Lec - 37

Recall: Covariance.

Example ①: Let X and Y be two independent $N(0,1)$ random variables and $Z = 1 + X + XY^2$, $W = 1 + X$. Find $\text{Cov}(Z, W)$.

Soln:

$$\text{Cov}(Z,W) = \text{Cov}(1 + X + XY^2, 1 + X)$$

$$= \text{Cov}(1 + X + XY^2, 1) + \text{Cov}(1 + X + XY^2, X)$$

$$= \text{Cov}(1 + X + XY^2, X) \quad [\because 1 \text{ is a const.}]$$

$$= \text{Cov}(1, X) + \text{Cov}(X, X) + \text{Cov}(XY^2, X)$$

$$= \text{Var}(X) + E[XY^2X] - E[XY^2]E[X]$$

$$= 1 + E[X^2Y^2] \quad \left[\because \begin{array}{l} E(X) = 0 \\ \text{Var}(X) = 1 \end{array}\right]$$

$$= 1 + E[X^2]E[Y^2]$$

$$= 1 + 1 \times 1 = 2$$

Example ②: For any ran. variables X and Y, show that $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\text{Cov}(X,Y)$.

Soln:

$$\text{Var}(X+Y) = \text{Cov}(X+Y, X+Y)$$

$$= \text{Cov}(X+Y, X) + \text{Cov}(X+Y, Y)$$

$$= \text{Cov}(X, X) + \text{Cov}(Y, X) + \text{Cov}(X, Y) + \text{Cov}(Y, Y)$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y).$$

Remark: If X and Y are independent, then from the last example it follows that

$$Var(X+Y) = Var(X) + Var(Y) \quad [\because \text{ Then } Cov(X,Y)=0]$$

Correlation

The sign of $Cov(X,Y)$ gives information about the linear relationship of X and Y. However, its actual magnitude does not provide us with much information about ~~~~ the behavior of X & Y relative to each other as the covariance depends on the variability of X and Y.

But the correlation coefficient removes in a sense the individual variability of each X and Y by dividing the covariance by the product of the standard deviations. Thus the correlation coefficient is a better measure of the linear relationship of X and Y than the covariance. Also the correlation coefficient is unit less.

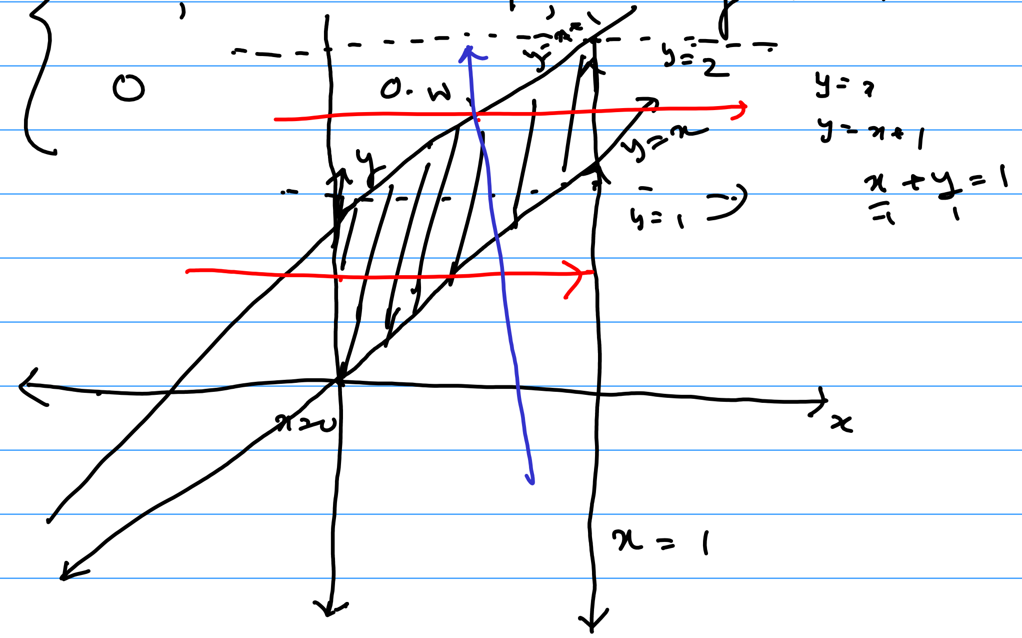
Defn: The correlation coefficient of two ran. var. X & Y denoted by $\rho(X,Y)$ is defined as:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\, Var(Y)}}$$

