Lec-36

Defn: Let X and Y be random variables with conditional pdf $f_{X|Y}$ of X given Y. The conditional expectation of X given $\{Y=y\}$ is defined as

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

provided $\int_{-\infty}^{\infty} |x|\, f_{X|Y}(x|y)\, dx < \infty$

Theo: ① Let X, Y be discrete random variables with joint pmf f. If Y has finite mean then

$$E[Y] = \sum_x E[Y \mid X=x]\, f_X(x)$$

② Let X, Y be random variables with joint pdf f. If Y has finite mean, then

$$E[Y] = \int_{-\infty}^{\infty} E[Y \mid X=x]\, f_X(x)\, dx$$

Proof: ①

$$\sum_x E[Y \mid X=x]\, f_X(x) = \sum_x \left(\sum_y y\, f_{Y|X}(y|x) \right) \times f_X(x)$$

$$= \sum_x \sum_y y\, f(x,y)$$

$$= \sum_y y \sum_x f(x,y) = \sum_y y\, f_Y(y) = E[Y]$$

② HW.

<u>Remark</u>: The above theorem is called "total expectation theorem". It expresses the fact that "the unconditional average can be obtained by averaging the conditional averages".

They can be used to calculate the unconditional expectation $E(X)$ from the conditional expectation.

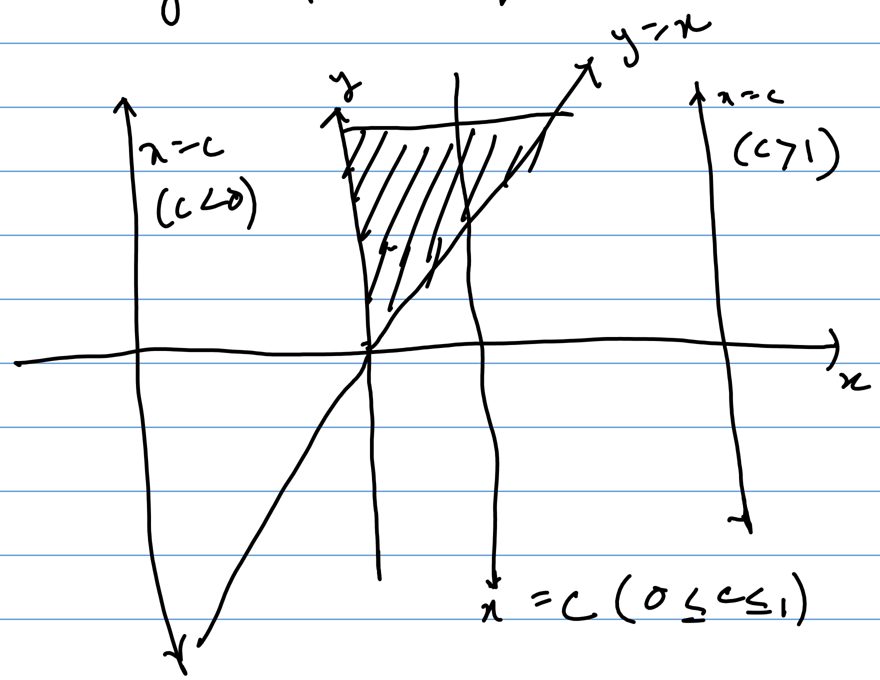
<u>Example (1)</u>: Let X, Y be continuous ran. var. with joint pdf given by

$$f(x,y) = \begin{cases} 6(y-x); & 0 \le x \le y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E[Y \mid X = x]$ and hence calculate $E[Y]$.

<u>Soln</u>: In order to calculate $E[Y \mid X=x]$ we need to find $f_{Y|X}$, which is by defn equal to

$$\frac{f(x,y)}{f_X(x)}.$$



Note that

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_x^1 f(x,y)\,dy & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \int_x^1 6(y-x)\,dy = 6\left[\frac{y^2}{2} - xy\right]_x^1 = 6\left[\frac{x^2}{2} - x + \frac{1}{2}\right]$$

$$= \begin{cases} 3(x-1)^2, & 0 \le x < 1 \\ 0 & \text{o.w.} \end{cases}$$

This implies that

$$f_{Y|X}(y|x) = \frac{\cancel{6}(y-x)}{3(x-1)^2} = \begin{cases} \frac{2(y-x)}{(x-1)^2}, & 0 \le x \le y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Hence $E[Y \mid X=x]$ would be non-zero only if $0 \le x < 1$.

