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Unit - 03. lecture - 19.

Topic :- Bita and gamma function Properties of Beta f^n.

Definitions of Beta function :- The for positive value of m & n an improper integral.

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

i.e Called Bita f^n it is denoted by $\beta(m,n)$ where m & n are perimeter of the integrals it is also calle. **Euler's Integral of first kind.**

$\Rightarrow$ Thus, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \; m>0, n>0$

Gamma f^n :- for positive value of n an improper.

$$\int_0^\infty e^{-x}x^{n-1}\,dx.$$

Integral is Called Gamma f^n it is denoted by Γn — Gamma(n)

$\rightarrow$ where n is parameter of the integral it is also Called. Euler's integral of second kind.

* Properties of Beta f^n.

(1) Property - Symmetry
i.e. $\beta(m,n) = \beta(n,m)$

Proof :- By definition of Beta f^n. we have
$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx. \quad m>0, n>0, \sim (1)$

Put $1-x = y \Rightarrow -dx = dy$
$dx = -dy$.

x	0	1
y	1	0

(1) become to :-

$$\beta(m,n) \Rightarrow \int_1^0 (1-y)^{m-1}\, y^{n-1} (-dy)$$

$$\int_0^1 y^{n-1} (1-y)^{m-1}\, dy$$

$\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

$\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

$$\Rightarrow \int_0^1 y^{n-1}(1-y)^{m-1}\,dy.$$

$$\int_0^1 x^{n-1}(1-x)^{m-1}\,dx.$$ (By definate integral property)

$\Rightarrow$ $\beta(n,m)$ H.P.

(2) Property :- Trigonometric form of β funct^n.

i.e $\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$. $(m>0, n>0)$

Proof :- we have,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{--- (1)}$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$.

x	0	1
θ	0	$\pi/2$

(1) become to

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta.$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot \sin\theta \cos\theta\, d\theta.$$

$$\Rightarrow 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$$

(3) Property :- β function in the form of an improper integral of first kind.

$\rightarrow$ Prove that :- $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$, $(m>0, n>0)$

Proof :- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $\{m>0, n>0\}$ $\rightsquigarrow$ (1)

Put $x = \frac{1}{1+y} \Rightarrow dx = \frac{-1}{(1+y)^2}\,dy$

$y = 1/x - 1$

x	0	1
y	∞	0

(1) become to :

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \cdot \left(1-\frac{1}{1+y}\right)^{n-1} \left(-\frac{1}{(1+y)^2}\,dy\right)$$

$$\Rightarrow \int_\infty^0 \frac{1}{(1+y)^{m-1}} \left(\frac{1+y-1}{1+y}\right)^{n-1}$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \left(\frac{1+y-1}{1+y}\right)^{n-1} \left(\frac{1}{(1+y)^2}\right) dy.$$

$$\int_0^\infty \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}\,dy.$$

$$\Rightarrow \int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1+n-1+2}}$$

Using property of d.I

$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$ in $\rightsquigarrow$ (2)

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \rightsquigarrow (3)$$

we get. $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \rightsquigarrow$ (4)

By similarly $\beta(m,n) = \beta(n,m)$ $\rightsquigarrow$ (5)

from eq (4) & (5) we get.

$$\beta(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \rightsquigarrow (6)$$

* deduction :-

(1) from (3),(4) & (5) we have

$$\beta(m,n) = \beta(n,m) = \int_0^\infty \frac{x^{m-1}}{(1-x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

(2) Adding (3) & (4) we get.

$\Gamma(n+1) = n\Gamma(n)$ $(n>0)$

$\Gamma(n+1) = n!$ (n is positive integer)

$\beta(m,n) + \beta(n,m)$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx \rightsquigarrow (7)$$

from (6) & (7) we get.

$$\beta(m,n) + \beta(n,m) = \int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$\Rightarrow 2\beta(m,n) = \int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$\beta(m,n) = \frac{1}{2}\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+1}}\,dx.$$

$(m>0, n>0)$

Lecture-20

(4) **Property** :- Gamma f^n for zero and negative value (Γn for $n \le 0$)

$\Rightarrow$ Thus, Gamma 0 is undefined & negative integer are also undefined.

we have. $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$n=0;\ \Gamma(0) = \frac{\Gamma(0+1)}{0} = \frac{\Gamma(1)}{0} = \frac{1}{0} = \pm\infty$$

$$n=-1;\ \Gamma(-1) = \frac{\Gamma(-1+1)}{-1} = \frac{\Gamma(0)}{-1} = \frac{\pm\infty}{-1} = \pm\infty$$

$\Gamma(1) = 1$, $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

(5) property - Existing on gamma function.

$\Rightarrow$ Gamma function is defined for every value of. $n>0$ and negative fractional value.

Some other forms of Gamma function

(1) Prove that

$$\Gamma(n) = \int_0^1 \left(\log \tfrac{1}{y}\right)^{n-1} dy \quad , n>0.$$

Proof: we have.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \rightsquigarrow (1) \quad , n>0$$

put $x = \log \frac{1}{y} \Rightarrow \frac{1}{y} = e^{-x}$

$\Rightarrow y = e^{-x}$

$dy = -e^{-x}\,dx$; $-dy = e^{-x}\,dx$

(1) become to

$$\Gamma(n) = \int_1^0 \left(\log \tfrac{1}{y}\right)^{n-1}(-dy)$$

$$\Rightarrow -\int_1^0 \left(\log \tfrac{1}{y}\right)^{n-1} dy$$

$$\Gamma(n) = \int_0^1 \left(\log \tfrac{1}{y}\right)^{n-1} dy.$$

x	0	∞
y	1	0

(2) Prove that $\Gamma(n+1) = \int_0^\infty e^{-y^{1/n}}\,dy = n\Gamma(n)$, $n>0$.

