The LNM Institute of Information Technology
Jaipur, Rajasthan

MTH 222 Probability and Statistics

Tutorial-2

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1. Consider a sequence of independent Bernoulli trials each of which ~~is~~ has a success with probability p. Let X_1 be the number of failures preceding the first success and let X_2 be the number of failures between the first two successes. Find the joint mass function of X_1 and X_2.

$$P(X_1 = m) = (1-p)^m p$$

$$\underbrace{FF\ldots F}_{X_1 = m}S \qquad m = 0, 1, 2, \ldots$$

$$P(X_1 = m, X_2 = n) = (1-p)^m p (1-p)^n p$$

$$\underbrace{FF\ldots F}_{m}S\,\underbrace{FF\ldots F}_{X_2 = n}S$$

Joint pmf:

$$f(m,n) = \begin{cases} (1-p)^{m+n} p^2 & m, n = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

2. The joint probability density of X and Y is given by

$$f(x,y) = c(y^2 - x^2)e^{-y}, \qquad -y \le x \le y, 0 < y < \infty.$$

$|x| \le y$,

(a) Find c.

(b) Find the marginal densities of X and Y.

(c) Find $E[X]$.

(a) $\because f$ is a pdf $\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$\int_{y=0}^{\infty} ce^{-y}\left(\int_{x=-y}^{y}(y^2-x^2)\,dx\right)dy = \int_{y=0}^{\infty} ce^{-y}\left[y^2x - \frac{x^3}{3}\right]_{-y}^{y}dy = \int_{y=0}^{\infty} ce^{-y}\left(\frac{4}{3}y^3\right)dy$$

$$= \frac{4c}{3}\int_0^{\infty} y^3 e^{-y}\,dy = \ldots = 8c = 1$$

$\Rightarrow \boxed{c = \frac{1}{8}}$

(b) $$f_X(x) = \frac{1}{8}\int_{y=|x|}^{\infty}(y^2-x^2)e^{-y}\,dy = \frac{1}{8}\int_{|x|}^{\infty} y^2 e^{-y}\,dy + \left[\frac{1}{8}x^2 e^{-y}\right]_{|x|}^{\infty}$$

$$= \frac{1}{8}\left\{\left[-y^2e^{-y}\right]_{|x|}^{\infty} + 2\int_{|x|}^{\infty} ye^{-y}\,dy\right\} - \frac{1}{8}x^2e^{-|x|}$$

$$= \ldots = \frac{1}{4}(|x|+1)e^{-|x|}$$

For $y > 0$, $f_Y(y) = \frac{1}{8}\int_{x=-y}^{y}(y^2-x^2)e^{-y}\,dx = \frac{1}{8}\cdot\frac{1}{6}y^3e^{-y}$

For $y < 0$, $f_Y(y) = 0$ for $y \le 0$

(c) $$E(X) = \int_{|x|}^{\infty} x\,\frac{|x|+1}{4}e^{-|x|}\,dx = \ldots = 0 \quad \text{(Check)}$$

by breaking into two parts when $x > 0$ & $x < 0$.

OR Note: $f_X(x) = f_X(-x)$, $x > 0$.

$\because f_X(x)$ is symmetric, we have $E(X) = 0$.

3. The joint density function of X and Y is given by

$$f(x, y) = \frac{6}{5}(x + y^2), 0 < x < 1, 0 < y < 1.$$

(a) Verify that $f(x, y)$ is a valid PDF.

(b) Find the marginal distributions of X and Y.

(c) Find $P\{0 < X < \frac{1}{2}, 0 < Y < \frac{1}{2}\}$.

(d) Find $E[X]$ and $E[Y]$.

