Lec - 32

Recall: PMF of a fnth of two random variables.

Example (i): Let X and Y be RVs with the joint pmf given by:

X \ Y	-1	0	2	6
-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
1	$\frac{1}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

(Q) Find the pmf of $|Y-X|$

Soln: $g(x,y) = |y-x|$.

Range of the RV $Z = g(X,Y) = |Y-X|$.

Fix $x=-2$, then $z = |y-(-2)| = |y+2|$.

By running all the y-values, we get $z = 1, 2, 4, 8$

... $x=1$ ....

$x=3$ ....

$R(Z) = \{1, 2, 3, 4, 5, 8\}$.

$$P(Z=1) = \sum_{(x,y):|y-x|=1} f(x,y) = f(-2,-1) + f(1,0) + f(1,2) + f(3,2)$$

$$= \frac{1}{9} + 0 + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$$

$$P(Z=2) = \sum_{(x,y):|y-x|=2} f(x,y) = f(-2,0) + f(1,-1) = \frac{1}{27} + \frac{2}{9} = \frac{7}{27}$$

$P(Z=3) = \cdots \quad \frac{4}{27}$

$P(Z=4) = \dots = \frac{1}{27}$ $\qquad P(Z=5) = \dots = \frac{1}{9}$

$P(Z=8) = \dots = \frac{1}{9}$

<u>Example ②</u>: Let X and Y be iid (independent & identically distributed) discrete uniform random variables with parameter N. Find the pmf of the RV $\min\{X, Y\}$.

<u>Soln</u>: Let $Z := \min\{X, Y\}$. Also it is clear that range of r.v. Z would be $\{1, 2, \dots, N\}$. Now we find its pmf.

$$P(Z=i) = \sum_{(x,y)\,:\,\min\{x,y\}=i} f(x,y)$$

Now both x and y ranges over the set $\{1, 2, \dots, N\}$.

Since X and Y are both independent, hence the joint pmf is:

$X \backslash Y$	1	2	- - -	N
1	$\frac{1}{N^2}$	$\frac{1}{N^2}$	- - -	$\frac{1}{N^2}$
2	$1/N^2$	$\frac{1}{N^2}$		$1/N^2$
⋮	⋮	⋮	- - -	⋮
N	$\frac{1}{N^2}$	$1/N^2$	- - -	$\frac{1}{N^2}$

$P(X=1, Y=1)$
$= P(X=1)\,P(Y=1)$
$= \frac{1}{N} \times \frac{1}{N} = \frac{1}{N^2}$

Hence for given $i \in \{1, 2, \dots, N\}$

$$P(Z=i) = \sum_{(x,y)=\min\{x,y\}=i} f(x,y) = \sum_{y=i}^{N} f(i, y) + \sum_{x=i+1}^{N} f(x, i)$$

$$= \underbrace{\left(\frac{1}{N^2} + \cdots + \frac{1}{N^2}\right)}_{N-(i-1)} + \underbrace{\left(\frac{1}{N^2} + \cdots + \frac{1}{N^2}\right)}_{N-i}$$

$$= \frac{N-(i-1)}{N^2} + \frac{N-i}{N^2} = \frac{2N-2i+1}{N^2}$$

Density of Functions of two Random Variables.

If X and Y have joint pdf and g is a fn s.t. $Z = g(X,Y)$ is absolutely continuous ran. var.

Then, how to compute the pdf of Z?

<u>Example ③</u>: Let X and Y be independent and exponential random variables with parameters λ and μ respectively. Find the density of $\max\{X,Y\}$ (if it exists).

<u>Soln:</u> Set $Z := \max\{X,Y\}$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$g(y) = \cdots$

Then for each $z \in \mathbb{R}$, we have

$$\{Z \leq z\} = \{X \leq z\} \cap \{Y \leq z\}$$

$\therefore$ CDF of Z is

$$F_Z(z) = P(Z \leq z) = P(X \leq z, Y \leq z)$$
$$= P(X \leq z) P(Y \leq z)$$

$$= F_X(z)\, F_Y(z)$$

CDF of X:

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$F_X(z) = \begin{cases} 1 - e^{-\lambda z} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

$$F_Y(z) = \begin{cases} 1 - e^{-\mu z} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

$$\therefore \quad F_Z(z) = \begin{cases} (1 - e^{-\lambda z})(1 - e^{-\mu z}) = 1 - e^{-\mu z} - e^{-\lambda z} + e^{-(\lambda+\mu) z} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

At $z=0$, $F_Z(0) = 0$. Hence CDF of Z is continuous everywhere. We may differentiate it to get the density (i.e, pdf).

$$f_Z(z) = \begin{cases} \mu e^{-\mu z} + \lambda e^{-\lambda z} - (\lambda+\mu)\, e^{-(\lambda+\mu) z} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

Check: $F_Z'(0) = 0$

Example (4) The joint density of fn^n of X and Y is given by:

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } 0 < x, y < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find the density fn^n of the RV $\frac{X}{Y}$.

