Lec-30

Recall: Joint CDF

Joint CDF and Joint Density

The joint cdf is usually not very handy for a discrete random vector. But for a r. vec. with density, we have the following important relationship.

$$F(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x,y)\, dx\, dy \quad \forall\ x_1, x_2 \in \mathbb{R}. \qquad \text{--- } (1)$$

(1) As in the one-dimensional case, joint density $f(x,y)$ is not uniquely defined by (1). We can change f at a finite no. of points or even over a finite no. of smooth curves in the plane without affecting integrals ~~for~~ of f over sets in the plane.

(2) Given joint CDF $F(x,y)$, we can determine the joint PDF $f(x,y)$ through the following formula:

$$f(x,y) = \frac{\partial^2 F}{\partial x\, \partial y} \qquad \text{--- } (2)$$

for every (x,y) at which the joint PDF f is continuous. This relationship is useful when in situations where an expression for $F(x,y)$ can be found. The mixed partial derivative can be computed to find joint pdf.

Example ①   Suppose a joint cdf is given by:

$$F(x,y)=\begin{cases} 6xy+y^3-3y^2-3x^2y, & 0<y<x,\ 0<x<1 \\ 3x^2-2x^3, & 0<x<1,\ y\geq x \\ 3y+y^3-3y^2, & 0<y<1,\ x\geq 1 \\ 1, & y\geq 1,\ x\geq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the joint density (if it exists).

Soln: ① Instead of checking that the $F(x,y)$ is continuous everywhere on the plane $\mathbb{R}^2$ and then obtaining mixed partial derivatives to obtain density, it is usually simpler to compute the mixed partials first and show that the fnⁿ f obtained from ② satisfies both the conditions of the thᵐ done in Lec-28.

② We may avoid the boundary points of various regions and compute the mixed partials in the interior points.

$$\frac{\partial F}{\partial x}(x,y)=\begin{cases} 6y-6xy, & 0<y<x,\ 0<x<1 \\ 6x-6x^2, & 0<x<1,\ y>x \\ 0, & 0<y<1,\ x>1 \\ 0 & y<1,\ x>1 \\ 0 & \text{o.w.} \end{cases}$$

Further $\frac{\partial^2 F}{\partial x \partial y}(x,y) = \begin{cases} 6-6x, & 0<y<x, \ 0<x<1 \\ 0, & 0<x<1, \ y>x \\ 0, & 0<y<1, x>1 \\ 0, & y<1, \ x>1 \\ 0 & \text{o.w.} \end{cases}$

$\therefore$ We obtain the possible joint pdf as:

$$f(x,y) = \begin{cases} 6(1-x), & 0<y<x, \ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Now let us check the reqd properties for f to be the joint pdf

(a) Clearly, $f(x,y) \geq 0 \quad \forall \ (x,y) \in \mathbb{R}^2$

(b) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1 \left[\int_{x=y}^{1} 6(1-x)\,dx\right] dy$$

$$= \ldots = 1$$

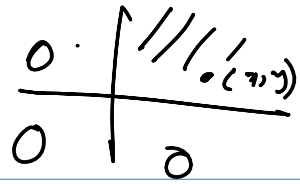
Hence f is the desired joint pdf.

Example (2) Let (X, Y) be a random vector with joint pdf given by

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0<x<\infty, \ 0<y<\infty \\ 0 & \text{o.w.} \end{cases}$$

Determine the joint cdf.

Soln: If either $x \le 0$ or $y \le 0$, the joint cdf $F \equiv 0$ as the joint pdf is 0 in this region.



Let (x,y) be an interior point of the 1st quadrant.

Then

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\, ds\, dt = \int_{0}^{x} \int_{0}^{y} e^{-(s+t)}\, ds\, dt$$

$$= \left[\int_{0}^{x} e^{-s}\, ds \right] \left[\int_{0}^{y} e^{-t}\, dt \right] = \left(1 - e^{-x}\right)\left(1 - e^{-y}\right)$$

$\therefore$ The joint cdf is

$$F(x,y) = \begin{cases} \left(1 - e^{-x}\right)\left(1 - e^{-y}\right), & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{o.w.} \end{cases}$$

Independent Random Variables

→ Analogous to the concept of independence between events.

They are developed by simply introducing suitable events involving the possible values of various random var. & considering the independance of these events.

Defn: Let $(\Omega, \mathcal{F}, P)$ be a probability space and (X, Y) be a random vector defined on it. We say that the random variables X and Y are <u>independent</u> if events $\{X \in A\}$ and $\{Y \in B\}$ are independent for every Borel subset A and B of $\mathbb{R}$.

