cases}$$

You can easily understand this by following the diagrams below.

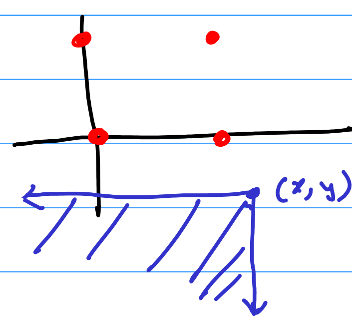
Case (I)

$x < 0$



Case (II)

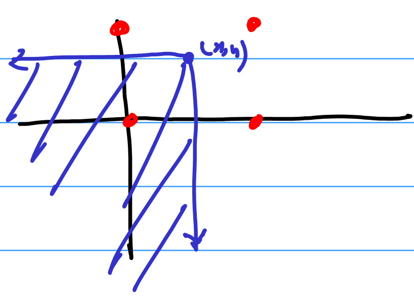
$y < 0$



Case III

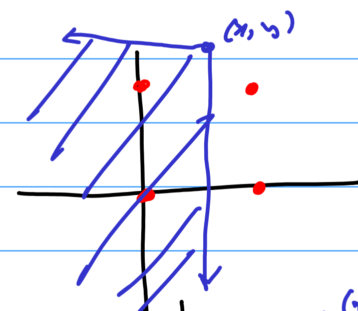
$0 \le x < 1$,

$0 \le y < 1$



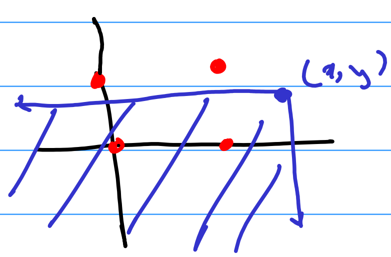
Case IV

$0 \le x < 1, y \geq 1$



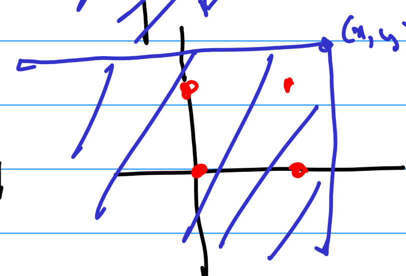
Case V

$x \geq 1, 0 \le y < 1$



Case VI

$x \geq 1, y \geq 1$



Properties of Joint CDF:

Thm: Let F be the joint distribution fn of a random vector (X,Y). Then F satisfies the following:

1.(a) $F(x,y) \to 0$ as $x \to -\infty$ or $y \to -\infty$

That is:

(i) $\lim_{x \to -\infty} F(x,y) = 0 \quad \forall\, y \in \mathbb{R}$

(ii) $\lim_{y \to -\infty} F(x,y) = 0 \quad \forall\, x \in \mathbb{R}$

(iii) $\lim_{(x,y) \to (-\infty,-\infty)} F(x,y) = 0$

Recall:
$f : \mathbb{R} \to \mathbb{R}$.
$\lim_{h \to 0^+} f(x+h) = f(x)$
$= \lim_{h \to 0^-} f(x+h)$

1.(b) $\lim_{(x,y) \to (\infty,\infty)} F(x,y) = 1$

(2) F is right continuous in each argument.

That is, $\lim_{h \to 0^+} F(x+h, y) = \lim_{h \to 0^+} F(x, y+h) = F(x,y)$

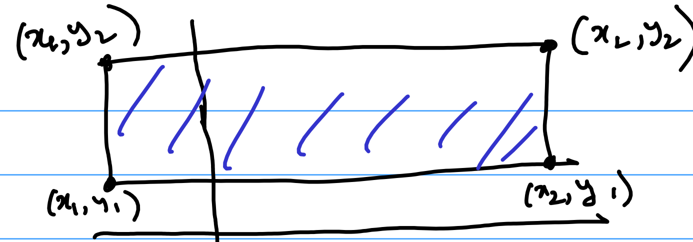
(3) F is non-decreasing in each argument. That is,

$F(x,y) \leq F(x+h, y) \quad \forall\, h > 0$

$F(x,y) \leq F(x, y+k) \quad \forall\, k > 0$

(4) For every (x_1,y_1), (x_2,y_2) with $x_1 < x_2$ and $y_1 < y_2$, the following inequality holds:

$$F(x_2,y_2) - F(x_2,y_1) + F(x_1,y_1) - F(x_1,y_2) \geq 0 .$$



Proof of property (4):

$$0 \leq P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\}$$ (Blue region)

$$= P\{X \leq x_2,\ Y \leq y_2\} + P\{X \leq x_1,\ Y \leq y_1\} - P\{X \leq x_1,\ Y \leq y_2\} - P\{X \leq x_2,\ Y \leq y_1\}$$

$$= F(x_2,y_2) + F(x_1,y_1) - F(x_1,y_2) - F(x_2,y_1)$$

$\square$

Recall: (The FAMOUS Example! ☺)

$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 . \\ \frac{1}{6}, & 0 \leq x < 1,\ 0 \leq y < 1 \\ \frac{2}{6}, & 0 \leq x < 1,\ y \geq 1 . \\ \frac{1}{2}, & x \geq 1,\ 0 \leq y < 1 \\ 1, & x \geq 1,\ y \geq 1 . \end{cases}$$

Let us verify the above properties for this example.

