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Recall: Characteristic Function of a r.v. X :

$$\Phi_X(t) = E[e^{itx}], \quad t \in \mathbb{R}.$$

Discrete, Cont., Ex: $X \sim \text{Bernoulli}(p)$

$$\Phi_X(t) = e^{it} p + (1-p)$$

Example ①: Find the characteristic function of the Poisson(λ) distribution.

Soln:
$$\Phi_X(t) = E[e^{itx}] = \sum_{k=0}^{\infty} e^{itk} P(X=k)$$

$$= \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{it}\lambda)^k}{k!} = e^{-\lambda} \exp(e^{it}\lambda)$$

$$= \exp(\lambda(e^{it}-1)).$$

Example ②: Let $X \sim N(0,1)$. Find its characteristic function.

Soln:
$$\Phi_X(t) = E[\cancel{\cos} \, e^{itx}] = E[\cos tx + i \sin tx]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cos tx \, e^{-\frac{x^2}{2}} \, dx + i \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sin tx \, e^{-\frac{x^2}{2}} \, dx .$$

Since characteristic function exists for every random variable, therefore both the improper integrals exist. So value of both improper integrals agrees with their Cauchy principle value.

We have:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sin tx \, e^{-\frac{x^2}{2}} \, dx = \lim_{a \to \infty} \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} \underbrace{\sin tx \, e^{-\frac{x^2}{2}}}_{\text{odd fn}} \, dx$$

$$= 0$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cos tx \, e^{-\frac{x^2}{2}} \, dx = \lim_{a \to \infty} \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} \underbrace{\cos tx \, e^{-\frac{x^2}{2}}}_{\text{even fn}} \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \cos tx \, e^{-\frac{x^2}{2}} \, dx = e^{-\frac{t^2}{2}},$$

where the last integral can be computed using differentiation under integration.

[Let $t \in \mathbb{R}$ be given. Define

$$I(t) = \int_{0}^{\infty} \cos tx \, e^{-\frac{x^2}{2}} \, dx$$

$$\Rightarrow I'(t) = -\int_0^\infty x \sin tx \, e^{-\frac{x^2}{2}} dx$$

$$= -\left[-\sin tx \, e^{-\frac{x^2}{2}} \right]_0^\infty + \int_0^\infty t \cos tx \, e^{-\frac{x^2}{2}} dx \Big]$$

$$= 0 - t I(t)$$

$$\Rightarrow \frac{I'(t)}{I(t)} = -t$$

$$\Rightarrow \ln I(t) = -\frac{t^2}{2} + c$$

$$\Rightarrow I(t) = k e^{-\frac{t^2}{2}}.$$

$$\text{Also } I(0) = \int_0^\infty e^{-x^2/2} dx = \frac{\sqrt{2\pi}}{2}. \text{ So } \boxed{k = \sqrt{\frac{\pi}{2}}}$$

<u>Example</u> 3: Let X be a random variable and a and b are real constants, then

$$\Phi_{a+bX}(t) = E\left[e^{it(a+bX)}\right] = E\left[e^{ita} e^{itbX}\right]$$

$$= e^{ita} E\left[e^{itbX}\right] = e^{ita} \Phi_X(bt)$$

<u>Example</u> 4: Let $X \sim N(\mu, \sigma^2)$. Then it is implicit that $\sigma > 0$. Then $Y = \frac{X-\mu}{\sigma}$ has mean 0 and variance 1. Also $Y \sim N(0,1)$.

Hence by example ③, $X = \sigma Y + \mu$ has the characteristic function:

$$\Phi_X(t) = \Phi_{\sigma Y + \mu}(t) = e^{it\mu}\,\Phi_Y(\sigma t)$$

$$= e^{it\mu}\, e^{-\frac{\sigma^2 t^2}{2}} \quad \text{[By example ②]}.$$

Example ⑤: Let X and Y be independent random variables. Show that

$$\Phi_{X+Y}(t) = \Phi_X(t)\,\Phi_Y(t)$$

Proof: $\Phi_{X+Y}(t) = E\left[e^{it(X+Y)}\right] = E\left[e^{itX} e^{itY}\right]$

$$= E\left[e^{itX}\right] E\left[e^{itY}\right] = \Phi_X(t)\,\Phi_Y(t).$$

More generally, if $X_1, X_2, \ldots, X_n$ are n independent random variables, then

$$\Phi_{X_1+X_2+\cdots+X_n}(t) = \Phi_{X_1}(t)\,\Phi_{X_2}(t)\cdots\Phi_{X_n}(t).$$

Example ⑥: Compute the characteristic function of a Binomial (n, p) random variable.

Soln: Note that a Binomial (n,p) random variable is a sum of n independent Bernoulli (p) random variables. Therefore its characteristic fn is:

$$\left[e^{it} p + (1-p)\right]^n.$$

Theo: (Uniqueness Theorem) Let X_1 and X_2 be two random variables s.t. $\Phi_{X_1} = \Phi_{X_2}$. Then X_1 and X_2 have same distribution.

Example (7): Let $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$ be two independent Binomial random variables. Show that $X+Y$ is a Binomial (n_1+n_2, p) random variable.

Soln: The characteristic fntn of $X+Y$ is:

$$\Phi_{X+Y}(t) = \Phi_X(t)\,\Phi_Y(t) \quad \text{(By example (5))}$$

$$= \left[e^{it}p + (1-p)\right]^{n_1}\left[e^{it}p + (1-p)\right]^{n_2} \quad \text{(By example (6))}$$

$$= \left[e^{it}p + (1-p)\right]^{n_1+n_2}$$

So RHS is a characteristic fntn of a Binomial (n_1+n_2, p) random variable. Hence by unique theorem,

$$X+Y \sim \text{Binomial}(n_1+n_2, p).$$

Example (8): Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ be two independent Poisson random variables. Show that $X+Y$ is a Poisson $(\lambda+\mu)$ random variable.

Soln: The characteristic fn of $X+Y$ is:

$$\Phi_{X+Y}(t) = \Phi_X(t)\,\Phi_Y(t) = \exp\left(\lambda(e^{it}-1)\right)\exp\left(\mu(e^{it}-1)\right)$$
$$= \exp\left[(\lambda+\mu)(e^{it}-1)\right].$$

RHS is a characteristic fn of a Poisson$(\lambda+\mu)$ random variable. Therefore by uniqueness theorem,

$$X+Y \sim \text{Poisson}(\lambda+\mu).$$

Example (9): Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent normal random variables. Then show that $X+Y$ is a $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ random variable.

Soln: We have
$$\Phi_X(t) = e^{it\mu_1}\, e^{-\frac{\sigma_1^2 t^2}{2}}$$
$$\Phi_Y(t) = e^{it\mu_2}\, e^{-\frac{\sigma_2^2 t^2}{2}}.$$

Now,
$$\Phi_{X+Y}(t) = \Phi_X(t)\,\Phi_Y(t)$$
$$= e^{it(\mu_1+\mu_2)}\, e^{-\frac{(\sigma_1^2+\sigma_2^2)}{2}t^2}$$

Now RHS is the characteristic function of a normal random variable with mean $\mu_1+\mu_2$ and variance $\sigma_1^2+\sigma_2^2$. Therefore by uniqueness theorem, we conclude that

$$X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$$

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