THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
PROBABILITY AND STATISTICS: MTH221
END SEMESTER EXAM

Time: 150 Minutes | Date: 26/04/2021 | Maximum Marks: 45

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1. (a) Let X be a random variable with $P(X=2)=\dfrac{1}{4}$ and its commulative distribution function (CDF) is

$$F_X(x)=\begin{cases}0, & x<-3\\ \alpha(x+3), & -3\le x<2,\\ \frac{3}{4}, & 2\le x<4,\\ \beta x^2, & 4\le x<\frac{8}{\sqrt{3}},\\ 1, & x\ge \frac{8}{\sqrt{3}}.\end{cases}$$

(i) Find α, β if 2 is the only jump point in CDF $F_X(x)$.

(ii) Compute $P(X<3|X\ge 2)$. **[2 marks]**

Sol. (i) Given $P(X=2)=\dfrac{1}{4}$. Therefore by using $p_X(x_i)=P(X=x_i)=F_X(x_i)-F_X(x_i^-)$, we have $\dfrac{3}{4}-5\alpha=\dfrac{1}{4}\Rightarrow \alpha=\boxed{1/10}$.Also, by continuity of $F_X(x)$ at $x=4$, we have $\beta 4^2=\dfrac{3}{4}\Rightarrow\beta=\boxed{3/64}$.

(ii) Now $P(X<3/X\ge 2)=\dfrac{P(X<3,X\ge 2)}{P(X\ge 2)}=\dfrac{P(2\le X<3)}{P(X\ge 2)}$

$$=\frac{F_X(3)-F_X(2)}{1-F_X(2)}=\frac{\frac{3}{4}-\frac{1}{10}(2+3)}{1-\frac{1}{10}(2+3)}=\boxed{1/2.}$$

(b) Let A be any event of a probability model such that $P(A)=p$. The probabilistic experiment is repeated, independently, until the event A is observed m times. Let X_m denote the random variable which counts the number of times the experiment is repeated. For the random variable X_m, determine the $E(X_m)$ and $\text{Var}(X_m)$. **[3 marks]**.

Sol. Given that $P(A)=p$

Define the random variable Y_j : Number of times the experiment repeated between j-th and $(j+1)-$th observation of the event A, $\quad j=0,1,\dots,m-1$. Its clear that

$$X_m=Y_0+y_1+\cdots+Y_{m-1}$$

and Y_j is a geometric random variable with the parameter p i.e., $Y_j\sim G(p)$.

$$\Rightarrow P(Y_j=k)=p(1-p)^{k-1}$$
$$E(Y_j)=\frac{1}{p}$$
$$Var(Y_j)=\frac{1-p}{p^2}$$

Since Y_j are independent, we have

$$\begin{aligned}E(X_m)&=E(Y_0+Y_1+\cdots+Y_{m-1})\\&=E(Y_0)+E(Y_1)+\cdots+E(Y_{m-1})\\&=\boxed{\frac{m}{p}}.\end{aligned}$$

$$\begin{aligned}\text{and } Var(X_m)&=Var(Y_0+Y_1+\cdots+Y_{m-1})\\&=Var(Y_0)+Var(Y_1)+\cdots+Var(Y_{m-1})\\&=\boxed{m\left(\frac{1-p}{p^2}\right)}.\end{aligned}$$

2. (a) Let X be a Binomial random variable with parameters n and p. If $n \to \infty$ and $p \to 0$ so that $np = \lambda = E(X)$ remains constant then show that X can be approximated by a Poisson distribution with parameter λ. **[2.5 marks]**

Sol. We know that $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$

$= \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$

$= \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$

$\to 1. \frac{\lambda^k}{k!}.e^{-\lambda}.1 = \frac{e^{-\lambda}\lambda^k}{k!}$, $k = 0, 1, 2, 3, \cdots$
as $n \to \infty$.

(b) Let E_1, E_2, E_3 and E_4 be four independent events such that $P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{3}, P(E_3) = \frac{1}{4}$ and $P(E_4) = \frac{1}{5}$. Let p be the probability that at most two events among E_1, E_2, E_3 and E_4 occur. Then find the value of $240p$. **[2.5 marks]** .

Solution: Event that at most two events among E_1, E_2, E_3 and E_4 occur can be written as disjoint union of the following events:

(i) none of E_1, E_2, E_3 and E_4 occur, i.e., $E_1^c \cap E_2^c \cap E_3^c \cap E_4^c$.