1.(a) For this CDF F, we have:

for any given $y \in \mathbb{R}$, $F(x,y) = 0 \quad \forall x < 0$

Hence, $\lim_{x \to -\infty} F(x,y) = 0 \quad \forall y \in \mathbb{R}$.

(b) Similarly, for any given $x \in \mathbb{R}$, $F(x,y) = 0 \quad \forall y < 0$

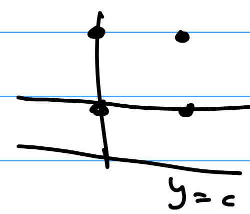
Hence, $\lim_{y \to -\infty} F(x,y) = 0 \quad \forall x \in \mathbb{R}$.

(c) Also since $F(x,y) = 0$ in the 3^{rd} quadrant,

we have $\lim_{(x,y) \to (-\infty, -\infty)} F(x,y) = 0$.

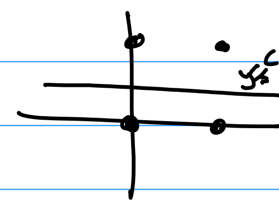
2.(a) (Right continuity w.r.t. x)

(i) If $c < 0$, then along the line $y = c$,



we have $F = 0$, which is continuous $\forall x \in \mathbb{R}$.

(ii) If $0 \le c < 1$, then along the line $y = c$

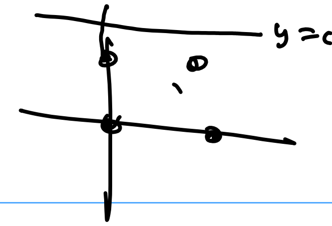


$$F(x,c) = \begin{cases} 0, & x < 0 \\ \frac{1}{6} & 0 \le x < 1 \\ \frac{1}{2} & x \geq 1 \end{cases}$$

$\lim_{h \to 0^+} F(x+h, y) = F(x,y)$ $\therefore$ F is right continuous everywhere w.r.t. x.

(for all the above x)

(iii) If $c \geq 1$, then along the line $y = c$

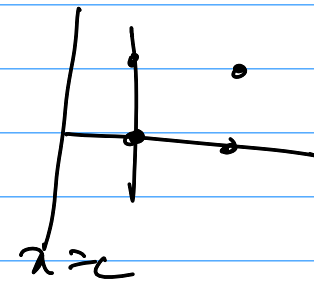


$$F(x,c) = \begin{cases} 0, & x < 0 \\ \frac{1}{3}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

which is right continuous everywhere.

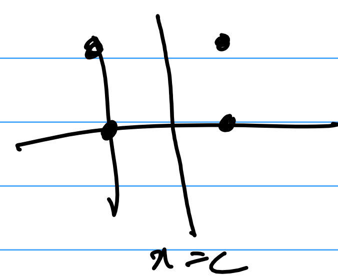
2.(b) (Right continuity w.r.t. y)

(i) If $c < 0$, then along the line $x = c$, $F = 0$ which is continuous $\forall y \in \mathbb{R}$.



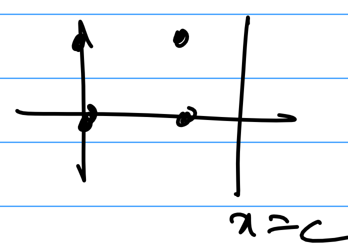
(ii) If $0 \leq c < 1$, then along $x = c$

$$F(c,y) = \begin{cases} 0, & y < 0 \\ \frac{1}{6}, & 0 \leq y < 1 \\ \frac{1}{3}, & y \geq 1 \end{cases}$$



which is right continuous everywhere.

(iii) If $c \geq 1$, then along $x = c$



$$F(c,y) = \begin{cases} 0, & y < 0 \\ \frac{1}{2}, & 0 \leq y < 1 \\ 1, & y \geq 1 \end{cases}$$

which is right continuous everywhere.

(3) Non-decreasing in each argument is also clear from the discussion about right continuity in each co-ordinate.

(4) For every (x_1, y_1) & (x_2, y_2) with $x_1 < x_2$ & $y_1 < y_2$, we have

$$F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1)$$
$$= P\{x_1 < X \le x_2, y_1 < Y \le y_2\} \geq 0.$$

Th^m: (Characterization of Joint CDF).

Any function F defined on $\mathbb{R}^2$ and satisfying conditions 1–4 in the above th^m, can be identified as the joint distribution fn^n of some 2-dimensional random vector.