(a) (i) $f(x,y) \geq 0 \;\; \forall\, (x,y) \in \mathbb{R}^2$ (ii) $\int_{y=0}^{1}\int_{x=0}^{1} \frac{6}{5}(x+y^2)\,dx\,dy$

$$= \frac{6}{5}\int_{x=0}^{1}\left[xy + \frac{y^3}{3}\right]_0^1 dx = \cdots = 1$$

(b) $f_X(x) = \int_{y=0}^{1} \frac{6}{5}(x+y^2)\,dy = \cdots = \frac{6x}{5} + \frac{\cancel{6}^{2}}{\cancel{15}_{5}} = \frac{6x}{5} + \frac{2}{5}$

$f_Y(y) = \int_{x=0}^{1} \frac{6}{5}(x+y^2)\,dx = \cdots = \frac{6}{5}y^2 + \frac{3}{5}$

(c) $P\{0 < X < \frac{1}{2}, 0 < Y < \frac{1}{2}\} = \int_{x=0}^{\frac{1}{2}}\left\{\int_{y=0}^{\frac{1}{2}} \frac{6}{5}(x+y^2)\,dy\right\}dx = \cdots = \frac{1}{10}$

(d) $E(X) = \int_{y=0}^{1} x\left(\frac{6x}{5} + \frac{2}{5}\right)dy = \frac{3}{5}$

(e) $E(Y) = \int_{x=0}^{1} y\left(\frac{6}{5}y^2 + \frac{3}{5}\right)dx = \frac{3}{5}$

4. Find the joint probability density of of the two random variables X and Y whose joint distribution function is given by

$$F(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y}, & \text{for } x > 0,\ y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Use the joint probability density obtained to find $P(X+Y>3)$. Are X and Y independent?

$$\frac{\partial F}{\partial x}(x,y) = \begin{cases} e^{-x} - e^{-x-y}, & x>0,\ y>0 \\ 0 & \text{o.w.} \end{cases}$$

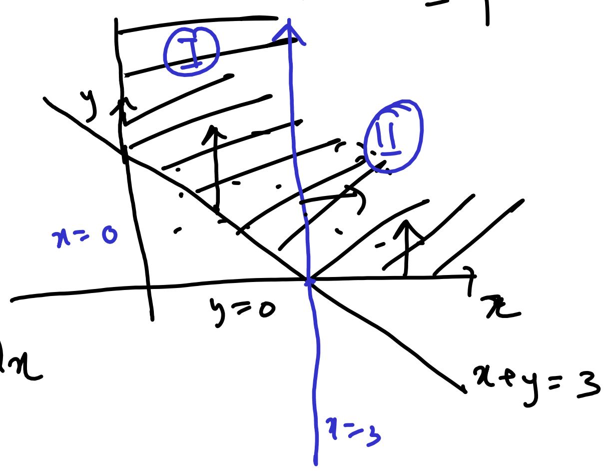
$$\frac{\partial^2 F}{\partial x \partial y}(x,y) = \begin{cases} e^{-(x+y)}, & x>0,\ y>0 \\ 0 & \end{cases}$$

We get the following candidate for our probable pdf : $f(x,y) = \begin{cases} e^{-(x+y)}, & x>0,\ y>0 \\ 0 & \text{o.w.} \end{cases}$

<u>Check</u>: (i) $f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$

(ii) $\displaystyle\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x+y)}\,dx\,dy = -\int_0^{\infty} e^{-y}\left[e^{-x}\right]_0^{\infty} dy = -\int_0^{\infty} e^{-y}(-1)\,dy = 1$

$\therefore f$ is the desired pdf.



$P(X+Y>3) = ?$

$$= \underbrace{\int_{x=0}^{3}\left(\int_{y=3-x}^{\infty} e^{-(x+y)}\,dy\right)dx}_{\text{I}} + \underbrace{\int_{x=3}^{\infty}\left(\int_{y=0}^{\infty} e^{-(x+y)}\,dy\right)dx}_{\text{II}}$$

$$= -\int_0^3 e^{-x}\left[e^{-y}\right]_{3-x}^{\infty} dx - \int_{x=3}^{\infty} e^{-x}\left[e^{-y}\right]_0^{\infty} dx$$

$$= \ldots = 4e^{-3}$$

5. Let $f(x,y,z) = kxyz^2, 0 < x < 1, 0 < y < 1, 0 < z < 2$ be the joint density function of three random variables X,Y and Z. Find $P(Z > X + Y)$.

