Lecture-18B

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Lecture-18

Poisson Distribution :- Poisson r.v. represent the number of occurences of a rare event in a large number of trials i.e. it is a limiting case of $B(n, p)$, when n is too large and p is small.

A r.v. X is said to be a Poisson r.v. with parameter λ if X takes the values $0, 1, 2, \text{---}, \infty$ with p.m.f.

$$p_X(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{\lfloor x}, \quad x = 0, 1, 2, \text{---} \infty$$

$$\sum_x \frac{e^{-\lambda} \lambda^x}{\lfloor x} = 1$$

$$X \sim p(\lambda)$$

$$R_X : 0, 1, 2, \text{------}$$

Also note that

$$\frac{p_X(k-1)}{p_X(k)} = \frac{\cancel{e^{-\lambda}} \lambda^{k-1}}{\lfloor k-1} \times \frac{\lfloor k}{\cancel{e^{-\lambda}} \lambda^k} = \frac{k}{\lambda}$$

$\Rightarrow$ if $k < \lambda$ then $p_X(k-1) < p_X(k)$

i.e. $P(X = k-1) < P(X = k)$

if $k > \lambda$ then $p_X(k-1) > p_X(k)$ i.e.

$P(X = k-1) > P(X = k)$

if $k = \lambda$ then $P(X = k-1) = P(X = k)$

Thus $P(X = k)$ increases with k from 0 till $k \leq \lambda$ and falls off beyond λ. If λ is an integer then $P(X = k)$ has two maximal value at $k = \lambda - 1$ and λ.

Typical examples includes

- Number of telephone calls at an telephone exchange over a fixed duration
- the number of winning tickets among those purchased in a large lottery
- Number of printing errors in a book.

$$B(\underset{\substack{\downarrow \\ \text{large}}}{n}, \underset{\substack{\downarrow \\ \text{small}}}{p}) \rightsquigarrow P(\lambda)$$

& $np = \lambda$

$$P(X = i) = \frac{\lfloor n}{\lfloor n-i \lfloor i} p^i (1-p)^{n-i}$$

$$P(X=i) = \frac{\lfloor\underline{n}}{\lfloor\underline{n-i}\ \lfloor\underline{i}}\, p^{i} (1-p)^{n-i}$$

$$= \frac{\lfloor\underline{n}}{\lfloor\underline{n-i}\ \lfloor\underline{i}} \left(\frac{\lambda}{n}\right)^{i} \left(1-\frac{\lambda}{n}\right)^{n-i}$$

Now for large n and moderate λ.

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}$$

$$\Rightarrow \frac{n(n-1) \text{———} (n-i+1)\,\cancel{\lfloor\underline{n-i}}}{\cancel{\lfloor\underline{n-i}}\ \lfloor\underline{i}} \left(\frac{\lambda}{n}\right)^{i} \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1) \text{———} (n-i+1)}{n^{i}}\ \frac{\lambda^{i}}{\lfloor\underline{i}}\ \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{i}}$$

$$= \frac{\cancel{n^{i}}\, 1 \cdot \left(1-\frac{1}{n}\right) \text{———} \left(1-\frac{i-1}{n}\right)}{\cancel{n^{i}}}\ \frac{\lambda^{i}}{\lfloor\underline{i}}\ \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{\cancel{i}}}$$

as $n \to \infty$, we get

$$P(X=i) = \frac{\lambda^{i}}{\lfloor\underline{i}}\ \frac{e^{-\lambda}}{1}$$

$$= \frac{e^{-\lambda}\lambda^{i}}{\lfloor\underline{i}}$$

$$\Longrightarrow B(n,p) \longrightarrow P(\lambda)$$

$$\Longrightarrow B(\underset{\text{large}}{n}, \underset{\text{small}}{p}) \longrightarrow P(\lambda)$$

large small and $np = \lambda$.

<u>Ex.</u>:- A typesetter makes on the average 1 mistake per 1000 words. Assume that he is setting a book with 100 words to a page. Let X be the number of mistakes that he make on a single page. The exact probability dist^n for X would be obtained by considering X as a result of 100 Bernoulli trials with $p = \frac{1}{1000}$ i.e.

$$P(X=j) = {}^{100}C_j \, p^j (1-p)^{100-j}$$

While Poisson approximation is given by

$$P(X=j) = \frac{e^{-\lambda} \lambda^j}{\lfloor j}, \text{ where } \lambda = np = 100 \times \frac{1}{1000} = 0.1$$

j	Poisson $\lambda = 0.1$	$B(n, p)$ $n = 100$, $p = \frac{1}{1000}$
0	0.9048	0.9048
1	0.0905	0.0905
2	0.0045	0.0045
3	0.0002	0.0002

2	0.0045	0.0045
3	0.0002	0.0002
4	0.0000	0.0000

Ex:- Assume that you live in a city of size 10 blocks by 10 block so that whole city is divided into 100 small squares. During a war, missiles hit the city. How likely is it that the square in which you live receive no hits if the city is hit by 400 missiles?

$$p = \frac{1}{100}, \quad n = 400$$

$$\lambda = np = \frac{400}{100} = 4$$

$$\boxed{P(X=0) = e^{-4}}$$

The Uniform Random Variable:- A r.v. X is said to be uniformly distributed over the interval $[a, b]$ if its probability density function (pdf) is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

Note that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1$$

and $f_X(x) \geq 0 \quad \forall\, x \in \mathbb{R}$

and in particular,

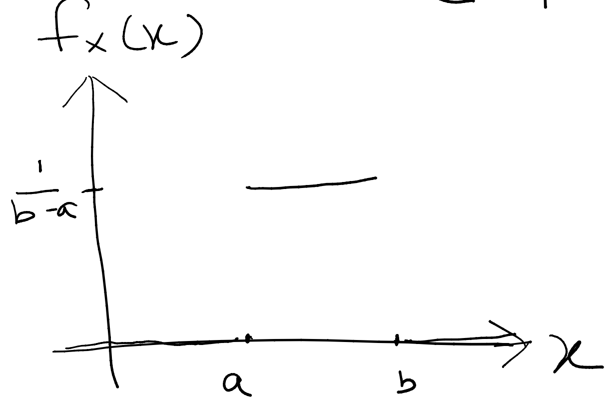
$f_X(x) > 0$ when $x \in [a, b]$.

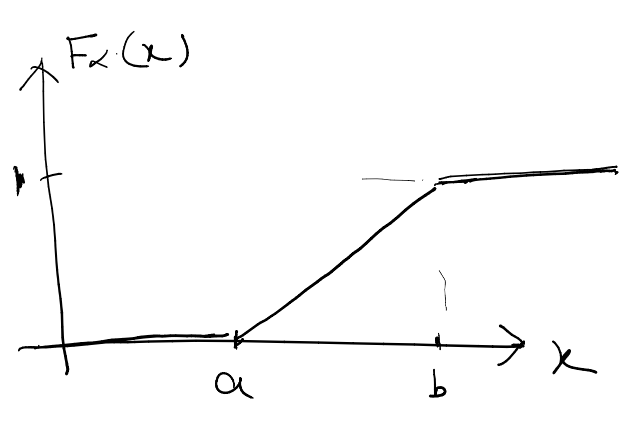
X is uniformly distributed then

$$X \sim U[a, b]$$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$





A particular important application is in the quantization of signals samples prior to encoding in digital communicat-
-ion systems.