$$E[Y \mid X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\,dy$$

$$= \int_x^1 y\,\frac{2(y-x)}{(x-1)^2}\,dy = \frac{2}{(x-1)^2}\int_x^1 (y^2 - xy)\,dy$$

$$= \frac{2}{(x-1)^2}\left[\frac{y^3}{3} - x\frac{y^2}{2}\right]_x^1$$

$$= \frac{2}{(x-1)^2}\left[\frac{1}{3} - \frac{x}{2} + \frac{x^3}{6}\right]$$

$$= \frac{2(x^3 - 3x + 2)}{6(x-1)^2} = \frac{x^2 + x - 2}{3(x-1)}$$

$$\therefore E[Y] = \int_{-\infty}^{\infty} E[Y \mid X = x]\, f_X(x)\,dx$$

$$= \int_0^1 \frac{x^2 + x - 2}{3(x-1)} \times 3(x-1)^2\,dx$$

$$= \int_0^1 (x^2 + x - 2)(x-1)\,dx = \int_0^1 (x^3 - 3x + 2)\,dx = \frac{3}{4}$$

Theo: (1) Let X and Y be discrete random variables with joint pmf f. If g is a function, then

$$E[g(X) \mid Y = y] = \sum_x g(x)\, f_{X|Y}(x|y),$$

provided $\sum_x |g(x)|\, f_{X|Y}(x|y) < \infty$

(2) If X and Y be ran. var. with joint pdf f, and if $g: \mathbb{R} \to \mathbb{R}$ is a Borel fn, then

$$E[g(X) \mid Y = y] = \int_{-\infty}^{\infty} g(x)\, f_{X|Y}(x|y)\,dx,$$

provided $\displaystyle\int_{-\infty}^{\infty} |g(x)|\, f_{X|Y}(x|y)\, dx < \infty$.

It is immediate that conditional expectation satisfies the usual properties of an expectation. The following properties are easy to prove. (We assume the existence of indicated expectations).

Theo: (Properties of Conditional Expectation)

Let X and Y be two random variables on a prob. sp. $(\Omega, \mathcal{F}, P)$ and let a, b and c be real nos. Suppose g_1 and g_2 are real-valued function of one real variable s.t. $E[g_1(X)] < \infty$ and $E[g_2(X)] < \infty$. Then

(a) $E[a g_1(X) + b g_2(X) + c \mid Y = y]$

$= a E[g_1(X) \mid Y = y] + b E[g_2(X) \mid Y = y] + c.$

(b) If $g_1(x) \geq g_2(x)$ $\forall x$, then

$E[g_1(X) \mid Y = y] \geq E[g_2(X) \mid Y = y].$

(c) Let Z be another ran. var, then

$E[X + Y \mid Z = z] = E[X \mid Z = z] + E[Y \mid Z = z].$

(d) If X and Y are independent, then
$$E[X \mid Y=y] = E[X] \text{ and } E[Y \mid X=x] = E[Y].$$

Covariance.

Defn: Let X and Y be jointly distributed on the prob. sp. $(\Omega, \mathcal{F}, P)$. The covariance of two random variables X and Y, denoted by $Cov(X,Y)$ is defined by

$$Cov(X,Y) = E\left[(X-E(X))(Y-E(Y))\right],$$

provided
$$E\left(\left|(X-E(X))(Y-E(Y))\right|\right) < \infty$$

When $Cov(X,Y)=0$, we say that X and Y are uncorrelated.

Remark: The covariance gives information about how random variables X and Y are linearly related.

Intuitively, the covariance betn X & Y indicates how the values of X and Y move relative to each other.

Alternate expression for Covariance

By applying linearity of the expectation,

$$\mathrm{Cov}(X,Y) = E\left[XY - E(X)Y - X\,E(Y) + E(X)E(Y)\right]$$

$$= E(XY) - E(X)E(Y) - \cancel{E(X)E(Y)} + \cancel{E(X)E(Y)}$$

$$= E(XY) - E(X)E(Y).$$

Independence & Covariance: If X and Y are independent, then $E(XY) = E(X)\,E(Y)$.

$$\therefore \mathrm{Cov}(X,Y) = 0$$

But the converse is NOT true in general.

Example ②: Let the joint probabilities of random variables X and Y are given by:

$X \backslash Y$	-1	0	1
-1	0	$\frac{1}{4}$	0
0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	0	$\frac{1}{4}$	0

Then X and Y are identically distributed and X has the following pmf:

$$P(X=-1) = P(X=1) = \frac{1}{4} \text{ and } P(X=0) = \frac{1}{2}$$

$$E[X] = E[Y] = 0 \quad \text{(Check)}.$$

Furthermore, random var. XY takes values $\{-1, 0, 1\}$ with the pmf

$$P(XY = 1) = 0 = P(XY = -1) \text{ and } P(XY = 0) = 1$$

$$\therefore E(XY) = 0 \Rightarrow \operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y)$$
$$= 0 - 0 \times 0$$
$$= 0$$

However, X and Y are not independent since

$$P(X=-1, Y=-1) = 0 \neq$$
$$P(X=-1) \times P(Y=-1) = \frac{1}{16}$$