(ii) Exactly one of E_1, E_2, E_3 and E_4 occur, i.e., $(E_1 \cap E_2^c \cap E_3^c \cap E_4^c) \cup (E_1^c \cap E_2 \cap E_3^c \cap E_4^c) \cup (E_1^c \cap E_2^c \cap E_3 \cap E_4^c) \cup (E_1^c \cap E_2^c \cap E_3^c \cap E_4)$

(iii) Exactly two of E_1, E_2, E_3 and E_4 occur, i.e., $(E_1 \cap E_2 \cap E_3^c \cap E_4^c) \cup (E_1 \cap E_2^c \cap E_3 \cap E_4^c) \cup (E_1 \cap E_2^c \cap E_3^c \cap E_4) \cup (E_1^c \cap E_2 \cap E_3 \cap E_4^c) \cup (E_1^c \cap E_2 \cap E_3^c \cap E_4) \cup (E_1^c \cap E_2^c \cap E_3 \cap E_4)$

Hence

$$\begin{aligned}
P(i) &= P(E_1^c \cap E_2^c \cap E_3^c \cap E_4^c) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5} \\
P(ii) &= P(E_1 \cap E_2^c \cap E_3^c \cap E_4^c) + P(E_1^c \cap E_2 \cap E_3^c \cap E_4^c) + P(E_1^c \cap E_2^c \cap E_3 \cap E_4^c) + P(E_1^c \cap E_2^c \cap E_3^c \cap E_4) \\
&= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} \\
&= \frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{20} = \frac{12+6+4+3}{60} = \frac{5}{12} \\
P(iii) &= P(E_1 \cap E_2 \cap E_3^c \cap E_4^c) + P(E_1 \cap E_2^c \cap E_3 \cap E_4^c) + P(E_1 \cap E_2^c \cap E_3^c \cap E_4) \\
&\quad + P(E_1^c \cap E_2 \cap E_3 \cap E_4^c) + P(E_1^c \cap E_2 \cap E_3^c \cap E_4) + P(E_1^c \cap E_2^c \cap E_3 \cap E_4) \\
&= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \\
&= \frac{1}{10} + \frac{1}{15} + \frac{1}{20} + \frac{1}{30} + \frac{1}{40} + \frac{1}{60} = \frac{12+8+6+4+3+2}{120} = \frac{35}{120} = \frac{7}{24} \\
p &= P(i) + P(ii) + P(iii) = \frac{1}{5} + \frac{5}{12} + \frac{7}{24} = \frac{24+50+35}{120} = \frac{109}{120} \implies 240p = 218.
\end{aligned}$$

3. (a) Let E_1, E_2, E_3 and E_4 be four events such that

$$P(E_i|E_4) = \frac{2}{3}, i = 1, 2, 3; P(E_i \cap E_j^c|E_4) = \frac{1}{6}, i, j = 1, 2, 3; i \neq j \text{ and } P(E_1 \cap E_2 \cap E_3^c|E_4) = \frac{1}{6}$$

Then find $P(E_1 \cup E_2 \cup E_3|E_4)$. **[3 marks]**

Solution: Note that conditional probability is a probability measure, hence

$$\begin{aligned}
P(E_1 \cup E_2 \cup E_3|E_4) &= P(E_1|E_4) + P(E_2|E_4) + P(E_3|E_4) - P(E_1 \cap E_2|E_4) - P(E_1 \cap E_3|E_4) - P(E_3 \cap E_2|E_4) \\
&\quad + P(E_1 \cap E_2 \cap E_3|E_4) \\
&= 3 \cdot \frac{2}{3} - 3 \cdot \frac{1}{2} + \frac{1}{3} = \frac{5}{6}
\end{aligned}$$

$$P(E_i \cap E_j|E_4) = P(E_i|E_4) - P(E_i \cap E_j^c|E_4) = \frac{1}{2} \text{ for } i, j = 1, 2, 3; i \neq j$$

$$P(E_1 \cap E_2|E_4) = P(E_1 \cap E_2 \cap E_3|E_4) + P(E_1 \cap E_2 \cap E_3^c|E_4) \implies P(E_1 \cap E_2 \cap E_3|E_4) = \frac{1}{3}$$

(b) A four digit positive integer is chosen at random. Find the probability that there are exactly two zeros in that number. **[2 marks]** .