provided $Var(X) > 0$ and $Var(Y) > 0$.

Example ③ : Let the joint pdf of (X, Y) be

$$f(x,y) = \begin{cases} 1 ; & 0 < x < 1, \quad x < y < x+1 \\ 0 & \end{cases}$$



Blue : y

Red : x

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} 0 & x \le 0 \\ \int_x^{x+1} 1\,dy = 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$\therefore \quad X \sim U(0,1)$ [Uniform dist. in the interval $(0,1)$]

$E[X] = \frac{1}{2}$

$\mathrm{Var}(X) = \frac{1}{12}$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 1\,dx = y \qquad 0 < y < 1$$

$$\begin{cases} = \int_{y-1}^{1} 1\,dx = 2-y & 1 \le y < 2 \\ \\ 0 & \text{o.w.} \end{cases}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^1 y^2\,dy + \int_1^2 y(2-y)\,dy$$

$$= \frac{y^3}{3}\Big]_0^1 + y^2 - \frac{y^3}{3}\Big]_0^1 = \frac{1}{3} + \frac{2}{3} = 1$$

$$\text{Var}(Y) = E[Y^2] - (E(Y))^2 = E(Y^2) - 1$$

$$= \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy - 1 = \int_0^1 y^3\,dy + \int_1^2 y^2(2-y)\,dy - 1$$

$$= \frac{7}{6} - 1 = \frac{1}{6}$$

$$\left[\text{Cov}(X,Y) = E(XY) - E(X)E(Y) \right.$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \left[\int_x^{x+1} xy\, 1\, dy \right] dx = \int_0^1 x \left[\frac{y^2}{2}\right]_x^{x+1} dx = \int_0^1 \frac{x(2x+1)}{2}\,dx$$

$$= \cdots = \frac{7}{12}$$

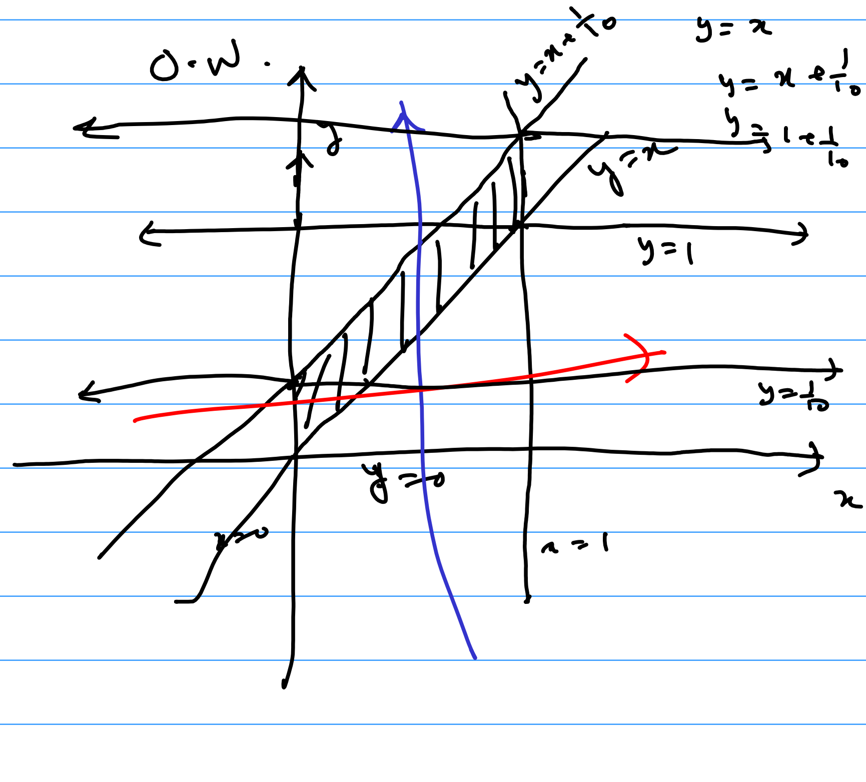
$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$
$$= \frac{7}{12} - \frac{1}{2} \times 1 = \frac{1}{12}.$$

