Lec-33

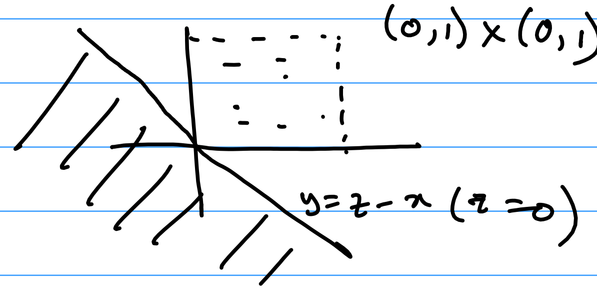
Recall: Last Example (from Lec-32)

$X, Y \to$ iid uniform r.v $(0,1)$

Density of $X+Y = ?$

Soln: $Z := X+Y$ $\qquad A_z = \{(x,y) \in \mathbb{R}^2 \mid x+y \le z\}$

① $-\infty < z \le 0$

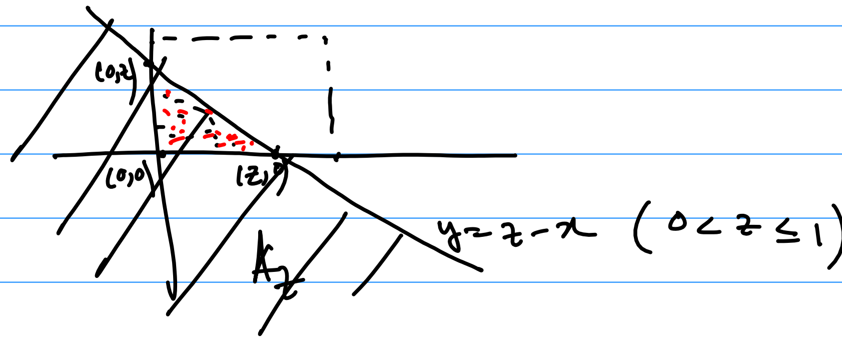


$$F_Z(z) = \iint_{A_z} f(x,y)\,dx\,dy = 0$$

$\quad \shortparallel$

$P(Z \le z)$

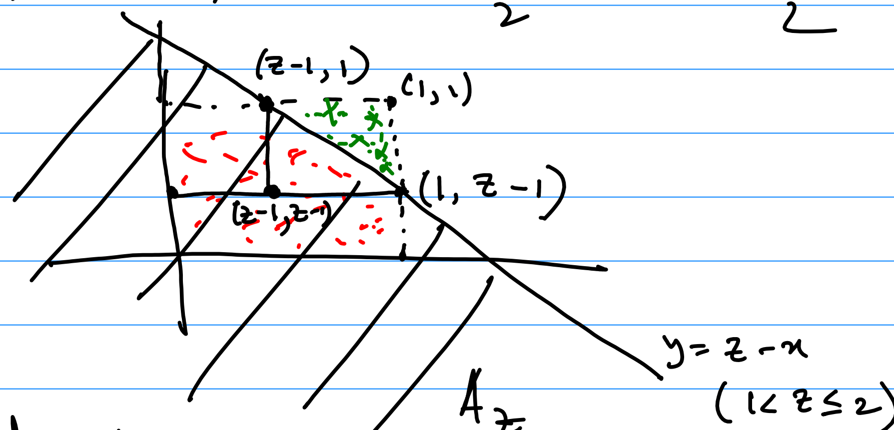
② $0 \le z \le 1$



Area of the red dotted Δ^{le} $= \iint_{A_z} f(x,y)\,dx\,dy$

$=$ Area of the Δ^{le} with vertices $(0,0), (z,0), (0,z)$ $= \frac{1}{2} \times z \times z = \frac{z^2}{2}$

