Lecture-19

$X \sim U[a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$$

Ex: $X \sim U(0, 10)$

$P(X < 3) = ?$

$P(X > 6) = ?$

$P(3 < X < 8) = ?$

$$f_X(x) = \begin{cases} \frac{1}{10} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X \leq x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$P(X < 3) = \underbrace{\int_{-\infty}^{0} f_X(x)\, dx} + \int_{0}^{3} f_X(x)\, dx$$

$$= 0 + \int_{0}^{3} \frac{1}{10}\, dx$$

$$= \boxed{\frac{3}{10}}$$

$= \boxed{\dfrac{\text{[illegible]}}{10}}$

$$P(X > 6) = P(6 \leq X < 10)$$

$$= \int_6^{10} \frac{1}{10}\, dx = \boxed{\frac{4}{10}}$$

$$= 1 - P(X \leq 6)$$

$$= 1 - \int_0^6 \frac{1}{10}\, dx = 1 - \frac{6}{10} = \boxed{\frac{4}{10}}$$

Ex.:- Buses arrive at a specified stop at 15 minutes intervals starting at 7:00 AM. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30 AM. Find the probability that he waits

(a) less than 5 minutes for a bus

(b) more than 10 minutes for a bus.

X : Count the number of minutes past 7 that the passenger arrives at the stop.

$$\Rightarrow \quad X \sim U(0, 30)$$

$\Rightarrow$ Passenger will have to wait less than 5 minutes iff he arrives between 7:10 and 7:15 or between 7:25 and 7:30.

$\Rightarrow$ P (Passenger waits less than 5 minutes)

$$= P(10 < X < 15) + P(25 < X < 30)$$

$$= \int_{10}^{15} \frac{1}{30}\, dx + \int_{25}^{30} \frac{1}{30}\, dx = \boxed{\frac{1}{3}}.$$

(b) P (Passenger waits more than 10 minutes)

$$= P(0 < X < 5) + P(15 < X < 20)$$

$$= \boxed{\frac{1}{3}}$$

Exponential Random Variable :- A continuous r.v. whose p.d.f. is given for some $\lambda > 0$ by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is said to be Exponentially distributed with parameter λ.

$X \sim E(\lambda)$, $\lambda > 0$

parameter λ.

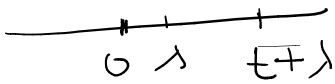
$$R_X : [0, \infty)$$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$F_X(\infty) = \int_0^\infty \lambda e^{-\lambda x}\,dx = 1$$

$$P(X \geq t + s \mid X \geq s)$$



$t + s > 0$

$t > 0$

$s > 0$

$$= \frac{P\{(X \geq t+s) \cap (X \geq s)\}}{P(X \geq s)}$$

$$= \frac{P(X \geq t + s)}{P(X \geq s)} = \frac{1 - P(X < t+s)}{1 - P(X < s)}$$

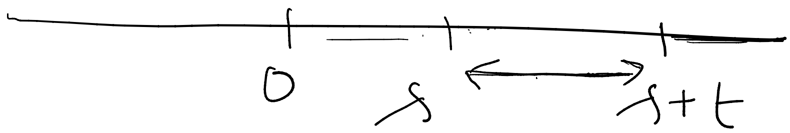
$$= \frac{1 - [1 - e^{-\lambda(t+s)}]}{1 - [1 - e^{-\lambda s}]}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= P(X \geq t)$$

Thus

$$P\left[X \geq t+s \,\middle|\, X \geq s\right] = P(X \geq t)$$



i.e. Exponential r.v. X with parameter λ follow the memoryless property.

The Exponential r.v. often arises in practice as being the distribution of the amount of time until some specific event occurs.

For instance

- the amount of time (starting from now) until an earthquake occurs.
- until a new war breaks out.

X : life time of an instrument
i.e. amount of time until it becomes non-functional.

The Memoryless property of exponential r.v. states that the prob. that the instrument survives for at least $t+s$

r.v. states that the prob. that the instrument survives for at least $t+s$ time units, given that it has survived s time units, is the same as the initial prob. that it survives at least t-time units.

i.e. equipment which has not failed, so far is as good as new equipment.

Ex:- Suppose the life length of an airplane has an exponential distⁿ with parameter $\lambda = t_0$ years. A used airplane is bought by some one. What is the prob. that it will not fail in the next 5 years?

Ans:- X: life time of the airplane and if t_0 years is its actual life ~~do~~ at the present time instant.

Then

$$P(X > t_0 + 5 \mid X > t_0)$$

$$= P(X > 5) = e^{-\lambda x}$$

$= P(X > 5) = e$

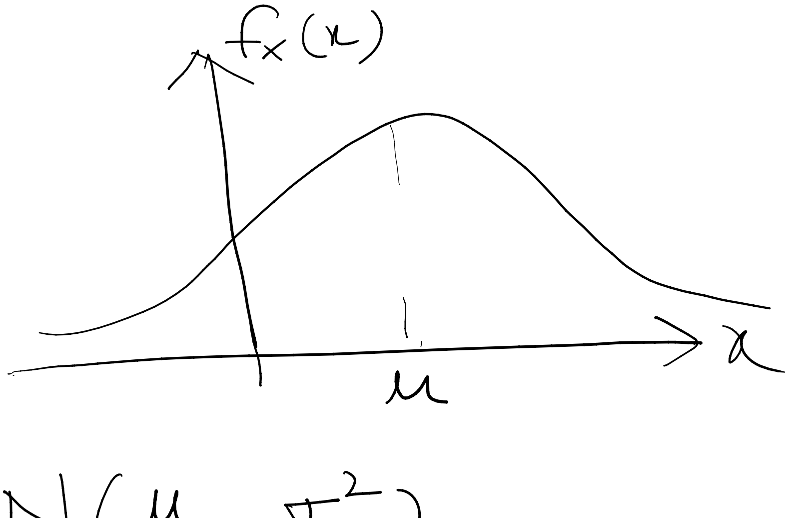
$$= e^{-\frac{1}{10}\times 5} = e^{-\frac{1}{2}}$$

$$= \boxed{0.368}$$

Normal Random Variable:-

We say that X is a normal r.v. (or simply X is normally distributed) with parameters μ and σ^2, if the p.d.f. of X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$



$$X \sim N(\mu, \sigma^2)$$

The constant $\sigma\sqrt{2\pi}$ is the normalizing constant that maintain the area under $f_X(x)$ to be unity.

under $f_X(x)$ to be unity.

If $X \sim N(\mu, \sigma^2)$, then its distⁿ fⁿ is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$\overset{\shortparallel}{P(X \leq x)}$$

$$= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

$$= G\left(\frac{x-\mu}{\sigma}\right)$$

where the function

$$G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$ is available in tabular form.

$$X \sim N(\mu, \sigma^2)$$

$$P(X \leq 5) = ?$$