Lec - 31

Recall: Independent random variables.

Thm ① Let X and Y be two discrete random variables. Then X and Y are independent iff joint pmf can be written as product of marginal pmf's, i.e.,

$$P\{X=x, Y=y\} = P\{X=x\} P\{Y=y\} \quad \forall x \in R(X), \; y \in R(Y)$$

Remark: Let us recall that we are only given marginal distributions of RVs X & Y. In general, it is impossible to define the joint distribution of X and Y. But in a very special situation, knowledge about marginal distributions is enough to construct the joint distribution, viz, when RVs X & Y are indep. (By Theo).

Example: Let the r. vector (X, Y) has joint prob. dist. as follows:

X \ Y	-1	0	1
-1	0	$\frac{1}{4}$	0
0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	0	$\frac{1}{4}$	0

Q. Determine if X and Y are independent.

Soln: Note that X & Y are identically distributed.

$$P(X=-1) = P(X=1) = P(Y=-1) = P(Y=1)$$
$$= \frac{1}{4} \;\&\; P(X=0) = P(Y=0) = \frac{1}{2}$$

$$P(X=-1, Y=-1) = 0$$

$$P(X=-1)\,P(Y=-1) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \neq 0$$

$\therefore$ X & Y are not independent.

Theo 2 Let random vector (X, Y) has joint pdf $f(x,y)$ and $f_X(x)$ & $f_Y(y)$ are pdf corresponding to r. var. X & Y respectively. Then X & Y are indep. iff joint pdf can be written as product of marginal pdf's, i.e., $\cancel{f(x,y)}$

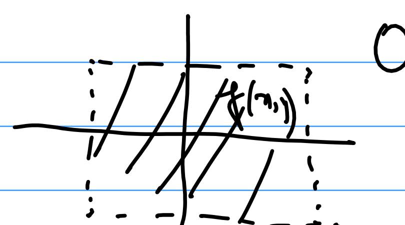
$$f(x,y) = f_X(x)\, f_Y(y) \quad \text{for each } (x,y) \in \mathbb{R}^2$$

where both $f(x,y)$ and $g(x,y) := f_X(x)\, f_Y(y)$ are continuous.

Example: Let X and Y be jointly distributed with the joint density

$$f(x,y) = \begin{cases} \dfrac{1+xy}{4}, & |x|<1,\ |y|<1 \\ 0 & \text{o.w.} \end{cases}$$

Q) Determine whether X & Y are independent.



P.T.O.

Soln:

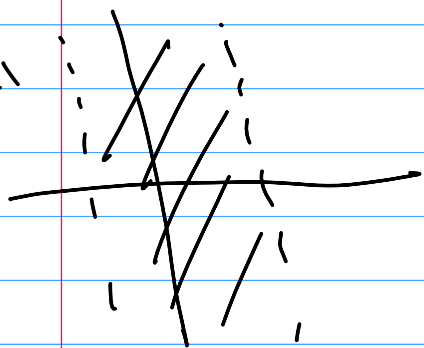
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \displaystyle\int_{-1}^{1} \frac{1+xy}{4}\,dy = \frac{1}{2}, & |x|<1 \\ 0 & \text{o.w.} \end{cases}$$

Since joint pdf is symmetric in x and y, hence

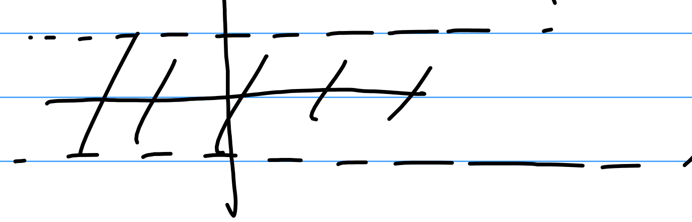
$$f_Y(y) = \begin{cases} \frac{1}{2}, & |y|<1 \\ 0 & \text{o.w.} \end{cases}$$

[Check: $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-1}^{1} \frac{1+xy}{4}\,dx$]

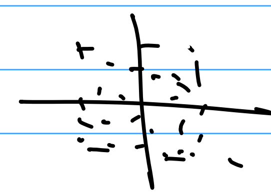
$|x|<1$



$|y|<1$



$f(x,y) \neq f_X(x)\, f_Y(y)$ at many points in the square $(-1,1)\times(-1,1)$ where both the joint density f and the product $f_X f_Y$ are continuous.



Hence, r. var. X & Y are NOT independent.

Theo ③: Let X & Y be independent r.var and f and g be Borel-measurable fn^{n}s. Then $f(X)$ and $g(Y)$ are also independent.