Sol^n By definition of Gamma function

we have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \rightsquigarrow (1)$, $n>0$

Put $x = y^{1/n} \Rightarrow y = x^n$

$dy = n x^{n-1}\,dx$

$x^{n-1} = \frac{dy}{n}$

x	0	∞
y	0	∞

(1) become to $\Gamma(n) = \int_0^\infty e^{-y} \frac{dy}{n}$

$$\Rightarrow \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\,dy$$

$$\Rightarrow n\Gamma(n) = \int_0^\infty e^{-y^{1/n}}\,dy \rightsquigarrow (2)$$

we know that,

$$\Gamma(n+1) = n\Gamma(n) \rightsquigarrow (3)$$

from (2) & (3)

$$\Gamma(n+1) = n\Gamma(n) = \int_0^\infty e^{-y^{1/n}}\,dy$$

Q1. show that

$$\Gamma\left(\frac{m+1}{n}\right) = n a^{\frac{m+1}{n}} \int_0^\infty t^m e^{-at^n}\,dt \quad (m>0, n>0, a>0).$$

(2) Prove that; $\Gamma(n+1) = (m+n)^{n+1}(-1)^n \int_0^1 x^m (\log x)^n\,dy$ $(m>-1, n>-1)$

(7) show that $\int_0^1 x^{m-1} (\log 1/x)^{n-1} dx = \frac{\Gamma n}{n^m}$, $m>0, n>0$

(2) Proof - RHS : $(m+1)^{n+1}(-1)^n \int_0^1 x^m (\log x)^n dx$ ~> (1)

Put $\log x = -y \Rightarrow \frac{1}{x} dx = -dy$.

$dx = -x dy = -e^{-y} dy$

x	0	1
y	∞	0

RHS = $(m+1)^{n+1}(-1)^n \int_\infty^0 (-e^{-y})^m (-y)^n (-e^{-y} dy)$

$= (m+1)^{n+1}(-1)^n \int_0^\infty e^{-my} (-1)^n y^n e^{-y} dy$

$\Rightarrow (m+1)^{n+1}(-1)^n(-1)^n \int_0^\infty e^{-my-y} y^n dy$

$\Rightarrow (m+1)^{n+1} (-1)^{2n} \int_0^\infty e^{-(m+1)y} \cdot y^{n+1-1} \cdot dy$.

$(m+1)^{n+1} \cdot \frac{\Gamma n+1}{(m+1)^{n+1}}$

$\Gamma n+1 = LHS$.

LHS = RHS. hence proved.

(1). proof RHS $= n a^{\frac{m+1}{n}} \int_0^\infty t^m e^{-at^n} dt$ ~> (1)

Put $at^n = x$ in (1)

$\Rightarrow t^n = \frac{x}{a}$

$t = \left(\frac{x}{a}\right)^{1/n}$

$dt = \frac{1}{n}\left(\frac{x}{a}\right)^{1/n - 1} \cdot \frac{dx}{a}$

t	0	∞
x	0	∞

(1) become to

RHS = $n a^{\frac{m+1}{n}} \int_0^\infty e^{-x} \frac{1}{n} \left(\frac{x}{a}\right)^{\frac{1-n}{n}} \frac{dx}{a}$

$\therefore n a^{\frac{m+1}{n}} \int_0^\infty \frac{x^{m/n}}{a^{m/n}} \cdot \frac{e^{-x}}{n} \cdot \frac{x}{a^{1/n}} \frac{dx}{a}$

$\therefore n a^{\frac{m+1}{n}} \int_0^\infty \frac{x^{\frac{m}{n} + \frac{1-n}{n}} \cdot e^{-x}}{a^{m/n + 1} + \frac{1-n}{n} \cdot n}$

$\therefore n a^{\frac{m+1}{n}} \int_0^\infty \frac{x^{\left(\frac{m+1}{n} - 1\right)} e^{-x} \cdot dx}{n a^{\frac{m+1}{n}}}$

$n a^{\frac{m+1}{n}} \int_0^\infty x^{\left(\frac{m+1}{n} - 1\right)} e^{-x} dx$

$\frac{\Gamma m+1}{n} = LHS$

LHS = RHS hence proved.

Q. Prove that:

$\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$. $m>0, n>0$

sol^n proof we have

$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Put $x = \frac{ay}{b}$

$dx = \frac{a}{b} dy$

x	0	∞
y	0	∞

(1) become to, $\beta(m,n) = \int_0^\infty \frac{(ay/b)^{m-1}}{\left(1+\frac{ay}{b}\right)^{m+n}} \cdot \left(\frac{a}{b} dy\right)$

$\int_0^\infty \frac{\frac{a^{m-1} y^{m-1}}{b^{m-1}}}{\frac{(b+ay)^{m+n}}{b^{m+n}}} \times \left(\frac{a}{b} dy\right)$

$\int_0^\infty \frac{a^{m-1} y^{m-1}}{b^{m-1}} \times \frac{b^{m+n}}{(b+ay)^{m+n}} \times \left(\frac{a}{b} dy\right)$

$\int_0^\infty \frac{a^m y^{m-1} b^{m+n-m+1-1}}{(b+ay)^{m+n}} dy$

$\Rightarrow a^m b^n \int_0^\infty \frac{y^{m-1}}{(b+ay)^{m+n}} dy$

$a^m b^n \int_0^\infty \frac{x^{m-1}}{(b+ax)^{m+n}} dx$. [by using property of integr.] $\int_a^b f(x) dx = \int_a^b f(y) dy$.

Lecture :- 21

Relation b/w beta & Gamma f^n.