First find cnt k.

$$\int_{z=0}^{2}\int_{y=0}^{1}\int_{x=0}^{1} kxyz^2\,dx\,dy\,dz = 1 \implies k\left[\frac{z^3}{3}\right]_0^2\left[\frac{y^2}{2}\right]_0^1\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\implies k = \frac{3}{2}$$

$P(Z > X+Y)$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{1}\left(\int_{x+y}^{2} kxyz^2\,dz\right)dy\right)dx$$

$$= \frac{3}{2}\int_{x=0}^{1} x\int_{y=0}^{1} y\left[\frac{2^3-(x+y)^3}{3}\right]dy\,dx$$

$$= \frac{3}{2}\int_{x=0}^{1} x\left(4 - x^3 - \frac{1}{4} - \frac{3x^2}{2} - x\right)x\,dx = \cdots = \frac{9}{10}$$

6. Suppose that A, B, C are independent random variables, each being uniformly distributed over $(0, 1)$.

(a) What is the joint distribution function of A, B, C?

(b) What is the probability that all of the roots of the equation $Ax^2 + Bx + C$ are real?

(a) For $(a,b,c) \in (0,1)\times(0,1)\times(0,1) = (0,1)^3$, we have

$$F_{A,B,C}(a,b,c) = P(A \le a, B \le b, C \le c)$$
$$= P(A \le a)\, P(B \le b)\, P(C \le c) \quad [\because A, B, C \text{ are indep.}]$$
$$= abc$$

Joint CDF

$$F_{A,B,C}(a,b,c) = \begin{cases} abc, & (a,b,c) \in (0,1)^3 \\ 0 & \text{o.w.} \end{cases}$$

Joint pdf:

$$f_{A,B,C}(a,b,c) = \begin{cases} 1, & (a,b,c) \in (0,1)^3 \\ 0 & \text{o.w.} \end{cases}$$

(b) The roots of the eqn $Ax^2 + Bx + C = 0$ are real iff $B^2 - 4AC \geq 0 \Rightarrow B^2 \geq 4AC$. We need to find $P(B^2 \geq 4AC) = ?$

Define $Z := AC$ Note that $Z \in (0,1)$ $[\because A, C \in (0,1)]$

$\therefore$ For $z \in (0,1)$, CDF of $Z := F_Z(z) = P(Z \le z) = P(AC \le z)$

$$= \iint_{A_z} 1 \cdot da\, dc, \quad \text{where } A_z := \{(a,c) \in (0,1)^2 \mid ac \le z\}$$

(I) $0 < a < z$

(II) $z < a < 1$

$$= \underbrace{\int_{a=0}^{z}\left(\int_{c=0}^{1} 1\, dc\right)da}_{\text{I}} + \underbrace{\int_{a=z}^{1}\left(\int_{c=0}^{z/a} 1\, dc\right)da}_{\text{II}}$$

$$= z + z\int_z^1 \frac{da}{a} = z - z\ln z$$

$\therefore$ PDF of $Z := f_Z(z) = F_Z'(z) = 1 - \ln z - 1 = -\ln z$

$\therefore$ Now $P(B^2 \geq 4AC) = P(B^2 \geq 4Z) = \iint_{b^2 \geq 4z} f_B(b)\, f_Z(z)\, db\, dz$ $[\because B, Z \text{ are indep}]$

$$= \int_{b=0}^{1}\left(\int_{z=0}^{b^2/4} (-\ln z)\, dz\right)db = \int_0^1 \left[z - z\ln z\right]_0^{b^2/4} db = \frac{\ln 2}{6} + \frac{5}{36}$$

Ans

7. If X and Y are jointly continuous with joint density function $f_{X,Y}(x,y)$, show that $X+Y$ is continuous with density function

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_{X,Y}(x, t-x).$$

Continuity of $X+Y$, follows from the defn.

$Z := X+Y$

CDF of $Z :=$ $F_Z(z) = P(X+Y \leq z)$

$$= \int_{x=-\infty}^{\infty} \left(\int_{y=-\infty}^{z-x} f(x,y)\, dy \right) dx$$

(This is differentiable by Fund. Thm of Calculus)

$\therefore$ Density: $F_Z'(z) = \displaystyle\int_{-\infty}^{\infty} \left[\frac{d}{dz} \int_{-\infty}^{z-x} f(x,y)\, dy \right] dx$

$$= \int_{-\infty}^{\infty} f(x, z-x)\, dx.$$

8. The trivariate probability density of X_1, X_2 and X_3 is given by:

$$f(x_1,x_2,x_3) = \begin{cases} (x_1+x_2)e^{-x_3}, & \text{for } 0<x_1<1,\ 0<x_2<1,\ x_3>0 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find $P\{(X_1,X_2,X_3 \in A)\}$, where A is the region

$$\left\{(x_1,x_2,x_3)\middle|\ 0<x_1<\frac{1}{2},\ \frac{1}{2}<x_2<1, x_3<1\right\}.$$

(b) Find the joint marginal density of X_1 and X_3.

