Lec-24B

16 February 2021 16:52

Lecture-24

X : r.v.

$E(X)$ Expectation of X

Mathematical Expectation of X

Expected value of X

the mean value of X

Expectation of Discrete r.v. :-

Consider prob. model (S, F, P).

X : discrete r.v.

$R_X : x_1, x_2, x_3, \ldots$

p.m.f. $p_X(x_j) = P(X = x_j) = p_j$

$j = 1, 2, 3, \ldots$

then Expectation of X is defined as

$$\boxed{E(X) = \sum_{x_j \in R_X} x_j p_j = \sum_{x_j \in R_X} x_j P(X = x_j)}$$

provided the right hand sum converges

provided the right hand sum converges absolutely i.e. $\sum_{x_j \in R_X} |x_j| \, p_X(x_j) < \infty$.

Absolute converge implies that the sum $\sum_{x_j \in R_X} x_j \, p_X(x_j)$ converges to a finite value.

Ex :- $\sum (-1)^n \frac{1}{n}$ converges but not converges absolutely.

(*) If $R_X \subset [0, \infty)$ then convergence of infinite series $\sum_{x_j \in R_X} x_j \, p_X(x_j)$ is equivalent to absolute convergence.

Ex :- Let X be a Bernoulli r.v.

$R_X : 0, 1$

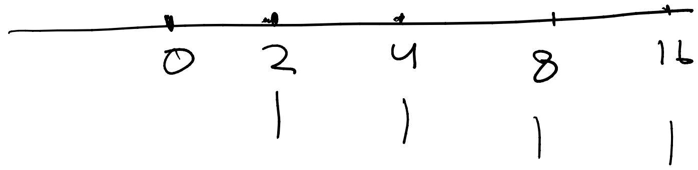
$P(X = 0) = 1 - p$,

$P(X = 1) = p$

$\sum_{x_j \in R_X} x_j \, p_X(x_j)$ will have a finite sum.

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1)$$
$$= p$$

Ex:- Let X be a random variable, that take the value 2^k for each $k \in \mathbb{N}$. $P(X=2^k) = 2^{-k}$



$$P(X=2) = 2^{-1}$$
$$P(X=2^2) = 2^{-2}$$

$$E(X) = \sum_{x_j \in R_X} x_j \, P(X=x_j)$$
$$= \sum_{k=1}^{\infty} 2^k \, P(X=2^k)$$
$$= \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \sum_{k=1}^{\infty} 1 = \infty$$

$\Rightarrow$ Infinite series does not converge to a finite value

$\Rightarrow$ $E(X)$ does not exists.

Ex:- Let X be a r.v. takes values $\frac{(-1)^{n+1} 3^n}{n}$ for each $n \in \mathbb{N}$.

$$x_n = \frac{(-1)^{n+1} 3^n}{n}$$

$$P(X = x_n) = P\left(X = \frac{(-1)^{n+1} 3^n}{n}\right)$$

$$= \frac{2}{3^n}, \quad n = 1, 2, 3, \ldots$$

$$E(X) = \sum_{x_n \in R_X} x_n P(X = x_n)$$

provided $\sum_{x_n \in R_X} |x_n| P(X = x_n) < \infty$

$$\sum_{x_n \in R_X} |x_n| P(X = x_n) = \sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n} \cdot \frac{2}{\cancel{3^n}}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$\Rightarrow$ $E(X)$ does not exist. Although the series

$$\sum x_n P(X = x_n) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{n} \cdot \frac{2}{3^n}$$

$$\sum_{x_n \in R_X} x_n P(X = x_n) = \sum_{n=1} \frac{\cdots}{n} \cdot \frac{\cdots}{3^n}$$

$$= 2\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$$

$$< \infty$$

$\sum (-1)^n u_n$

u_n is strictly decreasing seq^n i.e.

$u_n > u_{n+1} > u_{n+2} > \cdots$

and $\lim_{n \to \infty} u_n = 0$

H.W. $X \sim B(n, p)$

Show that $E(X) = np$

$R_X : 0, 1, 2, 3, \cdots n$

$$E(X) = \sum_{k=0}^{n} x_k P(X = x_k)$$

$$= \sum_{k=0}^{n} x_k \, {}^{n}C_k \, p^k (1-p)^{n-k}$$

$$= np$$

Expectation of a continuous r.v.s

Let X be a r.v. with pdf $f_X(x)$.
Then

$$\underline{E(X)} = \underline{\int_{x \in R_X} x f_X(x)\, dx}$$

provided the r.h.s. integral converges absolutely, i.e. $\underline{\int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty}$.