Solution: First place has to be non-zero number hence 9 choices. For rest three places for each one we have 10 choices. Hence total number of four digit positive integer are 9000.

Now we want to count those numbers which has exactly two zeros. For first place is anyway nonzero, so 9 choices. Now if second and third place is zero then 9*9. If second and fourth place is zero then 9*9. If third and fourth place is zero then 9*9. So desired probability is

$$\frac{3*9*9}{9000} = 0.027$$

4. Let the joint probability density function of X and Y be given by

$$f(x,y) = \begin{cases} c|x-y| & 0 \le x, y \le 1 \\ 0 & otherwise \end{cases}$$

where c is a constant.

(a). Find the value of c.

(b). Compute the marginal densities of X and Y.

(c). Compute the joint cumulative density function of X and Y. (1+2+3)

Solution. (a). As f is a joint probability density function, we have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$. Thus,

$$\begin{aligned} c\int_0^1\int_0^1 |x-y| &= c\int_0^1\int_0^y (y-x)dxdy + c\int_0^1\int_y^1 (x-y)dxdy \\ &= c\int_0^1 y^2/2dy + c\int_0^1 (y^2-y)dy \\ &= c(\frac{1}{6}+\frac{1}{6}) = 1 \end{aligned}$$

Thus $c = 3$.

(b). The marginal density of X is given by

$$\begin{aligned} f_X(x) &= \int_0^1 3|x-y|dy \\ &= \int_0^x 3(x-y)dy + \int_x^1 3(y-x)dy \qquad = 3(x^2-\frac{x^2}{2}) + 3(\frac{1}{2}-x+\frac{x^2}{2}) \\ &= \frac{3}{2} - 3x + 3x^2 \end{aligned}$$

By symmetry in X and Y (or similar computation), we have $f_Y(y) = \frac{3}{2} - 3y + 3y^2$.

(c) If either x or y is less than or equal to 0, then $F(x,y) = 0$. If both $x, y \ge 1$, the $F(x,y) = 1$. Let $1 \le x, y \le 1$, and $y \ge x$, the

$$\begin{aligned} F(x,y) &= P(X \le x, Y \le y) \\ &= \int_0^x\int_0^y 3|x-y|dydx \\ &= \int_0^x\int_0^x 3(x-y)dydx + \int_0^x\int_x^y 3(y-x)dydx \\ &= \int_0^x 3(\frac{x^2}{2})dx + \int_0^x 3(\frac{y^2}{2} - xy + \frac{x^2}{2})dx \\ &= x^3 - \frac{3x^2y}{2} + \frac{3xy^2}{2} \end{aligned}$$

If $1 \le x, y \le 1$, and $x \ge y$, then by symmetry or similar computation as above $F(x,y) = y^3 + \frac{3x^2y}{2} - \frac{3y^2x}{2}$.
If $x > 1$ and $0 \le y \le 1$ then

$$F(x,y) = P(X \le x, Y \le y) \qquad = \int_0^1 \int_0^y 3(1-y)dydx = y - \frac{y^2}{2}$$

If $y > 1$ and $0 \le x \le 1$ then by symmetry or similar computation, $F(x,y) = x - \frac{x^2}{2}$. Thus we can write $F(x,y) =$
$$\begin{cases} 0 & when\ x < 0\ or\ y < 0 \\ x^3 - \frac{3x^2y}{2} + \frac{3xy^2}{2}\ when\ 1 \le x, y \le 1, y \ge x \\ y^3 + \frac{3x^2y}{2} - \frac{3y^2x}{2}, & when\ 1 \le x, y \le 1, x \ge y \\ y - \frac{y^2}{2}\ when\ x > 1, 0 \le y \le 1 \\ x - \frac{x^2}{2}\ when\ y > 1, 0 \le x \le 1 \\ 1\ when\ x \ge 1, y \ge 1 \end{cases}$$

5. (a) The joint density function of X and Y is

$$f(x,y) = xe^{-x(y+1)}, \qquad x > 0, y > 0.$$

Let S be an exponential random variable with parameter 1 such that the random variables XY and S are independent. Find $P(XY > 1|S = 2)$ **2 marks**

(b) If two cards are randomly drawn (without replacement) from an ordinary deck of 52 playing cards, Z is the number of aces obtained in the first draw and W is the total number of aces obtained in both draws, find the joint pmf of Z and W. Also find $P(W = 1|Z = 1)$. **3 + 1 marks**.