Prove that $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$, $m>0, n>0$

Proof; we have; $\dfrac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$

$$\Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1} dx \rightsquigarrow (1)$$

Multiplying both side by $z^{m-1} e^{-z}$ and integrating w respect to z from 0 to ∞ we get.

$$\Gamma n \int_0^\infty z^{m-1} e^{-z} dz = \int_0^\infty \left\{ z^n \int_0^\infty x^{n-1} e^{-zx} dx \right\} z^{m-1} e^{-z} dz$$

$$\Gamma n \, \Gamma m = \int_0^\infty \int_0^\infty \left[z^{m+n-1} e^{-zx} e^{-z} dx \right] dz.$$

$$\Gamma m \, \Gamma n = \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-(1+x)z} dx \right] dz.$$

changing the order

$$\int_0^\infty x^{n-1} \left[\int_0^\infty e^{-(1+x)z} z^{m+n-1} dz \right] dx.$$

$$\int_0^\infty \frac{\Gamma m+n}{(1+x)^{m+n}} x^{n-1} dx.$$

$$\Gamma m+n \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$

$$\Gamma m+n \;\; \beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Gamma formula :-

Prove that. $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma \frac{p+1}{2} \cdot \Gamma \frac{q+1}{2}}{2 \, \Gamma \frac{p+q+2}{2}}$

Proof :- we have.

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \rightsquigarrow (1)$$

and $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \rightsquigarrow (2)$

from (1) & (2) we get.

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta. \rightsquigarrow (3)$$

Putting $2m-1 = p \Rightarrow m = \frac{p+1}{2}$

$2n-1 = q \Rightarrow n = \frac{q+1}{2}$ in (3).

$$\frac{\Gamma \frac{p+1}{2} \; \Gamma \frac{q+1}{2}}{\Gamma \frac{p+q+2}{2}} = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta.$$

Imp $\Rightarrow \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma \frac{p+1}{2} \, \Gamma \frac{q+1}{2}}{2 \, \Gamma \frac{p+q+2}{2}}$

Prove that. $\Gamma \frac{1}{2} = \sqrt{\pi}$

Proof :- we have.

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma \frac{p+1}{2} \; \Gamma \frac{q+1}{2}}{2 \, \Gamma \frac{p+q+2}{2}} \rightsquigarrow (1)$$

Putting $p=0, q=0$ in (1) we get

$$\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta = \frac{\Gamma \frac{0+1}{2} \; \Gamma \frac{0+1}{2}}{2 \, \Gamma \frac{0+0+2}{2}}$$

$$\int_0^{\pi/2} d\theta = \frac{\Gamma \frac{1}{2} \, \Gamma \frac{1}{2}}{2 \, \Gamma \frac{2}{2}}$$

$$[\theta]_0^{\pi/2} = \frac{(\Gamma \frac{1}{2})^2}{2 \, \Gamma 1}$$

{ for negative $\Gamma n = \dfrac{\Gamma n+1}{n}$

$$\left[\frac{\pi}{2} - 0\right] = \left(\frac{\Gamma \frac{1}{2}}{\cancel{2}}\right)^2$$

$$2 \times \frac{\pi}{2} = \left(\Gamma \tfrac{1}{2}\right)^2$$

$$\Gamma \frac{1}{2} = \sqrt{\pi} \quad \text{hence prove}$$

Ex(1). Evaluate :- Γn for $n = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$

Sol^n we have, $\Gamma n+1 = n \Gamma n$

for $n = \frac{3}{2}$

$$\Gamma \tfrac{3}{2} = \Gamma \tfrac{1}{2}+1 = \tfrac{1}{2} \Gamma \tfrac{1}{2} = \frac{\sqrt{\pi}}{2}$$

similarly, $\Gamma\frac{5}{2} = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{2}\times\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$

similarly, $\Gamma\frac{7}{2} = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\frac{5}{2} = \frac{5}{2}\times\frac{3\sqrt{\pi}}{4} = \frac{15\sqrt{\pi}}{8}$

Imp

Ex-3 prove that

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} = \pi \operatorname{cosec} n\pi$$

Proof:- we have; $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \rightsquigarrow (1)$

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \rightsquigarrow (2)$$

from (1) & (2).

$$\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \rightsquigarrow (3)$$

Put $m+n=1 \Rightarrow m=1-n$ in (3), we get.

$$\frac{\Gamma(1-n)\,\Gamma n}{\Gamma 1} = \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx$$

$$\frac{\Gamma n\,\Gamma(1-n)}{1} = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \rightsquigarrow (4)$$

we know that by standard result

$$\int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx = \frac{\pi}{\sin n\pi} \rightsquigarrow (5)$$

from (4) & (5)

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} = \pi \operatorname{cosec} n\pi$$

Ex-4 Evaluate Γn for $n = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}$

Sol? we have,

$$\Gamma n = \frac{\Gamma(n+1)}{n}, \text{ for } n = -\frac{1}{2}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\frac{1}{2}}{-\frac{1}{2}} = \frac{\sqrt{\pi}}{-\frac{1}{2}} \Rightarrow -2\sqrt{\pi}$$

for $n = -\frac{3}{2}$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{-2\sqrt{\pi}}{-\frac{3}{2}} = \frac{4}{3}\sqrt{\pi}$$

for $n = -\frac{5}{2}$.

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = \frac{\Gamma\left(-\frac{3}{2}\right)}{-\frac{5}{2}} = \frac{\frac{4\sqrt{\pi}}{3}}{-\frac{5}{2}} \Rightarrow -\frac{8}{15}\sqrt{\pi}$$

for $n = -\frac{7}{2}$.

$$\Gamma\left(-\frac{7}{2}\right) = \frac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}} = \frac{\Gamma\left(-\frac{5}{2}\right)}{-\frac{7}{2}} = \frac{16}{105}\sqrt{\pi}$$

evaluate Γn for $n = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}$

$\frac{\pi}{\sin m\pi}$

(7) legendre's Duplication formula:-

$$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m, \quad m>0$$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta, \quad m>0,\ n>0 \rightsquigarrow (1)$$

Putting $n = \frac{1}{2}$

$$\frac{1}{2}\frac{\Gamma m\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)} = \int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta, \quad m>0,\ m>0 \rightsquigarrow (2)$$

Putting $n = m$

$$\left(\frac{(\Gamma m)^2}{2\Gamma 2m}\right) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\,d\theta$$

$$\Rightarrow \int_0^{\pi/2} \left(\frac{2}{2}\sin\theta\cos\theta\right)^{2m-1} d\theta$$

$$\Rightarrow \int_0^{\pi/2} \frac{(\sin 2\theta)^{2m-1}}{2^{2m-1}}\,d\theta \rightsquigarrow (3)$$

