Lecture-9&10:A2

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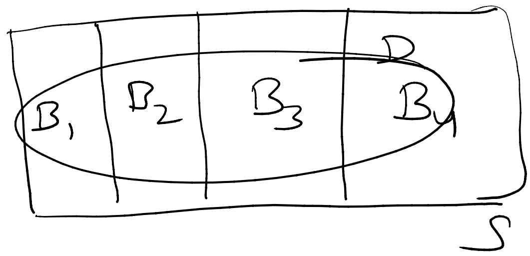
Lecture-9

Ex.:- We have four box.

Box 1 :	1900 g,	100 d	95 %	5 %
Box 2 :	300 g,	200 d	60 %	40 %
Box 3 :	900 g,	100 d	90 %	10 %
Box 4 :	900 g,	100 d	90 %	10 %

We select at random one of boxes and we remove at random a single component. What is the prob. that the selected component is defective?

B_i : event consisting of all equipments in the i-th box.



D : event consisting of all defective components.

$$P(B_i) = \frac{1}{4}, \quad i = 1, 2, 3, 4.$$

$$P(D|B_1) = \frac{100}{2000} = 0.05$$

$$P(D|B_2) = \frac{200}{500} = 0.4$$

$$P(D|B_3) = P(D|B_4) = \frac{100}{1000} = 0.1$$

By Total Prob. thm, we have

$$P(D) = \sum_{i=1}^{4} P(D|B_i)\,P(B_i)$$

$$= 0.05 \times \frac{1}{4} + 0.4 \times \frac{1}{4} + 0.1 \times \frac{1}{4} + 0.1 \times \frac{1}{4}$$

$$= \boxed{0.1625}$$

$= 0.05 \times \frac{1}{4} + \ldots \times \frac{1}{4} + \ldots \frac{1}{4} \ldots \times \frac{1}{4}$

$= \boxed{0.1625}$

i.e. 16.25% chance that chosen component is defective.

Ex:- Suppose we have 3 coin. A coin chosen "at random" shows H.

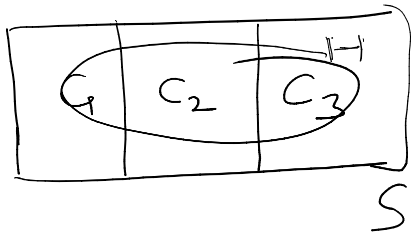
	H	T
Coin 1 :	1	0
Coin 2 :	$\frac{1}{2}$	$\frac{1}{2}$
Coin 3 :	$\frac{3}{4}$	$\frac{1}{4}$

What is the prob. that the coin chosen was coin 1. $P(\text{Coin1}/H) = ?$

C_i : i-th coin is chosen

H : Head shown

T : Tail shown



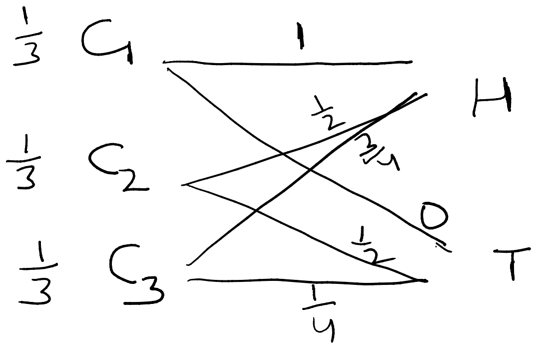
$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$

$P(H|C_1) = 1$

$P(T/C_1) = 0$

$P(H|C_2) = P(T/C_2) = \frac{1}{2}$

$P(H|C_3) = \frac{3}{4}, P(T/C_3) = \frac{1}{4}.$



$$P(C_1/H) = \frac{P(H|C_1)\,P(C_1)}{P(H/\ \ )P(C\ \ ) + P(H/\ \ )P(C\ \ ) + P(H/\ \ ) \ldots}$$

$$P(C_1/H) = \frac{P(H/C_1)P(C_1)}{P(H/C_1)P(C_1)+P(H/C_2)P(C_2)+P(H/C_3)P(C_3)}$$

$$= \frac{1\times\frac{1}{3}}{1\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}+\frac{3}{4}\times\frac{1}{3}}$$

$$= \boxed{\frac{4}{9}}$$

$$P(C_2|H) = \frac{2}{9}, \quad P(C_3/H) = \frac{1}{3}$$

A is indebendent of B if

$$P(A|B) = \underline{P(A)}$$

i.e. A is independent of B if occurence of B does not to change the prob. of occurence of A.

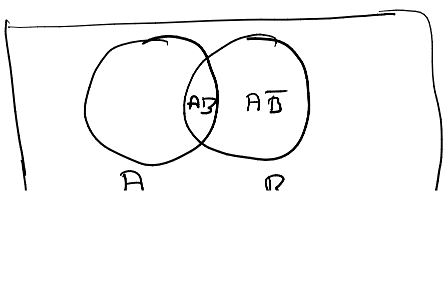
We also shown that A is independent of B if

$$\boxed{P(AB) = P(A)\,P(B)}$$ or vice-versa.

