Lec-44

Convolution: $f * g \to$ pdf of $X + Y$ $\quad$ $f, g \to$ pdf X & Y resp. (X, Y are ind.).

$$(f*g)(s) = \int_{-\infty}^{\infty} f(t)\, g(t-s)\, ds$$

$X_1, X_2, \ldots$

$f_1 * f_2 * \ldots$

$$((f*g)*h) = f*(g*h).$$

Example (1): CLT. Let $X_1, X_2, \ldots$ be iid Poisson(λ) RVs. Then we know $E(X_1) = Var(X_1) = \lambda$.

Hence by CLT, $S_n = X_1 + X_2 + \cdots + X_n$ has approximately an $N(n\lambda, n\lambda)$ distribution for large n.

Let $n = 64$, $\lambda = 0.125 \Rightarrow n\lambda = 8$

Also sum of independent Poisson is again Poisson. Hence exact distribution of $S_{64} \sim$ Poisson$(\underbrace{64 \times 0.125}_{=8})$ & from Poisson distribution tables

$$P(S_{64} = 10) = \frac{8^{10} e^{-8}}{10!} = 0.09926153\text{4}$$

Using normal approximation,

$$P(S_n = 10) \simeq P(9.5 < S_n < 10.5)$$

$$= P\left(\frac{9.5 - 8}{\sqrt{0.125 \times 8}} < \frac{S_n - n\lambda}{\lambda\sqrt{n}} < \frac{10.5 - 8}{\sqrt{0.125 \times 8}}\right)$$

$= P(0.530330086 < Z < 0.8838834776)$

$= 0.1087$

Here we have used the "continuity correction" to compute the $P(S_n = 10)$ which would be zero if S_n is taken to be normal ran-var.

Recall: Characteristic function of a $\Phi(x)$

$$\Phi_X(t) = E[e^{iXt}]$$

$X + iY$

Moment Generating Function (MGF) of a r.v. X is a fn $M_X(t)$ defined as:

$$M_X(t) = E[e^{Xt}], \quad t \in \mathbb{R}, \text{ provided } E[e^{Xt}] < \infty.$$

We say MGF of X exists if $\exists$ a +ve const. a s.t. $M_X(t)$ is finite $\forall$ $t \in [-a, a]$.

Recall: Moments The n^{th} moment of $X :=$

$$E(X^n).$$

Finding moments from MGF:

$\forall x \in \mathbb{R}$, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} = \sum_{k=0}^{\infty} \frac{X^k t^k}{k!}.$$

$$M_X(t) = E\left[e^{tX}\right] = \sum_{k=0}^{\infty} E\left[X^k\right] \frac{t^k}{k!}$$

$\therefore$ The k-th moment of X is the coefficient of $\frac{t^k}{k!}$ in the Taylor series of $M_X(t)$.

Eg $\quad Y \sim U(0,1)$

$$M_Y(t) = E\left[e^{tY}\right] = \int_0^1 e^{ty}\,dy = \frac{e^t - 1}{t}$$

$$= \frac{1}{t}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} - 1\right) = \frac{1}{t} \sum_{k=1}^{\infty} \frac{t^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{t^k - 1}{k!}$$

Coeff of $\frac{t^k}{k!}$: $\frac{1}{k+1}$ $\qquad \therefore E(X^k) = \frac{1}{k+1}.$

Gamma Function

$X \rightarrow$ ran. var. s.t. its pdf is:

$$f(x) = \begin{cases} k x^{\alpha-1} e^{-x/\beta} & , \quad x > 0 \\ 0 & \quad \text{o.w.} \end{cases}$$

Where $\alpha > 0$, $\beta > 0$ and k must be s.t. the total area under the curve is equal to 1.

To evaluate k, we first substitute $y = \frac{x}{\beta}$.

$$\therefore \int_0^\infty k x^{\alpha-1} e^{-x/\beta} dx = k\beta^\alpha \int_0^\infty y^{\alpha-1} e^{-y} dy$$

The integral thus obtained depends on α alone and it defines the well-known gamma function

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy, \quad \text{for } \alpha > 0.$$

Integrating by parts:

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1).$$

For $\alpha > 1$, and since $\Gamma(1) = \int_0^\infty e^{-y} dy = 1$

it follows by repetition that $\Gamma(\alpha) = (\alpha - 1)!$ whenever α is a +ve integer.

