1) X and Y are $U(0,1)$

Find $E[|X-Y|]$

More generally, find $E[|X-Y|^\alpha]$.

Solution:-

$$E[|X-Y|^\alpha] = \iint_{x>y} (x-y)^\alpha \, dx\, dy + \iint_{y>x} (y-x)^\alpha \, dx\, dy$$

$$= 2\iint_{x>y} (x-y)^\alpha \, dx\, dy.$$

$$= 2\int_{y=0}^{1}\int_{x=y}^{1} (x-y)^\alpha \, dx\, dy.$$

$$= 2\int_{y=0}^{1} \left[\frac{(x-y)^{\alpha+1}}{\alpha+1}\right]_{y}^{1} dy$$

$$= 2\int_0^1 \frac{(1-y)^{\alpha+1}}{\alpha+1} \, dy.$$

$$= 2\left[-\frac{(1-y)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right]_0^1 = \frac{2}{(\alpha+1)(\alpha+2)}.$$

(2) If X and Y are independent binomial random variables with parameters (n,p) and (m,p) resp. what is the distribution of $X+Y$?

Solution:- $M_X(t) = (pe^t + 1-p)^n$

$M_Y(t) = (pe^t + 1-p)^m$

$\therefore$ Since X, Y are independent

$M_{X+Y}(t) = M_X(t)\, M_Y(t)$.

$= (pe^t + 1-p)^{m+n}$.

$\therefore \quad X+Y \sim Bin(m+n, p)$.

(3.) Let X be a positive random variable with $EX = 10$. Then which is true for $E[e^{1/X+1}]$.

Solution:- $f(x) = e^x$ $\quad g(x) = 1/x+1$

Here f is convex and non decreasing and g is convex.

So $f \circ g$ is convex (done in Lecture 48 Q. 10)

$\therefore$ By Jensen's inequality,

$E[e^{1/X+1}] \geqslant e^{1/EX+1} = e^{1/11}$.

(4) Suppose that joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-x/y} e^{-y} & 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X>1 \mid Y=y)$

<u>Solution</u>:- If $y \leq 0$ then $P(X>1 \mid Y=y] = 0$.

So let $y>0$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{e^{-x/y} e^{-y}}{e^{-y} \int_0^\infty e^{-x/y}\,dx}$$

$$= \frac{e^{-x/y}}{y\left(-e^{-x/y}\Big|_0^\infty\right)}$$

$$= \frac{e^{-x/y}}{y \times 1} = \frac{e^{-x/y}}{y}.$$

$$P(X>1 \mid Y=y) = \int_1^\infty \frac{1}{y} e^{-x/y}\,dy$$

$$= -e^{-x/y}\Big|_1^\infty = e^{-1/y}.$$

5. Let X_i, $1 \le i \le 10$, be independent random variables each uniformly distributed over $(0,1)$. Calculate an approximation of $P(\sum_{i=1}^{10} X_i > 6)$.

Solution:- $E X_i = 1/2$ $\quad$ $Var X_i = 1/12$.

By CLT,

$$P\left(\sum_{i=1}^{10} X_i > 6\right) = P\left(\frac{\sum_{i=1}^{10} X_i - 5}{\sqrt{10 \times \frac{1}{12}}} > \frac{6-5}{\sqrt{10 \times \frac{1}{12}}}\right)$$

$$= 1 - \Phi(\sqrt{1.2})$$

$$\approx 0.1367$$

(6.) A standard normal random variable X satisfies

$EX = 0$, $EX^2 = 1$, $EX^3 = 0$, $EX^4 = 3$

Let $Y = a + bX^3$. Find $\rho(X, Y)$.

Solution :-

$$\rho(X,Y) = \frac{E[XY] - EX\,EY}{\sigma_X \sigma_Y}$$

$$E[XY] = E[aX + bX^4] = a\,EX + b\,EX^4$$
$$= 3b$$

$EX = 0.$

$EY = a + b\,EX^3 = a.$

$\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{EX^2 - (EX)^2} = \sqrt{1 - 0} = 1$

$\sigma_Y = \sqrt{\operatorname{Var} Y} = \sqrt{EY^2 - (EY)^2}$

but we are not given enough data to compute EY^2. So the answer is "none of these."

(7.) Let X and Y be random variables such that $Y = X^2$ and $E[X^i] = i-1$. What is $\rho(X,Y)$.

<u>Solution:-</u> $EX = 0$. $\quad \mathrm{Var}\,X = EX^2 - (EX)^2 = 1$

$EY = EX^2 = 1 \quad \mathrm{Var}\,Y = EX^4 - (EX^2)^2 = 3-1 = 2$.

$EXY = EX^3 = 2$.

$$\rho(X,Y) = \frac{E[XY] - EX\,EY}{\sigma_X \sigma_Y}$$

$$= \frac{2-0}{\sqrt{2}} = \sqrt{2}.$$

(8.) Let X_i, $i=1,2,\ldots n$ be independent $N(0,1)$ random variables. Then $\sum_{i=1}^{n} X_i^2$ follows χ^2- distribution with n degrees of freedom.

(see class notes).