Solution 5(a) : Let $Z := XY$. Then the CDF of Z is:

$$\begin{aligned} F_Z(z) &= P\big(XY \le z\big) = \int_{xy \le z} xe^{-x(y+1)}dxdy \\ &= \int_0^\infty \int_0^{\frac{z}{x}} xe^{-x(y+1)}dydx \\ &= \int_0^\infty \big[-e^{-x(y+1)}\big]_0^{\frac{z}{x}} dx \\ &= \int_0^\infty \big(e^{-x} - e^{-z-x}\big)dx \\ &= 1 - e^{-z}. \end{aligned}$$

1 mark

On differentiating, we get the pdf as:

$$f_Z(z) = e^{-z}, \qquad z > 0. \qquad \text{0.5 mark}$$

Now as Z and S are given to be independent, $P(Z > 1|S = 2)$ is simply $P(Z > 1)$ which is:

$$\int_1^\infty e^{-z}dz = e^{-1}. \qquad \text{0.5 mark.}$$

Solution 5(b) : Let X be the number of aces obtained in the first draw and Y be the no. of aces obtained in the second draw. So we have $f(x,y)$ as the joint pmf:

$$f(0,0) = \frac{48}{52} \cdot \frac{47}{51} = \frac{188}{221}, \qquad f(1,0) = \frac{4}{52} \cdot \frac{48}{51} = \frac{16}{221},$$

$$f(0,1) = \frac{48}{52} \cdot \frac{4}{51} = \frac{16}{221}, \qquad f(1,1) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}. \qquad \text{2 marks}$$

So we have from the given conditions that $Z = X$ and $W = X + Y$. Then the joint pmf of Z and W is:

$Z \backslash W$	0	1	2
0	$f(0,0)$	$f(0,1)$	0
1	0	$f(1,0)$	$f(1,1)$

1 mark.

Hence $P(W = 1|Z = 1) = \frac{P(W=1,Z=1)}{P(Z=1)} = \frac{f(1,0)}{0+f(1,0)+f(1,1)} = \frac{16}{17}$. 1 mark

6. The conditional covariance of X and Y given Z is defined by

$$Cov(X, Y|Z) = E[(X - E[X|Z])(Y - E[Y|Z])].$$

(a). Let $X_i, Y_j, 1 \le i \le n, 1 \le j \le m$ be random variables then show that $Cov(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, Y_j)$.

(b). Show that $Cov(X, Y|Z) = E[XY|Z] - E[X|Z]E[Y|Z]$.

(c). Prove the conditional covariance formula

$$Cov(X, Y) = E[Cov(X, Y|Z)] + Cov(E[X|Z], E[Y|Z]).$$

(d). Obtain the conditional variance formula from (c). (1+2+2+1)

Solution. (a) Let $\mu_i = E[X_i]$ and $\nu_j = E[Y_j]$ for $1 \le i \le n, 1 \le j \le m$. Note that

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mu_i, E\left[\sum_{j=1}^m Y_j\right] = \sum_{j=1}^m \nu_j$$

Therefore

$$\begin{aligned}
Cov(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) &= E\left[\left(\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i\right)\left(\sum_{j=1}^m Y_j - \sum_{j=1}^m \nu_j\right)\right] \\
&= E\left[\left(\sum_{i=1}^n (X_i - \mu_i)\right)\left(\sum_{j=1}^m (Y_j - \nu_j)\right)\right] \\
&= E\left[\sum_{i=1}^n \sum_{j=1}^m (X_i - \mu_i)(Y_j - \nu_j)\right] \\
&= \sum_{i=1}^n \sum_{j=1}^m E\left[(X_i - \mu_i)(Y_j - \nu_j)\right] \\
&= \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, Y_j)
\end{aligned}$$

(b).

$$\begin{aligned}
Cov(X, Y|Z) &= E[(XY - E[X|Z])Y - XE[Y|Z] - E[X|Z]E[Y|Z])|Z] \\
&= E[XY|Z] - E[X|Z]E[Y|Z] - E[X|Z]E[Y|Z] + E[X|Z]E[Y|Z] \\
&= E[XY|Z] - E[X|Z]E[Y|Z]
\end{aligned}$$

where the next to last equality uses the fact that given Z, $E[X|Z]$ and $E[Y|Z]$ can be treated as constants.