Soln: Let $Z := \frac{X}{Y}$. Let $z \in \mathbb{R}$ be given.

Then $\{Z \le z\} = \left\{\frac{X}{Y} \le z\right\} = \{(X,Y) \in A_z\}$

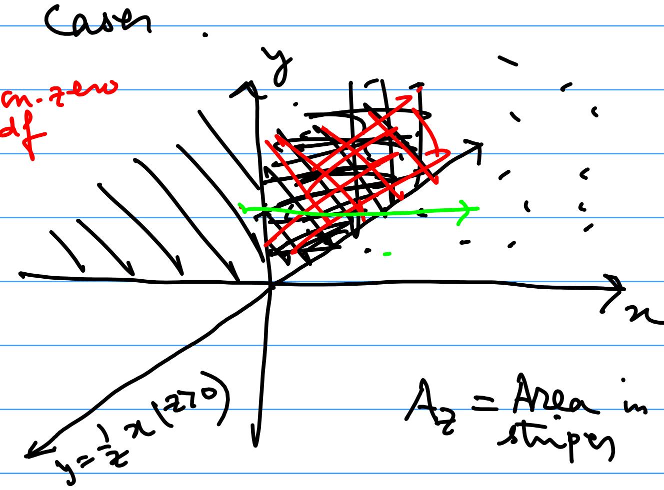
where $A_z = \left\{(x,y) \in \mathbb{R}^2 : \frac{x}{y} \le z\right\}$.

If $y > 0$, then $A_z = \{(x,y) \in \mathbb{R}^2 : x \le yz\}$.

If $y < 0$, then $A_z = \{(x,y) \in \mathbb{R}^2 : x \geq yz\}$

Now we plot the st. line $x = yz$, which we further divide into two cases.

Red part: Non-zero pdf



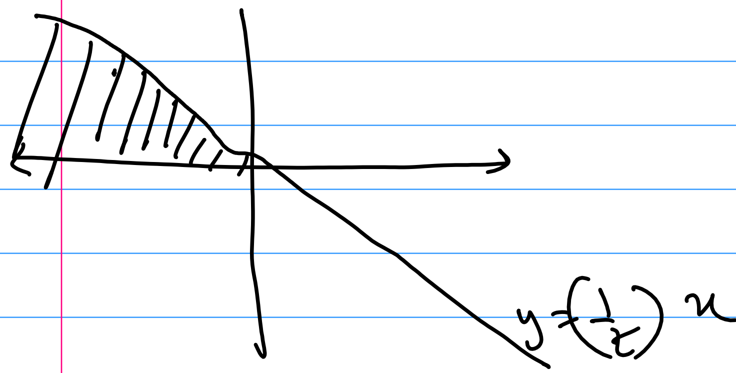
(1.) When $y > 0$:

(a) When $z \geq 0$:

$x = yz \Rightarrow y = \left(\frac{1}{z}\right)x$
positive slope

(b) When $z < 0$: $\quad x = yz \Rightarrow y = \left(\frac{1}{z}\right)x$.

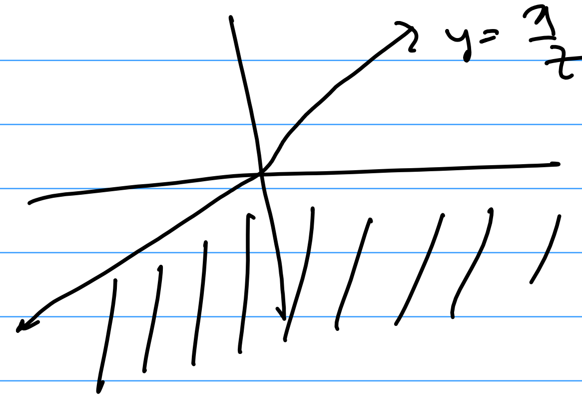
$\rightarrow$ negative slope



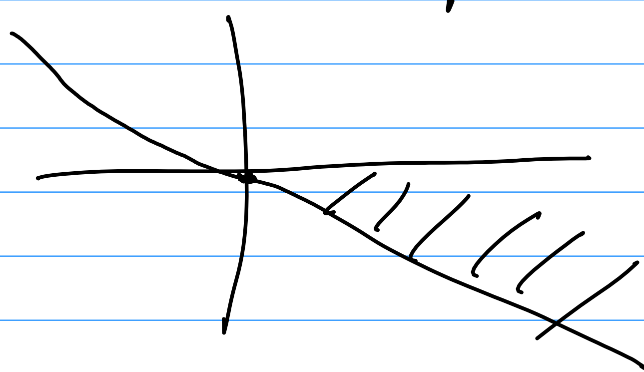
$y \leq \frac{x}{z}$

(2). When $y < 0$:

