Lecture-5:A2

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$A \cup B$, AB, $\bar{A}$, $\bar{B}$ etc corresponds to A & B also qualify to be events.

Field F.

We shall see that we will not consider as events of S but only a class F of subsets.

Field: A collection of subsets of a non-empty set S form a field F if

(i) $S \in F$

(ii) If $A \in F$ then $\bar{A} \in F$

(iii) If $A \in F$ and $B \in F$ then $A \cup B \in F$

Using (i) — (iii) it is easy to show that AB, $\bar{A}B$ etc also belong to F. For example from (ii), we have

If $A \in F$, $B \in F$ then $\bar{A} \in F$, $\bar{B} \in F$

and then using (iii), we have

$$\bar{A} \cup \bar{B} \in F$$

Applying (ii) again we get $\overline{\bar{A} \cup \bar{B}} \in F$

i.e. $A \cap B = AB \in F$.

Thus if $A \in F$, $B \in F$, then

$$F = \{S, A, B, \bar{A}, \bar{B}, A \cup B, AB, \bar{A} \cup B, A \cup \bar{B}, \overline{\bar{A} \cap \bar{B}} \text{---}\}$$

From here onwards, we shall reserve the term "event" only to members of field F.

From this it follows that all sets that can be written as ~~as~~ union or intersections of finitely many sets in F are ~~a~~ also in F. This is not necessarily the case for infinitely many sets.

<u>Borel Field (σ-algebra)</u> :- Suppose that

$A_1, A_2, \text{---} -$ is an infinite sequence of sets in F. If the union and intersection of these sets also ~~at~~ belong to F, then F is called a <u>Borel field</u>.

Axioms of Infinite Additivity:-

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If the events $A_1, A_2, \ldots$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Axiomatic Definition of an experiment: An experiment is specified in terms of the following concepts:

(1) The set S of all experimental outcomes.

(2) The Borel field F of events of S

(3) The probabily P of these events

(S, F, P) Probability model.

Countable Spaces: Suppose S is finite and consists of N possible outcomes where consists of N possible outcomes, where N is a finite number, i.e.,

$$S = \{s_1, s_2, \text{——} \text{——}, s_N\}$$

s_i : Elementary events

then the probabilities of all events can be expressed in terms of the probabilities:

$$P(s_i) = p_i$$

of the elementary events $\{s_i\}$.

Power set $\mathcal{P}(S) = \{A_1, A_2, \text{——}, A_{2^N}\}$

$$P(A_k) = ?$$

Specification of $P(s_i)$, $i = 1, 2, \text{——} N$ gives all $P(A_i)$.

From the axioms of the probability, it is clear that

$$p_i \geq 0, \quad P(S) = p_1 + p_2 + \text{——} + p_N = 1$$

Suppose an event A_k consisting of the r elementary events say s_{ki}, $i = 1, 2, \text{——} r$, i.e.;

$A_k = \{s_{k1}, s_{k2}, \text{——}, s_{kr}\}$, then

$A_k = \{ s_{k1}, s_{k2}, \text{———}, s_{kr} \}$, then

$$P(A_k) = P(s_{k1}) + P(s_{k2}) + \text{———} + P(s_{kr})$$

$$= p_{k1} + p_{k2} + \text{———} + p_{kr}$$

This is true even if S consists of an also infinite but <u>countable</u> number of elementary events or elements s_1, s_2, ——— — .

Ex – Consider the experiments of the toss of a fair coin three times

$$S = \{ \underset{s_1}{hhh}, \underset{s_2}{hht}, \underset{s_3}{hth}, \underset{s_4}{thh}, \underset{s_5}{htt}, \underset{s_6}{tht}, \underset{s_7}{tth}, \underset{s_8}{ttt} \}$$

fair coin $p_1 = \frac{1}{8}$ $p_2 = \frac{1}{8}$ ——— $p_8 = \frac{1}{8}$

A: heads at the first two tosses

$$A = \{ hhh; hht \}$$

$$P(A) = P(hhh) + P(hht)$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \boxed{\frac{1}{4}}$$

<u>Note</u> – In the case of countable infinite sample space, we have

$$\sum_{i}^{\infty} P(s_i) = 1 \qquad \underset{\text{Countable}}{S} = \{ s_1, s_2 \text{ ——— } \}$$

$$\sum_{i=1} P(s_i) = 1$$

Countable ... }

$$P(S) = 1 = P(s_1) + P(s_2) + \cdots$$

$$= \sum_{i=1}^{\infty} P(s_i)$$

$$\sum_{i=1}^{\infty} P(s_i) = 1$$

$$\boxed{\sum_{i=1}^{\infty} x_i = l \qquad \{x_i\} \to 0}$$

This is an infinite series converges to 1.

$$\Longrightarrow \{P(s_i)\} \longrightarrow 0$$

$(S, P(S), P) \longrightarrow$ Probability model finite

$(S, F, P) \longrightarrow$ Probability model, countable infinite.

Uncountable Sample space :- If sample space S is uncountable infinite, then its probabilities can not determined in terms of the probabilities of the elementary events.

To complete the specification of S, it is sufficient to assign the probability is sufficient to assign the probability to the events $\{x \le x_i\}$. All other probabilities can be determined from the axioms.

Suppose that $\alpha(x)$ is a f^n s.t.

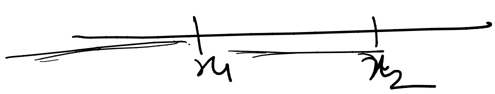
$$\int_{-\infty}^{\infty} \alpha(x)\,dx = 1, \quad \alpha(x) \geq 0$$

We define the probability of the event $\{x \le x_i\}$ by the following integral

$$P\{x \le x_i\} = \int_{-\infty}^{x_i} \alpha(x)\,dx$$

This specifies the probabilities of all events of S.

$\{x_1 < x \le x_2\}$ consisting of all points of $(x_1, x_2]$.



$\{x \le x_1\}$ and $\{x_1 < x \le x_2\}$ are mutually exclusive, and their union is $\{x \le x_2\}$. Hence

$$P\{x \le x_2\} = P\{x \le x_1\} + P\{x_1 < x \le x_2\}$$

$$\Rightarrow P\{x_1 < x \le x_2\} = P\{x \le x_2\} - P\{x \le x_1\}$$

$$= \int_{-\infty}^{x_2} \alpha(x)\,dx - \int_{-\infty}^{x_1} \alpha(x)\,dx$$

$$= \int_{x_1}^{x_2} \alpha(x)\,dx \quad \text{——— (1)}$$

if $\alpha(x)$ is bdd, and $x_1 \to x_2$

then/the integral (1) $\longrightarrow 0$.

$P\{x_2\} = 0, \quad x_2 \in \mathbb{R}$.

$P(S) = P(\mathbb{R}) = 1$.

This is not in conflict, because the total number of elements of S is not countable.

$(S, F, P) \longrightarrow$ Probability model.