Putting $2\theta = \phi$

$$\frac{(\Gamma m)^2}{2\Gamma 2m} = \frac{1}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\,d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin(\phi)^{2m}\,\frac{d\phi}{2}$$

$$= \frac{1}{2^{2m-1}}\,\frac{1}{2}\times\int \sin^{2m-1}\phi\,d\phi$$

$\int \sin^{2m-1}\theta \, d\theta = \frac{2^{2m-1} \Gamma m^2}{2\Gamma 2m}, m>0$

Putting in eqⁿ ①

$\frac{1}{2}\frac{\Gamma m \Gamma \frac{1}{2}}{\Gamma m+\frac{1}{2}} = \frac{2^{2m-1}(\Gamma m)^2}{2\Gamma 2m}$, $\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$

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Q.1. Evaluate $\beta(\frac{5}{2}, \frac{3}{2})$?

Solⁿ: $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n} = \frac{\Gamma \frac{5}{2} \Gamma \frac{3}{2}}{\Gamma \frac{5}{2}+\frac{3}{2}}$

$\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{6} = \frac{\frac{3}{8}\pi}{6} = \frac{\pi}{16}$

2) Evaluate $\Gamma\frac{1}{4}\Gamma\frac{3}{4}$?

Solⁿ ∵ $\Gamma n \, \Gamma 1-n = \pi \operatorname{cosec} n\pi$

Putting $n = \frac{1}{4}$

$\Gamma\frac{1}{4}\Gamma\frac{3}{4} = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\,\pi$

Q.3. Evaluate $\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta$.

Solⁿ: $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$

$= \frac{\Gamma\frac{6+1}{2}\Gamma\frac{7+1}{2}}{2\Gamma\frac{6+7+2}{2}} = \frac{\Gamma\frac{7}{2}\Gamma 4}{2\Gamma\frac{15}{2}}$ ⇒ $\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot 6^3}{2\cdot\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$

⇒ $\frac{3}{\frac{9009}{16}} = \frac{16}{3003}$

Q.4) Evaluate $\int_0^{\pi/2} \sin^{10}\theta \, d\theta$?

Solⁿ ∴ $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$

$\frac{\Gamma\frac{10+1}{2}\Gamma\frac{0+1}{2}}{2\Gamma\frac{10+0+2}{2}}$ ⇒ $\frac{\Gamma\frac{11}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{12}{2}}$ ⇒ $\frac{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{2\,(5!)}$

⇒ $\frac{\frac{945\pi}{32}}{240} = \frac{945\pi}{240\times 32} = \frac{63\pi}{512}$

Q5 Evaluate $\int_0^{\pi/2} \cos^9\theta \, d\theta$?

Solⁿ As we know. $\frac{\Gamma\frac{0+1}{2}\Gamma\frac{9+1}{2}}{2\Gamma\frac{0+9+2}{2}} = \frac{\Gamma\frac{1}{2}\Gamma\frac{10}{2}}{2\Gamma\frac{11}{2}} = \frac{\sqrt{\pi}\,(4!)}{2\Gamma\frac{11}{2}}$

⇒ $\frac{\sqrt{\pi}\,(4\times3\times2\times1)}{2(\frac{9}{2})(\frac{7}{2})(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\sqrt{\pi}}$

⇒ $\frac{12}{\frac{315}{32}} = \frac{128}{315}$

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6) Evaluate $\int_0^{\pi/2} \left(\frac{\sqrt[3]{\sin^8 x}}{\sqrt{\cos x}}\right) dx$?

Solⁿ $m = \frac{8}{3}$, $n = -\frac{1}{2}$

$= \frac{\Gamma\frac{\frac{8}{3}+1}{2}\Gamma\frac{-\frac{1}{2}+1}{2}}{2\Gamma\frac{\frac{8}{3}+2-\frac{1}{2}}{2}} = \frac{\Gamma\frac{11}{6}\Gamma\frac{1}{4}}{2\Gamma\frac{25}{12}}$

$= \frac{\frac{5}{6}\Gamma\frac{5}{6}\times\Gamma\frac{1}{4}}{2\cdot\frac{13}{12}\cdot\frac{1}{12}\Gamma\frac{1}{12}} = \frac{60\,\Gamma\frac{5}{6}\times\Gamma\frac{1}{4}}{13\,\Gamma\frac{1}{12}}$

Q.7) Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$. $\{m = \frac{1}{2}, n = -\frac{1}{2}$; $m = -\frac{1}{2}, n = \frac{1}{2}\}$

Q.8) " $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$.

Solⁿ $\frac{\Gamma\frac{\frac{1}{2}+1}{2}\Gamma\frac{-\frac{1}{2}+1}{2}}{2\Gamma\frac{\frac{1}{2}-\frac{1}{2}+2}{2}} = \frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{2\Gamma 1} = \frac{\Gamma 1-\frac{1}{4}\,\Gamma\frac{1}{4}}{2}$

$\frac{1}{2}\frac{\pi}{\sin\frac{1}{4}\pi} = \frac{\pi}{2(\frac{1}{\sqrt{2}})}$

Ex 9) Evaluate $\int_0^{\infty} \frac{x^2}{(1+x^4)^3}$ $\{A - Bx^n = Bx^n = A\sin^2\theta / A\cos^2\theta$; $A + Bx^n = Bx^n = A\tan^2\theta / A\cot^2\theta$

Solⁿ let $x^4 = \tan^2\theta$

$x = (\tan\theta)^{1/2}$

$dx = \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta \, d\theta$.

$\int_0^{\infty} \frac{x^2}{(1+x^4)^3} = \int_0^{\pi/2} \frac{\tan\theta}{(\sec^2\theta)^3} \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta \, d\theta$ $\left\{ \begin{array}{c|c|c} x = & 0 & \infty \\ \tan\theta = x^2 & 0 & \infty \\ \theta & 0 & \pi/2 \end{array} \right\}$

⇒ $\frac{1}{2}\int_0^{\pi/2} \frac{\tan\theta^{1/2}}{\sec\theta^4} d\theta$

⇒ $\frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, (\cos^4\theta) \, d\theta$.