(c) Find the marginal density of X_1 alone.

(d) Verify that X_1, X_2 and X_3 are not independent, but that the two random variables X_1 and X_3 and also the two random variables X_2 and X_3 are **pairwise independent**.

(a) $$\int_{x_3=0}^{1}\int_{x_2=\frac12}^{1}\int_{x_1=0}^{\frac12} (x_1+x_2)e^{-x_3}\,dx_1\,dx_2\,dx_3 = \cdots = \frac{1}{4}(1-e^{-1})$$

(b) $$f_{X_1,X_3}(x_1,x_3) = \int_{x_2=0}^{1} (x_1+x_2)e^{-x_3}\,dx_2 = \begin{cases} \left(x_1+\frac12\right)e^{-x_3}, & 0<x_1<1,\ x_3>0 \\ 0 & \text{o.w.} \end{cases}$$

(c) $$f_{X_1}(x_1) = \int_{x_3=0}^{\infty}\int_{x_2=0}^{1} f(x_1,x_2,x_3)\,dx_2\,dx_3 = \int_{x_3=0}^{\infty} f_{X_1,X_3}(x_1,x_3)\,dx_3$$

$$= \begin{cases} \left(x_1+\frac12\right), & 0<x_1<1 \\ 0 & \text{o.w.} \end{cases}$$

(d) Check: $f_{X_2}(x_2) = x_1+\frac12$

$f_{X_3}(x_3) = e^{-x_3}$ $\quad\therefore f(x_1,x_2,x_3) \neq f_{X_1}f_{X_2}f_{X_3}$

$\Rightarrow X_1, X_2, X_3$ are NOT indep.

But, $f(x_1,x_3) = (x_1+x_2)e^{-x_3} = f_{X_1}f_{X_3}$ $\quad\therefore X_1, X_3 \rightarrow$ indep

$\ldots f(x_2,x_3) \ldots$ Similar.

9. The joint density of X and Y is

$$f(x, y) = c(x^2 - y^2)e^{-x}, 0 \leq x < \infty, -x < y < x.$$

Find the conditional distribution of Y given $X = x$.

Similar to Qn. 2 (Just with x & y exchanged)

$$f_{Y|X} = \frac{f(x,y)}{f_X} \quad . \quad (x > 0).$$

10. The joint density function of X and Y is

$$f(x,y) = xe^{-x(y+1)}, \qquad x > 0, y > 0.$$

(a) Find the conditional density of X, given $Y = y$, and that of Y, given $X = x$.

(b) Find the density function of $Z = XY$.

ⓐ $f_{X|Y}(x|y) = \dfrac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}\,dx} = (y+1)^2\, xe^{-x(y+1)} \qquad x>0$

$\dfrac{f(x,y)}{f_Y(y)}$

$f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}\,dy} = xe^{-xy}, \quad y>0$

ⓑ $Z := XY \qquad F_Z(z) = P(XY \leq z)$

Ⅰ $0 < x < z$

Ⅱ $z < x < \infty$

$= \int_{x=0}^{z} \left(\int_{y=0}^{1} xe^{-x(y+1)}\,dy \right.$

$\left. + \int_{x=z}^{\infty} \left(\int_{y=1}^{z/x} xe^{-x(y+1)}\,dy \right) dx \right.$

$= \int_{x=0}^{\infty} \left(\int_{y=0}^{z/x} xe^{-x(y+1)}\,dy \right) dx = \int_0^\infty (1-e^{-z})e^{-x}\,dx = 1-e^{-z}$

$\therefore$ PDF $= f_Z(z) = \begin{cases} e^{-z}, & z > 0 \\ 0 & \text{o.w.} \end{cases}$