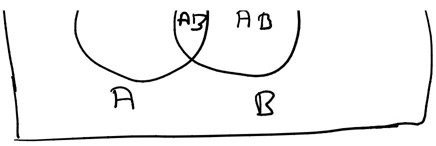
Few Properties

⊛ A and B are independent $\Longrightarrow$ A & $\overline{B}$ are also independent.

$B = AB \cup \overline{A}B$, $(AB)\cap(\overline{A}B) = \phi$



$\Rightarrow P(B) = P(AB) + P(\overline{A}B)$



$\Rightarrow P(B) = P(AB) + P(\bar{A}B)$

$\Rightarrow P(\bar{A}B) = P(B) - P(AB)$

$= P(B) - P(A)P(B)$

$= P(B)[1 - P(A)]$

$= P(B)P(\bar{A})$

$\Rightarrow P(\bar{A}B) = P(\bar{A})P(B)$

$\Rightarrow$ $\bar{A}$ & B are independent

Similary, we can show that if A & B are independent then A & $\bar{B}$ are also independent.

(*) A and B are independent $\Rightarrow$ $\bar{A}$ and $\bar{B}$ are also independent.

$$S = (A \cup B) \cup (\overline{A \cup B})$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$= (A \cup B) \cup (\bar{A}\bar{B})$$

$\Rightarrow P(S) = P(A \cup B) + P(\bar{A}\bar{B})$

$\Rightarrow P(\bar{A}\bar{B}) = P(S) - P(A \cup B)$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(AB)]$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= 1 - P(A) - P(B)[1 - P(A)]$

$= (1 - P(A))(1 - P(B))$

$= P(\bar{A})\,P(\bar{B})$

$\Rightarrow P(\bar{A}\bar{B}) = P(\bar{A})P(\bar{B})$

Ex:- We are given two experiments:

The first experiment is the rolling of a fair die.

die.

$S_1 = \{f_1, f_2, \text{——}, f_6\}, \quad P_1(f_i) = \frac{1}{6}.$

The second experiment is the tossing of a fair coin.

$S_2 = \{h, t\}, \quad P_2(h) = P_2(t) = \frac{1}{2}$

$$(S_1, F_1, P_1) \quad \text{and} \quad (S_2, F_2, P_2)$$

We ~~perform~~ perform both experiments and we want to find the prob. that we get "$\underbrace{\text{two}}_{A}$" on the die and "$\underbrace{\text{head}}_{B}$" on the coin.

If we make reasonable assumption that the outcome of the first experiment are independent of the outcomes of the second, we conclude that the desired prob. $= \frac{1}{6} \times \frac{1}{2}$.

This conclusion is reasonable, however, the notion of independence does not agree with the def^n of independent events. In the def^n, the events A and B were subsets of the same sample space.

So in order to fit this conclusion into our theory, we must therefore, construct a space S having as subsets the events "two" and "head".

$$S = S_1 \times S_2 = \{f_1 h, f_2 h, \text{——}, f_6 h, f_1 t, f_2 t, \text{——} f_6 t\}$$

$S = S_1 \times S_2 = \{f_1h, f_2h, \text{---}, f_6h, f_1t, f_2t, \text{---}, f_6t\}$

In this space, {two} is not an elementary event but a subset consisting of two elements

$$\{\text{two}\} = \{f_2h, f_2t\} \in S$$

$A = \{\text{two}\} \in S_1$, $\{f_2h, f_2t\} = A \times S_2$

Similarly {heads} is an event with six elements.

$$\{\text{heads}\} = \{f_1h, f_2h, \text{---}, f_6h\} = S_1 \times B$$

$B = \{\text{heads}\} \in S_2$

Lecture-10

In Continuation of Lecture-9

To complete the experiment, we must assign probabilities to all subsets of S. Clearly the event A = {two} occurs if the die shows "two" no matter what shows on the coin.

Hence $P(A) = P\{\text{two}\} = P_1\{f_2\} = \frac{1}{6}$

$= P(A \times S_2)$

Similarly $P(B) = P\{\text{head}\}$

$= \frac{1}{2} = P(S_1 \times B)$

$-\frac{}{2} - 1(\ldots)$

The intersection of the event A and B is the elementary event $\{f_2 h\}$

$$(S, F, P) = (S_1 \times S_2,\ F_1 \times F_2,\ P')$$ se

$$A \in (S_1, F_1, P_1),\quad B \in (S_2, F_2, P_2)$$

$$A \cap B = (A \times S_2) \cap (S_1 \times B)$$

Assuming that the events A and B are independent in (S, F, P) in the sense of the def^n,

$$\Rightarrow P\{f_2 h\} = \frac{1}{6} \times \frac{1}{2}$$

$$= P(A \times S_2) \times P(S_1 \times B)$$

Cartesian Product of Two Experiments :-

Based on the discussion above, the cartesian product of two experiment S_1 and S_2 is a new experiment $S = S_1 \times S_2$ whose events are all cartesian product of the form

$$A \times B$$

where A is an event of S_1 and B is an event of S_2 and their unions and intersections.