⇒ $\frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^{7/2} d\theta$

$\frac{\frac{1}{2}\Gamma\frac{\frac{1}{2}+1}{2}\Gamma\frac{\frac{7}{2}+1}{2}}{2\Gamma\frac{\frac{1}{2}+\frac{7}{2}+2}{2}} = \frac{\Gamma\frac{3}{4}\Gamma\frac{9}{4}}{4\Gamma 3}$

⇒ $\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}\cdot\frac{5}{4}\cdot\frac{1}{4}}{4\times 2!}$ ⇒ $\frac{5}{128}\Gamma\frac{1}{4}\Gamma 1-\frac{1}{4}$

⇒ $\frac{5}{128}\frac{\pi}{\sin\frac{1}{4}\pi}$ ⇒ $\frac{5\sqrt{2}\,\pi}{128}$

Ex-10 Evaluate $\int_0^{\infty} \frac{dx}{1+x^4}$

Q.13) Prove that $\frac{(\Gamma\frac{1}{3})^2}{\Gamma\frac{1}{6}} = \frac{\sqrt{\pi}}{\sqrt{3}} 2^{1/3}$?

Q 11. evaluate $\int \frac{x\,dx}{1+x^6}$

Q.12 evaluate $\int_0^{\pi/6} \sin^2 6\theta \cos^4 3\theta \, d\theta$

Q14. Evaluate a) $\int_0^\infty \sin x^2\,dx$ b) $\int_0^\infty x\cos x^3\,dx$.

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Euler's Theorm $e^{i\theta} = \cos\theta + i\sin\theta$.

$$e^{-ix^2} = \cos x^2 - i\sin x^2 \quad , x=y$$

Im part $e^{-ix^2} = -\sin x^2$

$$-Ie^{-ix^2} = \sin x^2$$

$$i = e^{i\pi/2} = 0$$

(ex) Evaluate $\int_0^\infty \frac{dz}{1+x^4}$; $x^4 = \tan^2\theta$

$x = (\tan\theta)^{1/2}$ + $dx = \frac{1}{2}$

$\tan^{-1/2}\theta \sec^2\theta\, d\theta$.

$$\frac{1}{2}\int_0^{\pi/2} \frac{\tan^{-1/2}\theta}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}\,d\theta \Rightarrow \frac{1}{2}\int_0^{\pi/2} \tan^{-1/2}\theta\, d\theta.$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} \frac{1}{\tan^{1/2}\theta}\, d\theta.$$

$$\frac{1}{2}\int_0^{\pi/2} \cot^{1/2}\theta\, d\theta \Rightarrow \frac{1}{2}\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$$

$$\frac{1}{2}\int_0^{\pi/2} \Rightarrow \frac{1}{2}\times\frac{\pi}{\sqrt{2}} \Rightarrow \frac{\pi}{2\sqrt{2}} \Rightarrow \frac{\sqrt{2}\pi}{4}$$

(b) Evaluate $\int_0^\infty \frac{x\,dx}{1+x^6}$

let $x^6 = \tan^2\theta$

$x = \tan^{1/3}\theta$.

$dx = \frac{1}{3}\tan^{-2/3}\theta\sec^2\theta\, d\theta$.

$$\frac{1}{3}\int_0^{\pi/2} \frac{\tan^{1/3}\theta}{\sec^2\theta}\tan^{-2/3}\theta\, d\theta$$

$$\frac{1}{3}\int_0^{\pi/2} \sin^{-1/3}\theta\cos^{1/3}\theta\, d\theta.$$

$$\frac{1}{3}\Gamma\tfrac{1}{3}\,\Gamma\tfrac{2}{3} \Rightarrow \frac{1}{3}\times\frac{2\pi}{\sqrt{3}}\times\frac{1}{2}$$

$$\Rightarrow \frac{\pi}{3\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} = \frac{\pi\sqrt{3}}{9}$$ Ans

other

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β, Γ function

Γ-function

$$\int_0^\infty e^{-x}x^{n-1}\,dx = \Gamma n$$

$$\Rightarrow \Gamma n = (n-1)!$$

∴ $\Gamma 0 = \infty$ (Gamma)

$\Gamma 1 = 1$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

formulas $\Gamma n = \frac{\Gamma n+1}{n}$

$$\int_0^\infty e^{-x}x^n\,dx = \Gamma n+1.$$

$$\Gamma n = (n-1)\Gamma n-1$$

how to solve. Γn.

Ist i) if n is positive integer.

eg- $\Gamma 5$

formula = $(n-1)!$

$\Gamma 5 = (5-1)!$

$= (4)!$

$4\times3\times2\times1$

$= 24$

eg- $\Gamma 6 = 5!$

$5\times4\times3\times2\times1 \Rightarrow 120$.

eg = $\Gamma 7 = 6!$

$\Rightarrow 6\times5\times4\times3\times2\times1 = 720$

ii) if n is negative integer.

eg = $\Gamma -4 =$

formula = $\Gamma n = \frac{\Gamma n+1}{n}$

$$\Rightarrow \frac{\Gamma -4+1}{-4} = \frac{\Gamma -3}{-4} \Rightarrow \frac{\Gamma -3+1}{-4(-3)} = \frac{\Gamma -2}{12}$$

Again

again. $\Rightarrow \frac{\Gamma -2+1}{12(-2)} = \frac{\Gamma -1}{-24} \Rightarrow \frac{\Gamma -1+1}{-24(-1)}$

$\Rightarrow \frac{\Gamma 0}{24} \Rightarrow \boxed{\infty}$

यदि Γ के अंदर तक value positive nahi आ जाती तब तक Gamma value करते रहेंगे

B.