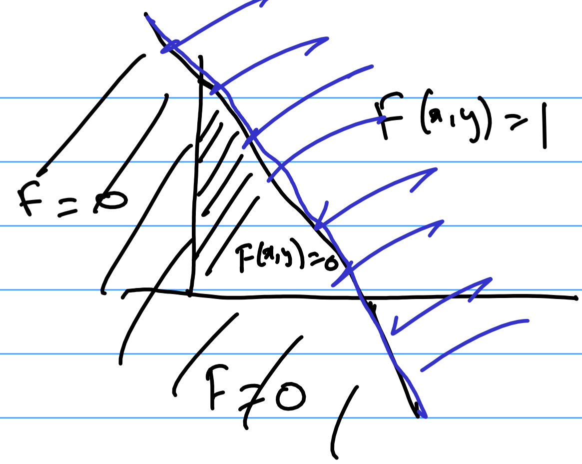
Application:= Example

Let F be a fn^n of two variables defined by

$$F(x, y) = \begin{cases} 0, & x < 0 \text{ or } x+y < 1 \text{ or } y < 0 \\ 1, & \text{o.w.} \end{cases}$$

Determine whether F is a joint CDF?

Soln: Let us verify properties 1-4 in the 1st theo:



1.(a) for any given $y \in \mathbb{R}$,
$F(x,y)=0 \quad \forall x<0$.

Hence, $\lim_{x \to -\infty} F(x,y)=0 \quad \forall y \in \mathbb{R}$.

(b) S^{lly}, for any given $x \in \mathbb{R}$, $F(x,y)=0 \quad \forall y<0$

Hence, $\lim_{y \to -\infty} F(x,y)=0 \quad \forall x \in \mathbb{R}$.

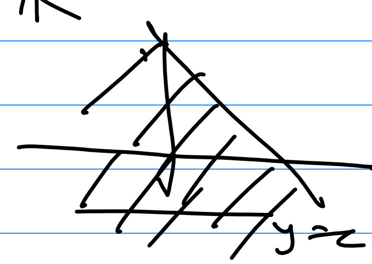
(c) Also since $F(x,y)=0$ in 3rd quadrant,

$\therefore \lim_{(x,y) \to (-\infty,-\infty)} F(x,y)=0$.

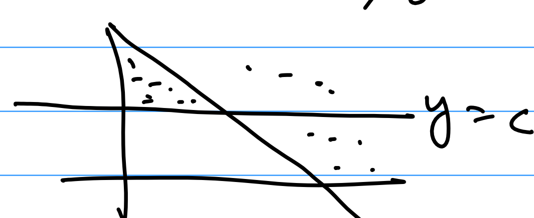
2.(a) (Right continuity w.r.t. x)

(i) If $c<0$, then along the line $y=c$,
$F=0$ which is continuous $\forall x \in \mathbb{R}$

(ii) If $0 \le c < 1$, then along line $y=c$,

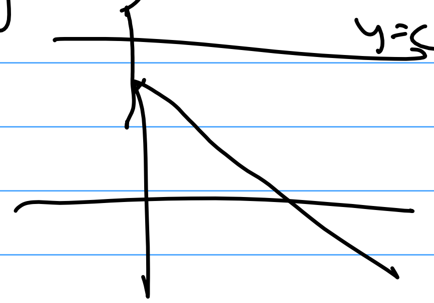


$$F(x,c)=\begin{cases} 0, & x<1-c \\ 1, & x \geq 1-c \end{cases}$$



It is right continuous everywhere.

(iii) If $c \geq 1$, then along line $y = c$,

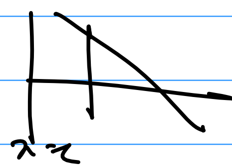


$$F(x,c) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

which is right continuous everywhere.

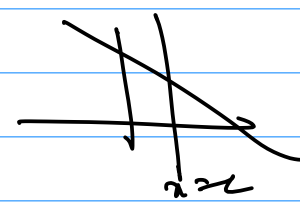
2. (b) (Right continuity w.r.t. y)

(i) If $c < 0$, then along line $x = c$, $F = 0$ which is continuous $\forall y \in \mathbb{R}$.



(ii) If $0 \leq c < 1$, then along line $x = c$,

$$F(c,y) = \begin{cases} 0, & y < 1-c \\ 1, & y \geq 1-c \end{cases}$$



which is right continuous everywhere.

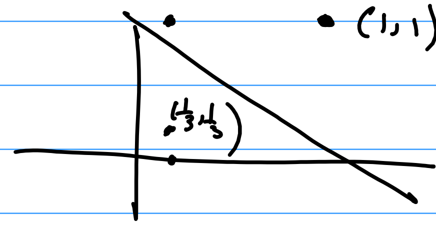
(iii) If $c \geq 1$, then along line $x = c$,

$$F(c,y) = \begin{cases} 0, & y < 0 \\ 1, & y \geq 0 \end{cases}$$

which is right continuous everywhere.

(3) Non-decreasing in each argument is also clear from the discussion about right continuity in each co-ordinate.

(4) Take $(x_1, y_1) = \left(\frac{1}{3}, \frac{1}{3}\right)$, $(x_2, y_2) = (1, 1)$



Then $F(x_2, y_2) - F(x_2, y_1)$
$+ F(x_1, y_1) - F(x_1, y_2)$
$= 1 - 1 + 0 - 1 = -1 < 0$

Hence, given F is NOT a CDF.

$\blacksquare$