The correlation is:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}} = \frac{\frac{1}{12}}{\sqrt{\frac{1}{6} \times \frac{1}{12}}}$$
$$= \frac{1}{12} \times 6\sqrt{2} = \frac{1}{\sqrt{2}}.$$

Example (4): Let the joint pdf of (X,Y) be

$$f(x,y) = \begin{cases} 10 ; & 0 < x < 1, \quad x < y < x + \frac{1}{10}. \\ 0 & \text{o.w.} \end{cases}$$



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \begin{cases} 0, & x \le 0 \\ \int_x^{x+\frac{1}{10}} 10\,dy = 1 & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$$

Then $X \sim U(0,1)$. Hence $E(X) = \frac{1}{2}$ & $\mathrm{Var}(X) = \frac{1}{12}$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \begin{cases} \int_0^y 10\,dx = 10y, & 0 < y < \frac{1}{10} \\ \int_{y-\frac{1}{10}}^{y} 10\,dx = 1, & \frac{1}{10} \le y < 1 \\ \int_{y-\frac{1}{10}}^{1} 10\,dx = 10(1-y+\frac{1}{10}) = 11-10y, & 1 \le y < 1+\frac{1}{10} \\ 0 & \text{o.w.} \end{cases}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^{\frac{1}{10}} y \times 10y\,dy + \int_{\frac{1}{10}}^{1} y\,dy$$

$$+ \int_1^{1+\frac{1}{10}} y(11-10y)\,dy = \cdots = \frac{11}{20}$$

$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^{\frac{1}{10}} 10y^3\,dy + \int_{\frac{1}{10}}^{1} y^2\,dy + \int_1^{1+\frac{1}{10}} (11y^2 - 10y^3)\,dy$$

$$= \cdots = \frac{29}{75}$$

$$\therefore \mathrm{Var}(Y) = \frac{29}{75} - \frac{121}{400} = \frac{101}{1200}$$

Also, $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = 10\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\,dx\,dy$

$$= 10\int_0^1 x\left[\int_x^{x+\frac{1}{10}} y\,dy\right]dx$$

$$= 10\int_0^1 x\left[\frac{y^2}{2}\right]_x^{x+\frac{1}{10}}dx = 10\int_0^1 x\,\frac{\left(x+\frac{1}{10}\right)^2 - x^2}{2}\,dx$$

$$= 10\int_0^1 \frac{x\left(2x+\frac{1}{10}\right)}{2}\times\frac{1}{10}\,dx = \int_0^1\left(x^2+\frac{x}{20}\right)dx$$

$$= \left[\frac{x^3}{3}+\frac{x^2}{40}\right]_0^1 = \frac{43}{120}$$

Hence, $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$$= \frac{43}{120} - \frac{11}{40} = \frac{1}{12}$$

The correlation is:

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \frac{\frac{1}{12}}{\sqrt{\frac{1}{12}\times\frac{101}{1200}}}$$

$$= \frac{1}{12}\times\frac{120}{\sqrt{101}} = \frac{10}{\sqrt{101}} = \sqrt{\frac{100}{101}}$$