eg ⇒ $\Gamma\frac{5}{2}$

$(n-1)\Gamma\,n-1$

$\left(\frac{5}{2}-1\right)\Gamma\,\frac{5}{2}-1$

$\frac{3}{2}\,\Gamma\frac{3}{2}$ → again

$\frac{3}{2}\left(\frac{3}{2}-1\right)\Gamma\,\frac{3}{2}-1$

⇒ $\frac{3}{2}\left(\frac{1}{2}\right)\Gamma\frac{1}{2}$ ⇒ $\sqrt{\pi}$

⇒ $\frac{3}{2}\left(\frac{1}{2}\right)\left(\sqrt{\pi}\right)$

$\frac{3}{4}\sqrt{\pi}$ or $\frac{3}{4}\Gamma\frac{1}{2}$

(2) evaluation $\int_0^{\pi/6} \sin^2 6\theta \cos^4 3\theta \, d\theta$

$\int_0^{\pi/2} \sin^2 6\theta \cos^4 3\theta \, d\theta$

let $3\theta = x$ ⇒ $3d\theta = dx$.

$\frac{1}{3}\int_0^{\pi/2} \sin^2 2t \cos^4 t \, dt$.

$\frac{1}{3}\int_0^{\pi/2} (2\sin t \cos t)^2 \cos^4 t \, dt$.

$\frac{4}{3}\int_0^{\pi/2} \sin^2 t \cos^2 t \, dt$.

$\frac{4}{3}\left[\frac{\Gamma\frac{2+1}{2}\,\Gamma\frac{6+1}{2}}{2\,\Gamma\frac{2+6+2}{2}}\right]$ ⇒

⇒ $\frac{4}{3}\left[\frac{\Gamma\frac{3}{2}\,\Gamma\frac{7}{2}}{2\,\Gamma 5}\right]$

$\frac{4}{3}\left[\frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\times4\times3\times2}\right]$

$\frac{4}{3}\left[\frac{15\pi}{16\times4\times4\times3}\right]$ ⇒ $\frac{4}{3}\left(\frac{5\pi}{256}\right)$

⇒ $\frac{20\pi}{768}$ ⇒ $\frac{5\pi}{192}$

Surface of Solid of Revolution

:- When a curve is rotated around a fixed straight line lying on plane, a surface is generated then it is called surface of revolution.

* Fixed line is called Axis of revolution.
* Rotating curve is generator of curve.

Area of surface of solid Revolution:-

1) When the curve is in Cartesian form -

(i) Axis of Revolution about x-axis

$$S = \int_A^B 2\pi y\,ds = 2\pi\int_{x_1}^{x_2} f(x)\frac{ds}{dx}\,dx.$$

s = length of arc of a curve

here, $ds^2 = dx^2 + dy^2$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+[f'(x)]^2}$$

(ii) Axis of Revolution about y-axis.

$$S = \int_A^B 2\pi x\,ds = 2\pi\int_{y_1}^{y_2} x\frac{ds}{dx}\,dx.$$

If the eqn of curve $x = f(y)$

$$S = \int_A^B 2\pi x\,ds = 2\pi\int_{y_1}^{y_2} f(y)\frac{ds}{dy}\,dy.$$

(iii) Revolution about any line.

Revolution of arc $y = f(x)$ b/w the point $A(x_1, y_1)$ and $B(x_2, y_2)$ about any given line -

$$S = \int_A^B 2\pi\, PM\,ds = 2\pi\int_{x_1}^{x_2} PM\frac{ds}{dx}\,dx,$$

where PM = length of $\perp$ on line L.

If $x = f(y)$

$$S = \int_A^B 2\pi\, PM\,ds = 2\pi\int_{y_1}^{y_2} PM\frac{ds}{dy}\,dy$$

2). When the curve is in parametric form:-

i) Axis of Revolution about x-axis.

$x = x(t),\ y = y(t)$

$$S = \int_A^B 2\pi y\,ds = 2\pi\int_{t_1}^{t_2} y(t)\frac{ds}{dt}\,dt.$$

$ds^2 = dx^2 + dy^2$

$$\frac{ds}{dt} = \sqrt{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}} = \sqrt{\{x'(t)\}^2+\{y'(t)\}^2}$$

ii) Axis of Revolution about y-axis:-

$$S = \int_A^B 2\pi x\,ds = 2\pi\int_{t_1}^{t_2} x(t)\frac{ds}{dt}\,dt.$$

(iii) Revolution about any line.

$$S = \int_A^B 2\pi\, PM\,ds = 2\pi\int_{t_1}^{t_2} PM\frac{ds}{dt}\,dt.$$

PM = length of $\perp$ on the line.

2 When the curve is in polar form $r = f(\theta)$, $\theta = f(r)$

(i) Revolution about initial line $\theta = 0$ (x-axis)

$$S = \int_A^B 2\pi y\,ds = 2\pi\int_{\theta_1}^{\theta_2} r\sin\theta\frac{ds}{d\theta}\,d\theta.$$

$ds^2 = dr^2 + (r\,d\theta)^2$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2},\quad \frac{ds}{dr} = \sqrt{1+\left(\frac{d\theta}{dr}\right)^2}$$

(ii) Revolution about line through pole, $\perp$ to initial line about line $\theta = \pi/2$ (y-axis)

$$S = \int_A^B 2\pi x\,ds = 2\pi\int_{\theta_1}^{\theta_2} r\cos\theta\frac{ds}{d\theta}\,d\theta$$

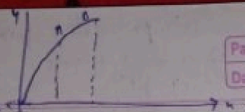
(iii) Revolution about Any line:-

$A(r_1, \theta_1)$ $B(r_2, \theta_2)$

$$S = \int_A^B 2\pi\, PM\,ds = 2\pi\int_{\theta_1}^{\theta_2} PM\frac{ds}{d\theta}\,d\theta.$$

Example 1) The Arc of parabola $y^2 = 4x$ b/w $x = 1/2$ and $x = 1$ is rotated about x-axis find area of surface generated?

Sol^n $y^2 = 4x$
$y = 2\sqrt{x}$
diff. $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$.