(c). From (b) part we have

$$E[Cov(X,Y|Z)] = E(E[XY|Z]) - E[X|Z]E[Y|Z]) = E[XY] - E(E[X|Z]E[Y|Z])$$

On the other hand,

$$Cov(E[X|Z], E[Y|Z]) = E(E[X|Z]E[Y|Z]) - E[X]E[Y].$$

Adding above two equations,

$$E[Cov(X,Y|Z)] + Cov(E[X|Z], E[Y|Z]) = E[XY] - E[X]E[Y]$$

(d). Noting that

$$Cov(X,X|Z) = Var(X|Z)$$

we obtain upon setting $Y = Z$ in (c) part that

$$Var(X) = E[Var(X|Z)] + Var(E[X|Z])$$

7. (a) A die is continually rolled untill the total sum of all rolls exceeds 300. What is the probability that at least 80 rolls are necessary? **3 marks**

   (b) If $X_1, X_2, \ldots, X_n$ are random variables each with mean μ and variance σ^2 such that $S = \frac{X_1+X_2+\cdots+X_n}{n}$ has mean μ and variance $\sigma_S^2 = \frac{\sigma^2}{n}$. If $\sigma^2 = 1$, how large must n be so that

$$P\Big(|S-\mu| < 1\Big) > 0.9?$$

2 marks.

   (c) Let X be a positive random variable with $E[X] = 10$. Find an upper bound for $E[ln\sqrt{X}]$. **1 mark**.

Solution 7(a) : Let X_i denote the outcome of the ith die. So we need to find

$$1 - P\big(X_1 + X_2 + \cdots + X_{79} > 300\big).$$

Note that $E[X_i] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{7}{2}$ and $E[X_i^2] = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + \cdots + 6^2 \times \frac{1}{6} = \frac{91}{6}$.

Hence, $Var[X_i] = \frac{91}{6} - (\frac{7}{2})^2 = \frac{35}{12}$.

Using Central Limit Theorem,

$$\begin{aligned} P\big(X_1 + \cdots + X_{79} > 300\big) &= P\big(X_1 + \cdots + X_{79} \geq 300.5\big) \qquad \text{(By continuity correction)} \\ &= P\Big(\frac{X_1 + \cdots + X_{79} - 79 \times \frac{7}{2}}{\sqrt{\frac{35}{12}} \times \sqrt{79}} \geq \frac{300.5 - 79 \times \frac{7}{2}}{\sqrt{\frac{35}{12}} \times \sqrt{79}}\Big) \\ &\approx P(Z \geq 1.58). \end{aligned}$$

Hence the required probability is: $1 - P(Z \geq 1.58) = P(Z < 1.58) = N(1.58)$ *or* $\Phi(1.58) \approx 0.9429$.

Solution 7(b) : By Chebyshev's inequality for any $\varepsilon > 0$, we have

$$P\Big(|S-\mu| \geq \varepsilon\Big) \leq \frac{\sigma_S^2}{\varepsilon^2}.$$

Taking $\varepsilon = 1$ and with the given condition that $\sigma^2 = 1$, we get:

$$P\Big(|S-\mu| \geq 1\Big) \leq \frac{1}{n}.$$

But according to the given condition, in order to have $P\Big(|S-\mu| \geq 1\Big) \leq 0.1$, we must have $\frac{1}{n} = 0.1$, which implies $n = 10$.

Solution 7(c) : Let $g(x) = ln\sqrt{x} = \frac{1}{2}lnx$ for $x \in (0, \infty)$. Now $g''(x) = -\frac{1}{2x^2} < 0$ for all $x \in (0, \infty)$. Hence, g is concave on $(0, \infty)$. Hence by jensen's inequality,

$$E[ln\sqrt{X}] = E[\frac{1}{2}lnX] \leq \frac{1}{2}lnE[X] = \frac{1}{2}ln10.$$

Thus $\frac{1}{2}Ln10$ is an upper bound for $E[ln\sqrt{X}]$.

8. Suppose X is a random variable having a χ^2-distribution with ν degrees of freedom. Recall that a random variable X has a χ^2-distribution with ν degrees of freedom iff its pdf is given by:

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma[\nu/2]} x^{\frac{\nu-2}{2}} e^{-\frac{x}{2}}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find its moment generating function $M_X(t)$. **2 marks**

   (b) Suppose Y is another random variable such that X and Y are independent, and also $X+Y$ has a χ^2-distribution with N degrees of freedom with $N > \nu$, then show that Y has a χ^2-distribution with $N-\nu$ degrees of freedom. **1 mark**.

