The LNM Institute of Information Technology
Jaipur, Rajsthan

P&S

Mid-Term Solution FEBRUARY 20, 2020

Time: 90 Minutes Maximum Marks: 25

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1. (a) The probability mass function $p(x)$ of a discrete random variable X satisfies $p(x+1) = \lambda p(x), x = 1, 2, 3, \cdots$, where $0 < \lambda < 1$. For positive integers m, n, find $P(X \geq m + n | X \geq m)$ **[2.5 Marks]**

Solution:

$$1 = \sum_{x=1}^{\infty} p(x) = p(1) + \lambda p(1) + \lambda p(2) + \cdots = p(1) + \lambda p(1) + \lambda^2 p(1) + \lambda^3 p(1) + \cdots$$

$$\implies 1 = p(1)\frac{1}{1-\lambda} \implies \lambda = 1 - p(1)$$

So X is Geometric random variable with parameter $p(1)$. By memoryless property of geometric random variable $P(X \geq m + n | X \geq m) = P(X \geq n) = [1 - p(1)]^{n-1} = \lambda^{n-1}$

(b) If X has the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{2} e^{-x}, & x > 0, \\ \\ 0, & \text{otherwise} \end{cases}$$

Then find variance of $\dfrac{1}{X}$. **[2.5 Marks]**

Solution:

$$E\left[\frac{1}{X}\right] = \int_0^{\infty} \frac{1}{x}\frac{x^2}{2} e^{-x} dx = \frac{1}{2}$$

$$E\left[\frac{1}{X^2}\right] = \int_0^{\infty} \frac{1}{x^2}\frac{x^2}{2} e^{-x} dx = \frac{1}{2}$$

$$Var\,(1/X) = 1/2 - 1/4 = 1/4.$$

2. Let $X \sim N(0,1)$ and $Y = X + |X|$. Find $E(Y^3)$. **[3 Marks]**

Solution:

$$\begin{aligned} E[Y^3] &= E[(X + |X|)^3] = E[X^3 + |X|^3 + 3X^2|X| + 3X|X|^2] \\ &= E[X^3 + |X|^3 + 3|X|^2 \cdot |X| + 3X \cdot X^2] \\ &= 4E[X^3 + |X|^3] \\ &= 4E[X^3] + 4E[|X|^3] = 4\sqrt{\frac{2}{\pi}} \end{aligned}$$

$$E[X^3] = \int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{2}} dx = 0$$

$$E[|X|^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^3 e^{-\frac{x^2}{2}} dx = 2\frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^3 e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u e^{-u} du = \sqrt{\frac{2}{\pi}}$$

3. The probability density function of a random variable X is

$$f(x) = \begin{cases} \dfrac{|x|}{2}, & -1 \leq x \leq 1, \\ \dfrac{3-x}{4}, & 1 < x \leq 3, \\ 0, & \text{otherwise} \end{cases}$$

Find the cumulative distribution function and the probability density function of $Y = |X|$. **[4 Marks]**

Solution: $g(x) = |x|$.

$$R(X) = [-1, 3] \implies R(Y) = [0, 3]$$

For $y \in (0, 1]$ there are two solutions $x_1 = y$ and $x_2 = -y$ of the equation $g(x) = y$. For $y \in (1, 3]$ we have unique solution $x = y$ of the equation $g(x) = y$. Therefore

$$f_Y(y) = \frac{|y|}{2} + \frac{|-y|}{2} = y, 0 < y \leq 1$$
$$f_Y(y) = \frac{3-y}{4}, 1 < y \leq 3$$
$$f_Y(y) = 0 \text{ otherwise}$$

Hence CDF of Y is given by

$$\begin{aligned} F_Y(y) &= 0, y \leq 0 \\ &= \frac{y^2}{2}, 0 < y \leq 1 \\ &= \frac{6y - y^2 - 1}{8}, 1 < y \leq 3 \\ &= 1, y > 3 \end{aligned}$$

4. In a probability space, let A, B and C and their complements be events such that: $P(A \cap B \cap C) = \frac{1}{16}$, $P(A \cap B^c \cap C) = \frac{5}{16}$, $P(A \cap B \cap C^c) = \frac{3}{16}$, $P(A \cap B^c \cap C^c) = \frac{2}{16}$, $P(A^c \cap B \cap C) = \frac{2}{16}$, $P(A^c \cap B \cap C^c) = \frac{1}{16}$, $P(A^c \cap B^c \cap C) = \frac{1}{16}$ and $P(A^c \cap B^c \cap C^c) = \frac{1}{16}$. Then

(a) determine the probabilities $P(A)$, $P(B)$ and $P(C)$. [3 Marks]

Solution: $A = (A \cap B \cap C) \cup (A \cap B^c \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C^c)$. Here $(A \cap B \cap C)$, $(A \cap B^c \cap C)$, $(A \cap B \cap C^c)$ and $(A \cap B^c \cap C^c)$ are mutually exclusive events, hence

$P(A) = P(A \cap B \cap C) + P(A \cap B^c \cap C) + P(A \cap B \cap C^c) + P(A \cap B^c \cap C^c) = \dfrac{1}{16} + \dfrac{5}{16} + \dfrac{3}{16} + \dfrac{2}{16} = \boxed{\dfrac{11}{16} = 0.6865}$.