③ $1 < z \le 2$



$$\iint_{A_z} f(x,y)\,dx\,dy$$

$=$ Area of the red region

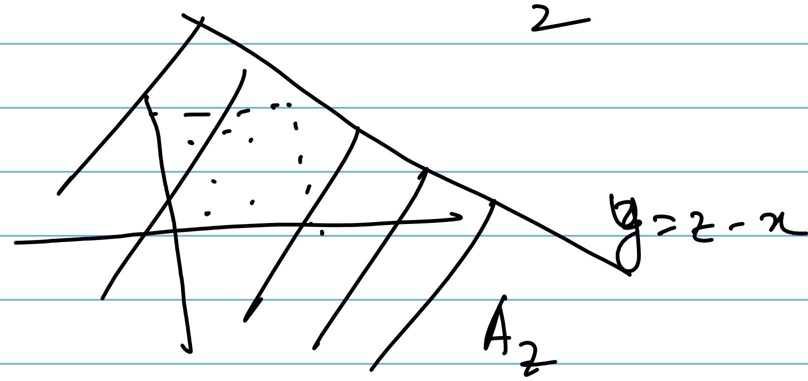
$=$ Area of the unit square $-$ Area of the triangle with green crosses.

= Area of the unit square − area of the triangle with vertices $(1,1)$, $(1, z-1)$, $(z-1,1)$.

$$= 1 - \frac{1}{2} \times (2-z) \times (2-z) = 1 - \frac{(2-z)^2}{2}$$

(4) $z \geq 2$:

$$\iint_{A_z} f(x,y)\, dx\, dy = 1$$



$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{z^2}{2} & \text{if } 0 < z \leq 1 \\ 1 - \frac{(2-z)^2}{2} & \text{if } 1 < z \leq 2 \\ 1 & \text{if } z > 2 \end{cases}$$

It is continuous everywhere. Hence we may differentiate to get the density.

$$F'_Z(z) = \begin{matrix} 0 & \text{if } z < 0 \\ z & \text{if } 0 < z < 1 \\ 2-z & \text{if } 1 < z < 2 \\ 0 & \text{if } z < 2 \end{matrix}$$

This tells us that F_Z is actually differentiable at $z = 0, 1, 2$ also. Hence we get the following pdf of the new r.v. Z:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ z & \text{if } 0 \leq z < 1 \\ 2-z & \text{if } 1 \leq z < 2 \\ 0 & \text{if } z \geq 2 \end{cases}$$

Expectation of function of two random variables:

Theo ①: Let X, Y be two discrete random variables with joint pmf $f(x,y)$. Let $g: \mathbb{R}^2 \to \mathbb{R}$ be a fnth.

Then $$E[g(X,Y)] = \sum_{x \in R(X)} \sum_{y \in R(Y)} g(x,y) f(x,y)$$

provided $$\sum_{x \in R(X)} \sum_{y \in R(Y)} |g(x,y)| f(x,y) < \infty$$

② Let X, Y be two absolutely continuous random variables with joint pdf $f(x,y)$. Let $g: \mathbb{R}^2 \to \mathbb{R}$ be a Borel-measurable fnth, then

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

provided $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x,y)| f(x,y)\, dx\, dy < \infty.$$

Example ①: (Recall last example from Lec-31).

X and Y were random variables with the joint pmf given by:

X \ Y	-1	0	2	6
-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

PMF of $Z = |Y - X|$.

$P(Z=1) = \frac{1}{3}$,

$P(Z=2) = \frac{7}{27}$

$P(Z=3)=\frac{4}{27}$, $P(Z=4)=\frac{1}{27}$, $P(Z=5)=\frac{1}{9}$,

$P(Z=8)=\frac{1}{9}$.

So $$E[Z] = \sum_{z\in R(Z)} z\, P(Z=z)$$

$$= 1\times\frac{1}{3} + 2\times\frac{7}{27} + 3\times\frac{4}{27} + 4\times\frac{1}{27} + 5\times\frac{1}{9} + 8\times\frac{1}{9} = \frac{26}{9}$$

~~Suppose we~~ are interested only ~~in~~

Example ② Let X and Y be independent and exponential random variables with parameters λ and μ respectively. Find the mean of $\max\{X,Y\}$ (if it exists).

$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0 & \text{o.w.}\end{cases}$

y - - .

