Lecture-25B

17 February 2021 12:50

Lecture - 25

$$E(h(x)) = \begin{cases} \sum_{x_j \in R_X} h(x_j) p_X(x_j) & \text{Discrete r.v.} \\ \int_{x \in R_X} h(x) f_X(x) & \text{Cont. r.v.} \end{cases}$$

Ex. 1 $X \sim \exp(5)$, $h(x) = \min\{x, 10\}$.

$Y = h(X)$ does not have pdf.

$E(h(x)) = ?$

$R_X : [0, \infty)$

$R_Y : [0, 10]$

$$E(h(x)) = E[\min\{X, 10\}]$$

$$= \int_{x \in R_X} \min\{x, 10\} f_X(x)\, dx$$

$$= \int_0^\infty \min\{x, 10\}\, 5 e^{-5x}\, dx$$

$$= 5\left[\int_0^{10} \min\{x, 10\} e^{-5x}\, dx + \int_{10}^\infty \min\{x, 10\} e^{-5x}\, dx\right]$$

$$= 5\left[\int_0^{10} x e^{-5x}\,dx + \int_{10}^{\infty} 10\, e^{-5x}\,dx\right]$$

$$= \frac{1-e^{-50}}{5}$$

We do'nt reasuired to check the condition of absolute convergence of the integrel in aborse extemple as $h(x)$ is non-negative on $[0,\infty)$.

Ex: Let $X \sim N(0,1)$

$$h(X) = \max\{X, 0\}$$

$$E(h(X)) = ?$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \text{for } x \in \mathbb{R}$$

$$R_X : (-\infty, \infty)$$

$$E[\max\{X,0\}] = \int_{x \in R_X} h(x) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \underbrace{\max\{x,0\}}\, \underbrace{f_X(x)}\,dx$$

$$= \int_{-\infty} \ldots$$

$$= \int_{-\infty}^{0} \max\{x, 0\}\, f_X(x)\, dx + \int_{0}^{\infty} \max\{x, 0\}\, f_X(x)\, dx$$

$$= \int_{-\infty}^{0} 0 \cdot f_X(x)\, dx + \int_{0}^{\infty} x\, f_X(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x\, e^{-\frac{x^2}{2}}\, dx$$

sub $x^2 = t$

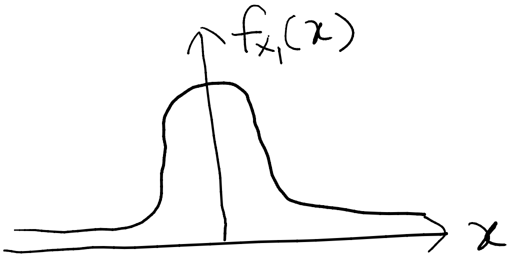
$\Rightarrow 2x\,dx = dt$

$$= \frac{1}{\sqrt{2\pi}}$$

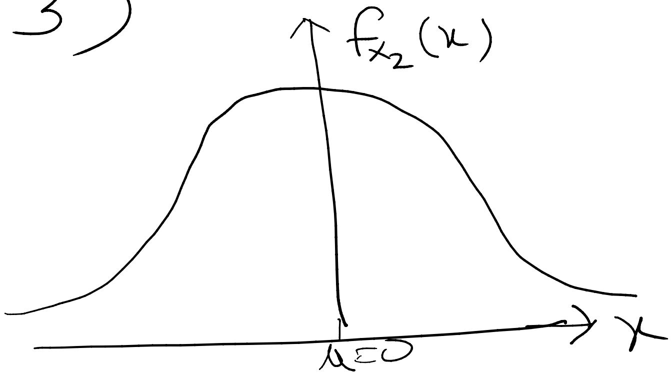
Variance $\vdash$

$X_1 \sim N(0, 1)$

$X_2 \sim N(0, 3)$



$\sigma^2 = 1$



$\sigma^2 = 3$

$$X \sim N(\mu, \sigma^2)$$

$E(X) = \mu$ | $Var(X) = \sigma^2$

Mean alone is not able to truly represent the pdf. of a r.v.

Clearly we need at least one parameter to measure the spread about mean.

$X - E(X)$: Deviation of r.v. X from its mean $E(X)$.

$X - E(X)$ may be +ive or -ive.

Consider the quantity $(X - E(X))^2$ and its average value

$$E\{(X - E(X))^2\}$$

Which represent the average square deviation of X around its mean.

We define

$$\boxed{Var(X) = \sigma_X^2 = E[(X - E(X))^2]}$$

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Note that

$$Var(X) \geq 0$$

$\sigma_X = \sqrt{Var(X)} = \sqrt{E[(X - E(X))^2]}$ = standard deviation of X = s.d. of X

From the definition,

$$Var(X) = \sigma_X^2 = E\left[(X - E(X))^2\right]$$

$$= E\left[X^2 + \{E(X)\}^2 - 2\underline{X}E(X)\right]$$

$$= E[X^2] + \{E(X)\}^2 - 2E(X)E(X)$$

$$= E[X^2] + \{E(X)\}^2 - 2\{E(X)\}^2$$

$$= E(X^2) - [E(X)]^2$$

$Y = aX + b$

$E(Y) = aE(X) + b$

Thus

$$\boxed{Var(X) = E[(X - E(X))^2]}$$

$$\boxed{\begin{aligned} Var(X) &= E[(X - E(X))] \\ &= \underline{E(X^2) - [E(X)]^2} \end{aligned}}$$

$$Var(X) \geq 0 \implies E(X^2) \geq \underline{[E(X)]^2}$$

One can also easily verify that

$$\boxed{Var(aX+b) = a^2 Var(X)}$$

$$Var(X) = \underline{E(X^2)} - [E(X)]^2$$

$$Var(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2$$

$$= E[a^2X^2 + b^2 + 2abX] - [aE(X) + b]^2$$

$$= a^2 E(X^2) + \cancel{b^2} + 2ab\cancel{E(X)} - [a^2[E(X)]^2 + \cancel{b^2} + 2ab\cancel{E(X)}]$$

$$= a^2 E(X^2) - a^2[E(X)]^2$$

$$= a^2 E(X^2) - a^2 [E(X)]^2$$

$$= a^2 \left[E(X^2) - [E(X)]^2 \right]$$

$$= a^2 \operatorname{Var}(X).$$

<u>Ex:-</u> Let $X \sim U[a, b]$

$$\operatorname{Var}(X) = ?$$

$$E(X) = \frac{a+b}{2}$$

$$\operatorname{Var}(X) = E\left[(X - E(X))^2 \right]$$

$$= E\left[\left(X - \frac{a+b}{2}\right)^2 \right]$$

$$= \int_{x \in R_X} \left(x - \frac{a+b}{2}\right)^2 f_X(x)\, dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\, dx$$

$$= \frac{(b-a)^2}{12}$$

R.V.	Mean	Variance
Bernoulli (p)	p	$p(1-p)$
Binomial $B(n,p)$	np	$np(1-p)$
Poisson $P(\lambda)$	λ	λ
Geometric (p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Normal (μ, σ^2)	μ	σ^2
Uniform $U[a,b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exp (λ)	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$