Similarly

$P(B) = P(A \cap B \cap C) + P(A^c \cap B \cap C) + P(A \cap B \cap C^c) + P(A^c \cap B \cap C^c) = \dfrac{1}{16} + \dfrac{2}{16} + \dfrac{3}{16} + \dfrac{1}{16} = \boxed{\dfrac{7}{16} = 0.4375}$.

and

$P(C) = P(A \cap B \cap C) + P(A^c \cap B \cap C) + P(A \cap B^c \cap C) + P(A^c \cap B^c \cap C) = \dfrac{1}{16} + \dfrac{2}{16} + \dfrac{5}{16} + \dfrac{1}{16} = \boxed{\dfrac{9}{16} = 0.5625}$.

(b) determine whether or not the events A, B, and C are independent. [1 Marks]

Solution: $0.0625 = \dfrac{1}{16} = P(A \cap B \cap C) \neq P(A)P(B)P(C) = 0.1691$. Thus A, B and C are not independent.

5. (a) Consider the experiment of rolling a pair of fair 4-sided dice. Let $S = \{(i, j) : i, j = 1, 2, 3, 4\}$ be the sample space of this experiment. We define the event $A = \{\max(i, j) = m\}$ and event $B = \{\min(i, j) = 2\}$. Then determine the conditional probability $P(A|B)$. [2.5 Marks]

Solution: Here, the conditioning event $B = \{\min(i, j) = 2\}$ is having 5 elements namely $(2, 2), (2, 3), (2, 4), (3, 2), (4, 2)$. The events $A = \{\max(i, j) = m\}$ and $B = \{\min(i, j) = 2\}$ are having one element say $(2, 2)$ in common if $m = 2$. Similarly, if $m = 3$ or $m = 4$, events A and B have 2 elements in common between them. Thus, we have

$$P(A|B) = \begin{cases} 2/5, & \text{if } m = 3 \text{ or } m = 4, \\ 1/5, & \text{if } m = 2, \\ 0, & \text{if } m = 1. \end{cases}$$

(b) A student believes that he has a negligible chance of being late in the class and assume that his chance of being late in the class is only one in five hundred. Assuming that he arrives the class throughout all 300 working days in a year. Then what is the probability that he arrives late in the class on at least one day out of these 300 days? We also assume that each arrival to the class is independent from any other arrival to class. [2.5 Marks]

Solution: Given that each arrival to the class is independent from any other arrival to class. Also, probability of being late in the class $= 1/500 = 0.002$. Thus the probability of not being late $= 1 - 0.02 = 0.998$. Let $A_i : i = 1, 2, 3, \ldots, 300$ be the event that student is not late to attend the class on i-th day with $P(A_i) = 0.998$. Thus

$$\begin{aligned} P(\text{Student arrives late at least one day}) &= 1 - P(\text{No late on any day out of 300 days}) \\ & 1 - P\left(\bigcap_{i=1}^{300} A_i\right) \\ &= 1 - \prod_{i=1}^{300} P(A_i) \quad \text{(by independence)} \\ &= 1 - (0.998)^{300} \\ &\approx 1 - 0.5485 = \boxed{0.4515}. \end{aligned}$$

6. A bag contains 3 red and 4 white balls, a second bag contains 1 red and 5 white balls. A bag is selected at random. What is the probability that a ball drawn from this bag is white? If the ball is white what is the probability that the first bag was selected? [2+2 Marks]

Solution: Let R be an event that we draw a red ball, W that we draw a white ball, F that we select a first bag and S that we select the second bag. Then we have the following information

$$P(R|F) = \frac{3}{7}, \quad P(W|F) = \frac{4}{7}, \quad P(R|S) = \frac{1}{6}, \quad P(W|S) = \frac{5}{6}$$

Also we select bag at random, so each back is equally likely to get selected *i.e.*,

$$P(F) = P(S) = \frac{1}{2}.$$

By Total probability theorem, we have

$$P(W) = P(W|F)P(F) + P(W|S)P(S) = \frac{4}{7} \times \frac{1}{2} + \frac{5}{6} \times \frac{1}{2} = \boxed{\frac{59}{84}}.$$

Also, by Baye's formula, we have

$$P(F|W) = \frac{P(W|F)P(F)}{P(W|F)P(F) + P(W|S)P(S)} = \frac{4/7 \times 1/2}{59/84} = \boxed{\frac{24}{59}}.$$