Lec-45

Recall: ① **Gamma-distn:** A r.v. X has the pdf

$$g(x;\alpha,\beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x>0$$

$$\alpha>0, \ \beta>0$$

② χ^2_ν **- distn:** Special case of the gamma distn with $\alpha = \frac{\nu}{2}$ & $\beta = 2$.

ν = no. of degrees of freedom

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\frac{\nu-2}{2}} e^{-x/2} & x>0 \\ 0 & \text{o.w.} \end{cases}$$

③ If $X \sim N(0,1)$ then $Z = X^2$ is χ^2_ν - with $\nu = 1$

④ **Moments** & MGF : $E(e^{Xt})$

rth moment $E(X^r)$

Examples of the above distributions :

Example ① : The r-th moment about the origin of the gamma distribution is given by:

$$\mu_r' = \frac{\beta^r \Gamma(\alpha+r)}{\Gamma(\alpha)}.$$

Proof: $$\mu_r' = E(X^r) = \int_0^\infty x^r \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{\beta^r}{\Gamma(\alpha)} \underbrace{\int_0^\infty y^{\alpha+r-1} e^{-y}\, dy}_{\Gamma(r+\alpha)} \qquad y = \frac{x}{\beta}$$

$$= \frac{\beta^r}{\Gamma(\alpha)} \Gamma(\alpha+r).$$

Example (2): Mean and variance of the gamma distribution are given by:

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2$$

Soln: $$\text{Mean} = E(X) = \mu_1' = \frac{\beta\,\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta\,\cancel{\Gamma(\alpha)}\,\alpha}{\cancel{\Gamma(\alpha)}}$$

$$= \alpha\beta.$$

$$[\because \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)]$$

$$E(X^2) = \mu_2' = \frac{\beta^2\,\Gamma(\alpha+2)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2$$

$$\therefore \underset{\text{Mean}}{\mu} = \alpha\beta \text{ and } \underset{\text{Variance}}{\sigma^2} = E(X^2) - E(X)^2$$

$$= \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2.$$

Example (3) Mean & variance of the χ^2-distr are:

$$\mu = \nu \text{ and } \sigma^2 = 2\nu$$

Soln: χ^2 is Gamma with $\alpha = \frac{\nu}{2}$ & $\beta = 2$.

$\therefore$ from example (2), we get

$$\mu = \frac{\nu}{2} \times 2 = \nu \quad \& \quad \sigma^2 = \frac{\nu}{2} \times 4 = 2\nu.$$

Theo: The moment-generating fnc^n of the gamma distribution is given by:

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

Corollary: MGF of χ^2-distr. is given by:

$$M_X(t) = (1 - 2t)^{-\nu/2}.$$

Example (4): If $X_1, X_2, \ldots, X_n$ are indep. ran. var. with $N(0,1)$ distr, then

$$Y = \sum_{i=1}^{n} X_i^2 = X_1^2 + X_2^2 + \cdots + X_n^2$$

has the χ^2-distr. with $\nu = n$ degrees of freedom.

Proof: $\because$ X_i^2 follows χ^2 with $\nu = 1$.

$\therefore$ We find (from the above corollary):

$$M_{X_i^2}(t) = (1-2t)^{-\frac{1}{2}}$$

$\therefore \cancel{M_X(t) =}$ Recall: MGF → special case of the characteristic fn $\Phi_X = E(e^{itX})$

If $X_1, X_2, \ldots, X_n$ are n-indep r.v. then

$$\Phi_{X_1+X_2+\cdots+X_n}(t) = \Phi_{X_1}(t)\,\Phi_{X_2}(t)\cdots\Phi_{X_n}(t)$$

S^{lly},

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\,M_{X_2}(t)\cdots M_{X_n}(t)$$

$$\therefore M_Y(t) = M_{X_1^2+X_2^2+\cdots+X_n^2}(t)$$

$$= M_{X_1^2}(t)\,M_{X_2^2}(t)\cdots M_{X_n^2}(t)$$

$$= (1-2t)^{-\frac{1}{2}} \cdots (1-2t)^{-1/2}$$

$$= (1-2t)^{-\frac{n}{2}}$$

$\therefore Y \sim \chi^2_\nu$ with $\nu = n$.

Statistics: It concerns itself mainly with conclusions and predictions from the chance outcomes that occur in carefully planned experiments / investigation

Recall: If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables, we say that they constitute a random sample from the 'infinite population' given by their common distn. [The word infinite here implies that there is logically speaking no limit to the no. of values that we could observe].

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \rightarrow \text{sample mean}$$

$$S^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \rightarrow \text{sample variance}$$

Distribution :- Sampling Distribution.

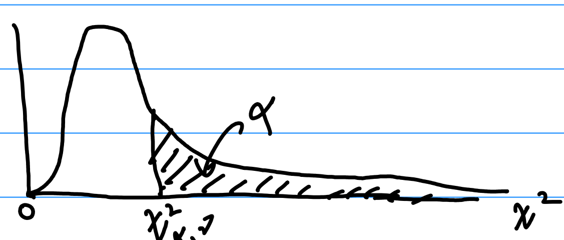
(Importance of χ^2-distn.)

Thm: If $\bar{X}$ and S^2 are the mean and variance of a random sample of size n from a normal population with the mean μ and the standard deviation σ, then

1. $\bar{X}$ and S^2 are independent.
2. The ran. var. $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2- distn. with $(n-1)$ degrees of freedom.

Notation: $\chi^2_{\alpha,\nu}$: If X is a ran. var. having a χ^2-distn with ν degrees of freedom, then

$$P(X \geq \chi^2_{\alpha,\nu}) = \alpha$$



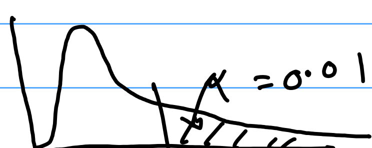
Example (5): Suppose that the thickness of a part used in a semiconductor is its critical dimension and that the process of manufacturing these parts is considered to be under control if the true variation among the thickness of the parts is given by a std. deviation not greater than $\sigma = 0.60$ thousandth of an inch. To keep a check on the process, random samples of size $n=20$ are taken periodically and it is regarded to be "out of control" if the probability that S^2 will take on a value greater than or equal to the observed sample value is 0.01 or less (even though $\sigma = 0.60$). What can one conclude about the process if the std. deviation of such a periodic random sample is $s = 0.84$ thousandth of an inch.

(One can assume that the sample is a random sample from a normal population}.

χ^2-tables: $\chi^2_{0.01,19} = 36.191$)

Soln: From the above thm, we know

$\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2$ with $(n-1)$ deg. of freedom.



$\chi^2_{0.01,19} \to$ out of control

$\therefore$ The process will be declared "out of control" if $\dfrac{(n-1)s^2}{\sigma^2}$ with $n=20$ and $\sigma = 0.60$ exceeds $\chi^2_{0.01,19} = 36.191$

Since $\frac{(n-1)S^2}{\sigma^2} = \frac{19(0.84)^2}{(0.60)^2} = 37.24 > 36.191$

$\therefore$ The process is declared out of control.