   (c) Now suppose $\nu = 2$ and $N = 3$ and Z is a $N(0,1)$ random variable such that Y and Z are independent too. Find the density of $\frac{Z}{\sqrt{Y}}$. (Do not just write out the answer, show all the intermediate steps.) **3 marks**

Solution 6(a) :

$$\begin{aligned} M_X(t) = E[e^{tx}] &= \int_0^\infty \frac{e^{tx}x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{2^{\nu/2}\Gamma[\nu/2]}dx \\ &= \frac{1}{2^{\nu/2}\Gamma[\nu/2]}\int_0^\infty x^{\frac{\nu}{2}-1}e^{-x(\frac{1}{2}-t)}dx \\ &= \frac{1}{2^{\nu/2}\Gamma[\nu/2]}\int_0^\infty \frac{y^{\frac{\nu}{2}-1}e^{-y}}{(\frac{1-2t}{2})^{\frac{\nu}{2}}}dy \qquad [\text{Substituting } x\Big(\frac{1}{2}-t\Big) = y] \\ &= \frac{1}{(1-2t)^{\nu/2}} \cdot \frac{1}{\Gamma(\frac{\nu}{2})}\int_0^\infty y^{\frac{\nu}{2}-1}e^{-y}dy \\ &= \frac{1}{(1-2t)^{\nu/2}} \cdot \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu}{2})} = (1-2t)^{-\frac{\nu}{2}} \end{aligned}$$

Solution 6(b) : Since X and Y are independent, we have $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.
As $X+Y$ is χ^2 with degrees of freedom N, we have

$$M_{X+Y}(t) = (1-2t)^{-\frac{N}{2}} = M_X(t) \cdot M_Y(t) = (1-2t)^{-\frac{\nu}{2}} \cdot M_Y(t).$$

Hence we get $M_Y(t) = (1-2t)^{-\frac{N-\nu}{2}}$. As the moment generating function determines a distribution uniquely, we get that Y has a χ^2-distribution with $N-\nu$ degrees of freedom.

Solution 6(c) : Now Y is χ^2 with degree of freedom 1 and Z is $N(0,1)$. Since Y and Z are independent their joint pdf is given by:

$$f(y,z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} \cdot \frac{1}{\Gamma(1/2)\sqrt{2}}y^{-\frac{1}{2}}e^{-\frac{y}{2}} = \frac{1}{2\pi}e^{-\frac{z^2}{2}}y^{-\frac{1}{2}}e^{-\frac{y}{2}},\ y > 0,\ -\infty < z < \infty.$$

Let $T := \frac{Z}{\sqrt{Y}}$. We need to find the density of T. So we have $z = t\sqrt{y}$. Hence $\frac{dz}{dt} = \sqrt{y}$.

Recall the change of variable theorem: Let X be a continuous random variable and $Y = u(X)$ be a function of X. Let $f(x)$ be the value of the pdf of the continuous random variable X at x. If the function given by $y = u(x)$ is differentiable and either increasing or decreasing for all values within the range of X for which $f(x) \neq 0$, then for these values of x, the equation $y = u(x)$ can be uniquely solved for x to give $x = \omega(y)$, and for the corresponding values of y the probability density of $Y = u(X)$ is given by

$$g(y) = f(\omega(y))|\omega'(y)|, \text{ provided } u'(x) \neq 0.$$

Elsewhere $g(y) = 0$.

So in this case we can treat the Y variable as constant and we have T is a function of Z, i.e., $T = u(Z)$. Also the function $t = \frac{z}{\sqrt{y}}$ is increasing for all values within the range of Z and the pdf of Z is non-zero everywhere. So we apply the theorem above directly to get the joint pdf of Y and T as:

$$g(y,t) = \frac{1}{2\pi}e^{-\frac{(t\sqrt{y})^2}{2}}y^{-\frac{1}{2}}e^{-\frac{y}{2}}\sqrt{y} = \frac{1}{2\pi}e^{-\frac{y}{2}(1+t^2)} \qquad y > 0. \ -\infty < t < \infty.$$

On integrating w.r.t. y we get the marginal density and hence the required density of T.

$$f(t) = \frac{1}{2\pi}\int_0^\infty e^{-\frac{1+t^2}{2}\cdot y}dy = \frac{1}{\pi(1+t^2)}, \qquad -\infty < t < \infty.$$

End of paper