$S = \int_A^B 2\pi y\, ds = 2\pi \int_{1/2}^{1} y \frac{ds}{dx} dx$

$\Rightarrow 2\pi \int_{1/2}^{1} y\sqrt{1+\left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_{1/2}^{1} 2\sqrt{x}\sqrt{1+\frac{1}{x}}\, dx$.

$2\pi(2) \int_{1/2}^{1} \sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_{1/2}^{1} \sqrt{x+1}\, dx$.

$4\pi \left[\frac{(x+1)^{3/2}}{3/2}\right]_{1/2}^{1} = 4\pi \left[\frac{2}{3}(2\sqrt{2}) - \frac{2}{3}\left(\frac{3}{2}\sqrt{\frac{3}{2}}\right)\right]$

$4\pi\left[\frac{4\sqrt{2}}{3} - \frac{\sqrt{3}\sqrt{3}}{\sqrt{2}\sqrt{2}}\right] = 4\pi\left[\frac{4\sqrt{2}}{3} - \frac{\sqrt{6}}{2}\right] \Rightarrow 4\pi\left[\frac{2\times 4\sqrt{2} - 3\sqrt{6}}{6}\right]$

$\Rightarrow 2\pi\left[\frac{8\sqrt{2} - 3\sqrt{6}}{3}\right] \Rightarrow \frac{2\pi}{3}(8\sqrt{2} - 3\sqrt{6})$

Example 2:- The portion of Curve $y = \log x$ for $1 \le x \le e$ is rotated about y-axis. Find area of surface generated?

sol^n $y = \log x$
$\frac{dy}{dx} = \frac{1}{x}$;
$S = \int_A^B 2\pi x\, ds = 2\pi \int_{x_1}^{x_2} x \frac{ds}{dx} dx$.
$2\pi \int_1^e x \frac{\sqrt{x^2+1}}{x} dx = 2\pi \int \sqrt{1+x^2}\, dx$
$2\pi \left[\frac{1}{2} x\sqrt{1+x^2} + \frac{1}{2}\log|x+\sqrt{1+x^2}|\right]_1^e$
$\pi\left[e\sqrt{1+e^2} + \log|e+\sqrt{1+e^2}| - \sqrt{1+1} - \log|1+\sqrt{1+1}|\right]$
$\Rightarrow \pi\left[e\sqrt{1+e^2} + \log(e+\sqrt{1+e^2}) - \sqrt{2} - \log(1+\sqrt{2})\right]$

Example 3 The spherical surface with unit radius is generated by rotating Curve $x = \cos t$, $y = \sin t$ about x-axis where $t \in [0, \pi]$ find area?

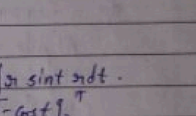
sol^n $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$

$\Rightarrow \left(\frac{ds}{dt}\right)^2 = (-\sin t)^2 + (\cos t)^2$
$\Rightarrow 1$

Area of spherical surface.
$S = \int_A^B 2\pi y\, ds = 2\pi \int_0^{\pi} \sin t\, dt$
$2\pi[-\cos t]_0^{\pi}$
$2\pi[-\cos\pi + \cos 0]$
$\Rightarrow \boxed{4\pi}$

Remark:- If spherical surface has radius r, it Can be generated the rotating Curve $x = r\cos t$, $y = r\sin t$, $0 \le t \le \pi$.
$(ds/dt)^2 = r^2\sin^2 t + r^2\cos^2 t$
$ds = r\,dt$
$S = \int_A^B 2\pi y\, ds = 2\pi \int r\sin t\, r\,dt$.
$2\pi r^2[-\cos t]_0^{\pi}$
$\Rightarrow 4\pi r^2$

Ex-4 Prove that Surface of oblate spheroid form by revolution of ellipse of semi-Major axis a and eccentricity e is $2\pi a^2\left[1 + \frac{1-e^2}{2e}\log\left(\frac{1+e}{1-e}\right)\right]$?



Sol^n Eq^n of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;
where $b = a\sqrt{1-e^2}$
$x = a\cos t$, $y = b\cos t$
$\left(\frac{ds}{dt}\right)^2 = (dx/dt)^2 + (dy/dt)^2$
$a^2\sin^2 t + b^2\cos^2 t \Rightarrow a^2\cancel{\sin^2 t} + a^2(1-\cos^2 t)$
$a^2(1 - e^2\cos^2 t)$
$ds/dt = a\sqrt{1-e^2\cos^2 t}$

here, ellipse is symmetrical to both axis. t varies form $0-\pi/2$
Req. Area $= 2\int_A^B 2\pi x\, ds$
$2\times 2\pi \int_0^{\pi/2} a\cos t \cdot a\sqrt{1-e^2\cos^2 t}\, dt$.
$\Rightarrow 4\pi a^2 \int_0^{\pi/2} \cos t\sqrt{1-e^2\cos^2 t}\, dt$
$\Rightarrow 4\pi a^2 \int \sqrt{(1-e^2)+z^2}\frac{dz}{e}$ $\left\{\begin{array}{l} e\sin t = z \\ dz = e\cos t\, dt \end{array}\right\}$
$\Rightarrow \frac{4\pi a^2}{e}\int \sqrt{c^2+z^2}\, dz$ $\{c = \sqrt{1-e^2}\}$
$\Rightarrow \frac{4\pi a^2}{e}\left[\frac{z}{2}\sqrt{c^2+z^2} + \frac{c^2}{2}\log|z+\sqrt{c^2+z^2}|\right]_0^e$
$= \frac{4\pi a^2}{2e}\left[z\sqrt{1-e^2+z^2} + (1-e^2)\log|z+\sqrt{1-e^2+z^2}|\right]_0^e$

Volume of solid of Revolution

1) when the Curve is given in Cartesian form :-

(i) Revolution about x-axis :-

$$V = \int_A^B \pi y^2\,dx = \pi\int_{x_1}^{x_2} [f(x)]^2\,dx$$

elementry Vol^m = πy^2

(ii) Revolution about y-axis :-

$$V = \int_A^B \pi x^2\,dy = \pi\int_{x_1}^{x_2} x^2 \frac{dy}{dx}\,dx = \pi\int_{x_1}^{x_2} x^2 f'(x)\,dx.$$