(a) $z \geq 0$



(b) When $z < 0$



Therefore if $z \geq 0$:

$$F_Z(z) = P\{(X, Y) \in A_z\} = \int_0^{\infty} e^{-y} \left(\int_{x=0}^{zy} e^{-x}\, dx \right) dy$$

$$= \int_0^{\infty} e^{-y} \left[-e^{-x} \right]_0^{zy} dy = \left[-e^{-y} + \frac{1}{z+1} e^{-(z+1)y} \right]_0^{\infty}$$

$$= 1 - \frac{1}{z+1}$$

$\therefore$ If $z < 0$, $F_Z(z) = P\{(X, Y) \in A_z\} = 0$

Hence, $F_Z(z) = \begin{cases} 1 - \dfrac{1}{z+1} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$

Since F_Z is a cont. fn, we may differentiate it to get the density.

$$F_Z'(z) = \begin{cases} \dfrac{1}{(z+1)^2} & \text{if } z > 0 \\ 0 & \text{if } z < 0 \end{cases}$$

F_Z is not differentiable at $z = 0$. So we set density to be equal to zero at this point.

Hence pdf of Z is

$$f_Z(z) = \begin{cases} \dfrac{1}{(z+1)^2} & \text{if } z > 0 \\ 0 & z \leq 0 \end{cases}$$

Example (4): Let X and Y be iid uniform random variables distributed over interval $(0, 1)$. Find the density of $X+Y$ (if it exists).

Soln: Define $Z := X + Y$.

For fixed $z \in \mathbb{R}$, the event $\{Z \le z\}$ is equivalent to the event $\{(X,Y) \in A_z\}$ where $A_z = \{(x,y) \in \mathbb{R}^2 \mid x+y \le z\}$.

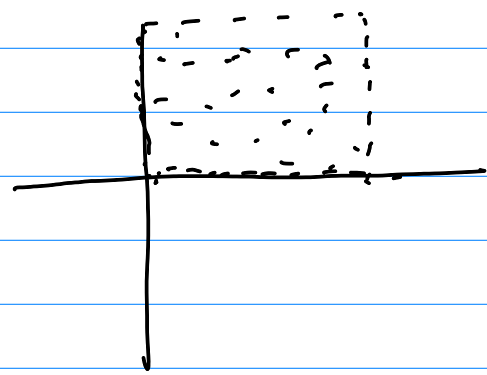
Thus

$$F_Z(z) = P(Z \le z)$$

$$= P\big((X,Y) \in A_z\big)$$

$$= \iint_{A_z} f(x,y)\,dx\,dy$$

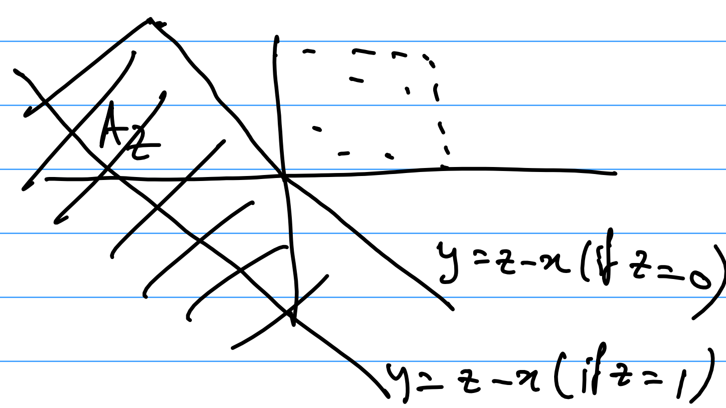
$(0,1)\times(0,1)$



Since our joint density is non-zero only on unit square, therefore we analyze the set A_z for various values of z.

1. If $-\infty < z \le 0$:

$$\iint_{A_z} f(x,y)\,dx\,dy = 0$$



2. If $0 < z \le 1$

$\iint_{A_z} f(x,y)\,dx\,dy$ = Area in Red $= \frac{1}{2} \times z \times z = \frac{z^2}{2}$