Soln: Since joint pdf is non-zero in 1^{st} quadrant only and function $\max\{x,y\}$ is also non-negative in first quadrant;

$$E[\max\{X,Y\}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max\{x,y\}\, f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max\{x,y\} f_X(x) f_Y(y)\,dx\,dy \quad [\because X \,\&\, Y \text{ are independent}]$$

$$= \int_{y=0}^{\infty}\int_{x=0}^{\infty} \max\{x,y\} f_X(x) f_Y(y)\,dx\,dy$$

$$= \iint_{(x\geq 0,\, y\geq 0)\cap(y\geq x)} \underline{\max\{x,y\}}\, \lambda\mu e^{-\lambda x} e^{-\mu y}\,dx\,dy$$

$$+ \iint_{(x\geq 0,\, y\geq 0)\cap(y< x)} \max\{x,y\}\, \lambda\mu e^{-\lambda x} e^{-\mu y}\,dx\,dy$$

$$= \lambda\mu\left[\int_0^{\infty} e^{-\lambda x}\left(\underline{\int_{y=x}^{\infty} y e^{-\mu y}\,dy}\right)dx + \int_0^{\infty} x e^{-\lambda x}\left(\int_0^{x} e^{-\mu y}\,dy\right)dx\right] \longrightarrow (*)$$

Now

$$\int_x^{\infty} y e^{-\mu y}\,dy = \left[\frac{y e^{-\mu y}}{-\mu}\right]_{y=x}^{\infty} + \int_x^{\infty}\frac{e^{-\mu y}}{\mu}\,dy \quad \text{--- (1)}$$

$$= \frac{x e^{-\mu x}}{\mu} - \frac{1}{\mu^2}\left[e^{-\mu y}\right]_x^{\infty} = \frac{x e^{-\mu x}}{\mu} + \frac{e^{-\mu x}}{\mu^2}$$

$$\int_0^{x} e^{-\mu y}\,dy = \left[\frac{e^{-\mu y}}{-\mu}\right]_0^{x} = \frac{1-e^{-\mu x}}{\mu} \quad \text{--- (2)}$$

Substituting ① & ② in ⊛ :-

$$E\left[\max\{X,Y\}\right] = \lambda\mu\left[\int_0^\infty e^{-\lambda x}\left(\frac{x e^{-\mu x}}{\mu} + \frac{e^{-\mu x}}{\mu^2}\right)dx + \int_0^\infty x e^{-\lambda x}\left(1 - \frac{e^{-\mu x}}{\mu}\right)dx\right]$$

$$= \lambda\mu\left[\int_0^\infty e^{-\lambda x}\frac{e^{-\mu x}}{\mu^2}\,dx + \int_0^\infty x e^{-\lambda x}\frac{1}{\mu}\,dx\right]$$

$$= \lambda\mu\left[\frac{1}{\mu^2}\left[-\frac{e^{-(\lambda+\mu)x}}{\lambda+\mu}\right]_0^\infty + \frac{1}{\mu}\frac{1}{\lambda^2}\right]$$

$$\left[\int_0^\infty x e^{-\lambda x}\,dx = -\frac{1}{\lambda}\left[x e^{-\lambda x}\right]_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\,dx = -\frac{1}{\lambda^2}\left[e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda^2}\right]$$

$$= \frac{\lambda}{\mu(\lambda+\mu)} + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}.$$

$\blacksquare$

Theo ②: Let X and Y be two R.V. on a prob. sp. $(\Omega, \mathcal{F}, P)$, s.t. both have finite mean. Then

(a) $E[X+Y] = E[X] + E(Y)$

(b) If X and Y are independent, then $E(XY) = E(X)E(Y)$

Proof: We shall prove it only in the case when (X,Y) have joint density, though both the results are true even if X is discrete and Y has pdf (i.e.,

(X,Y) is a mixed ran. vec.).

(a) $$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)\, f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)}\,dx + \int_{-\infty}^{\infty} y \underbrace{\left(\int_{-\infty}^{\infty} f(x,y)\,dx\right)}\,dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\,dx + \int_{-\infty}^{\infty} y\, f_Y(y)\,dy$$

$$= E(X) + E(Y).$$

(b) $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\,dx\,dy$$

$$= \left(\int_{-\infty}^{\infty} x\, f_X(x)\,dx\right)\left(\int_{-\infty}^{\infty} y\, f_Y(y)\,dy\right) = E(X)E(Y)$$