Also $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

So we get $\int_0^\infty k x^{\alpha-1} e^{-x/\beta} dx = k \beta^\alpha \Gamma(\alpha) = 1$

$$\Rightarrow k = \frac{1}{\beta^\alpha \Gamma(\alpha)}.$$

Gamma distribution:

A ran. var. X has a gamma distribution and it is referred to as a gamma random variable iff its pdf is given by:

$$g(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

Defn: A random variable X has a chi-square or χ^2_ν - distribution and it is referred to as a chi-square random variable iff its pdf is given by:

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2} \Gamma(\nu/2)} x^{\frac{\nu-2}{2}} e^{-x/2}, & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

The parameter ν is referred to as the number of degrees of freedom or simply the degrees of freedom.

Note: χ^2-dist. is a special case of the gamma-dist. with $\alpha = \frac{\nu}{2}$, $\beta = 2$.

Importance (Th^m:) If $X \sim N(0,1)$, then $Z = X^2$ follows χ^2-dist. with $\nu = 1$.

Before going to the proof of the above th^m, let us understand the following result:

Th^m (A): Let X be a cont. ran. var. & $Y = u(X)$ be a fn^n of X. Let $f(x)$ be the value of the pdf of the cont. ran. var. X at x. If the fn^n given by $y = u(x)$ is differentiable and either increasing or decreasing for all values within the range of X for which $f(x) \neq 0$, then for these values of x, the equation $y = u(x)$ can be uniquely solved for x to give $x = w(y)$ and for the corresponding values of y the pdf of $Y = u(X)$ is given by

$$g(y) = \begin{cases} f[w(y)] \cdot |w'(y)| & \text{provided } u'(x) \neq 0 \\ 0 & \text{o.w.} \end{cases}$$

Example: If X has the exponential distr.

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

Find pdf of $Y = \sqrt{X}$.

Soln: The eqn $y = \sqrt{x}$ has the unique inverse

$x = y^2$ $\Rightarrow$ $w'(y) = \dfrac{dx}{dy} = 2y$.

$x = w(y)$

$$\therefore g(y) = f(w(y))\,|w'(y)| = e^{-y^2}\,|2y|$$

$$= 2y\,e^{-y^2}, \quad y > 0$$

pdf of Y:

$$g(y) = \begin{cases} 2y e^{-y^2}, & y > 0 \\ 0 & \text{o.w.} \end{cases}$$

$Y = \sqrt{X}$

CDF of $Y = P(Y \le y)$

$= P(\sqrt{X} \le y)$

$= P(X \le y^2)$

HW (check). $=$]

Back to the proof of our main important thm:

$Z := X^2$

Proof: $\because$ The function given $z = x^2$ is decreasing for $-$ve values of x and increasing for $+$ve values of x, the conditions of thm Ⓐ are not met. However, the transformation from X to Z can be made in two steps.

① First, we find the pdf of $Y = |X|$.

② Then we find the pdf of $Z = Y^2 (= X^2)$.

①: $Y = |X|$

CDF: $F_Y(y) = P(Y \le y) = P(|X| \le y)$

$$= P(-y \le X \le y)$$

$$= F_X(y) - F_X(-y).$$

Differentiate:

$$f_Y(y) = f_X(y) + f_X(-y).$$

$\because |x|$ cannot be negative, $f_Y(y) = 0$ for $y < 0$

$$\therefore f_Y(y) = f_X(y) + f_X(-y)$$

$$= \begin{cases} \dfrac{2}{\sqrt{2\pi}} e^{-y^2/2}, & y > 0 \\ 0 & \text{o-w}. \end{cases}$$

② $z = y^2$ is increasing for $y > 0$, that is, for all values of Y for which $f_Y(y) \neq 0$.

$\therefore$ We can use thm Ⓐ. Since $\frac{dy}{dz} = \frac{1}{2} z^{-\frac{1}{2}}$,

we get $h(z) = \frac{2}{\sqrt{2\pi}} e^{-z/2} \left| \frac{1}{2} z^{-\frac{1}{2}} \right|$

$$= \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-z/2}, \quad z > 0$$

& $h(z) = 0$ o.w.

$$h(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-z/2} & z > 0 \\ 0 & \text{o.w.} \end{cases}$$

$\because \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\therefore$ The distr. of Z:

$$h(z) = \begin{cases} \frac{1}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})} z^{-\frac{1}{2}} e^{-z/2}, & z > 0 \\ 0 & \text{o.w.} \end{cases}$$

$\therefore Z \sim \chi^2_{\nu}$ with $\nu = 1$.