(iii) Revolution about any line ... [which is one side of area of revolution]

$$V = \int_A^B \pi (PM)^2\,ds = \pi\int_{x_1}^{x_2} (PM)^2 \frac{ds}{dx}\,dx$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

(iv) Revolution about any line ... [not a side of area of Revolution]

$$V = \int_A^B \pi (h^2 - k^2)\,ds = \pi\int_{x_1}^{x_2} (h^2 - k^2)\frac{ds}{dx}\,dx.$$

(v) Rev about line $y=k$ (Parallel to x-axis)

$$V = \int_A^B \pi (y-k)^2\,dx = \pi\int_{x_1}^{x_2} [f(x)-k]^2\,dx \quad \{ y = f(x)$$

(vi) Rev about line $x=h$ (Parallel to y-axis)

$$V = \int_A^B \pi (x-h)^2\,dy = \pi\int_{x_1}^{x_2} (x-h)^2 \frac{dy}{dx}\,dx.$$

$$\pi\int (x-h)^2 f'(x)\,dx$$

2) when the Curve is given in parametric Form :-

$x = x(t)$, $y = y(t)$

(i) Rev about x-axis :-

$$V = \int_A^B \pi y^2\,dx = \pi\int_{t_1}^{t_2} [y(t)]^2 \frac{dx}{dt}\,dt$$

(ii) Rev about y-axis :-

$$V = \int_A^B \pi x^2\,dy = \pi\int_{t_1}^{t_2} [x(t)]^2 \frac{dy}{dt}\,dt$$

(iii) Rev. about any line :-

$$V = \int_A^B \pi (PM)^2\,ds = \pi\int_{t_1}^{t_2} (PM)^2 \frac{ds}{dt}\,dt$$

3) when the Curve is in polar form :-

$r = f(\theta)$ or $F(r,\theta) = 0$;

(i) Revolution about initial line $\theta = 0$ (x-axis)

$$V = \int_A^B \pi (r\sin\theta)^2\,d(r\cos\theta)$$

$$\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin\theta\,d\theta.$$

(ii) Rev. about line through pole --- About $\theta = \pi/2$.

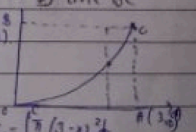
$$V = \int_A^B \pi (r\cos\theta)^2\,d(r\sin\theta)$$

$$\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \cos\theta\,d\theta$$

(iii) Rev. about any line $\theta = \alpha$;

Replace $\theta \rightarrow \theta + \alpha$; in whole System.

Ex-1 A(3,0) be a point on x-axis B(0,9) point on y-axis and C(3,9) be a point on parabola $y = x^2$ find Vol^m of solid of revolution generated when area OAC bounded by $y = x^2$, $x = 3$ and x-axis is rotated about a) x-axis b) y-axis c) line AC d) line OC



Sol) Given curve $y = x^2 \rightsquigarrow (1)$

a) About x-axis -

$$\int_O^A \pi y^2\,dx.$$

$$\pi\int_0^3 [x^2]^2\,dx \quad \text{by } (1).$$

$$\pi\left[\frac{x^5}{5}\right]_0^3 = \frac{243\pi}{5}$$

b) $$\int_A^C \pi (3^2 - x^2)\,dy = \pi\int_0^9 (9-y)\,dy$$

$$\pi\left[9y - \frac{y^2}{2}\right] = \pi\left(\frac{81}{2}\right)$$

c) About line AC $= \int_A^C \pi (3-x)^2\,dy$

$$\Rightarrow \pi\int_0^9 (3-\sqrt{y})^2\,dy \quad \text{by } (1)$$

$$\Rightarrow \pi\int_0^9 (9 - 6\sqrt{y} + y)\,dy \Rightarrow \pi\left[9y - \frac{6y^{3/2}}{3/2} + \frac{y^2}{2}\right]_0^9$$

$$\Rightarrow \pi\left[81 - 4(27) + \frac{81}{2}\right]$$

$$\Rightarrow 27\pi\left[3 - 4 + \tfrac{3}{2}\right] = \frac{27\pi}{2}$$

About line BC

$$= \int_0^3 \pi\left[9^2 - (9-y)^2\right]dx = \int_0^3 \pi\left[9^2 - (9-x^2)^2\right]dx$$

$$\pi\int_0^3\left[9^2 - (9^2 - 18x^2 + x^4)\right]dx = \pi\int_0^3\left[18x^2 - x^4\right]dx$$

$$\Rightarrow \pi\left[18\frac{x^3}{3} - \frac{x^5}{5}\right]_0^3 \Rightarrow \pi\left[6\times 3^3 - \frac{3^5}{5}\right]$$

$$\Rightarrow \frac{567}{5}\pi$$

Example 2:- Determine Vol^m of solid generated by revolving plane-4 about line $x = 4$?

Sol^n Given. Curve is $y^2 = 4x \rightsquigarrow$ ①

Req. Vol^m of solid

$$\Rightarrow \int_{-4}^{4} \pi(4-x)^2\,dy$$

$$\pi\int_{-4}^{4}\left(4 - \frac{y^2}{4}\right)^2 dy \longrightarrow \text{by ①}$$

$$\pi\int_{-4}^{4}\left(16 - 2y^2 + \frac{y^4}{16}\right)dy$$

$$2\pi\int_0^4\left(16 - 2y^2 + \frac{y^4}{16}\right)dy$$

$$2\pi\left[16y - \frac{2y^3}{3} + \frac{y^5}{16\times 5}\right]_0^4 \Rightarrow 2\pi\left[1-\right.$$

$$2\pi(4^3)\left[1 - \frac{2}{3} + \frac{1}{5}\right] \Rightarrow \frac{1024}{15}\pi$$

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$x + 2y + 3z = 12$, $x = 1$, $y = 1$, $x = 1$, $y = 1$