Intuitively, independence means that the value of Y provides no information on the value of X.

Example (3) Consider the experiment of tossing a fair coin and rolling a fair die simultaneously.

Let $X \to$ ran. var s.t. $\{X = 1\} \to$ Head &

$\{X = 0\} \to$ Tail.

Let $Y \to$ ran. var. s.t. $\{Y = 1\} \to$ 1 appears on the face of the die.

$\vdots$

$\{Y = 6\} \to$ 6 appear . . . die

So, we have:

$$\Omega = \{ (H, i), (T, i) \mid i = 1, 2 \dots, 6 \}, \quad \mathcal{F} = \mathcal{P}(\Omega)$$

$$P(\omega) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \quad \forall \omega \in \Omega$$

Define ran. var. $X : \Omega \to \mathbb{R}$ as

$$X(H, i) = 1, \quad X(T, i) = 0 \quad \forall i = 1, 2, \dots, 6.$$

Define ran. var. $Y : \Omega \to \mathbb{R}$ as

$$Y(H, i) = i = Y(T, i), \quad \forall i = 1, \dots 6.$$

Then $P(X = 1, Y \in \{3, 4\}) = P(\{H, i\} : i = 1, \dots 6\} \cap \{(H,3), (H,4), (T,3), (T,4)\})$

$$= P(\{(H,3), (H,4)\}) = \frac{2}{12} = \frac{1}{6}$$

$$P(X = 1) = P(\{(H, i) : i = 1, \dots 6\}) = \frac{1}{2}$$

$$P(Y \in \{3, 4\}) = P(\{(H,3), (H,4), (T,3), (T,4)\}) = \frac{1}{3}$$

We have: $P(X = 1, Y \in \{3, 4\}) = \frac{1}{6}$

$$= P(X = 1)\, P(Y \in \{3, 4\})$$

One can characterize the independence of X and Y in terms of its joint and marginal distr. fn^{n}s.

Theo. (1) Let (X, Y) be a ran. vec. with joint distr. fn^n F, and let F_X and F_Y be the distr. fn^{n}s of X and Y respectively. Then X and Y are independent iff

$$F(x,y) = F_X(x)\, F_Y(y) \quad \forall\ (x,y) \in \mathbb{R}^2.$$

Remark (1) The above defn & th^m do not assume any special structure on the random var. X or Y. In particular, we may take X as discrete and Y as absolutely continuous or vice-versa or one of them be a general (neither discrete ~~or~~ nor abs. cont.) ran. var.

The above th^m tells us that if X and Y are independent random variables, then marginal distributions of X and Y uniquely determine the joint distr. of X and Y. This suggests us one way to construct the joint cdf.

Example ④ Suppose $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ and $Y \sim$ Discrete Uniform over set $\{1, 2, \ldots, 6\}$.

So recall

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} \qquad F_Y(y) = \begin{cases} 0 & \text{if } y < 1 \\ \frac{1}{6} & \text{if } 1 \leq y < 2 \\ \vdots & \\ \frac{5}{6} & \text{if } 5 \leq y < 6 \\ 1 & \text{if } y \geq 6 \end{cases}$$

If we assume independence of X and Y then the joint cdf of X and Y is:

$$F(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 1 \\ \frac{1}{12}, & 0 \leq x < 1, \ 1 \leq y < 2 \\ \frac{2}{12}, & 0 \leq x < 1, \ 2 \leq y < 3 \\ \frac{3}{12}, & 0 \leq x < 1, \ 3 \leq y < 4 \\ \frac{4}{12}, & 0 \leq x < 1, \ 4 \leq y < 5 \\ \frac{5}{12}, & 0 \leq x < 1, \ 5 \leq y < 6 \\ \frac{6}{12}, & 0 \leq x < 1, \ y \geq 6 \\ \frac{1}{6}, & x \geq 1, \ 1 \leq y < 2 \\ \frac{2}{6}, & x \geq 1, \ 2 \leq y < 3 \\ \frac{3}{6}, & x \geq 1, \ 3 \leq y < 4 \\ \frac{4}{6}, & x \geq 1, \ 4 \leq y < 5 \\ \frac{5}{6}, & x \geq 1, \ 5 \leq y < 6 \\ \ldots & x \geq 1, \ y \geq 6 \end{cases}$$

Theo (2) Let X and Y be two discrete random variables. Then X and Y are independent iff joint pmf can be written as product of marginal pmf's, i.e.,

$$P\{X = x, Y = y\} = P\{X = x\} P\{Y = y\} \quad \forall\, x \in R(X),\ y \in R(Y).$$