$\underline{\text{Ex}}$:- Let $X \sim U[a, b]$. $E(X) = ?$

$$\underline{f_X(x)} = \begin{cases} \frac{1}{b-a}, & \underline{a \le x \le b} \\ 0, & \text{elsewhere.} \end{cases}$$

$$E(X) = \underline{\int_{x \in R_X} x f_X(x)\, dx}$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \int_a^b x\, dx$$

$$= \boxed{\frac{b+a}{2}}$$

$\underline{\text{Ex}}$:- X Cauchy's r.v.

$$f_X(x) = \frac{2}{\pi(1+x^2)}, \quad x > 0$$

$R_X : [0, \infty)$

Since R_X is +ive, hence we do not required to check the condition of absolute convergence.

Thus

$$E(X) = \int_0^\infty x \cdot \frac{2}{\pi(1+x^2)} \, dx$$

$$= \frac{1}{\pi} \int_0^\infty \frac{2x}{(1+x^2)} \, dx$$

$$= \frac{1}{\pi} \log(1+x^2) \Big|_0^\infty$$

$$= \frac{1}{\pi} \lim_{x \to \infty} \log(1+x^2) = \infty$$

Expectation does'nt exists.

Exp Let $X \sim \exp(\lambda)$ $R_X : (0, \infty)$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} \, dx$$

$$= \boxed{\frac{1}{\lambda}}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & \text{other} \end{cases}$$

Expectation of Function of Random Variable :-

Discrete Case :- Let X be a discrete r.v. with p.m.f. $p_X(x)$ and let $h: \mathbb{R} \to \mathbb{R}$ be any function.

Then
$$E[h(X)] = \sum_{x \in R_X} h(x)\, p_X(x)$$

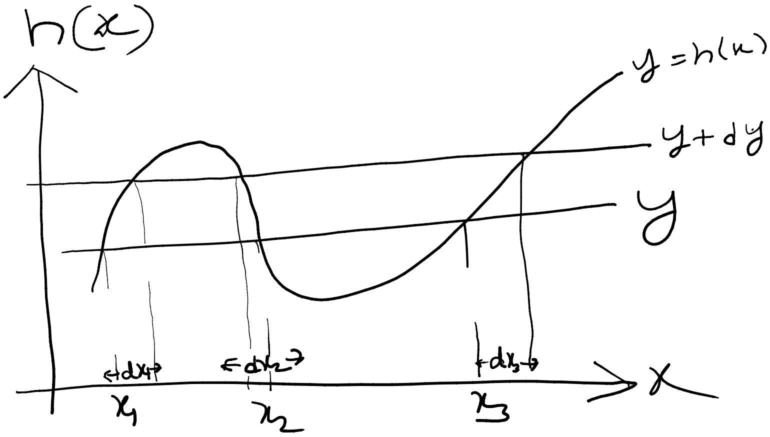
provided $\sum_{x \in R_X} |h(x)|\, p_X(x) < \infty$.

Continuous Case :- Let X be a r.v. with pdf f and $h: \mathbb{R} \to \mathbb{R}$ be (continuous, piecewise continuous) function, then

$$E(h(X)) = \int_{x \in D} h(x) f_X(x)\, dx$$

$$E(h(X)) = \int_{x \in R_X} h(x) f_X(x)\, dx$$

provided $\displaystyle\int_{x \in R_X} |h(x)|\, f_X(x)\, dx < \infty.$



from figure

$f_Y(y)\, dy$

$= f_X(x_1)\, dx_1 + f_X(x_2)\, dx_2 + f_X(x_3)\, dx_3$

$\Longrightarrow y\, f_Y(y)\, dy = h(x_1) f_X(x_1)\, dx_1 + h(x_2) f_X(x_2)\, dx_2 + h(x_3) f_X(x_3)\, dx_3$

$y \quad Y = h(X) \quad f_Y(y)$

$$E(Y) = \int_{y \in R_Y} y\, f_Y(y)\, dy$$

$$= \int_{x \in R_X} h(x) f_X(x)\, dx$$