In this experiment S, the probabilities of the events $A \times S_2$ and $S_1 \times B$ are s.t.

of the events $A \times S_2$ and $S_1 \times B$ are s.t.

$$P(A \times S_2) = P_1(A), \quad P(S_1 \times B) = P_2(B)$$

where $P_1(A)$ is the prob. of the event A in the experiment S_1 and $P_2(B)$ is the prob. of event B in S_2.

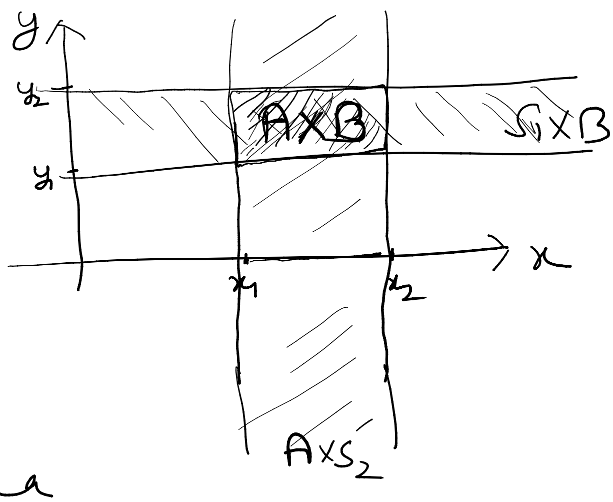
Ex:- Suppose $S_1 = x$ - axis

$S_2 = y$ - axis

$$A = \{x_1 \le x \le x_2\}, \quad B = \{y_1 \le y \le y_2\}$$

$A \times S_2$: Vertical strip

$S_1 \times B$: horizontal strip



$S = S_1 \times S_2$

$= xy$ - plane

$A \times B = (A \times S_2) \cap (S_1 \times B)$

In figure, rectangular area is $A \times B$.

$$P_1(A) = P(A \times S_2), \quad P_2(B) = P(S_1 \times B)$$

These equations determine the prob. of the events $A \times S_2$ and $S_1 \times B$. The prob. of events of the form $A \times B$ and of their union and intersections cannot determined in general in terms of P_1 and P_2. To determine them, we need additional information about experiments S_1 and S_2

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Independent Experiments :- In many applications the events $A \times S_2$ and $S_1 \times B$ of the experiment S are independent for any A and B.

$$P(A \times B) = P(A \times S_2) P(S_1 \times B) = P_1(A) P_2(B)$$

This complete the specification of the experiment S because all its events are union and intersections of the form $A \times B$.

(*) Repeat one experiment independently

$$(S, F, P) \longrightarrow (S \times S, F \times F, P')$$

$$S_1 = \{H, T\}$$

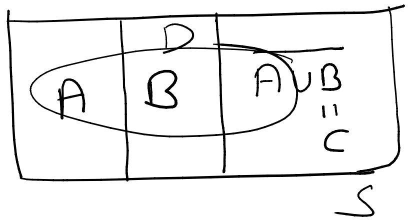
$$S_1 \times S_1 = \{HH, HT, TH, TT\}.$$

Ex :- Consider a prob. model (S, F, P).
Let $A \in F$ and $B \in F$ and both are mutually exclusive i.e. $AB = \phi$. Also $P(A) > 0$, $P(B) > 0$. What is the chance that A occurs before B if the model is repeated independently indefinitely.

Solⁿ:- $P\{A \text{ occurs before } B\} = ?$

D: event that A occurs before B.

Since the event $A, B, \overline{A \cup B} = C$ partition the prob. space, therefore conditioning on the outcome of the first experiment, we have, by Total prob. thm.



$$P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)$$

Clearly,

$$P(D|A) = 1$$

$$P(D|B) = 0$$

$$P(D|C) = P(D|\overline{A \cup B}) = P(D)$$

$\because$ The first experiment becomes irrelevant.

$$\Rightarrow P(D) = P(A) + P(D)\,P(C)$$

$$= P(A) + P(D)\left[1 - P(A \cup B)\right]$$

$$\Rightarrow P(D) = \frac{P(A)}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B)}$$

$\because AB = \phi$

$$\boxed{\frac{P(A)}{}}$$

P(A)

$$\boxed{P(D) = \frac{P(A)}{P(A)+P(B)}}$$

$$\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}} = \frac{1}{2}$$

<u>Another way:</u>

A ✓

B

C A ✓

C B

C C A ✓

C C B

C C C A ✓

|

D: event A occur before B if experiment is repeated independently infinitly.

Then D can occur in the following mutually exclusive ways:

A, CA, CCA, CCCA ———

$D = A \cup (CA) \cup (CCA) \cup (CCCA) - ——$

$\Rightarrow P(D) = P(A) + P(C)P(A) + P(C)P(C)P(A) + ——$

$= P(A)\left[1 + P(C) + P(C)P(C) + ——\right],$

$P(C) < 1$

$P(C) < 1$

$$= P(A) \cdot \frac{1}{1 - P(C)}$$

$$= \frac{P(A)}{1 - P(\overline{A \cup B})} = \frac{P(A)}{1 - [1 - P(A \cup B)]}$$

$$= \frac{P(A)}{P(A) + P(B)}$$