[Borel-measurable fn^{n}s: $f: \mathbb{R} \to \mathbb{R}$ is Borel-measurable when the inverse image $f^{-1}(U)$ is a Borel set for every open set U in the range of f].

Example:

(Try to prove directly without using the defn of Borel-measurable fn^{n}s)

(i) If X & Y are independent, then:

(i) X^2 and Y^2 are indep.

(ii) X and $\sin Y$ are indep.

(iii) $|X|$ and e^Y are indep.

Independence of Several Random Variables:

Defn: We say $X_1, X_2, --, X_n$ are independent if events $\{X_1 \in A_1\}, \{X_2 \in A_2\} \cdots \{X_n \in A_n\}$ are independent for all $A_1, A_2, \ldots, A_n$ Borel subsets of $\mathbb{R}$.

Theo (4): A collection of jointly distributed RVs $X_1, X_2, \ldots, X_n$ is said to be mutually or completely independent iff

$$F(x_1, \ldots, x_n) = F_{X_1}(x_1) F_{X_2}(x_2) \cdots F_{X_n}(x_n)$$

$\forall (x_1, \ldots, x_n) \in \mathbb{R}^n$ where F is the joint CDF of $X_1, X_2, \ldots, X_n$.

Theo (5) 1. Suppose X, Y, Z are discrete random variables. Then they are indep. iff

$$P(X=x, Y=y, Z=z) = P(X=x) P(Y=y) P(Z=z)$$

$\forall x \in R(X)$, $y \in R(Y)$ & $z \in R(Z)$.

2. Suppose X, Y, Z are random variables with joint pdf $f(x,y,z)$. Then they are independent iff

$$f(x,y,z) = \underbrace{f_X(x) f_Y(y) f_Z(z)}$$

$\forall (x,y,z) \in \mathbb{R}^3$ where both f and $g(x,y,z) := f_X(x) f_Y(y) f_Z(z)$

are continuous.

Analogously, for any finite no. of RVs we can write this defn. But a bit later in this course, we shall learn about the law of large numbers and central limit theorem. There we shall be dealing with sequence of independent random variables $X_1, X_2, \ldots \ldots$ So we need to understand the meaning of independence for countably infinite collection of random variables. In view of theo (4) we have:

<u>Defn</u>: We say that a sequence of random variables $(X_n)_{n \in \mathbb{N}}$ is independent if for every $n = 2, 3, \ldots$ the random variables $X_1, X_2, \ldots, X_n$ are independent.

<u>Real-Valued Functions of Random Variables</u>:

When there are multiple random variables of interest, it is possible to generate new random variables by considering functions involving several of these random variables. In particular, suppose we have two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be a fn.

Then $Z := g(X,Y) : \Omega \to \mathbb{R}$ defines another random variable.

If both X & Y are discrete, then Z is also discrete. Now next natural question is: What is the PMF of R.V. Z?

$$\left.\begin{array}{l} \Omega \xrightarrow{X} \mathbb{R} \\ \searrow_{Y} \ \mathbb{R}. \end{array}\right\} \quad \mathbb{R}^2 \xrightarrow{g} \mathbb{R}$$

$$\Omega \to \mathbb{R}.$$

PMF of fn of two random variables.

$$f_Z(z) = P\{Z = z\} = P\{g(X,Y) = z\}$$

$$= P\Big\{ \bigcup_{(x,y):\, g(x,y)=z} \{X = x, Y = y\} \Big\}$$

$$= \sum_{(x,y):\, g(x,y)=z} f(x,y)$$

Example: Let X and Y be random variables with the joint pmf given as follows:

$X \backslash Y$	-1	0	2	6
-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
1	$\frac{1}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

Find the PMF of $|Y - X|$.

Soln: Here $g(x,y) = |y-x|$.

First we compute the range of the random variable $Z = g(X,Y) = |Y-X|$.

Fix $x = -2$, then $z = |y-(-2)| = |y+2|$

Now run through all the y-values. We get ~~z = 1, 2, 0, 0, 8~~

$z = |-1+2| = 1$, $(y=-1)$

$z = |0+2| = 2$ $(y=0)$

$z = |2+2| = 4$ $(y=2)$

$z = |6+2| = 8$. $(y=6)$

Fix $x=1$, then $z = |y-1|$. We get $z = 2, 1, 1, 5$

Fix $x=3$, then $z = |y-3|$. We get $z = 4, 3, 1, 3$.

So, $R(Z) = \{1, 2, 3, 4, 5, 8\}$